1. Introduction

We consider the Cauchy problem for semilinear heat equations with singular initial data:

\begin{align}
(1) & \quad w_t = \Delta w + w^p \quad \text{in } \mathbb{R}^N \times (0, \infty), \\
(2) & \quad w(x, 0) = \lambda a(x/|x|)|x|^{-2/(p-1)} \quad \text{in } \mathbb{R}^N \setminus \{0\},
\end{align}

where $N > 2$, $p > 1$, $a : S^{N-1} \to \mathbb{R}$, and $\lambda > 0$ is a parameter. We assume that $a \in L^\infty(S^{N-1})$ and $a \geq 0$, $a \not\equiv 0$. A typical case is $a \equiv 1$.

The equation (1) is invariant under the similarity transformation

$$w(x, t) \mapsto w_\mu(x, t) = \mu^{2/(p-1)}w(\mu x, \mu^2 t) \quad \text{for all } \mu > 0.$$ 

In particular, a solution $w$ is said to be self-similar, when $w = w_\mu$ for all $\mu > 0$, that is,

$$w(x, t) = \mu^{2/(p-1)}w(\mu x, \mu^2 t) \quad \text{for all } \mu > 0.$$ 

Such self-similar solutions are global in time and often used to describe the large time behavior of global solutions to (1), see, e.g., [14, 15, 5, 21].

If $w(x, t)$ is a self-similar solution of (1.1) and has an initial value $A(x)$, then we easily see that $A$ has the form $A(x) = A(x/|x|)|x|^{-2/(p-1)}$. Then the problem of existence of self-similar solutions is essentially depend on the solvability of the Cauchy problem (1)-(2)\(\lambda\). In this talk we consider the existence of self-similar solutions of the problem (1)-(2)\(\lambda\). The idea of constructing self-similar solutions by solving the initial value problem for homogeneous initial data goes back to the study by Giga and Miyakawa [12] for the Navier-Stokes equation in vorticity form.

It is well known by Fujita [9] that if $1 < p \leq (N+2)/N$ then (1) has no time global solution $w$ such that $w \geq 0$ and $w \not\equiv 0$. (See also [25, 14].) Then the condition $p > (N+2)/N$ is necessary for the existence of positive self-similar solutions of (1).
We briefly review some results concerning the Cauchy problem for (1) with initial date in $L^q(\mathbb{R}^N)$. Weissler [23, 24] showed that the IVP (1) with $w(x, 0) = A \in L^q(\mathbb{R}^N)$ admits a unique time-local solution if $q \geq N(p - 1)/2$. He also showed in [25] that the solution exists time-globally if $q = N(p - 1)/2$ and if $\|A\|_{L^q(\mathbb{R}^N)}$ is sufficiently small. Giga [11] has constructed a unique local regular solution in $L^\alpha(0, T; L^\beta)$, where $\alpha$ and $\beta$ are chosen so that the norm of $L^\alpha(0, T; L^\beta)$ is invariant under scaling. On the other hand, for $1 \leq q < N(p - 1)/2$, Haraux and Weissler [13] constructed a solution $w_0 \in C([0, \infty); L^q(\mathbb{R}^N))$ of (1) satisfying $w_0(x, t) > 0$ for $t > 0$ and $\|w_0(\cdot, t)\|_{L^q(\mathbb{R}^N)} \to 0$ as $t \to 0$ when $(N + 2)/N < p < (N + 2)/(N - 2)$ by seeking solutions of self-similar form. Therefore, the Cauchy problem

$$(4) \quad w_t = \Delta w + w^p \quad \text{in } \mathbb{R}^N \times (0, \infty) \quad \text{and} \quad w(x, 0) = 0 \quad \text{in } \mathbb{R}^N$$

admits a non-unique solution in $C([0, \infty); L^q(\mathbb{R}^N))$ for $1 \leq q < N(p - 1)/2$ when $(N + 2)/N < p < (N + 2)/(N - 2)$.

Kozono and Yamazaki [16] constructed Besov-type function spaces based on the Morrey spaces, and then obtained global existence results for the equation (1) and the Navier-Stokes system with small initial data in these spaces. Cazenave and Weissler [5] proved the existence of global solutions, including self-similar solutions, to the nonlinear Schrödinger equations and the equation (1) with small initial data by using the weighted norms. By [16, 5] the problem (1)-(2) admits a time-global solution for sufficiently small $\lambda > 0$.

We note here that the equation (1) with $p > N/(N - 2)$ has a positive singular stationary solution $W(x) = L|x|^{-2/(p-1)}$, where

$$L = \left[\frac{2}{p-1} \left( N - 2 - \frac{2}{p-1} \right) \right]^{1/(p-1)}.$$

Galaktionov and Vazquez [10] investigated the uniqueness of solutions to the problem (1)-(2)$_{\lambda}$ in the case where $a \equiv 1$ and $\lambda = L$, and showed that the problem has a classical self-similar solution for $t > 0$ with certain values of $p$. In [10, p. 41] they also conjectured that the problem (1)-(2)$_{\lambda}$ has exactly two solutions (the minimal and maximal) when $N/(N - 2) < p \leq (N + 2)/(N - 2)$.

Letting $\mu = t^{-1/2}$ in (3), we see that the self-similar solution $w$ of (1) has the form

$$(5) \quad w(x, t) = t^{-1/(p-1)} u(x/\sqrt{t}),$$

where $u$ satisfies the elliptic equation

$$(6) \quad \Delta u + \frac{1}{2} x \cdot \nabla u + \frac{1}{p-1} u + u^p = 0 \quad \text{in } \mathbb{R}^N.$$
In addition, if \( w \) satisfies (2)\(_\lambda \) in the sense of \( L^1_{\text{loc}}(\mathbb{R}^N) \), then \( u \) satisfies

\[
(7)_{\lambda} \quad \lim_{r \to \infty} r^{2/(p-1)} u(r\omega) = \lambda a(\omega) \quad \text{for a.e. } \omega \in S^{N-1}.
\]

Conversely, if \( u \in C^2(\mathbb{R}^N) \) is a solution of (6) satisfying (7)\(_\lambda \), then the function \( w \) defined by (5) satisfies (1)-(2)\(_\lambda \) in the sense of \( L^1_{\text{loc}}(\mathbb{R}^N) \). (See Lemma B.1 in [17].)

In this talk we investigate the problem (6)-(7)\(_\lambda \) by making use of the methods for semilinear elliptic equations to derive the results for the Cauchy problem (1)-(2)\(_\lambda \). First, we show the existence of the minimal solution by employing the comparison results based on the maximum principle. Next we apply the variational method due to [1, 6, 4] to show the existence of the second solution of the problem (6)-(7)\(_\lambda \), which implies the non-uniqueness of solutions to the problem (1)-(2)\(_\lambda \).

2. Existence of the minimal solution  [17, Sec. 4]

For simplicity, we define \( \mathcal{L} u \) by

\[
\mathcal{L} u = \Delta u + \frac{1}{2} x \cdot \nabla u + \frac{1}{p-1} u
\]

for \( u \in C^2(\mathbb{R}^N) \). First we obtain the following results.

**Lemma 1.** Let \( p > (N+2)/N \). Assume that \( -\mathcal{L} u \geq 0 \) in \( \mathbb{R}^N \), and that

\[
\lim \inf_{|x| \to \infty} |x|^{2/(p-1)} u(x) \geq 0.
\]

Then \( u > 0 \) or \( u \equiv 0 \) in \( \mathbb{R}^N \). In particular, if \( -\mathcal{L} u \geq 0 \) and \( u \geq 0 \) in \( \mathbb{R}^N \) then \( u > 0 \) or \( u \equiv 0 \) in \( \mathbb{R}^N \).

**Lemma 2.** Assume that \( p > (N+2)/N \), and that \( \alpha, \beta \in L^\infty(S^{N-1}) \) satisfy

\[
0 \leq \alpha(\omega) \leq \beta(\omega) \quad \text{for a.e. } \omega \in S^{N-1}.
\]

Suppose that there exists a positive function \( v \) satisfying

\[
-\mathcal{L} v \geq v^p \quad \text{in } \mathbb{R}^N \quad \text{and} \quad \lim_{r \to \infty} r^{2/(p-1)} v(r\omega) = \beta(\omega), \quad \text{a.e. } \omega \in S^{N-1}.
\]

Then there exists a positive solution \( u \) of the problem

\[
-\mathcal{L} u = u^p \quad \text{in } \mathbb{R}^N \quad \text{and} \quad \lim_{r \to \infty} r^{2/(p-1)} u(r\omega) = \alpha(\omega), \quad \text{a.e. } \omega \in S^{N-1}.
\]

By using of Lemmas 1 and 2 we obtain the following:
Theorem 1. Assume that \( p > (N + 2)/N \). Then there exists a constant \( \lambda > 0 \) such that

(i) for \( 0 < \lambda < \lambda \), (6)-(7) has a positive minimal solution \( u_\lambda \in C^2(\mathbb{R}^N) \); the solution \( u_\lambda \) is increasing with respect to \( \lambda \) and satisfies \( \|u_\lambda\|_{L^\infty(\mathbb{R}^N)} \to 0 \) as \( \lambda \to 0 \);

(ii) for \( \lambda > \lambda \), there are no positive solutions \( u \in C^2(\mathbb{R}^N) \) of (6)-(7).

3. Weighted Sobolev space

Put \( \rho(x) = e^{\frac{|x|^2}{4}} \). Then the equation (6) can be written as

\[
\nabla \cdot (\rho \nabla u) + \rho \left( \frac{1}{p-1} u + u^p \right) = 0.
\]

Escobedo-Kavian [8] investigated the corresponding functional

\[
I_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx - \frac{1}{p+1} \int_{\mathbb{R}^N} u^{p+1} \rho dx
\]

on the weighted functional spaces

\[
L^q_\rho(\mathbb{R}^N) = \left\{ u \in L^q(\mathbb{R}^N) : \int_{\mathbb{R}^N} u^q \rho dx < \infty \right\} \quad \text{for } 1 \leq q < \infty
\]

and

\[
H^1_\rho(\mathbb{R}^N) = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) \rho dx < \infty \right\}.
\]

We recall here some results about the weighted Sobolev space \( H^1_\rho(\mathbb{R}^N) \).

Lemma 3 [8, 14]. (i) For every \( u \in H^1_\rho(\mathbb{R}^N) \),

\[
\frac{N}{2} \int_{\mathbb{R}^N} u^2 \rho dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 \rho dx.
\]

(ii) The embedding \( H^1_\rho(\mathbb{R}^N) \subset L^{p+1}_\rho(\mathbb{R}^N) \) is continuous for \( 1 \leq p \leq (N + 2)/(N - 2) \), and is compact for \( 1 \leq p < (N + 2)/(N - 2) \).

It was shown by [8, 24] that there exists a solution \( u_0 \) of the problem

\[
\begin{aligned}
\Delta u + \frac{1}{2} x \cdot \nabla u + \frac{1}{p-1} u + u^p &= 0 \quad \text{in } \mathbb{R}^N, \\
u \in H^1_\rho(\mathbb{R}^N) \quad \text{and } u > 0 \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]

with \( (N + 2)/N < p < (N + 2)/(N - 2) \). Moreover, it was shown in [8] that \( u_0 \in C^2(\mathbb{R}^N) \) and \( u_0(x) = O(e^{-|x|^2/8}) \) as \( |x| \to \infty \). The uniqueness of the solution to the problem (8) was obtained by combining the results [7, 27, 19].
Now put
\[
\tag{9} w_0(x, t) = t^{-1/(p-1)} u_0(x/\sqrt{t}),
\]
where \(u_0\) is the solution of the problem (8). We note that \(u_0 \in L^q(\mathbb{R}^N)\) for all \(q \geq 1\) and
\[
\|w_0(\cdot, t)\|_{L^q(\mathbb{R}^N)} = t^{-1/(p-1)+N/2q} \|u_0\|_{L^q(\mathbb{R}^N)}.
\]
Then \(w_0\) solves the the Cauchy problem (4) in \(C([0, \infty); L^q(\mathbb{R}^N))\) for \(1 \leq q < N(p-1)/2\). By the uniqueness result [19], we find that \(w_0\) defined by (9) coincides with the non-unique solution of (4) constructed by [13].

4. Existence of the second solution: subcritical case \([17, \text{Sec. 5}]\)

Let \(u_\lambda\) be the positive minimal solution of (6)-(7) \(\lambda\) obtained in Theorem 1. In order to find a second solution of (6)-(7) \(\lambda\) we introduce the following problem:
\[
\tag{10} \lambda \begin{cases} 
\Delta u + \frac{1}{2} x \cdot \nabla u + \frac{1}{p-1} u + g(u, u_\lambda) = 0 & \text{in } \mathbb{R}^N, \\
\ u \in H^1_\rho(\mathbb{R}^N) \text{ and } u > 0 & \text{in } \mathbb{R}^N,
\end{cases}
\]
where \(g(t, s) = (t+s)^p - s^p\). We easily see that, if (10)\(\lambda\) possesses a solution \(u_\lambda\), then we can get another positive solution \(\overline{u}_\lambda = u_\lambda + u_\lambda\) of (6)-(7)\(\lambda\).

In this section we will show the existence of solutions of (10)\(\lambda\) in the subcritical case \((N+2)/N < p < (N+2)/(N-2)\) by using the variational method. To this end we define the corresponding functional of (10)\(\lambda\) by
\[
I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx - \int_{\mathbb{R}^N} G(u, u_\lambda) \rho dx
\]
with \(u \in H^1_\rho(\mathbb{R}^N)\), where
\[
G(t, s) = \frac{1}{p+1} (t+s)^{p+1} - \frac{1}{p+1} s^{p+1} - s^p t.
\]
We see that the nontrivial critical point \(u \in H^1_\rho(\mathbb{R}^N)\) of the functional \(I_\lambda\) is a weak solution of the equation in (10)\(\lambda\). Moreover, we have \(u_\lambda \in C^2(\mathbb{R}^N)\) and \(u_\lambda > 0\) in \(\mathbb{R}^N\) by employing the bootstrap arguments and the maximum principle.

We will verify the existence of nontrivial solution of (10)\(\lambda\) by means of the Mountain Pass lemma ([1, 20]).

**Lemma 4.** For \(\lambda \in (0, \bar{\lambda})\) there exist some constants \(\delta = \delta(\lambda) > 0\) and \(\eta = \eta(\lambda) > 0\) such that \(I_\lambda(u) \geq \eta\) for all \(u \in H^1_\rho(\mathbb{R}^N)\) with \(\|\nabla u\|_{L^2_\rho} = \delta\).
Lemma 5. For any $v \in H^1_\rho(\mathbb{R}^N)$ with $v \geq 0$, $v \neq 0$, we have $I_\lambda(tv) \to -\infty$ as $t \to \infty$.

Lemma 6. The functional $I_\lambda$ satisfies the Palais-Smale condition, that is, any sequence $\{u_k\} \subset H^1_\rho(\mathbb{R}^N)$ such that
\[
\{I_\lambda(u_k)\} \text{ is bounded and } I'_\lambda(u_k) \to 0 \quad \text{as } k \to \infty
\]
contains a convergent subsequence.

In the proofs of Lemmas 4-6, the following results play a crucial role.

Lemma 7. Let $u_\lambda$ be the minimal solution obtained in Theorem 1 for $\lambda \in (0, \overline{\lambda})$. Then the linearized eigenvalue problem
\[
\begin{cases}
-\Delta w - \frac{1}{2} x \cdot \nabla w - \frac{1}{p-1} w = \mu p [u_\lambda]^{p-1} w & \text{in } \mathbb{R}^N, \\
w \in H^1_\rho(\mathbb{R}^N),
\end{cases}
\]
has the first eigenvalue $\mu = \mu(\lambda) > 1$. Moreover, $\mu(\lambda)$ is strictly decreasing in $\lambda \in (0, \overline{\lambda})$.

Lemma 7 follows from the fact that $u_\lambda$ is the positive minimal solution.

As a consequence of Lemmas 4-6 we obtain the following:

Theorem 2. Assume that $(N+2)/N < p < (N+2)/(N-2)$. Then, for $0 < \lambda < \overline{\lambda}$, there exists a positive solution $\overline{u}_\lambda$ of (6)-(7) satisfying $\overline{u}_\lambda > u_\lambda$,
\[
\overline{u}_\lambda - u_\lambda \in H^1_\rho(\mathbb{R}^N), \quad \text{and} \quad \overline{u}_\lambda(x) - u_\lambda(x) = O(e^{-|x|^2/4}) \quad \text{as } |x| \to \infty.
\]
Furthermore,
\[
\overline{u}_\lambda - u_\lambda \to u_0 \quad \text{in } H^1_\rho(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \quad \text{as } \lambda \to 0,
\]
where $u_0$ is the solution of the problem (8). In particular, $\overline{u}_\lambda \to u_0$ in $L^\infty(\mathbb{R}^N)$ as $\lambda \to 0$.

Now we consider the Cauchy problem (1)-(2)\_\lambda. Recall that, if $u$ is a solution of (6)-(7)\_\lambda, then the function $w$ defined by (5) is a solution of (1)-(2)\_\lambda in the sense of $L^1_{\text{loc}}(\mathbb{R}^N)$, and that $w_0$ defined by (9) coincides with the non-unique solution of (4) constructed by [13]. As a consequence of Theorems 1 and 2, we obtain the following results.
Corollary 1. Assume that \( p > (N + 2) / N \). Then there exists a constant \( \bar{\lambda} > 0 \) such that

(i) for \( 0 < \lambda < \bar{\lambda} \), \((1)-(2)_{\lambda}\) has a positive self-similar solution \( \varpi_{\lambda} \); the solution \( \varpi_{\lambda}(\cdot, t) \) satisfies, for each fixed \( t > 0 \),

\[
\| \varpi_{\lambda}(\cdot, t) \|_{L^\infty(\mathbb{R}^N)} \to 0 \quad \text{as} \; \lambda \to 0;
\]

(ii) for \( \lambda > \bar{\lambda} \), \((1)-(2)_{\lambda}\) has no positive self-similar solutions.

Assume, furthermore, that \( p < (N + 2) / (N - 2) \). Then \((1)-(2)_{\lambda}\) has a positive self-similar solution \( \varpi_{\lambda} \) satisfying \( \varpi_{\lambda} > w_0 \) in \( \mathbb{R}^N \times (0, \infty) \) for \( 0 < \lambda < \bar{\lambda} \). The solution \( \varpi_{\lambda} \) satisfies, for each fixed \( t > 0 \),

\[
\| \varpi_{\lambda}(\cdot, t) - w_0(\cdot, t) \|_{L^\infty(\mathbb{R}^N)} \to 0 \quad \text{as} \; \lambda \to 0,
\]

where \( w_0 \) is the non-unique solution of \((4)\) in \( C([0, \infty); L^q(\mathbb{R}^N)) \) for \( 1 \leq q < N(p - 1)/2 \), which is constructed by [13].

5. Existence and nonexistence of second solutions: critical case [18]

In this section we consider the existence and nonexistence of second solutions of the problem \((6)-(7)_{\lambda}\) in the critical case \( p = (N + 2) / (N - 2) \) by following the argument due to Brezis-Nirenberg [4].

For the critical growth case, there are serious difficulties in obtaining solutions by using variational methods because the Sobolev embedding \( H^1 \subset L^{p+1} \) is not compact. It is well known that this lack of compactness exhibits many interesting existence and nonexistence phenomena. See, e.g., [4, 2].

Let us denote by \( S \) the best Sobolev constant of the embedding \( H^1(\mathbb{R}^N) \subset L^{2N/(N+2)}(\mathbb{R}^N) \), which is given by

\[
S = \inf_{u \in H^1(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\left( \int_{\mathbb{R}^N} |u|^{2N/(N-2)} \, dx \right)^{(N-2)/N}}.
\]

In the critical case, the functional \( I_{\lambda} \) satisfies the following local Palais-Smale condition.

**Lemma 8.** Let \( p = (N + 2) / (N - 2) \). Then \( I_{\lambda} \) satisfies the \((PS)_c\) condition for \( c \in (0, S^{N/2}/N) \), that is, any sequence \( \{u_k\} \subset H^1_p(\mathbb{R}^N) \) such that

\[
I_{\lambda}(u_k) \to c \in \left(0, \frac{1}{N} S^{N/2}\right) \quad \text{and} \quad I'_{\lambda}(u_k) \to 0 \quad \text{as} \; k \to \infty
\]

contains a convergent subsequence.
By Lemma 8 and the Mountain Pass lemma, we obtain the following existence result.

**Lemma 9.** Let \( p = (N+2)/(N-2) \). Assume that there exists \( v_0 \in H_{\rho}^1(\mathbb{R}^N) \) with \( v_0 \geq 0, v_0 \not\equiv 0 \) such that

\[
\sup_{t>0} I_\lambda(t v_0) < \frac{1}{N} S_{N/2}^N.
\]

Then there exists a positive solution \( u_\lambda \in H_{\rho}^1(\mathbb{R}^N) \) of (10)\( _\lambda \).

Moreover, we have \( u_\lambda \in C^2(\mathbb{R}^N) \) by employing the estimate due to Brezis-Kato [3], based on the Moser's iteration technique.

In order to find a positive function \( v_0 \in H_{\rho}^1(\mathbb{R}^N) \) satisfying (11), we set

\[
\begin{align*}
u_\varepsilon(x) &= \frac{\phi(x)}{(\varepsilon + |x|^2)^{(N-2)/2}} \rho^{-1/2} \\
v_\varepsilon(x) &= \frac{\|u_\varepsilon\|}{\|u\|_{L_{\rho}^{p+1}}}
\end{align*}
\]

for \( \varepsilon > 0 \), where \( \phi \in C_0^{\infty}(\mathbb{R}^N) \) is a cut off function. We remark that the functional \( I_\lambda \) can be written as

\[
I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |\nabla u|^2 - \frac{1}{p-1} u^2 \right) \rho dx - \frac{1}{p+1} \int_{\mathbb{R}^N} u^{p+1} \rho dx
\]

\[
\equiv I_0(u) - \int_{\mathbb{R}^N} H(u, u_\lambda) \rho dx,
\]

where

\[
H(t, s) = G(t, s) - \frac{1}{p+1} t^{p+1}.
\]

**Lemma 10.** For sufficient small \( \varepsilon > 0 \), there exists \( t_\varepsilon > 0 \) such that

\[
\sup_{t>0} I_\lambda(t v_\varepsilon) = I_\lambda(t_\varepsilon v_\varepsilon).
\]

Moreover, as \( \varepsilon \to 0 \) we have

\[
I_0(t_\varepsilon v_\varepsilon) \leq \frac{1}{N} S_{N/2}^N + \begin{cases} O(\varepsilon), & N \geq 5 \\ O(\varepsilon \log \varepsilon), & N = 4 \\ O(\varepsilon^{1/2}), & N = 3 \end{cases}
\]

\[
\int_{\mathbb{R}^N} H(t_\varepsilon v_\varepsilon, u_\lambda) \rho dx \geq \begin{cases} C \varepsilon^{3/4}, & N = 5 \\ C \varepsilon^{1/2}, & N = 4 \\ C \varepsilon^{1/4}, & N = 3 \end{cases}
\]

with some constant \( C > 0 \).
As a consequence, we obtain the following:

**Theorem 3.** Let \( p = (N+2)/(N-2) \) and \( N = 3, 4, 5 \). Then, for \( 0 < \lambda < \overline{\lambda} \), the problem (6)-(7) has a positive solution \( u_\lambda \in C^2(\mathbb{R}^N) \) satisfying \( u_\lambda > u_\lambda \) and \( \pi_\lambda - u_\lambda \in H^1_\rho(\mathbb{R}^N) \).

On the other hand, for the case \( N \geq 6 \) we obtain the uniqueness result in the radial class by employing the Pohozaev type identity.

**Theorem 4.** Let \( p = (N+2)/(N-2) \) and \( N \geq 6 \). Assume that \( a \equiv 1 \) in (7)_\lambda. Then there exists a constant \( \lambda_0 \in (0, \overline{\lambda}) \) such that (6)-(7) has no positive radial solutions \( u \in C^2(\mathbb{R}^N) \) with \( u \not\equiv u_\lambda \) for \( \lambda \in (0, \lambda_0) \), that is, (6)-(7) has a unique positive radial solution \( u_\lambda \) for \( 0 < \lambda < \lambda_0 \).

**REFERENCES**


