SMOOTHING EFFECT AND LARGE TIME BEHAVIOR OF SOLUTIONS TO SCHRÖDINGER EQUATIONS WITH NONLINEARITY OF INTEGRAL TYPE

Dedicated to Professor Sigetoshi Kuroda
on the occasion of his seventieth birthday

T.Ozawa and Y.Yamazaki*)

Department of Mathematics
Hokkaido University
Sapporo 060-0810, Japan

Abstract. We study the smoothing effect in space and asymptotic behavior in time of solutions to the Cauchy problem for the nonlinear Schrödinger equation with interaction described by the integral of the intensity with respect to one direction in two space dimensions. A detailed description is given on the phase modification of scattering solutions by taking into account the long range effect of the interaction.

§1. Introduction

We study the nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = f(u), \tag{1.1}$$

where u is a complex-valued function of time and space variables denoted respectively by $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^2, \partial_t = \partial/\partial t, \Delta$ is the Laplacian in space $\mathbb{R}^2$, and $f(u)$ is the nonlinear interaction given by

$$(f(u))(t, x, y) = \lambda \left(\int_{-\infty}^{x} |u(t, x', y)|^2 dx' \right) u(t, x, y) \tag{1.2}$$

with $\lambda \in \mathbb{R}$. The equation (1.1) with integral type nonlinearity (1.2) appears as a model of the propagation of laser beams under the influence of a steady transverse wind along the x-axis [1,4,32] and as a special case of the Davey-Stewartson system where the velocity potential is independent of y-variable [2,6-8,13,14,16,22,28].

*) JSPS Fellow

In spite of a large literature on the nonlinear Schrödinger equations (see for instance [5,9,23,24] and references therein), there are few results on the equation (1.1) with a special nonlinearity (1.2) [1,4,21]. The existence and uniqueness of global solutions to the Cauchy problem for (1.1) is proved in the usual Sobolev spaces $H^m(\mathbb{R}^2)$ with $m \geq 1$ [4] and in the Lebesgue space $L^2(\mathbb{R}^2)$ [21]. It is also noticed that a smoothing effect takes place only in y-variable when measured by the spatial integrability properties [21]. The existence of modified wave operators is proved on a dense set of small and sufficiently regular asymptotic states [21] (see also [10-12,15,17,20,31]).

The purpose of this paper is to describe smoothing properties of solutions to the Cauchy problem for (1.1) in terms of the generators of Galilei transformations and large time asymptotics of small solutions in terms of the free propagator with phase modifications.

To state our results precisely, we introduce the following

Notation. $L^p_x L^q_y = L^p(\mathbb{R}_x; L^q(\mathbb{R}_y)), L^q_y L^p_x = L^q(\mathbb{R}_y; L^p(\mathbb{R}_x))$ with norms

$$\|u; L^p_x L^q_y\| = \|\|u; L^q_y\|; L^p_x\|, \ \|u; L^q_y L^p_x\| = \|\|u; L^p_x\|; L^q_y\|.$$

$L^p = L^p(\mathbb{R}^2) = L^p_x L^p_y = L^p_y L^p_x$. Similarly, $L^p_x \dot{H}^\alpha_y = L^p(\mathbb{R}_x; \dot{H}^\alpha(\mathbb{R}_y)), L^p_y \dot{H}^\alpha_x = L^p(\mathbb{R}_y; \dot{H}^\alpha(\mathbb{R}_x))$, where $\dot{H}^\alpha(\mathbb{R}_x) = (-\Delta_x)^{-\alpha/2} L^2(\mathbb{R}_x), \dot{H}^\alpha(\mathbb{R}_y) = (-\Delta_y)^{-\alpha/2} L^2(\mathbb{R}_y)$, and $(-\Delta_x)^{\alpha/2}$ and $(-\Delta_y)^{\alpha/2}$ are fractional powers of minus Laplacians $-\Delta_x$ in $\mathbb{R}_x$ and $-\Delta_y$ in $\mathbb{R}_y$ with $\alpha > 0$, respectively. $\dot{H}^{\alpha,\beta} = L^2_y \dot{H}^\alpha_x \cap L^2_x \dot{H}^\beta_y$ with norm $\|u; \dot{H}^{\alpha,\beta}\| = \mathrm{Max}(\|u; L^2_y \dot{H}^\alpha_x\|, \|u; L^2_x \dot{H}^\beta_y\|)$. Fourier transform of functions on $\mathbb{R}^2$ with the associated partial Fourier transforms $\mathcal{F}_x$ and $\mathcal{F}_y$ is given by

$$\mathcal{F}u(\xi,\eta) = \hat{u}(\xi,\eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(-i\xi x - i\eta y)\, u(x,y)\, dxdy$$
$$= (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-i\xi x)\, \mathcal{F}_y\, u\, dx = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \exp(-i\eta y)\, \mathcal{F}_x\, u\, dy.$$

$U(t) = \exp(i(t/2)\Delta) = \exp(i(t/2)\Delta_x) \exp(i(t/2)\Delta_y)$ denotes the free propagator acting on functions on $\mathbb{R}^2$, which is realized as the Fourier multiplier with symbol $\exp\left(-i(t/2)(\xi^2+\eta^2)\right)$ and is factorized as

$$U(t) = M(t)D(t)\mathcal{F}M(t) \tag{1.3}$$

for $t \neq 0$, where $M(t) = \exp\left(i(x^2+y^2)/(2t)\right) \cdot$ and $D(t)$ is the dilation operator defined by

$$(D(t)\psi)(x,y) = (it)^{-1}\psi(t^{-1}x, t^{-1}y).$$

A natural factorization associated with (1.3) in x and y directions is given respectively by means of partial multiplications $M_x(t) = \exp\left(ix^2/(2t)\right) \cdot, M_y(t) = \exp\left(iy^2/(2t)\right) \cdot$, partial dilations given by

$$(D_x(t)\psi)(x,y) = (it)^{-1/2}\psi(t^{-1}x, y),$$
$$(D_y(t)\psi)(x,y) = (it)^{-1/2}\psi(x, t^{-1}y),$$

and partial Fourier transforms $\mathcal{F}_x, \mathcal{F}_y$ as

$$U_x(t) = \exp\left(i(t/2)\Delta_x\right) = M_x(t)D_x(t)\mathcal{F}_x M_x(t),$$
$$U_y(t) = \exp\left(i(t/2)\Delta_y\right) = M_y(t)D_y(t)\mathcal{F}_y M_y(t).$$

The generators of Galilei transformations are denoted by $J = (J_x, J_y) = (x + it\partial_x, y + it\partial_y) = (x, y) + it\nabla$.

The operators J are represented as

$$\begin{aligned} J &= \left(U_x(t)xU_x(-t)\,,\, U_y(t)yU_y(-t)\right) \\ &= \left(M_x(t)it\partial_x M_x(-t)\,,\, M_y(t)it\partial_y M_y(-t)\right). \end{aligned}$$

We use explicit formulas for the fractional powers $|J_x|^\alpha$ and $|J_y|^\alpha$ as

$$|J_x|^\alpha = U_x(t)|x|^\alpha U_x(-t) = M_x(t)\left(-t^2\Delta_x\right)^{\alpha/2} M_x(-t),$$
$$|J_y|^\alpha = U_y(t)|y|^\alpha U_y(-t) = M_y(t)\left(-t^2\Delta_y\right)^{\alpha/2} M_y(-t)$$

with $\alpha > 0$ [19].

We consider the Cauchy problem for (1.1) with data $u(t_0) = \phi$ at time t_0 in the form of the corresponding integral equation

$$u(t) = U(t - t_0)\phi - i\int_{t_0}^{t} U(t - t')f\left(u(t')\right)dt'. \tag{1.4}$$

The integral equation (1.4) is studied by a contraction argument in a closed ball of the following function space

$$X(I) = C\left(I; L^2\right) \cap \bigcap_{0 \le 2/q = 1/2 - 1/r \le 1/2} L_t^q\left(I; L_x^2 L_y^r\right)$$

for $[t_0 - T, t_0 + T]$ with $T > 0$. We use related function spaces defined by

$$\begin{aligned} X &= C\left(\mathbb{R}; L^2\right) \cap \bigcap_{0 \le 2/q = 1/2 - 1/r \le 1/2} L^q\left(\mathbb{R}; L_x^2 L_y^r\right), \\ X_{loc} &= C\left(\mathbb{R}; L^2\right) \cap \bigcap_{0 \le 2/q = 1/2 - 1/r \le 1/2} L_{loc}^q\left(\mathbb{R}; L_x^2 L_y^r\right), \\ Y^{\alpha,\beta}(I) &= \{u \in X(I); |J_x|^\alpha u \in X(I),\ |J_y|^\beta u \in X(I)\}, \\ Y^{\alpha,\beta} &= \{u \in X; |J_x|^\alpha u \in X,\ |J_y|^\beta u \in X\}, \\ Y_{loc}^{\alpha,\beta} &= \{u \in X_{loc}; |J_x|^\alpha u \in X_{loc},\ |J_y|^\beta u \in X_{loc}\}. \end{aligned}$$

We now state basic existence and uniqueness results.

Theorem 1. *Let $t_0 \in \mathbb{R}$ and let $\phi \in L^2$. Then the equation (1.4) has a unique solution $u \in X_{loc}$. Moreover, u satisfies the conservation law of the L^2 norm : $\|u(t); L^2\| = \|\phi; L^2\|$ for all $t \in \mathbb{R}$.*

Theorem 2. *Let j, k be nonnegative integers. Let $\phi \in L^2$ satisfy $|x|^j U(-t_0)\phi$, $|y|^k U(-t_0)\phi \in L^2$. Then the solution u of (1.4) given by Theorem 1 satisfies $u \in Y^{j,k}_{loc}$.*

Remark 1. *The Cauchy problem in L^2 has been studied in [21]. The function space X_{loc} is smaller than that used in [21].*

Remark 2. *Theorem 2 describes the smoothing properties of solutions in terms of the generators of Galilei transforms. No regularity assumption is made on the Cauchy data.*

To describe the large time behavior of solutions of (1.4) with small Cauchy data, we introduce modified free dynamics for $\phi_\pm \in L^2 \cap \mathcal{F}(L^2_x L^\infty_y)$

$$\begin{aligned}
v_1^\pm(t) &= U(t) \exp\left(-iS_\pm(t, -i\nabla)\right) \phi_\pm, \\
v_2^\pm(t) &= U(t) M(-t) \exp\left(-iS_\pm(t, -i\nabla)\right) \phi_\pm \\
&= M(t) D(t) \exp\left(-iS_\pm(t, \cdot)\right) \hat{\phi}_\pm, \\
v_3^\pm(t) &= \exp\left(-iS_\pm(t, t^{-1}x, t^{-1}y)\right) U(t)\phi_\pm \\
&= M(t) D(t) \exp\left(-iS_\pm(t, \cdot)\right) \mathcal{F} M(t)\phi_\pm,
\end{aligned}$$

where

$$S_\pm(t, x, y) = \pm\lambda \int_{-\infty}^{x} |\hat{\phi}_\pm(x', y)|^2 \, dx' \, \log|t|.$$

For ρ, $\rho_0 > 0$, we define $B(\rho, \rho_0) = \{\phi \in L^2 \cap \mathcal{F}(\dot{H}^{\alpha,\beta}); \|\phi; L^2\| \le \rho_0, \|\hat{\phi}; \dot{H}^{\alpha,\beta}\| \le \rho\}$.

Theorem 3. *Let α and β satisfy $0 < \alpha < 1/2$, $\beta \ge 0$. Let $\phi \in L^2$ satisfy $|x|^\alpha U(-t_0)\phi$, $|y|^\beta U(-t_0)\phi \in L^2$. Then :*

(1)The equation (1.4) has a unique solution $u \in Y^{\alpha,\beta}_{loc}$.

(2)Let $0 < \alpha < 1/2 < \beta < 1$ and let $t_0 = 0$. For any $\rho_0 > 0$ there exists $\rho > 0$ with the following property: For any $\phi \in B(\rho, \rho_0)$ the solution given by (1) satisfies $u \in Y^{\alpha,\beta}$ and

$$\|u(t); L^2_x L^\infty_y\| = O(|t|^{-1/2}) \qquad as \qquad t \to \pm\infty.$$

(3)Let u be the solution given by Part (2). Then, there exist unique $\phi_\pm \in L^2 \cap \mathcal{F}(L^2_x L^\infty_y)$ such that for sufficiently small $\epsilon > 0$

$$\|\mathcal{F}U(-t)u(t) - \exp\left(-iS_\pm(t, \cdot)\right) \hat{\phi}_\pm; L^2 \cap L^2_x L^\infty_y\| = O(|t|^{-\epsilon}) \qquad as \qquad t \to \pm\infty.$$

Moreover, u satisfies

$$\|u(t) - v_j^{\pm}(t); L^2\| = O(|t|^{-\epsilon}) \qquad as \qquad t \to \pm\infty,$$

for $j = 1, 2$, and

$$\|u(t) - v_3^{\pm}(t); L^2\| \longrightarrow 0 \qquad as \qquad t \to \pm\infty.$$

Remark 3. *No regularity assumption is made on the Cauchy data.*

Remark 4. *In Part (2), smallness assumption on the Cauchy data is made with respect to its homogeneous weights. The L^2 norm of the Cauchy data need not be small, while its homogeneously weighted norm should be relatively and sufficiently small. For instance, data of the form $\epsilon^{-1}\psi(\epsilon^{-1}x, \epsilon^{-1}y)$ with $\epsilon > 0$ sufficiently small fall within the scope of Part (2). The available literature, however, does not cover those data since the L^2 norm of the data is also required to be small [15,17,18].*

We prove Theorem 1 in Section 3. The method is almost the same as in [21] except that $L^2_x L^\infty_y$ norm is used instead of weaker norm $L^\infty_y L^2_x$, since the former is necessary for Theorem 3. We prove Theorem 2 in Section 4. The method depends essentially on that of Theorem 1 with regularity in terms of Galilei transforms. We prove Theorem 3 in Sections 5,6, and 7, following basically the method of Hayashi and Naumkin [15,16,17] (see also [18]). The following ingredients are new and necessary to provide improvements, however. First, our method depends exclusively on a contraction argument and is independent of a contradiction argument in [15-18]. Secondly, our method depends exclusively on the generators of Galilei transforms and is independent of the usual regularity argument. This enables us not to impose any regularity assumption on the Cauchy data. Thirdly, our argument treats the L^2 norm and weighted norms separately for the Cauchy data as well as for solutions. This enables us not to impose smallness of the L^2 norm of the Cauchy data. Lastly, asymptotic formulas in Part (3) are simpler than those in [15] and uniqueness of asymptotic states in those formulas is also proved.

§2. Preliminary Estimates

In this section we collect some basic estimates of the free propagator $U(t)$ and the nonlinear term $f(u)$ in the anisotropic space.

Lemma 2.1. *$U(t)$ satisfies the following estimates:*

(1) Let r and δ satisfy $2 \le r \le \infty$, $\delta = 1/2 - 1/r$. Then for $t \neq 0$

$$\|U(t)\phi; L^2_x L^r_y\| \le (2\pi|t|)^{-\delta} \|\phi; L^2_x L^{r'}_y\|. \tag{2.1}$$

(2) For any (q,r) with $0 \le 2/q = 1/2 - 1/r \le 1/2$

$$\|U(\cdot)\phi; L^q(\mathbb{R}; L^2_x L^r_y)\| \le C\|\phi; L^2\|. \tag{2.2}$$

(3) For any (q_1, r_1) and (q_2, r_2) with $0 \le 2/q_j = 1/2 - 1/r_j \le 1/2$, $j = 1, 2$, for any interval $I \subset \mathbb{R}$ which may be unbounded, and for any $t_0 \in \overline{I}$ the operator G_{t_0} defined by

$$(G_{t_0} u)(t) = \int_{t_0}^{t} U(t - t')u(t')dt' \tag{2.3}$$

satisfies the estimate

$$\|G_{t_0} u; L^{q_1}\left(I; L^2_x L^{r_1}_y\right)\| \le C\|u; L^{q_2'}(I; L^2_x L^{r_2'}_y)\|, \tag{2.4}$$

where C is independent of I and t_0.

Proof. See [21], where the lemma is stated in a weak form, though the proof there works with slight modifications. For a general framework, see [5,9,23,24,34]. QED

Lemma 2.2. *Let r_j, $0 \le j \le 3$, satisfy $1 \le r_j \le \infty$ and $1/r_0 = 1/r_1 + 1/r_2 + 1/r_3$. Then*

(1)

$$\|\psi_1 \int_{-\infty}^{x} (\psi_2\psi_3)(x', y)dx'; L^{r_0}_y L^2_x\| \le \prod_{j=1}^{3} \|\psi_j; L^{r_j}_y L^2_x\|.$$

(2) For any α with $0 < \alpha < 1/2$

$$\|\psi_1 \int_{-\infty}^{x} (\psi_2\psi_3)(x', y)dx'; L^{r_0}_y \dot{H}^{\alpha}_x\| \le C\|\psi_1; L^{r_1}_y \dot{H}^{\alpha}_x\| \prod_{j=2}^{3} \|\psi_j; L^{r_j}_y L^2_x\|.$$

(3) For any β with $0 < \beta < 1$

$$\|\psi_1 \int_{-\infty}^{x} (\psi_2\psi_3)(x', y)dx'; L^2_x \dot{H}^{\beta}_y\| \le C \sum_{j=1}^{3} \|\psi_j; L^2_x \dot{H}^{\beta}_y\| \prod_{k \neq j} \|\psi_k; L^2_x L^{\infty}_y\|$$

Proof. For Part (1), see [21]. To prove Part (2), let

$$\varphi(x,y) = \int_{-\infty}^{x} (\psi_2\psi_3)(x',y)dx'$$

and we estimate the product $\psi_1\varphi$ in $\dot{H}^\alpha_x$ using Leibniz' rule [25] as

$$\begin{aligned}\|\psi_1\varphi; \dot{H}^\alpha_x\| &\le \|(-\Delta_x)^{\alpha/2}(\psi_1\varphi) - \psi_1(-\Delta_x)^{\alpha/2}\varphi - \varphi(-\Delta_x)^{\alpha/2}\psi_1; L^2_x\| \\ &\quad + \|\psi_1(-\Delta_x)^{\alpha/2}\varphi; L^2_x\| + \|\varphi(-\Delta_x)^{\alpha/2}\psi_1; L^2_x\| \\ &\le C\|\varphi; L^\infty_x\|\,\|(-\Delta_x)^{\alpha/2}\psi_1; L^2_x\| + \|\psi_1(-\Delta_x)^{\alpha/2}\varphi; L^2_x\|.\end{aligned}$$

The first term on the RHS of the last inequality is bounded by

$$\|\psi_2; L^2_x\|\,\|\psi_3; L^2_x\|\,\|\psi_1; \dot{H}^\alpha_x\|,$$

while the second term is represented in terms of the Hilbert transform H as

$$\|\psi_1 H(-\Delta_x)^{(\alpha-1)/2}(\psi_2\psi_3); L^2_x\|,$$

which is estimated by the generalized Hölder inequality in the Lorentz spaces [29,30] as

$$C\|\psi_1; L^{2/(1-2\alpha),2}_x\|\,\|H(-\Delta_x)^{(\alpha-1)/2}(\psi_2\psi_3); L^{1/\alpha,\infty}_x\|.$$

We now use the boundedness of the Hilbert transform and the Riesz potentials in the Lorentz spaces:

$$\begin{aligned}&(-\Delta_x)^{-\alpha/2} : L^2_x = L^{2,2}_x \longrightarrow L^{2/(1-2\alpha),2}_x, \\ &(-\Delta_x)^{-(\alpha-1)/2} : L^1_x \longrightarrow L^{1/\alpha,\infty}_x, \\ &H : L^{1/\alpha,\infty}_x \longrightarrow L^{1/\alpha,\infty}_x\end{aligned}$$

(see [27,35] for instance). Collecting these estimates above, we obtain

$$\|\psi_1\varphi; \dot{H}^\alpha_x\| \le C\|\psi_1; \dot{H}^\alpha_x\| \prod_{j=2}^{3} \|\psi_j; L^2_x\|.$$

Then, Part (2) follows from the Hölder inequality in y-variable. We now prove Part (3). We estimate $\psi_1\varphi$ in $\dot{H}^\beta_y$ using Leibniz' rule [25] as above to obtain

$$\|\psi_1\varphi; \dot{H}^\beta_y\| \le C\|\varphi; L^\infty_y\|\,\|\psi_1; \dot{H}^\beta_y\| + C\|\psi_1(-\Delta_y)^{\beta/2}\varphi; L^2_y\|$$

This implies that

$$\|\psi_1\varphi; L^2_x\dot{H}^\beta_y\| \le C\|\varphi; L^\infty\|\,\|\psi_1; L^2_x\dot{H}^\beta_y\| + C\|\psi_1(-\Delta_y)^{\beta/2}\varphi; L^2\|.$$

For the first term on the RHS of the last inequality, we estimate

$$\|\varphi; L^\infty\| \le \|\prod_{j=2}^{3} \|\psi_j; L^\infty_y\|; L^1_x\| \le \prod_{j=2}^{3} \|\psi_j; L^2_x L^\infty_y\|,$$

while the second term is estimated as

$$\begin{aligned}
&\|\psi_1(-\Delta_y)^{\beta/2}\varphi; L^2\| \le \|\|\psi_1; L^2_x\| \, \|(-\Delta_y)^{\beta/2}\varphi; L^\infty_x\|; L^2_y\| \\
&\le \|\psi_1; L^\infty_y L^2_x\| \, \|(-\Delta_y)^{\beta/2}(\psi_2\psi_3); L^2_y L^1_x\| \\
&\le \|\psi_1; L^2_x L^\infty_y\| \, \|(-\Delta_y)^{\beta/2}(\psi_2\psi_3); L^1_x L^2_y\| \\
&\le C\|\psi_1; L^2_x L^\infty_y\| \, \|\|\psi_2; L^\infty_y\| \, \|(-\Delta_y)^{\beta/2}\psi_3; L^2_y\|; L^1_x\| \\
&+ C\|\psi_1; L^2_x L^\infty_y\|\|\|\psi_3; L^\infty_y\|\|(-\Delta_y)^{\beta/2}\psi_2; L^2_y\|; L^1_x\| \\
&\le C\prod_{j=1}^{2} \|\psi_j; L^2_x L^\infty_y\| \cdot \|(-\Delta_y)^{\beta/2}\psi_3; L^2\| \\
&+ C\prod_{j\ne 2} \|\psi_j; L^2_x L^\infty_y\| \cdot \|(-\Delta_y)^{\beta/2}\psi_2; L^2\|,
\end{aligned}$$

where we have used the Hölder and Minkowski inequalities and Leibniz' rule [25]. Collecting these inequalities yields Part (3). QED

§3. Proof of Theorem 1

For $T > 0$ we define $X(I)$ with $I = [t_0 - T, t_0 + T]$ as in the introduction and equip $X(I)$ with norm

$$|||u||| = \|u; L^\infty(I; L^2)\| + \|u; L^4(I; L^2_x L^\infty_y)\|.$$

For $\phi \in L^2$ and $u \in X(I)$ we define

$$(\Phi(u))(t) = U(t - t_0)\phi - i\,(G_{t_0} f(u))(t). \tag{3.1}$$

By Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned}
&|||\Phi(u)||| \le C\|\phi; L^2\| + C\|f(u); L^{4/3}(I; L^2_x L^1_y)\| \\
&\le C\|\phi; L^2\| + C\|f(u); L^{4/3}(I; L^1_y L^2_x)\| \\
&\le C\|\phi; L^2\| + CT^{1/2}\|u; L^\infty(I; L^2_y L^2_x)\|^2 \cdot \|u; L^4(I; L^\infty_y L^2_x)\| \\
&\le C\|\phi; L^2\| + CT^{1/2}|||u|||^3,
\end{aligned} \tag{3.2}$$

where we have used the Minkowski and Hölder inequalities. Similarly, for $u, v \in X(I)$ we obtain

$$|||\Phi(u) - \Phi(v)||| \le CT^{1/2}(|||u|||^2 + |||v|||^2)\, |||u - v|||. \tag{3.3}$$

By a contraction argument with (3.2) and (3.3), for any $\phi \in L^2$ there exists $T > 0$ depending only on $\|\phi; L^2\|$ such that (3.1) has a unique fixed point $u \in X(I)$. The rest of the proof proceeds in the standard way as in [5,9,24,33] and is omitted. QED

§4. Proof of Theorem 2

Let $\phi \in L^2$ satisfy $|x|^j U(-t_0)\phi \in L^2$ with $j \ge 1$. For any T with $0 < T \le 1$ we define $Y^{j,0}(I)$ with $I = [t_0 - T, t_0 + T]$ as in the introduction and equip the closed ball B_R of $Y^{j,0}(I)$ with radius $R > 0$ and center at the origin with the metric associated with the norm $|||\cdot|||$ in the proof of Theorem 1. By the argument in [23,24] it suffices to show that Φ leaves B_R invariant for some $R > 0$. Let $u \in B_R$ and let

$$g(u) = \lambda \int_{-\infty}^{x} |u(t, x', y)|^2 dx'. \tag{4.1}$$

Then, by the relation $J_x^j = M_x (it\partial_x)^j M_x^{-1}$, we obtain

$$\begin{aligned} &J_x^j\,(f(u)) = g(u) J_x^j u \\ &+ it\lambda \sum_{k_1+k_2+k_3=j-1} \frac{j!\,(-1)^{k_2}}{(j-k_3)k_1!\,k_2!\,k_3!} J_x^{k_1} u \cdot \overline{J_x^{k_2} u} \cdot J_x^{k_3} u. \end{aligned} \tag{4.2}$$

We estimate the RHS of (4.2) in $L^{4/3}(I; L_y^1 L_x^2)$. By Lemma 2.2, the first term is estimated as

$$\begin{aligned} &\|g(u) J_x^j u; L^{4/3}(I; L_y^1 L_x^2)\| \\ &\le CT^{1/2} \|u; L^\infty(I; L^2)\|^2\, \|J_x^j u; L^4(I; L_y^\infty L_x^2)\| \le CT^{1/2} R^3. \end{aligned} \tag{4.3}$$

If $k_1 + k_2 + k_3 = j - 1 \ge 0$, then by the Hölder and Gagliardo-Nirenberg inequalities we have

$$\begin{aligned} &\|J_x^{k_1} u \cdot \overline{J_x^{k_2} u} \cdot J_x^{k_3} u; L_x^2\| \le \prod_{l=1}^{3} \|J_x^{k_l} u; L_x^{2(j-1)/k_l}\| \\ &\le C \prod_{l=1}^{3} \|u; L_x^2\|^{1-\delta_l}\, \|J_x^j u; L_x^2\|^{\delta_l} = C\|u; L_x^2\|^2\, \|J_x^j u; L_x^2\|, \end{aligned} \tag{4.4}$$

where $\delta_l = \frac{k_l}{j} - \frac{k_l}{2j\,(j-1)} + \frac{1}{2j}$. This yields

$$\begin{aligned} &\|J_x^{k_1} u \cdot \overline{J_x^{k_2} u} \cdot J_x^{k_3} u; L^{4/3}(I; L_y^1 L_x^2)\| \\ &\le CT^{1/2} \|u; L^\infty(I; L^2)\|^2\, \|J_x^j u; L^4(I; L_y^\infty L_x^2)\| \le CT^{1/2}\, R^3. \end{aligned} \tag{4.5}$$

By Lemma 2.1, (4.3), (4.5), we have

$$|||J_x^j\Phi(u)||| \le C\||x|^jU(-t_0)\phi; L^2\| + CT^{1/2}\,R^3, \tag{4.6}$$

which implies that B_R is invariant under Φ for some $R>0$, as was to be shown.

We now let $\phi\in L^2$ satisfy $|y|^kU(-t_0)\phi\in L^2$ with $k\ge 1$. For any $T>0$ we define $Y^{0,k}(I)$ with $I=[t_0-T,t_0+T]$ as in the introduction and equip the closed ball B_R of $Y^{0,k}(I)$ with the metric induced by $|||\cdot|||$. We prove that Φ leaves B_R invariant for some $R>0$. For $u\in B_R$ we have

$$J_y^k\left(f(u)\right) = \lambda \sum_{j_1+j_2+j_3=k} \frac{k!\,(-1)^{j_2}}{j_1!\,j_2!\,j_3!}\int_{-\infty}^{x} J_y^{j_1}u\cdot\overline{J_y^{j_2}u}\,dx'\cdot J_y^{j_3}u. \tag{4.7}$$

We estimate the RHS of (4.7) in $L^1(I;L^2)$. By the Gagliardo-Nirenberg and Hölder inequalities, we obtain with $\delta_l=j_l/k$

$$\begin{aligned}
&\|J_y^k\left(f(u)\right); L_y^2\| \le C\sum_{j_1+j_2+j_3=k}\|\prod_{l=1}^{2}\|J_y^{j_l}u; L_y^{2k/j_l}\|; L_x^1\|\,\|J_y^{j_3}u; L_y^{2k/j_3}\| \\
&\le C\sum_{j_1+j_2+j_3=k}\|\prod_{l=1}^{2}\|u; L_y^\infty\|^{1-\delta_l}\,\|J_y^ku; L_y^2\|^{\delta_l}; L_x^1\|\cdot\|u; L_y^\infty\|^{1-\delta_3}\,\|J_y^ku; L_y^2\|^{\delta_3} \\
&\le C\sum_{j_1+j_2+j_3=k}\prod_{l=1}^{2}\|u; L_x^2L_y^\infty\|^{1-\delta_l}\,\|J_y^ku; L^2\|^{\delta_l}\cdot\|u; L_y^\infty\|^{1-\delta_3}\,\|J_y^ku; L_y^2\|^{\delta_3},
\end{aligned}$$

from which we obtain

$$\begin{aligned}
&\|J_y^k\left(f(u)\right); L^1(I;L^2)\| \\
&\le C\sum_{j_1+j_2+j_3=k}\|\prod_{l=1}^{3}\|u; L_x^2L_y^\infty\|^{1-\delta_l}\,\|J_y^ku; L^2\|^{\delta_l}; L^1(I)\| \\
&\le C\|\|u; L_x^2L_y^\infty\|^2\|J_y^ku; L^2\|; L^1(I)\| \\
&\le CT^{1/2}\|u; L^4(I;L_x^2L_y^\infty)\|^2\,\|J_y^ku; L^\infty(I;L^2)\| \le CT^{1/2}\,R^3.
\end{aligned} \tag{4.8}$$

By Lemma 2.1 and (4.8), we have

$$|||J_y^k\Phi(u)||| \le C\||y|^kU(-t_0)\phi; L^2\| + CT^{1/2}\,R^3,$$

which implies that B_R is invariant under Φ for some $R>0$, as was to be shown. QED

§5. Proof of Theorem 3, Part (1)

For any $T > 0$ we define $Y^{\alpha,\beta}(I)$ with $I = [t_0 - T, t_0 + T]$ as in the introduction. As in the preceding sections it suffices to show that Φ leaves the closed ball B_R of $Y^{\alpha,\beta}(I)$ with radius $R > 0$ invariant for some R. Let $u \in B_R$. We estimate $|J_x|^\alpha (f(u))$ in $L^{4/3}(I; L^1_y L^2_x)$. By Lemma 2.2 with $\psi_1 = \psi_3 = M_x^{-1}\, u$ and $\psi_2 = \overline{M_x^{-1}\, u}$, we have

$$\begin{aligned}
&\||J_x|^\alpha (f(u))\,; L^{4/3}(I; L^1_y L^2_x)\| \\
&= \|(-t^2\Delta_x)^{\alpha/2}\ \big(f(M_x^{-1}\, u)\big)\,; L^{4/3}(I; L^1_y L^2_x)\| \\
&\le CT^{1/2}\|u; L^\infty(I; L^2)\|^2\, \||J_x|^\alpha u; L^4(I; L^\infty_y L^2_x)\| \le CT^{1/2}\, R^3. \qquad (5.1)
\end{aligned}$$

Similarly, we estimate $|J_y|^\beta (f(u))$ in $L^1(I; L^2)$ as

$$\begin{aligned}
&\||J_y|^\beta (f(u))\,; L^1(I; L^2)\| \\
&= \|(-t^2\Delta_y)^{\beta/2}\ \big(f(M_y^{-1}\, u)\big)\,; L^1(I; L^2)\| \\
&\le CT^{1/2}\, \|u; L^4(I; L^2_x L^\infty_y)\|^2\, \||J_y|^\beta u; L^\infty(I; L^2)\| \le CT^{1/2}\, R^3. \qquad (5.2)
\end{aligned}$$

Then Part (1) follows from Lemma 2.1, (5.1), and (5.2). QED

§6. Proof of Theorem 3, Part (2)

Throughout this section we fix α and β as $0 < \alpha < 1/2 < \beta < 1$ and we put $\theta = 1/2\beta$, so that $0 < \theta < 1$. Moreover, we consider the case $t > 0$ only since the case $t < 0$ may be treated similarly. We first prove the following lemma.

Lemma 6.1. *For any $t \ge 1$, the following estimates hold.*

$$\|\mathcal{F}u; L^2_x L^\infty_y\| \le C\|\mathcal{F}u; L^2_x \dot{H}^\beta_y\|^\theta \|u; L^2\|^{1-\theta}. \qquad (6.1)$$

$$\|\mathcal{F}U^{-1}u; L^2_x L^\infty_y\| \le C\|\mathcal{F}U^{-1}u; L^2_x \dot{H}^\beta_y\|^\theta \|u; L^2\|^{1-\theta}. \qquad (6.2)$$

$$\|\mathcal{F}MU^{-1}u; L^2_x L^\infty_y\| \le C\|\mathcal{F}U^{-1}u; L^2_x \dot{H}^\beta_y\|^\theta \|u; L^2\|^{1-\theta}. \qquad (6.3)$$

$$\|(M-1)U^{-1}u; L^2\| \le Ct^{-\alpha/2}\|\mathcal{F}U^{-1}u; \dot{H}^{\alpha,\beta}\|. \qquad (6.4)$$

$$\|\mathcal{F}(M-1)U^{-1}u; L^2_x L^\infty_y\| \le Ct^{-(1-\theta)\alpha/2}\|\mathcal{F}U^{-1}u; \dot{H}^{\alpha,\beta}\|. \qquad (6.5)$$

$$\|u; L^2_x L^\infty_y\| \le Ct^{-1/2}\|\mathcal{F}U^{-1}u; L^2_x L^\infty_y\| + Ct^{-1/2-(1-\theta)\alpha/2}\|\mathcal{F}U^{-1}u; \dot{H}^{\alpha,\beta}\|. \qquad (6.6)$$

Proof. For (6.1) we apply the Gagliardo-Nirenberg inequality in y-variable, the Hölder inequality in x-variable, and the unitarity of the Fourier transform in L^2 to obtain

$$\begin{aligned}
\|\mathcal{F}u; L^2_x L^\infty_y\| &\le C\|\|\mathcal{F}u; \dot{H}^\beta_y\|^\theta \|\mathcal{F}u; L^2_y\|^{1-\theta}; L^2_x\| \\
&\le C\|\mathcal{F}u; L^2_x \dot{H}^\beta_y\|^\theta \|u; L^2\|^{1-\theta}.
\end{aligned}$$

Estimate (6.2) follows from (6.1) and the unitarity of the free propagator $U(t)$. For (6.3), we apply (6.1) to obtain

$$\begin{aligned}
&\|\mathcal{F}MU^{-1}u; L^2_xL^\infty_y\| \\
&\le C\|\mathcal{F}MU^{-1}u; L^2_x\dot{H}^\beta_y\|^\theta\|MU^{-1}u; L^2\|^{1-\theta} \\
&= C\||y|^\beta MU^{-1}u; L^2\|^\theta\|u; L^2\|^{1-\theta} \\
&= C\||y|^\beta U^{-1}u; L^2\|^\theta\|u; L^2\|^{1-\theta} = C\|\mathcal{F}U^{-1}u; L^2_x\dot{H}^\beta_y\|^\theta\|u; L^2\|^{1-\theta}.
\end{aligned}$$

For (6.4), we use the estimate

$$\begin{aligned}
&|M(t)-1| = |M_x(t)(M_y(t)-1)+(M_x(t)-1)| \\
&\le |M_y(t)-1|+|M_x(t)-1| \\
&\le \mathrm{Min}\left(2, t^{-1}|y|^2\right)+\mathrm{Min}\left(2, t^{-1}|x|^2\right) \\
&\le Ct^{-\beta/2}|y|^\beta + Ct^{-\alpha/2}|x|^\alpha \\
&\le Ct^{-\alpha/2}\left(|x|^\alpha+|y|^\beta\right)
\end{aligned}$$

to obtain

$$\begin{aligned}
&\|(M-1)U^{-1}u; L^2\| \le Ct^{-\alpha/2}\left(\||x|^\alpha U^{-1}u; L^2\|+\||y|^\beta U^{-1}u; L^2\|\right) \\
&\le Ct^{-\alpha/2}\|\mathcal{F}U^{-1}u; \dot{H}^{\alpha,\beta}\|.
\end{aligned}$$

For (6.5), we use (6.1) and (6.4) to obtain

$$\begin{aligned}
&\|\mathcal{F}(M-1)U^{-1}u; L^2_xL^\infty_y\| \\
&\le C\|\mathcal{F}(M-1)U^{-1}u; L^2_x\dot{H}^\beta_y\|^\theta\|(M-1)U^{-1}u; L^2\|^{1-\theta} \\
&\le C\||y|^\beta U^{-1}u; L^2\|^\theta\|(M-1)U^{-1}u; L^2\|^{1-\theta} \\
&\le C\|\mathcal{F}U^{-1}u; \dot{H}^{\alpha,\beta}\|^\theta\left(t^{-\alpha/2}\|\mathcal{F}U^{-1}u; \dot{H}^{\alpha,\beta}\|\right)^{1-\theta} \\
&= Ct^{-(1-\theta)\alpha/2}\|\mathcal{F}U^{-1}u; \dot{H}^{\alpha,\beta}\|.
\end{aligned}$$

For (6.6), we estimate u in $L^2_xL^\infty_y$ in the form

$$u = MD\mathcal{F}U^{-1}u + MD\mathcal{F}(M-1)U^{-1}u$$

as

$$\|u; L^2_xL^\infty_y\| \le t^{-1/2}\|\mathcal{F}U^{-1}u; L^2_xL^\infty_y\| + t^{-1/2}\|\mathcal{F}(M-1)U^{-1}u; L^2_xL^\infty_y\|,$$

where we estimate the second term on the RHS by (6.5). QED

Let $\rho_0 > 0$ be given. For ϵ, $\rho > 0$ to be determined later, we define the following set $X_\epsilon(\rho, \rho_0)$ of functions over the time interval $[1, \infty)$ as

$$X_\epsilon(\rho, \rho_0) = \{u \in C\left([1,\infty); L^2\right); \mathcal{F}U(-t)u(t) \in L^2_x L^\infty_y \cap \dot{H}^{\alpha,\beta} \quad \text{a.e.} \quad t, \\ \|u; L^\infty(1,\infty; L^2)\| \le \rho_0,\ |||u|||_\epsilon \le \rho\},$$

where

$$|||u|||_\epsilon = \mathrm{Max}(\text{Ess. Sup}_{t\ge 1} \|\mathcal{F}U(-t)u(t); L^2_x L^\infty_y\|, \\ \text{Ess. Sup}_{t\ge 1} t^{-\epsilon}\|\mathcal{F}U(-t)u(t); \dot{H}^{\alpha,\beta}\|).$$

Let $\tilde{\phi} \in L^2$ satisfy $\|\tilde{\phi}; L^2\| \le \rho_0$ and $\mathcal{F}U(-1)\tilde{\phi} \in \dot{H}^{\alpha,\beta}$ and let $u \in X_\epsilon(\rho, \rho_0)$. We consider the integral equation

$$v(t) = U(t-1)\tilde{\phi} - i\int_1^t U(t-s)g(u(s))\,v(s)ds, \tag{6.7}$$

where

$$g(u(t)) = \lambda \int_{-\infty}^x |u(t, x', y)|^2 dx'.$$

As in the arguments in [34] and in the preceding section, the equation (6.7) has a unique solution $v \in Y^{\alpha,\beta}_{loc}$.

Proposition 6.1. *Let $\rho_0 > 0$. Let ϵ satisfy $0 < \epsilon < (1-\theta)\alpha/(2(2+\theta))$. Then there exists $\rho_1 > 0$ with the following property: For any ρ with $0 < \rho \le \rho_1$ and any $\tilde{\phi} \in L^2$ with $\|\tilde{\phi}; L^2\| \le \rho_0$ and $\|\mathcal{F}U(-1)\tilde{\phi}; \dot{H}^{\alpha,\beta}\| \le \rho/2$, $v \in X_\epsilon(\rho, \rho_0)$.*

Proof. Applying $U(-t)$ to both sides of (6.7), we have

$$U(-t)v(t) = U(-1)\tilde{\phi} - i\int_1^t U(-s)g(u(s))\,v(s)ds. \tag{6.8}$$

Differentiating both sides of (6.8) in t, we have

$$\partial_t(U(-t)v(t)) = -iU(-t)g(u(t))\,u(t).$$

This implies

$$\begin{aligned}
&\frac{d}{dt}\|U(-t)v(t); L^2\|^2 = 2\,\mathrm{Re}\,(\partial_t(U(-t)v(t)), U(-t)v(t)) \\
&= 2\,\mathrm{Im}\,(U(-t)g(u(t))\,v(t), U(-t)v(t)) \\
&= 2\,\mathrm{Im}\,(g(u(t))v(t), v(t)) \\
&= 0
\end{aligned}$$

and therefore

$$\begin{aligned}\|v(t);L^2\| &= \|U(-t)v(t);L^2\| \\ &= \|U(-1)v(1);L^2\| = \|\tilde{\phi};L^2\| \le \rho_0. \end{aligned} \tag{6.9}$$

Applying $U(-t)|J_x|^\alpha = |x|^\alpha U(-t)$ to both sides of (6.7), we have

$$U(-t)|J_x|^\alpha v(t) = |x|^\alpha U(-1)\tilde{\phi} - i\int_1^t U(-s)|J_x|^\alpha \left(g(u)v\right)(s)ds. \tag{6.10}$$

Differentiating both sides of (6.10) in t, we have

$$U(-t)|J_x|^\alpha v(t) = -iU(-t)|J_x|^\alpha \left(g(u)v\right)(t).$$

This implies

$$\begin{aligned}&\frac{d}{dt}\||J_x|^\alpha v(t);L^2\|^2 = \frac{d}{dt}\|U(-t)|J_x|^\alpha v(t);L^2\|^2 \\ &= 2\,\mathrm{Im}\left(|J_x|^\alpha\left(g(u)v\right), |J_x|^\alpha v\right) \\ &\le C\||J_x|^\alpha u;L^2\|\|u;L^2_xL^\infty_y\|\|v;L^2_xL^\infty_y\|\||J_x|^\alpha v;L^2\|,\end{aligned}$$

where we have used Part (2) of Lemma 2.2 with the relation $|J_x|^\alpha = M_x(t)(-t^2\Delta_x)^{\alpha/2}M_x(-t)$. Since $u \in X_\epsilon(\rho,\rho_0)$, we have

$$\frac{d}{dt}\||J_x|^\alpha v;L^2\| \le C\rho^2 t^{-1/2+\epsilon}\|v;L^2_xL^\infty_y\|$$

and therefore

$$\||J_x|^\alpha v(t);L^2\| \le \||x|^\alpha U(-1)\tilde{\phi};L^2\| + C\rho^2\int_1^t s^{-1/2+\epsilon}\|v(s);L^2_xL^\infty_y\|ds. \tag{6.11}$$

Similarly, we obtain

$$\||J_y|^\beta v(t);L^2\| \le \||y|^\beta U(-1)\tilde{\phi};L^2\| + C\rho^2\int_1^t s^{-1/2+\epsilon}\|v(s);L^2_xL^\infty_y\|ds. \tag{6.12}$$

By (6.11), (6.12), and (6.6), we have

$$\begin{aligned}\|\mathcal{F}U(-t)v(t);\dot{H}^{\alpha,\beta}\| &\le \|\mathcal{F}U(-1)\tilde{\phi};\dot{H}^{\alpha,\beta}\| \\ &\quad + C\rho^2\int_1^t s^{-1+\epsilon}\|\mathcal{F}U(-s)v(s);L^2_xL^\infty_y\|ds \\ &\quad + C\rho^2\int_1^t s^{-1+\epsilon-(1-\theta)\alpha/2}\|\mathcal{F}U(-s)v(s);\dot{H}^{\alpha,\beta}\|ds.\end{aligned}$$

Since $(1-\theta)\alpha/2-\epsilon>0$, Gronwall's inequality yields

$$\begin{aligned}
&\|\mathcal{F}U(-t)v(t);\dot{H}^{\alpha,\beta}\| \\
&\leq \exp\left(C\rho^2\right)\|\mathcal{F}U(-1)\tilde{\phi};\dot{H}^{\alpha,\beta}\| \\
&+C\exp\left(C\rho^2\right)\rho^2\int_1^t s^{-1+\epsilon}\|\mathcal{F}U(-s)v(s);L_x^2L_y^\infty\|ds.
\end{aligned} \tag{6.13}$$

We now consider $\mathcal{F}U(-t)u(t)$ in $L_x^2L_y^\infty$. Taking the Fourier transform on both sides of (6.8), we have

$$\mathcal{F}U(-t)v(t)=\mathcal{F}U(-1)\tilde{\phi}-i\int_1^t \mathcal{F}U(-s)g\left(u(s)\right)v(s)ds. \tag{6.14}$$

Differentiating both sides of (6.14) in t and making use of the factorization $U=MD\mathcal{F}M$, we compute

$$\begin{aligned}
i\partial_t\left(\mathcal{F}U^{-1}v\right) &= \mathcal{F}U^{-1}\left(g(u)v\right) \\
&= \mathcal{F}M^{-1}\mathcal{F}^{-1}D^{-1}M^{-1}\left(g(u)v\right) \\
&= \mathcal{F}M^{-1}\mathcal{F}^{-1}D^{-1}\left(g(M^{-1}u)M^{-1}v\right) \\
&= t^{-1}\mathcal{F}M^{-1}\mathcal{F}\left(g(D^{-1}M^{-1}u)D^{-1}M^{-1}v\right) \\
&= t^{-1}\mathcal{F}M^{-1}\mathcal{F}\left(g(\mathcal{F}MU^{-1}u)\mathcal{F}MU^{-1}v\right) \\
&= t^{-1}\left(g(\mathcal{F}U^{-1}u)\mathcal{F}U^{-1}v+I+I\!I\right),
\end{aligned} \tag{6.15}$$

where

$$I=g(\mathcal{F}MU^{-1}u)\mathcal{F}(M-1)U^{-1}v+\left(g(\mathcal{F}MU^{-1}u)-g(\mathcal{F}U^{-1}u)\right)\mathcal{F}U^{-1}v,$$

$$I\!I=\mathcal{F}(M-1)\mathcal{F}^{-1}\left(g(\mathcal{F}MU^{-1}u)\mathcal{F}MU^{-1}v\right).$$

By (6.15),

$$\begin{aligned}
&i\partial_t\left(\exp(i\int_1^t g(\mathcal{F}U^{-1}u)s^{-1}ds)\mathcal{F}U^{-1}v\right) \\
&=\exp\left(i\int_1^t g(\mathcal{F}U^{-1}u)s^{-1}ds\right)t^{-1}\left(I+I\!I\right).
\end{aligned} \tag{6.16}$$

By Lemmas 2.2 and 6.1, we estimate I in $L^2_xL^\infty_y$ as

$$\begin{aligned}
&\|I; L^2_xL^\infty_y\| \le \|\mathcal{F}MU^{-1}u; L^2_xL^\infty_y\|^2\|\mathcal{F}(M-1)U^{-1}v; L^2_xL^\infty_y\| \\
&+ \|\mathcal{F}(M-1)U^{-1}u; L^2_xL^\infty_y\|\|\mathcal{F}MU^{-1}u; L^2_xL^\infty_y\|\|\mathcal{F}U^{-1}v; L^2_xL^\infty_y\| \\
&+ \|\mathcal{F}U^{-1}u; L^2_xL^\infty_y\|\|\mathcal{F}(M-1)U^{-1}u; L^2_xL^\infty_y\|\|\mathcal{F}U^{-1}v; L^2_xL^\infty_y\| \\
&\le C\left(\|\mathcal{F}U^{-1}u; L^2_x\dot{H}^\beta_y\|^\theta\|u; L^2\|^{1-\theta}\right)^2\|\mathcal{F}(M-1)U^{-1}v; L^2_xL^\infty_y\| \\
&+ Ct^{-(1-\theta)\alpha/2}\|\mathcal{F}U^{-1}u; \dot{H}^{\alpha,\beta}\|\|\mathcal{F}U^{-1}u; L^2_x\dot{H}^\beta_y\|^\theta\|u; L^2\|^{1-\theta}\|\mathcal{F}U^{-1}v; L^2_xL^\infty_y\| \\
&\le C\rho^{2\theta}\rho_0^{2-2\theta}t^{2\theta\epsilon-(1-\theta)\alpha/2}\|\mathcal{F}U^{-1}v; \dot{H}^{\alpha,\beta}\| \\
&+ C\rho^{1+\theta}\rho_0^{1-\theta}t^{(1+\theta)\epsilon-(1-\theta)\alpha/2}\|\mathcal{F}U^{-1}v; L^2_xL^\infty_y\|. \qquad (6.17)
\end{aligned}$$

Similarly, we estimate $I\!I$ in $L^2_xL^\infty_y$ as

$$\|I\!I; L^2_xL^\infty_y\| \le Ct^{-(1-\theta)\alpha/2}\|g(\mathcal{F}MU^{-1}u)\mathcal{F}MU^{-1}v; \dot{H}^{\alpha,\beta}\|,$$

$$\begin{aligned}
&\|g(\mathcal{F}MU^{-1}u)\mathcal{F}MU^{-1}v; L^2_y\dot{H}^\alpha_x\| \\
&\le C\|\mathcal{F}MU^{-1}u; L^2_xL^\infty_y\|^2\|\mathcal{F}MU^{-1}v; L^2_y\dot{H}^\alpha_x\| \\
&\le C\rho^{2\theta}\rho_0^{2-2\theta}t^{2\theta\epsilon}\|\mathcal{F}U^{-1}v; L^2_y\dot{H}^\alpha_x\|,
\end{aligned}$$

$$\begin{aligned}
&\|g(\mathcal{F}MU^{-1}u)\mathcal{F}MU^{-1}v; L^2_x\dot{H}^\beta_y\| \\
&\le C\|\mathcal{F}MU^{-1}u; L^2_x\dot{H}^\beta_y\|\|\mathcal{F}MU^{-1}u; L^2_xL^\infty_y\|\|\mathcal{F}MU^{-1}v; L^2_xL^\infty_y\| \\
&+ C\|\mathcal{F}MU^{-1}u; L^2_xL^\infty_y\|^2\|\mathcal{F}MU^{-1}v; L^2_x\dot{H}^\beta_y\| \\
&\le C\rho^{1+\theta}\rho_0^{1-\theta}t^{(1+\theta)\epsilon}\|\mathcal{F}U^{-1}v; L^2_x\dot{H}^\beta_y\|^\theta\|v; L^2\|^{1-\theta} \\
&+ C\rho^{2\theta}\rho_0^{2-2\theta}t^{2\theta\epsilon}\|\mathcal{F}U^{-1}v; L^2_x\dot{H}^\beta_y\|
\end{aligned}$$

to obtain

$$\begin{aligned}
\|I\!I; L^2_xL^\infty_y\| &\le C\rho^{2\theta}\rho_0^{2-2\theta}t^{2\theta\epsilon-(1-\theta)\alpha/2}\|\mathcal{F}U^{-1}v; \dot{H}^{\alpha,\beta}\| \\
&\quad + C\rho^{1+\theta}\rho_0^{1-\theta}t^{(1+\theta)\epsilon-(1-\theta)\alpha/2}\|\mathcal{F}U^{-1}v; \dot{H}^{\alpha,\beta}\|^\theta\|v; L^2\|^{1-\theta}. \qquad (6.18)
\end{aligned}$$

Integrating (6.16) over the time interval $[1, t]$ and estimating the resulting time inte-

gral in $L^2_x L^\infty_y$ with (6.17) and (6.18), we obtain

$$
\begin{aligned}
&\|\mathcal{F}U^{-1}v; L^2_x L^\infty_y\| \\
&= \|\exp\left(i\int_1^t g(\mathcal{F}U^{-1}u)s^{-1}ds\right)\mathcal{F}U^{-1}v; L^2_x L^\infty_y\| \\
&\le \|\mathcal{F}U(-1)\tilde{\phi}; L^2_x L^\infty_y\| + \int_1^t s^{-1}\|I + I\!I; L^2_x L^\infty_y\|ds \\
&\le \|\mathcal{F}U(-1)\tilde{\phi}; L^2_x L^\infty_y\| \\
&+ C\rho^{1+\theta}\rho_0^{1-\theta}\int_1^t s^{-1-(1-\theta)\alpha/2+(1+\theta)\epsilon}\|\mathcal{F}U^{-1}v; L^2_x L^\infty_y\|ds \\
&+ C\rho^{2\theta}\rho_0^{2-2\theta}\int_1^t s^{-1-(1-\theta)\alpha/2+(1+\theta)\epsilon}\|\mathcal{F}U^{-1}v; \dot{H}^{\alpha,\beta}\|ds \\
&+ C\rho^{1+\theta}\rho_0^{1-\theta}\int_1^t s^{-1-(1-\theta)\alpha/2+(1+\theta)\epsilon}\|\mathcal{F}U^{-1}v; \dot{H}^{\alpha,\beta}\|^\theta\|v; L^2\|^{1-\theta}ds.
\end{aligned}
$$

Since $(1-\theta)\alpha/2-(1+\theta)\epsilon > 0$, Gronwall's inequality yields

$$
\begin{aligned}
&\|\mathcal{F}U^{-1}v; L^2_x L^\infty_y\| \\
&\le \exp\left(C\rho^{1+\theta}\rho_0^{1-\theta}\right)\|\mathcal{F}U(-1)\tilde{\phi}; L^2_x L^\infty_y\| \\
&+ C\exp\left(C\rho^{1+\theta}\rho_0^{1-\theta}\right)\rho^{2\theta}\rho_0^{2-2\theta}\int_1^t s^{-1-(1-\theta)\alpha/2+(1+\theta)\epsilon}\|\mathcal{F}U^{-1}v; \dot{H}^{\alpha,\beta}\|ds \\
&+ C\exp\left(C\rho^{1+\theta}\rho_0^{1-\theta}\right)\rho^{1+\theta}\rho_0^{1-\theta} \\
&\qquad\qquad \cdot\int_1^t s^{-1-(1-\theta)\alpha/2+(1+\theta)\epsilon}\|\mathcal{F}U^{-1}v; \dot{H}^{\alpha,\beta}\|^\theta\|v; L^2\|^{1-\theta}ds.
\end{aligned}
\tag{6.19}
$$

The last integral on the RHS of (6.19) is bounded by

$$
\begin{aligned}
&\int_1^t s^{-1-(1-\theta)\alpha/2+(1+\theta)\epsilon}\|\mathcal{F}U^{-1}v; \dot{H}^{\alpha,\beta}\|^\theta\,\rho_0^{1-\theta}ds \\
&\le C\rho_0 + C\int_1^t s^{-1-(1-\theta)\alpha/2+(1+\theta)\epsilon}\|\mathcal{F}U^{-1}v; \dot{H}^{\alpha,\beta}\|ds,
\end{aligned}
\tag{6.20}
$$

where we have used (6.9). The last integral on the RHS of (6.20) is the same as the first integral on the RHS of (6.19). We substitute (6.13) into the last integral on the RHS of (6.20) to obtain

$$
\begin{aligned}
&\int_1^t s^{-1-(1-\theta)\alpha/2+(1+\theta)\epsilon}\|\mathcal{F}U^{-1}v; \dot{H}^{\alpha,\beta}\|ds \\
&\le C\exp\left(C\rho^2\right)\|\mathcal{F}U(-1)\tilde{\phi}; \dot{H}^{\alpha,\beta}\| \\
&+ C\exp\left(C\rho^2\right)\rho^2\int_1^t s^{-1-(1-\theta)\alpha/2+(2+\theta)\epsilon}\|\mathcal{F}U^{-1}v; L^2_x L^\infty_y\|ds.
\end{aligned}
\tag{6.21}
$$

Collecting (6.19), (6.20), and (6.21), we have

$$\begin{aligned}
&\|\mathcal{F}U^{-1}v; L^2_xL^\infty_y\| \\
&\le C\exp\left(C\rho^{1+\theta}\rho_0^{1-\theta}\right)\left(\|\mathcal{F}U(-1)\tilde{\phi}; L^2_xL^\infty_y\| + \rho^{1+\theta}\rho_0^{2-\theta}\right) \\
&+ K\|\mathcal{F}U(-1)\tilde{\phi}; \dot{H}^{\alpha,\beta}\| \\
&+ K\rho^2\int_1^t s^{-1-\delta}\|\mathcal{F}U^{-1}v; L^2_xL^\infty_y\|ds,
\end{aligned} \tag{6.22}$$

where $\delta = (1-\theta)\alpha/2 - (2+\theta)\epsilon$ and $K = C\exp\left(C(\rho^2 + \rho^{1+\theta}\rho_0^{1-\theta})\right)(\rho^{2\theta}\rho_0^{2-2\theta} + \rho^{1+\theta}\rho_0^{1-\theta})$. Since $\delta > 0$, Gronwall's inequality yields

$$\begin{aligned}
&\|\mathcal{F}U^{-1}v; L^2_xL^\infty_y\| \\
&\le C\exp\left(C(K\rho^2 + \rho^{1+\theta}\rho_0^{1-\theta})\right)\left(\|\mathcal{F}U(-1)\tilde{\phi}; \dot{H}^{\alpha,\beta}\|^\theta\rho_0^{1-\theta} + \rho^{1+\theta}\rho_0^{2-\theta}\right) \\
&+ C\exp\left(CK\rho^2\right)\, K\,\|\mathcal{F}U(-1)\tilde{\phi}; \dot{H}^{\alpha,\beta}\|,
\end{aligned} \tag{6.23}$$

where we have used (6.1).

Substituting (6.23) into (6.13) and denoting by R the RHS of (6.23), we have

$$\|\mathcal{F}U^{-1}v; \dot{H}^{\alpha,\beta}\| \le \exp\left(C\rho^2\right)\|\mathcal{F}U(-1)\tilde{\phi}; \dot{H}^{\alpha,\beta}\| + C\exp\left(C\rho^2\right)\rho^2\, R\, t^\epsilon. \tag{6.24}$$

The proposition follows from (6.23) and (6.24) by taking ρ sufficiently small. QED

For $u_1, u_2 \in X_\epsilon(\rho, \rho_0)$ we define

$$d(u_1, u_2) = \mathrm{Sup}_{t\ge 1}\, t^{-\epsilon}\|u_1(t) - u_2(t); L^2\|.$$

Proposition 6.2. *$X_\epsilon(\rho, \rho_0)$ is a complete metric space with respect to d.*

Proof. Let $\{u_n\} \subset X_\epsilon(\rho, \rho_0)$ be a Cauchy sequence with respect to d. For any $\delta > 0$ there exists N such that for any m, n with $m, n \ge N$, $d(u_m, u_n) < \delta$. For any t, $\{u_n(t)\}$ is a Cauchy sequence and has a limit $u(t)$ in L^2. For any $n \ge N$, $t^{-\epsilon}\|u_n(t) - u(t); L^2\| \le \delta$. This implies that $d(u_n, u) \to 0$. Moreover, $\|u(t); L^2\| \le \rho_0$. Since $t^{-\epsilon}\|\mathcal{F}U(-t)u_n(t); \dot{H}^{\alpha,\beta}\| \le \rho$, for a.e. t, $t^{-\epsilon}\|\mathcal{F}U(-t)u(t); \dot{H}^{\alpha,\beta}\| \le \rho$. By Lemma 6.1, $\mathcal{F}U(-t)u(t) \in L^2_xL^\infty_y$ for a.e. t. Let $\psi \in L^2 \cap L^2_xL^1_y$ satisfy $\|\psi; L^2_xL^1_y\| \le 1$. Then $|(\mathcal{F}U(-t)u(t), \psi)| = \lim_{n\to\infty}|(\mathcal{F}U(-t)u_n(t), \psi)| \le \mathrm{Sup}_{n\ge 1}\|\mathcal{F}U(-t)u_n(t); L^2_xL^\infty_y\|$ $\le \rho$ and therefore $\|\mathcal{F}U(-t)u(t); L^2_xL^\infty_y\| \le \rho$. This proves $u \in X_\epsilon(\rho, \rho_0)$. QED

Proposition 6.3. *Let ϵ and ρ_1 be as in Proposition 6.1. Then for any ρ with $0 < \rho \le \mathrm{Min}\left(\rho_1, \epsilon^{1/2}/C_0\right)$ for some constant $C_0 > 0$, the map $u \mapsto v$ is a contraction on $X_\epsilon\left(\rho, \rho_0\right)$ with respect to d.*

Proof. Let $u_1, u_2 \in X_\epsilon\left(\rho, \rho_0\right)$ and let $v_1, v_2 \in X_\epsilon\left(\rho, \rho_0\right)$ be solutions of (6.7). Then

$$v_1(t) - v_2(t) = -i \int_1^t U(t-s)\left(g(u_1)v_1 - g(u_2)v_2\right)(s)ds.$$

In the same way as in the proof of Proposition 6.1, we obtain

$$\begin{aligned}
&\frac{d}{dt}\|v_1 - v_2; L^2\|^2 = \frac{d}{dt}\|U^{-1}(v_1 - v_2); L^2\|^2 \\
&= 2\,\mathrm{Re}\left(\partial_t(U^{-1}(v_1 - v_2)), U^{-1}(v_1 - v_2)\right) \\
&= 2\,\mathrm{Im}\left(g(u_1)v_1 - g(u_2)v_2, v_1 - v_2\right) \\
&= 2\,\mathrm{Im}\left((g(u_1) - g(u_2))v_1, v_1 - v_2\right) \\
&\le C\left(\|u_1; L^2_x L^\infty_y\| + \|u_2; L^2_x L^\infty_y\|\right)\|u_1 - u_2; L^2\|\|v_1; L^2_x L^\infty_y\|\|v_1 - v_2; L^2\| \\
&\le C\rho^2 t^{-1}\|u_1 - u_2; L^2\|\|v_1 - v_2; L^2\|.
\end{aligned}$$

This implies

$$\begin{aligned}
\|v_1(t) - v_2(t); L^2\| &\le C\rho^2 \int_1^t s^{-1}\|u_1(s) - u_2(s); L^2\|ds \\
&\le C\rho^2\,\epsilon^{-1}\,t^\epsilon\,d(u_1, u_2)
\end{aligned}$$

and therefore

$$d(v_1, v_2) \le C\rho^2\,\epsilon^{-1}\,d(u_1, u_2).$$

This proves the proposition with $C_0 > C^{1/2}$. QED

Proof of Theorem 3, *Part* (2).

It follows from Propositions 6.1, 6.2, and 6.3 that for any $\tilde{\phi} \in B\left(\rho/2, \rho_0\right)$ there exists a unique solution $\tilde{u} \in X_\epsilon\left(\rho, \rho_0\right)$ of the integral equation

$$\tilde{u}(t) = U(t-1)\tilde{\phi} - i\int_1^t U(t-s)g(\tilde{u})\tilde{u}(s)ds. \tag{6.25}$$

It follows from Part (1) of Theorem 3 that for any $\phi \in L^2 \cap \mathcal{F}(\dot{H}^{\alpha,\beta})$ there exists a unique solution $u \in Y^{\alpha,\beta}_{loc}$ of the integral equation

$$u(t) = U(t)\phi - i\int_0^t U(t-s)g(u)u(s)ds. \tag{6.26}$$

Let $\rho_0 = \|\phi; L^2\|$. In the same way as in the proof of Proposition 6.1, we see that the solution u of (6.26) satisfies

$$\frac{d}{dt}\|u(t); L^2\|^2 = \frac{d}{dt}\|U(-t)u(t); L^2\|^2 = 0, \tag{6.27}$$

$$\frac{d}{dt}\|\mathcal{F}U(-t)u(t); L^2_x \dot{H}^\beta_y\|^2 \le C\|u; L^2_x L^\infty_y\|^2 \|\mathcal{F}U(-t)u(t); L^2_x \dot{H}^\beta_y\|^2, \tag{6.28}$$

$$\frac{d}{dt}\|\mathcal{F}U(-t)u(t); L^2_y \dot{H}^\alpha_x\|^2 \le C\|u; L^2_x L^\infty_y\|^2 \|\mathcal{F}U(-t)u(t); L^2_y \dot{H}^\alpha_x\|^2. \tag{6.29}$$

By (6.27),

$$\|u(t); L^2\| = \rho_0. \tag{6.30}$$

By (6.28) and (6.29),

$$\|\mathcal{F}U(-t)u(t); \dot{H}^{\alpha,\beta}\| \le \|\hat{\phi}; \dot{H}^{\alpha,\beta}\| \exp\left(Ct^{1/2}\, \|u; L^4(0,t; L^2_x L^\infty_y)\|^2\right), \tag{6.31}$$

where we have used Gronwall's inequality and the Hölder inequality in t. In particular, we have from (6.30), (6.31)

$$\|u(1); L^2\| = \rho_0, \tag{6.32}$$

$$\|\mathcal{F}U(-1)u(1); \dot{H}^{\alpha,\beta}\| \le \|\hat{\phi}; \dot{H}^{\alpha,\beta}\| \exp\left(C\|u; L^4(0,1; L^2_x L^\infty_y)\|^2\right). \tag{6.33}$$

By the argument in the proof of Theorem 1, we note that $\|u; L^4(0,1; L^2_x L^\infty_y)\|$ depends on the data only through $\|\phi; L^2\| = \rho_0$. Hence there exists $\rho > 0$ such that for any $\phi \in B(\rho/4, \rho_0)$, $u(1) \in B(\rho/2, \rho_0)$. Taking ρ smaller if necessary, with data $\tilde{\phi} = u(1)$ we have the solution $\tilde{u}$ of (6.25). We define $v(t) = u(t)$ for $t \in [0,1]$ and $v(t) = \tilde{u}(t)$ for $t \in [1,\infty)$. Then v satisfies

$$\begin{aligned}
v(t) &= U(t-1)\tilde{\phi} - i\int_1^t U(t-s)g(v)v(s)ds \\
&= U(t-1)u(1) - i\int_1^t U(t-s)g(v)v(s)ds \\
&= U(t)\phi - i\int_0^1 U(t-s)g(u)u(s)ds - i\int_1^t U(t-s)g(v)v(s)ds \\
&= U(t)\phi - i\int_0^t U(t-s)g(v)v(s)ds.
\end{aligned}$$

By uniqueness for (6.26), we have $u = v$, so that $u \in X_\epsilon(\rho, \rho_0)$. This proves Part (2) by changing ρ in the statement of the theorem. QED

§7. Proof of Theorem 3, Part (3)

Let $\alpha, \beta, \theta, \epsilon$ be as in the preceding section. Let $\delta \equiv (1-\theta)\alpha/2 - (2\theta+1)\epsilon > 0$. Let u be the solution given by Part (2). Therefore $u \in X_\epsilon(\rho, \rho_0)$ on the time interval $[1, \infty)$. We consider the asymptotic behavior of $\mathcal{F}U(-t)u(t)$ in $L^2 \cap L^2_x L^\infty_y$. For that purpose we define

$$w(t) = \exp\left(i \int_1^t g(\mathcal{F}U^{-1}u)s^{-1}ds\right) \mathcal{F}U(-t)u(t). \tag{7.1}$$

In the same way as in (6.16), we have

$$i\partial_t w(t) = \exp\left(i \int_1^t g(\mathcal{F}U^{-1}u)s^{-1}ds\right) t^{-1}\left(I + I\!I\right), \tag{7.2}$$

where

$$\begin{aligned} I &= g(\mathcal{F}MU^{-1}u)\mathcal{F}MU^{-1}u - g(\mathcal{F}U^{-1}u)\mathcal{F}U^{-1}u, \\ I\!I &= \mathcal{F}(M-1)\mathcal{F}^{-1}g(\mathcal{F}MU^{-1}u)\mathcal{F}MU^{-1}u. \end{aligned}$$

In the same way as in (6.17) and (6.18), we have

$$\|I; L^2_x L^\infty_y\| + \|I\!I; L^2_x L^\infty_y\| \le Ct^{-\delta}. \tag{7.3}$$

Here and hereafter, we omit explicit dependence of constants on ρ and ρ_0. Similarly, we have

$$\|I; L^2\| + \|I\!I; L^2\| \le Ct^{-\delta}. \tag{7.4}$$

Integrating both sides of (7.2) in t and estimating the resulting time integral in $L^2 \cap L^2_x L^\infty_y$, we obtain from (7.3) and (7.4)

$$\|w(t) - w(s); L^2 \cap L^2_x L^\infty_y\| \le Cs^{-\delta} \tag{7.5}$$

for any t, s with $t > s \ge 1$. It follows from (7.5) that there exists $w_+ \in L^2 \cap L^2_x L^\infty_y$ such that

$$\|w(t) - w_+; L^2 \cap L^2_x L^\infty_y\| \le Ct^{-\delta} \tag{7.6}$$

for all $t \ge 1$. We now define

$$\psi(t) = \int_1^t \left(g(w(s)) - g(w(t))\right) s^{-1}ds. \tag{7.7}$$

Then we have for $t > s \ge 1$

$$\psi(t) - \psi(s) = \int_s^t \left(g(w(\tau)) - g(w(t))\right)\tau^{-1}d\tau - \left(g(w(t)) - g(w(s))\right)\log s \tag{7.6}$$

and therefore

$$
\begin{aligned}
&\|\psi(t)-\psi(s);L^\infty\| \\
&\le C\int_s^t \left(\|w(\tau);L^2_xL^\infty_y\|+\|w(t);L^2_xL^\infty_y\|\right)\|w(\tau)-w(t);L^2_xL^\infty_y\|\,\tau^{-1}d\tau \\
&+C\left(\|w(t);L^2_xL^\infty_y\|+\|w(s);L^2_xL^\infty_y\|\right)\|w(t)-w(s);L^2_xL^\infty_y\|\,\log s \\
&\le C\int_s^t \tau^{-1-\delta}d\tau + Cs^{-\delta}\,\log s\ \le Cs^{-\delta}(1+\log s),
\end{aligned}
\tag{7.7}
$$

where we have used (7.5). It follows from (7.7) that there exists $\psi_+\in L^\infty$ such that

$$
\|\psi(t)-\psi_+;L^\infty\|\le Ct^{-\delta}(1+\log t) \tag{7.8}
$$

for all $t\ge 1$. By (7.6), (7.7), and (7.8), we obtain

$$
\begin{aligned}
&\|\int_1^t g(w(s))s^{-1}ds-g(w_+)\log t-\psi_+;L^\infty\| \\
&=\|(\psi(t)-\psi_+)+(g(w(t))-g(w_+))\,\log t;L^\infty\| \\
&\le \|\psi(t)-\psi_+;L^\infty\|+C\left(\|w(t);L^2_xL^\infty_y\|+\|w_+;L^2_xL^\infty_y\|\right)\|w(t)-w_+;L^2_xL^\infty_y\|\,\log t \\
&\le Ct^{-\delta}(1+\log t)
\end{aligned}
\tag{7.9}
$$

for all $t\ge 1$. Since $g(\mathcal{F}U^{-1}u)=g(w)$, we have from (7.6) and (7.9)

$$
\begin{aligned}
&\|\mathcal{F}U(-t)u(t)-\exp\left(-ig(w_+)\,\log t-i\psi_+\right)w_+;L^2\cap L^2_xL^\infty_y\| \\
&\le \|w(t)-w_+;L^2\cap L^2_xL^\infty_y\| \\
&+\|w_+-\exp\left(i\left(\int_1^t g(w(s))s^{-1}ds-g(w_+)\,\log t-\psi_+\right)\right)w_+;L^2\cap L^2_xL^\infty_y\| \\
&\le Ct^{-\delta}+C\|w_+;L^2\cap L^2_xL^\infty_y\|\|\int_1^t g(w(s))s^{-1}ds-g(w_+)\log t-\psi_+;L^\infty\| \\
&\le Ct^{-\delta}(1+\log t)
\end{aligned}
\tag{7.10}
$$

for all $t\ge 1$. We define

$$
\phi_+=\mathcal{F}^{-1}\left(w_+\exp(-i\psi_+)\right). \tag{7.11}
$$

Then $\phi_+\in L^2\cap\mathcal{F}(L^2_xL^\infty_y)$ and (7.10) is rewritten as

$$
\|\mathcal{F}U(-t)u(t)-\exp\left(-ig(\hat\phi_+)\log t\right)\hat\phi_+;L^2\cap L^2_xL^\infty_y\|\le Ct^{-\delta}(1+\log t), \tag{7.12}
$$

from which existence of ϕ_+ for the first asymptotic formula follows. We now prove the uniqueness. let $\Phi_+\in L^2\cap\mathcal{F}(L^2_xL^\infty_y)$ satisfies the asymptotic formula

$$
\|\mathcal{F}U(-t)u(t)-\exp\left(-ig(\hat\Phi_+)\log t\right)\hat\Phi_+;L^2\cap L^2_xL^\infty_y\|\longrightarrow 0 \tag{7.13}
$$

as $t \to \infty$. Then by (7.12) and (7.13),

$$\| \exp\left(-ig(\hat{\phi}_+)\log t\right)\hat{\phi}_+ - \exp\left(-ig(\hat{\Phi}_+)\log t\right)\hat{\Phi}_+; L^2 \cap L^2_x L^\infty_y \| \longrightarrow 0 \tag{7.14}$$

as $t \to \infty$. This implies

$$\begin{aligned}
&\||\hat{\phi}_+|^2 - |\hat{\Phi}_+|^2; L^1\| \\
&= \||\exp\left(-ig(\hat{\phi}_+)\log t\right)\hat{\phi}_+|^2 - |\exp\left(-ig(\hat{\Phi}_+)\log t\right)\hat{\Phi}_+|^2; L^1\| \\
&\le \left(\|\hat{\phi}_+; L^2\| + \|\hat{\Phi}_+; L^2\|\right)\| \exp\left(-ig(\hat{\phi}_+)\log t\right)\hat{\phi}_+ - \exp\left(-ig(\hat{\Phi}_+)\log t\right)\hat{\Phi}_+; L^2\| \\
&\longrightarrow 0
\end{aligned}$$

as $t \to \infty$, which implies $|\hat{\phi}_+| = |\hat{\Phi}_+|$ and hence $g(\hat{\phi}_+) = g(\hat{\Phi}_+)$. Therefore, by (7.14)

$$\begin{aligned}
\|\hat{\phi}_+ - \hat{\Phi}_+; L^2\| &= \| \exp\left(-ig(\hat{\phi}_+)\log t\right)\left(\hat{\phi}_+ - \hat{\Phi}_+\right); L^2\| \\
&= \| \exp\left(-ig(\hat{\phi}_+)\log t\right)\hat{\phi}_+ - \exp\left(-ig(\hat{\Phi}_+)\log t\right)\hat{\Phi}_+; L^2\| \longrightarrow 0
\end{aligned}$$

and hence $\phi_+ = \Phi_+$, as required.

Finally, we prove other asymptotic formulas. By unitarity, we have

$$\begin{aligned}
&\|u(t) - v_1^+(t); L^2\| \\
&= \|u(t) - U(t)\mathcal{F}^{-1}\exp\left(-ig(\hat{\phi}_+)\log t\right)\mathcal{F}\phi_+; L^2\| \\
&= \|\mathcal{F}U(-t)u(t) - \exp\left(-ig(\hat{\phi}_+)\log t\right)\hat{\phi}_+; L^2\| \le Ct^{-\delta}(1+\log t),
\end{aligned}$$

where we have used (7.12). We use the last decomposition in the proof of Lemma 6.1 to write

$$\begin{aligned}
&u(t) - v_2^+(t) \\
&= MD\left(\mathcal{F}U(-t)u(t) - \exp\left(-ig(\hat{\phi}_+)\log t\right)\hat{\phi}_+\right) + MD\mathcal{F}(M-1)U(-t)u(t).
\end{aligned}$$

By (7.12) and (6.4), we obtain

$$\begin{aligned}
\|u(t) - v_2^+(t); L^2\| &\le \|\mathcal{F}U(-t)u(t) - \exp\left(-ig(\hat{\phi}_+)\log t\right)\hat{\phi}_+; L^2\| \\
&\quad + Ct^{-\alpha/2}\|\mathcal{F}U(-t)u(t); \dot{H}^{\alpha,\beta}\| \\
&\le Ct^{-\delta}(1+\log t) + Ct^{\epsilon-\alpha/2}.
\end{aligned}$$

We write $u(t) - v_3^+(t)$ as

$$u(t) - v_3^+(t) = U(1 - M^{-1})U^{-1}u + MD\left(\mathcal{F}U^{-1}u - \exp\left(-ig(\hat{\phi}_+)\log t\right)\hat{\phi}_+\right)$$
$$+ MD\exp\left(-ig(\hat{\phi}_+)\log t\right)\mathcal{F}(1 - M)\phi_+.$$

This implies

$$\|u(t) - v_3^+(t); L^2\| \leq \|(1 - M^{-1})U^{-1}u; L^2\|$$
$$+ \|\mathcal{F}U^{-1}u - \exp\left(-ig(\hat{\phi}_+)\log t\right)\hat{\phi}_+; L^2\| + \|(1 - M)\hat{\phi}_+; L^2\| \longrightarrow 0$$

as $t \to \pm\infty$, as was to be shown. QED

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