

The horospherical geometry of submanifolds in Hyperbolic space

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Abstract

We study some geometrical properties associated to the contact of submanifolds with hyperhorospheres in hyperbolic n -space as an application of the theory of Legendrian singularities.

1 Introduction

In [10] we have studied geometric properties of hypersurfaces in hyperbolic space associated to the contact with hyperhorospheres. We call this geometry the “horospherical geometry” of hypersurfaces in hyperbolic space. The main tool for the study of hypersurfaces is the hyperbolic Gauss mapping which has been originally introduced by Ch. Epstein in [4] for surfaces in the Poincaré ball model. The target of the hyperbolic Gauss map is the boundary sphere of the Poincaré ball in the original definition. In [10] we have studied hypersurfaces in the Minkowski space model of hyperbolic space. In this case the corresponding hyperbolic Gauss map is a mapping from the hypersurface to the spacelike sphere on the light-cone. However, we have defined the hyperbolic Gauss indicatrix on the light-cone whose singular set is the same as that of the hyperbolic Gauss map. We have shown that the hyperbolic Gauss indicatrix is the wave front set of a certain Legendrian submanifold in the projective cotangent bundle of the light-cone. Therefore we have been able to apply the theory of Legendrian singularities to this situation.

In this paper we study the analogous geometric properties of higher codimensional submanifolds in hyperbolic space. Instead of the hyperbolic Gauss indicatrix we introduce the notion of horospherical hypersurfaces of submanifolds which is a generalization of the notion of hyperbolic Gauss indicatrices to the higher codimension case. The singularity of the horospherical hypersurface of a submanifold describes the contact of the submanifold with hyperhorosphere. We show that the horospherical hypersurface of a submanifold is the wave front set of a certain

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Legendrian submanifold of the projective cotangent bundle of the light-cone. Moreover we consider the hyperbolic canal hypersurface of a submanifold which is the boundary of the tubular neighbourhood of the submanifold. Since the hyperbolic canal hypersurface is a hypersurface in hyperbolic space, we can apply the previous theory on the hyperbolic Gauss indicatrix of a hypersurface. We show that the corresponding Legendrian submanifolds for the horospherical hypersurface and the hyperbolic Gauss indicatrix of the hyperbolic canal surface of a submanifold are Legendrian equivalent (cf., Theorem 4.3). As a consequence, we can apply the theory of Legendrian singularities to study the contact of submanifolds with hyperhorospheres. In §2 we prepare some fundamental concepts on hyperbolic space as the Minkowski space model and review the previous results on hypersurfaces in hyperbolic space. We consider general submanifolds in hyperbolic space and study basic properties in §3. The main tools are the horospherical hypersurface and the hyperbolic canal hypersurface of a manifold. We define the horospherical height function (family) on a submanifold and show that the discriminant set is the horospherical hypersurface (cf., Proposition 3.4). Moreover we show that the horospherical hypersurface and the hyperbolic Gauss indicatrix of the hyperbolic canal hypersurface of a submanifold are diffeomorphic (cf., Lemma 3.9). In §4 we show that the horospherical height function of a submanifold is a Morse family (cf., Proposition 4.1). Therefore the horospherical hypersurface of a submanifold is the wave front set of a certain Legendrian submanifold. The main results in §4 is Theorem 4.3. In §5 we study the contact of submanifolds with hyperhorospheres as applications of the previous results and the theory of Legendrian singularities.

We shall assume throughout the whole paper that all the maps and manifolds are C^∞ unless the contrary is explicitly stated.

2 Hypersurfaces in hyperbolic space

In this section we give a brief review on the horospherical differential geometry of hypersurfaces in hyperbolic n -space which was established in [10]. We adopt the model of hyperbolic n -space in Minkowski $(n + 1)$ -space. Let $\mathbb{R}^{n+1} = \{(x_0, x_1, \dots, x_n) \mid x_i \in \mathbb{R} \ (i = 0, 1, \dots, n)\}$ be an $(n + 1)$ -dimensional vector space. For any $\mathbf{x} = (x_0, x_1, \dots, x_n)$, $\mathbf{y} = (y_0, y_1, \dots, y_n) \in \mathbb{R}^{n+1}$, the *pseudo scalar product* of \mathbf{x} and \mathbf{y} is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_0y_0 + \sum_{i=1}^n x_iy_i.$$

We call $(\mathbb{R}^{n+1}, \langle, \rangle)$ *Minkowski $(n + 1)$ -space*. We denote \mathbb{R}_1^{n+1} instead of $(\mathbb{R}^{n+1}, \langle, \rangle)$. We say that a non-zero vector $\mathbf{x} \in \mathbb{R}_1^{n+1}$ is *spacelike*, *lightlike* or *timelike* if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ or $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ respectively. For a vector $\mathbf{v} \in \mathbb{R}_1^{n+1}$ and a real number c , we define the *hyperplane with pseudo normal \mathbf{v}* by

$$HP(\mathbf{v}, c) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{v} \rangle = c\}.$$

We call $HP(\mathbf{v}, c)$ a *spacelike hyperplane*, a *timelike hyperplane* or a *lightlike hyperplane* if \mathbf{v} is timelike, spacelike or lightlike respectively.

We now define *hyperbolic n -space* by

$$H_+^n(-1) = \{\mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1, x_0 \geq 1\}$$

and de Sitter n -space by

$$S_1^n = \{ \mathbf{x} \in \mathbb{R}_1^{n+1} \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1 \}.$$

For any $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n \in \mathbb{R}_1^{n+1}$, we define a vector $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \dots \wedge \mathbf{a}_n$ by

$$\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \dots \wedge \mathbf{a}_n = \begin{vmatrix} -\mathbf{e}_0 & \mathbf{e}_1 & \cdots & \mathbf{e}_n \\ a_0^1 & a_1^1 & \cdots & a_n^1 \\ a_0^2 & a_1^2 & \cdots & a_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ a_0^n & a_1^n & \cdots & a_n^n \end{vmatrix},$$

where $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n$ is the canonical basis of \mathbb{R}_1^{n+1} and $\mathbf{a}_i = (a_0^i, a_1^i, \dots, a_n^i)$. We can easily check that

$$\langle \mathbf{a}, \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \dots \wedge \mathbf{a}_n \rangle = \det(\mathbf{a}, \mathbf{a}_1, \dots, \mathbf{a}_n),$$

so that $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \dots \wedge \mathbf{a}_n$ is pseudo orthogonal to any \mathbf{a}_i ($i = 1, \dots, n$). We also define a set

$$LC_+^* = \{ \mathbf{x} = (x_0, \dots, x_n) \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0, x_0 > 0 \}$$

and we call it the *future lightcone* at the origin. If $\mathbf{x} = (x_0, x_1, \dots, x_n)$ is a non-zero lightlike vector, then $x_0 \neq 0$. Therefore we have

$$\tilde{\mathbf{x}} = \left(1, \frac{x_1}{x_0}, \dots, \frac{x_n}{x_0} \right) \in S_+^{n-1} = \{ \mathbf{x} = (x_0, x_1, \dots, x_n) \mid \langle \mathbf{x}, \mathbf{x} \rangle = 0, x_0 = 1 \}.$$

Here, we call S_+^{n-1} the *spacelike* $(n-1)$ -*sphere*.

Let $\mathbf{x} : U \rightarrow H_+^n(-1)$ be an embedding, where $U \subset \mathbb{R}^{n-1}$ is an open subset. We denote that $M = \mathbf{x}(U)$ and identify M and U through the embedding \mathbf{x} . Since $\langle \mathbf{x}, \mathbf{x} \rangle \equiv -1$, we have $\langle \mathbf{x}_{u_i}, \mathbf{x} \rangle \equiv 0$ ($i = 1, \dots, n-1$), where $u = (u_1, \dots, u_{n-1}) \in U$ and we denote that $\mathbf{x}_{u_i} = (\partial \mathbf{x} / \partial u_i)$. Define a vector

$$\mathbf{e}(u) = \frac{\mathbf{x}(u) \wedge \mathbf{x}_{u_1}(u) \wedge \dots \wedge \mathbf{x}_{u_{n-1}}(u)}{\| \mathbf{x}(u) \wedge \mathbf{x}_{u_1}(u) \wedge \dots \wedge \mathbf{x}_{u_{n-1}}(u) \|},$$

then we have $\langle \mathbf{e}, \mathbf{x}_{u_i} \rangle \equiv \langle \mathbf{e}, \mathbf{x} \rangle \equiv 0$ and $\langle \mathbf{e}, \mathbf{e} \rangle \equiv 1$. Therefore the vector $\mathbf{x} \pm \mathbf{e}$ is lightlike. Since $\mathbf{x}(u) \in H_+^n(-1)$ and $\mathbf{e}(u) \in S_1^n$, we can show that $\mathbf{x}(u) \pm \mathbf{e}(u) \in LC_+^*$. We define a map

$$\mathbb{L}^\pm : U \rightarrow LC_+^*$$

by $\mathbb{L}^\pm(u) = \mathbf{x}(u) \pm \mathbf{e}(u)$ which is called the *hyperbolic Gauss indicatrix* (or the *lightcone dual*) of \mathbf{x} . In [10] we have shown that $D_v \mathbf{e} \in T_p M$ for any $p = \mathbf{x}(u_0) \in M$ and $\mathbf{v} \in T_p M$. Here, D_v denotes the *covariant derivative* with respect to the tangent vector \mathbf{v} . Therefore, we have $D_v \mathbb{L}^\pm \in T_p M$. Under the identification of U and M , the derivative $d\mathbf{x}(u_0)$ can be identified to the identity mapping $id_{T_p M}$ on the tangent space $T_p M$, where $p = \mathbf{x}(u_0)$. This means that $d\mathbb{L}^\pm(u_0) = id_{T_p M} \pm d\mathbf{e}(u_0)$. Thus, $d\mathbb{L}^\pm(u_0)$ can be regarded as a linear transformation on the tangent space $T_p M$. We call the linear transformation $S_p^\pm = -d\mathbb{L}^\pm(u_0) : T_p M \rightarrow T_p M$ the *hyperbolic shape operator* of $M = \mathbf{x}(U)$ at $p = \mathbf{x}(u_0)$. We denote the eigenvalue of S_p^\pm by $\bar{\kappa}_p^\pm$ which is called a *principal hyperbolic curvature* of $\mathbf{x}(U) = M$ at $p = \mathbf{x}(u_0)$. The *hyperbolic Gauss-Kronecker curvature* of $M = \mathbf{x}(U)$ at $p = \mathbf{x}(u_0)$ is defined to be

$$K_h^\pm(u_0) = \det S_p^\pm.$$

Since \mathbf{x}_{u_i} ($i = 1, \dots, n-1$) are spacelike vectors, we induce the Riemannian metric (the *hyperbolic first fundamental form*) $ds^2 = \sum_{i=1}^{n-1} g_{ij} du_i du_j$ on $M = \mathbf{x}(U)$, where $g_{ij}(u) = \langle \mathbf{x}_{u_i}(u), \mathbf{x}_{u_j}(u) \rangle$ for any $u \in U$. We also define the *hyperbolic second fundamental invariant* by $\bar{h}_{ij}^\pm(u) = \langle -\mathbb{L}_{u_i}^\pm(u), \mathbf{x}_{u_j}(u) \rangle$ for any $u \in U$. We have an explicit expression of the hyperbolic Gauss-Kronecker curvature by Riemannian metric and the hyperbolic second fundamental invariant as follows:

$$K_h^\pm = \frac{\det(\bar{h}_{ij}^\pm)}{\det(g_{\alpha\beta})}.$$

We say that a point $u \in U$ or $p = \mathbf{x}(u)$ is an *umbilic point* if $S_p^\pm = \bar{\kappa}^\pm(p) id_{T_p M}$. We also say that M is *totally umbilic* if all points of M are umbilic. A hypersurface given by the intersection of $H_+^n(-1)$ and a hyperplane in \mathbb{R}_1^{n+1} is a totally umbilic hypersurface in hyperbolic n -space. It can be classified as follows: a totally umbilic hypersurface is respectively called a hypersphere, an equidistant hypersurface or a hyperhorosphere if it is given by the intersection of $H_+^n(-1)$ and a spacelike hyperplane, a timelike hyperplane or a lightlike hyperplane. If the timelike hyperplane contains the origin, the equidistant hypersurface is simply called a *hyperplane*. By using the principal hyperbolic curvature, we have classified the totally umbilic hypersurfaces as follows:

Proposition 2.1 *Suppose that $M = \mathbf{x}(U)$ is totally umbilic, then $\bar{\kappa}^\pm(p)$ is constant $\bar{\kappa}^\pm$. Under this condition, we have the following classification:*

- 1) Suppose that $\bar{\kappa}^\pm \neq 0$.
 - a) If $\bar{\kappa}^\pm \neq -1$ and $|\bar{\kappa}^\pm + 1| < 1$, then M is a part of an equidistant hypersurface.
 - b) If $\bar{\kappa}^\pm \neq -1$ and $|\bar{\kappa}^\pm + 1| > 1$, then M is a part of a hypersphere.
 - c) If $\bar{\kappa}^\pm = -1$, then M is a part of a hyperplane.
- 2) If $\bar{\kappa}^\pm = 0$, then M is a part of a hyperhorosphere.

We say that a point $p = \mathbf{x}(u_0)$ is a (*positive or negative*) *horospherical parabolic point* (or, briefly a H^\pm -*parabolic point*) of $\mathbf{x} : U \rightarrow H_+^n(-1)$ if $K_h^\pm(u_0) = 0$. We also say that a point $p = \mathbf{x}(u_0)$ is a hyperhorospherical point if $\bar{h}_{ij}^\pm(u_0) = 0$ for each $i, j = 1, \dots, n-1$.

In [10] we have considered a family of functions on M as a fundamental tool for the study of hyperbolic Gauss indicatrix. We define a family of functions

$$H : U \times LC_+^* \rightarrow \mathbb{R}$$

by $H(u, \mathbf{v}) = \langle \mathbf{x}(u), \mathbf{v} \rangle + 1$. We call H a *horospherical height function* on $\mathbf{x} : U \rightarrow H_+^n(-1)$. We have the following fundamental properties:

Proposition 2.2 ([10]) *Let $H : U \times LC_+^* \rightarrow \mathbb{R}$ be a horospherical height function on $\mathbf{x} : U \rightarrow H_+^n(-1)$. Then we have the following:*

(1) $H(u_0, \mathbf{v}_0) = 0$ if and only if there exist real numbers $\mu, \xi_1, \dots, \xi_{n-1}$ such that $\mathbf{v}_0 = \mathbf{x}(u_0) + \mu \mathbf{e}(u_0) + \xi_1 \mathbf{x}_{u_1}(u_0) + \dots + \xi_{n-1} \mathbf{x}_{u_{n-1}}(u_0)$.

(2) $H(u_0, \mathbf{v}_0) = \partial H / \partial u_i(u_0, \mathbf{v}_0) = 0$ ($i = 1, \dots, n-1$) if and only if $\mathbf{v}_0 = \mathbf{x}(u_0) \pm \mathbf{e}(u_0) = \mathbb{L}^\pm(u_0)$.

Under the condition (2) (i.e., $\mathbf{v}_0^\pm = \mathbb{L}^\pm(u_0)$), we have the following:

(3) $p = \mathbf{x}(u_0)$ is a H^\pm -parabolic point if and only if $\det \text{Hess}(h_{\mathbf{v}_0^\pm})(u_0) = 0$.

(4) $p = \mathbf{x}(u_0)$ is a hyperhorospherical point if and only if $\text{rank Hess}(h_{\mathbf{v}_0^\pm})(u_0) = 0$.

Here, $\text{Hess}(h_{\mathbf{v}_0^\pm})(u_0)$ is the Hessian matrix of the horospherical height function $h_{\mathbf{v}_0^\pm}(u) = H(u, \mathbf{v}_0^\pm)$ at u_0 .

We have also naturally interpreted the hyperbolic Gauss indicatrix of a hypersurface as a wave front set in the framework of contact geometry in [10]. We consider a point $\mathbf{v} = (v_0, v_1, \dots, v_n) \in LC_+^*$, then we have the relation $v_0 = \sqrt{v_1^2 + \dots + v_n^2}$. So we adopt the coordinate system (v_1, \dots, v_n) of the manifold LC_+^* . Here, we consider the projective cotangent bundle

$$\pi : PT^*(LC_+^*) \longrightarrow LC_+^*$$

with the canonical contact structure. We now review geometric properties of this space. Consider the tangent bundle $\tau : TP^*(LC_+^*) \rightarrow PT^*(LC_+^*)$ and the differential map $d\pi : TP^*(LC_+^*) \rightarrow TPT^*(LC_+^*)$ of π . For any $X \in TP^*(LC_+^*)$, there exists an element $\alpha \in T^*(LC_+^*)$ such that $\tau(X) = [\alpha]$. For an element $V \in T_x(LC_+^*)$, the property $\alpha(V) = \mathbf{0}$ does not depend on the choice of representative of the class $[\alpha]$. Thus we can define the canonical contact structure on $PT^*(LC_+^*)$ by

$$K = \{X \in TP^*(LC_+^*) \mid \tau(X)(d\pi(X)) = 0\}.$$

In the coordinate system (v_1, \dots, v_n) , we have the trivialisation $PT^*(LC_+^*) \cong LC_+^* \times P(\mathbb{R}^{n-1})^*$ and we call

$$((v_1, \dots, v_n), [\xi_1 : \dots : \xi_n])$$

homogeneous coordinates, where $[\xi_1 : \dots : \xi_n]$ are homogeneous coordinates of the dual projective space $P(\mathbb{R}^{n-1})^*$. It is easy to show that $X \in K_{(x, [\xi])}$ if and only if $\sum_{i=1}^n \mu_i \xi_i = 0$, where $d\pi(X) = \sum_{i=1}^n \mu_i (\partial/\partial v_i)$. An immersion $i : L \rightarrow PT^*(LC_+^*)$ is said to be a *Legendrian immersion* if $\dim L = n$ and $di_q(T_q L) \subset K_{i(q)}$ for any $q \in L$. We also call the map $\pi \circ i$ the *Legendrian map* and the set $W(i) = \text{image } \pi \circ i$ the *wave front* of i . Moreover, i (or, the image of i) is called the *Legendrian lift* of $W(i)$. For extra results and notions on the theory of Legendrian singularities, please refer to the appendix.

For any hypersurface $\mathbf{x} : U \rightarrow H_+^n(-1)$, we denote $\mathbf{x}(u) = (x_0(u), \dots, x_n(u))$ and $\mathbb{L}^\pm(u) = (\ell_0^\pm(u), \dots, \ell_n^\pm(u))$ as coordinate representations. We now define a smooth mapping

$$\mathcal{L}^\pm : U \longrightarrow PT^*(LC_+^*)$$

by

$$\mathcal{L}^\pm(u) = (\mathbb{L}^\pm(u), [-\ell_1^\pm(u)x_0(u) + \ell_0^\pm(u)x_1(u) : \dots : -\ell_n^\pm(u)x_0(u) + \ell_0^\pm(u)x_n(u)]).$$

Then we have the following [10]:

Proposition 2.3 *For any hypersurface $\mathbf{x} : U \rightarrow H_+^n(-1)$, \mathcal{L}^\pm is a Legendrian immersion whose generating family is the horospherical height function $H : U \times LC_+^* \rightarrow \mathbb{R}$ of \mathbf{x} .*

Therefore, we have the Legendrian immersion \mathcal{L}^\pm whose wave front set is the hyperbolic Gauss indicatrix \mathbb{L}^\pm .

Actually we have shown in [10] that the horospherical height function $H : U \times LC_+^* \rightarrow \mathbb{R}$ is a Morse family.

3 General submanifolds in hyperbolic n -space

Let $\mathbf{x} : U \rightarrow H_+^n(-1)$ be an embedding of codimension $(r+1)$, where $U \subset \mathbb{R}^s$ ($s+r+1=n$) is an open subset. We also write that $M = \mathbf{x}(U)$ and identify M and U through the embedding \mathbf{x} .

For any $p = \mathbf{x}(u) \in M \subset H_+^n(-1)$, we have $\langle \mathbf{x}(u), \mathbf{x}(u) \rangle = -1$, so that $\langle \mathbf{x}_{u_i}(u), \mathbf{x}(u) \rangle = 0$, where $u = (u_1, u_2, \dots, u_s)$. Hence the tangent space of M at $p = \mathbf{x}(u)$ is

$$T_p M = \langle \mathbf{x}_{u_1}(u), \mathbf{x}_{u_2}(u), \dots, \mathbf{x}_{u_s}(u) \rangle_{\mathbb{R}}.$$

Let $N_p(M)$ be the normal space of M at $p = \mathbf{x}(u)$ in \mathbb{R}_1^{n+1} and we define $N_p^h(M) = N_p(M) \cap T_p H_+^n(-1)$. Since the normal bundle $N(M)$ is trivial, we can arbitrary choose a unit normal section $\mathbf{n}(u) \in S^r(N_p^h(M))$. We can consider differential geometry of general submanifolds in hyperbolic space which generalizes geometry of hypersurfaces in hyperbolic space in [10]. Since $\langle \mathbf{n}(u), \mathbf{n}(u) \rangle = 1$ and $\langle \mathbf{x}(u), \mathbf{n}(u) \rangle = 0$, $\mathbf{n}_{u_i}(u)$ ($i = 1, \dots, s$) are orthogonal to both of $\mathbf{n}(u)$ and $\mathbf{x}(u)$. Therefore we have $\mathbf{n}_{u_i}(u) \in T_p M \oplus N_p^h(M)$. We now consider the orthogonal projections $\pi^T : T_p M \oplus N_p^h(M) \longrightarrow T_p M$ and $\pi^N : T_p M \oplus N_p^h(M) \longrightarrow N_p^h(M)$. Let $d\mathbf{n}_u : T_u U \longrightarrow T_p M \oplus N_p^h(M)$ be the derivative of \mathbf{n} . We define that $d\mathbf{n}_u^T = \pi^T \circ d\mathbf{n}_u$ and $d\mathbf{n}_u^N = \pi^N \circ d\mathbf{n}_u$. We call the linear transformation $A_{p_0}(\mathbf{n}) = -d\mathbf{n}_{u_0}^T : T_{p_0} M \longrightarrow T_{p_0} M$ the \mathbf{n} -shape operator of $M = \mathbf{x}(U)$ at $p_0 = \mathbf{x}(u_0)$. Under the identification of U and M , the derivative $d\mathbf{x}_u$ can be identified with the identity mapping $id_{T_p M}$. We also call the linear transformation $S_{p_0}(\mathbf{n}) = -(id_{T_{p_0} M} + d\mathbf{n}_{u_0}^T) : T_{p_0} M \longrightarrow T_{p_0} M$ the horospherical (or, hyperbolic) \mathbf{n} -shape operator of $M = \mathbf{x}(U)$ at $p_0 = \mathbf{x}(u_0)$. We also call the linear mapping $d\mathbf{n}_u^N$ the normal connection with respect to \mathbf{n} of $M = \mathbf{x}(U)$ at $p_0 = \mathbf{x}(u_0)$. We denote the eigenvalue of $A_{p_0}(\mathbf{n})$ by $\kappa_{p_0}(\mathbf{n})$ and the eigenvalue of $S_{p_0}(\mathbf{n})$ by $\bar{\kappa}_{p_0}(\mathbf{n})$. By the relation of $A_{p_0}(\mathbf{n})$ and $S_{p_0}(\mathbf{n})$ we have a relation that $\bar{\kappa}_{p_0}(\mathbf{n}) = \kappa_{p_0}(\mathbf{n}) - 1$. We call $\bar{\kappa}_{p_0}(\mathbf{n})$ the principal horospherical curvature at p_0 with respect to \mathbf{n} . We now define the notion of curvature as follows. The horospherical (or hyperbolic) curvature with respect to \mathbf{n} at $p_0 = \mathbf{x}(u_0)$ is defined to be

$$K_h(\mathbf{n})(u_0) = K_h(\mathbf{n})_{p_0} = \det S_{p_0}(\mathbf{n}).$$

We say that a point $p_0 = \mathbf{x}(u_0)$ is \mathbf{n} -umbilic point if $S_{p_0}(\mathbf{n}) = \bar{\kappa}_{p_0}(\mathbf{n})id_{T_{p_0} M}$. Since the eigenvectors of $S_{p_0}(\mathbf{n})$ and $A_{p_0}(\mathbf{n})$ are the same, the above condition is equivalent to the condition $A_{p_0}(\mathbf{n}) = \kappa_{p_0}(\mathbf{n})id_{T_{p_0} M}$. We say that $M = \mathbf{x}(U)$ is totally \mathbf{n} -umbilic if all points on M are \mathbf{n} -umbilic. We say that the unit normal vector field \mathbf{n} is parallel at p_0 if $d\mathbf{n}_{u_0}^N = 0$. We simply say that \mathbf{n} is parallel if it is parallel at all points of M . Then we have the following result:

Proposition 3.1 *Suppose that $M = \mathbf{x}(U)$ is totally \mathbf{n} -umbilic and \mathbf{n} is a parallel unit normal vector field on M . Then $\bar{\kappa}_p(\mathbf{n})$ is constant $\bar{\kappa}$. Under this condition, we have the following four cases:*

- (1) Suppose that $\bar{\kappa} \neq 0$.
 - (a) If $\bar{\kappa} \neq -1$ and $|\bar{\kappa} + 1| < 1$, then M is contained in a equidistant hypersurface.
 - (b) If $\bar{\kappa} \neq -1$ and $|\bar{\kappa} + 1| > 1$, then M is contained in a hypersphere.
 - (c) If $\bar{\kappa} = -1$, then M is contained in a hyperplane.
- (2) If $\bar{\kappa} = 0$, then M is contained in a hyperhorosphere.

Proof. By definition, we have $-\pi^T \circ \mathbf{n}_{u_i} = \kappa \mathbf{x}_{u_i}$ for $i = 1, \dots, s$. Therefore, we have $-\pi^T \circ \mathbf{n}_{u_i u_j} = \kappa_{u_j} \mathbf{x}_{u_i} + \kappa \mathbf{x}_{u_i u_j}$. Since $\mathbf{n}_{u_i u_j} = \mathbf{n}_{u_j u_i}$ and $\kappa \mathbf{x}_{u_i u_j} = \kappa \mathbf{x}_{u_j u_i}$, we have $\kappa_{u_j} \mathbf{x}_{u_i} = \kappa_{u_i} \mathbf{x}_{u_j}$. By definition $\{\mathbf{x}_{u_1}, \dots, \mathbf{x}_{u_s}\}$ is linearly independent, so that $\bar{\kappa} = \kappa - 1$ is constant.

Since \mathbf{n} is a parallel unit normal vector field along M , we have $\pi^T \circ \mathbf{n}_{u_i} = \mathbf{n}_{u_i}$ for $i = 1, \dots, s$. We now assume that $\bar{\kappa} \neq 0$. If $\bar{\kappa} \neq -1$ then $\kappa \neq 0$, so that we have $-\mathbf{n}_{u_i} = \kappa \mathbf{x}_{u_i}$. Therefore, there exists a constant vector \mathbf{a} such that $\mathbf{x} = \mathbf{a} + (1/\kappa)\mathbf{n}$. Since $\langle \mathbf{x} - \mathbf{a}, \mathbf{x} - \mathbf{a} \rangle = (1/\kappa)^2$, we have $\langle \mathbf{a}, \mathbf{x} \rangle = -1$. This means that $M = \mathbf{x}(U) \subset HP(\mathbf{a}, -1) \cap H_+^n(-1)$. If $|\bar{\kappa} + 1| < 1$, then

\mathbf{a} is spacelike. If $|\bar{\kappa} + 1| > 1$, then \mathbf{a} is timelike. Then the assertion (1) (a), (b) follow. If $\bar{\kappa} = -1$, then $\mathbf{n}_{u_i} = 0$, so that \mathbf{n} is a constant vector \mathbf{a} . Since \mathbf{a} is a spacelike vector and we have $\langle \mathbf{x}, \mathbf{a} \rangle = 0$, $M = \mathbf{x}(U) \subset HP(\mathbf{a}, 0) \cap H_+^n(-1)$. The assertion (1), (c) follows.

Finally we assume that $\bar{\kappa} = 0$. In this case, we have $\mathbf{x} + \mathbf{n}$ is a constant vector \mathbf{a} , so that we have $\langle \mathbf{x}, \mathbf{a} \rangle = -1$. Since \mathbf{a} is a lightlike vector, this means that M is contained in a hyperhorosphere. \square

We now give the following generalized hyperbolic Weingarten formula. Since \mathbf{x}_{u_i} ($i = 1, \dots, s$) are spacelike vectors, we induce the Riemannian metric (the *hyperbolic first fundamental form*) $ds^2 = \sum_{i=1}^s g_{ij} du_i du_j$ on $M = \mathbf{x}(U)$, where $g_{ij}(u) = \langle \mathbf{x}_{u_i}(u), \mathbf{x}_{u_j}(u) \rangle$ for any $u \in U$. We also define the *horospherical (or, hyperbolic) second fundamental invariant with respect to the unit normal vector field \mathbf{n}* by $\bar{h}_{ij}(\mathbf{n})(u) = \langle -(\mathbf{x} + \mathbf{n})_{u_i}(u), \mathbf{x}_{u_j}(u) \rangle$ for any $u \in U$. If we define the *second fundamental invariant with respect to the normal vector field \mathbf{n}* by $h_{ij}(\mathbf{n})(u) = -\langle \mathbf{n}_{u_i}(u), \mathbf{x}_{u_j}(u) \rangle$, then we have the following relation:

$$\bar{h}_{ij}(\mathbf{n})(u) = -g_{ij}(u) + h_{ij}(\mathbf{n})(u), \quad (i, j = 1, \dots, s).$$

Proposition 3.2 *Under the above notations, we have the following horospherical (or, hyperbolic) Weingarten formula with respect to \mathbf{n} :*

$$\pi^T \circ (\mathbf{x} + \mathbf{n})_{u_i} = - \sum_{j=1}^s \bar{h}_i^j(\mathbf{n}) \mathbf{x}_{u_j},$$

where $(\bar{h}_i^j(\mathbf{n})) = (\bar{h}_{ik}(\mathbf{n})) (g^{kj})$ and $(g^{kj}) = (g_{kj})^{-1}$.

Proof. There exist real numbers Γ_i^j such that

$$\pi^T \circ (\mathbf{x} + \mathbf{n})_{u_i} = \sum_{j=1}^s \Gamma_i^j \mathbf{x}_{u_j}.$$

Since $\langle \pi^N \circ (\mathbf{x} + \mathbf{n})_{u_i}, \mathbf{x}_{u_j} \rangle = 0$, we have

$$-\bar{h}_{i\beta}(\mathbf{n}) = \sum_{\alpha=1}^s \Gamma_i^\alpha \langle \mathbf{x}_{u_\alpha}, \mathbf{x}_{u_\beta} \rangle = \sum_{\alpha=1}^s \Gamma_i^\alpha g_{\alpha\beta}.$$

Hence, we have

$$-\bar{h}_i^j(\mathbf{n}) = - \sum_{\beta=1}^s \bar{h}_{i\beta}(\mathbf{n}) g^{\beta j} = \sum_{\beta=1}^s \sum_{\alpha=1}^s \Gamma_i^\alpha g_{\alpha\beta} g^{\beta j} = \Gamma_i^j.$$

This completes the proof. \square

As a corollary of the above proposition, we have an explicit expression of the horospherical curvature by Riemannian metric and the horospherical second fundamental invariant.

Corollary 3.3 *Under the same notations as in the above proposition, the horospherical curvature with respect to \mathbf{n} is given by*

$$K_h(\mathbf{n}) = \frac{\det(\bar{h}_{ij}(\mathbf{n}))}{\det(g_{\alpha\beta})}.$$

Proof. By the horospherical Weingarten formula, the representation matrix of the horospherical shape operator with respect to the basis $\{\mathbf{x}_{u_1}, \dots, \mathbf{x}_{u_s}\}$ is $(\bar{h}_i^j(\mathbf{n})) = (\bar{h}_{i\beta}) (g^{\beta j})$. It follows from this fact that

$$K_h(\mathbf{n}) = \det S_p(\mathbf{n}) = \det (\bar{h}_i^j(\mathbf{n})) = \det (\bar{h}_{i\beta}(\mathbf{n})) (g^{\beta j}) = \frac{\det (\bar{h}_{ij}(\mathbf{n}))}{\det (g_{\alpha\beta})}.$$

□

Since $\langle -(\mathbf{x} + \mathbf{n})(u), \mathbf{x}_{u_j}(u) \rangle = 0$, we have $\bar{h}_{ij}(\mathbf{n})(u) = \langle \mathbf{x}(u) + \mathbf{n}(u), \mathbf{x}_{u_i u_j}(u) \rangle$. Therefore the horospherical second fundamental invariant at a point $p_0 = \mathbf{x}(u_0)$ depends only on $\mathbf{x}(u_0) + \mathbf{n}(u_0)$ and $\mathbf{x}_{u_i u_j}(u_0)$. By the above corollary, the horospherical curvature also depends only on $\mathbf{x}(u_0) + \mathbf{n}(u_0)$ and $\mathbf{x}_{u_i u_j}(u_0)$. It is independent on the choice of the normal vector field \mathbf{n} . We write $K_h(\mathbf{n}_0)(u_0)$ as the horospherical curvature at $p_0 = \mathbf{x}(u_0)$ with respect to $\mathbf{n}_0 = \mathbf{n}(u_0)$. We might also say that a point $p_0 = \mathbf{x}(u_0)$ is \mathbf{n}_0 -umbilic because the horospherical \mathbf{n} -shape operator is independent on the choice of the normal vector filed \mathbf{n} (it depends on the normal vector $\mathbf{n}_0 = \mathbf{n}(u_0)$).

We say that a point $p_0 = \mathbf{x}(u_0)$ is a *horospherical parabolic point with respect to \mathbf{n}_0* (or, briefly a $H(\mathbf{n}_0)$ -parabolic point) of $\mathbf{x} : U \longrightarrow H_+^n(-1)$ if $K_h(\mathbf{n}_0)(u_0) = 0$. We also say that a point $p_0 = \mathbf{x}(u_0)$ is a *horospherical point with respect to \mathbf{n}_0* (or, briefly an $\mathbf{e}(u_0, \mu_0)$ -horospherical point) if it is an \mathbf{n}_0 -umbilic point and $K_h(\mathbf{n}_0)(u_0) = 0$.

We now arbitrary choose unit orthonormal sections $\mathbf{n}_j(u) \in S^r(N_p^h(M))$ ($j = 1, \dots, r+1$). Therefore we have

$$N_p(M) = \langle \mathbf{x}(u), \mathbf{n}_1(u), \dots, \mathbf{n}_{r+1}(u) \rangle_{\mathbb{R}}.$$

Since $\{\mathbf{x}, \mathbf{x}_{u_1}, \dots, \mathbf{x}_{u_s}, \mathbf{n}_1, \dots, \mathbf{n}_{r+1}\}$ is a pseudo orthonormal frame of $T\mathbb{R}_1^{n+1}$ along M , we have $(\mathbf{n}_j)_{u_i} = \sum_{i=1}^s \lambda_i \mathbf{x}_{u_i} + \sum_{k=1}^{r+1} \mu_k \mathbf{n}_k + \mu_{r+2} \mathbf{x}$ for some $\lambda_i, \mu_k \in \mathbb{R}$, $i = 1, 2, \dots, s$, $j = 1, \dots, r+2$, where we denote $(\mathbf{n}_j)_{u_i} = (\partial \mathbf{n}_j / \partial u_i)(u)$. It follows from the fact $\langle \mathbf{n}_j, \mathbf{n}_j \rangle = 1$ that $\langle (\mathbf{n}_j)_{u_i}, \mathbf{n}_j \rangle = 0$. Thus we have $\mu_j = 0$. By the relation $\langle \mathbf{x}, \mathbf{n}_j \rangle = 0$, we have $\langle \mathbf{x}, (\mathbf{n}_j)_{u_i} \rangle = -\langle \mathbf{x}_{u_i}, \mathbf{n}_j \rangle = 0$. Hence $\mu_{r+2} = 0$. Therefore, we have a relation

$$(\mathbf{n}_j)_{u_i}(u) \in \langle \mathbf{x}_{u_1}(u), \dots, \mathbf{x}_{u_s}(u), \mathbf{n}_1(u), \dots, \mathbf{n}_{j-1}(u), \mathbf{n}_{j+1}(u), \dots, \mathbf{n}_{r+1}(u) \rangle_{\mathbb{R}},$$

The boundary of the tubular neighbourhood of M with sufficiently small radius is called the *hyperbolic canal hypersurface* of M . In general it is the image of an embedding from the unit normal bundle of M in $H_+^n(-1)$. Since we consider the local parameterization $\mathbf{x} : U \longrightarrow H_+^n(-1)$, we can explicitly write the embedding as follows: We define a mapping $\bar{\mathbf{x}} : U \times S^r \longrightarrow H_+^n(-1)$ by

$$\bar{\mathbf{x}}(u, \mu) = \cosh \theta \mathbf{x}(u) + \sinh \theta \sum_{j=1}^{r+1} \mu_j \mathbf{n}_j(u),$$

where $\mu = (\mu_1, \dots, \mu_{r+1}) \in \mathbb{R}^{r+1}$ with $\sum_{j=1}^{r+1} \mu_j^2 = 1$, $u = (u_1, \dots, u_s)$ and θ is a fixed real number. For sufficiently small $|\theta| > 0$ we can show that $\bar{\mathbf{x}}$ is an embedding. We write $CM = \bar{\mathbf{x}}(U \times S^r)$ and call it the *hyperbolic canal hypersurface of $M = \mathbf{x}(U)$* . Throughout the remainder in this paper we write that $\mathbf{e}(u, \mu) = \sum_{j=1}^{r+1} \mu_j \mathbf{n}_j(u)$ by using the fixed pseudo-orthonormal frame $\{\mathbf{x}(u), \mathbf{n}_1(u), \dots, \mathbf{n}_{r+1}(u)\}$ of $N_p(M)$ and $\mu = (\mu_1, \dots, \mu_{r+1})$ with $\sum_{j=1}^{r+1} \mu_j^2 = 1$. We now

consider the horospherical height function on a general submanifold in $H_+^n(-1)$ as follows: For any embedding $\mathbf{x} : U \longrightarrow H_+^n(-1)$ ($U \subset \mathbb{R}^s$), define a function

$$H : U \times LC_+^* \longrightarrow \mathbb{R}$$

by $H(u, \mathbf{v}) = \langle \mathbf{x}(u), \mathbf{v} \rangle + 1$, where $\mathbf{v} = (v_0, v_1, \dots, v_n) \in LC_+^*$. We call H a *horospherical height function* on M . We denote that $h_{\mathbf{v}_0}(u) = H(\mathbf{x}(u), \mathbf{v}_0)$, for any $\mathbf{v}_0 \in LC_+^*$, then the following proposition holds:

Proposition 3.4 *We have the following assertions:*

(1) $h_{\mathbf{v}}(u) = 0$ if and only if there exist real numbers $\lambda_1, \dots, \lambda_s, \mu_1, \dots, \mu_{r+1}$ such that $\mathbf{v} = \mathbf{x} + \sum_{i=1}^s \lambda_i \mathbf{x}_i + \sum_{j=1}^{r+1} \mu_j \mathbf{n}_j$.

(2) $h_{\mathbf{v}}(u) = (\partial h_{\mathbf{v}} / \partial u_i)(u) = 0$ ($i = 1, \dots, s$) if and only if $\mathbf{v} = \mathbf{x}(u) + \mathbf{e}(u, \mu)$.

Proof. (1) Since $\{\mathbf{x}, \mathbf{x}_{u_1}, \dots, \mathbf{x}_{u_s}, \mathbf{n}_1, \dots, \mathbf{n}_{r+1}\}$ is a basis of the vector space $T_p \mathbb{R}_1^{n+1}$ where $p = \mathbf{x}(u)$, there exist real numbers $\lambda, \lambda_1, \dots, \lambda_s, \mu_1, \dots, \mu_{r+1}$ such that $\mathbf{v} = \lambda \mathbf{x} + \sum_{i=1}^s \lambda_i \mathbf{x}_i + \sum_{j=1}^{r+1} \mu_j \mathbf{n}_j$.

Therefore $H(u, \mathbf{v}) = 0$ if and only if $-1 = \langle \mathbf{x}, \mathbf{v} \rangle = \lambda \langle \mathbf{x}, \mathbf{x} \rangle = -\lambda$.

(2) Since $(\partial H / \partial u_i)(u, \mathbf{v}) = \langle \mathbf{x}_{u_i}, \mathbf{v} \rangle$, we have $0 = \langle \mathbf{x}_{u_i}, \mathbf{v} \rangle = \lambda_i \langle \mathbf{x}_{u_i}, \mathbf{x}_{u_i} \rangle$. Since $\mathbf{v} \in LC_+^*$, the condition $H(u, \mathbf{v}) = (\partial H / \partial u_i)(u, \mathbf{v}) = 0$ holds if and only if $\mathbf{v} = \mathbf{x} + \sum_{j=1}^{r+1} \mu_j \mathbf{n}_j$ with $\sum_{j=1}^{r+1} \mu_j^2 = 1$. This completes the proof. \square

It follows that

$$\Sigma_*(H) = \left\{ (u, \mathbf{v}) \in U \times LC_+^* \mid \mathbf{v} = \mathbf{x}(u) + \mathbf{e}(u, \mu), \mu = (\mu_1, \dots, \mu_{r+1}) \text{ with } \sum_{j=1}^{r+1} \mu_j^2 = 1 \right\}$$

The set $\Sigma_*(H)$ is defined in the appendix and the discriminant set of H is

$$\mathcal{D}_H = \{ \mathbf{x}(u) + \mathbf{e}(u, \mu) \mid (u, \mu) \in U \times S^r \}.$$

We define a mapping

$$HS_{\mathbf{x}} : U \times S^r \longrightarrow LC_+^*$$

by $HS_{\mathbf{x}}(u, \mu) = \mathbf{x}(u) + \mathbf{e}(u, \mu)$. We call $HS_{\mathbf{x}}$ the *horospherical hypersurface* of M . Of course, $HS_{\mathbf{x}}$ depends on the choice of the pseudo-orthonormal frame $\{\mathbf{x}, \mathbf{n}_1, \dots, \mathbf{n}_{r+1}\}$ of $N(M)$. Let $\{\mathbf{x}, \mathbf{n}'_1, \dots, \mathbf{n}'_{r+1}\}$ be another pseudo-orthonormal frame of $N(M)$. Then we have $\mathbf{n}_i = \sum_{j=1}^{r+1} \lambda_{ij} \mathbf{n}'_j$, where $\lambda_{ij} = \langle \mathbf{n}_i(u), \mathbf{n}'_j(u) \rangle$. We now define a diffeomorphism $\Phi : U \times S^r \longrightarrow U \times S^r$ by

$$\Phi(u, \mu) = \left(u, \left(\sum_{j=1}^{r+1} \lambda_j^1(u) \mu_j, \dots, \sum_{j=1}^{r+1} \lambda_j^{r+1}(u) \mu_j \right) \right),$$

where $\mu = (\mu_1, \dots, \mu_{r+1})$. We also define that $\mathbf{e}'(u, \mu) = \sum_{i=1}^{r+1} \mu_i \mathbf{n}'_i(u)$. It follows from the above definition that $\mathbf{e}(u, \mu) = \mathbf{e}' \circ \Phi(u, \mu)$. Therefore we have

$$HS_{\mathbf{x}}(u, \mu) = HS'_{\mathbf{x}} \circ \Phi(u, \mu),$$

where $HS'_{\mathbf{x}}(u, \mu) = \mathbf{x}(u) + \mathbf{e}'(u, \mu)$. This means that $HS'_{\mathbf{x}}$ defines the same hypersurface as $HS_{\mathbf{x}}(U \times S^r)$ with a different parameterization.

Since we are interested in the singularity of $HS_{\mathbf{x}}(U \times S^r)$, we arbitrary fix a pseudo-orthonormal frame

$$\{\mathbf{x}(u), \mathbf{n}_1(u), \dots, \mathbf{n}_{r+1}(u)\}$$

of $N(M)$ throughout the remainder of this paper. We can show the following assertion:

Proposition 3.5 *Let $\mathbf{x} : U \longrightarrow H_+^n(-1)$ be a submanifold. Then there exists a smooth mapping $\mu : U \longrightarrow S^r$ such that $HS_{\mathbf{x}}(u, \mu(u))$ is a constant vector if and only if $M = \mathbf{x}(U)$ is contained in a hyperhorosphere. By Proposition 3.1, the above condition is equivalent to the condition that M is totally $\mathbf{e}(u, \mu(u))$ -umbilic, the normal vector field $\mathbf{e}(u, \mu(u))$ is parallel and $K_h(\mathbf{e}(u, \mu(u)))(u) = 0$.*

Proof. Suppose that $\mathbf{v}_0 = \mathbf{x}(u) + \mathbf{e}(u, \mu(u))$ is a constant lightlike vector. Since $\mathbf{e}(u, \mu(u))$ is a normal vector of M for any $u \in U$, we have $\langle \mathbf{v}_0, \mathbf{x}(u) \rangle = -1$ for any $u \in U$. This means that $M \subset HS^{n-1}(\mathbf{v}_0, -1)$. On the other hand if $M \subset HS^{n-1}(\mathbf{v}_0, -1)$ for some lightlike vector \mathbf{v}_0 , then $\langle \mathbf{v}_0, \mathbf{x}(u) \rangle = -1$ for any $u \in U$. It follows that we have $\langle \mathbf{v}_0 - \mathbf{x}(u), \mathbf{x}(u) \rangle = 0$ for any $u \in U$. Moreover, we have $\langle \mathbf{v}_0, \mathbf{x}_{u_i} \rangle = 0$. Therefore, $\mathbf{v}_0 - \mathbf{x}(u)$ is a normal vector of M . We define a smooth mapping $\mu : U \longrightarrow S^r$ by $\mu(u) = \sum_{i=1}^{r+1} \langle \mathbf{v}_0 - \mathbf{x}(u), \mathbf{n}_i(u) \rangle \mathbf{n}_i(u)$. Then we have $\mathbf{v}_0 - \mathbf{x}(u) = \mathbf{e}(u, \mu(u))$. This completes the proof. \square

Since the image of the horospherical hypersurface $HS_{\mathbf{x}}$ of M is the discriminant set of the horospherical height function H on M , the singular set of $HS_{\mathbf{x}}$ corresponds to the nondegenerate set of the Hessian matrix of the horospherical height function. Therefore we have the following proposition.

Proposition 3.6 *The singular set of $HS_{\mathbf{x}}$ is given by*

$$\Sigma(HS_{\mathbf{x}}) = \{(u, \mu) \in U \times S^r \mid K_h(\mathbf{e}(u, \mu))(u) = 0\}.$$

Proof. By a straight forward calculation, the Hessian matrix of the horospherical height function $h_{\mathbf{v}}$ at $p = \mathbf{x}(u)$ is given by $(\langle \mathbf{x}_{u_i u_j}(u), \mathbf{v} \rangle)$, where \mathbf{v} is a unit normal vector of M at p . Since $(u, \mathbf{v}) \in \Sigma_*(H)$, we have $\mathbf{v} = \mathbf{x}(u) + \mathbf{e}(u, \mu)$ for some $\mu \in S^r$. By the remark after Corollary 3.3, $K_h(\mathbf{e}(u, \mu))(u) = \det(\langle \mathbf{x}_{u_i u_j}(u), \mathbf{x}(u) + \mathbf{e}(u, \mu) \rangle)$. This completes the proof. \square

By the proof of the above proposition, we have the following proposition.

Proposition 3.7 *For any submanifold $\mathbf{x} : U \longrightarrow H_+^n(-1)$ and a lightlike vector $\mathbf{v}_0 = \mathbf{x}(u_0) + \mathbf{e}(u_0, \mu_0)$, we have the following assertions:*

- (1) $p_0 = \mathbf{x}(u_0)$ is an $H(\mathbf{e}(u_0, \mu_0))$ -parabolic point if and only if $\det \text{Hess}(h_{\mathbf{v}_0})(u_0) = 0$,
- (2) $p_0 = \mathbf{x}(u_0)$ is an $\mathbf{e}(u_0, \mu_0)$ -horospherical point if and only if $\text{Hess}(h_{\mathbf{v}_0})(u_0) = 0$.

Here $\text{Hess}(h_{\mathbf{v}_0})(u_0) = 0$ is the Hessian matrix of $h_{\mathbf{v}_0}(u)$ at u_0 .

On the other hand we now consider the hyperbolic canal hypersurface of M . Since CM is a hypersurface in $H_+^n(-1)$, we can apply previous calculations on the horospherical height function of a hypersurface (cf., §2). We consider the horospherical height function on the hyperbolic canal hypersurface CM :

$$\bar{H} : CM \times LC_+^* \longrightarrow \mathbb{R}; \quad \bar{H}((u, \mu), \mathbf{v}) = \langle \bar{\mathbf{x}}((u, \mu), \mathbf{v}) \rangle + 1.$$

Denote that $\bar{h}_v(u) = \bar{H}((u, \mu), \mathbf{v})$ for any $\mathbf{v} \in LC_+^*$ and $\mathbf{N}(u, \mu) = \sinh \theta \mathbf{x}(u) + \cosh \theta \mathbf{e}(u, \mu)$, then we have the following proposition.

Proposition 3.8 *We have the following assertions:*

- (1) $\bar{h}_v(u, \mu) = 0$ if and only if there exist real numbers $\lambda_1, \dots, \lambda_{n-1}, \rho$ such that $\mathbf{v} = \bar{\mathbf{x}} + \sum_{i=1}^{n-r-1} \lambda_i \bar{\mathbf{x}}_{u_i}(u, \mu) + \sum_{j=1}^r \lambda_{n-r-1+j} \bar{\mathbf{x}}_{\mu_j}(u, \mu) + \rho \mathbf{N}(u, \mu)$ and $\sum_{i=1}^{r+1} \lambda_i^2 + \rho^2 = 1$.
- (2) $\bar{h}_v(u, \mu) = (\partial \bar{h}_v / \partial u_i)(u, \mu) = (\partial \bar{h}_v / \partial \mu_j)(u, \mu_j) = 0$ ($i = 1, \dots, n-r-1$; $j = 1, \dots, r$) if and only if $\mathbf{v} = \bar{\mathbf{x}}(u, \mu) \pm \mathbf{N}(u, \mu) = (\sinh \theta \pm \cosh \theta)(\mathbf{x}(u) \pm \mathbf{e}(u, \mu))$.

Proof. Since $\bar{\mathbf{x}}(u, \mu) = \cosh \theta \mathbf{x}(u) + \sinh \theta \mathbf{e}(u, \mu)$, we have

$$\bar{\mathbf{x}}_{u_i}(u, \mu) = \cosh \theta \mathbf{x}_{u_i}(u) + \sinh \theta \mathbf{e}_{u_i}(u, \mu) \quad (i = 1, \dots, s),$$

$$\bar{\mathbf{x}}_{\mu_j}(u, \mu) = \sinh \theta \mathbf{e}_{\mu_j}(u, \mu) \quad (j = 1, \dots, r),$$

where $\mathbf{e}_{\mu_j} = \partial \mathbf{e} / \partial \mu_j$. Without the loss of generality we consider the case that $\mu_{r+1} > 0$, so that $\mu_{r+1} = \sqrt{1 - \sum_{j=1}^r \mu_j^2}$. Since $\mathbf{e} = \sum_{i=1}^{r+1} \mu_i \mathbf{n}_i$, we have $\mathbf{e}_{\mu_j} = \mathbf{n}_j - \frac{\mu_j}{\mu_{r+1}} \mathbf{n}_{r+1}$. By definition, $\langle \mathbf{N}, \bar{\mathbf{x}} \rangle = 0$. It follows that

$$\langle \mathbf{N}, \bar{\mathbf{x}}_{u_i} \rangle = \langle \sinh \theta \mathbf{x} + \cosh \theta \mathbf{e}, \cosh \theta \mathbf{x}_{u_i} + \sinh \theta \mathbf{e}_{u_i} \rangle = \langle \cosh \theta \mathbf{e}, \cosh \theta \mathbf{x}_{u_i} \rangle = 0.$$

On the other hand, we have

$$\begin{aligned} \langle \mathbf{N}, \bar{\mathbf{x}}_{\mu_j} \rangle &= \langle \sinh \theta \mathbf{x} + \cosh \theta \mathbf{e}, \sinh \theta \frac{\partial \mathbf{e}}{\partial \mu_j} \rangle \\ &= \langle \sinh \theta \mathbf{x} + \cosh \theta \sum_{j=1}^{r+1} \mu_j \mathbf{n}_j, \sinh \theta (\mathbf{n}_j - \frac{\mu_j}{\mu_{r+1}} \mathbf{n}_{r+1}) \rangle \\ &= \cosh \theta \sinh \theta \langle \sum_{j=1}^r \mu_j \mathbf{n}_j + \mu_{r+1} \mathbf{n}_{r+1}, \mathbf{n}_j - \frac{\mu_j}{\mu_{r+1}} \mathbf{n}_{r+1} \rangle \\ &= \cosh \theta \sinh \theta (\mu_j - \mu_j) = 0. \end{aligned}$$

This means that $\mathbf{N}(u, \mu) = \sinh \theta \mathbf{x}(u) + \cosh \theta \mathbf{e}(u, \mu)$ is a unit normal vector of CM at $\bar{\mathbf{x}}(u, \mu) = \bar{p}$. Therefore

$$\{\bar{\mathbf{x}}, \bar{\mathbf{x}}_{u_1}, \dots, \bar{\mathbf{x}}_{u_s}, \bar{\mathbf{x}}_{\mu_1}, \dots, \bar{\mathbf{x}}_{\mu_r}, \mathbf{N}\}$$

is a basis of $T_{\bar{p}} \mathbb{R}_1^{n+1}$. Hence there exist real numbers λ, λ_i, ρ ($i = 1, \dots, n-1$) such that

$$\mathbf{v} = (\lambda \bar{\mathbf{x}} + \sum_{i=1}^s \lambda_i \bar{\mathbf{x}}_{u_i} + \sum_{j=1}^r \lambda_{s+j} \bar{\mathbf{x}}_{\mu_j} + \rho \mathbf{N})(u, \mu).$$

(1) The condition $\bar{h}_v(u, \mu) = 0$ is equivalent to the condition that $-1 = \langle \mathbf{x}, \mathbf{v} \rangle = \lambda \langle \bar{\mathbf{x}}, \bar{\mathbf{x}} \rangle = -\lambda$. Hence, we have the assertion (1).

(2) The condition $\bar{h}_v(u, \mu) = (\partial \bar{h}_v / \partial u_i)(u, \mu) = (\partial \bar{h}_v / \partial \mu_j)(u, \mu_j) = 0$ ($i = 1, \dots, s$; $j = 1, \dots, r$) is equivalent to the conditions that $\lambda = 1$, $\langle \bar{\mathbf{x}}_{u_i}(u, \mu), \mathbf{v} \rangle = \langle \bar{\mathbf{x}}_{\mu_j}(u, \mu), \mathbf{v} \rangle = 0$ and $\mathbf{v} \in LC_+^*$. Therefore we have the condition that $\lambda = 1$, $\lambda_i = 0$, $\lambda = \pm \rho$ and $\mathbf{v} = (\bar{\mathbf{x}} \pm \mathbf{N})(u, \mu) \in$

LC_+^* By definition, we have $(\bar{\mathbf{x}} \pm \mathbf{N})(u, \mu) = (\sinh \theta \pm \cosh \theta)(\mathbf{x} \pm \mathbf{e})(u, \mu)$. This completes the proof of the assertion (2). \square

Since the unit normal vector field of the hyperbolic canal hypersurface CM is given by \mathbf{N} , the hyperbolic Gauss indicatrix of the hyperbolic canal hypersurface CM is a mapping

$$\mathbb{L}_{CM} : U \times S^r \longrightarrow LC_+^*$$

given by

$$\mathbb{L}_{CM}(u, \mu) = \bar{\mathbf{x}}(u, \mu) + \mathbf{N}(u, \mu) = (\sinh \theta + \cosh \theta)(\mathbf{x}(u) + \mathbf{e}(u, \mu)).$$

We now define a diffeomorphism

$$\mathcal{M}_c : LC_+^* \longrightarrow LC_+^*$$

by $\mathcal{M}_c(\mathbf{v}) = c\mathbf{v}$ for a fixed positive real number c . Then we have the following lemma:

Lemma 3.9 *Under the above notations, we have*

$$\mathcal{M}_c \circ HS\mathbf{x}(u, \mu) = \mathbb{L}_{CM}(u, \mu),$$

where $c = \cosh \theta + \sinh \theta$.

By Lemma 3.3, the horospherical hypersurface of M is diffeomorphic to the hyperbolic indicatrix of the hyperbolic canal surface CM of M .

4 Horospherical hypersurfaces as wave fronts

In this section we naturally interpret the horospherical hypersurfaces of M in the future light-cone LC_+^* as a wavefront set in the framework of contact geometry. We also refer to the appendix for basic notions and results on the theory of Legendrian singularities. For an s -dimensional embedding $\mathbf{x} : U \longrightarrow H_+^n(-1)$, we have defined the horospherical height function H in §3 and shown that the discriminant set is the horospherical hypersurface of $\mathbf{x}(U) = M$ in LC_+^* . Moreover, we have the following proposition:

Proposition 4.1 *The horospherical height function $H : U \times LC_+^* \longrightarrow \mathbb{R}$ is a Morse family.*

Proof. For any $\mathbf{v} = (v_0, v_1, \dots, v_n) \in LC_+^*$, we have $v_0 = \sqrt{v_1^2 + \dots + v_n^2}$, so that

$$H(u, \mathbf{v}) = \langle \mathbf{x}(u), \mathbf{v} \rangle + 1 = -x_0(u)\sqrt{v_1^2 + \dots + v_n^2} + x_1(u)v_1 + \dots + x_n(u)v_n,$$

where $\mathbf{x}(u) = (x_0(u), \dots, x_n(u))$. We now prove that the mapping

$$\Delta^* H = \left(H, \frac{\partial H}{\partial u_1}, \dots, \frac{\partial H}{\partial u_s} \right)$$

is non-singular at any point. The Jacobian matrix of $\Delta^* H$ is given as follows:

$$\begin{pmatrix} \langle \mathbf{x}_{u_1}, \mathbf{v} \rangle & \cdots & \langle \mathbf{x}_{u_s}, \mathbf{v} \rangle & -x_0 \frac{v_1}{v_0} + x_1 & \cdots & -x_0 \frac{v_n}{v_0} + x_n \\ \langle \mathbf{x}_{u_1 u_1}, \mathbf{v} \rangle & \cdots & \langle \mathbf{x}_{u_1 u_s}, \mathbf{v} \rangle & -x_{0u_1} \frac{v_1}{v_0} + x_{1u_1} & \cdots & -x_{0u_1} \frac{v_n}{v_0} + x_{nu_1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \langle \mathbf{x}_{u_s u_1}, \mathbf{v} \rangle & \cdots & \langle \mathbf{x}_{u_s u_s}, \mathbf{v} \rangle & -x_{0u_s} \frac{v_1}{v_0} + x_{1u_s} & \cdots & -x_{0u_s} \frac{v_n}{v_0} + x_{nu_s} \end{pmatrix},$$

where $\mathbf{x}_{u_i} = \partial \mathbf{x} / \partial u_i$ and $\mathbf{x}_{u_j u_k} = \partial^2 \mathbf{x} / \partial u_j \partial u_k(u)$. We now show that the rank of the matrix

$$X = \begin{pmatrix} -x_0 \frac{v_1}{v_0} + x_1 & \cdots & -x_0 \frac{v_n}{v_0} + x_n \\ -x_{0u_1} \frac{v_1}{v_0} + x_{1u_1} & \cdots & -x_{0u_1} \frac{v_n}{v_0} + x_{nu_1} \\ \vdots & \vdots & \vdots \\ -x_{0u_s} \frac{v_1}{v_0} + x_{1u_s} & \cdots & -x_{0u_s} \frac{v_n}{v_0} + x_{nu_s} \end{pmatrix}$$

is $s + 1$ at $(u, \mathbf{v}) \in \Sigma_*(H)$. Since $(u, \mathbf{v}) \in \Sigma_*(H)$, we have $\mathbf{v} = \mathbf{x}(u) + \sum_{j=1}^{r+1} \mu_j \mathbf{n}_j(u)$ with $\sum_{j=1}^{r+1} \mu_j^2 = 1$. Without the loss of generality, we assume that $\mu_{r+1} \neq 0$. We denote that $\mathbf{n}_i(u) = (m_0^i(u), \dots, m_n^i(u))$ for $i = 1, \dots, r+1$. Then it is enough to show that the rank of the matrix

$$A = \begin{pmatrix} -x_0 \frac{v_1}{v_0} + x_1 & \cdots & -x_0 \frac{v_n}{v_0} + x_n \\ -x_{0u_1} \frac{v_1}{v_0} + x_{1u_1} & \cdots & -x_{0u_1} \frac{v_n}{v_0} + x_{nu_1} \\ \vdots & \vdots & \vdots \\ -x_{0u_s} \frac{v_1}{v_0} + x_{1u_s} & \cdots & -x_{0u_s} \frac{v_n}{v_0} + x_{nu_s} \\ -m_0^1 \frac{v_1}{v_0} + m_1^1 & \cdots & -m_0^1 \frac{v_n}{v_0} + m_n^1 \\ \vdots & \vdots & \vdots \\ -m_0^r \frac{v_1}{v_0} + m_1^r & \cdots & -m_0^r \frac{v_n}{v_0} + m_n^r \end{pmatrix}$$

is n at $(u, \mathbf{v}) \in \Sigma_*(H)$. We denote that $\mathbf{a}_i = {}^t(x_i, x_{iu_1}, \dots, x_{iu_s}, m_i^1, \dots, m_i^r)$ for $i = 0, \dots, n$.

Then we have

$$A = \left(-\mathbf{a}_0 \frac{v_1}{v_0} + \mathbf{a}_1, \dots, -\mathbf{a}_0 \frac{v_n}{v_0} + \mathbf{a}_n \right)$$

and

$$\det A = \frac{v_0}{v_0} \det(\mathbf{a}_1, \dots, \mathbf{a}_n) - \frac{v_1}{v_0} \det(\mathbf{a}_0, \mathbf{a}_2, \dots, \mathbf{a}_n) - \cdots - \frac{v_n}{v_0} \det(\mathbf{a}_1, \dots, \mathbf{a}_{n-1}, \mathbf{a}_0).$$

On the other hand, we have

$$\begin{aligned} \mathbf{a} &= \mathbf{x} \wedge \mathbf{x}_1 \wedge \cdots \wedge \mathbf{x}_s \wedge \mathbf{n}_1 \wedge \cdots \wedge \mathbf{n}_r \\ &= \left(-\det(\mathbf{a}_1, \dots, \mathbf{a}_n), -\det(\mathbf{a}_0, \mathbf{a}_2, \dots, \mathbf{a}_n), \dots, (-1)^n \det(\mathbf{a}_0, \dots, \mathbf{a}_{n-1}) \right). \end{aligned}$$

Therefore we have

$$\begin{aligned}\det A &= \left\langle \left(\frac{v_0}{v_0}, \dots, \frac{v_n}{v_0} \right), \mathbf{a} \right\rangle = \frac{1}{v_0} \langle \mathbf{x} + \mathbf{e}(u, \mu), \|\mathbf{a}\| \mathbf{n}_{r+1} \rangle \\ &= \frac{1}{v_0} \|\mathbf{a}\| \mu_{r+1} \neq 0\end{aligned}$$

for $(u, \mathbf{v}) \in \Sigma_*(H)$. This complete the proof of proposition. \square

By the above proposition, we can define the Legendrian lift of the horospherical hypersurface as follows: We denote that $\mathbf{x}(u) = (x_0(u), \dots, x_n(u))$ and $HS\mathbf{x}(u, \mu) = (\bar{\ell}_0(u, \mu), \dots, \bar{\ell}_n(u, \mu))$ as coordinate representations. Define a map

$$\mathcal{L}\mathbf{x} : U \times S^r \longrightarrow PT^*(LC_+^*)$$

by

$$\mathcal{L}\mathbf{x}(u, \mu) = (HS\mathbf{x}(u, \mu), [\bar{\ell}(u, \mu)]).$$

where

$$[\bar{\ell}(u, \mu)] = [-\bar{\ell}_1(u, \mu)x_0(u) + \bar{\ell}_0(u, \mu)x_1(u) : \dots : -\bar{\ell}_n(u, \mu)x_0(u) + \bar{\ell}_0(u, \mu)x_n(u)].$$

By definition, we have the following corollary of the above proposition:

Corollary 4.2 *For an s -dimensional embedding $\mathbf{x} : U \longrightarrow H_+^n(-1)$, $\mathcal{L}\mathbf{x} : U \times S^r \longrightarrow PT^*(LC_+^*)$ is a Legendrian immersion such that the horospherical height function $H : U \times LC_+^* \longrightarrow \mathbb{R}$ of $\mathbf{x}(U) = M$ is a generating family of $\mathcal{L}\mathbf{x}$.*

On the other hand, we define a contact diffeomorphism

$$\widetilde{\mathcal{M}}_c : PT^*(LC_+^*) \longrightarrow PT^*(LC_+^*)$$

by $\widetilde{\mathcal{M}}_c(\mathbf{v}, [\xi]) = (c\mathbf{v}, [\xi])$ for a fixed positive number c , which is the unique contact lift of the diffeomorphism $\mathcal{M}_c : LC_+^* \longrightarrow LC_+^*$. By definition we have the following theorem:

Theorem 4.3 *For an s -dimensional embedding $\mathbf{x} : U \longrightarrow H_+^n(-1)$, we have*

$$\widetilde{\mathcal{M}}_c \circ \mathcal{L}\mathbf{x}(u, \mu) = \mathcal{L}_{CM}(u, \mu),$$

where $c = \cosh \theta + \sinh \theta$ and \mathcal{L}_{CM} is the Legendrian lift of the Gaussian indicatrix \mathbb{L}_{CM} of the hyperbolic canal hypersurface CM .

In other words, the Legendrian lift of the Gaussian indicatrix $\mathbb{L}\mathbf{x}$ of the hyperbolic canal hypersurface CM is Legendrian equivalent to the Legendrian lift $\mathcal{L}\mathbf{x}$ of the horospherical hypersurface $HS\mathbf{x}$ of $M = \mathbf{x}(U)$.

Corollary 4.4 *Under the same assumption of the above theorem, the horospherical height function germ H of \mathbf{x} at (u_0, \mathbf{v}_0) and the horospherical height function germ \bar{H} at $(u_0, \mu_0, \bar{\mathbf{v}}_0)$ are stably P - \mathcal{K} -equivalent, where $\mathbf{v}_0 = \mathbf{x}(u_0) + \mathbf{e}(u_0, \mu_0)$ and $\bar{\mathbf{v}}_0 = \bar{\mathbf{x}}(u_0, \mu_0) + \mathbf{N}(u_0, \mu_0)$.*

5 Contact with hyperhorospheres

In this section we consider the contact of submanifolds with hyperhorospheres. Before we start to consider the contact between hypersurfaces and hyperhorospheres, we briefly review the theory of contact due to Montaldi [17]. Let X_i, Y_i ($i = 1, 2$) be submanifolds of \mathbb{R}^n with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. We say that the *contact of X_1 and Y_1 at y_1 is same type as the contact of X_2 and Y_2 at y_2* if there is a diffeomorphism germ $\Phi : (\mathbb{R}^n, y_1) \rightarrow (\mathbb{R}^n, y_2)$ such that $\Phi(X_1) = X_2$ and $\Phi(Y_1) = Y_2$. In this case we write $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$. It is clear that in the definition \mathbb{R}^n could be replaced by any manifold. In his paper [17], Montaldi gives a characterization of the notion of contact by using the terminology of singularity theory.

Theorem 5.1 *Let X_i, Y_i ($i = 1, 2$) be submanifolds of \mathbb{R}^n with $\dim X_1 = \dim X_2$ and $\dim Y_1 = \dim Y_2$. Let $g_i : (X_i, x_i) \rightarrow (\mathbb{R}^n, y_i)$ be immersion germs and $f_i : (\mathbb{R}^n, y_i) \rightarrow (\mathbb{R}^p, 0)$ be submersion germs with $(Y_i, y_i) = (f_i^{-1}(0), y_i)$. Then $K(X_1, Y_1; y_1) = K(X_2, Y_2; y_2)$ if and only if $f_1 \circ g_1$ and $f_2 \circ g_2$ are \mathcal{K} -equivalent.*

Define a function $\mathfrak{H} : H_+^n(-1) \times LC_+^* \rightarrow \mathbb{R}$ by $\mathfrak{H}(\mathbf{v}_1, \mathbf{v}_2) = \langle \mathbf{v}_1, \mathbf{v}_2 \rangle + 1$. For any $\mathbf{v}_0 \in LC_+^*$, we define that $\mathfrak{h}_{\mathbf{v}_0}(\mathbf{u}) = \mathfrak{H}(\mathbf{u}, \mathbf{v}_0)$ and we have a hyperhorosphere $\mathfrak{h}_{\mathbf{v}_0}^{-1}(0) = HP(\mathbf{v}_0, -1) \cap H_+^n(-1) = HS^{n-1}(\mathbf{v}_0, -1)$. Let $\mathbf{x} : U \rightarrow H_+^n(-1)$ be an embedding of codimension $(r+1)$. For any $u_0 \in U$ and $\mu_0 \in S^r$, we consider a lightlike vector $\mathbf{v}_0 = \mathbf{x}(u_0) + \mathbf{e}(u_0, \mu_0) \in LC_+^*$, then we have

$$\mathfrak{h}_{\mathbf{v}_0} \circ \mathbf{x}(u_0) = \mathfrak{H} \circ (\mathbf{x} \times id_{LC_+^*})(u_0, \mathbf{v}_0) = H(u_0, \mathbf{v}_0) = 0$$

by Proposition 3.1, (1). It also follows from Proposition 3.1, (2) that we have

$$\frac{\partial \mathfrak{h}_{\mathbf{v}_0} \circ \mathbf{x}}{\partial u_i}(u_0) = \frac{\partial H}{\partial u_i}(u_0, \mathbf{x}(u_0) + \mathbf{e}(u_0, \mu_0)) = 0,$$

for $i = 1, \dots, n - r - 1$. This means that the hyperhorospheres $\mathfrak{h}_{\mathbf{v}_0}^{-1}(0) = HS^{n-1}(\mathbf{v}_0, -1)$ is tangent to $M = \mathbf{x}(U)$ at $p = \mathbf{x}(u_0)$. In this case, we call $HS^{n-1}(\mathbf{v}_0, -1)$ the *tangent hyperhorosphere* of $M = \mathbf{x}(U)$ at $p_0 = \mathbf{x}(u_0)$ (or, u_0) with respect to $\mathbf{x}(u_0) + \mathbf{e}(u_0, \mu_0)$, which we write $HS(\mathbf{x}, (u_0, \mu_0))$. In the case when $s = n - 1$ (i.e., the hypersurface case), we have exactly two normal direction at each point. Therefore, there are two tangent hyperhorospheres of a hypersurface at each point. For the higher codimensional case, we have the hyperbolic canal hypersurface $\bar{\mathbf{x}} : U \times S^r \rightarrow H_+^n(-1)$ of $\mathbf{x}(U) = M$. We denote $HS(\bar{\mathbf{x}}, (u_0, \mu_0))$ as the tangent hyperhorosphere of $CM = \bar{\mathbf{x}}(U \times S^r)$ at (u_0, μ_0) with respect to $\bar{\mathbf{x}}(u_0) + \mathbf{N}(u_0, \mu_0)$.

We now consider the contact of M with tangent hyperhorospheres at $p_0 \in M$ as an application of Legendrian singularity theory. Let $f_i : (N_i, x_i) \rightarrow (P_i, y_i)$ ($i = 1, 2$) be C^∞ map germs. We say that f_1, f_2 are \mathcal{A} -equivalent if there exist diffeomorphism germs $\phi : (N_1, x_1) \rightarrow (N_2, x_2)$ and $\psi : (P_1, y_1) \rightarrow (P_2, y_2)$ such that $\psi \circ f_1 = f_2 \circ \phi$. Let $HS\mathbf{x}_i : (U \times S^r, (u_i, \mu_i)) \rightarrow (LC_+^*, \mathbf{v}_i)$ ($i = 1, 2$) be horospherical hypersurface germs of submanifold germs $\mathbf{x}_i : (U, u_i) \rightarrow (H_+^n(-1), \mathbf{x}_i(u_i))$. If both the regular sets of $HS\mathbf{x}_i$ are dense in $(U \times S^r, (u_i, \mu_i))$, it follows from Proposition A.2 that $HS\mathbf{x}_1$ and $HS\mathbf{x}_2$ are \mathcal{A} -equivalent if and only if the corresponding Legendrian immersion germs $\mathcal{L}\mathbf{x}_1 : (U \times S^r, (u_1, \mu_1)) \rightarrow (LC_+^*, \mathbf{v}_1)$ and $\mathcal{L}\mathbf{x}_2 : (U \times S^r, (u_2, \mu_2)) \rightarrow (LC_+^*, \mathbf{v}_2)$ are Legendrian equivalent. This condition is also equivalent to the condition that two generating families H_1 and H_2 are P - \mathcal{K} -equivalent by Theorem A.3. Here, $H_i : (U \times LC_+^*, (u_i, \mathbf{v}_i)) \rightarrow \mathbb{R}$ is the horospherical height function germ of \mathbf{x}_i . By Lemma 3.9, the above condition is equivalent to that $\mathbb{L}_{CM_1}, \mathbb{L}_{CM_2}$ are \mathcal{A} -equivalent, where $CM_i = \bar{\mathbf{x}}_i(U \times S^r)$ ($i = 1, 2$) is the hyperbolic canal hypersurfaces of \mathbf{x}_i .

On the other hand, we define that $h_{i,v_i}(u) = H_i(u, \mathbf{v}_i)$, then we have $h_{i,v_i}(u) = \mathfrak{h}_{v_i} \circ \mathbf{x}_i(u)$. By Theorem 5.1, $K(\mathbf{x}_1(U), HS(\mathbf{x}_1, (u_1, \mu_1)), \mathbf{v}_1) = K(\mathbf{x}_2(U), HS(\mathbf{x}_2, (u_2, \mu_2)), \mathbf{v}_2)$ if and only if h_{1,v_1} and h_{1,v_2} are \mathcal{K} -equivalent. Therefore, we can apply the arguments in the appendix to our situation.

Theorem 5.2 *Let $\mathbf{x}_i : (U, u_i) \longrightarrow (H_+^n(-1), \mathbf{x}_i(u_i))$ ($i = 1, 2$) be hypersurface germs such that $\Sigma(HS\mathbf{x}_i)$ ($i = 1, 2$) have no interior points as subspaces of $U \times S^r$. Then we have the following assertions:*

(A) *The following conditions are equivalent:*

- (1) *Horospherical hypersurface germs $HS\mathbf{x}_1$ and $HS\mathbf{x}_2$ are \mathcal{A} -equivalent.*
- (2) *$\mathcal{L}\mathbf{x}_1$ and $\mathcal{L}\mathbf{x}_2$ are Legendrian equivalent.*
- (3) *H_1 and H_2 are $P\text{-}\mathcal{K}$ -equivalent.*
- (4) *Hyperbolic Gauss indicatrix germs \mathbb{L}_{CM_1} and \mathbb{L}_{CM_2} are \mathcal{A} -equivalent.*
- (5) *\mathcal{L}_{CM_1} and \mathcal{L}_{CM_2} are Legendrian equivalent.*
- (6) *\bar{H}_1 and \bar{H}_2 are $P\text{-}\mathcal{K}$ -equivalent.*

(B) *If one of the above conditions hold for \mathbf{x}_i ($i = 1, 2$),*

$$K(\mathbf{x}_1(U), HS(\mathbf{x}_1, (u_1, \mu_1)), \mathbf{v}_1) = K(\mathbf{x}_2(U), HS(\mathbf{x}_2, (u_2, \mu_2)), \mathbf{v}_2).$$

In this case, $(\mathbf{x}_1^{-1}(HS(\mathbf{x}_1, (u_1, \mu_1))), u_1)$ and $(\mathbf{x}_2^{-1}(HS(\mathbf{x}_2, (u_2, \mu_2))), u_2)$ are diffeomorphic as set germs.

Proof. (A) By the assumption, the corresponding Legendrian lifts $\mathcal{L}\mathbf{x}_i$ satisfy the hypothesis of Proposition A.2. It follows from Proposition A.2 and Theorem A.3 that the conditions (1), (2) and (3) are equivalent. By Theorem 4.3, the condition (2) is equivalent to the condition (5). It also follows from Proposition A.2 and Theorem A.3 that the conditions (4), (5) and (6) are equivalent.

(B) Suppose that H_1 and H_2 are $P\text{-}\mathcal{K}$ -equivalent. Then h_{1,v_1} and h_{1,v_2} are \mathcal{K} -equivalent. By Theorem 5.1, we have

$$K(\mathbf{x}_1(U), HS(\mathbf{x}_1, (u_1, \mu_1)), \mathbf{v}_1) = K(\mathbf{x}_2(U), HS(\mathbf{x}_2, (u_2, \mu_2)), \mathbf{v}_2).$$

On the other hand, we have $(\mathbf{x}_i^{-1}(HS(\mathbf{x}_i, (u_i, \mu_i))), u_i) = h_{i,v_i}^{-1}(0)$. It follows that

$$(\mathbf{x}_1^{-1}(HS(\mathbf{x}_1, (u_1, \mu_1))), u_1) \text{ and } (\mathbf{x}_2^{-1}(HS(\mathbf{x}_2, (u_2, \mu_2))), u_2)$$

are diffeomorphic as set germs because the \mathcal{K} -equivalence preserves the zero level sets. \square

For a submanifold germ $\mathbf{x} : (U, u_0) \longrightarrow (H_+^n(-1), \mathbf{x}(u_0))$, we call $(\mathbf{x}^{-1}(HS(\mathbf{x}, (u_0, \mu_0))), u_0)$ the *tangent hyperhorospherical indicatrix germ* of \mathbf{x} with respect to $\mathbf{e}(u_0, \mu_0)$. By Theorem 5.2, the diffeomorphism type of the tangent hyperhorospherical indicatrix germ is an invariant of the \mathcal{A} -classification among the horospherical hypersurface germs for generic submanifolds.

In the case when the corresponding Legendrian immersion $\mathcal{L}\mathbf{x}$ is Legendrian stable, we have more detailed assertions. We now denote $Q(\mathbf{x}, (u_0, \mu_0))$ the local ring of the function germ $h_{v_0} : (U, u_0) \longrightarrow \mathbb{R}$, where $\mathbf{v}_0 = \mathbf{x}(u_0) + \mathbf{e}(u_0, \mu_0)$. We remark that we can explicitly write the local ring as follows:

$$Q(\mathbf{x}, u_0; \mu_0) = \frac{C_{u_0}^\infty(U)}{\langle \langle \mathbf{x}(u), \mathbf{x}(u_0) + \mathbf{e}(u_0, \mu_0) \rangle + 1 \rangle_{C_{u_0}^\infty(U)}}$$

where $C_{u_0}^\infty(U)$ is the local ring of function germs at u_0 with the unique maximal ideal $\mathfrak{M}_{u_0}(U)$. We also denote $Q(\bar{\mathbf{x}}, (u_0, \mu_0))$ as the local ring of the function germ $\bar{h}_{\bar{\mathbf{v}}_0} : (U \times S^r, (u_0, \mu_0)) \rightarrow \mathbb{R}$, where $\bar{\mathbf{v}}_0 = \bar{\mathbf{x}}(u_0, \mu_0) + \mathbf{N}(u_0, \mu_0)$.

Theorem 5.3 *Let $\mathbf{x}_i : (U, u_i) \rightarrow (H_+^n(-1), \mathbf{x}_i(u_i))$ ($i = 1, 2$) be hypersurface germs such that the corresponding Legendrian immersion germs $\mathcal{L}\mathbf{x}_i : (U \times S^r, (u_i, \mu_i)) \rightarrow (PT^*(LC_+^*), z_i)$ are Legendrian stable. Then the following conditions are equivalent:*

- (1) *Horospherical hypersurface germs $HS\mathbf{x}_1$ and $HS\mathbf{x}_2$ are \mathcal{A} -equivalent.*
- (2) *$\mathcal{L}\mathbf{x}_1$ and $\mathcal{L}\mathbf{x}_2$ are Legendrian equivalent.*
- (3) *H_1 and H_2 are P - \mathcal{K} -equivalent.*
- (4) *h_{1,v_1} and h_{1,v_2} are \mathcal{K} -equivalent.*
- (5) *$K(\mathbf{x}_1(U), HS(\mathbf{x}_1, u_1), \mathbf{v}_1) = K(\mathbf{x}_2(U), HS(\mathbf{x}_2, u_2), \mathbf{v}_2)$.*
- (6) *$Q(\mathbf{x}_1, u_1; \mu)$ and $Q(\mathbf{x}_2, u_2; \mu_2)$ are isomorphic as \mathbb{R} -algebras.*
- (7) *Hyperbolic Gauss indicatrix germs \mathbb{L}_{CM_1} and \mathbb{L}_{CM_2} are \mathcal{A} -equivalent.*
- (8) *\mathcal{L}_{CM_1} and \mathcal{L}_{CM_2} are Legendrian equivalent.*
- (9) *\bar{H}_1 and \bar{H}_2 are P - \mathcal{K} -equivalent.*
- (10) *\bar{h}_{1,v_1} and \bar{h}_{1,v_2} are \mathcal{K} -equivalent.*
- (11) *$K(\bar{\mathbf{x}}_1(U \times S^r), HS(\bar{\mathbf{x}}_1, (u_1, \mu_1), \mathbf{v}_1) = K(\bar{\mathbf{x}}_2(U \times S^r), HS(\bar{\mathbf{x}}_2, (u_2, \mu_2), \mathbf{v}_2)$.*
- (12) *$Q(\bar{\mathbf{x}}_1, (u_1, \mu_1))$ and $Q(\bar{\mathbf{x}}_2, (u_2, \mu_2))$ are isomorphic as \mathbb{R} -algebras.*

Proof. We remark that if $\mathcal{L}\mathbf{x}_i$ is Legendrian stable then the singular set $\Sigma(HS\mathbf{x}_i)$ of the corresponding horospherical hypersurface has no interior points as a subspace of $U \times S^r$. By Theorem 5.2, the conditions (1), (2), (3), (7), (8) and (9) are equivalent. It follows from Propositions A.3 and A.4 that the conditions (2),(4) and (6) are equivalent. Since $\mathcal{L}\mathbf{x}_i$ and \mathcal{L}_{CM_i} are Legendrian equivalent, \mathcal{L}_{CM_i} is also Legendrian stable. Therefore the conditions (8), (10) and (12) are also equivalent. By Theorem 5.1, the conditions (4) and (5) (respectively, (10) and (11)) are equivalent. \square

In the next section, we will prove that the assumption of the theorem is generic in the case when $n \leq 6$. For general dimension, we need the topological theory (cf., Proposition A.5).

Theorem 5.4 *Let $\mathbf{x}_i : (U, u_i) \rightarrow (H_+^n(-1), \mathbf{x}_i(u_i))$ ($i = 1, 2$) be submanifold germs such that the map germ given by $\pi_{H_i} : (H_i^{-1}(\mathbf{v}_i), (u_i, \mathbf{v}_i)) \rightarrow (LC_+^*, \mathbf{v}_i)$ at any point $u_i \in U$ is an MT-stable map germ, where H_i is the horospherical height function of \mathbf{x}_i and $\mathbf{v}_i = \mathbf{x}_i(u_i) + \mathbf{e}(u_i, \mu_i)$. If $Q(\mathbf{x}_1, (u_1, \mu_1))$ and $Q(\mathbf{x}_2, (u_2, \mu_2))$ are isomorphic as \mathbb{R} -algebras, then $HS\mathbf{x}_1$ and $HS\mathbf{x}_2$ are stratified equivalent as set germs.*

By the above results, we can borrow some basic invariants from the singularity theory on function germs. We need \mathcal{K} -invariants for function germs. The local ring of a function germ is a complete \mathcal{K} -invariant for generic function germs. It is, however, not a numerical invariant. The \mathcal{K} -codimension (or, Tyurina number) of a function germ is a numerical \mathcal{K} -invariant of function germs [13]. We denote that

$$\text{H-ord}(\mathbf{x}, (u_0, \mu_0)) = \dim \frac{C_{u_0}^\infty(U)}{\langle \langle \mathbf{x}(u), \mathbf{v}_0 \rangle + 1, \langle \mathbf{x}_{u_i}(u), \mathbf{v}_0 \rangle \rangle_{C_{u_0}^\infty}},$$

where $\mathbf{v}_0 = \mathbf{x}(u_0) + \mathbf{e}(u_0, \mu_0)$. Usually $\text{H-ord}(\mathbf{x}, (u_0, \mu_0))$ is called the \mathcal{K} -codimension of $h_{\mathbf{v}_0}$. However, we call it the *order of contact with the tangent hyperhorosphere* at $\mathbf{x}(u_0)$ with respect

to $\mathbf{e}(u_0, \mu_0)$. We also have the notion of corank of function germs.

$$\text{H-corank}(\mathbf{x}, (u_0, \mu_0)) = s - \text{rank Hess}(h_{v_0}(u_0)).$$

By Proposition 3.7, $\mathbf{x}(u_0)$ is a $H(\mathbf{e}(u_0, \mu_0))$ -parabolic point if and only if

$$\text{H-corank}(\mathbf{x}, (u_0, \mu_0)) \geq 1.$$

Moreover $\mathbf{x}(u_0)$ is a $\mathbf{e}(u_0, \mu_0)$ -horospherical point if and only if $\text{H-corank}(\mathbf{x}, (u_0, \mu_0)) = s$.

On the other hand, a function germ $f : (\mathbb{R}^{n-1}, \mathbf{a}) \rightarrow \mathbb{R}$ has the A_k -type singularity if f is \mathcal{K} -equivalent to the germ $\pm u_1^2 \pm \cdots \pm u_{n-2}^2 + u_{n-1}^{k+1}$. If $\text{H-corank}(\mathbf{x}, (u_0, \mu_0)) = 1$, the horospherical height function h_{v_0} has the A_k -type singularity at u_0 in generic. In this case we have $\text{H-ord}(\mathbf{x}, (u_0, \mu_0)) = k$. This number is equal to the order of contact in the classical sense (cf., [2]). This is the reason why we call $\text{H-ord}(\mathbf{x}, (u_0, \mu_0))$ the order of contact with the tangent hyperhorosphere with the polar vector $\mathbf{v}_0 = \mathbf{x}(u_0) + \mathbf{e}(u_0, \mu_0)$ at $\mathbf{x}(u_0)$.

6 Generic properties

In this section we consider generic properties of submanifolds in $H_+^n(-1)$. The main tool is a kind of transversality theorems. We consider the space of embeddings $\text{Emb}(U, H_+^n(-1))$ with Whitney C^∞ -topology for an open subset $U \subset \mathbb{R}^s$. We also consider the function $\mathfrak{H} : H_+^n(-1) \times LC_+^* \rightarrow \mathbb{R}$ which is given in §5. We claim that \mathfrak{H}_u is a submersion for any $\mathbf{u} \in LC_+^*$, where $\mathfrak{H}_u(\mathbf{v}) = \mathfrak{H}(\mathbf{u}, \mathbf{v})$. For any $\mathbf{x} \in \text{Emb}(U, H_+^n(-1))$, we have $H = \mathfrak{H} \circ (\mathbf{x} \times id_{LC_+^*})$. We also have the ℓ -jet extension

$$j_1^\ell H : U \times LC_+^* \rightarrow J^\ell(U, \mathbb{R})$$

defined by $j_1^\ell H(u, \mathbf{v}) = j^\ell h_{\mathbf{v}}(u)$. We consider the trivialisation $J^\ell(U, \mathbb{R}) \equiv U \times \mathbb{R} \times J^\ell(s, 1)$. For any submanifold $Q \subset J^\ell(s, 1)$, we denote that $\tilde{Q} = U \times \{0\} \times Q$. Then we have the following proposition as a corollary of Lemma 6 in Wassermann [21]. (See also Montaldi [18]).

Proposition 6.1 *Let Q be a submanifold of $J^\ell(s, 1)$. Then the set*

$$T_Q = \{\mathbf{x} \in \text{Emb}(U, H_+^n(-1)) \mid j_1^\ell H \text{ is transversal to } \tilde{Q}\}$$

is a residual subset of $\text{Emb}(U, H_+^n(-1))$. If Q is a closed subset, then T_Q is open.

On the other hand, we already have the canonical stratification $\mathcal{A}_0^\ell(U, \mathbb{R})$ of $J^\ell(\mathbb{R}^s, \mathbb{R}) \setminus W^\ell(\mathbb{R}^s, \mathbb{R})$ (cf., the appendix). By the above proposition and arguments in the appendix, we have the following theorem.

Theorem 6.2 *There exists an open dense subset $\mathcal{O} \subset \text{Emb}(U, H_+^n(-1))$ such that for any $\mathbf{x} \in \mathcal{O}$, the germ of the corresponding horospherical hypersurface $HS_{\mathbf{x}}$ at each point is the critical part of an MT-stable map germ.*

In the case when $n \leq 6$, for any $\mathbf{x} \in \mathcal{O}$, the germ of the Legendrian lift $\mathcal{L}_{\mathbf{x}}$ of the hyperbolic horospherical hypersurface $HS_{\mathbf{x}}$ at each point is Legendrian stable.

We remark that we can also prove the multi-jet version of Proposition 6.1. As an application of such a multi-jet transversality theorem, we can show that the horospherical hypersurface $HS_{\mathbf{x}}$ is the critical part of a (global) MT-stable map for a generic submanifold $x : U \rightarrow H_+^n(-1)$ (cf., Appendix). However, the arguments are rather tedious and we only consider local phenomenon in this paper, so that we omit it.

Appendix Generating families

In which we give a brief survey on the theory of Legendrian singularities mainly developed by Arnol'd and Zakalyukin [1, 24, 25]. Let $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be a function germ. We say that F is a *Morse family* if the mapping

$$\Delta^* F = \left(F, \frac{\partial F}{\partial q_1}, \dots, \frac{\partial F}{\partial q_k} \right) : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R} \times \mathbb{R}^k, \mathbf{0})$$

is non-singular, where $(q, x) = (q_1, \dots, q_k, x_1, \dots, x_n) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$. In this case we have a smooth $(n-1)$ -dimensional submanifold $\Sigma_*(F) = \Delta^* F^{-1}(\mathbf{0})$ and a map germ $\Phi_F : (\Sigma_*(F), \mathbf{0}) \longrightarrow PT^*\mathbb{R}^n$ defined by

$$\Phi_F(q, x) = \left(x, \left[\frac{\partial F}{\partial x_1}(q, x) : \dots : \frac{\partial F}{\partial x_n}(q, x) \right] \right)$$

is a Legendrian immersion. Then we have the following fundamental theorem of the theory of Legendrian singularities.

Proposition A.1 *All Legendrian submanifold germs in $PT^*\mathbb{R}^n$ are constructed by the above method.*

We call F a *generating family* of Φ_F . Therefore the wave front is

$$W(\Phi_F) = \left\{ x \in \mathbb{R}^n \mid \text{there exists } q \in \mathbb{R}^k \text{ such that } F(q, x) = \frac{\partial F}{\partial q_1}(q, x) = \dots = \frac{\partial F}{\partial q_k}(q, x) = 0 \right\}.$$

We sometime denote $\mathcal{D}_F = W(\Phi_F)$ and call it the *discriminant set* of F .

We now introduce an equivalence relation among Legendrian immersion germs. Let $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs. Then we say that i and i' are *Legendrian equivalent* if there exists a contact diffeomorphism germ $H : (PT^*\mathbb{R}^n, p) \longrightarrow (PT^*\mathbb{R}^n, p')$ such that H preserves fibres of π and that $H(L) = L'$. A Legendrian immersion germ into $PT^*\mathbb{R}^n$ at a point is said to be *Legendrian stable* if for every map with the given germ there is a neighbourhood in the space of Legendrian immersions (in the Whitney C^∞ topology) and a neighbourhood of the original point such that each Legendrian immersion belonging to the first neighbourhood has in the second neighbourhood a point at which its germ is Legendrian equivalent to the original germ.

Since the Legendrian lift $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ is uniquely determined on the regular part of the wave front $W(i)$, we have the following simple but significant property of Legendrian immersion germs:

Proposition A.2 *Let $i : (L, p) \subset (PT^*\mathbb{R}^n, p)$ and $i' : (L', p') \subset (PT^*\mathbb{R}^n, p')$ be Legendrian immersion germs such that regular sets of $\pi \circ i, \pi \circ i'$ are dense respectively. Then i, i' are Legendrian equivalent if and only if wave front sets $W(i), W(i')$ are diffeomorphic as set germs.*

This result has been firstly pointed out by Zakalyukin [25]. The assumption in the above proposition is a generic condition for i, i' . Specially, if i, i' are Legendrian stable, then these satisfy the assumption.

We can interpret the Legendrian equivalence by using the notion of generating families. We denote \mathcal{E}_n the local ring of function germs $(\mathbb{R}^n, \mathbf{0}) \longrightarrow \mathbb{R}$ with the unique maximal ideal

$\mathfrak{M}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$. Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ be function germs. We say that F and G are P - \mathcal{K} -equivalent if there exists a diffeomorphism germ $\Psi : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ of the form $\Psi(x, u) = (\psi_1(q, x), \psi_2(x))$ for $(q, x) \in (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0})$ such that $\Psi^*(\langle F \rangle_{\mathcal{E}_{k+n}}) = \langle G \rangle_{\mathcal{E}_{k+n}}$. Here $\Psi^* : \mathcal{E}_{k+n} \longrightarrow \mathcal{E}_{k+n}$ is the pull back \mathbb{R} -algebra isomorphism defined by $\Psi^*(h) = h \circ \Psi$.

Let $F : (\mathbb{R}^k \times \mathbb{R}^3, \mathbf{0}) \longrightarrow (\mathbb{R}, \mathbf{0})$ a function germ. We say that F is a \mathcal{K} -versal deformation of $f = F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$ if

$$\mathcal{E}_k = T_e(\mathcal{K})(f) + \left\langle \frac{\partial F}{\partial x_1} |_{\mathbb{R}^k \times \{\mathbf{0}\}}, \dots, \frac{\partial F}{\partial x_n} |_{\mathbb{R}^k \times \{\mathbf{0}\}} \right\rangle_{\mathbb{R}},$$

where

$$T_e(\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial q_1}, \dots, \frac{\partial f}{\partial q_k}, f \right\rangle_{\mathcal{E}_k}.$$

(See [13].)

The main result in the theory of Legendrian singularities is the following:

Theorem A.3 *Let $F_i : (\mathbb{R}^{k_i} \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families ($i = 1, 2$). Then*

- (1) Φ_{F_1} and Φ_{F_2} are Legendrian equivalent if and only if F_1, F_2 are stably P - \mathcal{K} -equivalent.
- (2) Φ_F is Legendrian stable if and only if F is a \mathcal{K} -versal deformation of $F|_{\mathbb{R}^k \times \{\mathbf{0}\}}$. By the

uniqueness result of the \mathcal{K} -versal deformation of a function germ, Proposition 5.2 and Theorem 5.3, we have the following classification result of Legendrian stable germs: For any map germ $f : (\mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}^p, \mathbf{0})$, we define the *local ring of f* by $Q(f) = \mathcal{E}_n / f^*(\mathfrak{M}_p)\mathcal{E}_n$.

Proposition A.4 *Let $F_i : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families ($i = 1, 2$) such that Φ_{F_i} are Legendrian stable. Then the following conditions are equivalent.*

- (1) $(W(\Phi_{F_1}), \mathbf{0})$ and $(W(\Phi_{F_2}), \mathbf{0})$ are diffeomorphic as germs.
- (2) Φ_{F_1} and Φ_{F_2} are Legendrian equivalent.
- (3) $Q(f_1)$ and $Q(f_2)$ are isomorphic as \mathbb{R} -algebras, where $f_i = F_i|_{\mathbb{R}^k \times \{\mathbf{0}\}}$.

We have another characterization of \mathcal{K} -versal deformations of function germs. Let $J^\ell(\mathbb{R}^k, \mathbb{R})$ be the ℓ -jet bundle of n -variable functions which has the canonical decomposition: $J^\ell(\mathbb{R}^k, \mathbb{R}) \cong \mathbb{R}^k \times \mathbb{R} \times J^\ell(k, 1)$. For any Morse family $F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$, we define a map germ

$$j_1^\ell F : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow J^\ell(\mathbb{R}^k, \mathbb{R})$$

by $j_1^\ell F(q, x) = j^\ell F_x(q)$, where $F_x(q) = F(q, x)$. We denote $\mathcal{K}^\ell(z)$ the \mathcal{K} -orbit through $z = j^\ell f(\mathbf{0}) \in J^\ell(k, 1)$. (cf., [13]). If $f(q) = F(q, \mathbf{0})$ is ℓ -determined relative to \mathcal{K} , then F is a \mathcal{K} -versal deformation of f if and only if $j_1^\ell F$ is transversal to $\mathbb{R}^k \times \{0\} \times \mathcal{K}^\ell(z)$ (cf., [13])

We now consider the stratification of the ℓ -jet space $J^\ell(\mathbb{R}^k, \mathbb{R})$ such that the discriminant set of \mathcal{K} -versal deformations has the corresponding canonical stratification. By Theorem 5.3, such a stratification should be \mathcal{K} -invariant, where we have the \mathcal{K} -action on $J^\ell(k, 1)$ (cf., [13, 14]). By this reason, we use Mather's canonical stratification here [6, 15]. Let $\mathcal{A}^\ell(k, 1)$ be the canonical stratification of $J^\ell(k, 1) \setminus W^\ell(k, 1)$, where

$$W^\ell(k, 1) = \{j^\ell f(0) \mid \dim_{\mathbb{R}} \mathcal{E}_k / ((T_e \mathcal{K})(f) + \mathfrak{M}_k^\ell) \geq \ell\}.$$

We now define the stratification $\mathcal{A}_0^\ell(\mathbb{R}^k, \mathbb{R})$ of $J^\ell(\mathbb{R}^k, \mathbb{R}) \setminus W^\ell(\mathbb{R}^k, \mathbb{R})$ by

$$\mathbb{R}^k \times (\mathbb{R} \setminus \{0\}) \times (J^\ell(k, 1) \setminus W^\ell(k, 1)), \quad \mathbb{R}^k \times \{0\} \times \mathcal{A}^\ell(k, 1),$$

where $W^\ell(\mathbb{R}^k, \mathbb{R}) \equiv \mathbb{R}^k \times \mathbb{R} \times W^\ell(k, 1)$. In [23], Y.-H. Wan has shown that if $j_1^\ell F(0) \notin W^\ell(k, 1)$ and $j_1^\ell F$ is transversal to $\mathcal{A}_0^\ell(\mathbb{R}^k, \mathbb{R})$ then $\pi_F : (F^{-1}(0), \mathbf{0}) \longrightarrow (\mathbb{R}^n, \mathbf{0})$ is a MT-stable map germ. (See also [8]). Here, we call a map germ *MT-stable* if it is transversal to the canonical stratification of a jet space which is introduced in [6].

Proposition A.5 *Let $F, G : (\mathbb{R}^k \times \mathbb{R}^n, \mathbf{0}) \longrightarrow (\mathbb{R}, 0)$ be Morse families such that π_F and π_G are MT-stable map germs. If $Q(f)$ and $Q(g)$ are isomorphic as \mathbb{R} -algebras, then π_F and π_G are topological equivalent. Moreover, in this case, \mathcal{D}_F and \mathcal{D}_G are stratified equivalent.*

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