ATTRACTORS OF ASYMPTOTICALLY PERIODIC
MULTIVALUED DYNAMICAL SYSTEMS GOVERNED
BY TIME-DEPENDENT SUBDIFFERENTIALS

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Abstract. Let us consider a nonlinear evolution equation associated with time-dependent
subdifferential in a separable Hilbert space. In this paper we treat an asymptotically pe-
riodic system which means that time-dependent terms converge to some time-periodic
ones as time goes to $+\infty$. Then we consider the large-time behaviour of solutions without
uniqueness. In such a situation the corresponding dynamical systems are multivalued. In
fact we discuss the stability of multivalued semiflows from the view-point of attractors.
Namely, the main object of this paper is to construct a global attractor for the asymptot-
ically periodic multivalued dynamical system, and to discuss the relationship to one for
the limiting periodic system.

1 Introduction

In this paper let us consider a non-autonomous system in a real separable Hilbert space
$H$ of the form

$$v'(t) + \partial \varphi^t(v(t)) + G(t, v(t)) \ni f(t) \quad \text{in } H, \quad t > s \geq 0,$$

(1.1)

where $v' = \frac{dv}{dt}$, $\partial \varphi^t$ is a subdifferential of time-dependent proper lower semicontinuous
(l.s.c.) convex function $\varphi^t$ on $H$, $G(t, \cdot)$ is a multivalued perturbation small relative to
$\varphi^t$, and $f$ is a forcing term.

In the case when $G(t, \cdot) \equiv 0$, many mathematicians studied the existence-uniqueness,
asymptotic stability, time periodic and almost periodic problem for (1.1) (cf. [7], [8], [13],
[14], [15], [16], [18], [23], [24]).

For the multivalued nonmonotone perturbation $G(t, \cdot)$, Ôtani has already shown the
existence of solution for (1.1) in [21]. The large-time behavior of solutions for (1.1) was

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discussed by [28] from the viewpoint of attractors. For the time periodic case, assuming the periodicity conditions with same period $T_0$, $0 < T_0 < +\infty$, i.e.
\[
\varphi^t = \varphi^{t+T_0}, \quad G(t, \cdot) = G(t + T_0, \cdot), \quad f(t) = f(t + T_0), \quad \forall t \in R_+ := [0, \infty),
\]
the existence of periodic solution for (1.1) was proved in [22]. Moreover, the periodic stability was discussed in [29]. In fact, the author showed the existence and characterization of time-periodic global attractors for (1.1).

In this paper, for a given positive number $T_0 > 0$ let us treat the case when $\varphi^t$, $G(t, \cdot)$ and $f(t)$ are asymptotically $T_0$-periodic in time. Namely we assume that
\[
\varphi^t - \varphi_p^t \longrightarrow 0, \quad G(t, \cdot) - G_p(t, \cdot) \longrightarrow 0, \quad f(t) - f_p(t) \longrightarrow 0 \quad (1.2)
\]
in appropriate senses as $t \rightarrow +\infty$, where $\varphi_p^t = \varphi^{t+T_0}_p$, $G_p(t, \cdot) = G_p(t + T_0, \cdot)$ and $f_p(t) = f_p(t + T_0)$ for any $t \in R_+$. By the asymptotically $T_0$-periodic stability (1.2), we have the limiting $T_0$-periodic system for (1.1) of the form:
\[
u'(t) + \partial \varphi_p^t(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in} \; H, \quad t > s \; (\geq 0). \quad (1.3)
\]
In the case when $G(t, \cdot)$ and $G_p(t, \cdot)$ are single-valued, the asymptotically $T_0$-periodic problem has already been discussed in [11]. In order to guarantee the uniqueness of solutions for the Cauchy problem of (1.1) and (1.3), they assumed some conditions on $\varphi^t$, $\varphi_p^t$, $G(t, \cdot)$ and $G_p(t, \cdot)$. Then, they discussed the asymptotically $T_0$-periodic stability for (1.1) from the viewpoint of attractors (cf. [11]). The main object of this paper is to develop the result obtained in [11] in order to consider the large-time behaviour of solution for (1.1) without uniqueness. Namely, we would like to construct the attractor for the asymptotically $T_0$-periodic multivalued flows associated with (1.1). Moreover we shall discuss the relationship to the $T_0$-periodic attractor for (1.3) obtained in [29].

In the next Section 2, we recall the known results for the Cauchy problem of (1.1). In Section 3 we consider the limiting $T_0$-periodic problem (1.3) and recall the abstract results obtained in [29]. In Section 4, we introduce the notion of a metric topology on the family \{ $\varphi^t; t \geq 0$ \} which was constructed in [16]. And we present and prove the main results in this paper. In proving main results, we generalize the results obtained in [11] and [30].

In the final section we apply our abstract results to the parabolic variational inequality with asymptotically $T_0$-periodic double obstacles. Then we can discuss the asymptotic stability for the asymptotically $T_0$-periodic double obstacle problem without uniqueness of solutions.

**Notation.** Throughout this paper, let $H$ be a (real) separable Hilbert space with norm $| \cdot |_H$ and inner product $\langle \cdot, \cdot \rangle_H$. For a proper l.s.c. convex function $\varphi$ on $H$ we use the notation $D(\varphi)$, $\partial \varphi$ and $D(\partial \varphi)$ to indicate the effective domain, subdifferential and its domain of $\varphi$, respectively; for their precise definitions and basic properties see [4].

For two non-empty sets $A$ and $B$ in $H$, we define the so-called Hausdorff semi-distance
\[
\text{dist}_H(A, B) := \sup_{x \in A} \inf_{y \in B} |x - y|_H.
\]
2 Preliminaries

In this section let us recall the known results for a nonlinear evolution equation in $H$ of the form:

$$u'(t) + \partial \varphi^t(u(t)) + G(t, u(t)) \ni f(t) \quad \text{in } H, \quad t \in J,$$

(2.1)

where $J$ is an interval in $R_+$, $\partial \varphi^t$ is the subdifferential of a time-dependent proper l.s.c. and convex function $\varphi^t$ on $H$, $G(t, \cdot)$ is a multivalued operator from a subset $D(t, \cdot) \subset H$ into $H$ for each $t \in R_+$ and $f$ is a given function in $L^1_{loc}(J; H)$.

We begin with the definition of solution for (2.1).

**Definition 2.1.**

(i) For a compact interval $J := [t_0, t_1] \subset R_+$ and $f \in L^2(J; H)$, a function $u : J \to H$ is called a solution of (2.1) on $J$, if $u \in C(J; H) \cap W^{1,2}_{loc}((t_0, t_1]; H)$, $\varphi^t(u(\cdot)) \in L^1(J)$, $u(t) \in D(\partial \varphi^t)$ for a.e. $t \in J$, and if there exists a function $g \in L^2_{loc}(J; H)$ such that $g(t) \in G(t, u(t))$ for a.e. $t \in J$ and

$$f(t) - g(t) - u'(t) \in \partial \varphi^t(u(t)), \quad \text{a.e. } t \in J.$$

(ii) For any interval $J$ in $R_+$ and $f \in L^2_{loc}(J; H)$, a function $u : J \to H$ is called a solution of (2.1) on $J$, if it is a solution of (2.1) on every compact subinterval of $J$ in the sense of (i).

(iii) Let $J$ be any interval in $R_+$ with initial time $s \in R_+$. For $f \in L^2_{loc}(J; H)$, a function $u : J \to H$ is called a solution of the Cauchy problem for (2.1) on $J$ with given initial value $u_0 \in H$, if it is a solution of (2.1) on $J$ satisfying $u(s) = u_0$.

Throughout this paper, let $\{a_r\} := \{a_r; r \geq 0\}$ and $\{b_r\} := \{b_r; r \geq 0\}$ be families of real functions in $W^{1,2}_{loc}(R_+)$ and $W^{1,1}_{loc}(R_+)$, respectively, such that

$$\sup_{t \in R_+} |a'_r|_{L^2(t,t+1)} + \sup_{t \in R_+} |b'_r|_{L^1(t,t+1)} < +\infty \quad \text{for each } r \geq 0.$$

Now we define the class $\Phi(\{a_r\}, \{b_r\})$ of time-dependent convex function $\varphi^t$.

**Definition 2.2.**

$\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})$ if and only if $\varphi^t$ is a proper l.s.c. convex function on $H$ satisfying the following properties ($\Phi1$)-($\Phi3$):

($\Phi1$) For each $r > 0$, $s$, $t \in R_+$ and $z \in D(\varphi^s)$ with $|z|_H \leq r$, there exists $\tilde{z} \in D(\varphi^t)$ such that

$$|\tilde{z} - z|_H \leq |a_r(t) - a_r(s)|(1 + |\varphi^s(z)|)^{\frac{1}{2}}$$

and

$$\varphi^t(\tilde{z}) - \varphi^s(z) \leq |b_r(t) - b_r(s)|(1 + |\varphi^s(z)|).$$

($\Phi2$) There exists a positive constant $C_1 > 0$ such that

$$\varphi^t(z) \geq C_1 |z|_H^2, \quad \forall t \in R_+, \forall z \in D(\varphi^t).$$

($\Phi3$) For each $k > 0$ and $t \in R_+$, the level set $\{z \in H; \varphi^t(z) \leq k\}$ is compact in $H$. 

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Next, we introduce the class $\mathcal{G} (\{ \varphi^i \})$ of time-dependent multivalued perturbation $G(t, \cdot)$ associated with $\{ \varphi^i \} \in \Phi (\{ a_r \}, \{ b_r \})$.

**Definition 2.3.** $\{ G(t, \cdot) \} \in \mathcal{G} (\{ \varphi^i \})$ if and only if $G(t, \cdot)$ is a multivalued operator from $D(G(t, \cdot)) \subset H$ into $H$ which fulfills the following conditions (G1)-(G5):

(G1) $D(\varphi^i) \subset D(G(t, \cdot)) \subset H$ for any $t \in R_+$. And for any interval $J \subset R_+$ and $v \in L^2_{loc}(J; H)$ with $v(t) \in D(\varphi^i)$ for a.e. $t \in J$, there exists a strongly measurable function $g(\cdot)$ on $J$ such that

$$g(t) \in G(t, v(t)) \text{ for a.e. } t \in J.$$

(G2) $G(t, z)$ is a convex subset of $H$ for any $z \in D(\varphi^i)$ and $t \in R_+$.

(G3) There are positive constants $C_2, C_3$ such that

$$|g|^2_H \leq C_2 \varphi^i(z) + C_3, \quad \forall t \in R_+, \forall z \in D(\varphi^i), \forall g \in G(t, z).$$

(G4) (demi-closedness) If $z_n \in D(\varphi^{i_n}), g_n \in G(t_n, z_n), \{ t_n \} \subset R_+, \{ \varphi^{i_n}(z_n) \}$ is bounded, $z_n \to z$ in $H$, $t_n \to t$ and $g_n \to g$ weakly in $H$ as $n \to +\infty$, then $g \in G(t, z)$.

(G5) For each bounded subset $B$ of $H$, there exist positive constants $C_4(B)$ and $C_5(B)$ such that

$$\varphi^i(z) + (g, z - b)_H \geq C_4(B) |z|^2_H - C_5(B),$$

$$\forall t \in R_+, \forall g \in G(t, z), \forall z \in D(\varphi^i), \forall b \in B.$$

For given $\{ \varphi^i \} \in \Phi (\{ a_r \}, \{ b_r \}), \{ G(t, \cdot) \} \in \mathcal{G} (\{ \varphi^i \})$ and a forcing term $f \in L^2_{loc}(R_+; H)$, we consider the following evolution equation

$$(E)_s \quad u'(t) + \partial \varphi^i(u(t)) + G(t, u(t)) \ni f(t)\quad \text{in } H, \quad t > s$$

for each $s \in R_+$.

Now let us recall the known results on the existence and global estimates of solutions for the Cauchy problem of $(E)_s$:

(A) [Existence of solution for $(E)_s$] (cf. [21, Theorem II, III])

The Cauchy problem for $(E)_s$ has at least one solution $u$ on $J = [s, +\infty)$ such that $(\cdot - s)^{\frac{1}{2}} u' \in L^2_{loc}(J; H), (\cdot - s)^{\varphi^i(u(\cdot))} \in L^\infty_{loc}(J)$ and $\varphi^i(u(\cdot))$ is absolutely continuous on any compact subinterval of $(s, +\infty)$, provided that $u_0 \in D(\varphi^s)$. In particular, if $u_0 \in D(\varphi^s)$, then the solution $u$ satisfies that $u' \in L^2_{loc}(J; H)$ and $\varphi^i(u(\cdot))$ is absolutely continuous on any compact interval in $J$.

(B) [Global boundedness of solutions for $(E)_s$] (cf. [25, Theorem 2.2])

Suppose that

$$S_f := \sup_{t \in R_+} |f|^2_{L^2(t, t + 1; H)} < +\infty.$$
Then, the solution \( u \) of the Cauchy problem for \((E)_s\) on \([s, +\infty)\) satisfies the following global estimate:

\[
\sup_{t \geq s} |u(t)|_H^2 + \sup_{t \geq s} \int_t^{t+1} \varphi'(u(\tau))d\tau \leq N_1(1 + S_f^2 + |u_0|_H^2),
\]

where \( N_1 \) is a positive constant independent of \( f, s \in R_+ \) and \( u_0 \in \overline{D}(\varphi^s) \). Moreover, for each \( \delta > 0 \) and each bounded subset \( B \) of \( H \), there is a constant \( N_2(\delta, B) > 0 \), depending only on \( \delta > 0 \) and \( B \), such that

\[
\sup_{t \geq s+\delta} |u'|_L^2(\varphi^s) + \sup_{t \geq s+\delta} \varphi'(u(t)) \leq N_2(\delta, B)
\]

for the solution \( u \) of the Cauchy problem for \((E)_s\) on \([s, +\infty)\) with \( s \in R_+ \) and \( u_0 \in \overline{D}(\varphi^s) \cap B \).

Next, let us remember a notion of convergence of convex functions.

**Definition 2.4.** (cf. [20]) Let \( \psi, \psi_n \ (n \in N) \) be proper l.s.c. and convex functions on \( H \). Then we say that \( \psi_n \) converges to \( \psi \) on \( H \) as \( n \to +\infty \) in the sense of Mosco, if the following two conditions (i) and (ii) are satisfied:

(i) for any subsequence \( \{\psi_{n_k}\} \subset \{\psi_n\} \), if \( z_k \to z \) weakly in \( H \) as \( k \to +\infty \), then

\[
\liminf_{k \to +\infty} \psi_{n_k}(z_k) \geq \psi(z).
\]

(ii) for any \( z \in D(\psi) \), there is a sequence \( \{z_n\} \) in \( H \) such that

\[
z_n \to z \text{ in } H \text{ as } n \to +\infty, \quad \lim_{n \to +\infty} \psi_n(z_n) = \psi(z).
\]

Now, we recall a convergence result (cf. [25, Lemma 4.1]) as follows.

(C) Let \( \{\varphi_n^t\} \in \Phi(\{a_r\}, \{b_r\}), \ {G_n(t, \cdot)} \in G(\{\varphi_n^t\}) \) with common positive constants \( C_1, C_2, C_3, C_4(B) \) and \( C_5(B) \), \( \{f_n\} \subset L^2(J; H) \), \( J = [s, t_1] \subset R_+ \) and \( u_{0,n} \in \overline{D}(\varphi_n^s) \) for \( n = 1, 2, \cdots \). Assume that

(i) \( \varphi_n^t \) converges to \( \varphi^t \) on \( H \) in the sense of Mosco [20] for each \( t \in J \) (as \( n \to +\infty \)) and \( \bigcup_{n=1}^{+\infty} \{z \in H; \varphi_n^t(z) \leq k\} \) is relatively compact in \( H \) for every real \( k > 0 \) and \( t \in J \), where \( \varphi^t = \varphi^t \) if \( n = +\infty \).

(ii) if \( z_n \in D(\varphi_n^s) \), \( g_n \in G_n(t_n, z_n) \), \( \{t_n\} \subset R_+ \), \( \{\varphi_n^s(z_n)\} \) is bounded, \( z_n \to z \) in \( H \), \( t_n \to t \) and \( g_n \to g \) weakly in \( H \) as \( n \to +\infty \), then \( g \in G(t, z) \), where \( \{G(t, \cdot)\} \in G(\{\varphi^t\}) \).

(iii) \( f_n \to f \) weakly in \( L^2(J; H) \) for some \( f \in L^2(J; H) \) and \( u_{0,n} \to u_0 \) in \( H \) for some \( u_0 \in \overline{D}(\varphi^s) \).
Denote by $u$ the solution of the Cauchy problem for $(E)_s$ on $J$ with $u(s) = u_0$ and by $u_n$ the solution of the Cauchy problem for $(E)_s$ with $\varphi^t, G, f$ replaced by $\varphi^t_n, G_n, f_n$, and with $u_n(s) = u_{0,n}$. Then $u_n$ converges to $u$ on $J$ in the sense that
\[ u_n \to u \text{ in } C(J;H), \quad (\cdot - s)^{\frac{1}{2}}u'_n \to (\cdot - s)^{\frac{1}{2}}u' \text{ weakly in } L^2(J;H), \]
\[ \int_J \varphi^t_n(u_n(t))dt \to \int_J \varphi^t(u(t))dt \quad \text{as } n \to +\infty. \]

3 Attractor for periodic multivalued dynamical system

In this section let us recall the known results obtained in [29] for a $T_0$-periodic system in $H$, of the form:
\[ (P)_s \quad u'(t) + \partial \varphi^t_p(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in } H, \quad t > s \]
for each $s \in R_+$, where $\varphi^t_p, G_p(t, \cdot)$ and $f_p(t)$ are $T_0$-periodic, namely periodic in time with the same period $T_0, 0 < T_0 < +\infty$.

**Definition 3.1.** Let $T_0$ be a positive number. Then
(i) $\Phi_p(\{a_r\}, \{b_r\}; T_0)$ is the set of all $\{\varphi^t_p\} \in \Phi(\{a_r\}, \{b_r\})$ satisfying $T_0$-periodicity condition:
\[ \varphi^{t+T_0}_p(\cdot) = \varphi^t_p(\cdot) \quad \text{on } H, \quad \forall t \in R_+. \quad (3.1) \]
(ii) $G_p(\{\varphi^t_p\}; T_0)$ is the set of all $\{G_p(t, \cdot)\} \in G(\{\varphi^t_p\})$ satisfying $T_0$-periodicity condition:
\[ G_p(t + T_0, \cdot) = G_p(t, \cdot) \quad \text{in } H, \quad \forall t \in R_. \quad (3.2) \]

Throughout this section we assume that $\{\varphi^t_p\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0), \{G_p(t, \cdot)\} \in G_p(\{\varphi^t_p\}; T_0)$ and $f_p \in L^2_{\text{loc}}(R_+; H)$ is $T_0$-periodic in time, namely
\[ f_p(t + T_0) = f_p(t) \quad \text{in } H, \quad \forall t \in R_. \quad (3.3) \]

Here we note that $(P)_s$ can be considered as $(E)_s$ in Section 2. So, by the result (A) in Section 2, the Cauchy problem for $(P)_s$ has at least one solution $u$ on $[s, +\infty)$. Hence we can define the multivalued dynamical process associated with $(P)_s$ as follows:

**Definition 3.2.** For every $0 \leq s \leq t < +\infty$ we denote by $U(t, s)$ the mapping from $\overline{D(\varphi^t_p)}$ into $\overline{D(\varphi^t_p)}$ which assigns to each $u_0 \in \overline{D(\varphi^t_p)}$ the set
\[ U(t, s)u_0 := \left\{ z \in H \mid \begin{array}{l} \text{There is a solution } u \text{ of } (P)_s \text{ on } [s, +\infty) \text{ such that } \smallbreak u(s) = u_0 \text{ and } u(t) = z. \end{array} \right\}. \quad (3.4) \]

Then we easily see the following properties of $\{U(t, s)\} := \{U(t, s); 0 \leq s \leq t < +\infty\}$:
(U1) \(U(s, s) = I\) on \(\overline{D(\varphi_p^r)}\) for any \(s \in R_+\);

(U2) \(U(t_2, s)z = U(t_2, t_1)U(t_1, s)z\) for any \(0 \leq s \leq t_1 \leq t_2 < +\infty\) and \(z \in \overline{D(\varphi_p^s)}\);

(U3) \(U(t + T_0, s + T_0)z = U(t, s)z\) for any \(0 \leq s \leq t < +\infty\) and \(z \in \overline{D(\varphi_p^s)}\), that is, \(U\) is \(T_0\)-periodic.

(U4) \(\{U(t, s)\}\) has the following demi-closedness:

- If \(0 \leq s_n \leq t_n < +\infty\), \(s_n \to s\), \(t_n \to t\), \(z_n \in \overline{D(\varphi_p^{s_n})}\), \(z \in \overline{D(\varphi_p^s)}\), \(z_n \to z\) in \(H\) and a element \(w_n \in U(t_n, s_n)z_n\) converges to some element \(w \in H\) as \(n \to +\infty\), then \(w \in U(t, s)z\).

Next we define the discrete dynamical system in order to construct a global attractor for \((P)_s\).

**Definition 3.3.** Let \(U(\cdot, \cdot)\) be the solution operator for \((P)_s\) defined by Definition 3.2. Then

(i) For each \(\tau \in R_+\), we denote by \(U_\tau\) the \(T_0\)-step mapping from \(\overline{D(\varphi_p^\tau)}\) into \(\overline{D(\varphi_p^{\tau+T_0})}\), namely,

\[U_\tau := U(\tau + T_0, \tau)\]

(ii) For any \(k \in Z_+ := N \cup \{0\}\), we define

\[U_\tau^k := U_\tau \circ U_\tau \circ \cdots \circ U_\tau\]

\(k\)-th iteration

Clearly we have \(U_\tau^k = U(\tau + kT_0, \tau)\) for any \(\tau \in R_+\) and \(k \in Z_+\).

Now, let us recall the known result on the existence of global attractors for discrete multivalued dynamical systems \(U_\tau\) associated with \((P)_s\).

**Theorem 3.1.** (cf. [29, Theorem 3.1]) Assume that \(\{\varphi_p^r\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)\), \(\{G_p(t, \cdot)\} \in G_p(\{a_r^s\}; T_0)\), \(f_p \in L^{2}_\text{loc}(R_+; H)\) satisfies the \(T_0\)-periodicity condition (3.3). Then, for each \(\tau \in R_+\), there exists a subset \(A_\tau\) of \(\overline{D(\varphi_p^\tau)}\) such that

(i) \(A_\tau\) is non-empty and compact in \(H\);

(ii) for each bounded set \(B\) in \(H\) and each number \(\epsilon > 0\) there exists \(N_{B, \epsilon} \in N\) such that

\[\text{dist}_H(U_\tau^kz, A_\tau) < \epsilon\]

for all \(z \in \overline{D(\varphi_p^\tau)} \cap B\) and all \(k \geq N_{B, \epsilon}\);

(iii) \(U_\tau^kA_\tau = A_\tau\) for any \(k \in N\).

**Remark 3.1.** By [29, Lemma 3.1] we can get the compact absorbing set \(B_{0, \tau}\) of \(\overline{D(\varphi_p^\tau)}\) for \(U_\tau\) such that for each bounded subset \(B\) of \(H\) there is a positive integer \(n_B\) (independent of \(\tau \in R_+\)) satisfying

\[U_\tau^n(\overline{D(\varphi_p^\tau)} \cap B) \subset B_{0, \tau}\]

for all \(n \geq n_B\).
Then we observe that the global attractor $A_\tau$ is given by the $\omega$-limit set of the absorbing set $B_0, \tau$ for $U_\tau$, i.e.

$$A_\tau = \bigcap_{n \in \mathbb{Z}_+} \bigcup_{k \geq n} U_\tau^k \overline{B_0, \tau}.$$ 

The next theorem is concerned with a relationship between two global attractors $A_s$ and $A_\tau$. For detail proof, see [29].

**Theorem 3.2.** (cf. [29, Theorem 3.2]) Suppose the same assumptions are made as in Theorem 3.1. Let $A_s$ and $A_\tau$ be the global attractors for $U_s$ and $U_\tau$, with $0 \leq s \leq \tau \leq T_0$, respectively. Then, we have

$$A_\tau = U(\tau, s) A_s,$$

where $U(\tau, s)$ is the $T_0$-periodic process given in Definition 3.2.

**Remark 3.2.** By Theorem 3.1 (iii) and Theorem 3.2, we see that the global attractor $A_\tau$ for $U_\tau$ is $T_0$-periodic in $\tau$. In fact, for each $\tau \in \mathbb{R}_+$ choose $m_\tau \in \mathbb{Z}_+$ and $\sigma_\tau \in [0, T_0)$ so that $\tau = \sigma_\tau + m_\tau T_0$. Then, we have $A_\tau = A_{\sigma_\tau}.$

The third known result is the existence of a global attractor for the $T_0$-periodic multivalued dynamical system $(P)_s$.

**Theorem 3.3.** (cf. [29, Theorem 3.3]) Under the same assumptions as Theorem 3.1, put

$$A := \bigcup_{0 \leq \tau \leq T_0} A_\tau,$$

where $A_\tau$ is as obtained in Theorem 3.1. Then, $A$ has the following properties:

(i) $A$ is non-empty and compact in $H$;

(ii) for each bounded set $B$ in $H$ and each number $\epsilon > 0$ there exists a finite time $T_{B, \epsilon} > 0$ such that

$$\text{dist}_H(U(t + \tau, \tau) z, A) < \epsilon$$

for all $\tau \in \mathbb{R}_+$, all $z \in D(\varphi_0^\tau) \cap B$ and all $t \geq T_{B, \epsilon}$.

**Remark 3.3.** In [29, Section 4] the characterization of the $T_0$-periodic global attractor was discussed. The author proved that for each time $\tau \in \mathbb{R}_+$ the global attractor $A_\tau$ for the discrete multivalued dynamical system $U_\tau$ coincides with the cross-section of the family of all global bounded complete trajectories for the $T_0$-periodic system $(P)_s$.

### 4 Attractors of asymptotically periodic multivalued dynamical system

Throughout this section, let $M > 0$ be a fixed (sufficiently) large positive number. Now we put

$$\Psi_M := \left\{ \psi; \psi \text{ is proper, l.s.c. and convex on } H, \exists z \in D(\psi) \text{ s.t. } |z|_H \leq M, \psi(z) \leq M \right\}.$$
Then let us introduce the notion of a metric topology on $\Psi_M$ which was introduced in [16].

Given $\varphi, \psi \in \Psi_M$, we define $\rho(\varphi, \psi; \cdot): D(\varphi) \to \mathbb{R}$ by putting

$$\rho(\varphi, \psi; z) = \inf \{ \max \{ |y - z|_H, \psi(y) - \varphi(z) \}; y \in D(\psi) \}$$

for each $z \in D(\varphi)$, and for each $r \geq M$

$$\rho_r(\varphi, \psi) := \sup_{z \in L_\varphi(r)} \rho(\varphi, \psi; z),$$

where $L_\varphi(r) := \{ z \in D(\varphi); |z|_H \leq r, \varphi(z) \leq r \}$. Moreover, for each $r \geq M$, we define the functional $\pi_r(\cdot, \cdot)$ on $\Psi_M \times \Psi_M$ by

$$\pi_r(\varphi, \psi) := \rho_r(\varphi, \psi) + \rho_r(\psi, \varphi) \quad \text{for } \varphi, \psi \in \Psi_M.$$

Then, according to [16, Proposition 3.1], we can define a complete metric topology on $\Psi_M$ so that the convergence $\psi_n \to \psi$ in $\Psi_M$ (as $n \to +\infty$) if and only if

$$\pi_r(\psi_n, \psi) \to 0 \quad \text{for every } r \geq M.$$

Now by using the above topology on $\Psi_M$, we consider an asymptotically $T_0$-periodic system as follows.

**Definition 4.1.** Assume \{ \varphi_t \} $\in \Phi(\{a_r\}, \{b_r\}) \cap \Psi_M$, \{ G(t, \cdot) \} $\in G(\{ \varphi_t \})$ and $f \in L_{loc}^2(R_+; H)$. Then the system

$$(AP)_s \quad v'(t) + \partial\varphi'(v(t)) + G(t, v(t)) \ni f(t) \quad \text{in } H, \ t > s \ (s \geq 0)$$

is asymptotically $T_0$-periodic, if there are \{ \varphi_t \} $\in \Phi_p(\{a_r\}, \{b_r\}; T_0) \cap \Psi_M$, \{ G_\varphi(t, \cdot) \} $\in G_\varphi(\{ \varphi_t \}; T_0)$ and a $T_0$-periodic function $f_p \in L_{loc}^2(R_+; H)$ such that

**A1** (Convergence of $\varphi_t - \varphi'_p \to 0$ as $t \to +\infty$) For each $r \geq M$,

$$J_m^{(r)} := \sup_{\sigma \in [0, T_0]} \pi_r(\varphi_{mT_0 + \sigma}, \varphi'_p) \to 0 \quad \text{as } m \to +\infty;$$

**A2** (Convergence of $G(t, \cdot) - G_p(t, \cdot) \to 0$ as $t \to +\infty$) If \{ $\tau_n$ \} $\subset [0, T_0]$, \{ $m_n$ \} $\subset Z_+$, $m_n \to +\infty$, $z_n \in D(\varphi^{m_nT_0 + \tau_n})$, $g_n \in G(m_nT_0 + \tau_n, z_n)$, \{ $\varphi^{m_nT_0 + \tau_n}(z_n)$ \} is bounded, $z_n \to z$ in $H$, $\tau_n \to \tau$ and $g_n \to g$ weakly in $H$ (as $n \to +\infty$), then

$$g \in G_p(\tau, z);$$

**A3** (Convergence of $f(t) - f_p(t) \to 0$ as $t \to +\infty$)

$$\| f(mT_0 + \cdot) - f_p \|_{L^2(0, T_0; H)} \to 0 \quad \text{as } m \to +\infty.$$
By Definition 4.1 we easily see that a limiting system for (AP)$_s$ is a $T_0$-periodic one of the form:

\[(P)_s \quad u'(t) + \partial\varphi_p'(u(t)) + G_p(t, u(t)) \ni f_p(t) \quad \text{in } H, \ t > s \ (\geq 0).
\]

Here we note that (AP)$_s$ is also considered as (E)$_s$. So, by the result (A) in Section 2, the Cauchy problem for (AP)$_s$ has at least one solution $v$ on $[s, +\infty)$. Hence we can define the multivalued dynamical system associated with (AP)$_s$ as follows:

**Definition 4.2.** For every $0 \leq s \leq t < +\infty$ we denote by $E(t, s)$ the mapping from $D(\varphi^s)$ into $D(\varphi^t)$ which assigns to each $v_0 \in D(\varphi^s)$ the set

\[
E(t, s)v_0 := \left\{ z \in H \mid \text{There is a solution } v \text{ of (AP)$_s$ on } [s, +\infty) \text{ such that } v(s) = v_0 \text{ and } v(t) = z. \right\}.
\]

Then we easily see that $\{E(t, s)\} := \{E(t, s); 0 \leq s \leq t < +\infty\}$ has the following evolution properties:

1. \textbf{(E1)} $E(s, s) = I$ on $\overline{D(\varphi^s)}$ for any $s \in R_+$;
2. \textbf{(E2)} $E(t_2, s)z = E(t_2, t_1)E(t_1, s)z$ for any $0 \leq s \leq t_1 \leq t_2 < +\infty$ and $z \in \overline{D(\varphi^s)}$;
3. \textbf{(E3)} $\{E(t, s)\}$ has the following demi-closedness:
   - If $0 \leq s_n \leq t_n < +\infty$, $s_n \rightarrow s$, $t_n \rightarrow t$, $z_n \in \overline{D(\varphi^{s_n})}$, $z \in \overline{D(\varphi^t)}$, $z_n \rightarrow z$ in $H$ and a element $w_n \in E(t_n, s_n)z_n$ converges to some element $w \in H$ as $n \rightarrow +\infty$, then $w \in E(t, s)z$.

We begin with the definition of a discrete $\omega$-limit set for $E(\cdot, \cdot)$.

**Definition 4.3.** (Discrete $\omega$-limit set for $E(\cdot, \cdot)$) Let $\tau \in R_+$ be fixed. Let $\mathcal{B}(H)$ be a family of bounded subsets of $H$. Then for each $B \in \mathcal{B}(H)$, the set

\[
\omega_\tau(B) := \bigcap_{n \in Z_+, m \geq n } \bigcup_{k \geq m} E(kT_0 + mT_0 + \tau, mT_0 + \tau)\overline{(\overline{D(\varphi^{mT_0+\tau}) \cap B})}
\]

is called the discrete $\omega$-limit set of $B$ under $E(\cdot, \cdot)$.

**Remark 4.1.** By definition of the discrete $\omega$-limit set $\omega_\tau(B)$, it is easy to see that $x \in \omega_\tau(B)$ if and only if there exist sequences $\{k_n\} \subset Z_+$ with $k_n \uparrow +\infty$, $\{m_n\} \subset Z_+$, $\{z_n\} \subset B$ with $z_n \in \overline{D(\varphi^{m_nT_0+\tau})}$ and $\{x_n\} \subset H$ with $x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n$ such that $x_n \rightarrow x$ in $H$ as $n \rightarrow +\infty$.

Now let us mention main theorems in this paper.

**Theorem 4.1.** (Discrete attractors of (AP)$_{\tau}$) For each $\tau \in R_+$, let $\mathcal{A}_\tau$ be the global attractor of $T_0$-periodic dynamical systems $U_\tau$, which is obtained in Section 3. For $\{\varphi^s\} \in$
\[ \Phi(\{a_r\}, \{b_r\}) \cap \Psi_M, \{G(t, \cdot)\} \in \mathcal{G}(\{\phi^t\}) \text{ and } f \in L^2_{loc}(R_+; H), \]
we assume that the system \((AP)s\) is asymptotically \(T_0\)-periodic. Here we put
\[ A^*_\tau := \bigcup_{B \in \mathcal{B}(H)} \omega_\tau(B). \]  
(4.1)

Then, we have

(i) \(A^*_\tau(\subset D(\phi^\tau))\) is non-empty and compact in \(H\);

(ii) for each bounded set \(B \in \mathcal{B}(H)\) and each number \(\epsilon > 0\) there exists \(N_{B,\epsilon} \in \mathbb{N}\) such that
\[ \text{dist}_H(E(kT_0 + \tau, \tau)z, A^*_\tau) < \epsilon \]
for all \(z \in D(\phi^\tau) \cap B\) and all \(k \geq N_{B,\epsilon}\);

(iii) \(A^*_\tau \subset U_\tau A^*_\tau \subset A_\tau\) for any \(l \in \mathbb{N}\), where \(U_\tau\) is the discrete dynamical system for \((P)_\tau\) given in Definition 3.3.

**Remark 4.2.** By the definition of the discrete \(\omega\)-limit set \(\omega_\tau(B)\) and \(A^*_\tau\), we easily see that
\[ A^*_\tau = A^*_{\tau + nT_0}, \quad \forall n \in \mathbb{N}. \]
Hence \(A^*_\tau\) is \(T_0\)-periodic in time in a sense of the above.

The second main theorem is concerned with a relationship between two attractors \(A^*_s\) and \(A^*_\tau\).

**Theorem 4.2.** Suppose the same assumptions are made as in Theorem 4.1. Let \(A^*_s\) and \(A^*_\tau\) be discrete attractors for \(E(\cdot, s)\) and \(E(\cdot, \tau)\) with \(0 \leq s \leq \tau < +\infty\), respectively. Then,
\[ A^*_\tau \subset U(\tau, s) A^*_s. \]
where \(U(\tau, s)\) is the \(T_0\)-periodic process for \((P)_s\) which is given in Definition 3.2.

By Theorems 4.1-4.2, we can get the attractor for asymptotic \(T_0\)-periodic system \((AP)\tau\).

**Theorem 4.3.** (Global attractor for \((AP)\tau\)) Suppose the same assumptions are made as in Theorem 4.1. For any \(\tau \in R_+\), let \(A^*_\tau\) be the discrete attractor for \(E(\cdot, \tau)\) obtained in Theorem 4.1. Here we put
\[ A^* := \bigcup_{\tau \in [0, T_0]} A^*_\tau. \]  
(4.2)

Then, for any bounded set \(B \in \mathcal{B}(H)\),
\[ \bigcap_{s \geq 0} \bigcup_{t \geq s, \tau \in R_+} E(t + \tau, \tau)(D(\phi^\tau) \cap B) \subset A^*. \]  
(4.3)
By Theorem 4.3, the set $A^*$ can be called the global attractor of $(AP)_\tau$.
Here we give some key lemmas.

**Lemma 4.1.** If $\{s_n\} \subset R_+, \{\tau_n\} \subset R_+, s \in R_+, \tau \in R_+, s_n \to s, \tau_n \to \tau$, $\{m_n\} \subset Z_+$ with $m_n \to +\infty$, $z_n \in \overline{D(\varphi_{m_nT_0+s_n})}$, $z \in \overline{D(\varphi_p)}$, $z_n \to z$ in $H$ and a element $w_n \in E(m_nT_0 + \tau_n + s_n, m_nT_0 + s_n)z_n$ converges to some element $w \in H$ as $n \to +\infty$, then $w \in U(\tau + s, s)z$.

**Proof.** Since $\tau_n \to \tau$, without loss of generality we may assume that there exists a finite time $T > 0$ such that $\{\tau_n\} \subset [0, T]$ and $\tau \in [0, T]$. By $w_n \in E(m_nT_0 + \tau_n + s_n, m_nT_0 + s_n)z_n$, there is a solution $v_n$ of $(AP)_{m_nT_0+s_n}$ on $[m_nT_0 + s_n, +\infty)$ such that
\[
v_n(m_nT_0 + \tau_n + s_n) = w_n \text{ and } v_n(m_nT_0 + s_n) = z_n.
\]

Now we put $u_n(t) := v_n(t + m_nT_0 + s_n)$, then we easily see that $u_n$ is the solution for
\[
\left\{
\begin{aligned}
u_n'(t) + \partial \varphi_{t+m_nT_0+s_n}(u_n(t)) + G(t + m_nT_0 + s_n, u_n(t)) &\geq f(t + m_nT_0 + s_n), \quad t > 0, \\
u_n(0) &= z.
\end{aligned}
\right.
\]

Let $\delta \in (0, 1)$ be fixed. Since $z_n \to z$ in $H$ as $n \to +\infty$, $\{z_n\}$ is bounded in $H$. Hence, from global estimates of solutions (cf. (B) in Section 2) it follows that there is a positive constant $M_\delta > 0$ (independent of $n$) satisfying
\[
\sup_{t \geq \delta} |u_n(t)|^2_H + \sup_{t \geq \delta} u_n'(t)^2_{L^2(t, t+1; H)} + \sup_{t \geq \delta} \varphi_{t+m_nT_0+s_n}(u_n(t)) \leq M_\delta. \tag{4.4}
\]

By [16, Lemma 4.1] we note that the convergence assumption (A1) implies
\[
\varphi_{t+m_nT_0+s_n} \to \varphi_p^{t+s} \text{ in the sense of Mosco} \tag{4.5}
\]
for each $t \geq 0$ as $n \to +\infty$. Moreover by the same argument in [10, Lemma 3.1] we can prove that
\[
\bigcup_{n=1}^{+\infty} \{z \in H; \varphi_{t+m_nT_0+s_n}(z) \leq k\} \text{ is relatively compact in } H \tag{4.6}
\]
for every real $k > 0$ and $t \geq 0$, where $\varphi_{t+m_nT_0+s_n} = \varphi_p^{t+s}$ if $n = +\infty$. Therefore, by (4.4)-(4.6), (A2), (A3) and the convergence result (C) in Section 2, (by taking a subsequence of $\{n\}$, if necessary) we see that there is a function $u_\delta$ such that
\[
u_\delta'(t) + \partial \varphi_p^{t+s}(u_\delta(t)) + G_p(t + s, u_\delta(t)) \geq f_p(t + s), \quad t > \delta.
\]

By the standard diagonal process and the same argument in [21, Lemma 3.10], we can construct the solution $u$ on $[0, +\infty)$ satisfying
\[
\left\{
\begin{aligned}
u'(t) + \partial \varphi_p^{t+s}(u(t)) + G_p(t + s, u(t)) &\geq f_p(t + s), \quad t > 0, \\
u(0) &= z
\end{aligned}
\right.
\]
and
\[
u_n \to u \text{ in } C([0, T]; H) \text{ as } n \to +\infty. \tag{4.7}
\]
Then, by (4.7) and \( u_n(\tau_n) = w_n \) we have \( u(\tau) = w \), which implies that \( w \in U(\tau + s, s)z \). \( \square \)

By (B) in Section 2, for each \( B \in \mathcal{B}(H) \) we can choose constants \( r_B > 0 \) and \( M_B > 0 \) so that

\[
|v|_H \leq r_B \quad \text{and} \quad \varphi^{t+s}(v) \leq M_B, \tag{4.8}
\]

for any \( s \in R_+, \ t \geq T_0, \ z \in \overline{D(\varphi^t)} \cap B \) and \( v \in E(t+s,s)z \). Hence it follows from condition (A1) that for each \( m \in Z_+, \ \tau \in [0,T_0], \ n \in N \) and \( z \in \overline{D(\varphi^{mT_0+\tau})} \cap B \) there is \( \bar{z} := \bar{z}_{mT_0+\tau,z,nT_0} \in D(\varphi^\tau) \) such that

\[
|\bar{z} - v|_H \leq J_{m+n}^{(r_B+M_B+M)},
\]

(hence \( |\bar{z}|_H \leq r_B + J_{m+n}^{(r_B+M_B+M)} \))

and

\[
\varphi_p^\tau(\bar{z}) - \varphi^{nT_0+mT_0+\tau}(v) \leq J_{m+n}^{(r_B+M_B+M)},
\]

(hence \( \varphi_p^\tau(\bar{z}) \leq M_B + J_{m+n}^{(r_B+M_B+M)} \)).

where \( v \in E(nT_0 + mT_0 + \tau, mT_0 + \tau)z \).

Since \( J_k^{(r_B+M_B+M)} \to 0 \) as \( k \to +\infty \), there is a number \( N_0 \in N \) such that

\[
J_k^{(r_B+M_B+M)} \leq 1, \quad \forall k > N_0.
\]

Now, put \( J_0 := 1 + \sup_{1 \leq k \leq N_0} J_k^{(r_B+M_B+M)} < +\infty \). Then, we define the bounded set \( \overline{B}_\tau \) by

\[
\overline{B}_\tau := \{ z \in H; |z|_H \leq r_B + J_0 \} \cap \overline{D(\varphi^\tau_p)}.
\]

Let \( B_{0,\tau} \) be the compact absorbing set for \( U_\tau \) introduced by Remark 3.1. Then, we see that there exists a number \( \tilde{N} \in N \) so that

\[
U_\tau^l \overline{B}_\tau \subset B_{0,\tau}, \quad \forall l \geq \tilde{N}. \tag{4.9}
\]

The next lemma is very important to prove Theorem 4.1 (iii).

**Lemma 4.2.** Let \( \tau \in R_+ \) and \( B_{0,\tau} \) be the compact absorbing set for \( U_\tau \). Then we have

\[
\omega_\tau(B) \subset B_{0,\tau}, \quad \forall B \in \mathcal{B}(H).
\]

**Proof.** At first we assume \( \tau \in [0,T_0] \).

For each \( B \in \mathcal{B}(H) \), let \( x \) be any element of \( \omega_\tau(B) \). Then, it follows from Remark 4.1 that there exist sequences \( \{k_n\} \subset Z_+ \) with \( k_n \to +\infty \), \( \{m_n\} \subset Z_+ \), \( \{z_n\} \subset B \) with \( z_n \in \overline{D(\varphi^{m_nT_0+\tau})} \) and \( \{x_n\} \subset H \) with \( x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n \) such that

\[
x_n \longrightarrow x \text{ in } H \quad \text{as } n \to +\infty. \tag{4.10}
\]

Let \( \tilde{N} \) be the positive integer obtained in (4.9). Then by (E2) we have

\[
x_n \in E(k_nT_0 + m_nT_0 + \tau, k_nT_0 - \tilde{N}T_0 + m_nT_0 + \tau)
\]

for some \( k_n \in Z_+ \) and \( m_n \in Z_+ \). Then, we have

\[
x_n \longrightarrow x \text{ in } H \quad \text{as } n \to +\infty.
\]

Therefore, \( x \) is the limit point of \( \{x_n\} \) and \( x \in \overline{B}_\tau \). Hence \( \omega_\tau(B) \subset B_{0,\tau} \).
\( \circ E(k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n \)  
(4.11)

for any \( n \) with \( k_n \geq \tilde{N} + 1 \).

Hence, there exists an element \( y_n \in E(k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n \) such that

\[
x_n \in E(k_n T_0 + m_n T_0 + \tau, k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau) y_n.
\]

(4.12)

Since \( \{ z_n \} \subset B \), we see that

\[
|y_n|_H \leq r_B \quad \text{and} \quad \varphi^{k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau}(y_n) \leq M_B \quad \text{for any } n \text{ with } k_n \geq \tilde{N} + 1,
\]

where \( r_B \) and \( M_B \) are same positive constants in (4.8).

From the convergence condition (A1) it follows that for \( y_n \in E(k_n T_0 - \tilde{N} T_0 + m_n T_0 + \tau, m_n T_0 + \tau) z_n \) there is \( \tilde{z}_n \in D(\varphi^\tau_p) \) such that

\[
|\tilde{z}_n - y_n|_H \leq J^{(r_B + M_B + M)}_{k_n - \tilde{N} + m_n},
\]

(hence \( |\tilde{z}_n|_H \leq r_B + J^{(r_B + M_B + M)}_{k_n - \tilde{N} + m_n} \))

and

\[
\varphi^\tau_p(\tilde{z}_n) \leq M_B + J^{(r_B + M_B + M)}_{k_n - \tilde{N} + m_n}.
\]

Since \( \{ \tilde{z}_n \in D(\varphi^\tau_p) ; n \in N \text{ with } k_n \geq \tilde{N} + 1 \} \subset \tilde{B}_\tau \) is relatively compact in \( H \), we may assume that

\[
\tilde{z}_n \longrightarrow \tilde{z}_\infty \text{ in } H \quad \text{as } n \rightarrow +\infty
\]

for some \( \tilde{z}_\infty \in H \). Then we easily see that \( \tilde{z}_\infty \in \tilde{B}_\tau \) and

\[
y_n \longrightarrow \tilde{z}_\infty \text{ in } H \quad \text{as } n \rightarrow +\infty. \quad (4.13)
\]

By Lemma 4.1 and (4.10)-(4.13), we observe that

\[
x \in U(\tilde{N} T_0 + \tau, \tau) \tilde{z}_\infty,
\]

which implies that

\[
x \in U(\tilde{N} T_0 + \tau, \tau) \tilde{B}_\tau = U^{\tilde{N}} \tilde{B}_\tau \subset B_{0, \tau}.
\]

Hence we have

\[
\omega_\tau(B) \subset B_{0, \tau}.
\]

For the general case of \( \tau \in R_+ \), choose positive numbers \( i_\tau \in N \) and \( \tau_0 \in [0, T_0] \) so that \( \tau = \tau_0 + i_\tau T_0 \). Then, we can show \( \omega_\tau(B) \subset B_{0, \tau} \) by the same argument as above. \( \diamond \)

**Proof of Theorem 4.1.** On account of Lemma 4.2 we can get \( A_\tau^* \subset B_{0, \tau} \). Hence, Theorem 4.1 (i) holds. Also, by (4.1) and Remark 4.1 we observe that Theorem 4.1 (ii) holds.

Now, we prove Theorem 4.1 (iii). At first, let us prove that \( A_\tau^* \subset U_\tau^l A_\tau^* \) for any \( l \in N \).
Let $x$ be any element of $\mathcal{A}_r^*$. By the definition of $\mathcal{A}_r^*$, there are sequences $\{B_n\} \subset \mathcal{B}(H)$ and $\{x_n\} \subset H$ with $x_n \in \omega_r(B_n)$ such that

$$x_n \longrightarrow x \text{ in } H \quad \text{as } n \rightarrow +\infty. \quad (4.14)$$

Then, for each $n$ it follows from Remark 4.1 that there exist sequences $\{k_{n,j}\} \subset \mathbb{Z}_+$ with $k_{n,j} \rightarrow +\infty$, $\{m_{n,j}\} \subset \mathbb{Z}_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in \overline{D(\varphi^{m_{n,j}T_0+\tau})}$ and $\{v_{n,j}\} \subset H$ with $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \longrightarrow x_n \text{ in } H \quad \text{as } j \rightarrow +\infty. \quad (4.15)$$

Let $l$ be any number in $N$, then we see that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau)$$

$$\circ E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$$

for $j$ with $k_{n,j} \geq l + 1$. So, there exists an element $w_{n,j} \in E(k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 - lT_0 + m_{n,j}T_0 + \tau)w_{n,j}. \quad (4.16)$$

By global estimates (B) in Section 2, $\{w_{n,j} \in H : j \in N \text{ with } k_{n,j} \geq l+1\}$ is relatively compact in $H$ for each $n$. Therefore we may assume that the element $w_{n,j}$ converges to some element $\bar{w}_{n,\infty} \in H$ as $j \rightarrow +\infty$. Clearly, $\bar{w}_{n,\infty} \in \omega_r(B_n)$. Moreover, it follows from Lemma 4.1 and (4.15)-(4.16) that

$$x_n \in U(lT_0 + \tau, \tau)\bar{w}_{n,\infty} \subset U(lT_0 + \tau, \tau)\omega_r(B_n),$$

hence, we have

$$x_n \in \bigcup_{n \geq 1} U_{l}^\tau \omega_r(B_n), \quad \forall n \geq 1. \quad (4.17)$$

Here, by the closedness of $U(\cdot, \tau)$ we note that for each subset $X$ of $B_{0,\tau}$,

$$\overline{U_{l}^\tau X} \subset U_{l}^\tau \overline{X}, \quad \forall l \in N. \quad (4.18)$$

Taking account of Lemma 4.2, (4.14), (4.17) and (4.18), we observe that

$$x \in \overline{\bigcup_{n \geq 1} U_{l}^\tau \omega_r(B_n)}$$

$$= U_{l}^\tau \bigcup_{n \geq 1} \omega_r(B_n)$$

$$\subset U_{l}^\tau \bigcup_{n \geq 1} \omega_r(B_n)$$

$$\subset U_{l}^\tau \mathcal{A}_r^*,$$

which implies that $\mathcal{A}_r^*$ is semi-invariant under the $T_0$-periodic dynamical systems $U_{\tau}$, i.e.

$$\mathcal{A}_r^* \subset U_{l}^\tau \mathcal{A}_r^*, \quad \forall l \in N. \quad (4.19)$$
Next we shall prove that $U^l_r \mathcal{A}^*_r \subset \mathcal{A}_r$ for any $l \in N$. By (4.19), for each $l \in N$

\[ U^l_r \mathcal{A}^*_r \subset U^l_r U^n_r \mathcal{A}^*_r = U^{l+n}_r \mathcal{A}^*_r, \quad \forall n \in N. \quad (4.20) \]

By $\mathcal{A}^*_r \subset B_{0,\tau}$, (4.20) and the attractive property of $\mathcal{A}_r$, we have

\[ U^l_r \mathcal{A}^*_r \subset \mathcal{A}_r, \quad \forall l \in N. \]

Therefore we conclude that

\[ \mathcal{A}^*_r \subset U^l_r \mathcal{A}^*_r \subset \mathcal{A}_r, \quad \forall l \in N. \]

\[ \diamond \]

**Proof of Theorem 4.2.** Let $x$ be any element of $\mathcal{A}^*_r$. Then by the definition of $\mathcal{A}^*_r$, there exist sequences $\{B_n\} \subset B(H)$ and $\{x_n\} \subset H$ with $x_n \in \omega_\tau(B_n)$ such that

\[ x_n \longrightarrow x \text{ in } H \quad \text{as } n \to +\infty. \quad (4.21) \]

From Remark 4.1 it follows that for each $n$, there are sequences $\{k_{n,j}\} \subset Z_+$ with $k_{n,j} \to +\infty$, $\{m_{n,j}\} \subset Z_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in D(\varphi^{m_{n,j}T_0+\tau})$ and $\{v_{n,j}\} \subset H$ with $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, m_{n,j}T_0 + \tau)z_{n,j}$ such that

\[ v_{n,j} \longrightarrow x_n \text{ in } H \quad \text{as } j \to +\infty. \quad (4.22) \]

Note that for given $s, \tau \in R_+$ with $s \leq \tau$ there is a positive number $l_s \in N$ satisfying

\[ s \leq \tau \leq l_s T_0 + s. \]

By using the property (E2) we see that

\[ v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s) \]

\[ \circ E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_s T_0 + s) \]

\[ \circ E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j} \]

for any $j \in Z_+$ with $k_{n,j} \geq l_s + 2$. Here we can take elements $w_{n,j} \in H$ and $y_{n,j} \in H$ so that

\[ v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau, k_{n,j}T_0 + m_{n,j}T_0 + s)w_{n,j}, \quad (4.23) \]

\[ w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_s T_0 + s)y_{n,j} \quad (4.24) \]

and

\[ y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j}. \quad (4.25) \]

By $\{z_{n,j}\} \subset B_n$ and the global boundedness result (B) in Section 2, we can get a positive constant $C_n := C_n(B_n) > 0$ satisfying

\[ |y_{n,j}|_H \leq C_n, \quad \forall y_{n,j} \in E(T_0 + m_{n,j}T_0 + l_s T_0 + s, m_{n,j}T_0 + \tau)z_{n,j}. \quad (4.26) \]
Here we define the bounded set $B_{C_n}$ by
\[ B_{C_n} := \{ b \in H : |b|_H \leq C_n \}. \]
From (4.26) and the result (B) in Section 2 it follows that the set
\[ \left\{ w_{n,j} \in H : w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s, T_0 + m_{n,j}T_0 + l_nT_0 + s)y_{n,j} \right\} \]
for any $j \in \mathbb{Z}_+$ with $k_{n,j} \geq l_n + 2$ is relatively compact in $H$. Hence, we may assume that the element $w_{n,j}$ converges to some element $\tilde{w}_{n,\infty} \in H$ as $j \to +\infty$. Clearly, $\tilde{w}_{n,\infty} \in \omega_s(B_{C_n})$, and it follows from Lemma 4.2 that
\[ \omega_s(B_{C_n}) \subset B_{0,s} \subset D(\varphi^{s}). \]
Moreover, by Lemma 4.1 and (4.22)-(4.23) we have
\[ x_n \in U(\tau, s)\tilde{w}_{n,\infty} \subset U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \geq 1, \]
hence, we see that
\[ x_n \in \bigcup_{n \geq 1} U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \geq 1. \quad (4.27) \]
Here, by the closedness of $U(\cdot, \cdot)$, we note that for each subset $X$ of $B_{0,s}$,
\[ \overline{U(\tau, s)X} \subset U(\tau, s)X. \quad (4.28) \]
On account of Lemma 4.2, (4.21), (4.27) and (4.28), we observe that
\[ x \in \bigcup_{n \geq 1} U(\tau, s)\omega_s(B_{C_n}) \]
\[ = U(\tau, s) \bigcup_{n \geq 1} \omega_s(B_{C_n}) \]
\[ \subset U(\tau, s) \bigcup_{n \geq 1} \omega_s(B_{C_n}) \]
\[ \subset U(\tau, s)A^*_s, \]
which implies that $A^*_s$ is the subset of $U(\tau, s)A^*_s$, namely
\[ A^*_s \subset U(\tau, s)A^*_s. \]
\[ \diamond \]
\textbf{Proof of Theorem 4.3.} For any $B \in \mathcal{B}(H)$, let $z_0$ be any element of the $\omega$-limit set $\omega_E(B)$ which is define by
\[ \omega_E(B) := \bigcap_{s \geq 0} \bigcup_{t \geq s, \tau \in \mathbb{R}_+} E(t + \tau, \tau)(\overline{D(\varphi^{s}) \cap B}). \]
Then we easily see that there exist sequences \( \{t_n\} \subset R_+ \) with \( t_n \uparrow +\infty \), \( \{\tau_n\} \subset R_+ \), \( \{y_n\} \subset B \) with \( y_n \in \overline{D}(\varphi^\tau) \) and \( \{z_n\} \subset H \) with \( z_n \in E(t_n + \tau_n, \tau_n)y_n \) such that
\[
\begin{align*}
t_n &:= k_n T_0 + t'_n, \ k_n \in \mathbb{Z}_+, \ k_n \not\to +\infty, \ t'_n \in [T_0, 2T_0], \ t'_n \to t'_0, \\
\tau_n &:= l_n T_0 + \tau'_n, \ l_n \in \mathbb{Z}_+, \ \tau'_n \in [0, T_0], \ \tau_n \to \tau'_0
\end{align*}
\]
and
\[
z_n \rightarrow z_0 \quad \text{in } H \quad (4.29)
\]
as \( n \to +\infty \). Without loss of generality, we may assume that
\[
(a) \quad t'_n + \tau_n \not\to t'_0 + \tau'_0 \quad \text{or} \quad (b) \quad t'_n + \tau_n \not\to t'_0 + \tau'_0.
\]
Now, assume that (a) holds. Then let us consider the multivalued semiflow
\[
v_n \in E(1 + k_n T_0 + l_n T_0 + t'_0 + \tau'_0, \ k_n T_0 + l_n T_0 + t'_n + \tau'_n)z_n. \quad (4.30)
\]
Then, there is a solution \( u_n \) on \( [k_n T_0 + l_n T_0 + t'_n + \tau'_n, +\infty) \) for
\[
\begin{cases}
  u_n'(t) + \partial_{\varphi^t + k_n T_0 + l_n T_0 + t'_0 + \tau'_0}(u_n(t)) + G(t + k_n T_0 + l_n T_0 + t'_n + \tau'_n, u_n(t)) \\
  \quad \ni f(t + k_n T_0 + l_n T_0 + t'_n + \tau'_n), \quad t > 0,
  \\
  u_n(0) = z_n \quad \text{and} \quad u_n(1 + t'_0 + \tau'_0 - t'_n - \tau'_n) = v_n.
\end{cases}
\]
Since \( z_n \to z_0 \) in \( H \), \( \{z_n\} \) is bounded in \( H \). Therefore by the global estimate \( (B) \) in Section 2, we see that
\[
\{v_n \in H; \ v_n \in E(1 + k_n T_0 + l_n T_0 + t'_0 + \tau'_0, \ k_n T_0 + l_n T_0 + t'_n + \tau'_n)z_n \}
\]
is relatively compact in \( H \). Hence we may assume that
\[
v_n \rightarrow v \quad \text{in } H \quad \text{for some } v \in H. \quad (4.31)
\]
Now applying Lemma 4.1 with (4.29)-(4.31), we can get
\[
v \in U(1 + t'_0 + \tau'_0, \ t'_0 + \tau'_0)z_0,
\]
more precisely, (taking the subsequence of \( \{n\} \) if necessary) we observe that
\[
[u_n \rightarrow u \quad \text{in } C([0, 2]; H) \quad \text{as } n \to +\infty, \quad (4.32)
\]
where \( u \) is the solution \( [t'_0 + \tau'_0, +\infty) \) satisfying
\[
\begin{cases}
  u'(t) + \partial_{\varphi^t + \tau'_0}(u(t)) + G_{(t + \tau'_0)}(t + \tau'_0, u(t)) \ni f_{(t + \tau'_0)}(t + \tau'_0, v(t)), \quad t > 0, \\
  u(0) = z_0 \quad \text{and} \quad u(1) = v.
\end{cases}
\]
By (4.32) we easily see that
\[
[u_n(t'_0 + \tau'_0 - t'_n - \tau'_n) \rightarrow z_0 \quad \text{as } n \to +\infty. \quad (4.33)
\]
Note that
\[ u_n(t_n' + \tau_n' - t_n' - \tau_n') \]
\[ \in E(k_n T_0 + l_n T_0 + t_0' + \tau_n', k_n T_0 + l_n T_0 + t_n' + \tau_n') z_n \]
\[ = E(k_n T_0 + l_n T_0 + t_0' + \tau_n', l_n T_0 + \tau_n') y_n \]
\[ = E(k_n T_0 + l_n T_0 + t_0' + \tau_0', l_n T_0 + t_0' + \tau') E(l_n T_0 + t_0' + \tau' + l_n T_0 + \tau') y_n. \]

So, we can take an element \( x_n \in E(l_n T_0 + t_0' + \tau_0', l_n T_0 + \tau_0') y_n \) such that
\[ u_n(t_0' + \tau_n' - t_n' + \tau_n') \in E(k_n T_0 + l_n T_0 + t_0' + \tau_n'; l_n T_0 + t_0' + \tau_0') x_n. \] (4.34)

By \( \{y_n\} \subset B \) and the global estimate (B) in Section 2, we easily see that \( \{x_n\} \) is bounded, i.e.
\[ \{x_n\} \subset \tilde{B} \] for some \( \tilde{B} \in \mathcal{B}(H). \) (4.35)

Therefore, from Remarks 4.1-4.2 and (4.33)-(4.35) we observe that

\[ z_0 \in \omega_{t_0' + \tau_0'}(\tilde{B}) \subset \mathcal{A}_{t_0' + \tau_0'}^* \subset \mathcal{A}^*. \]

Thus (4.3) holds.

In the case (b) when \( t_n' + \tau_n' \searrow t_0' + \tau_0' \), we can prove (4.3) by the slight modification of the proof as above. \( \Diamond \)

Theorem 4.1 implies that the attracting set \( \mathcal{A}^*_\tau \) for \( (AP)_\tau \) is semi-invariant under \( U_\tau \) associated with the limiting \( T_0 \)-periodic system \( (P)_s \), in general. Moreover, from Theorem 4.2 we observe that

\[ \mathcal{A}^*_\tau \subset U(\tau, s)\mathcal{A}^*_s \] for any \( 0 \leq s \leq \tau < +\infty. \]

In order to get the invariance of \( \mathcal{A}^*_\tau \) under \( U_\tau \) and \( \mathcal{A}^*_\tau = U(\tau, s)\mathcal{A}^*_s \), let us use a concept of a regular approximation, which was introduced in [17].

**Definition 4.4.** (Regular approximation) Let \( s \in \mathbb{R} \) be fixed. Let \( \varphi \in D(\varphi^s) \). Then, we say that \( U(t, s) z \) is regularly approximated by \( E(t + k T_0 + s, k T_0 + s) \) as \( k \to +\infty \), if for each finite \( T > 0 \) there are sequences \( \{k_n\} \subset \mathbb{Z} \) with \( k_n \to +\infty \) and \( \{z_n\} \subset H \) with \( z_n \in D(\varphi^{k_n T_0 + s}) \) and \( z_n \to z \) in \( H \) satisfying the following property: for any function \( u \in W^{1,2}(0, T; H) \) satisfying \( u(t) \in U(t + s, s) z \) for all \( t \in [0, T] \) there is a sequence \( \{u_n\} \subset W^{1,2}(0, T; H) \) such that \( u_n(t) \in E(t + k_n T_0 + s, k_n T_0 + s) z_n \) for all \( t \in [0, T] \) and \( u_n \to u \) in \( C([0, T]; H) \) as \( n \to +\infty \).

Using the above concept, we can show that the invariance of \( \mathcal{A}^*_\tau \) under \( U_\tau \). Moreover we can get

\[ \mathcal{A}^*_\tau = U(\tau, s)\mathcal{A}^*_s. \]

**Theorem 4.4** Suppose all assumptions in Theorem 4.1. Let \( \mathcal{A}^*_\tau \) and \( \mathcal{A}^*_\tau \) be discrete attractors for \( E(\cdot, s) \) and \( E(\cdot, \tau) \), with \( 0 \leq s \leq \tau < +\infty \), respectively. Assume that for
any point \( z \) of \( \mathcal{A}_n^* \), \( U(t + s, s)z \) is regularly approximated by \( E(t + kT_0 + s, kT_0 + s) \) as \( k \to +\infty \). Then we have
\[
\mathcal{A}_n^* = U(\tau, s) \mathcal{A}_n^*.
\]

**Proof.** By Theorem 4.2, we have only to show that
\[
U(\tau, s) \mathcal{A}_n^* \subset \mathcal{A}_n^*.
\]

To do so, let \( x \) be any element of \( U(\tau, s) \mathcal{A}_n^* \).

At first, taking account of Definitions 3.2-3.3 and Theorem 4.1 (iii), we see that for each \( n \in \mathbb{N} \)
\[
\begin{align*}
U^n_\tau U(\tau, s) \mathcal{A}_n^* &= U(nT_\tau + \tau, \tau)U(\tau, s) \mathcal{A}_n^* \\
&= U(nT_\tau + \tau, nT_0 + s)U(nT_0 + s, s) \mathcal{A}_n^* \\
&= U(\tau, s)U^n_\tau \mathcal{A}_n^* \\
&\supset U(\tau, s) \mathcal{A}_n^*.
\end{align*}
\]

Hence, there exists a element \( y_n \in \mathcal{A}_n^* \) such that
\[
x \in U^n_\tau U(\tau, s)y_n = U(nT_\tau + \tau - s + s, s)y_n.
\]

By using our assumption as \( t = nT_\tau + \tau - s \), we observe that for each \( n \), there are sequences \( \{k_{n,j}\} \subset \mathbb{Z}_+ \), \( \{x_{n,j}\} \subset H \) and \( \{y_{n,j}\} \subset H \) such that
\[
k_{n,j} \to +\infty, \quad y_{n,j} \in D(\varphi^{k_{n,j}T_\tau + s}), \quad y_{n,j} \to y_n \quad \text{in} \quad H
\]
and
\[
x_{n,j} \in E(nT_\tau + \tau - s + k_{n,j}T_\tau + s, k_{n,j}T_\tau + s)y_{n,j}, \quad x_{n,j} \to x \quad \text{in} \quad H
\]
as \( j \to +\infty \). Therefore, by the usual diagonal argument, we can find a subsequence \( \{j_n\} \) of \( \{j\} \) such that \( \tilde{x}_n := x_{n,j_n}, \tilde{y}_n := y_{n,j_n} \) and \( \tilde{k}_n := k_{n,j_n} \) satisfy
\[
|\tilde{x}_n - x|_H < \frac{1}{n}, \quad \tilde{x}_n \in E(nT_\tau + \tau - s + \tilde{k}_nT_\tau + s, \tilde{k}_nT_\tau + s)y_n, \quad |\tilde{y}_n - y_n|_H < \frac{1}{n}
\]
for every \( n = 1, 2, \ldots \). Since \( \{\tilde{y}_n\} \) is bounded in \( H \), there is a bounded set \( B \in \mathcal{B}(H) \) so that \( \{\tilde{y}_n\} \subset B \).

By (E2), we see that
\[
\tilde{x}_n \in E(nT_\tau + \tau - s + \tilde{k}_nT_\tau + s, \tilde{k}_nT_\tau + s)y_n
\]
\[
= E(nT_0 + \tilde{k}_nT_0 + \tau, T_0 + \tilde{k}_nT_0 + \tau)E(T_0 + \tilde{k}_nT_0 + \tau, \tilde{k}_nT_0 + s)y_n,
\]
hence there is an element \( \tilde{z}_n \in E(T_0 + \tilde{k}_nT_0 + \tau, \tilde{k}_nT_0 + s)y_n \) such that
\[
\tilde{x}_n \in E(nT_0 + \tilde{k}_nT_0 + \tau, T_0 + \tilde{k}_nT_0 + \tau)\tilde{z}_n.
\]

Since \( \{\tilde{y}_n\} \subset B \) and the global estimate (B) in Section 2, we see that \( \{\tilde{z}_n\} \) is also bounded in \( H \). Hence, there is a bounded set \( \tilde{B} \in \mathcal{B}(H) \) so that \( \{\tilde{z}_n\} \subset \tilde{B} \). The above fact (4.37)-(4.39) implies (cf. Remark 4.1) that \( x \in \omega_\tau(\tilde{B}) \subset \mathcal{A}_n^* \). Thus we have \( U(\tau, s) \mathcal{A}_n^* \subset \mathcal{A}_n^* \). \( \diamond \)
By the same argument in Theorem 4.4, we can get the following corollary:

**Corollary.** (i) Suppose the same assumptions of Theorem 4.4. Then, by Remark 4.2 we observe that \( A^*_s \) is invariant under the \( T_0 \)-periodic dynamical system \( U_s(:=U(T_0+s,s)) \). Namely,

\[
A^*_s = U_s^l A^*_s \quad \text{for any } l \in N.
\]

(ii) Assume that for any point \( z \) of \( A_r \), \( U(t + \tau, \tau)z \) is regularly approximated by \( E(t + kT_0 + \tau, kT_0 + \tau) \) as \( k \to +\infty \). Then, we have \( A^*_r \supset A_r(= U_r A_r) \). Hence by Theorem 4.1 (iii) we conclude that

\[
A^*_r = A_r.
\]

**Remark 4.3.** If the solution operator \( U(t,s) \) is singlevalued, namely the solution for the Cauchy problem of (P) is unique, the assumptions of Theorem 4.4 always hold. Thus, Theorems 4.1-4.4 contain the abstract results obtained in [11], which was concerned with the asymptotic \( T_0 \)-periodic stability for the singlevalued dynamical system associated with time-dependent subdifferentials.

## 5 Application to obstacle problems for PDE’s

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) (\( 1 \leq N < +\infty \)) with smooth boundary \( \Gamma = \partial \Omega \), \( q \) be a fixed number with \( 2 \leq q < +\infty \) and \( T_0 \) be a fixed positive number. We use the notation

\[
a_q(v, z) := \int_{\Omega} |v|^q - 2v \cdot \nabla z dx, \quad \forall v, \ z \in W^{1,q}(\Omega)
\]

and denote by \( (\cdot, \cdot) \) the usual inner product in \( L^2(\Omega) \).

For prescribed obstacle functions \( \sigma_0 \leq \sigma_1 \) and each \( t \in \mathbb{R}_+ \) we define the set

\[
K(t) := \{ z \in W^{1,q}(\Omega); \ \sigma_0(t, \cdot) \leq z \leq \sigma_1(t, \cdot) \ \text{a.e. on } \Omega \}.
\]

Let \( f \) be a function in \( L^2_{loc}(\mathbb{R}_+; L^2(\Omega)) \) and \( h \) be a non-negative function on \( \mathbb{R}_+ \times \mathbb{R} \). Then for given \( b \in L^\infty(\Omega)^N \) we consider an interior asymptotically \( T_0 \)-periodic double obstacle problem \((\text{OP})^{AP}_s \) \((s \in \mathbb{R}_+) \):

- Find functions \( v \in C([s, +\infty); L^2(\Omega)) \) and \( \theta \in L^2_{loc}((s, +\infty); L^2(\Omega)) \) such that

\[
\begin{cases}
\begin{align*}
v &\in L^q_{loc}((s, +\infty); W^{1,q}(\Omega)) \cap W^{1,2}_{loc}((s, +\infty); L^2(\Omega)); \\
v(t) &\in K(t) \text{ for a.e. } t \geq s; \\
0 &\leq \theta(t,x) \leq h(t, v(t,x)) \text{ a.e. on } (s, +\infty) \times \Omega; \\
(v'(t) + \theta(t) + b \cdot \nabla v(t) - f(t), v(t) - z) + a_q(v(t), v(t) - z) &\leq 0 \\
&\text{for any } z \in K(t) \text{ and a.e. } t \geq s.
\end{align*}
\end{cases}
\]

\( (\text{OP})^{AP}_s \)
The main object of this section is to consider the large-time behaviour of solution for \( (OP)^s_{AP} \) assuming asymptotically \( T_0 \)-periodicity conditions
\[
\sigma_i(t) - \sigma_{i,p}(t) \to 0 \quad (i = 0, 1), \quad h(t, \cdot) - h_p(t, \cdot) \to 0, \quad f(t) - f_p(t) \to 0
\]
as \( t \to \infty \) in the sense specified below, where \( \sigma_i(t), h_p(t, \cdot), f_p(t) \) are periodic in time with the same period \( T_0 \). By the above assumptions, the limiting system of \( (OP)^s_{AP} \) is a \( T_0 \)-periodic one \( (OP)^p_s \) as follows:

- Find functions \( u \in C((s, +\infty); L^2(\Omega)) \) and \( \theta \in L^2_{loc}((s, +\infty); L^2(\Omega)) \) such that

\[
(\text{OP})^p_s\begin{cases}
    u \in L^2_{loc}((s, +\infty); W^{1,q}(\Omega)) \cap W^{1,2}_{loc}((s, +\infty); L^2(\Omega)); \\
    u(t) \in K_p(t) \text{ for a.e. } t \geq s; \\
    0 \leq \theta(t, x) \leq h_p(t, u(t, x)) \text{ a.e. on } (s, +\infty) \times \Omega; \\
    (u'(t) + \theta(t) + b \cdot \nabla u(t) - f_p(t), u(t) - z) \leq 0 \\
\end{cases}
\]

where \( K_p(t) := \{ z \in W^{1,q}(\Omega); \sigma_{0,p}(t, \cdot) \leq z \leq \sigma_{1,p}(t, \cdot) \text{ a.e. on } \Omega \} \).

Now we suppose the following conditions:

- \( \sigma_i \) and \( \sigma_{i,p} \) are functions on \( R_+ \times \Omega \) such that

\[
\sup_{t \in R_+} \left| \frac{d \sigma_i}{dt} \right|_{L^2(t,t+1; W^{1,q}(\Omega))} + \sup_{t \in R_+} \left| \frac{d \sigma_{i,p}}{dt} \right|_{L^2(t,t+1; L^\infty(\Omega))} < +\infty,
\]

and \( \sigma_{i,p} \) is a \( T_0 \)-periodic obstacle function, i.e.

\[
\sigma_{i,p}(t + T_0, x) = \sigma_{i,p}(t, x) \quad \text{for a.e. } x \in \Omega \text{ and any } t \in R_+
\]

for \( i = 0, 1 \). Moreover, there are positive constants \( k_1 > 0 \) and \( k_2 > 0 \) such that

\[
\sigma_1 - \sigma_0 \geq k_1 \quad \text{and} \quad \sigma_{1,p} - \sigma_{0,p} \geq k_1 \quad \text{a.e. on } R_+ \times \Omega
\]

and

\[
|\sigma_i|_{L^\infty(R_+; W^{1,q}(\Omega))} + |\sigma_i|_{L^\infty(R_+ \times \Omega)} + |\sigma_{i,p}|_{L^\infty(R_+; W^{1,q}(\Omega))} + |\sigma_{i,p}|_{L^\infty(R_+ \times \Omega)} \leq k_2
\]

for \( i = 0, 1 \).

- \( h \) and \( h_p \) are non-negative continuous functions on \( R_+ \times R \). There is a positive constant \( L \) such that

\[
|h(t, z_1) - h(t, z_2)| \leq L|z_1 - z_2| \\
|h_p(t, z_1) - h_p(t, z_2)| \leq L|z_1 - z_2|
\]

for all \( t \in R_+, \ z_i \in R \) and \( i = 1, 2 \). Moreover, \( h_p \) is a \( T_0 \)-periodic function, i.e. for any \( z \in R \), \( h_p(t + T_0, z) = h_p(t, z) \) for any \( t \in R_+ \).
• $f, f_p \in L^2_{\text{loc}}(R_+; L^2(\Omega))$, and $f_p$ is a $T_0$-periodic function, i.e.

$$f_p(t + T_0) = f_p(t) \quad \text{in } L^2(\Omega), \quad \forall t \in R_+.$$  

Moreover, we suppose the following convergence conditions:

• (Convergence of $\sigma_i(t) - \sigma_{i,p}(t) \longrightarrow 0$ as $t \rightarrow +\infty$) Put

$$I_m := \sup_{t \in [0,T_0]} |\sigma_0(mT_0 + t) - \sigma_{0,p}(t)|_{W^{1,q}(\Omega)} + \sup_{t \in [0,T_0]} |\sigma_1(mT_0 + t) - \sigma_{1,p}(t)|_{W^{1,q}(\Omega)}$$

$$+ \sup_{t \in [0,T_0]} |\sigma_0(mT_0 + t) - \sigma_{0,p}(t)|_{L^\infty(\Omega)} + \sup_{t \in [0,T_0]} |\sigma_1(mT_0 + t) - \sigma_{1,p}(t)|_{L^\infty(\Omega)}$$

Then,

$$I_m \longrightarrow 0 \quad \text{as } m \rightarrow +\infty;$$

• (Convergence of $h(t, \cdot) - h_p(t, \cdot) \longrightarrow 0$ as $t \rightarrow +\infty$) For any $z \in R$,

$$\sup_{t \in [0,T_0]} |h(mT_0 + t, z) - h_p(t, z)| \longrightarrow 0 \quad \text{as } m \rightarrow +\infty; \quad (5.1)$$

• (Convergence of $f(t) - f_p(t) \longrightarrow 0$ as $t \rightarrow +\infty$)

$$|f(mT_0 + \cdot) - f_p|_{L^2(0,T_0;L^2(\Omega))} \longrightarrow 0 \quad \text{as } m \rightarrow +\infty. \quad (5.2)$$

Under the above assumptions, let us consider problems (OP)$_s^{AP}$ and (OP)$_s^P$.

In order to apply the abstract results in Sections 2-4, we choose $L^2(\Omega)$ as a real separable Hilbert space $H$. And we define a family $\{\varphi^t\}$ of proper l.s.c. convex functions $\varphi^t$ on $L^2(\Omega)$ by

$$\varphi^t(z) = \begin{cases} 
\frac{1}{q} \int_\Omega |\nabla z|^q dx & \text{if } z \in K(t), \\
+\infty & \text{if } z \in L^2(\Omega) \setminus K(t),
\end{cases} \quad (5.3)$$

and define $\varphi^t_p$ by replacing $K(t)$ by $K_p(t)$ in (5.3).

Also, we define a multivalued operator $G(\cdot, \cdot)$ from $R_+ \times H^1(\Omega)$ into $L^2(\Omega)$ by

$$G(t, z) := \left\{ g \in L^2(\Omega); \begin{array}{lcl}
g &=& l + b \cdot \nabla z \quad \text{in } L^2(\Omega) \\
0 &\leq& l(x) \leq h(t, z(x)) \quad \text{a.e. on } \Omega
\end{array} \right\} \quad (5.4)$$

for all $t \in R_+$ and $z \in H^1(\Omega)$. And we define $G_p(\cdot, \cdot)$ by replacing $h(t, \cdot)$ by $h_p(t, \cdot)$ in (5.4).

By the same argument in [27, Lemma 5.1], we can get the following lemmas.

**Lemma 5.1.** (cf. [27, Lemma 5.1]) Put for any $r > 0$ and $t \in R_+$

$$a_r(t) = b_r(t) := k_3 \int_0^t \{ |\sigma_{0,p}|_{L^\infty(\Omega)} + |\sigma_{0,p}'|_{W^{1,q}(\Omega)} + |\sigma_{1,p}'|_{L^\infty(\Omega)} + |\sigma_{1,p}'|_{W^{1,q}(\Omega)} \} \, dt$$

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where \(k_3\) is a (sufficiently large) positive constant. Then, \(\{\varphi^t\} \in \Phi(\{a_r\}, \{b_r\})\) and \(\{\varphi^T_p\} \in \Phi_p(\{a_r\}, \{b_r\}; T_0)\).

Moreover we have \(\{G(t, \cdot)\} \in \mathcal{G}(\{\varphi^t\})\) and \(\{G_p(t, \cdot)\} \in \mathcal{G}_p(\{\varphi^T_p\}; T_0)\).

**Lemma 5.2.** The convergence assumptions (A1)-(A3) hold.

**Proof.** We easily see that (A2) and (A3) hold by assumptions (5.1) and (5.2).

Now let us show (A1). For each \(t \in \mathbb{R}_+\) there are \(m \in \mathbb{Z}_+\) and \(\tau \in [0, T_0]\) so that \(t = mT_0 + \tau\).

For each \(z_p \in D(\varphi^T_p) = K_p(t)\), we put  
\[
z := (z_p - \sigma_{0,p}(t)) \frac{\sigma_1(t) - \sigma_0(t)}{\sigma_{1,p}(t) - \sigma_{0,p}(t)} + \sigma_0(t).
\]

Then we easily see that \(z \in D(\varphi^t) = K(t)\). Moreover, by the same argument in [27, Lemma 5.1], we see that

\[
|z - z_p|_{L^2(\Omega)} \leq k_4 I_m \quad \text{and} \quad |\nabla z - \nabla z_p|_{L^2(\Omega)} \leq k_4 I_m (1 + |\nabla z_p|_{L^2(\Omega)})
\]

for some constant \(k_4 > 0\). Hence we have

\[
\varphi^t(z) - \varphi^T_p(z_p) \leq k_5 I_m (1 + \varphi^T_p(z_p)),
\]

for a sufficiently large \(k_5 > 0\).

Conversely, let \(z \in D(\varphi^t) = K(t)\) and we put

\[
z_p := (z - \sigma_0(t)) \frac{\sigma_{1,p}(t) - \sigma_{0,p}(t)}{\sigma_1(t) - \sigma_0(t)} + \sigma_{0,p}(t).
\]

Then, we observe that \(z_p \in D(\varphi^T_p) = K_p(t)\) and

\[
|z_p - z|_{L^2(\Omega)} \leq k_4 I_m \quad \text{and} \quad \varphi^T_p(z_p) - \varphi^t(z) \leq k_5 I_m (1 + \varphi^t(z)).
\]

Therefore by (5.5)-(5.7) we see that the convergence assumption (A1) holds. \(\triangle\)

Clearly, the obstacle problem \((\text{OP})^s_{AP}\) can be reformulated as an evolution equation \((\text{AP})_s\) involving the subdifferential of \(\varphi^t\) given by (5.3) and the multivalued operator \(G(t, \cdot)\) defined by (5.4). Also, the limiting \(T_0\)-periodic problem \((\text{OP})^P_s\) can be reformulated as an evolution equation \((P)_s\). Therefore, by Lemmas 5.1-5.2 we can apply abstract results in Section 2-4. Namely, we can obtain an attractor \(\mathcal{A}_s^*\) for \((\text{OP})^s_{AP}\), a \(T_0\)-periodic attractor \(\mathcal{A}_s\) for \((\text{OP})^P_s\) and the relationships between \((\text{OP})^s_{AP}\) and \((\text{OP})^P_s\).

Additionally, we assume that \(f(t) \equiv f_p(t)\) for any \(t \in \mathbb{R}_+\) and

\[
\sigma_0(t, z) \equiv \sigma_{0,p}(t, z), \quad \sigma_{1,p}(t, z) \equiv \sigma_1(t, z), \quad h_p(t, z) \leq h(t, z)
\]
for any $0 \leq t < +\infty$ and $z \in R$. Then we easily see that the assumptions of Theorem 4.4 and its Corollary hold. Hence we can get $A^*_s = A_s$ by the same argument in [30, Theorem 5.4].

Unfortunately we do not give assumptions for $\sigma_i(t, \cdot)$, $h(t, \cdot)$ and $f(t)$ in order to get

$$U(\tau, s)A^*_s = A^*_r \subset A_r$$

for any $0 \leq s \leq \tau < +\infty$. (5.8)

It seems difficult to show (5.8), so it is the open problem.

References


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