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Large Time Behavior of the Vorticity of Two-dimensional Viscous Flow and its Application to Vortex Formation

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Abstract

We consider the Cauchy problem for the two-dimensional vorticity equation. We show that the solution ω behaves like a constant multiple of the Gauss kernel having the same total vorticity as time tends to infinity. No particular structure of initial data $\omega_0 = \omega(x,0)$ is assumed except the restriction that the Reynolds number $R = \int|\omega_0|dx/\nu$ is small, where ν is the kinematic viscosity. Applying a time-dependent scale transformation, we show a stability of Burgers' vortex, which physically implies formation of a concentrated vortex.

1. Introduction

This paper studies the large time asymptotic behavior of the vorticity distribution of two-dimensional viscous incompressible flow. We consider the two-dimensional vorticity equation

$$\frac{\partial\omega}{\partial t} - \nu\Delta\omega + (v\cdot\nabla)\omega = 0 , \quad v = K*\omega \tag{1}$$

which is known to be equivalent to the Navier-Stokes equations. Here $\omega = \omega(x,t)$ and $v = (v^1(x,t),v^2(x,t))$ represent the scalar vorticity distribution and the velocity field, respectively ; $\nu > 0$ is the kinematic viscosity and $v\cdot\nabla = \sum_{j=1}^{2} v^j\partial/\partial x_j$. The second equation involving the convolution $*$ is called the Biot-Savart law. Its explicit form is

$$v(x,t) = \int_{\mathbb{R}^2} K(x-y)\omega(y,t)dy \ ,$$

where K is the vector function

$$K(x_1,x_2) = (-x_2,x_1)/2\pi|x|^2 \quad , \quad x = (x_1,x_2) \ .$$

There is a special solution to (1) called Oseen's vortex [17]

$$(2) \qquad \omega_{*\kappa}(x,t) = \frac{\kappa}{4\pi\nu t} \exp(-\frac{|x|^2}{4\nu t}) \qquad (\kappa : \text{real number}) \ ,$$

where κ is the strength of the vortex. Since $\omega_{*\kappa}$ is a Gauss kernel, $\omega_{*\kappa}$ is regarded as a solution of (1) with the initial data $\omega_{*\kappa}(x,0) = \kappa\delta(x)$ where $\delta(x)$ is Dirac's delta function.

The main goal of this paper is to show that even if we start with a general initial vorticity distribution ω_0 the solution ω of (1) behaves like the above special solution $\omega_{*\kappa}$ as $t \to \infty$ with the total vorticity $\kappa = \int\omega_0 dx$ provided that the Reynolds number $R = \int|\omega_0|dx/\nu$ is sufficiently small. In fact, we prove

$$(3) \qquad \|\omega-\omega_{*\kappa}\|_p(t) \leq C(\nu t)^{-1+1/p-\delta} \quad t > 0 \ , \ 1 \leq p \leq \infty \ .$$

for $0 < \delta < 1/2$ provided that R is sufficiently small and that $\int|\omega_0|(|x|^2+1)dx$ is finite where C is a constant independent of t and ν ; $\|\cdot\|_p$ denotes the L^p-norm in space variables. Since $\|\omega_{*\kappa}\|_p(t) = C_p\kappa(\nu t)^{-1+1/p}$, (C_p:constant dependig only on p), our estimate (3) gives an asymptotic expression of ω as $t \to \infty$. No particular structure of initial vorticity ω_0 is assumed.

As an application of (3) we show a stability of Burgers' vortex [5], which physically implies formation of a concentrated vortex. We consider a three-dimensional viscous incompressible flow expressed as a superposition of two flows - an axisymmetric irrotational flow and a two-dimensional flow the vorticity of which directs to the symmetry axis. The axisymmetric flow is assumed to have an inward convection and axially stretching flow which is an incompressible flow with constant rate of strain. We show that the vorticity field tends to its equilibrium state called Burgers' vortex as the time tends to infinity, provided that Reynolds number R of the rotational part is sufficiently small. In fact the three-dimensional vorticity equations can be transformed to (1) by a time-dependent scale transformation due to Kambe [11,13] and Lundgren [16]. Such an asymptotic behavior is shown by Kambe [12] assuming that initial vorticity is axially symmetric but for arbitrary R since the governing equation (1) is reduced to the heat equation. Our results extend this because no particular structure of initial vorticity is assumed. Although we are forced to assume that R is small, we do not restrict the speed of the axisymmetric irrotational flow.

To prove (3) we study the integral form of (1) :

$$(4)\quad \omega(t) = e^{\nu t\Delta}\omega_0 + B(\omega,\omega)\ ,\ B(\omega,\omega) = -\int_0^t e^{\nu(t-s)\Delta}(v\cdot\nabla)\omega(s)ds$$

where $U = e^{\nu t\Delta}\omega_0$ solves the heat equation $\partial_t U - \nu\Delta U = 0$ with $U(x,0) = \omega_0$ and $\omega(t) = \omega(\cdot,t)$, Unfortunately, the term $B(\omega,\omega)$ can not be regarded as a minor term as $t \to \infty$, unless we use the

special structure of B. Since ω_{*K} defined in (2) is radially symmetric, as is easily seen ω_{*K} solves (1) with $(v\cdot\nabla)\omega = 0$. This implies $B(\omega_{*K},\omega_{*K}) = 0$. Applying this property to (4) we rewrite the equation for the difference $w = \omega - \omega_{*K}$ to obtain

$$(5)\quad w = W + B(w,w) + B(w,\omega_{*K}) + B(\omega_{*K},w)\ ,\ W = e^{\nu t\Delta}\omega_0 - \omega_{*K}\ .$$

We estimate the right hand side of (5). Based on the decay estimate

$$(6)\qquad \|\omega\|_p(t) \le CRt^{-1+1/p}$$

obtained by Giga, Miyakawa and Osada [8], one can regard B's in (5) as perturbation terms provided that R is sufficiently small. Thus the estimate for w in (5) is reduced to the estimate for W which is easy to derive. As is seen above, our result (3) heavily depends on the particular structure of the nonlinear term in (1) and the estimate (6).

It turns out that the estimate (3) is still valid even if we allow to choose a finite Radon measure as initial vorticity rather than integrable functions. Since $\omega(x,t)$ is regularized instanteneously, this is not a substantial improvement of the results. However, all estimates are parallel and there appears no extra difficulty. So we rather start with a finite Radon measure because the initial value of ω_{*K} is $\kappa\delta(x)$, which is a typical example of a finite Radon measure but not an integrable function.

We note that vortex sheets of finite length is another example of finite Radon measures.

There are many works on the large time behavior of solutions of the Navier-Stokes equations on $\mathbb{R}^n$(cf.[2,10,14,18,20,21,22]) . However, when $n = 2$ it is usually assumed that initial velocity $v_0 = K*\omega_0$ is in $L^2(\mathbb{R}^2)$, in other words the initial total energy is finite. Our assumption does not imply $v_0 \in L^2(\mathbb{R}^2)$ even if we assume $\omega_0 \in L^1(\mathbb{R}^2)$. So our situation is not included in those treated in the literature. In our setting even the existence of solution of (1) is recently proved in [8] with a decay estimate (6). The decay results in the literature is mostly not for the vorticity but the velocity especially its L^2-norm. For more detailed comparison with the literature, see Remark 4.4.

We study in Section 2 the asymptotic behavior of the solutions of the heat equation so that we estimate the decay of W in (5). In Section 3 we recall the estimate (6) and prepare estimates for B . In Section 4 we state our main results including (3), which are proved in Section 5. The final section is devoted to an application of our results in Section 3, which is mentioned in the third paragraph of Introduction.

2. The heat equation

The goal of this section is to prepare various estimates for the solution of the heat equation as time t tends to infinity. We are especially concerned with estimates for the second term in the asymptotic expansion of the solution as $t \to \infty$.

Let G be the Gauss kernel on $\mathbb{R}^n$ defined by

$$G(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp(-\frac{|x|^2}{4t}) \ , \ x \in \mathbb{R}^n \ , \ t > 0 \ .$$

As is well known the function

$$U(x,t) = \int_{\mathbb{R}^n} G(x-y,t)a(y)dy \ (= G*a)$$

solves the heat equation

$$\partial_t U - \Delta U = 0 \qquad \text{for} \quad t > 0$$

with

$$U(x,0) = \lim_{t \to +0} U(x,t) = a(x) \ .$$

The meaning of the convergence of $U(x,t)$ as $t \to +0$ depends on a class of functions a we consider. We shall write U by $e^{t\Delta}a$. The semigroup $e^{t\Delta}$ is used only for the convenience of notation ; we shall not use the abstract theory of semigroups in this paper.

We collect various estimates for $G(= e^{t\Delta}\delta)$ and $e^{t\Delta}a$. A direct calculation shows

$$\text{(2.1a)} \qquad \|G\|_r = (\int_{\mathbb{R}^n} |G|^r dx)^{1/r} = C_r t^{-\sigma(r)} \ , \ \sigma(r) = \frac{n}{2}(1-\frac{1}{r})$$

$$\text{(2.1b)} \qquad \|\partial_j G\|_r = C'_r t^{-\sigma(r)-1/2} \ , \ \partial_j = \partial/\partial x_j$$

where C_r and C_r' are constants depending only on n and r. When $r = \infty$, (2.1a) and (2.1b) still hold if we regard $\|f\|_\infty$ as the superimum of $|f|$ on $\mathbb{R}^n$ and $\sigma(\infty) = n/2$. Applying the Young inequality.

$$(2.2) \qquad \|f*g\|_p \leq \|f\|_r \, \|g\|_q$$

for $1/p = 1/r + 1/q - 1$, $1 \leq p, q, r \leq \infty$ with $f = G$, $g = a$, we see (2.1a,b) yields

$$(2.3a) \qquad \|e^{t\Delta}a\|_p \leq Ct^{-(n/q-n/p)/2}\|a\|_q \,, \quad t > 0 \qquad 1 \leq q \leq p \leq \infty$$

$$(2.3b) \qquad \|\partial_j e^{t\Delta}a\|_p \leq Ct^{-(n/q-n/p)/2-1/2}\|a\|_q \,, \quad t > 0 \,, \qquad 1 \leq q \leq p \leq \infty$$

where $C = \max(\sup_r C_r, \sup_r C_r')$ depends only on n. This gives a decay estimate for $e^{t\Delta}a$ as $t \to \infty$ provided a is in $L^q(\mathbb{R}^n)$. In particular, (2.3a) yields

$$(2.4) \qquad \|e^{t\Delta}a\|_p \leq Ct^{-\sigma(p)}\|a\|_1 \quad t > 0 \,, \; 1 \leq p \leq \infty \,.$$

This estimate extends to a finite Radon measure a on $\mathbb{R}^n$.
A finite Radon measure μ is a Schwartz distribution which is a bounded linear functional on the Banach space $BC(\mathbb{R}^n)$ of bounded continuous functions on $\mathbb{R}^n$. In other words μ is an element of $M(\mathbb{R}^n)$, the dual space of $BC(\mathbb{R}^n)$. The norm of μ in $M(\mathbb{R}^n)$ is

called the total variation of μ denoted by $\|\mu\|_1$. Its explicit definition is

$$\|\mu\|_1 = \sup_{\substack{\phi\in BC(\mathbb{R}^n) \\ \|\phi\|_\infty\le 1}} \left|\int_{\mathbb{R}^n}\phi(x)\mu(dx)\right| ,$$

where $\int\phi(x)\mu(dx)$ denotes the canonical duality pairing. The estimate (2.4) holds for $a \in M(\mathbb{R}^n)$ if we regard $\|a\|_1$ as the total variation of a . In fact, (2.2) is valid for a Radon measure g by setting $q = 1$ and $p = r$.

We next approximate $e^{t\Delta}a$ by αG with $\alpha = \int a(x)dx$ for large t . Formally,

$$\begin{aligned}(e^{t\Delta}a)(x) &= \int_{\mathbb{R}^n} G(x-y,t)a(y)dy \\ &= G(x)\int_{\mathbb{R}^n}\exp\left(\frac{2x\cdot y-|y|^2}{4t}\right)a(y)dy \\ &= \alpha G + O(t^{-\frac{n}{2}-1}) \qquad \text{as} \quad t\to\infty \quad \text{(pointwise)} ,\end{aligned}$$

since $\exp\frac{2x\cdot y-y^2}{4t} = 1 + \frac{2x\cdot y-y^2}{4t} + \cdots$. We give a rigorous meaning for this approximation. Since we are interested in uniform estimates in x , we lose $t^{-1/2}$ in estimating the error term.

Lemma 2.1. Assume that $1 \le p \le \infty$ and that a is a finite Radon measure on $\mathbb{R}^n$. Let $\alpha = \int_{\mathbb{R}^n} a(dx)$. Then

(2.5) $\|e^{t\Delta}a - \alpha G\|_p \le Ct^{-\sigma(p)}(t^{-1/2}\||x|a\|_1 + t^{-1}\||x|^2a\|_1)$, $t > 0$,

where $\sigma(p) = n(1-1/p)/2$ and C depends only on n .
In particular,

(2.6) $$\|e^{t\Delta}a-\alpha G\|_p \le Ct^{-\sigma(p)-1/2}\|(|x|^2+1)a\|_1 \quad \text{for} \quad t > 0 .$$

Proof. The idea of the proof is standard. Estimates similar to those used in the proof often appear in constructing the fundamental solutions of parabolic equations (see eg. [6,15]). However, since the proof is short we give if here for completeness. We write the proof pretending that the measure has the density, i.e., $a \in L^1(\mathbb{R}^n)$ only to use a standard calculus notation. The proof is the same for general $a \in M(\mathbb{R}^n)$ if we modify notation by replacing $a(y)dy$, $|a(y)|dy$ by $a(dy)$, $|a|(dy)$, respectively, since all functions appearing below are continuous in each variables for $t > 0$.

Since $\int Gdx = 1$ we see

$$e^{t\Delta}a - \alpha G = \int h(x,y,t)a(y)dy$$

with

$$h(x,y,t) = G(x-y,t) - G(x,t)$$

where the integration is over $\mathbb{R}^n$. By the mean value theorem we have

(2.7) $$(4\pi t)^{n/2}h(x,y,t) = (\exp(\frac{2x\cdot y-|y|^2}{4t})-1)\exp(-\frac{|x|^2}{4t})$$

$$= \frac{2x\cdot y-|y|^2}{4t} \cdot \exp(-\frac{|x|^2-\zeta}{4t})$$

where ζ is a number between zero and $2x\cdot y - |y|^2$.
If $2x\cdot y - |y|^2 \geq 0$, then $\zeta \leq 2x\cdot y - |y|^2$ which yields

$$|x|^2 - \zeta \geq |x-y|^2 .$$

Applying this to (2.7) gives

$$4(4\pi t)^{n/2}|h| \leq t^{-1}(2(x-y)\cdot y+|y|^2)\exp(-\frac{|x-y|^2}{4t}) .$$

Since $A_\beta = \sup_{z>0} z \exp(-\beta z^2/4) < \infty$ for $\beta > 0$,

we see

$$(|x-y|/t^{1/2})\exp(-\frac{\beta|x-y|^2}{4t}) \leq A_\beta .$$

We take β such that $0 < \beta < 1$ and fix β to obtain

(2.8) $$|h(x,y,t)| \leq k(y,t)G(x-y,\gamma t) ,$$
$$k(y,t) = (A_\beta|y|/2t^{1/2}+|y|^2/4t)\gamma^{n/2} , \quad \gamma = 1/(1-\beta) ,$$

provided $2x\cdot y - |y|^2 \geq 0$. If $2x\cdot y - |y|^2 < 0$,
we see $\zeta < 0$. This time (2.7) yields

$$4(4\pi t)^{n/2}|h| \leq (|y|^2/t+2|x||y|/t)\exp(-\frac{|x|^2}{4t})$$

As is before we obtain

$$(2.9) \qquad |h(x,y,t)| \leq k(y,t)G(x,\gamma t) \ .$$

provided $2x\cdot y - |y|^2 < 0$.

The estimates (2.8) and (2.9) yield

$$F(x) = \left|\int h a dy\right| \leq \max(e^{\gamma t\Delta}(k|a|) \ , \ G(x,\gamma t)\cdot\int k|a|dy) \ .$$

Applying (2.4) and (2.1a) gives

$$\|F(x)\|_p \leq Ct^{-\sigma(p)}\|k|a|\|_1$$

with C depending only on n . This is the same as (2.5), since $k \leq C(|y|/t^{1/2}+|y|^2/t)$. The estimate (2.6) for $t \geq 1$ follows from (2.5). Since $|\alpha| \leq \|a\|_1$, the estimate (2.6) for $t \leq 1$ follows from (2.1a) and (2.3a) with $t \leq 1$. We thus obtain (2.6) . □

3. The two-dimensional vorticity equation

The first part of this section reviews the existence of solutions to the two-dimensional vorticity equation with measures as initial vorticity. We also recall decay estimates for the vorticity as time tends to infinity.

The second part is devoted to the study of properties of the nonlinear term in the vorticity equation which is useful in the sequel.

Existence results mentioned above are not classical because the initial velocity may not be locally square summable even if initial vorticity is in $L^1(\mathbb{R}^2)$. The first attack is done by Benfatto, Esposito and Pulvirenti [3], where they prove the global existence for small initial vorticity consisting of a linear combination of δ-functions supported at a point of the plane. Recently, Miyakawa, Osada and the first author [8] improve their results. Without the smallness assumption they [8] constructed a global solution even if initial vorticity is a finite Radon measure, and also give decay estimates for the vorticity as time tends to infinity.

We consider the two-dimensional vorticity equation :

$$\omega_t - \Delta\omega + (v\cdot\nabla)\omega = 0 \tag{3.1}$$

$$v = K*\omega \ , \tag{3.2}$$

where K is the vector function

$$K(x_1,x_2) = (-x_2,x_1)/2\pi|x|^2 \ , \quad x = (x_1,x_2) \ ;$$

we shall always assume the space dimension $n = 2$.
Here the vorticity $\omega = \omega(x,t)$ is scalar since $n = 2$. The convolution operator $K*$ improves differentiability of order one. Another expression of this regularizing property of $K*$ is

(3.3) $\|K*f\|_p \le C\|f\|_q \quad 1/q = 1/p + 1/2 \ , \quad 1 < q < 2$

obtained by the Hardy-Littlewood-Sobolev inequality (see eg.[19, p.28]), where $\|f\|_q$ denotes the norm of f in $L^q(\mathbb{R}^2)$ and C depends only on q . We consider (3.1) and (3.2) with initial condition

(3.4) $\omega(x,0) = \lim_{t\to+0} \omega(x,t) = \omega_0(x) \ .$

If ω_0 is a finite Radon measure on $\mathbb{R}^2$, i.e., $\omega_0 \in M(\mathbb{R}^2)$, the convergence should be understood under the weak topology of measures, that is,

$$\lim_{t\to+0} \int_{\mathbb{R}^2} \phi(x)\omega(x,t)dx = \int_{\mathbb{R}^2} \phi(x)\omega_0(dx)$$

for every bounded continuous functions ϕ on $\mathbb{R}^2$. We write $\omega(x,t)dx$ rather than $\omega(dx,t)$ because the solution ω we handle is smooth for $t > 0$.

Proposition 3.1. Suppose that $\omega_0 \in M(\mathbb{R}^2)$, i.e. ω_0 is a Radon measure on $\mathbb{R}^2$ such that the total variation $\|\omega_0\|_1$ is finite. There is a global solution of (3.1),(3.2) and (3.4) such that

(3.5) $\|\omega\|_q(t) \le Ct^{-1+1/q}\|\omega_0\|_1 \qquad t > 0 \ , \quad 1 \le q \le \infty$

(3.6) $\|v\|_p(t) \le C't^{-1/2+1/p}\|\omega_0\|_1 \quad t > 0 \ , \quad 2 < p \le \infty$

(3.7) $\sup_{t\le\tau\le T}\|\nabla^k\partial_t^h\omega\|_\infty(\tau) < \infty \qquad k,h = 0,1,2,\cdots \quad t > 0$

(3.8) $\int_{\mathbb{R}^2}\omega(x,t)dx = \int_{\mathbb{R}^2}\omega_0(dx) \ , \quad t > 0$

<u>with</u> $C = C(m)$, $C' = C'(m,p)$ <u>for</u> $\|\omega_0\|_1 \le m$. <u>The estimate</u> (3.7) <u>in particular implies that</u> ω <u>is smooth for</u> $t > 0$.

Proof. This is essentially a combination of Theorems 4.2 and 4.3 in [8] with initial velocity $u_0 = K*\omega_0$. In fact, Theorem 4.2 in [8] asserts that there is a smooth solution $u(x,t)$ for $t > 0$ to the Navier-Stokes system :

$$u_t - \Delta u + (u\cdot\nabla)u + \nabla p = 0 \ , \ \nabla\cdot u = 0$$

with initial velocity u_0 and initial vorticity ω_0 . Taking $\nabla\times$ of the first equation shows that $\omega = \nabla\times u$ with $v = u$ solves (3.1) for $t > 0$. Since Theorem 4.2 (4.1) in [8] implies $\|u\|_r(t) < \infty$ for $t > 0$, $r > 2$, applying Lemma 2.2(ii) in [8] , we see $\omega = \nabla\times u$ solves (3.2) with $v = u$. We thus conclude that $\omega = \nabla\times u$, $v = u$ solves (3.1)-(3.2) for $t > 0$, where u is in Theorem 4.2 [8] .

The initial condition (3.4) is contained in Theorem 4.2(ii) of [8]. The estimate (3.5) is the same as (4.7) of Theorem 4.3 in [8]. This together with (3.3) yields (3.6). The estimate (3.7) immediately follows from Theorem 4.2(iv) in [8]. Theorem 4.3 in [8] also gives a representation for ω

$$\omega(x,t) = \int \Gamma(x,t;y,0)\omega_0(dy)$$

with $\Gamma(x,t,y,s) > 0$ and $\int \Gamma(x,t,y,s)dx = 1$, where the integration is over $\mathbb{R}^2$. This yields the conservation of the vorticity (3.8). (We note that the representation for ω together with estimates for Γ yields (3.5)).

Remark 3.2. If point vortex part of initial measure ω_0 is small we have the uniqueness of solution in a certain class of functions [8]. In particular, if $\|\omega_0\|_1$ is small or ω_0 contains no point vortices, one can claim the uniqueness with additional conditions. (cf. Theorem 4.5. in [8]). For general initial data in $M(\mathbb{R}^2)$ we do not know what conditions guarantee the uniqueness of solutions. However, since our ω in Proposition 3.1 has as physically reasonble properties like (3.8), by a solution of the vorticity equation, we shall always mean $\omega(x,t)$ in Proposition 3.1.

It is convenient to write the equations (3.1),(3.2) and (3.4) in its integral form. Formally the system (3.1),(3.2),(3.4) is equivalent to

$$\omega(t) = e^{t\Delta}\omega_0 + B(\omega,\omega) \tag{3.9}$$

$$B(\omega,\omega) = \int_0^t b_t(\omega,\omega)(s)ds \tag{3.10}$$

$$b_t(\omega_1,\omega_2)(s) = -\nabla\cdot e^{(t-s)\Delta}(v_1\omega_2)(s) \ , \ v_1 = K*\omega_1$$

$$= -\sum_{j=1}^{2} \partial_j e^{(t-s)\Delta} v_1^j(s)\ \omega_2(s)$$

since $\nabla\cdot v = 0$ implies $\nabla\cdot(v\omega) = (v\cdot\nabla)\omega$ and since ∇ commutes with $e^{t\Delta}$. Here $e^{t\Delta}$ is the solution operator of the heat equation defined in section 2, i.e., $e^{t\Delta}f = G*f$, and $\omega(t) = \omega(\cdot,t)$.

Proposition 3.3. Suppose that $\omega(x,t)$ is a solution of the vorticity equation where $\omega(x,0) = \omega_0 \in M(\mathbb{R}^2)$. Then $\|b_t(\omega,\omega)\|_1(s)$ is integrable on $(0,t)$ where b_t is defined in (3.10). Moreover $\omega(t) = \omega(\cdot,t)$ solves the integral equation (3.9).

Proof. Applying (2.1b) yields

$$\|b_t(\omega,\omega)(s)\|_1 \leq C(t-s)^{-1/2}\|v\omega\|_1 .$$

The estimates (3.5) and (3.6) now show that $\|b_t(\omega,\omega)(s)\|_1$ is integrable on $(0,t)$.

It remains to prove that ω solves (3.9). By (3.7) a classical uniqueness theorem of solutions of the heat equation [6] shows that

$$\omega(t) = e^{(t-\varepsilon)\Delta}\omega(\varepsilon) + \int_\varepsilon^t b_t(\omega,\omega)(s)ds \tag{3.11}$$

for $\varepsilon > 0$. We shall show that for $t > 0$

$$e^{(t-\varepsilon)\Delta}\omega(\varepsilon) \to e^{t\Delta}\omega_0 \quad \text{weakly in } M(\mathbb{R}^2) \text{ as } \varepsilon \to 0 \text{ i.e.}$$

$$(3.12)\qquad \lim_{\varepsilon\to 0} (\phi, e^{(t-\varepsilon)\Delta}\omega(\varepsilon)) = (\phi, e^{t\Delta}\omega_0)$$

for every bounded continuous function ϕ, where $(\phi,\psi) = \int \phi\psi dx$. We have

$$\begin{aligned}
&(e^{(t-\varepsilon)\Delta}\omega(\varepsilon) - e^{t\Delta}\omega_0 \ , \ \phi)\\
&= ((e^{(t-\varepsilon)\Delta} - e^{t\Delta})\omega(\varepsilon), \phi) + (e^{t\Delta}(\omega(\varepsilon) - \omega_0), \phi)\\
&= (e^{(t-\varepsilon)\Delta}\omega(\varepsilon), (I - e^{\varepsilon\Delta})\phi) + (\omega(\varepsilon) - \omega_0, e^{t\Delta}\phi) = I_1 + I_2 \ .
\end{aligned}$$

Using (2.4) and (3.5), the first term is estimated as

$$|I_1| < C\|\omega_0\|_1 \ \|(I - e^{\varepsilon\Delta})\phi\|_\infty \ .$$

Since $e^{\varepsilon\Delta}\phi \to \phi$ uniformly as $\varepsilon \to 0$, $I_1 \to 0$ as $\varepsilon \to 0$. Since $\omega(\varepsilon) \to \omega_0$ weakly in $M(\mathbb{R}^2)$ as $\varepsilon \to 0$ we see $I_2 \to 0$ as $\varepsilon \to 0$. We thus obtain (3.12). Since $\|b_t(\omega,\omega)(s)\|_1$ is integrable on $(0,t)$, (3.12) now yields (3.9) by letting $\varepsilon \to 0$ in (3.11). □

The remaining part of this section is devoted to the study of B defined in (3.10).

Lemma 3.4. *Let* b_t *and* B *be the bilinear forms defined in* (3.10). *Then* $(K*f\cdot\nabla)g = 0$ *so* $b_t(f,g) = 0$ *and* $B(f,g) = 0$ *provided that* f *and* g *are radially symmetric, where* $K = (-x_2, x_1)/2\pi|x|^2$.

Proof. This is elementary and well known. If f is radial, the derivative $(K*f\cdot\nabla)$ contains only the angular derivative. If g is also radial, this implies $(K*f\cdot\nabla)g = 0$. We thus conclude $b_t(f,g) = 0$ so $B(f,g) = 0$. □

The second property of B we need is the estimate for B. As we see later, it is convenient to divide the integral in B into two parts - the region where s is close to t and s is close to zero. We write

$$
\begin{aligned}
B_1(w_1,w_2)(t) &= \int_0^{t/2} b_t(w_1,w_2)(\tau)d\tau \ , \\
B_2(w_1,w_2)(t) &= \int_{t/2}^{t} b_t(w_1,w_2)(\tau)d\tau
\end{aligned}
\tag{3.13}
$$

with

$$b_t(w_1,w_2)(\tau) = -\nabla\cdot e^{(t-\tau)\Delta}(v_1 w_2)(\tau) \ , \ v_1 = K*w_1$$

so that

$$B(w_1,w_2) = B_1(w_1,w_2) + B_2(w_1,w_2) \ .$$

To simplify notation, for a function f on $\mathbb{R}^2 \times (0,T)$ we encode the decay in t in norms :

$$
\begin{aligned}
[f]_{pT} &= \sup_{0<t<T} t^{1-1/p}\|f\|_p \\
[f]_{p\delta T} &= \sup_{0<t<T} t^{1-1/p+\delta}\|f\|_p \ .
\end{aligned}
\tag{3.14}
$$

For example $[f]_{pT} \le 1$ for all T is equivalent to $\|f\|_p(t) \le 1/t^{1-1/p}$ for all $t > 0$. We estimate the decay rate of B by using the decay rate of each variables in B.

Lemma 3.5. *Suppose that* $1 \le s \le p \le \infty$, $1 < q < 2$, $\delta \ge 0$ *and* $1 \le r \le \infty$ *with* $1/s = 1/q + 1/r - 1/2$. *There is a positive constant* C *depending only on* p,q,r *and* δ *such that*

$$[B_1(w_1,w_2)]_{p\delta T} \le C \min([w_1]_{q\delta T}[w_2]_{rT}\ ,\ [w_1]_{qT}[w_2]_{r\delta T}) \tag{3.15}$$

with $\delta < 1/s - 1/2$ *and that*

$$[B_2(w_1,w_2)]_{p\delta T} \le C \min([w_1]_{q\delta T}[w_2]_{rT}\ ,\ [w_1]_{qT}[w_2]_{r\delta T}) \tag{3.16}$$

with $1/s < 1/2 + 1/p$. *Here* w_1 *and* w_2 *are functions on* $\mathbb{R}^2 \times (0,T)$; B_1 *and* B_2 *are defined by* (3.13).

Proof. Since $v_1 = K*w_1$, we have by (3.3)

$$\|v_1\|_\theta \le C_1\|w_1\|_q \quad 1/\theta = 1/q - 1/2\ ,\quad 1 < q < 2\ .$$

Using Hölder's inequality, we have

$$\|v_1w_2\|_s \le \|v_1\|_\theta\ \|w_2\|_r \le C_1\|w_1\|_q\ \|w_2\|_r\ ,\ 1/s = 1/\theta + 1/r \le 1\ .$$

Applying (2.3b) now yields

$$\begin{aligned} \|b_t(w_1,w_2)\|_p(\tau) &\le C_2(t-\tau)^{-1/2-1/s+1/p}\|v_1w_2\|_s(\tau) \qquad p \ge s \ge 1 \\ &\le C_3M(t-\tau)^{-1/2-1/s+1/p_\tau-\sigma(q)-\sigma(r)-\delta}\ , \qquad 0 \le \tau \le T \end{aligned} \tag{3.17}$$

with $M = [w_1]_{q\delta T}[w_2]_{rT}$ or $[w_1]_{qT}[w_2]_{r\delta T}$, where C_j is a constant depending only on p,q and r and $\sigma(q) = 1 - 1/q$. The restrictions on exponents p,q,r so far we use are

(3.18) $\quad 1 \leq s < p < \infty$, $1 < q < 2$, $1 \leq r \leq \infty$ with $1/s = 1/q + 1/r - 1/2$.

Since we see

$$\int_0^{t/2} (t-\tau)^{-\alpha}\tau^{-\beta} d\tau = c(\alpha,\beta)t^{-\alpha-\beta+1} \quad \text{for} \quad \beta < 1$$

($c(\alpha,\beta)$: constant independent of t)

by setting $\tau = t\tilde{\tau}$, the estimate (3.17) now yields

$$\text{(3.19)} \qquad \|B_1(w_1,w_2)\|_p(t) \leq \int_0^t \|b_t(w_1,w_2)\|_p(\tau)d\tau \leq C_4 M t^{-\sigma(p)-\delta}, \quad t > 0$$

provided that

$$\sigma(q) + \sigma(r) + \delta < 1,$$

which is equivalent to

(3.20) $\quad \delta < 1/s - 1/2$.

The estimate (3.19) yields (3.15) under the restrictions (3.18) and (3.20).

It remains to prove (3.16). Since, as is before,

$$\int_{t/2}^{t} (t-\tau)^{-\alpha}\tau^{-\beta}d\tau = c'(\alpha,\beta)t^{-\alpha-\beta+1} \quad \text{for} \quad \alpha < 1$$

the estimate (3.17) now yields

$$\|B_2(w_1,w_2)\|_p(t) \leq C_5 M t^{-\sigma(p)-\delta} \ , \ t > 0 \tag{3.21}$$

provided that

$$1/2 + 1/s - 1/p < 1 \ ,$$

which is equivalent to

$$1/s < 1/2 + 1/p \ . \tag{3.22}$$

The estimate (3.21) yields (3.16) under (3.18) and (3.22) which completes the proof.

4. Large time asymptotics of the vorticity

This section states our main results on the large time behavior of solutions of the vorticity equation.

Let ω_0 be a finite Radon measure on $\mathbb{R}^2$, i.e. $\omega_0 \in M(\mathbb{R}^2)$. We denote the total vorticity (at time zero) by

(4.1) $$\kappa = \int_{\mathbb{R}^2} \omega_0(dx) \ .$$

We define the Reynolds number (at time zero) by

(4.2) $$R = \nu^{-1} \int_{\mathbb{R}^2} |\omega_0|(dx) = \|\omega_0\|_1/\nu$$

where $\nu > 0$ is the kinematic viscosity which is assumed one in (3.1). The function

(4.3) $$\omega_{*\kappa}(x,t) = \frac{\kappa}{4\pi\nu t} \exp(-|x|^2/4\nu t) = \kappa G(x,\nu t) \ , \ x \in \mathbb{R}^2$$

is a solution of the heat equation

$$\omega_t - \nu\Delta\omega = 0$$

with $\omega(x,0) = \kappa\delta(x)$. Since $\omega_{*\kappa}$ is radially symmetric, Lemma 3.4 implies that $(v_{*\kappa}\cdot\nabla)\omega_{*\kappa} = 0$ where $v_{*\kappa} = K*\omega_{*\kappa}$. We thus see $\omega_{*\kappa}$ solves the vorticity equation with $\nu > 0$

(4.4) $$\omega_t - \nu\Delta\omega + (v\cdot\nabla)\omega = 0$$

(4.5) $$v = K*\omega$$

with initial data $\kappa\delta(x) \in M(\mathbb{R}^2)$. To avoid later confusion by a solution of the vorticity equation with ν we shall always mean a solution of (4.4)-(4.5) with (3.4) which satisfies (3.5)-(3.8) with $\omega(x,0) = \omega_0 \in M(\mathbb{R}^2)$. Since one can reduce (4.4)-(4.5) to

(3.1)-(3.2) by a normalization, Proposition 3.1 guarantees the existence of such a solution for $\omega(x,0) = \omega_0 \in M(\mathbb{R}^2)$. For a solution ω , (3.8) implies that the total vorticity is conserved for all time, namely

$$\kappa = \int_{\mathbb{R}^2} \omega(x,t)dx \ , \ t \geq 0$$

where κ is defined in (4.1). This says that the total vorticity is defined independent of time.

We claim that a solution ω of the vorticity equation with ν behaves like $\omega_{*\kappa}$ with the same total vorticity κ provided the Reynolds number R is small no matter what initial vorticity $\omega_0 \in M(\mathbb{R}^2)$ is.

Theorem 4.1. Suppose that the kinematic viscosity ν equals 1 and that $\omega(x,t)$ is a solution of the vorticity equation for $\omega(x,0) = \omega_0 \in M(\mathbb{R}^2)$. For every δ , $0 < \delta < 1/2$, there is $\varepsilon > 0$ depending only on δ such that if the Reynolds number R in (4.2) is smaller than ε , then

$$\|\omega - \omega_{*\kappa}\|_p(t) \leq CNt^{-1+1/p-\delta} \tag{4.6a}$$

$$\|\omega - e^{t\Delta}\omega_0\|_p(t) \leq CNt^{-1+1/p-\delta} \ , \ t > 0 \ , \ 1 \leq p \leq \infty \tag{4.6b}$$

with a universal constant C and $N = \|(|x|^2+1)\omega_0\|_1$, where κ is the total vorticity of ω and $\omega_{*\kappa}$ is defined in (4.3).

We postpone to prove this theorem in the next section. Admitting Theorem 4.1, we derive various results. First, we observe that Theorem 4.1 gives the large time behavior of a solution $\omega(x,t)$ of the vorticity equation with $\nu > 0$ just by a normalization. In fact $\bar{\omega}(x,t) = \nu^{-1}\omega(x,t/\nu)$ is a solution of the vorticity equation with $\nu = 1$ where $\bar{\omega}(x,0) = \nu^{-1}\omega(x,0)$. Using this relation one can rewrite Theorem 4.1 for general $\nu > 0$.

Theorem 4.2. *Suppose that* $\omega(x,t)$ *is a solution of the vorticity equation with* $\nu > 0$ *for* $\omega(x,0) = \omega_0 \in M(\mathbb{R}^2)$. *For every* δ, $0 < \delta < 1/2$, *there is* $\varepsilon > 0$ *depending only on* δ *such that* $R < \varepsilon$ *implies*

$$\|\omega - \omega_{*K}\|_p(t) \le CN(t\nu)^{-1+1/p-\delta} \tag{4.7a}$$

$$\|\omega - e^{\nu t\Delta}\omega_0\|_p(t) \le CN(t\nu)^{-1+1/p-\delta}, \quad t > 0, \quad 1 \le p \le \infty \tag{4.7b}$$

with a universal constant C *and* $N = \|(|x|^2+1)\omega_0\|_1$, *where* K *is the total vorticity of* ω *and* ω_{*K} *is defined in* (4.3).

Since $\|\omega_{*K}\|_p(t) = C_p K(t\nu)^{-1+1/p}$ by (2.1a), Theorems 4.1 and 4.2 give an asymptotic expression of a solution as $t \to \infty$. We at least observe that ω_{*K} is the main term in the asymptotics as $t \to \infty$ and that the solution of the linear equation mainly describes the behavior as $t \to \infty$. We now discuss an asymptotic expansion of the velocity $v = K*\omega$ corresponding to (4.7a),(4.7b).

Theorem 4.3. *Suppose that* $\omega(x,t)$ *is a solution of the vorticity equation with* $\nu > 0$ *for* $\omega(x,0) = \omega_0 \in M(\mathbb{R}^2)$. *For every* δ , $0 < \delta < 1/2$ *there is* $\varepsilon > 0$ *depending only on* δ *such that* $R < \varepsilon$ *implies*

$$\|\nabla v - \nabla v_{*K}\|_q(t) \leq C' N (t\nu)^{-1+1/q-\delta} \quad , \quad N = \|(|x|^2+1)\omega_0\|_1 \tag{4.8a}$$

$$\|\nabla v - \nabla e^{\nu t\Delta} v_0\|_q(t) \leq C' N (t\nu)^{-1+1/q-\delta} \tag{4.8b}$$

$$\|v - v_{*K}\|_r(t) \leq C'' N (t\nu)^{-1/2+1/r-\delta} \tag{4.9a}$$

$$\|v - e^{\nu t\Delta} v_0\|_r(t) \leq C'' N (t\nu)^{-1/2+1/r-\delta} \quad t > 0 \tag{4.9b}$$

with C' *depending only on* q , $1 < q < \infty$ *and* C'' *depending only on* r , $2 < r < \infty$, *where* $v_{*K} = K*\omega_{*K}$ *and* $v_0 = K*\omega_0$.

Proof. Since ∇K is the Calderon-Zygmund kernel, we have

$$\|\nabla(K*f)\|_q \leq C \|f\|_q \quad , \quad 1 < q < \infty$$

by applying the Calderon-Zygmund inequality [9, Chap. 9]. The estimate (4.7a,b) now yields (4.8a,b). Estimate (4.9a,b) for $2 < r < \infty$ follows directly from (4.7a,b) and (3.3).

It remains to prove (4.9a,b) for $r = \infty$. Applying the Gagliardo-Nirenberg inequality (see eg. [7, p.24]) $\|g\|_\infty \leq C\|g\|_p^{1-2/p} \|\nabla g\|_p^{2/p}$ for $p > 2$ to $g = v - v_{*K}$ or $v - e^{\nu t\Delta} v_0$,

we see (4.8a,b) and (4.9a,b) for $2 < r < \infty$ yield (4.9a,b) for $r = \infty$. □

Remark 4.4. There are several results [2,10,14,18,20,21,22] on the decay of the velocity v for the n-dimensional Navier-Stokes equations assuming that the initial velocity v_0 is in some L^p . When $n = 2$, their main results read :

$$\lim_{t\to\infty} \|v\|_2(t) = 0 \quad \text{if} \quad v_0 \in L^2(\mathbb{R}^2) \ , \ [10,14,18]$$

and

$$\|v\|_2(t) \leq Ct^{-1/q+1/2} \ , \ \|v-e^{t\Delta}v_0\|_2(t) \leq Ct^{-1/q+1/2-\delta} \ , \tag{4.10}$$
$$v_0 \in L^2(\mathbb{R}^2) \cap L^q(\mathbb{R}^2) \ , \ \delta = 1/q-1/2 \ , \ 1 \leq q < 2$$
$$[10,20,21,22] \ ,$$

where the viscosity ν is assumed one. The estimates (4.10) gives an asymptotic expansion. Since our assumptions do not in general imply $v_0 \in L^2(\mathbb{R}^2) \cap L^q(\mathbb{R}^2)$, (4.9b) is not included in (4.10). Also, (4.10) is not included in (4.9b) since $r = 2$ is excluded in (4.9b). Among them Kato [14] studied the decay of $\|v\|_r(t)$, $r > 2$ other than energy. His results yield

$$\|v\|_r(t) < Ct^{-1/2+1/r} \quad 2 < r \leq \infty$$
$$\|\nabla v\|_q(t) \leq Ct^{-1+1/q} \quad 2 \leq q < \infty$$

provided $v_0 \in L^2(\mathbb{R}^2)$. Our results (4.8a) (4.9a) claims the faster decay if $\kappa = \int \omega_0(dx) = 0$.

5. Proof of Theorem 4.1

We study the integral equation (3.9) :

$$\omega(t) = e^{t\Delta}\omega_0 + B(\omega,\omega) \ .$$

A naive idea to prove (4.6b) would be to estimate $B(\omega,\omega)$. If we were successful to prove $\|B(\omega,\omega)\|_p(t) \le Ct^{-1+1/p-\delta}$, we would obtain (4.6b) and using the estimate for linear part (2.6), (4.6b) would yield (4.6a). Unfortunately this idea appears not to work. In fact the optimal decay rate estimate for $\|\omega\|_p(t)$ is $t^{-1+1/p}$ even if ω_0 decays rapidly at infinity unless the total vorticity $\kappa = 0$; the simplest example is $\omega_{*\kappa}$ where $\omega_{*\kappa}(x,0) = \kappa\delta(x)$. Using Lemma 3.5, all we can derive from the estimate of $\|\omega\|_p$ is $\|B(\omega,\omega)\|_p(t) \le Ct^{-1+1/p}$ in general, which is too weak to derive (4.6b). To overcome this difficulty we rather study the difference $w = \omega - \omega_*$ where $\omega_* = \omega_{*\kappa}$ in (4.3).

Proposition 5.1. *Suppose that* $\omega(x,t)$ *is a solution of the vorticity equation for* $\omega(x,0) = \omega_0 \in M(\mathbb{R}^2)$. *Then the difference* $w = \omega - \omega_*$ *solves*

$$w = W + B(w,w) + B(w,\omega_*) + B(\omega_*,w) \tag{5.1}$$

with

$$W = e^{t\Delta}\omega_0 - \omega_*$$

$$\omega_* = \frac{\kappa}{4\pi t}\exp\left(-\frac{|x|^2}{4t}\right) ,$$

where

$$\kappa = \int_{\mathbb{R}^2}\omega_0(dx) \ .$$

Proof. By Proposition 3.3 we may assume ω solves (3.9). Since B is bilinear, plugging $\omega = w + \omega_*$ yields

$$w = W + B(w,w) + B(w,\omega_*) + B(\omega_*,w) + B(\omega_*,\omega_*) .$$

Since ω_* is radial, Lemma 3.4 implies that the last term vanishes identically. We thus obtain (5.1). □

We shall prove (4.6a) by estimating W and B in (5.1). Roughly speaking, we appeal to a perturbation method. We estimate $[w]_{p\delta T}$ by using the right hand side of (5.1). Here $[w]_{p\delta T}$ is defined by (3.14) and is finite since $T < \infty$ and (3.5) holds ; this is why we take $T < \infty$ rather than $T = \infty$.
We eventually have

$$[w]_{p\delta T} \leq [W]_{p\delta T} + CR[w]_{p\delta T}, \quad (C : \text{constant})$$

where R is the Reynolds number. If R is small enough, one get

$$[w]_{p\delta T} \leq C'[W]_{p\delta T} \quad , \quad C' > 1 .$$

Since $[W]_{p\delta T}$ is bounded independent of T by (2.6), we conclude the desired estimate. Unfortunately, since there are restrictions of exponents in Lemma 3.5 such idea only works for p, $1 < p < 2$. After we prove (4.6a) for $1 < p < 2$, we shall prove it for $p = \infty$ and $p = 1$ and interpolate these results.

Even if we are interested only in the case $p = \infty$ or $p = 1$, we should check the result for $1 < p < 2$.

Proposition 5.2. *Suppose that* $\omega(x,t)$ *is a solution of the vorticity equation for* $\omega(x,0) = \omega_0 \in M(\mathbb{R}^2)$ *and that* $1 < p < 2$. *Then for* δ, $0 < \delta < 1/2$ *there is a constant* $\varepsilon = \varepsilon(p,\delta) > 0$ *such that* $R < \varepsilon$ *implies*

$$[w]_{p\delta T} \le cN \ , \quad N = \|(|x|^2 + 1)\omega_0\|_1 \tag{5.2}$$

with a universal constant $c > 0$, *where* R *is the Reynolds number and* $w = \omega - \omega_*$, $\omega_* = \omega_{*\kappa}$. *Here* κ *is the total vorticity ;* $\kappa = \int \omega_0(dx)$.

Proof. We may assume $N < \infty$ otherwise the result is trivial. Since $1 < p < 2$, one can handle B_1 and B_2 simultaneously and estimates look like symmetric for both variables in B. In fact, we take $s = 1$ in (3.15) so that $\delta < 1/2$. Since $1 < p < 2$ implies $1/s = 1 < 1/2 + 1/p$, we can also apply (3.16). Taking $q = p$ or $r = p$ in (3.15) and (3.16) yields

$$[B(w_1,w_2)]_{p\delta T} \le 2C \min\left([w_1]_{p\delta T}[w_2]_{\theta T} \ , \ [w_1]_{\theta T}[w_2]_{p\delta T}\right)$$

with $3/2 = 1/\theta + 1/p$. Applying this to (5.1) to get

$$[w]_{p\delta T} \le [W]_{p\delta T} + 2C[w]_{p\delta T}[w]_{\theta T} + 4C[w]_{p\delta T}[\omega_*]_{\theta T} \ . \tag{5.3}$$

Since $T < \infty$, we note $[w]_{p\delta T}$ is finite. By (2.1a) we have

$$(5.4) \qquad [\omega_*]_{rT} = C_r \kappa \leq c_1 R \qquad 1 \leq r \leq \infty$$

where c_j $(j = 1,2\cdots)$ is a universal constant. Since $w = \omega - \omega_*$, for fixed $m > 0$ the estimates (3.5) and (5.4) yield

$$(5.5) \qquad [w]_{rT} < [\omega]_{rT} + [\omega_*]_{rT} \leq (c_1 + c_2)R \quad \text{for} \quad R \leq m,$$

where c_2 may depend on m. Applying (5.4) and (5.5) to (5.3) now yields

$$(5.6) \qquad [w]_{p\delta T} \leq [W]_{p\delta T} + C'R[w]_{p\delta T}$$

with $C' = 2C(3c_1 + c_2)$ depending only on p and δ. We take ε sufficiently small, say,

$$(5.7) \qquad 0 < \varepsilon = \varepsilon(p,\delta) = \min\,(1/2C',1) .$$

If $R < \varepsilon$, then (5.6) gives

$$[w]_{p\delta T} \leq 2[W]_{p\delta T}$$

The estimate (5.2) now follows from (2.6). □

If p is large, one cannot take $s = 1$ in (3.16) so we should split the integral in B into two parts, B_1 and B_2. We next prove (5.2) for $p = \infty$.

Proposition 5.3. *Suppose the same situation as in Proposition 5.2. For δ, $0 < \delta < 1/2$, there is a constant $\varepsilon_\infty > 0$ such that $R < \varepsilon_\infty$ implies*

$$[w]_{\infty\delta T} \le cN \tag{5.8}$$

where c is a universal constant.

Proof. We take $s = 1$ and $r = q = 4/3$ in (3.15), which gives

$$\begin{aligned} &[B_1(w,w)]_{p\delta T} \le C[w]_{q\delta T}[w]_{qT} \qquad q = 4/3 \\ &[B_1(w,\omega_*)]_{p\delta T},\ [B_1(\omega_*,w)]_{p\delta T} \le C[w]_{q\delta T}[\omega_*]_{qT} \end{aligned} \tag{5.9}$$

where $\delta < 1/2$. To estimate B_2, we take $s = 4$ in (3.16) so that $1/s < 1/2 \le 1/2 + 1/p$. Applying (3.16) with $q = 4/3$ and $r = \infty$ yields

$$\begin{aligned} &[B_2(w,w)]_{\infty\delta T} \le C[w]_{q\delta T}[w]_{\infty T} \\ &[B_2(w,\omega_*)]_{\infty\delta T} \le C[w]_{q\delta T}[\omega_*]_{\infty T} \\ &[B_2(\omega_*,w)]_{\infty\delta T} \le C[\omega_*]_{qT}[w]_{\infty\delta T}, \quad q = 4/3 . \end{aligned} \tag{5.10}$$

We apply (5.9) and (5.10) to (5.1) and use (5.4) and (5.5). Similarly to (5.6), we obtain

(5.11) $$[w]_{\infty\delta T} \le [W]_{\infty\delta T} + C_1 R[w]_{q\delta T} + C_2 R[w]_{\infty\delta T}$$

where $q = 4/3$, $C_1 = 2C(3c_1 + c_2)$, $C_2 = c_1 C$. We take

$$\varepsilon_\infty = \min(\varepsilon(4/3,\delta) , 1/2C_2 , 1/2C_1) .$$

If $R < \varepsilon_\infty$, the estimate (5.11) together with (5.2) now yields

$$[w]_{\infty\delta T} \le 2[W]_{\infty\delta T} + cN .$$

Applying (2.6), we now obtain

$$[w]_{\infty\delta T} \le 2cN$$

which is the same as (5.8) by replacing $2c$ by c . □

Unfortunately we are forced to treat the case $p = 1$ separately because one cannot take $s = 4$ in (3.16) which requires $p \ge s = 4$.

Proposition 5.4. *Suppose the same situation as in Proposition 5.2. For δ , $0 < \delta < 1/2$, there is a constant $\varepsilon_1 > 0$ such that $R < \varepsilon_1$ implies*

(5.12) $$[w]_{1\delta T} \le cN$$

where c is a universal constant.

Proof. We observe that (5.9) holds even for $p = 1$. However, we need modification to (5.10). We take $s = 1$ and $r = q = 4/3$ in (3.16) which yields the estimate (5.9) where B_1 is replaced by B_2. Similarly to deriving (5.6), one get

$$[w]_{1\delta T} \le [w]_{1\delta T} + 2C_1'R[w]_{q\delta T}, \quad q = 4/3$$

with $C_1' = 2C(3c_1 + c_2)$. We take

$$\varepsilon_1 = \min(\varepsilon(4/3,\delta), 1/2C_1').$$

Applying (5.2) with (2.6), we observe

$$[w]_{1\delta T} \le [W]_{1\delta T} + cN \le 2cN$$

provided $R < \varepsilon_1$. This is the same as (5.12) by replacing $2c$ by c. □

We now obtain similar estimates like (5.2) for all $1 \le p \le \infty$ just by an interpolation.

Proposition 5.5. *Suppose the same situation as in Proposition 5.2. For δ, $0 < \delta < 1/2$, there is ε depending only on δ such that $R < \varepsilon$ implies*

$$(5.13) \qquad [w]_{r\delta T} \le cN, \quad 1 \le r \le \infty,$$

where c is a universal constant.

Proof. The Riesz-Thorin interpolation (see eg. [4]) yields

$$[w]_{r\delta T} \leq [w]_{1\delta T}^{1/r}[w]_{\infty\delta T}^{1-1/r} , \quad 1 \leq r \leq \infty .$$

Interpolating (5.8) and (5.12) now shows that $[w]_{r\delta T} \leq cN$ provided $R < \varepsilon = \min(\varepsilon_1, \varepsilon_\infty)$. □

Proof of Theorem 4.1. Since c is independent of T , (5.13) yields (4.6a) by the definition of the norm (3.14).

It remains to prove (4.6b). Since

$$\omega - e^{t\Delta}\omega_0 = \omega - \omega_* - W = w - W$$

(4.6b) follows immediately from (4.6a) and (2.6).

6. Stability of Burgers' vortex - Formation of a concentrated vortex

This section is devoted to an application of our large time asymptotic expression for the vorticity (Theorem 4.2). We consider a three-dimensional viscous incompressible flow written as a superposition of an axisymmetric irrotational flow and a two-dimensional flow whose vorticity vector directs to the symmetry axis. We study the large time behavior of such a flow when the axisymmetric flow is a inward convection-axially constant

stretching flow. We shall show the vorticity tends to Burgers' vortex [1,5] as the time tends to infinity provided the Reynolds number of two-dimensional flow is small. No particular structure of initial vorticity is assumed. There are no assumptions on the speed of the axisymmetric flow. Our asymptotic results physically imply formation of a concentrated vortex.

We consider the Navier-Stokes equations in the three dimensional space $\mathbb{R}^3$:

$$(6.1) \qquad \frac{\partial u}{\partial \tau} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = 0 \ , \quad \nabla \cdot u = 0 \ ,$$

where $u = u(y,\tau)$, $p = p(y,\tau)$, $y = (y_1, y_2, y_3)$. Suppose that our velocity field u is expressed as

$$(6.2) \qquad u = U + V$$

U : an axisymmetric irrotational divergence-free velocity field

V : a two-dimensional rotational velocity field ,

where the vorticity of u(or V) directs to the symmetry axis. To fix the idea, we take y_3 - axis as the symmetry axis. The vector field U is assumed to have the form

$$(6.3) \qquad U(y_1, y_2, y_3, \tau) = (-\alpha y_1, -\alpha y_2, 2\alpha y_3) \quad \alpha = \alpha(\tau)$$

which evidently satisfies $\nabla \cdot U = 0$ and $\nabla \times U = 0$. If α is a positive constant, U is a steady inward convection-axially stretching flow. The vector field V has the form

$$(6.4) \qquad V(y_1, y_2, y_3, \tau) = (V^1(y_1, y_2, \tau),\ V^2(y_1, y_2, \tau),\ 0)$$

so that the vorticity is

$$(6.5) \qquad (0, 0, \Omega(y_1, y_2, \tau)) \ ,$$

where $\Omega = \nabla \times V = (\partial/\partial y_1)V^2 - (\partial/\partial y_2)V^1$. We first derive the vorticity equation for Ω.

Proposition 6.1. The equations (6.1) with (6.2) - (6.5) are formally equivalent to

$$(6.6) \qquad \frac{\partial \Omega}{\partial \tau} - \nu \Delta \Omega - \alpha (y \cdot \nabla)\Omega - 2\alpha\Omega + (V \cdot \nabla)\Omega = 0$$

$$(6.7) \qquad V = K * \Omega \ , \quad K = (-y_2, y_1)/2\pi |y|^2 \ ,$$

provided that α is a constant and that V decays at space infinity.

Proof. This is very similar to the proof of the equivalence of the vorticity equation and the Navier-Stokes equation. Plugging $u = U + V$ in (6.1) and noting

$$(U \cdot \nabla)U = \nabla P \ , \quad P = \alpha^2 (y_1^2 + y_2^2 + 4y_3^2)/2$$

$$(V\cdot\nabla)U = -\alpha V\ ,\quad (U\cdot\nabla)V = -\alpha(y\cdot\nabla)V\ ,$$

we obtain

$$(6.8)\qquad \frac{\partial V}{\partial\tau} - \nu\Delta V - \alpha(y\cdot\nabla)V - \alpha V + (V\cdot\nabla)V + \nabla(P+p) = 0\ ,\ \nabla\cdot V = 0\ .$$

Taking $\nabla\times$ of (6.8) yields (6.6) since

$$\nabla\times((y\cdot\nabla)V) = (y\cdot\nabla)\Omega + \Omega\ .$$

Since $\nabla\cdot V = 0$ and V decays at $|y| = \infty$, we have (6.7). This shows that (6.1) with (6.2) - (6.5) yields (6.6) and (6.7). Since the above calculation shows that (6.6) implies that $\partial u/\partial\tau - \nu\Delta u + (u\cdot\nabla)u$ is irrotational, (6.1) now follows from (6.7). □

Our main goal is to study the large time asymptotic behavior of the vorticity Ω of (6.6)-(6.7) with arbitrary initial data $\Omega(y,0) = \Omega_0 \in M(\mathbb{R}^2)$. If $\alpha = 0$, Theorem 4.2 already gives an answer, since (6.6)-(6.7) is nothing but (4.4)-(4.5). The vorticity $\Omega(y,\tau)$ is asymptotically equivalent to $\omega_{*\kappa}$ in (4.3) called the diffused vortex filament of Oseen [17] with the total vorticity κ provided that the Reynolds number R is sufficiently small. We shall always assume that α is a positive constant unless otherwise claimed. As is before, we define the Reynolds number R and the total vorticity κ by

$$(6.9)\qquad R = \nu^{-1}\int|\Omega_0|(dy) = \nu^{-1}\|\Omega_0\|_1\ ,\quad \kappa = \int\Omega_0(dy)\ .$$

We observe that (6.6)-(6.7) has a special steady (circular symmetric) solution called Burgers' vortex [1,5] :

$$(6.10) \qquad \bar{\Omega}_\kappa = \frac{\kappa}{\pi \mathfrak{L}^2} e^{-|y|^2/\mathfrak{L}^2} , \quad \mathfrak{L} = (2\nu/\alpha)^{1/2} .$$

We shall claim below that Ω in (6.6)-(6.7) converges to $\bar{\Omega}_\kappa$ as $\tau \to \infty$ provided the Reynolds number R is small.

Theorem 6.2. *For a general initial vorticity $\Omega_0 \in M(\mathbb{R}^2)$, there is a solution $\Omega(y,\tau)$ of (6.6)-(6.7) having following properties. For every δ, $0 < \delta < 1/2$, there is $\varepsilon_\delta > 0$ (independent of α, p and τ) such that*

$$(6.11) \qquad \|\Omega(y,\tau) - \bar{\Omega}_\kappa(y)\|_p = O(e^{-2\alpha\delta\tau}) \quad \text{as} \quad \tau \to \infty ,$$

where $1 \le p \le \infty$, provided $R < \varepsilon_\delta$. Here $\bar{\Omega}_\kappa$ is Burgers' vortex defined in (6.10) and κ is the total vorticity defined in (6.9). Moreover, the total vorticity is conserved, i.e.,

$$(6.12) \qquad \kappa = \int \Omega(y,\tau)dy \qquad \text{for all} \quad \tau \ge 0 .$$

Proof. We first observe that (6.6)-(6.7) can be reduced to (4.4)-(4.5) by a time-dependent scaling transformation which is introduced by Lundgren [16] and by Kambe [11,13] for axially symmetric case. We introduce x, t, $\omega(x,t)$ such that

(6.13) $$x = A(\tau)y\ ,\quad t = \int_0^\tau A^2(\sigma)d\sigma\ ,\quad \omega(x,t) = A^{-2}(\tau)\Omega(y,\tau)$$

$$A' = \frac{d}{d\tau}A(\tau) = \alpha A\ ,\quad A(0) = 1\ .$$

Since $dt = A^2(\tau)d\tau$, $dx = A(\tau)dy$, we have

$$\begin{aligned}\partial_t\omega(x,t) &= A^{-2}\partial_\tau(\Omega(x/A(\tau),\tau)\cdot A^{-2})\\ &= A^{-4}(\partial_\tau\Omega - A'A^{-1}(y,\nabla)\Omega - 2A'A^{-1}\Omega)\ ,\\ \Delta\omega(x,t) &= A^{-4}\Delta_y\Omega\\ (v\cdot\nabla)\omega &= A^{-4}(V\cdot\nabla_y)\Omega \quad \text{with} \quad v = V/A(\tau)\ .\end{aligned}$$

Observing $A' = \alpha A$, we see ω solves (4.4). By a dilation of the variable of the integration, we obtain (4.5) by putting $v = V/A(\tau)$. (The transformation (6.13) reduces (6.6)-(6.7) to (4.4)-(4.5) even if α is time dependent.)

Suppose that $\omega(x,t)$ is a solution of the vorticity equation with ν whose existence is proved in Proposition 3.1 with an appropriate scaling. Since $A(0) = 1$ implies $\omega_0 = \omega_0(x,0) = \Omega_0(x,0) = \Omega_0(y,0)$, our asymptotic result (4.7a) yields

$$\|\omega - \omega_{*K}\|_p(t) \le CN(t\nu)^{-1+1/p-\delta},\quad 0 < \delta < 1/2,\quad 1 \le p \le \infty$$

provided that $R = \nu^{-1}\|\omega_0\|_1 = \nu^{-1}\|\Omega_0\|_1$ is sufficiently small, say, $R < \varepsilon_\delta$ where $\varepsilon = \varepsilon_\delta$ is the same as in Theorem 4.2. Using original variables y,τ,Ω in (6.13), this estimate yields

$$\|\Omega - \Omega_{*K}\|_p(\tau)(= (\int_{\mathbb{R}^2}|\Omega(y,\tau) - \Omega_{*K}(y,\tau)|^p dy)^{1/p})$$

$$\leq CN(A(\tau)^2/\nu t)^{1-1/p}(\nu t)^{-\delta}$$

where

$$(6.14)\qquad \Omega_{*K}(y,\tau) = \frac{K}{\pi\mathcal{Q}^2(1-e^{-2\alpha\tau})}\exp\left(-\frac{|y|^2}{\mathcal{Q}^2(1-e^{-2\alpha\tau})}\right),$$

$$\mathcal{Q} = (2\nu/\alpha)^{1/2},$$

since $t = (e^{2\alpha\tau} - 1)/2\alpha$, $A = e^{\alpha\tau}$. Since

$$\lim_{\tau\to\infty} A(\tau)^2/\nu t = 4/\mathcal{Q}^2$$

we have

$$\limsup_{\tau\to\infty} e^{2\alpha\delta\tau}\|\Omega - \Omega_{*K}\|_p(\tau) \leq \gamma CN, \quad \gamma = (4\mathcal{Q}^{-2})^{1-1/p+\delta}.$$

In particular,

$$(6.15)\qquad \|\Omega - \Omega_{*K}\|_p(\tau) = O(e^{-2\alpha\delta\tau}), \quad 0 < \delta < 1/2$$

as $\tau \to \infty$. As is pointed out by Kambe [12],

$$\lim_{\tau\to\infty} \Omega_{*K}(y,t) = \bar{\Omega}_K(y)$$

where $\bar{\Omega}_K$ is Burgers' vortex in (6.10). More precisely, a direct calculation to (6.14) shows that

$$(6.16)\qquad \|\Omega_{*K} - \bar{\Omega}_K\|_p = O(e^{-2\alpha\tau}) \quad \text{as} \quad \tau \to \infty$$

The estimates (6.15) and (6.16) now yield (6.11).

It remains to prove (6.12). Since

$$\int \omega(x,t)dx = \int \Omega(y,\tau)dy$$

(3.8) now yields (6.12). □

Remark 6.3. For radial initial data Ω_0 the estimate (6.11) is pointed out by Kambe [12] at least for $p = \infty$ without the assumption on the Reynolds number. In this case by Lemma 3.4, $(V\cdot\nabla)\Omega$ in (6.6) vanishes so the problem is reduced to the heat equation. For the heat equation (2.6) shows that (4.7a) holds even for large R. Parallelly to the above proof, we see (6.11) holds without the assumption on R which extends results in [12].

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