Factorizations Of Functions In $H^p(T^n)$

By

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Abstract. We are interested in extremal functions in a Hardy space $H^p(T^n)$ $(1 \leq p \leq \infty)$. For example, we study extreme points of the unit ball of $H^1(T^n)$ and give a factorization theorem. In particular, we show that any rational function can be factorized.

§1. Introduction

Let D^n be the open unit polydisc in $\mathbb{C}^n$ and T^n be its distiguished boundary. The normalized Lebesgue measure on T^n is denoted by dm. For $0 < p \le \infty$, $H^p(D^n)$ is the Hardy space and $L^p(T^n)$ is the Lebesgue space on T^n. Let $N(D^n)$ denote the Nevanlinna class. Each f in $N(D^n)$ has radial limits f^* defined on T^n $a.e.dm.$ Moreover, there is a singular measure $d\sigma_f$ on T^n determined by f such that the least harmonic majorant $u(\log|f|)$ of $\log|f|$ is given by $u(\log|f|)(z) = P_z(\log|f^*| + d\sigma_f)$ where P_z denotes Poisson integration and $z = (z_1, z_2, \cdots, z_n) \in D^n$. Put $N_*(D^n) = \{f \in N(D^n)\ ;\ d\sigma_f \le 0\}$, then $H^p(D^n) \subset N_*(D^n) \subset N(D^n)$ and $H^p(D^n) = N_*(D^n) \cap L^p(T^n) \subsetneq N(D^n) \cap L^p(T^n)$. These facts are shown in [5, Theorem 3.3.5].

Let $\mathcal{L}$ be a subset of $L^\infty(T^n)$. For a function f in H^p, put

$$\mathcal{L}_p^f = \{\phi \in \mathcal{L}\ ;\ \phi f \in H^p\}.$$

When $\mathcal{L}_p^f \subseteq H^\infty$, f is called an $\mathcal{L}$-extremal function for H^p. When $\mathcal{L} = L^\infty(T^n)$, $\mathcal{L} = L^\infty_R(T^n)$ or $\mathcal{L} = L^\infty_U(T^n)$ is the set of all unimodular functions, such $\mathcal{L}$-extremal functions have been considered in [3]. In [3], the author studied functions which have harmonic properties (A),(B),(C). For example, the property (A) is the following : If $f \in H^p$ and $|f| \ge |g|$ $a.e.$ on T^n, then $|f| \ge |g|$ on D^n. It is easy to see that f is an $\mathcal{L}$-extremal function for H^p and $\mathcal{L} = L^\infty(T^n)$ if and only if f has the property (A). The propertis (B) and (C) are related to $\mathcal{L} = L^\infty_R(T^n)$ and $\mathcal{L} = L^\infty_U(T^n)$, respectively. In this paper, as $\mathcal{L}$ we consider only the above three sets.

Definition. *When f is not $\mathcal{L}$-extremal for H^p, if there exists a function ϕ in $\mathcal{L}$ such that $\phi f = h$ is an $\mathcal{L}$-extremal function for H^p, we say that f is factorized as $f = \phi^{-1}h$.*

In this paper, we are interested in when f is factorized for $\mathcal{L} = L^\infty(T^n)$ or $\mathcal{L} = L^\infty_R(T^n)$. The function h in $N(D^n)$ is called outer function if

$$\int_{T^n} \log|h| dm = \log\left|\int_{T^n} h dm\right| > -\infty.$$

The function q in $N_*(D^n)$ is called inner function if $|q| = 1$ $a.e.dm$ on T^n. When $\mathcal{L} \subset \mathcal{L}'$, a $\mathcal{L}'$-extremal function is always $\mathcal{L}$-extremal. If f is an outer function, then f is $\mathcal{L}$-extremal for $\mathcal{L} = L^\infty(T^n)$. In fact, if ϕf is in H^p then ϕ belongs to $f^{-1}H^p$ and $f^{-1}H^p \subset N_*$. Hence if ϕ is bounded then ϕ belongs to H^∞ because $N_* \cap L^\infty(T^n) = H^\infty$. When $n = 1$, f is $\mathcal{L}$-extremal if and only if f is an outer function. This is known because f has an inner outer factorization.

In this paper, for a subset S in L^∞ we say that S is of finite dimension if the linear span of S is of finite dimension. We use the follwing notations.

$z = (z_j, z'_j)$, $z'_j = (z_1, \cdots, z_{j-1}, z_{j+1}, \cdots, z_n)$.

$D^n = D_j \times D'_j,\ D'_j = \prod_{\ell \neq j} D_\ell$ where $D^n = \prod_{\ell=1}^n D_\ell$ and $D_\ell = D$.

$T^n = T_j \times T'_j,\ T'_j = \prod_{\ell \neq j} T_\ell$ where $T^n = \prod_{\ell=1}^n T_\ell$ and $T_\ell = T$.

$m = m_j \times m'_j,\ m'_j = \prod_{\ell \neq j} m_\ell$ where $m = \prod_{\ell=1}^n m_\ell$ and m_ℓ is the normarized Lebesgue measure on T_ℓ.

§2. $\mathcal{L} = L_R^\infty$

In this section, we assume that $\mathcal{L} = L_R^\infty$. When $n = 1$, any nonzero function in H^p has a L_R^∞-factorization in H^p by Proposition 1. Even if $n > 1$, we have a lot of L_R^∞-extremal functions for H^p.

Proposition 1. *If $f = qh$ where q is inner and h is L_R^∞-extremal for H^p, then f has a L_R^∞-factorization in H^p : $f = \phi^{-1}k$ where $\phi = q+\bar{q}$ and $k = (1+q^2)h$ is L_R^∞-extremal for H^p.*

Proof. It is enough to show that $(1+q^2)h$ is L_R^∞-extremal for H^p. If $\psi \in L_R^\infty(T^n)$ and $\psi(1+q^2)h \in H^p$ then $\psi(1+q^2)$ belongs to H^∞ because h is L_R^∞-extremal for H^p. Since $1+q^2$ is outer and so $1+q^2$ is L_R^∞-extremal for H^p, ψ belongs to H^∞. □

Proposition 2. *Suppose f is a nonzero function in H^1. f is L_R^∞-extremal for H^1 if and only if $f/\|f\|_1$ is an extreme point of the unit ball of H^1.*

Proof. It is well known. □

The degree of a monomial $z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ (where $\alpha_i \in Z_+$) is $\alpha_1+\cdots+\alpha_n$. The degree of a polynomial P is the maximum of the degrees of the monomials which occur in P with non-zero coefficient. The degree of a rational function $f = P/Q$ is the maximum of deg P, deg Q, provided that all common factors of positive degree have first been cancelled.

Theorem 3. *Let $0 < p \leq \infty$ and $\mathcal{L} = L_R^\infty$. If f is a nonzero function in H^p and $\mathcal{L}_p^f$ is of finite dimension then there exists a function ϕ in $\mathcal{L}$ such that $f = \phi^{-1}h$ and h is $\mathcal{L}$-extremal for H^p.*

Proof. Suppose that $\mathcal{L}_p^f$ is of finite dimension. Then there exist $s_1, s_2, \cdots, s_n$ in L_R^∞ such that $\{s_j\}_{j=1}^n$ is a basis of $\mathcal{L}_p^f$, $s_1 = 1$ and $s_n^{-1} \notin L^\infty$. For if $s_n^{-1} \in L^\infty$ then there exists a real number λ such that $(s_n - \lambda)^{-1} \notin L^\infty$. Then $\{s_1, s_2, \cdots, (s_n - \lambda)\}$ is also a basis.

When $\mathcal{L}_p^{s_n f} = R$, put $\phi = s_n$ and $h = s_n f$, then the theorem is proved. Suppose that $\mathcal{L}_p^{s_n f} \neq R$. If ℓ_1 is a nonconstant function in $\mathcal{L}_p^{s_n f}$ then $\ell_1 s_n$ is nonconstant because $s_n^{-1} \notin L^\infty$. We may assume that $\ell_1^{-1} \notin L^\infty$. When $\mathcal{L}_p^{\ell_1 s_n f} = R$, put $\phi = \ell_1 s_n$ and $h = \ell_1 s_n f$, then the theorem is proved. Suppose that $\mathcal{L}_p^{\ell_1 s_n f} \neq R$. Then there exists ℓ_2 in $\mathcal{L}_p^{\ell_1 s_n f}$ such that $\ell_2 \ell_1 s_n$ is nonconstant and $\ell_2^{-1} \notin L^\infty$. When $\mathcal{L}_p^{\ell_2 \ell_1 s_n f} \neq R$, we can proceed similarly. Put

$$k_j = \ell_j \ell_{j-1} \cdots \ell_1 \quad (j = 1, 2, \cdots, n)$$

where $\ell_i^{-1} \notin L^\infty \ \ (1 \le i \le j)$. Suppose that $\mathcal{L}_p^{k_j s_n f} \neq R$ for $j = 1, 2, \cdots, n$. Hence

$$k_j s_n = \sum_{i=1}^{n} \alpha_{ij} s_i \ \ (j = 1, 2, \cdots, n)$$

and so for $j = 1, 2, \cdots, n$

$$\sum_{i=1}^{n-1} \alpha_{ij} s_i + (\alpha_{nj} - k_j) s_n = 0.$$

Hence

$$\begin{vmatrix} \alpha_{11} & \cdots & \alpha_{n-1\ 1} & \alpha_{n1} - k_1 \\ \alpha_{12} & \cdots & \alpha_{n-1\ 2} & \alpha_{n2} - k_2 \\ \vdots & & \vdots & \vdots \\ \alpha_{1n} & \cdots & \alpha_{n-1\ n} & \alpha_{nn} - k_n \end{vmatrix} = 0$$

and so there exist $\gamma_1, \cdots, \gamma_n$ in $\mathbb{C}$ such that

$$\gamma_1(\alpha_{n1} - k_1) + \gamma_2(\alpha_{n2} - k_2) + \cdots + \gamma_n(\alpha_{nn} - k_n) = 0$$

where

$$\gamma_j = \begin{vmatrix} \alpha_{11} & \cdots & \alpha_{n-1\ 1} \\ \cdots & & \\ \alpha_{1\ j-1} & \cdots & \alpha_{n-1\ j-1} \\ \alpha_{1\ j+1} & \cdots & \alpha_{n-1\ j+1} \\ \cdots & & \\ \alpha_{1n} & \cdots & \alpha_{n-1\ n} \end{vmatrix} < j$$

Hence $\sum_{j=1}^{n} \gamma_j \alpha_{nj} = \sum_{j=1}^{n} \gamma_j k_j$. Here we need the following claim.

Claim *For any t $(1 \le t \le n)$, if $(\delta_1, \cdots, \delta_t) \neq (0, \cdots, 0)$ then $\delta = \sum_{j=1}^{t} \delta_j k_j$ can not be constant.*

Proof. Let s be the smallest integer such that $\delta_s \neq 0$ and $1 \le s \le t$. Then $\delta = \sum_{j=s}^{t} \delta_j k_j$. Hence

$$\delta = \delta_s(\ell_1 \cdots \ell_s) + \cdots + \delta_t(\ell_1 \cdots \ell_s)\ell_{s+1} \cdots \ell_t.$$

If $\delta = 0$, then $0 = \delta_s + \delta_{s+1}\ell_{s+1} + \cdots + \delta_t \ell_{s+1} \cdots \ell_t$ and this contradicts that $\ell_{s+1}^{-1} \notin L^\infty$ because $\delta_s \neq 0$. If δ is a nonzero constant, then this contradicts that $(\ell_1 \cdots \ell_s)^{-1} \notin L^\infty$.

Now we will prove that the equality : $\sum_{j=1}^{n} \gamma_j \alpha_{nj} = \sum_{j=1}^{n} \gamma_j k_j$ contradicts the definition of k_j $(1 \le j \le n)$. If $\gamma_n = 0$, then there exist $(\delta_1, \cdots, \delta_{n-1}) \neq (0, \cdots, 0)$ such that

$\delta_1(\alpha_{11}, \cdots, \alpha_{n-1\ 1}) + \cdots + \delta_{n-1}(\alpha_{1\ n-1}, \cdots, \alpha_{n-1\ n-1}) = (0, \cdots, 0)$. Hence

$$\left(\sum_{j=1}^{n-1} \delta_j \alpha_{nj}\right) s_n = \left(\sum_{j=1}^{n-1} \delta_j k_j\right) s_n$$

because $\sum_{i=1}^{n-1} \alpha_{ij} s_i + \alpha_{nj} s_n = k_j s_n$ for $j = 1, 2, \cdots, n$. Hence $\sum_{j=1}^{n-1} \delta_j \alpha_{nj} = \sum_{j=1}^{n-1} \delta_j k_j$ because $|s_n| > 0$. This contradicts the claim. Hence $\gamma_n \neq 0$. Thus $(\gamma_1, \cdots, \gamma_n) \neq (0, \cdots, 0)$ and $\sum_{j=1}^{n} \gamma_j k_j$ is constant. This also contradicts the claim. Thus $\mathcal{L}^{k_n s_n f} = R$ and so the theorem is proved.

Lemma 1. *Let $p \geq 1$ and f be in H^p. If $f_z(\zeta) = f(\zeta z)$ is a rational function (of one variable) of degree $\leq k_0 < \infty$, for almost all $z \in T^n$ then f is a rational function (of n variables) of degree k and $k \leq k_0$.*

Proof. There exist a nonnegative integer $k \leq k_0$ and a closed set E_k such that $f_z(\zeta)$ is a rational function (of one variable) of degree k for all $z \in E_k$ and E_k is a nonempty interior. We will use [5, Theorem 5.2.2]. In Theorem 5.2.2 in [5], we put $\Omega = D^n$ and $E = E_k$. If $f_z(\zeta)$ is a rational function (of one variable) of degree k for all $z \in E$, then f belongs to Y in Theorem 5.2.2 in [5]. For f_z is in $H^p(D)$, $p \geq 1$ and so f_z is continuous on ∂D. Now Theorem 5.2.2 in [5] implies the lemma. □

Proposition 4. *Suppose $1 \leq p \leq \infty$. $\mathcal{L}_p^f$ is of finite dimension if f is a rational function.*

Proof. Suppose $f = P/Q$ is a nonzero function in H^p where P and Q are polynomials. If $s \in \mathcal{L}_p^f$ then $sP/Q \in H^p$ and so $sP \in H^p$. Hence s belongs to $\mathcal{L}_p^P$ and so $\mathcal{L}_p^f \subseteq \mathcal{L}_p^P$. It is enough to prove that $\mathcal{L}_p^P$ is of finite dimension.

Case $n = 1$. *We have the inner outer factorization for $n = 1$, that is, $P = qh$ where q is a finite Blaschke product and h is an outer function in H^p. Then it is easy to see that $\mathcal{L}_p^P = \mathcal{L}_p^q$. Since $\mathcal{L}_p^q \subset \bar{q}H^p \cap q\bar{H}^p$ and $\mathcal{L}_p^q \subset L^\infty$, $\mathcal{L}_p^q \subset \bar{q}H^2 \cap q\bar{H}^2 = \bar{q}(H^2 \cap q^2\bar{H}^2) = \bar{q}(H^2 \ominus q^2 H_0^2)$. $H^2 \ominus q^2 H_0^2$ is of finite dimension because q is a finite Blaschke product.*

Case $n \neq 1$. *If $s \in \mathcal{L}_p^P$ then $sP \in H^p$ and by Case $n = 1$ $(sP)_z(\zeta)$ is a rational function (of one variable) of degree $\leq k_0$ for almost all $z \in T^n$. By Lemma 1, sP is a rational function (of n variables) of degree k and $k \leq k_0$. This implies that $\mathcal{L}_p^P$ is of finite dimension.* □

When f is a rational function in H^p, by Theorem 3 and Proposition 4 f has our factorization. The function h in $N(D^n)$ is called z_j-outer if

$$\int_{T_j \times T_j'} \log |h(z_j, z_j')| dm = \int_{T_j'} (\log |\int_{T_j} h(z_j, z_j') dm_j|) dm_j' > -\infty.$$

Proposition 5. *Fix $1 \le j \le n$. If $f(z_1, \cdots, z_n)$ is z_i-outer in H^p for $i \ne j$ and $1 \le i \le n$, then f has a factorization in H^p.*

Proof. We will generalize Theorem 2 in [3]. That is, when h is z_i-outer in H^p for $i \ne j$, $\mathcal{L}_p^h = R$ if and only if the common inner divisor of $\{h_\alpha(z_j)\}_\alpha$ is constant, where

$$h_\alpha(z_j) = \int_{T'_j} h(z_j, z'_j)\overline{z'}^\alpha_j dm'_j$$

, $\alpha = (\alpha_1, \cdots, \alpha_{j-1}, \alpha_{j+1}, \cdots, \alpha_n)$ and $\overline{z'}^\alpha_j = \bar{z}_1^{\alpha_1} \cdots \bar{z}_{j-1}^{\alpha_{j-1}} \bar{z}_{j+1}^{\alpha_{j+1}} \cdots \bar{z}_n^{\alpha_n}$. Note that $h_\alpha(z_j)$ belongs to $H^p(T_j)$. For the proof, we use the following notation : $\mathbf{H}^p_{(j)} = \{f \in L^p(T^n)$; $\hat{f}(m_1, \cdots, m_n) = 0$ if $m_i < 0$ for all $i \ne j\}$ and $\mathbf{H}^p_{(j)} \cap \bar{\mathbf{H}}^p_{(j)} = \mathcal{L}^p_j =$ the Lebesgue space on T_j.

If $\phi \in \mathcal{L}_p^h$ then $g = \phi h$ and ϕ belongs to $\mathbf{H}^p_{(j)}$ because h is z_i-outer in H^p for $i \ne j$. Since ϕ is real-valued, $\phi \in \mathcal{L}^p_j$ and so $\phi = \phi(z_j)$. If the common inner divisor of $\{h_\alpha(z_j)\}_\alpha$ is constant, then for each α

$$\phi(z_j)h_\alpha(z_j) = \int_{T'_j} \phi(z_j)h(z_j, z'_j)\overline{z'}^\alpha_j dm'_j = \int_{T'_j} g(z_j, z'_j)\overline{z'}^\alpha_j dm'_j$$

belongs to $H^p(T_j)$ and hence $\phi \in H^\infty(T_j)$. Therefor ϕ is constant. This implies that $\mathcal{L}_p^h = R$. Conversely suppose that $\mathcal{L}_p^h = R$. If $\{h_\alpha(z_j)\}_\alpha$ has a non-constant common inner divisor $q(z_j)$, put $\phi(z_j, z'_j) = \overline{q(z_j)} + q(z_j)$, then $g = \phi h$ belongs to H^p. This contradiction shows the 'only if' part.

Now we will prove that f has a factorization in H^p. If $\{f_\alpha(z_j)\}_\alpha$ does not have common inner divisors, then by what was just prove $\mathcal{L}_p^f = R$ and so we need not prove. If $\{f_\alpha(z_j)\}_\alpha$ have common inner divisors, let $q(z_j)$ be the greatest common inner divisor. Put $\phi = \bar{q}(z_j) + q(z_j)$ and $h = \phi f$, then h belongs to H^p and $\mathcal{L}_p^h = R$. This completes the proof.

§3 $\mathcal{L} = L^\infty$

In this section, we assume that $\mathcal{L} = L^\infty$. If f is a L^∞-extremal function for H^p then f is also a L_R^∞-extremal function for H^p. When $n = 1$, the converse is true. However this is not true for $n \ne 1$. For example, $z - 2w$ is a L_R^∞-extremal fnction but not a L^∞-extremal. We can prove an analogy of Proposition 1 for L^∞. Let M_f be an invariant closed subspace generated by f in H^p and $\mathcal{M}(M_f)$ the set of multipliers of M_f (see [1]). Then $\mathcal{M}(M_f) = \mathcal{L}_p^f$ for $\mathcal{L} = L^\infty$. It is easy to see that

$$(L^\infty)_p^f \cap \overline{(L^\infty)_p^f} = (L_R^\infty)_p^f + i(L_R^\infty)_p^f.$$

It is easy to see that $(L^\infty)_p^f$ is a weak $*$ closed invariant subspace which contains H^∞. Then $(L^\infty)_p^f/H^\infty$ is of infinite dimension (see [4, Theorem 1]). Thus we can not expect the analogy of Theorem 3.

Proposition 6. *Let $1 \leq p \leq \infty$ and f a nonzero function in H^p. Suppose ϕ is in $\mathcal{L}_p^f$.*

(1) $\mathcal{L}_p^f \supseteq \phi \mathcal{L}_p^{\phi f} \supseteq \phi H^\infty$.

(2) ϕ^{-1} is in L^∞ if and only if $\mathcal{L}_p^f = \phi \mathcal{L}_p^{\phi f}$.

(3) If $\mathcal{L}_p^{\phi f} \supseteq \mathcal{L}_p^f$ then ϕ belongs to H^∞. If $\mathcal{L}_p^f \supseteq \mathcal{L}_p^{\phi f}$ and ϕ^{-1} is in L^∞ then ϕ^{-1} belongs to H^∞.

Proof. (1) If $g \in \mathcal{L}_p^{\phi f}$ then $\phi g f = g \phi f \in H^p$ and so $\phi g \in \mathcal{L}_p^f$. (2) If $\phi^{-1} \in L^\infty$ then $\mathcal{L}_p^{\phi f} \supseteq \phi^{-1} \mathcal{L}_p^f \supseteq \mathcal{L}_p^{\phi f}$ by (1). If $\mathcal{L}_p^f = \phi \mathcal{L}_p^{\phi f}$ then $\phi^{-1} \mathcal{L}_p^f = \mathcal{L}_p^{\phi f}$ and so $\phi^{-1} \in \mathcal{L}_p^{\phi f}$. This implies that $\phi^{-1} \in L^\infty$. (3) Suppose $\mathcal{L}_p^{\phi f} \supseteq \mathcal{L}_p^f$. If $k \in \mathcal{L}_p^f$ then $k \in \mathcal{L}_p^{\phi f}$ and so $k \phi f \in H^p$. Hence $\phi^2 f \in H^p$. Repeating this process, $\phi^n \in \mathcal{L}_p^f$ and so $\phi^n f \in H^p$ for all $n \geq 1$. Thus ϕ belongs to H^∞. If $\mathcal{L}_p^f \supseteq \mathcal{L}_p^{\phi f}$ and $\phi^{-1} \in L^\infty$, then ϕ^{-1} belongs to H^∞. For apply what was proved above for ϕ^{-1} assuming $\phi^{-1}(\phi f) = f$.

When f is a nonzero function in H^p, f is factorable in H^p if and only if there exists a nonzero function h in H^p such that $|f| \geq |h|$ *a.e.* on T^2 and $\mathcal{L}_p^h = H^\infty$.

Proposition 7. *Let $1 \leq p \leq \infty$ and f be a nonzero function in H^p. Suppose ϕ is a nonzero function in $\mathcal{L}_p^f$.*

(1) If ϕ^{-1} is in L^∞ and $\mathcal{L}_p^{\phi f} = H^\infty$, then ϕ^{-1} belongs to H^∞ and $\mathcal{L}_p^f = \phi H^\infty$.

(2) If $\mathcal{L}_p^f = \phi H^\infty$, then ϕ^{-1} belongs to H^∞ and $\mathcal{L}_p^{\phi f} = H^\infty$.

(3) If $\mathcal{L}_p^f$ is the weak $*$ closure of ϕH^∞ and $|\phi| = |h|$ a.e. for some function h in H^∞, then $\mathcal{L}_p^{\phi_0 f} = H^\infty$ for some inner function $\bar{\phi}_0$ and so f is factorable.

(4) There exist f and ϕ such that ϕ is not the quotient of any two menbers of $H^\infty(T^n)$.

Proof. (1) By (2) of Proposition 6, $\phi \mathcal{L}_p^{\phi f} = \mathcal{L}_p^f$. Since $\mathcal{L}_p^{\phi f} = H^\infty$, $\phi H^\infty = \mathcal{L}_p^f \supset H^\infty$ and so ϕ^{-1} belongs to H^∞. (2) Since $\mathcal{L}_p^f \supseteq \phi\ \mathcal{L}_p^{\phi f} \supseteq \phi H^\infty$ by (1) of Proposition 6 and $\mathcal{L}_p^f = \phi H^\infty$, $\mathcal{L}_p^{\phi f} = H^\infty$. It is clear that $\phi^{-1} \in H^\infty$. (3) Since $|\phi| = |h|$ *a.e.*, $\phi = \phi_0 h$ and $|\phi_0| = 1$ *a.e.*. Then $\mathcal{L}_p^f = [\phi H^\infty]_* = \phi_0 [h H^\infty]_* \supset H^\infty$ and so $\bar{\phi}_0$ is an inner function where $[S]_*$ is the weak $*$ closure of S. (4) This is a result of [6].

Proposition 8. *Let $1 \leq p \leq \infty$. Suppose f and g are nonzero functions in H^p.*

(1) If $\mathcal{L}_p^f = H^\infty$ and $|f| \geq |g|$ a.e., then there exists a function ϕ in H^∞ such that $g = \phi f$.

(2) If $\mathcal{L}_p^f = H^\infty$ and $|f| = |g|$ a.e., then there exists an inner function ϕ such that $g = \phi f$.

Proof. (1) Let $\phi = g/f$, then $\phi \in L^\infty$ because $|f| \geq |g|$ *a.e.*. By (1) of Proposition 6, ϕ belongs to H^∞ because $H^\infty = \mathcal{L}_p^f$. (2) follows from (1).

Proposition 9. *Let $1 \leq p \leq \infty$. If f is homogeneous polynomial such that $f(z_1, \cdots, z_n) = g(z, w)$ where $z = z_i$, $w = z_j$ and $i \neq j$ then f is factorable in H^p.*

Proof. Since $f(z_1, \cdots, z_n) = \sum_{j=0}^{\ell} a_j z^{\ell - j} w^j$, $f(z_1, \cdots, z_n) = z^\ell \sum_{j=0}^{\ell} a_j \left(\frac{w}{z}\right)^j =$

$c\prod_{j=0}^{\ell}(b_j w - c_j z)$ where $b_j = 1$ or $c_j = 1$, and $|b_j| \leq 1$, $|c_j| \leq 1$. It is easy to see that $\mathcal{L}_p^f = \phi H^\infty$ where $\phi = \prod(\alpha z - \beta w)^{-1}$ and $(\alpha, \beta) \in (\partial D \times D) \cup (D \times \partial D)$ (cf. [2],[4]). By (2) of Proposition 7, f is factorable.

Question

(1) For any nonzero function f in H^p, does there exists a function ϕ such that $\mathcal{L}_p^f \underset{\neq}{\supset} \mathcal{L}_p^{\phi f}$?

(2) Describe ϕ in L^∞ such that $\mathcal{L}_p^f \underset{\neq}{\supset} \mathcal{L}_p^{\phi f}$.

(3) Describe ϕ in L^∞ such that $[\phi H^\infty]_* \underset{\neq}{\supset} H^\infty$.

References

1. T.Nakazi, Certain invariant subspaces of H^2 and L^2 on a bidisc, Can.J.Math. XL(1988), 1272-1280.

2. T.Nakazi, Slice maps and multipliers of invariant subspaces, Can.Math.Bull. 39(1996), 219-226.

3. T.Nakazi, An outer function and several important functions in two variables, Arch.Math., 66(1996), 490-498.

4. T.Nakazi, On an invariant subspace whose common zero set is the zeros of some function, Nihonkai Math.J., 11(2000), 1-9.

5. W.Rudin, Function Theory in Polydisks, Benjamin, New York (1969)

6. W.Rudin, Invariant subspaces of H^2 on a torus, J.Funct.Anal. 61(1985), 378-384.

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