THE DEGREE OF THE DUAL OF A HOMOGENEOUS SPACE

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1. Introduction

This study is a joint work with A. Némethi and R. Rimányi. Suppose that $X \subset \mathbb{P}^n$ is a smooth algebraic variety. Then

$$\check{X} := \{H \in \mathbb{P}^{n*} : H \text{ is tangent to } X\}$$

is called the dual of X. (If X is not smooth then the dual can still be defined: apply the previous construction to the smooth part and close it.) Duality is a beautiful chapter of projective algebraic geometry connected with singularity theory, discriminants etc.

In "generic" cases the dual of X is a hypersurface. Giving the defining equation is usually a too ambitious task. So most results are about the degree of this hypersurface. Kleiman [Kle76] and Katz [Kat76] found the following beautiful formula:

$$\deg(\check{X}) = \int_X c_n(J(L))$$

where L is the tautological line bundle restricted to X and $J(L)$ is its first jet bundle. The proof is quite simple. The reader might get the impression that after this formula there is nothing to do. However already Kleiman noticed that the simplicity of the formula is misleading. Let us look at the example of the Plücker embedding $Gr_k(\mathbb{C}^n) \to \mathbb{P}(\Lambda^k\mathbb{C}^n)$. The reader in encouraged to try the calculations for small values of k and n. The difficulty is that we don't have a nice description of the Chern classes of the (tangent bundle of the) Grassmannian. Lascoux [Las81] translated the problem into K-theory and gave an algorithm to calculate the degree of the dual of the Grassmannian.

The more general setup is the following: Let $\rho : G \to GL(V)$ be an irreducible representation of the Lie group G on the complex vector space V. For technical reasons we assume that:

Assumption 1.1. $\mathrm{Im}(\rho)$ *contains the scalars.*

ρ induces an action of G on the projective space $\mathbb{P}(V)$. This action has a single closed orbit, the orbit of the weight vector of the highest weight λ. E.g. for the standard representation of $GL(n)$ on $\Lambda^k\mathbb{C}^n$ this orbit is just the Grassmannian $Gr_k(\mathbb{C}^n)$. The dual D_λ of this orbit is called the *discriminant* of ρ since it generalizes the classical discriminant. The goal of this paper is to give a computable formula for the degree of the discriminant.

In several special cases formulas were obtained ([Tev01] [CW97] and for a near complete list see [Tev01]), however in some cases computability is only a theoretical possibility.

Using localization techniques in equivariant cohomology we derived a formula given purely in terms of the representation ρ. In fact we give a formula (Theorem 2.1) for the the equivariant cohomology class (with rational coefficients)

$$[C_\lambda] \in H^2_G(V) \cong H^2(BG)$$

represented by $C_\lambda \subset V$—the cone of the discriminant D_λ. (You might call $[C_\lambda]$ the equivariant Poincaré dual or Thom polynomial of C_λ.) The cohomology class $[C_\lambda]$ is in $H^2(BG)$ because we assume that C_λ is a hypersurface. Then we show in Theorem 2.2 how to calculate the degree from this information.

2. The equivariant cohomology ring of V

Let $T \subset G$ be a maximal torus of G. Then by a theorem of Borel the induced map $H^*(BG) \to H^*(BT)$ is injective and its image is the subring of $H^*(BT)$ invariant for the Weyl group W. So we consider $H^*(BG)$ to be a subring of $H^*(BT)$. For every weight $w : T \to U(1)$ we can associate an element in $H^2(BT)$ by taking $c_1(L_w)$ where c_1 is the first Chern class of the line bundle over BT associated to w. We will use the same notation for the weights and the corresponding cohomology classes. $H^*(BT) \cong \mathbb{Q}[x_1, \dots, x_r]$ where $\{x_1, \dots, x_r\}$ is a basis for the weights.

Theorem 2.1. *Let $\rho : G \to GL(V)$ be an irreducible representation of the Lie group G such that* $\mathrm{Im}(\rho)$ *contains the scalars. Let λ be the highest weight of ρ. Then*

$$[C_\lambda] = \sum_{\mu \in W\lambda} \mu \prod_{\beta \in T_\lambda} \frac{\beta + \mu}{\beta}$$

where $W\lambda$ is the orbit of λ under the Weyl group and

$$T_\lambda = \{\beta \in R^-(G) : B(\beta, \lambda) < 0\},$$

where $R^-(G)$ denotes the set of negative roots of G and B denotes the Killing form.

Since we assume that the image of ρ contains the scalars there is a homomorphism $\varphi : GL(1) \to G$ and a non zero integer q such that $\rho \circ \varphi(z) = z^q v$ for all $v \in V$, $z \in GL(1)$. For the inclusion $m : T \to G$ we can assume that $\mathrm{Im}\, m \supset \mathrm{Im}\, \varphi|_{U(1)}$ so we have a homomorphism $\tilde{\varphi} : U(1) \to U(1)^n$ such that $\varphi|_{U(1)} = m \circ \tilde{\varphi}$. The homomorphism $\tilde{\varphi}$ is necessarily of the form $\tilde{\varphi}(t) = (t^{w_1}, \dots, t^{w_n})$ where $t \in U(1)$ and w_i are integers.

Theorem 2.2. *Suppose that $X \subset V$ is a G-invariant subvariety. Then using the Borel theorem $[X] = p(x_1, \dots, x_r)$ where p is a polynomial and*

$$\deg(X) = p(\frac{w_1}{q}, \dots, \frac{w_r}{q}).$$

Theorem 2.1 and Theorem 2.2 gives a practical way to calculate these degrees.

3. Examples

If we take $G = GL(n)$ then the weights are of the form $\lambda = \sum_{i=1}^{n} a_i L_i$. A highest weight is always dominant i.e. the sequence a_i is non-increasing. $R^-(G) = \{L_i - L_j : i > j\}$ and $T_\lambda = \{L_i - L_j : i > j,\ a_i < a_j\}$. The Weyl group is the symmetric group acting by permuting the L_i's. If we apply Theorem 2.2 to $G = GL(n)$ we get that

Proposition 3.1.

$$\deg(D_\lambda) = \frac{n}{\sum a_i} \cdot \frac{[C_\lambda]}{\sum L_i}$$

Example 3.2. A simple induction argument gives that $[C_{kL_1}] = k(k-1)^{n-1} \sum_{i=1}^{n} L_i$. So by Proposition 3.1 $\deg(D_{kL_1}) = n(k-1)^{n-1}$ which is known as the *Boole* formula. (Notice that $\lambda = kL_1$ corresponds to the Veronese embedding.)

Example 3.3. For $\lambda = \sum_{i=1}^{k} L_i$ we get the Plücker embedding.

$$[C_{\sum_{i=1}^{k} L_i}] = \sum_{S \in \binom{n}{k}} L_S \prod_{i \notin S, j \in S} \frac{L_S + L_i - L_j}{(L_i - L_j)}$$

where $\binom{n}{k}$ denotes the k-element subsets of $\{1, 2, \ldots, n\}$ and $L_S = \sum_{i \in S} L_i$. Applying Proposition 3.1 and substituting $L_i = i$ we get an expression for the degree of the dual of the Grassmannians

$$\deg(\check{G}r(k,n)) = \frac{2k}{n+1} \sum_{S \in \binom{n}{k}} w(S) \prod_{i \notin S, j \in S} \frac{w(S) + i - j}{(i-j)},$$

where $w(S) = \sum_{k \in S} k$.

Our formula has about the same complexity as of the formula of Lascoux. On a personal computer you can calculate e.g. the case of $k = 3$ and $n = 30$ in 5 minutes. But for small values of k and n you can calculate by hand noticing that for an appropriate choice of x the substitutions $L_i = i - x$ work even better since in most terms of the formula there will be a 0 factor.

Example 3.4. Let $k = 3$ and $n = 8$. Apply the substitution $L_i = i - \frac{13}{3}$. Then all terms—labelled by 3-element subsets of $\{1, \ldots, 8\}$—are zero except for the last one $(6, 7, 8)$, so

$$\deg(\check{G}r_k(\mathbb{C}^n)) = \frac{8}{3} \cdot \frac{1}{\binom{9}{2} - 8 \cdot \frac{13}{3}} \cdot 8 \cdot \frac{(5\cdot6\cdot7)(4\cdot5\cdot6)(3\cdot4\cdot5)(2\cdot3\cdot4)(1\cdot2\cdot3)}{(1\cdot2\cdot3)(2\cdot3\cdot4)(3\cdot4\cdot5)(4\cdot5\cdot6)(5\cdot6\cdot7)} = \frac{8}{3} \cdot \frac{3}{4} \cdot 8 = 16$$

We reproved some other existing formulas using Theorem 2.1 and conjectured some new ones (generalizing formulas of Tevelev for $\lambda = aL_1 + L_2$). We spent a considerable effort on the case of $\lambda = L_1 + L_2 + L_3$ (i.e. $Gr_3(\mathbb{C}^n)$) since we can easily get the degrees for $n < 30$ but couldn't find even a conjectural formula.

4. About the proof

An idea from [FNR] reduces the problem to the calculation of an integral over a resolution of the dual. But the incidence correspondence is a resolution if the dual is a hypersurface. Then we apply the Atiyah-Bott localization theorem and the integral is reduced to a sum over the fixed points of the T-action on the incidence correspondence. Some miraculous cancellations simplify the sum.

References

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