Brown-Halmos Type Theorems Of Weighted Toeplitz Operators II

By

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Abstract. The spectra of the Toeplitz operators on the weighted Hardy space $H^p(W d\theta/2\pi)$ are studied. For example, the theorems of Brown-Halmos type and Hartman-Wintner type are proved. These generalize results in the previous paper which were proved for $p = 2$.

§1. Introduction

Let $m = d\theta/2\pi$ be the normalized Lebesgue measure on the unit circle T and let W be a non-negative integrable function on T which does not vanish identically. Suppose $1 \le p \le \infty$. Let $L^p(W) = L^p(Wdm)$ and $L^p(W) = L^p$ when $W \equiv 1$. Let $H^p(W)$ denote the closure in $L^p(W)$ of the set $\mathcal{P}$ of all analytic polynomials when $p \neq \infty$. We will write $H^p(W) = H^p$ when $W \equiv 1$, and then this is a usual Hardy space. H^∞ denotes the weak $*$ closure of $\mathcal{P}$ in L^∞. P denotes the projection from the set $\mathcal{C}$ of all trigonometric polynomials to $\mathcal{P}$. For $1 < p < \infty$, P can be extended to a bounded map of $L^p(W)$ onto $H^p(W)$ if and only if W satisfies the condition (A_p) (see [3, Theorem 6.2 of Chapter VI]). This is the well known theorem of Hunt, Muckenhoupt and Wheeden, which is a generalization of the theorem of Helson and Szegő (see [3, Theorem 3.2 of Chapter IV]).

Assuming that a weight W satisfies the condition (A_p) for $1 < p < \infty$, we define a Toeplitz operator $T_\phi^{W,p}$ on $H^p(W)$ as follows. For ϕ in L^∞, suppose that

$$T_\phi^{W,p} f = P(\phi f) \quad (f \in H^p(W)).$$

If $W \equiv 1$, we will write $T_\phi^{W,p} = T_\phi^p$.

In this paper, we study the spectrum $\sigma(T_\phi^{W,p})$ of a Toeplitz operator $T_\phi^{W,p}$. For any weights W in (A_p) and for any ϕ in L^∞, the symbol ϕ for invertible $T_\phi^{W,p}$ was completely described by H.Widom, A.Devinatz and R.Rochberg (see Theorem WDR in this section). This is one of our main tools. In the previous paper [7, (1) of Theorem 1], for $p = 2$ we gave a generalization of a theorem of Brown and Halmos [2, Propsition 7.19] to arbitrary weight in (A_2). In §2 we generalize this theorem for arbitrary p. I.Spitkovsky [10] showed that the set of all weights W for which $\sigma(T_\phi^{W,p}) = \sigma(T_\phi^p)$ for all ϕ in L^∞ does not depend on p. In §2 we give another proof of this result. In fact we describe such a set of weights by using [4, Theorem 2.12]. This also generalizes (1) of Theorem 2 of the previous paper [7].

When ϕ is a continuous function and $W \equiv 1$, the spectrum of T_ϕ^p was completely described (cf. [2, Corollary 7.28]). In §3 we prove $\sigma(T_\phi^{W,p}) = \sigma(T_\phi^p)$ for any continuous function ϕ whenever W satisfies the condition (A_p). In the previous paper [7, (2) of Theorem 1], for $p = 2$ we gave a generalization of a theorem of Hartman and Wintner (cf. [2, Theorem 7.20]) to arbitrary weight in (A_2). In §3 we improve this theorem for $p = 2$ and we generalize this theorem for arbitrary p and arbitrary weight in (A_p). For each inner function q, sing q denotes the subset of ∂D on which q can not be analytically extended. For two inner functions q_1 and q_2, M.Lee and D.Sarason [5] showed that $\sigma(T_\phi) = \bar{D}$ if $\phi = \bar{q}_1 q_2$ and sing $q_1 \neq$ sing q_2.

For $\alpha = \alpha_1 + i\alpha_2 \in \mathbb{C}$ and $\beta = \beta_1 + i\beta_2 \in \mathbb{C}$, put $\langle \alpha, \beta \rangle = \alpha_1\beta_1 + \alpha_2\beta_2$ and $\theta(\alpha, \beta) = \arccos(\langle \alpha, \beta \rangle / |\alpha||\beta|)$ for $\alpha \neq 0$ and $\beta \neq 0$. Set

$$\ell_\alpha^+ = \{z \in \mathbb{C} \ ; \ \langle z, \alpha \rangle \ge 1\} \text{ and } \ell_\alpha^- = \{z \in \mathbb{C} \ ; \ \langle z, \alpha \rangle \le 1\}$$

and let $\mathcal{E}_{\alpha\beta}^{ij}$ denote $\ell_\alpha^i \cap \ell_\beta^j$ where $i = +$ or $-$ and $j = +$ or $-$. For each pair (α, β)

$$\mathbb{C} = \mathcal{E}_{\alpha\beta}^{++} \cup \mathcal{E}_{\alpha\beta}^{+-} \cup \mathcal{E}_{\alpha\beta}^{-+} \cup \mathcal{E}_{\alpha\beta}^{--}$$

and if $\ell = -i$ and $m = -j$, then

$$\overline{(\mathcal{E}^{\ell m}_{\alpha\beta})^c} = \overline{\mathbb{C}\setminus\mathcal{E}^{\ell m}_{\alpha\beta}} \supset \mathcal{E}^{ij}_{\alpha\beta}.$$

For any bounded subset E in $\mathbb{C}$, there exists a pair (α, β) such that $\mathcal{E}^{ij}_{\alpha\beta} \supseteq E$ for some (i, j). When $0 \le t \lneq \pi/2$, put

$$h^t(E) = \bigcap \left\{ \overline{(\mathcal{E}^{\ell m}_{\alpha\beta})^c} \ ; \ \mathcal{E}^{ij}_{\alpha\beta} \supseteq E \text{ and } \ell = -i, \ m = -j, |\theta(\alpha, \beta)| = \pi - 2t \right\}$$

for a subset E in $\mathbb{C}$. If $t = 0$, then $h^0(E)$ is the closed convex hull of E. If E is a simple set such that $E = [a, b]$ or $E = \{z \in \mathbb{C} \ ; \ |z| \le 1\}$, then we can describe $h^t(E)$ for $0 \le \ t \lneq \pi/2$.

If a weight W satifies the condition (A_p) then $\log W$ belongs to BMO and so there exist two real valued function u and v in L^∞_R such that $\log W = u + \tilde{v}$ where $\tilde{v}$ denotes the harmonic conjugate with $\tilde{v}(0) = 0$. For $W = e^{u+\tilde{v}}$, put

$$t_W = \|v\|' = \inf\{\|v - \tilde{s} - a\|_\infty \ ; \ s \in L^\infty_R, \ a \in R\}.$$

In the previous paper [7, (1) of Theorem 1], we showed that $\sigma(T^{W,2}_\phi) \subseteq h^t(R(\phi))$ for $t = t_W$. This implies a theorem of Brown and Halmos (cf. [2, Corollary 7.19]) for $W \equiv 1$, that is, $\sigma(T^2_\phi) \subseteq h^0(R(\phi))$. In this paper, we generalize this result for $T^{W,p}_\phi$, that is, if $t = \dfrac{\pi}{2}\left(1 - \dfrac{2}{\max(p, q)}\right) + \dfrac{2}{p} t_W$ then $\sigma(T^{W,p}_\phi) \subseteq h^t(R(\phi))$ because $t = t_W$ for $p = 2$.

In this paper, we use the following theorems about the invertibility of Toeplitz operators on $H^p(W)$ or H^p. The first one is due to H.Widom, A.Devinatz and R.Rochberg (cf. [1, Theorem 5.3], [6]). The second one is due to N.Krupnik (cf. [1, Theorem 5.22]).

Theorem WDR. *Suppose $1 < p < \infty$ and $W = |h|^p$ satisfies the condition (A_p), where h is an outer function in H^p. Then the following conditions on ϕ and W are equivalent.*

(1) $T^{W,p}_\phi$ is an invertible operator on $H^p(W)$.

(2) $\phi = k(\bar{h}_0/h_0)(h/\bar{h})$, where k is an invertible function in H^∞ and h_0 is an outer funcction in H^p with $|h_0|^p$ satisfying the condition (A_p).

(3) $\phi = \gamma \exp(U - i\tilde{V})$, where γ is constant with $|\gamma| = 1$, U is a bounded real function in L^1 and $W \exp\left(\dfrac{p}{2}V\right)$ satisfies (A_p).

Theorem K. *Suppose $1 < p < \infty$ and $1/p + 1/q = 1$, and ϕ is a function in L^∞. The following are equivalent.*

(1) Both T^p_ϕ and T^q_ϕ are invertible on H^p and H^q, respectively.

(2) T^ℓ_ϕ is invertible for all ℓ with $\min\{p, q\} \le \ell \le \max\{p, q\}$.

(3) $\phi = ke^{U+iV}$, where k is an invertible function in H^∞, U and V are bounded real functions and $\|V\|_\infty < \pi/\max\{p, q\}$.

In this paper, $W \in (A_p)$ means that W satisfies the condition (A_p).

§2. Arbitrary symbols

Corollary 1 was proved in the previous paper [7, Theorem 1]. Corollary 2 was proved for $p \geq 2$ in [7, Theorem 3]. Corollaries 1 and 2 are just the generalizations of a theorem of Brown and Halmos (cf. [2, Proposition 7.19]). Theorem 2 for $p = 2$ was proved in [7, (1) of Theorem 2]. I.Spitkovsky [10] showed that the set of all weights W for which $\sigma(T_\phi^W) = \sigma(T_\phi)$ for any ϕ in L^∞ does not depend on p. Hence Theorem 2 for $1 < p < \infty$ follows. We give another proof.

Theorem 1. *Suppose W satisfies the condition $(A_p) \cap (A_q)$ where $1 < p < \infty$ and $1/p + 1/q = 1$, and $t = \dfrac{\pi}{2}\left(1 - \dfrac{2}{\max(p,q)}\right) + \dfrac{2}{p} t_W$. If ϕ is a function in L^∞, then*

$$\mathcal{R}(\phi) \subseteq \sigma(T_\phi^{W,p}) \subseteq h^t(\mathcal{R}(\phi)).$$

Proof. By Theorem WDR, it is clear that $\mathcal{R}(\phi) \subseteq \sigma(T_\phi^{W,p})$. We will show that $\sigma(T_\phi^{W,p}) \subseteq h^t(\mathcal{R}(\phi))$. Suppose $\lambda \notin h^t(\mathcal{R}(\phi))$. Then by definition $\lambda \in \cup\{(\mathcal{E}_{\alpha\beta}^{\ell m})^0 \ ; \ \mathcal{E}_{\alpha\beta}^{ij} \supseteq \mathcal{R}(\phi)$ and $\ell = -i,\ m = -j,\ |\theta(\alpha, \beta)| = \pi - 2t\}$. Then $(\phi - \lambda)/|\phi - \lambda| = e^{is_\lambda}$ where $0 \leq s_\lambda \leq \pi - 2t - 2\varepsilon$ *a.e.* or $-\pi + 2t + 2\varepsilon \leq s_\lambda \leq 0$ *a.e.* for some $\varepsilon > 0$ Hence $|s_\lambda - \frac{\pi}{2} + t + \varepsilon| \leq \frac{\pi}{2} - t - \varepsilon$ *a.e.* or $|s_\lambda + \frac{\pi}{2} - t - \varepsilon| \leq \frac{\pi}{2} - t - \varepsilon$ *a.e.* Let $W = |h|^p$ and $h^p = \exp(u + \tilde{v} + i(\tilde{u} - v))$. Then

$$\frac{\phi - \lambda}{|\phi - \lambda|} \frac{\bar{h}}{h} = \exp i(s_\lambda + \frac{2}{p}(v - \tilde{u}))$$

and

$$\begin{aligned} & \|s_\lambda + \frac{2}{p}(v - \tilde{u})\|' \\ = \ & \|s_\lambda + \frac{2}{p} v\|' \leq \frac{\pi}{2} - t - \varepsilon + \frac{2}{p}\|v\|' \\ = \ & \frac{\pi}{2} - \frac{\pi}{2}(1 - \frac{2}{\max(p,q)}) - \frac{2}{p} t_W - \varepsilon + \frac{2}{p} t_W = \frac{\pi}{\max(p,q)} - \varepsilon. \end{aligned}$$

By Theorem K, $T^p_{\frac{\phi-\lambda}{|\phi-\lambda|}\frac{\bar{h}}{h}}$ is invertible and so by Theorem WDR $T^{W,p}_{\phi-\lambda}$ is invertible. Thus $\lambda \notin \sigma(T_\phi^{W,p})$.

Corollary 1. *Suppose $W = e^{u+\tilde{v}}$ is a Helson-Szegő weight and $t = t_W$. If ϕ is a function in L^∞, then $\mathcal{R}(\phi) \subseteq \sigma(T_\phi^{W,2}) \subseteq h^t(\mathcal{R}(\phi))$.*

Corollary 2. *Suppose* $W \equiv 1,\ 1<p<\infty$ *and* $1/p+1/q=1$ *and* $t=|p-2|\pi/2p$. *If* ϕ *is a function in* L^∞, *then* $\mathcal{R}(\phi) \subseteq \sigma(T_\phi^p) \subseteq h^t(\mathcal{R}(\phi))$.

Proof. Since $W \equiv 1,\ t=\dfrac{\pi}{2}\left(1-\dfrac{2}{\max(p,q)}\right)$. If $p \geq 2$, then $t=\dfrac{\pi}{2}\left(1-\dfrac{2}{p}\right)=\dfrac{\pi(p-2)}{2p}$. If $1<p<2$, then $t=\dfrac{\pi}{2}\left(1-\dfrac{2}{q}\right)=\dfrac{\pi(2-p)}{2p}$ because $q=\dfrac{p}{p-1}$.

Theorem 2. *Suppose* W *satisfies the condition* (A_p) *for some* p *with* $1<p<\infty$. *Then,* $t_W=0$ *if and only if* $\sigma(T_\phi^{W,p})=\sigma(T_\phi^p)$ *for any* ϕ *in* L^∞.

Proof. Suppose that $\sigma(T_\phi^{W,p})=\sigma(T_\phi^p)$ for any ϕ in L^∞. If $\phi=\bar{h}_0/h_0$ and h_0 is an outer function with $|h_0|^p \in (A_p)$, then T_ϕ^p is invertible and so $T_\phi^{W,p}$ is invertible. Put $h_0=\exp\dfrac{1}{p}(u_0+\tilde{v}_0+i(\tilde{u}_0-v_0))$ where $u_0 \in L_R^\infty$ and $v_0 \in L_R^\infty$. Then

$$\phi=\frac{\bar{h}_0}{h_0}=\exp i\frac{2}{p}(v_0-\tilde{u}_0).$$

Since $T_\phi^{W,p}$ is invertible, by Theorem WDR $W|h_0|^p=W\exp(\tilde{v}_0+u_0)$ belongs to (A_p). Thus $W(A_p) \subseteq (A_p)$ and so by [4, Theorem 2.12] $t_W=0$.

Conversely if $t_W=0$ then $\log W$ belongs to the closure of L^∞ in BMO. Hence $W(A_p)=(A_p)$ by [4, Theorem 2.12]. Let $W=|h|^p$ and h an outer function in H^p. By Theorem WDR in Introduction, $T_\phi^{W,p}$ is invertible if and only if $T_{\phi/|\phi|}^{W,p}$ is invertible and ϕ is invertible in L^∞. If $T_{\phi/|\phi|}^{W,p}$ is invertible then by Theorem WDR

$$\frac{\phi}{|\phi|}=\frac{h}{\bar{h}}\,\frac{\bar{h}_0}{h_0}$$

for some outer function h_0 with $|h_0|^p \in (A_p)$. Since $W(A_p)=(A_p)$, $|h_0|^p|h|^{-p} \in (A_p)$ and $\phi=\overline{h_0h^{-1}}/h_0h^{-1}$. This implies that $T_{\phi/|\phi|}^p$ is invertible. Thus $\sigma(T_\phi^{W,p}) \supseteq \sigma(T_\phi^p)$ for any ϕ in L^∞. If $T_{\phi/|\phi|}^p$ is invertible then $\phi/|\phi|=\bar{h}_1/h_1$ for some outer function h_1 with $|h_1|^p \in (A_p)$. Since $W(A_p)=(A_p)$, $|h|^p|h_1|^p \in (A_p)$ and so

$$\frac{\phi}{|\phi|}=\frac{h}{\bar{h}}\cdot\frac{\overline{h_1h}}{h_1h}\ .$$

Hence $T_{\phi/|\phi|}^{W,p}$ is invertible. Thus $\sigma(T_\phi^{W,p}) \subseteq \sigma(T_\phi^p)$ for any ϕ in L^∞. Therefore $\sigma(T_\phi^{W,p})=\sigma(T_\phi^p)$ for any ϕ in L^∞.

§3. Special symbols

In this section, we study the spectrum of a Toeplitz operator whose symbol is continuous, real-valued or the quotient of two inner functions. Theorem 3 generalizes

(3) of Theorem 3 in the previous paper [7]. Theorem 4 generalizes (2) of Theorem 1 in [7]. Theorem 5 generalizes and improves Corollary 1 in [7]. Corollary 3 improves (3) of Theorem 1 in [7].

Theorem 3. *Let $1 < p < \infty$. If ϕ is a continuous function on T then*

$$\sigma(T_\phi^{W,p}) = \mathcal{R}(\phi) \cup \{\lambda \in \mathbb{C} \ ; \ i_t(\phi, \lambda) \neq 0\}$$

for any W in (A_p), where $i_t(\phi, \lambda)$ is the winding number of the curve determined by ϕ with respect to λ.

Proof. If $\lambda \notin \mathcal{R}(\phi)$ and $i_t(\phi, \lambda) = 0$ then $(\phi - \lambda)/|\phi - \lambda| = e^{is_\lambda}$ where $s_\lambda \in C$ and so $W \exp \frac{p}{2}(-\tilde{s}_\lambda)$ belongs to (A_p). By Theorem WDR this implies that $\lambda \notin \sigma(T_\phi^{W,p})$. Conversely if $\lambda \notin \sigma(T_\phi^{W,p})$ then $\lambda \notin \mathcal{R}(\phi)$. Hence

$$\frac{\phi - \lambda}{|\phi - \lambda|} = z^\ell e^{is_\lambda}$$

where ℓ is an integer and $s_\lambda \in C$. Since $T_{\phi-\lambda}^{W,p}$ is invertible, by Theorem WDR there exists an outer function h_1 such that

$$\frac{\phi - \lambda}{|\phi - \lambda|} = \frac{h}{\bar{h}} \frac{\bar{h}_1}{h_1}$$

where $|h_1|^p \in (A_p)$ and $W = |h|^p \in (A_p)$ and h is an outer function. Then $|h|^{-q} \in (A_q)$ where $1/p + 1/q = 1$ and so $f = h^{-1}h_1$ belongs to H^t for some $t > 1$. Put $g^2 = \exp(-s_\lambda + is_\lambda)$ then $g \in \bigcap_{1 \leq s < \infty} H^s$ and so gf belongs to H^1. Similary we can show that $(gf)^{-1}$ belongs to H^1. Then if $\ell \geq 0$ then $z^\ell gf = \overline{gf}$ and $z^\ell (gf)^2$ is nonnegative in $H^{1/2}$. Hence $\ell = 0$ because $H^{1/2}$ does not contain any nonconstant nonnegative functions. If $\ell \leq 0$ then $\bar{z}^\ell (gf)^{-2}$ is nonnegative in $H^{1/2}$ and so $\ell = 0$. Thus $(\phi - \lambda)/|\phi - \lambda| = e^{is_\lambda}$ and so $\lambda \notin \sigma(T_\phi^p)$ because $e^{is_\lambda} = g/\bar{g}$ and $|g^{-1}|^p \in (A_p)$.

Theorem 4. *Suppose W satisfies the condition $(A_p) \cap (A_q)$ where $1 < p < \infty$ and $1/p + 1/q = 1$, and $t = \frac{\pi}{2}\left(1 - \frac{2}{\max(p,q)}\right) + \frac{2}{p} t_W$. If ϕ is real valued, $a = \operatorname{ess\,inf} \phi$ and $b = \operatorname{ess\,sup} \phi$, then*

$$\mathcal{R}(\phi) \subseteq \sigma(T_\phi^{W,p}) \subseteq \triangle(c, r) \cap \triangle(\bar{c}, r)$$

where $c = \frac{a+b}{2} - i\frac{a-b}{2}\cos 2t$ and $r = -\frac{a-b}{2\sin 2t}$. If $t_W = 0$ then $[a, b] \subseteq \sigma(T_\phi^{W,p})$.

Proof. By Theorem 1, $\sigma(T_\phi^{W,p}) \subseteq h^t(\mathcal{R}(\phi)) \subseteq h^t([a, b])$ for $t = \frac{\pi}{2}\left(1 - \frac{2}{\max(p,q)}\right) + \frac{2}{p} t_W$. It is elementary to see that $h^t([a, b]) \subseteq \triangle(c, r) \cap \triangle(\bar{c}, r)$. Suppose $t_W = 0$.

Then $t = \frac{\pi}{2}\left(1 - \frac{2}{\max(p,q)}\right)$ and by Theorem 2 $\sigma(T_\phi^{W,p}) = \sigma(T_\phi^p)$. We will show that $[a,b] \subseteq \sigma(T_\phi^p)$. Suppose $\lambda \in [a,b]$ and $\lambda \notin \mathcal{R}(\phi)$, then $\psi = (\phi-\lambda)/|\phi-\lambda| = 2\chi_E - 1$ for some measurable set E in ∂D. If $\lambda \notin \sigma(T_\phi^p)$, then by Theorem WDR there exists an outer function h_1 in H^p with h_1^{-1} in H^q such that $\psi = \bar{h}_1/h_1$. Since $T_\psi^q = T_{\bar{\psi}}^q$ is also invertible, there exists an outer function h_2 in H^q with h_2^{-1} in H^p such that $\psi = \bar{h}_2/h_2$. Hence

$$\frac{\bar{h}_1}{h_1} = \frac{h_1}{\bar{h}_1} = \frac{\bar{h}_2}{h_2} = \frac{h_2}{\bar{h}_2}$$

because ψ is a real valued function. Hence $h_1^2 = \bar{h}_1^2 \in H^{p/2}$ and $h_2^2 = \bar{h}_2^2 \in H^{q/2}$. Therefore h_1 or h_2 is constant because $\max(p/2, q/2) \geq 1$ and the only real function in H^1 is constant. Thus ψ is constant and this contradicts that ϕ is not constant. Thus $[a,b] \subseteq \sigma(T_\phi^p)$.

For a weight W in (A_p) and a measurable set E, put

$$\gamma_+(E,W,p) = \sup\{t > 0 \ ; \ W\exp(t\tilde{\chi}_E) \text{ satisfies } (A_p)\}$$

and

$$\gamma_-(E,W,p) = \inf\{t < 0 \ ; \ W\exp(t\tilde{\chi}_E) \text{ satisfies } (A_p)\}.$$

Theorem 5. *Let W satisfy the condition (A_p) and $1 < p < \infty$. Suppose $\phi = a\chi_E + b\chi_{E^c}$ where a, b are real numbers and E is measurable set in ∂D with $0 < d\theta(E) < 2\pi$. Then*

$$\sigma(T_\phi^{W,p}) = \{\lambda \in \mathbb{C} \ ; \ \pi \geq \ \operatorname{Arg} \frac{a-\lambda}{b-\lambda} \geq \frac{2}{p}\gamma_+(E,W,p)$$
$$\text{or} \ -\pi \leq \ \operatorname{Arg} \frac{a-\lambda}{b-\lambda} \leq \frac{2}{p}\gamma_-(E,W,p)\}$$

where $-\pi \leq \ \operatorname{Arg} z \leq \pi$. In particular, $\sigma(T_\phi^{W,p}) \supseteq [a,b]$.

Proof. If $\lambda \neq a, b$, and λ is a real number then

$$\frac{\phi-\lambda}{|\phi-\lambda|} = \frac{a-\lambda}{|a-\lambda|}\chi_E + \frac{b-\lambda}{|b-\lambda|}\chi_{E^c}.$$

There exist $a(\lambda)$ and $b(\lambda)$ such that $-\pi \leq a(\lambda), \ b(\lambda) \leq \pi$ and

$$\frac{a-\lambda}{|a-\lambda|} = e^{ia(\lambda)}, \ \frac{b-\lambda}{|b-\lambda|} = e^{ib(\lambda)}.$$

Hence

$$\frac{\phi-\lambda}{|\phi-\lambda|} = \exp i\{(a(\lambda) - b(\lambda))\chi_E + b(\lambda)\}$$

where $0 \leq a(\lambda) - b(\lambda) \leq \pi$ or $-\pi \leq a(\lambda) - b(\lambda) \leq 0$. If $\lambda \notin \sigma(T_\phi^{W,p})$, then by Theorem WDR $W \exp\left\{\frac{p}{2}(a(\lambda) - b(\lambda))\tilde{\chi}_E\right\}$ belongs to (A_p). Hence

$$\sigma(T_\phi^{W,p}) \subseteq \left\{\lambda \in \mathbb{C} \ ; \ \pi \geq a(\lambda) - b(\lambda) \geq \frac{2}{p}\gamma_+(E, W, p)\right\}$$
$$\bigcup \left\{\lambda \in \mathbb{C} \ ; \ \frac{2}{p}\gamma_-(E, W, p) \geq a(\lambda) - b(\lambda) \geq -\pi\right\}.$$

If $\pi \geq a(\lambda) - b(\lambda) > \frac{2}{p}\gamma_+(E, W, p)$ or $-\pi \leq a(\lambda) - b(\lambda) < \frac{2}{p}\gamma_-(E, W, p)$, then $W \exp\left\{\frac{p}{2}(a(\lambda) - b(\lambda))\tilde{\chi}_E\right\}$ does not belong to (A_p) and so $\lambda \in \sigma(T_\phi^{W,p})$. Since $\sigma(T_\phi^{W,p})$ is closed,

$$\sigma(T_\phi^{W,p}) = \left\{\lambda \in \mathbb{C} \ ; \ \pi \geq a(\lambda) - b(\lambda) \geq \frac{2}{p}\gamma_+(E, W, p)\right\}$$
$$\bigcap \left\{\lambda \in \mathbb{C} \ ; \ \frac{2}{p}\gamma_-(E, W, p) \geq a(\lambda) - b(\lambda) \geq -\pi\right\}.$$

Lemma 1. *For a measurable set E in T with $0 < m(E) < 1$, $\|\pi\chi_E - v\|' \geq \pi/2$ for any v in L_R^∞ with $\|v\|_\infty < \pi/2$.*

Proof. Suppose $\phi = a\chi_E + b\chi_{E^c}$ where a and b are real numbers, $a \neq b$ and $0 < m(E) < 1$. For $W = e^{u+\tilde{v}}$ where $u, v \in L_R^\infty$ and $\|v\|_\infty < \pi/2$, $\sigma(T_\phi^{W,2}) \supseteq [a, b]$ if and only if $\|\pi\chi_E - v\|' \geq \pi/2$. This is proved in [7, Corollary 1]. Now Theorem 5 shows Lemma 1.

Corollary 3. *Suppose W satisfies the condition (A_2). If ϕ is real valued, $a = \operatorname{ess\,inf} \phi$ and $b = \operatorname{ess\,sup} \phi$ then $[a, b] \subseteq \sigma(T_\phi^{W,2})$.*

Proof. Since $W \in (A_2)$, $W = e^{u+\tilde{v}}$ where $u, v \in L_R^\infty$ and $\|v\|_\infty < \pi/2$. For $\lambda \in [a, b] \cap \mathcal{R}(\phi)^c$, $\frac{\phi - \lambda}{|\phi - \lambda|} = e^{i\ell}$ and $\ell = \pi(1 - \chi_E)$ for some measurable set E in T with $0 < m(E) < 1$. Then, in [7, (3) of Theorem 1], it is proved that $\lambda \in \sigma(T_\phi^{W,2})$ if and only if $\|\pi\chi_E - v\|' \geq \pi/2$. Now Lemma 1 implies this corollary.

Lemma 2. *If q_1 and q_2 are inner functions and $\bar{q}_1 q_2 = f/|f| = |g|/g$ where both f and g are in $\cap_{p>1/2} H^p$, then* $\operatorname{sing} q_1 \neq \operatorname{sing} q_2$.

Proof. See the proof of [8, Corollary 5].

Theorem 6. *Suppose W satisfies the condition (A_p) where $1 < p < \infty$. If $\phi = \bar{q}_1 q_2$ where q_1 and q_2 are inner functions with* $\operatorname{sing} q_1 \neq \operatorname{sing} q_2$ *then $\sigma(T_\phi^{W,p}) = \bar{D}$.*

Proof. Suppose $W = |h|^p$ for some outer function in H^p. If $\lambda \in D$ then

$$\bar{q}_1 q_2 - \lambda = \bar{q}_1(q_2 - \lambda q_1) = \bar{q}_1 q_3 k$$

where q_3 is inner and k is invertible in H^∞. By the proof of [8, Theorem 2(2)] $\bar{q}_2 q_3 = \dfrac{f}{|f|} = \dfrac{|g|}{g}$ where both f and g are in H^1. By Lemma 2 singq_2 = singq_3 and so sing$q_1 \neq$ singq_3. If $\lambda \notin \sigma(T^{W,p}_{\bar{q}_1 q_2})$ then $0 \notin \sigma(T^{W,p}_{\bar{q}_1 q_3})$ because k is invertible in H^∞. By Theorem WDR

$$\bar{q}_1 q_3 = \frac{h}{\bar{h}} \frac{\bar{h}_0}{h_0}$$

where h_0 is an outer function in H^p with $|h_0|^p \in (A_p)$. Hence $\bar{q}_1 q_3 = f/|f| = |g|/g$ where $f = (h/h_0)^2$ and $g = (h_0/h)^2$. Since $|h|^p$ and $|h_0|^p$ are in (A_p), both f and g belong to $H^{1/2}$. This contradicts Lemma 2. Hence $\lambda \in \sigma(T^{W,p}_{\bar{q}_1 q_2})$ and so $\sigma(T^{W,p}_{\bar{q} q_2}) = \bar{D}$.

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References

1. A.Böttcher and B.Silbermann, Analysis Of Toeplitz Operators, Springer-Verlag

2. R.G.Douglas, Banach Algebra Techniques In Operator Theory (Academic Press, New York, 1972)

3. J.B.Garnett, Bounded analytic functions. Academic Press, New York, 1981.

4. R.Johnson and C.J.Neugebauer, Homeomorphisms preserving A_p, Revista Math. Iberoamericana 3(1987), 249-273.

5. M.Lee and D.E.Sarason, The spectra of some Toeplitz operators, J.Math.Anal.Appl. 33(1971), 529-543.

6. T.Nakazi, Toeplitz operators and weighted norm inequalities, Acta Sci. Math. (Szeged) 58(1993), 443-452.

7. T.Nakazi, Brown-Halmos type theorems of weighted Toeplitz operators, Canad.Math. Bull. 41(1998), 196-206.

8. T.Nakazi, The spectra of Toeplitz operators with unimodular symbols, Proc. Edinburgh Math.Soc. 41(1998), 133-139.

9. R.Rochberg, Toeplitz operators on weighted H^p spaces, Indiana Univ.Math.J. 26(1977), 291-298.

10. I.Spitkovsky, Multipliers that do not influence factorability, Math.Notes 27(1980), 145-149.

11. H.Widom, Toeplitz operators on H_p, Pacific J.Math. 19(1966), 573-582.

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