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A variational approach to self-similar solutions for semilinear heat equations

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1. Introduction

We consider the Cauchy problem for semilinear heat equations with singular initial data:

$$w_t = \Delta w + w^p \qquad \text{in } \mathbf{R}^N \times (0, \infty), \tag{1}$$

$$w(x,0) = \lambda a\,(x/|x|)\,|x|^{-2/(p-1)} \qquad \text{in } \mathbf{R}^N \setminus \{0\}, \tag{2_λ}$$

where $N > 2$, $p > 1$, $a : S^{N-1} \to \mathbf{R}$, and $\lambda > 0$ is a parameter. We assume that $a \in L^\infty(S^{N-1})$ and $a \geq 0$, $a \not\equiv 0$. A typical case is $a \equiv 1$.

The equation (1) is invariant under the similarity transformation

$$w(x,t) \mapsto w_\mu(x,t) = \mu^{2/(p-1)} w(\mu x, \mu^2 t) \quad \text{for all } \mu > 0.$$

In particular, a solution w is said to be *self-similar*, when $w = w_\mu$ for all $\mu > 0$, that is,

$$w(x,t) = \mu^{2/(p-1)} w(\mu x, \mu^2 t) \quad \text{for all } \mu > 0. \tag{3}$$

Such self-similar solutions are global in time and often used to describe the large time behavior of global solutions to (1), see, e.g., [14, 15, 5, 21].

If $w(x,t)$ is a self-similar solution of (1.1) and has an initial value $A(x)$, then we easily see that A has the form $A(x) = A(x/|x|)|x|^{-2/(p-1)}$. Then the problem of existence of self-similar solutions is essentially depend on the solvablity of the Cauchy problem (1)-$(2)_\lambda$. In this talk we consider the existence of self-similar solutions of the problem (1)-$(2)_\lambda$. The idea of constructing self-similar solutions by solving the initial value problem for homogeneous initial data goes back to the study by Giga and Miyakawa [12] for the Navier-Stokes equation in vorticity form.

It is well known by Fujita [9] that if $1 < p \leq (N+2)/N$ then (1) has no time global solution w such that $w \geq 0$ and $w \not\equiv 0$. (See also [25, 14].) Then the condition $p > (N+2)/N$ is necessary for the existence of positive self-similar solutions of (1).

We briefly review some results concerning the Cauchy problem for (1) with initial date in $L^q(\mathbf{R}^N)$. Weissler [23, 24] showed that the IVP (1) with $w(x,0) = A \in L^q(\mathbf{R}^N)$ admits a unique time-local solution if $q \geq N(p-1)/2$. He also showed in [25] that the solution exists time-globally if $q = N(p-1)/2$ and if $\|A\|_{L^q(\mathbf{R}^N)}$ is sufficiently small. Giga [11] has constructed a unique local regular solution in $L^\alpha(0,T : L^\beta)$, where α and β are chosen so that the norm of $L^\alpha(0,T : L^\beta)$ is invariant under scaling. On the other hand, for $1 \leq q < N(p-1)/2$, Haraux and Weissler [13] constructed a solution $w_0 \in C([0,\infty); L^q(\mathbf{R}^N))$ of (1) satisfying $w_0(x,t) > 0$ for $t > 0$ and $\|w_0(\cdot,t)\|_{L^q(\mathbf{R}^N)} \to 0$ as $t \to 0$ when $(N+2)/N < p < (N+2)/(N-2)$ by seeking solutions of self-similar form. Therefore, the Cauchy problem

$$w_t = \Delta w + w^p \quad \text{in } \mathbf{R}^N \times (0,\infty) \quad \text{and} \quad w(x,0) = 0 \quad \text{in } \mathbf{R}^N \tag{4}$$

admits a non-unique solution in $C([0,\infty); L^q(\mathbf{R}^N))$ for $1 \leq q < N(p-1)/2$ when $(N+2)/N < p < (N+2)/(N-2)$.

Kozono and Yamazaki [16] constructed Besov-type function spaces based on the Morrey spaces, and then obtained global existence results for the equation (1) and the Navier-Stokes system with small initial data in these spaces. Cazenave and Weissler [5] proved the existence of global solutions, including self-similar solutions, to the nonlinear Schrödinger equations and the equation (1) with small initial data by using the weighted norms. By [16, 5] the problem (1)-$(2)_\lambda$ admits a time-global solution for sufficiently small $\lambda > 0$.

We note here that the equation (1) with $p > N/(N-2)$ has a positive singular stationary solution $W(x) = L|x|^{-2/(p-1)}$, where

$$L = \left[\frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right)\right]^{1/(p-1)}.$$

Galaktionov and Vazquez [10] investigated the uniqueness of solutions to the problem (1)-$(2)_\lambda$ in the case where $a \equiv 1$ and $\lambda = L$, and showed that the problem has a classical self-similar solution for $t > 0$ with certain values of p. In [10, p. 41] they also conjectured that the problem (1)-$(2)_\lambda$ has exactly two solutions (the minimal and maximal) when $N/(N-2) < p \leq (N+2)/(N-2)$.

Letting $\mu = t^{-1/2}$ in (3), we see that the self-similar solution w of (1) has the form

$$w(x,t) = t^{-1/(p-1)} u(x/\sqrt{t}), \tag{5}$$

where u satisfies the elliptic equation

$$\Delta u + \frac{1}{2} x \cdot \nabla u + \frac{1}{p-1} u + u^p = 0 \qquad \text{in } \mathbf{R}^N. \tag{6}$$

In addition, if w satisfies $(2)_\lambda$ in the sense of $L^1_{\rm loc}(\mathbf{R}^N)$, then u satisfies

$$(7)_\lambda \qquad \lim_{r\to\infty} r^{2/(p-1)}u(r\omega) = \lambda a(\omega) \qquad \text{for a.e.} \quad \omega \in S^{N-1}.$$

Conversely, if $u \in C^2(\mathbf{R}^N)$ is a solution of (6) satisfying $(7)_\lambda$, then the function w defined by (5) satisfies (1)-$(2)_\lambda$ in the sense of $L^1_{\rm loc}(\mathbf{R}^N)$. (See Lemma B.1 in [17].)

In this talk we investigate the problem (6)-$(7)_\lambda$ by making use of the methods for semilinear elliptic equations to derive the results for the Cauchy problem (1)-$(2)_\lambda$. First, we show the existence of the minimal solution by employing the comparison results based on the maximum principle. Next we apply the variational method due to [1, 6, 4] to show the existence of the second solution of the problem (6)-$(7)_\lambda$, which implies the non-uniqueness of solutions to the problem (1)-$(2)_\lambda$.

2. Existence of the minimal solution [17, Sec. 4]

For simplicity, we define $\mathcal{L}u$ by

$$\mathcal{L}u = \Delta u + \frac{1}{2}x\cdot\nabla u + \frac{1}{p-1}u$$

for $u \in C^2(\mathbf{R}^N)$. First we obtain the following results.

Lemma 1. *Let $p > (N+2)/N$. Assume that $-\mathcal{L}u \ge 0$ in $\mathbf{R}^N$, and that*

$$\liminf_{|x|\to\infty} |x|^{2/(p-1)}u(x) \ge 0.$$

Then $u > 0$ or $u \equiv 0$ in $\mathbf{R}^N$. In particular, if $-\mathcal{L}u \ge 0$ and $u \ge 0$ in $\mathbf{R}^N$ then $u > 0$ or $u \equiv 0$ in $\mathbf{R}^N$.

Lemma 2. *Assume that $p > (N+2)/N$, and that $\alpha, \beta \in L^\infty(S^{N-1})$ satisfy $0 \le \alpha(\omega) \le \beta(\omega)$ for a.e. $\omega \in S^{N-1}$. Suppose that there exists a positive function v satisfying*

$$-\mathcal{L}v \ge v^p \quad \text{in } \mathbf{R}^N \quad \text{and} \quad \lim_{r\to\infty} r^{2/(p-1)}v(r\omega) = \beta(\omega), \quad \text{a.e. } \omega \in S^{N-1}.$$

Then there exists a positive solution u of the problem

$$-\mathcal{L}u = u^p \quad \text{in } \mathbf{R}^N \quad \text{and} \quad \lim_{r\to\infty} r^{2/(p-1)}u(r\omega) = \alpha(\omega), \quad \text{a.e. } \omega \in S^{N-1}.$$

By using of Lemmas 1 and 2 we obtain the following:

Theorem 1. *Assume that* $p > (N+2)/N$. *Then there exists a constant* $\overline{\lambda} > 0$ *such that*

(i) *for* $0 < \lambda < \overline{\lambda}$, (6)-(7)$_\lambda$ *has a positive minimal solution* $\underline{u}_\lambda \in C^2(\mathbf{R}^N)$; *the solution* $\underline{u}_\lambda$ *is increasing with respect to* λ *and satisfies* $\|\underline{u}_\lambda\|_{L^\infty(\mathbf{R}^N)} \to 0$ *as* $\lambda \to 0$;

(ii) *for* $\lambda > \overline{\lambda}$, *there are no positive solutions* $u \in C^2(\mathbf{R}^N)$ *of* (6)-(7)$_\lambda$.

3. Weighted Sobolev space

Put $\rho(x) = e^{|x|^2/4}$. Then the equation (6) can be written as

$$\nabla \cdot (\rho \nabla u) + \rho \left(\frac{1}{p-1} u + u^p \right) = 0.$$

Escobedo-Kavian [8] investigated the corresponding functional

$$I_0(u) = \frac{1}{2} \int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx - \frac{1}{p+1} \int_{\mathbf{R}^N} u^{p+1} \rho dx$$

on the weighted functional spaces

$$L^q_\rho(\mathbf{R}^N) = \left\{ u \in L^q(\mathbf{R}^N) : \int_{\mathbf{R}^N} u^q \rho dx < \infty \right\} \quad \text{for } 1 \le q < \infty$$

and

$$H^1_\rho(\mathbf{R}^N) = \left\{ u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} (|\nabla u|^2 + u^2) \rho dx < \infty \right\}.$$

We recall here some results about the weighted Sobolev space $H^1_\rho(\mathbf{R}^N)$.

Lemma 3 [8, 14]. (i) *For every* $u \in H^1_\rho(\mathbf{R}^N)$,

$$\frac{N}{2} \int_{\mathbf{R}^N} u^2 \rho dx \le \int_{\mathbf{R}^N} |\nabla u|^2 \rho dx.$$

(ii) *The embedding* $H^1_\rho(\mathbf{R}^N) \subset L^{p+1}_\rho(\mathbf{R}^N)$ *is continuous for* $1 \le p \le (N+2)/(N-2)$, *and is compact for* $1 \le p < (N+2)/(N-2)$.

It was shown by [8, 24] that there exists a solution u_0 of the problem

$$\text{(8)} \qquad \begin{cases} \Delta u + \dfrac{1}{2} x \cdot \nabla u + \dfrac{1}{p-1} u + u^p = 0 & \text{in } \mathbf{R}^N, \\ u \in H^1_\rho(\mathbf{R}^N) \quad \text{and} \quad u > 0 & \text{in } \mathbf{R}^N, \end{cases}$$

with $(N+2)/N < p < (N+2)/(N-2)$. Moreover, it was shown in [8] that $u_0 \in C^2(\mathbf{R}^N)$ and $u_0(x) = O(e^{-|x|^2/8})$ as $|x| \to \infty$. The uniqueness of the solution to the problem (8) was obtained by combining the results [7, 27, 19].

Now put

$$w_0(x,t) = t^{-1/(p-1)} u_0(x/\sqrt{t}), \tag{9}$$

where u_0 is the solution of the problem (8). We note that $u_0 \in L^q(\mathbf{R}^N)$ for all $q \geq 1$ and

$$\|w_0(\cdot,t)\|_{L^q(\mathbf{R}^N)} = t^{-1/(p-1)+N/2q} \|u_0\|_{L^q(\mathbf{R}^N)}.$$

Then w_0 solves the the Cauchy problem (4) in $C([0,\infty); L^q(\mathbf{R}^N))$ for $1 \leq q < N(p-1)/2$. By the uniqueness result [19], we find that w_0 defined by (9) coincides with the non-unique solution of (4) constructed by [13].

4. Existence of the second solution: subcritical case [17, Sec. 5]

Let $\underline{u}_\lambda$ be the positive minimal solution of (6)-(7)$_\lambda$ obtained in Theorem 1. In order to find a second solution of (6)-(7)$_\lambda$ we introduce the following problem:

$$(10)_\lambda \qquad \begin{cases} \Delta u + \dfrac{1}{2} x \cdot \nabla u + \dfrac{1}{p-1} u + g(u, \underline{u}_\lambda) = 0 \quad \text{in } \mathbf{R}^N, \\ u \in H^1_\rho(\mathbf{R}^N) \quad \text{and} \quad u > 0 \quad \text{in } \mathbf{R}^N, \end{cases}$$

where $g(t,s) = (t+s)^p - s^p$. We easily see that, if (10)$_\lambda$ possesses a solution u_λ, then we can get another positive solution $\overline{u}_\lambda = \underline{u}_\lambda + u_\lambda$ of (6)-(7)$_\lambda$.

In this section we will show the existence of solutions of (10)$_\lambda$ in the subcritical case $(N+2)/N < p < (N+2)/(N-2)$ by using the variational method. To this end we define the corresponding functional of (10)$_\lambda$ by

$$I_\lambda(u) = \frac{1}{2} \int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx - \int_{\mathbf{R}^N} G(u, \underline{u}_\lambda) \rho dx$$

with $u \in H^1_\rho(\mathbf{R}^N)$, where

$$G(t,s) = \frac{1}{p+1}(t+s)^{p+1} - \frac{1}{p+1} s^{p+1} - s^p t.$$

We see that the nontrivial critical point $u \in H^1_\rho(\mathbf{R}^N)$ of the functional I_λ is a weak solution of the equation in (10)$_\lambda$. Moreover, we have $u_\lambda \in C^2(\mathbf{R}^N)$ and $u_\lambda > 0$ in $\mathbf{R}^N$ by employing the bootstrap arguments and the maximum principle.

We will verify the existence of nontrivial solution of (10)$_\lambda$ by means of the Mountain Pass lemma ([1, 20]).

Lemma 4. *For $\lambda \in (0, \overline{\lambda})$ there exist some constants $\delta = \delta(\lambda) > 0$ and $\eta = \eta(\lambda) > 0$ such that $I_\lambda(u) \geq \eta$ for all $u \in H^1_\rho(\mathbf{R}^N)$ with $\|\nabla u\|_{L^2_\rho} = \delta$.*

Lemma 5. *For any $v \in H^1_\rho(\mathbf{R}^N)$ with $v \geq 0$, $v \not\equiv 0$, we have $I_\lambda(tv) \to -\infty$ as $t \to \infty$.*

Lemma 6. *The functional I_λ satisfies the Palais-Smale condition, that is, any sequence $\{u_k\} \subset H^1_\rho(\mathbf{R}^N)$ such that*

$$\{I_\lambda(u_k)\} \text{ is bounded} \quad \text{and} \quad I'_\lambda(u_k) \to 0 \text{ as } k \to \infty$$

contains a convergent subsequence.

In the proofs of Lemmas 4-6, the following results play a crucial role.

Lemma 7. *Let $\underline{u}_\lambda$ be the minimal solution obtained in Theorem 1 for $\lambda \in (0, \overline{\lambda})$. Then the linearized eigenvalue problem*

$$\begin{cases} -\Delta w - \frac{1}{2} x \cdot \nabla w - \frac{1}{p-1} w = \mu p [\underline{u}_\lambda]^{p-1} w \quad \text{in } \mathbf{R}^N, \\ w \in H^1_\rho(\mathbf{R}^N), \end{cases}$$

has the first eigenvalue $\mu = \mu(\lambda) > 1$. Moreover, $\mu(\lambda)$ is strictly decreasing in $\lambda \in (0, \overline{\lambda})$.

Lemma 7 follows from the fact that $\underline{u}_\lambda$ is the positive minimal solution.

As a consequence of Lemmas 4-6 we obtain the following:

Theorem 2. *Assume that $(N+2)/N < p < (N+2)/(N-2)$. Then, for $0 < \lambda < \overline{\lambda}$, there exists a positive solution $\overline{u}_\lambda$ of (6)-(7)$_\lambda$ satisfying $\overline{u}_\lambda > \underline{u}_\lambda$,*

$$\overline{u}_\lambda - \underline{u}_\lambda \in H^1_\rho(\mathbf{R}^N), \quad \text{and} \quad \overline{u}_\lambda(x) - \underline{u}_\lambda(x) = O(e^{-|x|^2/4}) \quad \text{as } |x| \to \infty.$$

Furthermore,

$$\overline{u}_\lambda - \underline{u}_\lambda \to u_0 \quad \text{in } H^1_\rho(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N) \quad \text{as } \lambda \to 0,$$

where u_0 is the solution of the problem (8). In particular, $\overline{u}_\lambda \to u_0$ in $L^\infty(\mathbf{R}^N)$ as $\lambda \to 0$.

Now we consider the Cauchy problem (1)-(2)$_\lambda$. Recall that, if u is a solution of (6)-(7)$_\lambda$, then the function w defined by (5) is a solution of (1)-(2)$_\lambda$ in the sense of $L^1_{\text{loc}}(\mathbf{R}^N)$, and that w_0 defined by (9) coincides with the non-unique solution of (4) constructed by [13]. As a consequence of Theorems 1 and 2, we obtain the following results.

Corollary 1. *Assume that* $p > (N+2)/N$. *Then there exists a constant* $\overline{\lambda} > 0$ *such that*

(i) *for* $0 < \lambda < \overline{\lambda}$, (1)-(2)$_\lambda$ *has a positive self-similar solution* $\underline{w}_\lambda$; *the solution* $\underline{w}_\lambda(\cdot, t)$ *satisfies, for each fixed* $t > 0$,

$$\|\underline{w}_\lambda(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} \to 0 \quad \text{as } \lambda \to 0;$$

(ii) *for* $\lambda > \overline{\lambda}$, (1)-(2)$_\lambda$ *has no positive self-similar solutions.*

Assume, furthermore, that $p < (N+2)/(N-2)$. *Then* (1)-(2)$_\lambda$ *has a positive self-similar solution* $\overline{w}_\lambda$ *satisfying* $\overline{w}_\lambda > \underline{w}_\lambda$ *in* $\mathbf{R}^N \times (0, \infty)$ *for* $0 < \lambda < \overline{\lambda}$. *The solution* $\overline{w}_\lambda$ *satisfies, for each fixed* $t > 0$,

$$\|\overline{w}_\lambda(\cdot, t) - w_0(\cdot, t)\|_{L^\infty(\mathbf{R}^N)} \to 0 \quad \text{as } \lambda \to 0,$$

where w_0 *is the non-unique solution of* (4) *in* $C([0, \infty); L^q(\mathbf{R}^N))$ *for* $1 \leq q < N(p-1)/2$, *which is constructed by* [13].

5. Existence and nonexistence of second solutions: critical case [18]

In this section we consider the existence and nonexistence of second solutions of the problem (6)-(7)$_\lambda$ in the critical case $p = (N+2)/(N-2)$ by following the argument due to Brezis-Nirenberg [4].

For the critical growth case, there are serious difficulties in obtaining solutions by using variational methods because the Sobolev embedding $H^1 \subset L^{p+1}$ is not compact. It is well known that this lack of compactness exhibits many interesting existence and nonexistence phenomena. See, e.g., [4, 2].

Let us denote by S the best Sobolev constant of the embedding $H^1(\mathbf{R}^N) \subset L^{2N/(N+2)}(\mathbf{R}^N)$, which is given by

$$S = \inf_{u \in H^1(\mathbf{R}^N) \setminus \{0\}} \frac{\displaystyle\int_{\mathbf{R}^N} |\nabla u|^2 dx}{\left(\displaystyle\int_{\mathbf{R}^N} |u|^{2N/(N-2)} dx\right)^{(N-2)/N}}.$$

In the critical case, the functional I_λ satisfies the following local Palais-Smale condition.

Lemma 8. *Let* $p = (N+2)/(N-2)$. *Then* I_λ *satisfies the* $(PS)_c$ *condition for* $c \in (0, S^{N/2}/N)$, *that is, any sequence* $\{u_k\} \subset H^1_\rho(\mathbf{R}^N)$ *such that*

$$I_\lambda(u_k) \to c \in \left(0, \frac{1}{N} S^{N/2}\right) \quad \text{and} \quad I'_\lambda(u_k) \to 0 \quad \text{as } k \to \infty$$

contains a convergent subsequence.

By Lemma 8 and the Mountain Pass lemma, we obtain the following existence result.

Lemma 9. *Let* $p = (N+2)/(N-2)$*. Assume that there exists* $v_0 \in H^1_\rho(\mathbf{R}^N)$ *with* $v_0 \geq 0$*,* $v_0 \not\equiv 0$ *such that*

$$\sup_{t>0} I_\lambda(tv_0) < \frac{1}{N} S^{N/2}. \tag{11}$$

Then there exists a positive solution $u_\lambda \in H^1_\rho(\mathbf{R}^N)$ *of* $(10)_\lambda$*.*

Moreover, we have $u_\lambda \in C^2(\mathbf{R}^N)$ by employing the estimate due to Brezis-Kato [3], based on the Moser's iteration technique.

In order to find a positive function $v_0 \in H^1_\rho(\mathbf{R}^N)$ satisfying (11), we set

$$u_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon + |x|^2)^{(N-2)/2}} \rho^{-1/2} \quad \text{and} \quad v_\varepsilon(x) = \frac{u_\varepsilon(x)}{\|u_\varepsilon\|_{L^{p+1}_\rho}}$$

for $\varepsilon > 0$, where $\phi \in C_0^\infty(\mathbf{R}^N)$ is a cut off function. We remark that the functional I_λ can be written as

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2} \int_{\mathbf{R}^N} \left(|\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx - \frac{1}{p+1} \int_{\mathbf{R}^N} u^{p+1} \rho dx \\ &\qquad\qquad\qquad\qquad - \int_{\mathbf{R}^N} H(u, \underline{u}_\lambda) \rho dx \\ &\equiv I_0(u) - \int_{\mathbf{R}^N} H(u, \underline{u}_\lambda) \rho dx, \end{aligned}$$

where

$$H(t,s) = G(t,s) - \frac{1}{p+1} t^{p+1}.$$

Lemma 10. *For sufficient small* $\varepsilon > 0$*, there exists* $t_\varepsilon > 0$ *such that* $\sup_{t>0} I_\lambda(tv_\varepsilon) = I_\lambda(t_\varepsilon v_\varepsilon)$*. Moreover, as* $\varepsilon \to 0$ *we have*

$$I_0(t_\varepsilon v_\varepsilon) \leq \frac{1}{N} S^{N/2} + \begin{cases} O(\varepsilon), & N \geq 5 \\ O(\varepsilon |\log \varepsilon|), & N = 4 \\ O(\varepsilon^{1/2}), & N = 3 \end{cases}$$

$$\int_{\mathbf{R}^N} H(t_\varepsilon v_\varepsilon, \underline{u}_\lambda) \rho dx \geq \begin{cases} C\varepsilon^{3/4}, & N = 5 \\ C\varepsilon^{1/2}, & N = 4 \\ C\varepsilon^{1/4}, & N = 3 \end{cases}$$

with some constant $C > 0$*.*

As a consequence, we obtain the following:

Theorem 3. *Let* $p = (N+2)/(N-2)$ *and* $N = 3, 4, 5$. *Then, for* $0 < \lambda < \overline{\lambda}$, *the problem* (6)-$(7)_\lambda$ *has a positive solution* $\overline{u}_\lambda \in C^2(\mathbf{R}^N)$ *satisfying* $\overline{u}_\lambda > \underline{u}_\lambda$ *and* $\overline{u}_\lambda - \underline{u}_\lambda \in H^1_\rho(\mathbf{R}^N)$.

On the other hand, for the case $N \geq 6$ we obtain the uniqueness result in the radial class by employing the Pohozaev type identity.

Theorem 4. *Let* $p = (N+2)/(N-2)$ *and* $N \geq 6$. *Assume that* $a \equiv 1$ *in* $(7)_\lambda$. *Then there exists a constant* $\lambda_0 \in (0, \overline{\lambda})$ *such that* (6)-$(7)_\lambda$ *has no positive radial solutions* $u \in C^2(\mathbf{R}^N)$ *with* $u \not\equiv \underline{u}_\lambda$ *for* $\lambda \in (0, \lambda_0)$, *that is,* (6)-$(7)_\lambda$ *has a unique positive radial solution* $\underline{u}_\lambda$ *for* $0 < \lambda < \lambda_0$.

REFERENCES

[1] A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, J. Functional Analysis **14** (1973), 349-381.

[2] H. Brezis, Elliptic equations with limiting Sobolev exponents—the impact of topology, Comm. Pure Appl. Math. **39** (1986), 17-39.

[3] H. Brezis and T. Kato, Remarks on the Schrödinger operator with singular complex potentials, J. Math. Pures Appl. **58** (1979), 137-151.

[4] H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. **36** (1983), 437-477.

[5] T. Cazenave and F. B. Weissler, Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations, Math. Z. **228** (1998), 83-120.

[6] M. G. Crandall and P. H. Rabinowitz, Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, Arch. Rational Mech. Anal. **58** (1975), 207-218.

[7] C. Dohmen and M. Hirose, Structure of positive radial solutions to the Haraux-Weissler equation, Nonlinear Anal. TMA **33** (1998), 51-69.

[8] M. Escobedo and O. Kavian, Variational problems related to self-similar solutions for the heat equation, Nonlinear Anal. TMA **11** (1987), 1103-1133.

[9] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo, Sect.I **13** (1966), 109-124.

[10] V. A. Galaktionov and J. L. Vazquez, Continuation of blowup solutions of nonlinear heat equations in several space dimensions, Comm. Pure Appl. Math. **50** (1997) 1-67.

[11] Y. Giga, Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system, J. Differential Equations **62** (1986), 186-212.

[12] Y. Giga and T. Miyakawa, Navier-Stokes flow in $\mathbf{R}^3$ with measures as initial vorticity and Morrey spaces, Comm. Partial Differential Equations **14** (1989), 577-618.

[13] A. Haraux and F. B. Weissler, Non-uniqueness for a semilinear initial value problem, Indiana Univ. Math. J. **31** (1982), 167-189.

[14] O. Kavian, Remarks on the large time behavior of a nonlinear diffusion equation, Ann. Inst. H. Poincaré Anal. Non Linéaire **4** (1987), 423-452.

[15] T. Kawanago, Asymptotic behavior of solutions of a semilinear heat equation with subcritical nonlinearity, Ann. Inst. H. Poincaré Anal. Non Linéaire **13** (1996), 1-15.

[16] H. Kozono and M. Yamazaki Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data, Comm. Partial Differential Equations **19** (1994), 959-1014.

[17] Y. Naito, Non-uniqueness of solutions to the Cauchy problem for semilinear heat equations with singular initial data, Math. Ann. **329** (2004), 161-196.

[18] Y. Naito, Self-similar solutions for a semilinear heat equation with critical Sobolev exponent, preprint.

[19] Y. Naito and T. Suzuki, Radial symmetry of self-similar solutions for semilinear heat equations, J. Differential Equations **163** (2000), 407-428.

[20] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, Amer.Math. Soc., Providence, 1986.

[21] S. Snoussi and S. Tayachi, and F. B. Weissler, Asymptotically self-similar global solutions of a general semilinear heat equation, Math. Ann. **321** (2001), 131-155.

[22] P. Souplet and F. B. Weissler, Regular self-similar solutions of the nonlinear heat equation with initial data above the singular steady state, Ann. Inst. H. Poincare Anal. Non Lineaire **20** (2003), 213-235.

[23] F. B. Weissler, Semilinear evolution equations in Banach spaces, J. Funct. Anal. **32** (1979), 277-296.

[24] F. B. Weissler, Local existence and nonexistence for semilinear parabolic equations in L^p, Indiana Univ. Math. J. **29** (1980), 79-102.

[25] F. B. Weissler, Existence and non-existence of global solutions for a semilinear heat equation, Israel J. Math. **38** (1981), 29-40.

[26] F. B. Weissler, Rapidly decaying solutions of an ordinary differential equation with applications to semilinear elliptic and parabolic partial differential equations, Arch. Rational Mech. Anal. **91** (1985), 247-266.

[27] E. Yanagida, Uniqueness of rapidly decaying solutions to the Haraux-Weissler equation, J. Differential Equations **127** (1996), 561-570.