BRAID MONODROMY AND TOPOLOGY OF COMPLEXIFIED REAL ARRANGEMENTS

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There is a close relationship between topology and combinatorics of complex line arrangements. A complex line arrangement is a finite set $\mathscr{L}$ of lines of the complex projective space $\mathbb{CP}^2 =: \mathbb{P}^2$. By *topology*, we mean the oriented homeomorphism type of the pair $(\mathbb{P}^2, \bigcup \mathscr{L})$. By *combinatorics*, we mean the intersection pattern of the lines.

This problem is in the intersection of two generalizations: either we consider hyperplanes in higher dimension or we consider plane curves of higher degree. In the case of arrangement of hyperplanes it is well-known that the associated matroid (which is a combinatorial invariant) determines the cohomology ring of the complement of the arrangement. In the case of plane curves, there is a natural notion of combinatorics, and Zariski showed that these *combinatorics* do not determine the topology.

Taking into account the general results of hyperplane arrangements, it is a natural question to decide whether combinatorics determine the topology of line arrangements.

In a famous preprint [6], G. Rybnikov proves the existence of two line arrangements $\mathscr{L}_1, \mathscr{L}_2$ in $\mathbb{P}^2 := \mathbb{CP}^2$ which have the same combinatorics but whose pairs, $(\mathbb{P}^2, \bigcup \mathscr{L}_1)$ and $(\mathbb{P}^2, \bigcup \mathscr{L}_2)$, are not homeomorphic. Rybnikov claims that the fundamental groups of the complements are not homeomorphic, see [1] for a further discussion of his work. We recall that Rybnikov's examples do not admit real equations.

In the present talk we are interested in the topology of complexified real arrangements which form a very important class of arrangements. By a *complexified real arrangement* we mean an arrangement such that there exists a coordinate system for which all lines admit real equations. These arrangements belong to a class of plane projective curves, which may be called *totally real curves*. A curve is *totally real* if all its topological properties can be deduced from the type of their singular points and from its real picture in a suitable coordinate system where its equation has real coefficients.

This is part of a join work [2] with J. Carmona, J.I. Cogolludo and M. Marco.

In this talk we will indicate how to prove the existence of complexified real arrangements with the same combinatorics but different embeddings in $\mathbb{P}^2$. Moreover we produce counterexamples with an additional property: they admit Galois-conjugated equations on the ring of polynomials over $\mathbb{Q}(\sqrt{5})$. We set some definitions.

Definition 1. A *line combinatorics* is a couple $(\mathscr{L}, \mathscr{P})$ where $\mathscr{L}$ is a finite set and $\mathscr{P} \subset \mathcal{P}(\mathscr{L})$ verifies:

- $\forall \ell, \ell' \in \mathscr{L}$, $\ell \neq \ell'$, $\exists!\, p \in \mathscr{P}$ such that $\ell, \ell' \in p$.
- $\forall p \in \mathscr{P}$ the *multiplicity* $\nu(p) := \# p$ of p verifies $\nu(p) \geq 2$.

Analogously *ordered line combinatorics* are defined. An *automorphism* of $\mathscr{L}$ is a permutation of $\mathscr{L}$ preserving $\mathscr{P}$. The group $\operatorname{Aut} \mathscr{L}$ of such automorphisms is the *automorphism group* of $\mathscr{L}$.

A simple way to obtain line combinatorics is via point arrangements in $\mathbb{F}_q\mathbb{P}^2$ for some finite field $\mathbb{F}_q$. For example, the starting point in [6] is the MacLane arrangement. This arrangement can be defined as follows. Consider $\mathbb{F}_3\mathbb{P}^2$ as the union of $\mathbb{F}_3^2$ and the line at infinity. Consider also the points of $\mathbb{F}_3^2 \setminus \{0\}$: the MacLane combinatorics is the abstract line combinatorics corresponding to the dual of this 8 point arrangement. It is easily seen that the automorphism group of MacLane combinatorics is naturally isomorphic to $\mathrm{GL}(2; \mathbb{F}_3)$.

In this talk, we consider a point arrangement in $\mathbb{F}_4\mathbb{P}^2$. The starting point of this work is a similar idea. Let us consider $\mathbb{F}_4\mathbb{P}^2 = \mathbb{F}_4^2 \cup L_\infty$, where L_∞ is the line at infinity. Let us consider $A := \left(\begin{smallmatrix} 0 & 1 \\ 1 & \zeta \end{smallmatrix}\right) \in GL(2; \mathbb{F}_4)$, where ζ is any of the elements of $\mathbb{F}_4 \setminus \mathbb{F}_2$; note that $A^5 = I_2$. The orbit of $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ and L_∞ is 10-point arrangement which produces a line combinatorics $\mathscr{C}$ which has an automorphism group of order 20. Note that deleting one point of the line at infinity produces the Falk-Sturmfels line combinatorics, see [4].

Definition 2. Let $\mathscr{C}$ be a line combinatorics. An arrangement $\mathscr{L}$ of $\mathbb{P}^2 = \mathbb{C}\mathbb{P}^2$ is a *complex realization of* $\mathscr{C}$ if its combinatorics agrees with $\mathscr{C}$. An *ordered complex realization* of an ordered line combinatorics is defined accordingly. The space of all complex realizations,resp. ordered, of a line combinatorics $\mathscr{C}$ is denoted by $\Sigma(\mathscr{C})$, resp. $\Sigma^{\mathrm{ord}}(\mathscr{C})$.

The *moduli space* of a combinatorics $\mathscr{C}$ is the quotient $\mathscr{M}(\mathscr{C}) := \Sigma(\mathscr{C}) / \operatorname{PGL}(3; \mathbb{C})$. The *ordered moduli space* $\mathscr{M}^{\mathrm{ord}}(\mathscr{C})$ of an ordered combinatorics $\mathscr{C}$ is defined accordingly.

We come back to MacLane line combinatorics $\mathscr{N}$. It is well known that $\#\mathscr{M}(\mathscr{N}) = 1$ and that $\#\mathscr{M}^{\mathrm{ord}}(\mathscr{N}) = 2$. This is a crucial step in Rybnikov's

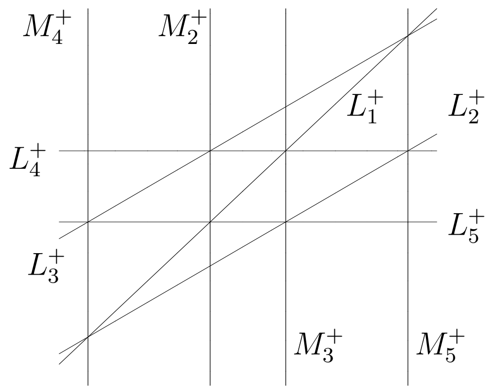

FIGURE 1. Arrangement $\mathscr{C}^+$

work. One can find representatives having equations in the polynomial ring over the field of cubic roots of unity. Moreover, the MacLane line combinatorics has no real realization. For our combinatorics $\mathscr{C}$ we have an analogous result.

Proposition 3. *The space $\mathscr{M}^{\mathrm{ord}}(\mathscr{C})$ has two elements. Moreover, representatives can be chosen to have the following equations:*

$$M_1 : z = 0, \quad M_2 : x = 0, \quad M_3 : x = z, \quad M_4 : x = -(\gamma+1)z, \quad M_5 : x = (\gamma+2)z,$$

$$L_1 : y = x, \quad L_2 : y = \gamma(x - z), \quad L_3 : y = \gamma x + z, \quad L_4 : y = z, \quad L_5 : y = 0,$$

where $\gamma^2 + \gamma - 1 = 0$. The space $\mathscr{M}(\mathscr{C})$ has only one element.

Let us state the first main result of Rybnikov's work [6]: if $\mathscr{N}^{\pm}$ are two representatives of the elements of $\mathscr{M}^{\mathrm{ord}}(\mathscr{N})$, hen there is no isomorphism of the fundamental groups of $\mathbb{P}^2 \setminus \bigcup \mathscr{N}^+$ and $\mathbb{P}^2 \setminus \bigcup \mathscr{N}^-$ inducing the identity in the homology groups. In particular, there is is no orientation-preserving homeomorphism between $(\mathbb{P}^2, \bigcup \mathscr{N}^+)$ and $(\mathbb{P}^2, \bigcup \mathscr{N}^-)$.

The next step in our work is to consider the two ordered arrangements $\mathscr{C}^{\pm}$ representing the elements of $\mathscr{M}^{\mathrm{ord}}(\mathscr{C})$, determined by the equations given in Proposition 3 and by the choice of $\gamma^{\pm} := \frac{-1\pm\sqrt{5}}{2}$. Let us denote by $L_i^{\pm}$ and $M_i^{\pm}$, $i = 1, \dots, 5$ the lines as in section §2. Affine pictures are shown in Figures 1 and 2, where $M_1^{\pm}$ are the corresponding lines at infinity.

Rybnikov's approach does not work with $\mathscr{C}$. Nevertheless, we are able to prove:

Theorem 4. *There is no order-preserving homeomorphism between $(\mathbb{P}^2, \bigcup \mathscr{C}^+)$ and $(\mathbb{P}^2, \bigcup \mathscr{C}^-)$.*

The main point for the proof of this theorem is the use of the non-generic braid monodromy with respect to the vertical projection of figures 1 and 2. We prove

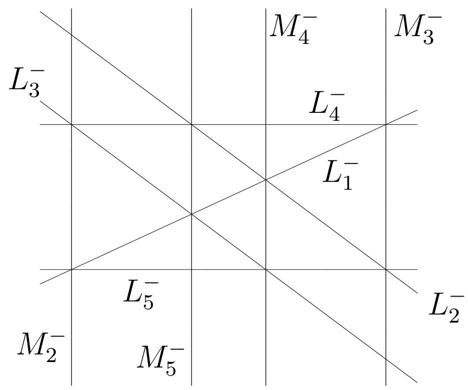

FIGURE 2. Arrangement $\mathscr{C}^-$

that these braid monodromies are not equivalent (see [2] for the precise definition) using a GAP4 [5] program. Then, we adapt the main result of [3] to prove theorem 4.

Since the line combinatorics $\mathscr{N}$ and $\mathscr{C}$ have a great symmetry, they verify $\#\mathscr{M}(\mathscr{N}) = \#\mathscr{M}(\mathscr{C}) = 1$ and hence will not provide pairs of arrangements having the same combinatorics but different topologies. We will add a suitable line which will *break* the symmetries of $\mathscr{C}$.

Let $\mathscr{H}^\pm := \mathscr{C}^\pm \cup \{N^\pm\}$ be two new arrangements, where $N^\pm$ is the line joining the points $L_3^\pm \cap L_5^\pm \cap M_4^\pm = [1 : 0 : -\gamma^\pm]$ and $L_2^\pm \cap M_2^\pm = [0 : 1 : -(\gamma^\pm + 1)]$. In particular, $N^\pm : \gamma^\pm x + (\gamma^\pm + 1)y + z = 0$. Note that these two arrangements have the same combinatorics, say $\mathscr{H}$, since their equations are conjugate in $\mathbb{Q}(\sqrt{5})$. The new abstract line of $\mathscr{H}$ will be denoted by N. It can be proved that $\operatorname{Aut}\mathscr{H}$ is trivial due to the choice of the line N. This fact implies that $\#\mathscr{M}(\mathscr{H}) = 2$ and $\mathscr{H}^\pm$ are representatives of the two elements and provides the main result of the work:

Theorem 5. *There is no homeomorphism between* $(\mathbb{P}^2, \bigcup \mathscr{H}^+)$ *and* $(\mathbb{P}^2, \bigcup \mathscr{H}^-)$.

Remark 6. An important feature of this pair of arrangements is the fact that they are defined by conjugate equations in $\mathbb{Q}(\sqrt{5})$. This fact implies that they cannot be distinguished by algebraic methods. Note for example that the fundamental groups of $\mathbb{P}^2 \setminus \bigcup \mathscr{H}^+$ and $\mathbb{P}^2 \setminus \bigcup \mathscr{H}^-$ have isomorphic profinite completions.

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