

LETTER TO THE EDITOR

Remarks on the Inverse Scattering Problem for Ocean Acoustics

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Abstract. We propose two new formulations of inverse scattering problems for ocean acoustics and give the reconstruction formula for them. Both of them use near field data instead of the far field pattern of the scattered wave.

1. Introduction

Let $\mathbf{R}_H^3 := \{x = (x', z), x' = (x_1, x_2) ; 0 < z < H\}$ be a finite depth ocean with flat bottom $z = H$ and surface $z = 0$. We consider as a direct problem the scattering of acoustic wave in \mathbf{R}_H^3 by scatterer which can be either an obstacle or inhomogeneity of the medium. The usual inverse scattering problem uses the far field pattern of the scattered wave as a measured data to identify the unknown scatterer. The first mathematical works in this direction were done by Xu ([X]), Gilbert and Xu ([GX]). However, there is the so called evanescent mode for the scattered wave which attenuates very rapidly away from the scatterer and it cannot be seen in the far field pattern. This means some information about the scatterer carried by the evanescent mode may be lost. From the mathematical point of view, this evanescent modes are the eigenfunction associated to the eigenvalues of the direct scattering problem created by the scatterer.

We propose to use a near field instead of the far field pattern of the scattered wave as a measured data. In this letter two kind of measured data are proposed. More precisely, the first one is to use $G(x, y)$ ($x \in E_1, y \in E_2$) as a measured data, where $G(x, y)$ is the outgoing Green function defined in the next section and $E_j \subset \mathbf{R}_H^3$ ($j = 1, 2$) are open hypersurfaces which can be very small. The inverse problem using this kind of measurement is introduced in ([IMN]). The second one is to use the inverse S^{-1} of the single layer potential S defined on the lateral surface of a cylinder which contains the scatterer and has the bottom and the top surfaces on $z = 0$ and $z = H$, respectively. The more detailed definition of S is given in the next section.

The standard measurement in ocean acoustics is a near field measurement which can be mathematically formulated as giving $G(x, y)$ for $x \in c_1, y \in c_2$ on curves $c_j \subset E_j$ ($j = 1, 2$) ($j = 1, 2$). Hence, the first measured data we proposed above is the first good step to model the real measurement.

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Assuming that we know that the scatterer is totally inside a cylinder B_R , we can reduce these inverse problems associated with these near fields measurements to an inverse boundary value problem in the cylinder ([IMN]). For these reductions, we need to assume that the wave number k in the equation $\Delta u + k^2 n^2 u = 0$ in B_R with the refraction index n for the scattering by the inhomogeneity of the medium and $\Delta u + k^2 u = 0$ in $B_R \setminus \overline{\mathcal{O}}$ for the scattering by obstacle \mathcal{O} is not an eigenvalue for the boundary value problem with homogeneous Dirichlet condition on the lateral surface of this cylinder. We will refer this assumption as the eigenvalue assumption.

This eigenvalue assumption is an artificial assumption. In fact, due to the continuity and the strict monotonicity of the eigenvalue k with respect to changing the size of B_R , we can always choose another smaller cylinder $B_{R'}$ very close to B_R such that one of these cylinders satisfies the eigenvalue assumption. Then, as shown in [IMN] for the scattering by the inhomogeneity of the medium, we can further reduce this inverse boundary value problem to that for a region Ω with smooth boundary $\partial\Omega$ totally contained in B_R . This reduction can be possible also for the scattering by obstacle. Hence, applying Nachman's reconstruction formula ([N]) to identify n and the probe method ([I]) to reconstruct \mathcal{O} , we do have a reconstruction formula for identifying the scatterer from each of our near field measurements. Of course, we have to test the reduction arguments for $\{B_R, B_{R'}\}$ and the same for Ω . The appropriate combinations of the cylinder and the region should give a good reconstruction of the scatterer.

Note that for the first measurement which uses $G(x, y)$ we still have to assume that the wave number k is not the embedded eigenvalue of the direct scattering problem. On the other hand, the advantage of the second measurement using S^{-1} is that we can even avoid the eigenvalue problem for the direct scattering problem created by the scatterer. However, we don't know the feasibility of measuring S^{-1} .

2. Statement of the problem and the results

Let $\mathbf{R}_H^3 := \{x = (x', z) ; x' := (x_1, x_2) \in \mathbf{R}^2, 0 < z < H\}$ be a finite depth ocean with flat hard bottom $z = H$ and pressure-release surface $z = 0$.

In order to simplify our description, we only consider the scattering due to the inhomogeneity of the medium. By obvious changes, all the statements and results given here also hold for the scattering by obstacles. Also for further simplification, we restrict to the case where our two kind of measurements are done on small open subsets of the lateral surface of the cylinder totally containing the inhomogeneity of the medium.

2.1. The first measured data

The total wave field u of the ocean acoustics is governed by the following boundary value problem:

$$\begin{cases} (\Delta + k^2 n^2(x))u = 0 & \text{in } \mathbf{R}_H^3 \\ u|_{z=0} = \frac{\partial u}{\partial z} u|_{z=H} = 0 & \text{(abbreviated by BC from now on)} \\ \text{RC} . \end{cases} \quad (2.1)$$

Here, $k > 0$ is the wave number, $n \in L^\infty(\mathbf{R}_H^3)$ is the refraction index and u describes the pressure of the acoustic sound. The radiation condition abbreviated by RC in (2.1) means

that

$$\lim_{|x'| \rightarrow \infty} |x'|^{1/2} \left| \frac{\partial u_\ell}{\partial |x'|} (x') - ik_0 u_\ell(x') \right| = 0, \quad (2.2)$$

where

$$u_\ell(x') = \frac{2}{H} \int_0^H u(x', z) \overline{\phi_\ell(z)} dz \quad (\ell = 0, 1, \dots),$$

$$\phi_\ell(z) = \sin\{k^2(1 - a_\ell^2)^{1/2}z\} \text{ and } a_\ell = \left(1 - \frac{(2\ell+1)^2\pi^2}{4k^2H^2}\right)^{1/2} \text{ for } \ell = 0, 1, 2, \dots.$$

We assume the following conditions:

(A-1) The refraction index n is constant, say equal to one, outside a cylinder $B_R := \{x = (x', z) ; |x'| < R, 0 < z < H\}$.

(A-2) $k \neq \frac{(2\ell+1)\pi}{2H}$ ($\ell = 0, 1, 2, \dots$).

(A-3) Zero is not an eigenvalue of the boundary value problem (2.1).

By the outgoing Green function for (2.1) with $n = 1$ constructed in [AK], the assumption (A-3) and limiting absorption principle, there exists an outgoing Green function $G(x, y)$ which is the solution to the boundary value problem:

$$\begin{cases} (\Delta + k^2n^2)G + \delta(x - y) = 0 & \text{in } \mathbf{R}_H^3, \\ G|_{z=0} = \frac{\partial}{\partial z}G|_{z=H} = 0, \\ \text{RC,} \end{cases} \quad (2.3)$$

where RC is the previous radiation condition for the function $G(x, y)$ in x fixing y . Moreover, for each fixed $y \in \mathbf{R}_H^3$, $G(\cdot, y)$ is locally $H^2(\mathbf{R}_H^3 \setminus \{y\})$ and $G(\cdot, y) = O(|x - y|^{-1})$ as $x \rightarrow y$.

Theorem 2.1 *Let Γ_R be the lateral surface of the cylinder B_R . Also, let $E_j \subset \Gamma_R$ ($j = 1, 2$) be small open sets. If $G(x, y)$ ($x \in E_1, y \in E_2, x \neq y$) is given as measured data, there is a reconstruction formula for identifying the unknown refraction index n from this data.*

2.1.1. Proof of Theorem 2.1 First of all, by the analyticity of $G(x, y)$ ($x, y \notin B_R, x \neq y$), we can continue our given measured data to the whole Γ_R . Hence, we can suppose that $G(x, y)$ ($x, y \in \Gamma_R, x \neq y$) is given.

Let $R' > R$ and $f \in H^{-\frac{1}{2}}(\Gamma_{R'})$ and define $T_f \in H^{-1}(\mathbf{R}_H^3)$ by

$$\langle T_f, \phi \rangle = \langle f, \phi \rangle \quad (\phi \in \dot{H}^1(\mathbf{R}_H^3)).$$

Also, let $u(f) \in \dot{H}_{-\delta}^1(\mathbf{R}_H^3)$ be the solution of the problem:

$$\begin{cases} (\Delta + k^2n^2)u = T_f & \text{in } \mathbf{R}_H^3, \\ \text{BC,} \\ \text{RC,} \end{cases} \quad (2.4)$$

where $\dot{H}^1(\mathbf{R}_H^3) := \{\phi \in H^1(\mathbf{R}_H^3); \phi|_{z=0} = 0\}$ with the induced norm $\|\phi\|_{\dot{H}^1(\mathbf{R}_H^3)}$ from the standard Sobolev space $H^1(\mathbf{R}_H^3)$, $\dot{H}_{-\delta}^1(\mathbf{R}_H^3) := \{u ; \langle x \rangle^{-\delta} u \in H^1(\mathbf{R}_H^3), u|_{z=0} = 0\}$ with the norm $\|u\|_{\dot{H}_{-\delta}^1(\mathbf{R}_H^3)} := \|\langle x \rangle^{-\delta} u\|_{H^1(\mathbf{R}_H^3)}$ where $\langle x \rangle := \sqrt{1 + |x|^2}$, and $H^{-1}(\mathbf{R}_H^3)$ is the dual space of $\dot{H}^1(\mathbf{R}_H^3)$.

Then, define the operator S by

$$S(f) := u(f)|_{\Gamma_{R'}}. \quad (2.5)$$

Next, we prove that the data given by $G(x, y)$ ($x, y \in \Gamma_R, x \neq y$) determine S . As we mentioned in the introduction, we can assume that k^2 is not an eigenvalue of the boundary value problem $\Delta u + k^2 n^2 u = 0$ in $B_{R'}$ and the homogeneous Dirichlet condition on $\Gamma_{R'}$. This implies that S is injective and there is a reconstruction formula identifying n from S^{-1} (see [IMN]).

Fix $y \in \Gamma_R$ and let $\alpha(x) \in C_0^\infty(\mathbf{R}^3)$ satisfy $\alpha(y) = 0$ and $\text{supp } \alpha \subset \{0 < z < H\}$. Define $\beta(x) := 1 - \alpha(x)$, $\Phi(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}$ and $K(x, y) := G(x, y) - \beta(x)\Phi(x, y)$. Then, $K = K(\cdot, y)$ satisfies

$$\begin{cases} (\Delta + k^2)K = -2\nabla\beta \cdot \nabla\Phi + \Delta\beta\Phi & \text{in } \mathbf{R}_H^3 \setminus \overline{B_R}, \\ K|_{z=0} = \frac{\partial}{\partial z}K|_{z=H} = 0, \\ K = (G(\cdot, y) - \beta(\cdot)\Phi(\cdot, y)) & \text{on } \Gamma_R, \\ \text{RC.} \end{cases} \quad (2.6)$$

Since (2.6) is a well-posed boundary value problem (see [X]), $K \in \dot{H}_{-\delta}^1(\mathbf{R}_H^3 \setminus B_R)$ with $\delta > 1/2$ can be defined in $\mathbf{R}_H^3 \setminus B_R$. We also take $\dot{H}_{-\delta}^1(\mathbf{R}_H^3 \setminus B_R) := \{u ; \langle x \rangle^{-\delta} u \in H^1(\mathbf{R}_H^3 \setminus B_R), u|_{z=0} = 0\}$ where $H^1(\mathbf{R}_H^3 \setminus B_R)$ is the standard Sobolev space. Hence, $G(\cdot, y)$ with fixed $y \in \Gamma_R$ given on Γ_R can be extended to $\mathbf{R}_H^3 \setminus B_R$. By the selfadjointness of the boundary value problem (2.3), we can easily prove that $G(x, y) = G(y, x)$ ($x, y \in \mathbf{R}_H^3 \setminus B_R, x \neq y$). Therefore, $G(x, y)$ given for $x, y \in \Gamma_R, x \neq y$ can be extended to $G(x, y)$ ($x, y \in \mathbf{R}_H^3 \setminus B_R, x \neq y$). Hence, the proof will be completed if we show that the operator S can be represented using $G(x, y)$ ($x, y \in \mathbf{R}_H^3 \setminus B_R$).

Let $g_n \in L^2(\mathbf{R}_H^3)$ ($n = 1, 2, \dots$) be a sequence of functions such that $\text{supp } g_n \subset \mathbf{R}_H^3 \setminus B_{R'}$ and g_n tends to T_f in $H^{-1}(\mathbf{R}_H^3)$. Define $u_n := u(g_n)$ as the solution of (2.4) replacing T_f by g_n . Then using the Green function $G(x, y)$, we have the following representation of u_n in terms of $G(x, y)$ ($x, y \in \mathbf{R}_H^3 \setminus B_R$):

$$u_n(x) = \int_{\mathbf{R}_H^3} G(x, y)g_n(y)dy \quad (x \in \mathbf{R}_H^3).$$

Since k satisfies the eigenvalue assumption for the boundary value problem (2.4) replacing T_f by 0, it is possible to prove that there exists a subsequence $\{u'_n\}$ of $\{u_n\}$ which converges to $u(f)$ in $H_{-\delta}^1(\mathbf{R}_H^3)$ (see [IMN]). By the continuity of the trace operator, $u'_n|_{\Gamma_{R'}}$ tends to $S(f) = u(f)|_{\Gamma_{R'}}$ in $H^{1/2}(\Gamma_{R'})$. Thus we have shown that the operator S can be represented using $G(x, y)$ ($x, y \in \mathbf{R}_H^3 \setminus B_R$).

2.2. The second measured data

In this section we assume the following conditions:

(A-1) The refraction index n is constant, say equal to one, outside a cylinder $B_R := \{x = (x', z) ; |x'| < R, 0 < z < H\}$.

(A-2) $k \neq \frac{(2\ell+1)\pi}{2H}$ ($\ell = 0, 1, 2, \dots$).

(A-4) Zero is not an eigenvalue for the problem (2.7).

For $g \in H^{1/2}(\Gamma_R)$, let $v_i \in \dot{H}^1(B_R)$ and $v_e \in H_{-\delta}^1(\mathbf{R}_H^3 \setminus B_R)$ be the solutions to the following two problems:

$$\begin{cases} (\Delta + k^2 n^2(x))v_i = 0 & \text{in } B_R, \\ \text{BC,} \\ v_i = g & \text{on } \Gamma_R \end{cases} \quad (2.7)$$

and

$$\begin{cases} (\Delta + k^2)v_e = 0 & \text{in } \mathbf{R}_H^3 \setminus B_R, \\ \text{BC}, \\ v_e = g \text{ on } \Gamma_R, \\ \text{RC}, \end{cases} \quad (2.8)$$

respectively. The condition $(A - 4)$ implies that the problem (2.7) is well posed while the problem (2.7) is well posed for every value of k satisfying the condition $(A - 2)$, as we noticed it for the problem (2.6). Then, we define the operator $J : H^{1/2}(\Gamma_R) \rightarrow H^{-1/2}(\Gamma_R)$ by $J(g) := \frac{\partial v_e}{\partial \nu} - \frac{\partial v_i}{\partial \nu}$, where ν is the unit normal vector directed outside B_R . Hence the operator J is defined for all k satisfying the conditions $(A - 2)$ and $(A - 4)$. In practice one can avoid the last condition $(\mathbf{A} - 4)$ by choosing another radius R' very close to R .

Now we prove the following lemma which gives the injectivity of S and the representation of its inverse.

Lemma 2.2 $JS = I$, where I is the identity operator on $H^{-1/2}(\Gamma_R)$. Hence, $J = S^{-1}$.

Proof.

Let $f \in H^{-1/2}(\Gamma_R)$ and $u = u(f) \in \dot{H}_{-\delta}^1(\mathbf{R}_H^3)$ be the solution to (2.4). Also, let $v_i \in H^1(B_R)$ and $v_e \in H_{-\delta}^1(\mathbf{R}_H^3 \setminus \overline{B_R})$ be the solutions to (2.7) and (2.8) with $g = u(f)|_{\Gamma_R} = S(f)$, respectively. Then, by the well-posedness of the boundary value problems (2.7) and (2.8), we have

$$v_i = u \quad \text{in } B_R, \quad v_e = u \quad \text{in } \mathbf{R}_H^3 \setminus \overline{B_R}. \quad (2.9)$$

For any $\phi \in \dot{H}_{-\delta}^1(\mathbf{R}_H^3)$, we have

$$\int_{B_R} (\nabla v_i \cdot \nabla \phi - k^2 n^2 v_i \phi) dx = \int_{\Gamma_R} \frac{\partial v_i}{\partial \nu} \phi d\sigma \quad (2.10)$$

and

$$\begin{aligned} \int_{\mathbf{R}_H^3 \setminus \overline{B_R}} (\nabla v_e \cdot \nabla \phi - k^2 n^2 v_e \phi) dx &= \int_{\mathbf{R}_H^3 \setminus \overline{B_R}} (\nabla v_e \cdot \nabla \phi - k^2 v_e \phi) dx \\ &= - \int_{\Gamma_R} \frac{\partial v_e}{\partial \nu} \phi d\sigma. \end{aligned} \quad (2.11)$$

Hence, from (2.9), we have

$$\int_{\mathbf{R}_H^3} (\nabla u \cdot \nabla \phi - k^2 n^2 u \phi) dx = - \int_{\Gamma_R} J(g) \phi d\sigma = - \int_{\Gamma_R} J(S(f)) \phi d\sigma. \quad (2.12)$$

On the other hand, we have

$$\int_{\mathbf{R}_H^3} (\nabla u \cdot \nabla \phi - k^2 n^2 u \phi) dx = - \langle T_f, \phi \rangle = - \int_{\Gamma_R} f \phi d\sigma. \quad (2.13)$$

Therefore, we have $J(S(f)) = f$ for any $f \in H^{-1/2}(\Gamma_R)$. \square

Remark 2.3 We want to point out that the measured data $J = S^{-1}$ do not require the assumption $(A-3)$ even if the measurement S does. This is an immediate consequence of the well-posedness of the boundary value problem (2.8).

Likewise the end of the proof of Theorem 2.1, there is a reconstruction formula identifying n from S^{-1} . We have the following theorem.

Theorem 2.4

If we take $j = S^{-1}$ as measured data, there is a reconstruction formula for identifying the unknown refraction index n from this data. Moreover, this measured data essentially avoids any eigenvalue assumption.

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