STABILITY FOR ASYMPTOTICALLY PERIODIC MULTIVALUED DYNAMICAL SYSTEMS GENERATED BY DOUBLE OBSTACLE PROBLEMS

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Abstract. In this paper let us consider double obstacle problems, which includes regional economic growth models. By prescribed time-dependent obstacles, our problems are non-autonomous systems and it is impossible to show the uniqueness of solutions. Therefore the associated dynamical systems are multivalued. In this paper from the viewpoint of attractors we shall consider the periodic stability for the double obstacle problem with asymptotically periodic data. Namely, assuming that time-dependent data converges to time-periodic ones as time goes to infinity, we shall construct the global attractor for the asymptotically periodic multivalued dynamical system. Moreover we shall discuss the relationship to the attractor for the limiting periodic problem.

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1 Introduction

Let Ω be a bounded domain in R^N $(1 \leq N < +\infty)$ with smooth boundary $\Gamma := \partial\Omega$ and q be a fixed number with $2 \leq q < +\infty$. Then, for each $s \geq 0$ let us consider the following double obstacle problem $(\mathrm{P})_s$: Find functions $u \in C([s, +\infty); L^2(\Omega))$ and $\theta \in L^2_{loc}((s, +\infty); L^2(\Omega))$ such that

$$(\mathrm{P})_s \begin{cases} u'(t) - \mathbf{div}(|\nabla u(t)|^{q-2}\nabla u(t)) - g(u(t)) = \theta(t,x) \quad \text{in } Q_s := [s, +\infty) \times \Omega; \\ 0 \leq \theta(t,x) \leq h(t, u(t,x)) \quad \text{a.e. on } (s, +\infty) \times \Omega; \\ u(t) = l(t) \quad \text{a.e. on } (s, +\infty) \times \Gamma; \\ u(s) = u_0 \quad \text{in } \Omega. \end{cases}$$

Where $g(\cdot)$, $h(\cdot,\cdot)$, $l(\cdot)$ are given functions. Here we note that $(\mathrm{P})_s$ with $q = 2$ is a regional economic growth model, in which the unknown function u represents a stock of available capital, the unknown function θ is a rate of investment and $-g(u)$ is a recursive depreciation of capital.

In the case that $q = 2$ and the boundary condition $l(t) \equiv 0$ for any $t > 0$, the existence of solution for $(\mathrm{P})_s$ was proved in [2, 9] and Papageorgiou [9] studied the optimal control problem. Unfortunately, by given double obstacles, $(\mathrm{P})_s$ loses the uniqueness of solutions for a given initial value. Recently, from the viewpoint of attractors Kapustian and Valero [6] considered the asymptotic behaviour of solutions for $(\mathrm{P})_s$ without uniqueness in the case that $q = 2$ and time-independent given functions $h(t,\cdot) \equiv h(\cdot)$, $l(t) \equiv 0$ for any $t \geq 0$. Namely they constructed the global attractor for the multivalued autonomous dynamical system associated with $(\mathrm{P})_s$.

In the general case $2 \leq q < +\infty$, the existence of solution for $(\mathrm{P})_s$ was proved in [12]. Moreover, assuming that the given functions $h(t,\cdot)$ and $l(t)$ converge to time-independent ones $h^\infty(\cdot)$ and l^∞ as $t \to +\infty$ in appropriate senses, the author [12] constructed the global attractor for $(\mathrm{P})_s$ and discussed the relationship to the one for the limiting autonomous system of $(\mathrm{P})_s$.

In this paper for a given period $T_0 > 0$ let us consider an asymptotically T_0-periodic problem $(\mathrm{AP})_s$ for $(\mathrm{P})_s$. Namely we assume that $h(t,\cdot) - h_p(t,\cdot) \to 0$, $l(t,\cdot) - l_p(t,\cdot) \to 0$ in appropriate senses as $t \to +\infty$, where $h_p(t,\cdot)$ and $l_p(t)$ are T_0-periodic in time, i.e.

$$h_p(t,\cdot) = h_p(t + T_0,\cdot), \quad l_p(t) = l_p(t + T_0), \qquad \forall t \in R_+ := [0, +\infty).$$

Then, by the above asymptotically T_0-periodic stability conditions we have a limiting non-autonomous T_0-periodic double obstacle problem $(\mathrm{PP})_{T_0}$ of $(\mathrm{AP})_s$ as follows: Find functions $u \in C([0, +\infty); L^2(\Omega))$ and $\theta \in L^2_{loc}((0, +\infty); L^2(\Omega))$ such that

$$(\mathrm{PP})_{T_0} \begin{cases} u'(t) - \mathbf{div}(|\nabla u(t)|^{q-2}\nabla u(t)) - g(u(t)) = \theta(t,x) \quad \text{in } Q_0 = [0, +\infty) \times \Omega; \\ 0 \leq \theta(t,x) \leq h_p(t, u(t,x)) \quad \text{a.e. on } (0, +\infty) \times \Omega; \\ u(t) = l_p(t) \quad \text{a.e. on } (0, +\infty) \times \Gamma; \\ u(0) = u_0 \quad \text{in } \Omega. \end{cases}$$

The main object of this paper is to investigate the large-time behaviour of solutions for $(\mathrm{AP})_s$ and $(\mathrm{PP})_{T_0}$ without uniqueness from the viewpoint of attractors. In fact, we shall show the existence of attractors for $(\mathrm{AP})_s$ and $(\mathrm{PP})_{T_0}$ and discuss the relationship between them.

Throughout this paper, $|\cdot|_{L^q(\Omega)}$ (resp. $|\cdot|_{W^{1,q}(\Omega)}$) is a standard norm of $L^q(\Omega)$ (resp. $W^{1,q}(\Omega)$) for each $q \geq 2$. For the subset A of $L^2(\Omega)$, $\overline{A}$ denotes the closure of A in $L^2(\Omega)$. For two sets A and B in $L^2(\Omega)$, we define the so-called Hausdorff semi-distance

$$\mathrm{dist}_{L^2(\Omega)}(A, B) := \sup_{x\in A} \inf_{y \in B} |x-y|_{L^2(\Omega)}.$$

2 Assumptions and weak formulation

In this paper we consider the asymptotically T_0-periodic double obstacle problem $(\mathrm{AP})_s$ under the following assumptions:

(A1) $g(\cdot)$ is a Lipschitz continuous function on R satisfying the following property:

$$\min\left\{ \liminf_{z\to-\infty} \frac{-g(z)}{z},\ \liminf_{z\to+\infty} \frac{-g(z)}{z} \right\} =: g_0 > 0;$$

(A2) $h(\cdot,\cdot)$ and $h_p(\cdot,\cdot)$ are non-negative continuous functions on $R_+ \times R$. $h_p(t,z)$ is T_0-periodic in t for each $z \in R$. And there exists a positive constant L with $0 < L < \frac{g_0}{2}$ such that

$$|h(t,z_1) - h(t,z_2)| \leq L|z_1 - z_2|, \quad \forall t \in R_+,\ z_i \in R\ (i=1,2),$$
$$|h_p(t,z_1) - h_p(t,z_2)| \leq L|z_1 - z_2|, \quad \forall t \in R_+,\ z_i \in R\ (i=1,2).$$

Moreover, for any $z\in R$, $\sup_{t\in[0,T_0]} |h(mT_0+t,z) - h_p(t,z)| \longrightarrow 0$ as $m \to +\infty$;

(A3) $l, l_p \in L^\infty(R_+; W^{1,q}(\Omega))$ with $\sup_{t\in R_+} |l'|_{L^2(t,t+1;W^{1,q}(\Omega))} + \sup_{t\in R_+} |l'_p|_{L^2(t,t+1;W^{1,q}(\Omega))} < +\infty$. Moreover l_p is T_0-periodic in time and

$$J_m := \sup_{t\in[0,T_0]} |l(mT_0+t) - l_p(t)|_{W^{1,q}(\Omega)} \longrightarrow 0 \quad \text{as } m \to +\infty;$$

Now we give weak formulations of $(\mathrm{AP})_s$ and $(\mathrm{PP})_{T_0}$. To do so, we define a closed convex subset $K(t)$ of $W^{1,q}(\Omega)$ for each $t \in R_+$ by

$$K(t) := \{ z \in W^{1,q}(\Omega)\ ;\ z = l(t) \text{ a.e. on } \Gamma \}. \tag{2.1}$$

Also the set $K_p(t)$ is also defined by replacing l by $l_p(t)$ in (2.1).

Definition 2.1. (i) For each $s \geq 0$ and $u_0 \in L^2(\Omega)$, a couple of functions $\{u, \theta\}$ is called a solution of $(\mathrm{AP})_s$ if $u \in C([s,+\infty); L^2(\Omega)) \cap L^2_{loc}((s,+\infty); W^{1,q}(\Omega))$, $u' \in L^2_{loc}((s,+\infty); L^2(\Omega))$, $\theta \in L^2_{loc}((s,+\infty); L^2(\Omega))$, $u(0) = u_0$ in $L^2(\Omega)$,

$$u(t) \in K(t) \quad \text{for a.e. } t \geq s,$$

$$0 \leq \theta(t,x) \leq h(t, u(t,x)) \quad \text{a.e. on } (s,+\infty)\times\Omega,$$

and

$$\int_\Omega (u'(t,x) - \theta(t,x) - g(u(t,x)))(u(t,x) - z(x))dx$$
$$+ \int_\Omega |\nabla u(t,x)|^{q-2}\nabla u(t,x) \cdot (\nabla u(t,x) - \nabla z(x))dx = 0$$
$$\text{for any } z \in K(t) \text{ and a.e. } t \geq s.$$

(ii) A solution of $(\mathrm{PP})_{T_0}$ is similarly defined by replacing $h(t), l(t)$, $K(t)$ by $h_p(t), l_p(t)$, $K_p(t)$ in (i).

3 Existence of global solutions

In this section we shall show the existence and global boundedness of solutions for $(\mathrm{AP})_s$ and $(\mathrm{PP})_{T_0}$.

By the same argument in [11, 12], we can get the following the result.

Theorem 3.1. (cf. [11, 12]) *Assume that (A1)-(A3) hold. Then, for each $s \geq 0$ and $u_0 \in \overline{K(s)}$ the double obstacle problem $(AP)_s$ has at least one solution $\{u, \theta\}$ with initial value $u(s) = u_0$. Moreover, for each $\delta > 0$ and the bounded set $B \subset L^2(\Omega)$ there is a positive constant N_δ such that*

$$\sup_{t\geq s} |u(t)|^2_{L^2(\Omega)} + \sup_{t\geq 0} \int_t^{t+1} |\nabla u(\tau)|^q_{L^q(\Omega)} d\tau$$
$$+ \sup_{t\geq s+\delta} |u'|^2_{L^2(t,t+1;L^2(\Omega))} + \sup_{t\geq s+\delta} |\nabla u(t)|^q_{L^q(\Omega)} \leq N_\delta$$

for all $s \geq 0$ and $u_0 \in \overline{K(s)} \cap B$.

In fact, by applying the abstract theory of nonlinear evolution equations governed by time-dependent subdifferential of convex functions, we can prove Theorem 3.1. For detail proofs, see [11, 12].

Here note that the limiting T_0-periodic double obstacle problem $(\mathrm{PP})_{T_0}$ can be considered as the special case of $(\mathrm{AP})_s$ by taking $h_p(t,\cdot)$ and $l_p(t)$ as $h(t,\cdot)$ and $l(t)$. Therefore, by Theorem 3.1 we can get the similar result on the existence and global boundedness of solutions for $(\mathrm{PP})_{T_0}$ on $[0, +\infty)$.

4 Attractor for the limiting periodic problem

In this section we shall construct a global attractor for the limiting T_0-periodic double obstacle problem $(\mathrm{PP})_{T_0}$. To do so, let us define a solution operator for $(\mathrm{PP})_{T_0}$. In fact, by Theorem 3.1 we can define a family $\{U(t,s); 0 \leq s \leq t < +\infty\}$ of solution operators. But we cannot get the uniqueness of solution for $(\mathrm{PP})_{T_0}$. Hence the solution operator $U(t,s)$ from $\overline{K_p(s)}$ into $\overline{K_p(t)}$ is multivalued. Namely, for each $s, t \in R_+$ with $s \leq t$, $U(t,s)$ assigns to any $u_0 \in \overline{K_p(s)}$ the set

$$U(t,s)u_0 := \left\{ z \in \overline{K_p(t)} \;\middle|\; \begin{array}{c} \text{There is a solution } \{u,\ \theta\} \text{ of } (\mathrm{PP})_{T_0} \\ \text{such that} \\ u(s) = u_0 \text{ and } u(t) = z. \end{array} \right\}.$$

Then, it is easy to check the following properties of $\{U(t,s)\}$:

(U1) $U(s,s) = I \quad \text{on } \overline{K_p(s)} \quad \text{for any } s \in R_+$;

(U2) $U(t_2,s)z = U(t_2,t_1)U(t_1,s)z \quad \text{for any } 0 \le s \le t_1 \le t_2 < +\infty \text{ and } z \in \overline{K_p(s)}$;

(U3) $U(t+T_0, s+T_0) = U(t,s) \quad \text{for any } 0 \le s \le t < +\infty$, that is, U is T_0-periodic.

(U4) $\{U(t,s)\}$ has the following demiclosedness:

- If $0 \le s_n \le t_n < +\infty$, $s_n \to s$, $t_n \to t$, $z_n \in \overline{K_p(s_n)}$, $z \in \overline{K_p(s)}$, $z_n \to z$ in $L^2(\Omega)$ and a element $w_n \in U(t_n, s_n)z_n$ converges to some element $w \in L^2(\Omega)$ as $n \to +\infty$, then $w \in U(t,s)z$

Therefore $\{U(t,s)\}$ forms a multivalued T_0-periodic dynamical process. For some properties of the multivalued mapping, see [1], for instance.

Clearly, the limiting T_0-periodic obstacle problem $(\mathrm{PP})_{T_0}$ can be reformulated as an evolution equation

$$(\mathrm{E})_{T_0} \qquad u'(t) + \partial\varphi_p^t(u(t)) + G_p(t,u(t)) \ni 0 \ \text{ in } L^2(\Omega), \qquad t > s,$$

where φ_p^t is a T_0-periodic proper lower semicontinuous convex functions on $L^2(\Omega)$ defined by

$$\varphi_p^t(z) = \begin{cases} \dfrac{1}{q}\displaystyle\int_\Omega |\nabla z|^q dx & \text{if } z \in K_p(t), \\ +\infty & \text{if } z \in L^2(\Omega) \setminus K_p(t). \end{cases}$$

Also, $G_p(t,\cdot)$ is a T_0-periodic multivalued operator in $L^2(\Omega)$ defined by

$$G_p(t,z) := \left\{ w \in L^2(\Omega); \quad \begin{array}{c} w = -g(z) - v \quad \text{in } L^2(\Omega) \\ 0 \le v(x) \le h_p(t, z(x)) \quad \text{a.e. on } \Omega \end{array} \right\}.$$

The author [13] showed the existence of T_0-periodic attractor for $(\mathrm{E})_{T_0}$. So, by applying the abstract results to $(\mathrm{PP})_{T_0}$, we can get the T_0-periodic stability results for $(\mathrm{PP})_{T_0}$ as follows:

Theorem 4.1. (cf. [13]) *Suppose (A1)-(A3). For each $\tau \ge 0$, we define the T_0-step mapping $U_\tau := U(\tau + T_0, \tau)$ and $U_\tau^k := U(\tau + kT_0, \tau)$ for each $k \in N$. Then, there exists a subset $\mathcal{A}_\tau$ of $K_p(\tau)$ such that*

(i) *$\mathcal{A}_\tau$ is non-empty and compact in $L^2(\Omega)$;*

(ii) *for each bounded set B in $L^2(\Omega)$ and each number $\epsilon > 0$ there exists a positive number $N_{B,\epsilon} \in N$ such that*

$$dist_{L^2(\Omega)}(U_\tau^k z, \mathcal{A}_\tau) < \epsilon \qquad \text{for all } z \in \overline{K_p(\tau)} \cap B \text{ and all } k \ge N_{B,\epsilon};$$

(iii) *$U_\tau^k \mathcal{A}_\tau = \mathcal{A}_\tau$ for any $k \in N$.*

In fact, we can construct the compact absorbing set $B_{0,\tau}$ for the discrete multivalued dynamical system U_τ. Here, we define the set $\mathcal{A}_\tau := \bigcap_{n\in Z_+} \overline{\bigcup_{k\geq n} U_\tau^k B_{0,\tau}}$ where $Z_+ := N \cup \{0\}$. Then we see that the set $\mathcal{A}_\tau$ has the properties (i)-(iii) in Theorem 4.1. For detail proofs, see [13].

Theorem 4.2. (cf. [13]) *Suppose (A1)-(A3). Let $\mathcal{A}_s$ and $\mathcal{A}_\tau$ be the global attractors of U_s and U_τ, with $0 \leq s \leq \tau \leq T_0$, respectively. Then, we have $\mathcal{A}_\tau = U(\tau, s)\mathcal{A}_s$.*

Theorem 4.3. (cf. [13]) *Under the assumptions (A1)-(A3), let $\mathcal{A}_\tau$ be the global attractor of U_τ for each $\tau \geq 0$. We put the set $\mathcal{A} := \bigcup_{0\leq\tau\leq T_0} \mathcal{A}_\tau$. Then, $\mathcal{A}$ has the following properties:*

(i) *$\mathcal{A}$ is non-empty and compact in $L^2(\Omega)$;*

(ii) *for each bounded set B in $L^2(\Omega)$ and each number $\epsilon > 0$ there exists a finite time $T_{B,\epsilon} > 0$ such that*

$$dist_{L^2(\Omega)}(U(t+\tau, \tau)z, \mathcal{A}) < \epsilon$$

for all $\tau \in R_+$, all $z \in \overline{K_p(\tau)} \cap B$ and all $t \geq T_{B,\epsilon}$.

5 Attractor of asymptotically periodic problems

In this section we shall construct a global attractor for the asymptotically T_0-periodic double obstacle problems $(\mathrm{AP})_s$.

In section 3 we see that $(\mathrm{AP})_s$ has at least one solution on $[s, +\infty)$. So we can define a solution operator $E(t, s)$ $(0 \leq s \leq t < +\infty)$ for $(\mathrm{AP})_s$. But we cannot show the uniqueness of solutions for $(\mathrm{AP})_s$ on $[s, +\infty)$. Therefore $E(t, s)$ is multivalued, that is, $E(t, s)$ $(0 \leq s \leq t < +\infty)$ is the operator from $\overline{K(s)}$ into $\overline{K(t)}$ which assigns to each $u_0 \in \overline{K(s)}$ the set

$$E(t,s)u_0 := \left\{ z \in L^2(\Omega) \;\middle|\; \begin{array}{c} \text{There is a solution } \{u,\ \theta\} \text{ of } (\mathrm{AP})_s \text{ on } [s,+\infty) \\ \text{such that} \\ u(s) = u_0 \text{ and } u(t) = z. \end{array} \right\}.$$

Then we easily see that $\{E(t, s)\ ;\ 0 \leq s \leq t < +\infty\}$ satisfies the following evolution properties :

(E1) $E(s, s) = I$ on $\overline{K(s)}$ for any $s \geq 0$.

(E2) $E(t_2, s)z = E(t_2, t_1)E(t_1, s)z$ for any $0 \leq s \leq t_1 \leq t_2 < +\infty$ and $z \in \overline{K(s)}$.

(E3) $E(t, s)$ has the following demiclosedness:

- Assume that $s_n, s, t_n, t \in R_+$ with $s_n \to s$ and $t_n \to t$, $u_{0n} \in \overline{K(s_n)}$, $u_0 \in \overline{K(s)}$ with $u_{0n} \to u_0$ in $L^2(\Omega)$ and a element $z_n \in E(t_n + s_n, s_n)u_{0n}$ converges to some element z in $L^2(\Omega)$ as $n \to +\infty$. Then, $z \in E(t + s, s)u_0$.

In order to construct a global attractor for $\{E(t,s)\ ;\ 0 \le s \le t < +\infty\}$ associated with $(\mathrm{AP})_s$, we give a definition of a discrete ω-limit set under $E(t,s)$.

Definition 5.1. (Discrete ω-limit set for $E(\cdot,\cdot)$) Let $\mathcal{B}(L^2(\Omega))$ be a family of bounded subsets of $L^2(\Omega)$. Let $\tau \in R_+$ be fixed. Then for each $B \in \mathcal{B}(L^2(\Omega))$, the set

$$\omega_\tau(B) := \bigcap_{n \in Z_+} \overline{\bigcup_{k \ge n, m \in Z_+} E(kT_0 + mT_0 + \tau, mT_0 + \tau)(\overline{K(mT_0 + \tau)} \cap B)}$$

is called the discrete ω-limit set of B under $E(\cdot,\cdot)$.

Remark 5.1. By definition of the discrete ω-limit set $\omega_\tau(B)$, it is easy to see that $x \in \omega_\tau(B)$ if and only if there exist sequences $\{k_n\} \subset Z_+$ with $k_n \uparrow +\infty$, $\{m_n\} \subset Z_+$, $\{z_n\} \subset B$ with $z_n \in \overline{K(m_nT_0 + \tau)}$ and $\{x_n\} \subset L^2(\Omega)$ with $x_n \in E(k_nT_0 + m_nT_0 + \tau, m_nT_0 + \tau)z_n$ such that $x_n \longrightarrow x$ in $L^2(\Omega)$ as $n \to +\infty$.

Now let us mention main theorems in this paper.

Theorem 5.1. (Discrete attractors of $(\mathrm{AP})_\tau$) *Suppose the conditions (A1)-(A3). For each $\tau \in R_+$, let $\mathcal{A}_\tau$ be the global attractor of T_0-periodic dynamical systems U_τ, which is obtained in section 4. Here we put*

$$\mathcal{A}^*_\tau := \overline{\bigcup_{B \in \mathcal{B}(L^2(\Omega))} \omega_\tau(B)}. \tag{5.1}$$

Then, we have

(i) $\mathcal{A}^*_\tau (\subset K_p(\tau))$ *is non-empty and compact in* $L^2(\Omega)$*;*

(ii) *for each bounded set $B \in \mathcal{B}(L^2(\Omega))$ and each number $\epsilon > 0$ there exists a positive number $N_{B,\epsilon} \in N$ such that*

$$dist_{L^2(\Omega)}(E(kT_0 + \tau, \tau)z, \mathcal{A}^*_\tau) < \epsilon$$

for all $z \in \overline{K(\tau)} \cap B$ and all $k \ge N_{B,\epsilon}$;

(iii) $\mathcal{A}^*_\tau \subset U^i_\tau \mathcal{A}^*_\tau \subset \mathcal{A}_\tau$ *for any $i \in N$.*

Remark 5.2. By the definition of $\omega_\tau(B)$ and $\mathcal{A}^*_\tau$, we easily see that $\mathcal{A}^*_\tau = \mathcal{A}^*_{\tau+nT_0}$ for any number $n \in N$. Hence $\mathcal{A}^*_\tau$ is T_0-periodic in time.

Our second main theorem gives a relationship between global attractors $\mathcal{A}^*_s$ and $\mathcal{A}^*_\tau$.

Theorem 5.2. *Suppose the conditions (A1)-(A3). Let $\mathcal{A}^*_s$ and $\mathcal{A}^*_\tau$ be discrete attractors for $E(\cdot,s)$ and $E(\cdot,\tau)$ with $0 \le s \le \tau < +\infty$, respectively. Then, we have $\mathcal{A}^*_\tau \subset U(\tau,s)\mathcal{A}^*_s$, where $U(\tau,s)$ is the T_0-periodic process given in section 4.*

By Theorems 5.1 and 5.2, we can construct the attractor for asymptotic T_0-periodic problems $(\mathrm{AP})_\tau$.

Theorem 5.3. (Global attractor of $(\mathrm{AP})_\tau$) *Assume (A1)-(A3). For any $\tau \in R_+$, let $\mathcal{A}^*_\tau$ be the discrete attractors for $E(\cdot,\tau)$ obtained in Theorem 5.1. Here we put*

$$\mathcal{A}^* := \bigcup_{\tau \in [0,T_0]} \mathcal{A}^*_\tau. \tag{5.2}$$

Then, for any bounded set $B \in \mathcal{B}(H)$,

$$\omega_E(B) := \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s, \tau\in R_+} E(t+\tau,\tau)(\overline{K(\tau)} \cap B)} \subset \mathcal{A}^*. \tag{5.3}$$

By Theorem 5.3, we can say that the set $\mathcal{A}^*$ can be called the attractor of $(\mathrm{AP})_\tau$.

In order to prove Theorems 5.1-5.3, we prepare some lemmas.

Lemma 5.1. *If $\{s_n\} \subset R_+$, $\{\tau_n\} \subset R_+$, $s \in R_+$, $\tau \in R_+$, $s_n \to s$, $\tau_n \to \tau$, $\{m_n\} \subset Z_+$ with $m_n \to +\infty$, $z_n \in \overline{K(m_nT_0+s_n)}$, $z \in \overline{K_p(s)}$, $z_n \to z$ in $L^2(\Omega)$ and a element $w_n \in E(m_nT_0+\tau_n+s_n, m_nT_0+s_n)z_n$ converges to some element $w \in L^2(\Omega)$ as $n \to +\infty$, then $w \in U(\tau+s,s)z$*

Proof. Since $\tau_n \to \tau$, we may assume that $\{\tau_n\} \subset [0,T]$ for some $T>0$.

By $w_n \in E(m_nT_0+\tau_n+s_n, m_nT_0+s_n)z_n$, there exists a solution $\{v_n,\theta_n\}$ of $(\mathrm{AP})_{m_nT_0+s_n}$ such that

$$v_n(m_nT_0+\tau_n+s_n) = w_n \text{ and } v_n(m_nT_0+s_n) = z_n.$$

Here we put $u_n(t) := v_n(t+m_nT_0+s_n)$. Then, we easily see that the function u_n is the solution for

$$(\mathrm{AP})_0 \begin{cases} u_n'(t) - \mathbf{div}(|\nabla u_n(t)|^{q-2}\nabla u_n(t)) - g(u_n(t)) = \theta_n(t+m_nT_0+s_n,x) \quad \text{in } Q_0; \\ 0 \leq \theta_n(t+m_nT_0+s_n,x) \leq h(t+m_nT_0+s_n,u_n(t,x)) \quad \text{a.e. on } (0,+\infty)\times\Omega; \\ u_n(t) = l(t+m_nT_0+s_n) \quad \text{a.e. on } (0,+\infty)\times\Gamma; \\ u_n(0) = z_n \quad \text{in } \Omega. \end{cases}$$

Let $\delta \in (0,1)$ be fixed. Since $z_n \to z$ in $L^2(\Omega)$ as $n \to +\infty$, $\{z_n\}$ is bounded in $L^2(\Omega)$. Therefore, by Theorem 3.1 there exists a positive constant $N_\delta > 0$ such that

$$\begin{aligned} &\sup_{t\geq\delta} |u_n(t)|^2_{L^2(\Omega)} + \sup_{t\geq\delta}\int_t^{t+1} |\nabla u_n(\tau)|^q_{L^q(\Omega)} d\tau \\ &+ \sup_{t\geq\delta} |u_n'|^2_{L^2(t,t+1;L^2(\Omega))} + \sup_{t\geq\delta} |\nabla u_n(t)|^q_{L^q(\Omega)} \leq N_\delta. \end{aligned} \tag{5.4}$$

Here it follows from the convergence assumption (A2), (A3) and (5.4) that (by taking a subsequence of $\{n\}$, if necessary) there are functions u_δ and θ_δ such that

$$\begin{cases} u_\delta'(t) - \mathbf{div}(|\nabla u_\delta(t)|^{q-2}\nabla u_\delta(t)) - g(u_\delta(t)) = \theta_\delta(t+s,x) \quad \text{in } [\delta,+\infty)\times\Omega; \\ 0 \leq \theta_\delta(t+s,x) \leq h_p(t+s,u_\delta(t,x)) \quad \text{a.e. on } (\delta,+\infty)\times\Omega; \\ u_\delta(t) = l_p(t+s) \quad \text{a.e. on } (\delta,+\infty)\times\Gamma. \end{cases}$$

By the standard diagonal process, we can get the solution $\{u,\theta\}$ for $(\mathrm{PP})_{T_0}$ such that

$$(\mathrm{PP})_{T_0}\begin{cases} u'(t)-\mathbf{div}(|\nabla u(t)|^{q-2}\nabla u(t))-g(u(t))=\theta(t+s,x) \quad \text{in } Q_0;\\ 0\le \theta(t+s,x)\le h_p(t+s,u(t,x)) \quad \text{a.e. on } (0,+\infty)\times\Omega;\\ u(t)=l_p(t+s) \quad \text{a.e. on } (0,+\infty)\times\Gamma;\\ u(0)=z\end{cases}$$

and

$$u_n \longrightarrow u \text{ in } C([0,T];H) \text{ as } n\to+\infty. \tag{5.5}$$

Therefore, it follows from (5.5) and $u_n(\tau_n)=w_n$ that $u(\tau)=w$. Hence we have $w\in U(\tau+s,s)z$. ♢

Lemma 5.2. *Let $\tau\in R_+$ and $B_{0,\tau}$ be the compact absorbing set for U_τ. Then*

$$\omega_\tau(B)\subset B_{0,\tau},\quad \forall B\in\mathcal{B}(L^2(\Omega)). \tag{5.6}$$

Proof. For simplicity, at first let us consider the case of $\tau\in[0,T_0]$. Let us fix a bounded subset $B\in\mathcal{B}(L^2(\Omega))$. By the global boundedness result obtained in Theorem 3.1, there is a constant $N_B>0$ such that

$$\begin{aligned}&\sup_{t\ge s}|u(t)|^2_{L^2(\Omega)}+\sup_{t\ge 0}\int_t^{t+1}|\nabla u(\tau)|^q_{L^q(\Omega)}d\tau\\ &+\sup_{t\ge s+T_0}|u'|^2_{L^2(t,t+1;L^2(\Omega))}+\sup_{t\ge s+T_0}|\nabla u(t)|^q_{L^q(\Omega)}\le N_B,\end{aligned} \tag{5.7}$$

for the solution u of $(\mathrm{AP})_s$ on $[s,+\infty)$ with initial value z as long as $s\ge 0$ and $z\in\overline{K(s)}\cap B$.

Here for each $m\in Z_+$, $\tau\in[0,T_0]$, $n\in N$, $z\in\overline{K(mT_0+\tau)}\cap B$ and $w\in E(nT_0+mT_0+\tau,mT_0+\tau)z$, we put $\widetilde{w}:=w-l(nT_0+mT_0+\tau)+l_p(\tau)$. Then $\widetilde{w}\in K_p(\tau)$ and

$$\begin{aligned}&|\widetilde{w}-w|_{L^2(\Omega)}\le C_1J_{m+n},\\ &(\text{hence } |\widetilde{w}|_{L^2(\Omega)}\le\sqrt{N_B}+C_1J_{m+n})\end{aligned} \tag{5.8}$$

and

$$|\nabla\widetilde{w}|_{L^q(\Omega)}\le N_B^{\frac{1}{q}}+J_{m+n}, \tag{5.9}$$

where $C_1:=\mathrm{meas.}(\Omega)^{\frac{q-2}{2q}}$.

Since J_k converges to 0 as $k\to+\infty$, there exists a positive number $N_0\in N$ such that

$$J_k\le 1,\quad \forall k>N_0.$$

Here we put $J_0:=1+\sup_{1\le k\le N_0}J_k<+\infty$.

Now, we denote the set $\widetilde{B_\tau}$ by

$$\widetilde{B_\tau}:=\{z\in L^2(\Omega);|z|_{L^2(\Omega)}\le\sqrt{N_B}+C_1J_0\}\cap\overline{K_p(\tau)} \tag{5.10}$$

Since $B_{0,\tau}$ is the absorbing set for U_τ, there is a positive number $\widetilde{N}\in N$ such that

$$U_\tau^l\widetilde{B_\tau}\subset B_{0,\tau},\quad \forall l\ge\widetilde{N}. \tag{5.11}$$

Now, let us prove (5.6). Let x be any element of $\omega_\tau(B)$. Then, by Remark 5.1 we see that there exist sequences $\{k_n\} \subset Z_+$ with $k_n \uparrow +\infty$, $\{m_n\} \subset Z_+$, $\{z_n\} \subset B$ with $z_n \in \overline{K(m_nT_0+\tau)}$ and $\{x_n\} \subset L^2(\Omega)$ with $x_n \in E(k_nT_0+m_nT_0+\tau, m_nT_0+\tau)z_n$ such that

$$x_n \longrightarrow x \text{ in } L^2(\Omega) \quad \text{as } n \to +\infty. \tag{5.12}$$

Let $\widetilde{N}$ be the positive number obtained in (5.11). It follows from (E2) that

$$\begin{aligned} x_n \in\ & E(k_nT_0+m_nT_0+\tau, k_nT_0+m_nT_0+\tau-\widetilde{N}T_0) \\ & \circ E(k_nT_0+m_nT_0+\tau-\widetilde{N}T_0, m_nT_0+\tau)z_n \\ & \text{for any } n \text{ with } k_n \geq \widetilde{N}+1. \end{aligned} \tag{5.13}$$

By (5.13) there is a element $y_n \in E(k_nT_0+m_nT_0+\tau-\widetilde{N}T_0, m_nT_0+\tau)z_n$ such that

$$x_n \in E(k_nT_0+m_nT_0+\tau, k_nT_0+m_nT_0+\tau-\widetilde{N}T_0)y_n. \tag{5.14}$$

Here we note that

$$|y_n|^2_{L^2(\Omega)} \leq N_B \text{ and } |\nabla y_n|^q_{L^q(\Omega)} \leq N_B \text{ for any } n \text{ with } k_n \geq \widetilde{N}+1,$$

where N_B is the same positive constant in (5.7).

It follows from (5.8)-(5.9) that for $y_n \in E(k_nT_0+m_nT_0+\tau-\widetilde{N}T_0, m_nT_0+\tau)z_n$ we can take $\widetilde{y}_n := y_n - l(k_nT_0+m_nT_0+\tau-\widetilde{N}T_0) + l_p(\tau) \in K_p(\tau)$ satisfying

$$|\widetilde{y}_n|_{L^2(\Omega)} \leq \sqrt{N_B} + C_1 J_{k_n+m_n-\widetilde{N}} \quad \text{and} \quad |\nabla \widetilde{y}_n|_{L^q(\Omega)} \leq N_B^{\frac{1}{q}} + J_{k_n+m_n-\widetilde{N}}.$$

Clearly, $\{\widetilde{y}_n \in K_p(\tau)\ ;\ n \in N \text{ with } k_n \geq \widetilde{N}+1\} (\subset \widetilde{B_\tau})$ is relatively compact in $L^2(\Omega)$, hence we may assume that

$$\widetilde{y}_n \longrightarrow \widetilde{y}_\infty \text{ in } L^2(\Omega) \quad \text{as } n \to +\infty$$

for some $\widetilde{y}_\infty \in L^2(\Omega)$; it is easily see that $\widetilde{y}_\infty \in \widetilde{B_\tau}$ and

$$y_n \longrightarrow \widetilde{y}_\infty \text{ in } L^2(\Omega) \quad \text{as } n \to +\infty. \tag{5.15}$$

Here, applying Lemma 5.1, it follows from (5.12)-(5.15) that

$$x \in U(\widetilde{N}T_0+\tau,\tau)\widetilde{y}_\infty \subset U(\widetilde{N}T_0+\tau,\tau)\widetilde{B_\tau} = U_\tau^{\widetilde{N}}\widetilde{B_\tau} \subset B_{0,\tau}.$$

Therefore we observe that $\omega_\tau(B) \subset B_{0,\tau}$.

For the general case of $\tau \in R_+$ there are positive numbers $i_\tau \in N$ and $\tau_0 \in [0, T_0]$ such that $\tau = \tau_0 + i_\tau T_0$. Therefore, by the same argument as above, we can prove (5.6). ♢

Proof of Theorem 5.1. By Lemma 5.2 we easily see that $\mathcal{A}^*_\tau \subset B_{0,\tau}$, hence, Theorem 5.1 (i) holds. Also, it follows from (5.1) and Remark 5.1 that Theorem 5.1 (ii) holds.

Now, let us prove Theorem 5.1 (iii). At first, we show that $\mathcal{A}^*_\tau \subset U^i_\tau \mathcal{A}^*_\tau$ for any $i \in N$. To do so, let x be any element of $\mathcal{A}^*_\tau$. By the definition of $\mathcal{A}^*_\tau$, we may assume that there exist sequences $\{B_n\} \subset \mathcal{B}(L^2(\Omega))$ and $\{x_n\} \subset L^2(\Omega)$ with $x_n \in \omega_\tau(B_n)$ such that

$$x_n \longrightarrow x \text{ in } L^2(\Omega) \quad \text{as } n \to +\infty. \tag{5.16}$$

It follows from Remark 5.1 that for each n, there exist sequences $\{k_{n,j}\} \subset Z_+$ with $k_{n,j} \to +\infty$, $\{m_{n,j}\} \subset Z_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in \overline{K(m_{n,j}T_0+\tau)}$ and $\{v_{n,j}\} \subset L^2(\Omega)$ with $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau,\ m_{n,j}T_0 + \tau) z_{n,j}$ such that

$$v_{n,j} \longrightarrow x_n \text{ in } L^2(\Omega) \quad \text{ as } j \to +\infty. \tag{5.17}$$

Let i be any number in N. We note that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau,\ k_{n,j}T_0 + m_{n,j}T_0 + \tau - iT_0)$$

$$\circ E(k_{n,j}T_0 + m_{n,j}T_0 + \tau - iT_0,\ m_{n,j}T_0 + \tau) z_{n,j}$$

for j with $k_{n,j} \geq i+1$. Hence there is a $w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau - iT_0,\ m_{n,j}T_0 + \tau) z_{n,j}$ such that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau,\ k_{n,j}T_0 + m_{n,j}T_0 + \tau - iT_0) w_{n,j}. \tag{5.18}$$

For each n, by Theorem 3.1, the set $\{w_{n,j} \in L^2(\Omega)\ ;\ j \in N \text{ with } k_{n,j} \geq i+1\}$ is relatively compact in $L^2(\Omega)$. So, we may assume that the element $w_{n,j}$ converges to some element $\widetilde{w}_{n,\infty} \in L^2(\Omega)$ as $j \to +\infty$. Clearly, $\widetilde{w}_{n,\infty} \in \omega_\tau(B_n)$. Moreover, from Lemma 5.1 and (5.17)-(5.18), we observe that

$$x_n \in U(iT_0+\tau, \tau)\widetilde{w}_{n,\infty} \subset U(iT_0+\tau,\tau)\omega_\tau(B_n),$$

which implies that

$$x_n \in \bigcup_{n\geq 1} U_\tau^i \omega_\tau(B_n), \quad \forall n \geq 1. \tag{5.19}$$

Moreover, by the closedness of $U(\cdot,\cdot)$, we observe that for each subset X of $B_{0,\tau}$,

$$\overline{U_\tau^i X} \subset U_\tau^i \overline{X}, \quad \forall i \in N. \tag{5.20}$$

Since Lemma 5.2, (5.16), (5.19) and (5.20), we see that

$$x \in \overline{\bigcup_{n\geq 1} U_\tau^i \omega_\tau(B_n)} = \overline{U_\tau^i \bigcup_{n\geq 1} \omega_\tau(B_n)} \subset U_\tau^i \overline{\bigcup_{n\geq 1} \omega_\tau(B_n)} \subset U_\tau^i \mathcal{A}_\tau^*.$$

Hence we observe that $\mathcal{A}_\tau^*$ is semi-invariant under the T_0-periodic dynamical systems U_τ, namely

$$\mathcal{A}_\tau^* \subset U_\tau^i \mathcal{A}_\tau^*, \quad \forall i \in N. \tag{5.21}$$

Next we shall show that $U_\tau^i \mathcal{A}_\tau^* \subset \mathcal{A}_\tau$ for any $i \in N$. By (5.21), for each $i \in N$

$$U_\tau^i \mathcal{A}_\tau^* \subset U_\tau^i U_\tau^n \mathcal{A}_\tau^* = U_\tau^{i+n} \mathcal{A}_\tau^*, \quad \forall n \in N. \tag{5.22}$$

Since $\mathcal{A}_\tau^* \subset B_{0,\tau}$, from (5.22) and the attractive property of $\mathcal{A}_\tau$ it follows that

$$U_\tau^i \mathcal{A}_\tau^* \subset \mathcal{A}_\tau, \quad \forall i \in N,$$

hence we conclude that $\mathcal{A}_\tau^* \subset U_\tau^i \mathcal{A}_\tau^* \subset \mathcal{A}_\tau$ for any $i \in N$. $\diamondsuit$

Proof of Theorem 5.2. Let x be any element of $\mathcal{A}_\tau^*$. Then by (5.1), we see that there exist sequences $\{B_n\} \subset \mathcal{B}(L^2(\Omega))$ and $\{x_n\} \subset L^2(\Omega)$ with $x_n \in \omega_\tau(B_n)$ such that

$$x_n \longrightarrow x \text{ in } L^2(\Omega) \quad \text{ as } n \to +\infty. \tag{5.23}$$

It follows from Remark 5.1 that for each n, there exist sequences $\{k_{n,j}\} \subset Z_+$ with $k_{n,j} \to +\infty$, $\{m_{n,j}\} \subset Z_+$, $\{z_{n,j}\} \subset B_n$ with $z_{n,j} \in \overline{K(m_{n,j}T_0 + \tau)}$ and $\{v_{n,j}\} \subset L^2(\Omega)$ with $v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau,\ m_{n,j}T_0 + \tau)z_{n,j}$ such that

$$v_{n,j} \longrightarrow x_n \text{ in } L^2(\Omega) \quad \text{ as } j \to +\infty. \tag{5.24}$$

Note that for given $s, \tau \in R_+$ with $s \le \tau$, we can take a positive number $i_s \in N$ satisfying

$$s \le \tau \le s + i_s T_0.$$

From (E2) it follows that

$$\begin{aligned} v_{n,j} \in E(&k_{n,j}T_0 + m_{n,j}T_0 + \tau,\ k_{n,j}T_0 + m_{n,j}T_0 + s) \\ &\circ E(k_{n,j}T_0 + m_{n,j}T_0 + s,\ i_sT_0 + m_{n,j}T_0 + s + T_0) \\ &\circ E(i_sT_0 + m_{n,j}T_0 + s + T_0,\ m_{n,j}T_0 + \tau)z_{n,j} \end{aligned}$$

for any $j \in Z_+$ with $k_{n,j} \ge i_s + 2$. So, there are element $w_{n,j} \in L^2(\Omega)$ and $y_{n,j} \in L^2(\Omega)$ such that

$$v_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + \tau,\ k_{n,j}T_0 + m_{n,j}T_0 + s)w_{n,j}, \tag{5.25}$$

$$w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s,\ i_sT_0 + m_{n,j}T_0 + s + T_0)y_{n,j} \tag{5.26}$$

and

$$y_{n,j} \in E(i_sT_0 + m_{n,j}T_0 + s + T_0,\ m_{n,j}T_0 + \tau)z_{n,j}. \tag{5.27}$$

Since $\{z_{n,j}\} \subset B_n$, it follows from the global boundedness results in Theorem 3.1 that there is a positive constant $C_n := C_n(B_n) > 0$ satisfying

$$|y_{n,j}|_{L^2(\Omega)} \le C_n, \quad \forall y_{n,j} \in E(i_sT_0 + m_{n,j}T_0 + s + T_0,\ m_{n,j}T_0 + \tau)z_{n,j}. \tag{5.28}$$

By (5.28) and Theorem 3.1, the set

$$\left\{ w_{n,j} \in L^2(\Omega)\ ;\ \begin{array}{c} w_{n,j} \in E(k_{n,j}T_0 + m_{n,j}T_0 + s,\ i_sT_0 + m_{n,j}T_0 + s + T_0)y_{n,j} \\ \text{for any } j \in Z_+ \text{ with } k_{n,j} \ge i_s + 2 \end{array} \right\}$$

is relatively compact in $L^2(\Omega)$. So, we may assume that the element $w_{n,j}$ converges to some element $\widetilde{w}_{n,\infty} \in L^2(\Omega)$ as $j \to +\infty$. Clearly, $\widetilde{w}_{n,\infty} \in \omega_s(B_{C_n})$, where $B_{C_n} := \{b \in L^2(\Omega)\ ; |b|_{L^2(\Omega)} \le C_n\}$. Moreover, by Lemma 5.2, we see that

$$\omega_s(B_{C_n}) \subset B_{0,s} \subset \overline{K_p(s)},$$

where $B_{0,s}$ is the compact absorbing set for U_s. Also, by Lemma 5.1 and (5.24)-(5.25) we have

$$x_n \in U(\tau, s)\widetilde{w}_{n,\infty} \subset U(\tau, s)\omega_s(B_{C_n}), \quad \forall n \ge 1,$$

which implies that

$$x_n \in \bigcup_{n\geq 1} U(\tau,s)\omega_s(B_{C_n}), \quad \forall n \geq 1. \tag{5.29}$$

Moreover, by the closedness of $U(\cdot,\cdot)$, we observe that for each subset X of $B_{0,s}$,

$$\overline{U(\tau,s)X} \subset U(\tau,s)\overline{X}. \tag{5.30}$$

Since Lemma 5.2, (5.23), (5.29) and (5.30), we see that

$$x \in \overline{\bigcup_{n\geq 1} U(\tau,s)\omega_s(B_{C_n})} = \overline{U(\tau,s)\bigcup_{n\geq 1}\omega_s(B_{C_n})} \subset U(\tau,s)\overline{\bigcup_{n\geq 1}\omega_s(B_{C_n})} \subset U(\tau,s)\mathcal{A}_s^*.$$

Hence we observe that $\mathcal{A}_\tau^*$ is the subset of $U(\tau,s)\mathcal{A}_s^*$, namely $\mathcal{A}_\tau^* \subset U(\tau,s)\mathcal{A}_s^*$. ♢

Proof of Theorem 5.3. For any $B \in \mathcal{B}(L^2(\Omega))$, let z_0 be any element of $\omega_E(B)$. Then there exist sequences $\{t_n\} \subset R_+$ with $t_n \uparrow +\infty$, $\{\tau_n\} \subset R_+$, $\{y_n\} \subset B$ with $y_n \in \overline{K(\tau_n)}$ and $\{z_n\} \subset L^2(\Omega)$ with $z_n \in E(t_n+\tau_n,\ \tau_n)y_n$ such that

$$\begin{aligned} &t_n := k_nT_0 + t'_n,\ k_n \in Z_+,\ k_n \nearrow +\infty,\ t'_n \in [T_0, 2T_0],\ t'_n \to t'_0,\\ &\tau_n := i_nT_0 + \tau'_n,\ i_n \in Z_+,\ \tau'_n \in [0,T_0],\ \tau'_n \to \tau'_0 \end{aligned}$$

and

$$z_n \longrightarrow z_0 \quad \text{ in } L^2(\Omega) \tag{5.31}$$

as $n \to +\infty$; we may assume further that

$$\text{(a)} \quad t'_n + \tau'_n \nearrow t'_0 + \tau'_0 \quad \text{ or } \quad \text{(b)} \quad t'_n + \tau'_n \searrow t'_0 + \tau'_0.$$

Assume that (a) holds. Let us consider the semiflow

$$v_n \in E(1 + k_nT_0 + i_nT_0 + t'_0 + \tau'_0,\ k_nT_0 + i_nT_0 + t'_n + \tau'_n)z_n. \tag{5.32}$$

Then, there exists functions u_n and θ_n such that

$$\begin{cases} u'_n(t) - \mathbf{div}(|\nabla u_n(t)|^{q-2}\nabla u_n(t)) - g(u_n(t)) = \theta_n(t,x) \quad \text{ in } [0,+\infty)\times\Omega,\\ 0 \leq \theta_n(t,x) \leq h(t + k_nT_0 + i_nT_0 + t'_n + \tau'_n, u_n(t,x)) \quad \text{ a.e. on } (0,+\infty)\times\Omega,\\ u_n(t) = l(t + k_nT_0 + i_nT_0 + t'_n + \tau'_n) \quad \text{a.e. on } (0,+\infty)\times\Gamma,\\ u_n(0) = z_n \quad \text{ in } \Omega,\\ u_n(1 + t'_0 + \tau'_0 - t'_n - \tau'_n) = v_n. \end{cases}$$

Since $z_n \to z_0$ in $L^2(\Omega)$, $\{z_n\}$ is bounded in $L^2(\Omega)$, hence we see that

$$\left\{ v_n \in L^2(\Omega);\ \begin{array}{c} v_n \in E(1 + k_nT_0 + i_nT_0 + t'_0 + \tau'_0,\ k_nT_0 + i_nT_0 + t'_n + \tau'_n)z_n \\ \text{for any } n \in N \end{array} \right\}$$

is relatively compact in $L^2(\Omega)$. So we may assume that

$$v_n \longrightarrow v \text{ in } L^2(\Omega) \text{ for some } v \in L^2(\Omega). \tag{5.33}$$

Therefore, by Lemma 5.1 and (5.31)-(5.33), we have

$$v \in U(1 + t_0' + \tau_0',\ t_0' + \tau_0')z_0,$$

more precisely, (taking the subsequence of $\{n\}$ if necessary) there are functions u and θ such that

$$(\mathrm{PP})_{T_0} \quad \begin{cases} u'(t) - \mathbf{div}(|\nabla u(t)|^{q-2}\nabla u(t)) - g(u(t)) = \theta(t,x) \quad \text{in } [0,+\infty) \times \Omega, \\ 0 \le \theta(t,x) \le h_p(t + t_0' + \tau_0', u(t,x)) \quad \text{a.e. on } (0,+\infty) \times \Omega, \\ u(t) = l_p(t + t_0' + \tau_0') \quad \text{a.e. on } (0,+\infty) \times \Gamma, \\ u(0) = z_0 \quad \text{in } \Omega, \\ u(1) = v. \end{cases}$$

and

$$u_n \longrightarrow u \quad \text{in } C([0,2]; L^2(\Omega)) \quad \text{as } n \to +\infty. \tag{5.34}$$

By (5.34), we easily observe that

$$u_n(t_0' + \tau_0' - t_n' - \tau_n') \longrightarrow z_0 \quad \text{as } n \to +\infty. \tag{5.35}$$

Here, we note that

$$\begin{aligned} &u_n(t_0' + \tau_0' - t_n' - \tau_n') \\ &\in E(k_nT_0 + i_nT_0 + t_0' + \tau_0',\ k_nT_0 + i_nT_0 + t_n' + \tau_n')z_n \\ &= E(k_nT_0 + i_nT_0 + t_0' + \tau_0',\ i_nT_0 + t_0' + \tau_0')E(i_nT_0 + t_0' + \tau_0',\ i_nT_0 + \tau_n')y_n, \end{aligned}$$

hence there is a element $x_n \in E(i_nT_0 + t_0' + \tau_0',\ i_nT_0 + \tau_n')y_n$ such that

$$u_n(t_0' + \tau_0' - t_n' - \tau_n') \in E(k_nT_0 + i_nT_0 + t_0' + \tau_0',\ i_nT_0 + t_0' + \tau_0')x_n. \tag{5.36}$$

Clearly, by the global estimate of solutions, $\{x_n\}$ is bounded, i.e.

$$\{x_n\} \subset \widetilde{B} \text{ for some } \widetilde{B} \in \mathcal{B}(L^2(\Omega)). \tag{5.37}$$

Hence it follows from (5.35)-(5.37) and Remark 5.2 that

$$z_0 \in \omega_{t_0'+\tau_0'}(\widetilde{B}) \subset \mathcal{A}^*_{t_0'+\tau_0'} \subset \mathcal{A}^*.$$

Thus (5.3) holds. Assuming that (b) holds, we similarly get (5.3). $\diamondsuit$

Theorem 5.1 says that the attracting set $\mathcal{A}^*_\tau$ for $(\mathrm{AP})_s$ is semi-invariant under U_τ associated with the limiting T_0-periodic problem $(\mathrm{PP})_{T_0}$, in general. Moreover, in Theorem 5.2 we see that $\mathcal{A}^*_\tau \subset U(\tau,s)\mathcal{A}^*_s$.

In order to get the invariance of $\mathcal{A}^*_\tau$ under U_τ and $\mathcal{A}^*_\tau = U(\tau,s)\mathcal{A}^*_s$, we have to assume the additional conditions for l and h.

Theorem 5.4 *Suppose all conditions (A1)-(A3). Let $\mathcal{A}^*_s$ and $\mathcal{A}^*_\tau$ be discrete attractors*

for $E(\cdot,s)$ and $E(\cdot,\tau)$, with $0 \le s \le \tau < +\infty$, respectively. Furthermore we assume that the boundary condition $l(t)$ for $(AP)_s$ coincides with $l_p(t)$, namely $l(t) \equiv l_p(t)$ on Γ for any $t \ge 0$. And we suppose that $h_p(t,z) \le h(t,z)$ for any $0 \le t < +\infty$ and $z \in R$. Then,
(i) $\mathcal{A}^*_\tau = U(\tau,s)\mathcal{A}^*_s$ for any $0 \le s \le \tau < +\infty$.
(ii) $\mathcal{A}^*_\tau = \mathcal{A}_\tau$ for any $\tau \in R_+$, where $\mathcal{A}_\tau$ is the discrete attractor of U_τ for $(PP)_{T_0}$.

Proof. Let us show (i). By taking account of Theorem 5.2, we have only to show that $U(\tau,s)\mathcal{A}^*_s \subset \mathcal{A}^*_\tau$. To do so, let x be any element of $U(\tau,s)\mathcal{A}^*_s$.

At first, we note that for each $n \in N$

$$\begin{aligned} U^n_\tau U(\tau,s)\mathcal{A}^*_s &= U(nT_0+\tau,\tau)U(\tau,s)\mathcal{A}^*_s \\ &= U(nT_0+\tau,nT_0+s)U(nT_0+s,s)\mathcal{A}^*_s \\ &= U(\tau,s)U^n_s\mathcal{A}^*_s \\ &\supset U(\tau,s)\mathcal{A}^*_s. \end{aligned} \tag{5.38}$$

By (5.38), there is a element $y_n \in \mathcal{A}^*_s$ such that

$$x \in U^n_\tau U(\tau,s)y_n = U(nT_0+\tau,s)y_n.$$

Therefore, there is a solution $\{u,\theta\}$ of $(\mathrm{PP})_{T_0}$ on $[s,+\infty)$ such that $u(nT_0+\tau) = x$ and $u(s) = y_n$.

Let $\{k_n\} \subset N$ be a sequence with $k_n \to +\infty$ as $n \to +\infty$. Here, we put

$$u_n(\sigma,\cdot\,) := u(\sigma - k_nT_0,\cdot\,) \text{ and } \theta_n(\sigma,\cdot\,) := \theta(\sigma - k_nT_0,\cdot\,)$$

for any $\sigma \ge k_nT_0 + s$. Then, by the assumptions of Theorem 5.4 we see that

$$u_n(\sigma) = u(\sigma - k_nT_0) = l_p(\sigma - k_nT_0) = l_p(\sigma) = l(\sigma) \text{ on } \Gamma$$

and

$$\begin{aligned} 0 \le \theta_n(\sigma,x) = \theta(\sigma - k_nT_0,x) &\le h_p(\sigma - k_nT_0, u(\sigma - k_nT_0,x)) \\ &= h_p(\sigma,u_n(\sigma,x)) \le h(\sigma,u_n(\sigma,x)) \end{aligned}$$

for any $\sigma \ge k_nT_0 + s$ and $x \in \Omega$. Therefore, the pair of functions $\{u_n,\theta_n\}$ is the solution of $(\mathrm{AP})_{k_nT_0+s}$ such that $u_n(nT_0+k_nT_0+\tau) = u(nT_0+\tau) = x$ and $u_n(k_nT_0+s) = u(s) = y_n$, which implies that $x \in E(nT_0+k_nT_0+\tau, k_nT_0+s)y_n$ for any $n \ge 1$. By (E2), we see that

$$\begin{aligned} x &\in E(nT_0+k_nT_0+\tau,\ k_nT_0+s)y_n \\ &= E(nT_0+k_nT_0+\tau,\ T_0+k_nT_0+\tau)E(T_0+k_nT_0+\tau,\ k_nT_0+s)y_n. \end{aligned}$$

Hence there is an element $z_n \in E(T_0+k_nT_0+\tau,\ k_nT_0+s)y_n$ such that

$$x \in E(nT_0+k_nT_0+\tau,\ T_0+k_nT_0+\tau)z_n. \tag{5.39}$$

Since $\{y_n\} \subset \mathcal{A}^*_s$ and the global estimate obtained in Theorem 3.1, we see that $\{z_n\}$ is bounded in $L^2(\Omega)$, namely $\{z_n\} \subset \widetilde{B}$ for some $\widetilde{B} \in \mathcal{B}(L^2(\Omega))$. The above fact (5.39) implies (cf. Remark 5.1) that $x \in \omega_\tau(\widetilde{B}) \subset \mathcal{A}^*_\tau$. Thus $U(\tau,s)\mathcal{A}^*_s \subset \mathcal{A}^*_\tau$, which implies that (i) of Theorem 5.4 holds.

Since $\mathcal{A}_\tau$ is invariant under U_τ (cf. Theorem 4.1 (iii)), by the same argument in (i), we can show (ii). Therefore, Theorem 5.4 has been completed. ◇

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