PONTRYAGIN CLASSES OF TOPOLOGICAL MANIFOLDS

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1. CHARACTERISTIC CLASSES OF SMOOTH MANIFOLDS

There are various definitions of characteristic classes on smooth manifolds.

1.1. **Pontryagin's pioneering work.** Let V be a smooth manifold and $f : V \to BO$ the classifying map of the tangent bundle, and p_k the canonical generator of $H^{4k}(BO)$. Then by definition :

$$p_k(V) = f^*(p_k) \tag{1.1}$$

1.2. **Submanifolds with trivial normal bundle.** It is useful to consider $\mathcal{L}$-classes which are universal polynomials in Pontryagin classes. Independently in late 50's, Rohlin-Svac and Thom showed that $\mathcal{L}$-classes could be defined using inverse images of regular values of maps $f : V \to S^k$ and at the same time extended the definition to PL-manifolds.

1.3. **Index theorem.** Signature operator gives rise to a class $\Sigma(V)$ in the K-homology group $K_0(V)$, which maps to $H_*(V, \mathbf{Q})$ by the Chern character, and by Atiyah-Singer index theorem, one has

$$\mathrm{ch}(\Sigma(V)) = \mathcal{L}(V) \tag{1.2}$$

1.4. **Curvature of connection.** Characteristic classes in $H^*(V, \mathbf{R})$ may be obtained by universal polynomials of the curvature of a connection on TV.

2. TOPOLOGICAL MANIFOLDS

Novikov's theorem states that rational Pontryagin classes are topological invariants.

2.1. **Classifying spaces.** The space $BTop$ classifies topological microbundles, and $H^{4k}(BTop, \mathbf{Q}) \simeq H^{4k}(BO) \otimes \mathbf{Q}) = p_k\mathbf{Q}$. Thus if $f : V \to BTop$ is the classifying map of a topological manifold, then equation (1.1) works.

2.2. **Regular points of maps $V \to S^k$.** It has been worked out by N. Teleman by means of the signature operator on topological manifolds and transversality theorems of Marin, Kirby-Siebenmann.

2.3. **K-homology.** In 1980's, D. Sullivan proved that every manifold of dimension $\neq 4$ can be endowed with a Lipschitz structure, unique up isotopy.

Signature operator has be defined for non-smooth manifolds :

(1) Combinatorial manifolds (N. Teleman 1978)
(2) Lipschitz manifolds (N. Teleman 1982, M.H. 1983)
(3) Quasiconformal manifolds even dimensional (Connes-Sullivan-Teleman)
(4) L^p-manifolds (M.H. 1997)

An index theorem holds, and more generally signature operator defines a class $\Sigma(V) \in K_0(V)$, and formula (1.2) remains valid.

Moreover, this operator allows to define a functor in the category of topological manifold which generalizes Gysin maps ; the smooth case has been worked out by A. Connes and G. Skandalis. To an oriented map $f : V \to W$ of topological manifolds corresponds a homomorphism of K-theory groups :

$$\Sigma(f) : K^*(V) \otimes \mathbf{Z}[\frac{1}{2}] \to K^*(W) \otimes \mathbf{Z}[\frac{1}{2}]$$

It enjoys the following properties :

(1) $\Sigma(g \circ f) = \Sigma(g) \circ \Sigma(f)$
(2) $\Sigma(V \to .) = \Sigma(V)$
(3) $\Sigma(\text{immersion})$ is K-orientation of microbundle.

This construction uses basically Kasparov's bivariant K-theory.

3. Gromov's classes for topological manifolds

Here is a purely topological definition, which does not require transversality hypothesis. Evidently, characteristic classes obtained by this device are topological invariants.

Let V be a topological manifold of dimension $4k+1$, h a homology class in $H_{4k}(V)$ and E a flat symplectic vector bundle on V : there is a quadratic form $(\alpha, \beta) \to < \alpha \cup \beta, h >$ on $H_{2k}(V, E)$, and denote its signature by :

$$\text{Signature}(V, E, h)$$

There exists a flat symplectic vector bundle X on a riemann surface B such that $\mathrm{Signature}(B, X) = s \neq 0$ (W. Goldman).

Let V be a closed topological manifold of dimension $4k + 2l + 1$ and $f : V \to S^{2l+1}$ be a map representing a homology class $z \in H_{4k}(V, \mathbf{Q})$, *i.e.* $f^*([S^{2l+1}]^{co}) = \mathcal{P}z$ (such a map exists by Serre's finiteness theorem).

Let $j : B \to S^{2l+1}$ be an imbedding with tubular neighbourhood $U \simeq B^l \times \mathbf{R}$ and $\tilde{X}^l \to U$ the pull-back of X^l. Let $h \in H_{4k+2}(V)$ the homology class the Poincare dual of which is the pull back by f of the Poincare dual of B^l in $H_{2l}(U)$.

Then there exists a unique class $\tilde{L}^{4k}(V) \in H^{4k}(V, \mathbf{Q})$ such that :

$$< \tilde{\mathcal{L}}^{4k}(V), z >= \mathrm{Signature}(f^{-1}(U), f^*(\tilde{X}^l), h) \tag{3.1}$$

It is not difficult to see that this definition agrees with previous ones for a smooth manifold. However, the question remained unsolved for topological manifolds.

Theorem 3.1. *Let V be a closed oriented topological manifold, then :*

$$\mathcal{L}(V) = \tilde{\mathcal{L}}(V)$$

Proof. One shows that $\mathrm{ch}(\Sigma(V)) = \tilde{\mathcal{L}}(V)$.

□

4. PSEUDOMANIFOLDS

Intersection $\mathcal{L}$-classes were defined for admissible PL pseudomanifolds by M. Goresky and R. Mac Pherson, using intersection homology theory. Also a signature operator has been constructed by J. Cheeger.

Intersection homology is a topological invariant and is well defined for topological pseudomanifolds.

Gromov's construction can be extended to topological pseudomanifolds. This would establish topological invariance of intersection $\mathcal{L}$-classes, as well as their generalization to the topological context.

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