

# Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation<sup>1</sup>

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**Abstract.** We prove endpoint Strichartz estimates for the Klein-Gordon and wave equations in mixed norms on the polar coordinates in three spatial dimensions. As an application, global wellposedness of the nonlinear Dirac equation is shown for small data in the energy class with some regularity assumption for the angular variable.

## 1. INTRODUCTION

Let us consider the Klein–Gordon equation in three spatial dimensions :

$$\partial_t^2 u - \Delta u + m^2 u = 0, \quad (1.1)$$

where  $u : \mathbb{R}^{1+3} \rightarrow \mathbb{C}$  and  $m \geq 0$  is the mass constant. The endpoint Strichartz estimate

$$\|u\|_{L_t^2 L_x^\infty} \lesssim E(u)^{1/2} \quad (1.2)$$

is known to be false in general [4, 16], where  $E(u)$  is the conserved energy defined by

$$E(u) = E(u; t) := \int_{\mathbb{R}^3} |\partial_t u|^2 + |\nabla u|^2 + m^2 |u|^2 dx (= E(u; 0)). \quad (1.3)$$

Moreover, Montgomery-Smith [8] has shown that even if we replace the  $L^\infty$  norm in (1.2) by  $BMO$ , the estimate does not hold. On the other hand, Klainerman and Machedon [4] proved that the estimate (1.2) holds if  $u$  is radial and  $m = 0$ . Then a natural question arises: To what extent does the endpoint estimate depend on the radial symmetry? Our theorem below answers that it is very little. We denote the polar coordinates by  $x = r\theta$ ,  $r = |x|$ ,  $\theta \in S^2$ .

**Theorem 1.1.** (I) *For any  $m \geq 0$ , any  $1 \leq p < \infty$  and any finite energy solution  $u$  of (1.1), we have*

$$\|u\|_{L_t^2 L_r^\infty L_\theta^p} \leq C(pE(u))^{1/2}, \quad (1.4)$$

where  $C$  is a positive absolute constant.

(II) *The power  $p^{1/2}$  in (1.4) is optimal in the following sense: For any  $m \geq 0$  and*

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any  $\varepsilon > 0$ , there exists a finite energy solution  $u$  of (1.1) satisfying

$$\lim_{p \rightarrow \infty} \|u\|_{L_t^2 L_r^\infty L_\theta^p} / p^{1/2-\varepsilon} = \infty. \quad (1.5)$$

(III) For any finite energy solution  $u$  of the wave equation (1.1) with  $m = 0$ , we have

$$\|u\|_{L_t^2 L_r^\infty H_\theta^{3/4}} \leq CE(u)^{1/2}, \quad (1.6)$$

where  $C$  is a positive absolute constant.

The first statement implies that if the initial data has slight additional regularity for rotation  $H_\theta^\varepsilon$ ,  $\varepsilon > 0$ , then the endpoint Strichartz  $L_t^2 L_x^\infty$  is recovered by the Sobolev embedding  $H_\theta^{\varepsilon,p} \hookrightarrow L_\theta^\infty$ ,  $p > 2/\varepsilon$ . Notice that the optimal power  $p^{1/2}$  in (1.4) is the same as in the critical Sobolev embedding  $H_\theta^1 \hookrightarrow p^{1/2} L_\theta^p$ .

We do not know if  $H_\theta^{3/4}$  in (1.6) can be improved to higher Sobolev norm  $H_\theta^s$ . However we can show an upper bound  $s \leq 5/6$  (Theorem 5.1), and so the  $L_\theta^p$  estimate (I) for  $p > 12$  can not be recovered from  $H_\theta^s$  and the Sobolev embedding.

We remark that as for the Schrödinger equation in two spatial dimensions, Tao [15] proved the following endpoint estimate:

$$\|u\|_{L_t^2 L_r^\infty H_\theta^s} \lesssim \|u(0)\|_{L_x^2}, \quad (1.7)$$

for some small  $s > 0$ . In this case we have an upper bound  $s \leq 1/3$  (Theorem 5.1), and so  $L_\theta^p$  estimate for  $p > 6$  can not be obtained by the Sobolev embedding. It seems open if we can replace  $H_\theta^s$  by  $L_\theta^p$  for all  $p < \infty$  in the Schrödinger case (1.7).

Our primary motivation for the above endpoint estimates was application to nonlinear wave equations. Indeed, the lack of the endpoint estimate causes in some cases serious difficulties to prove wellposedness; the following Cauchy problem for the nonlinear Dirac equation is a good example.

$$\sum_{\alpha=0}^3 i\gamma^\alpha \partial_\alpha u - mu = \lambda(\gamma^0 u, u)u, \quad (1.8)$$

$$u(0, x) = \varphi(x),$$

where  $\varphi(x) : \mathbb{R}^3 \rightarrow \mathbb{C}^4$  is the given,  $u(t, x) : \mathbb{R}^{1+3} \rightarrow \mathbb{C}^4$  is the unknown function,  $m \geq 0$  and  $\lambda \in \mathbb{C}$  are given constants,  $(\partial_0, \partial_1, \partial_2, \partial_3) = (\partial_t, \nabla_x)$  is the space-time derivatives,  $(\cdot, \cdot)$  denotes the inner product on  $\mathbb{C}^4$ , and  $\gamma^\alpha \in GL(\mathbb{C}, 4)$  ( $\alpha = 0, 1, 2, 3$ ) denote the Dirac matrices given by

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}, \quad (1.9)$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.10)$$

In [7] we proved existence of global solution with small  $H^s$  data  $\varphi \in H^s$  for  $s > 1$  and  $m > 0$ . Local existence was proved by Escobedo and Vega in  $H^s$ ,  $s > 1$  [3].

Here the value  $s = 1$  is the scaling critical exponent for  $m = 0$ , see the introduction in [3].

There are similar situations for nonlinear wave equations with derivative nonlinearity. Lindblad [5] constructed counterexamples to disprove local wellposedness in  $H^1$  of the following equation:

$$\partial_t^2 u - \Delta u + (\partial_t u - \partial_1 u)u = 0, \quad (1.11)$$

while it is easy to prove its local wellposedness in  $H^{1+\varepsilon}$  by using non-endpoint Strichartz estimates (cf. [11]). Lindblad's counterexample of the initial data is concentrated in one direction. Our endpoint estimates imply that if the data had regularity for the angular variable  $H_\theta^\varepsilon$ ,  $\varepsilon > 0$ , then the blowup could not occur.

Notice that radial symmetry is not preserved for most equations including the above examples, and so the endpoint estimate (1.2) for radial solutions is not directly applicable. But our estimates can be applied without any consideration on special symmetries of given systems.

For the nonlinear Dirac equation, we have the following global existence for small  $H^1$  data with a slight regularity for angular variables.

**Theorem 1.2.** *Let  $m \geq 0$ ,  $\lambda \in \mathbb{C}$  and  $s > 0$ . Then there exists  $\delta > 0$  such that if  $\varphi \in H^1(\mathbb{R}^3)$  satisfies*

$$\|\varphi\|_{H^1(H_\theta^s)} := \|\varphi\|_{L_r^2(H_\theta^s)} + \|\nabla\varphi\|_{L_r^2(H_\theta^s)} < \delta \quad (1.12)$$

*then we have a unique global solution  $u$  of (1.8) satisfying  $u(0) = \varphi$  and*

$$u \in C_t(\mathbb{R}; H^1(H_\theta^s)) \cap L_t^2(\mathbb{R}; L^\infty). \quad (1.13)$$

*In the case of  $m = 0$ , we may replace the above norm of  $H^1(H_\theta^s)$  with its homogeneous version, namely  $\|\nabla\varphi\|_{L_r^2(H_\theta^s)}$ .*

We prove Theorem 1.2 by the standard fixed point arguments using the above endpoint estimates that hold uniformly on any time interval. Hence we can easily obtain global wellposedness and scattering for small data, as well as local existence for large data by the standard arguments (see, e.g., [3]).

The rest of this paper is organized as follows. In Section 2, we introduce the notations and basic estimates on the fractional Sobolev spaces on the sphere  $S^2$ . In Section 3, we prove our endpoint Strichartz estimates. In Section 4, we prove the global wellposedness for the nonlinear Dirac. In Section 5, we make a number of remarks.

Throughout this paper, we often use the notation  $A \lesssim B$  and  $D \sim E$  which mean  $A \leq CB$  and  $D/C \leq E \leq CD$ , respectively, where  $C$  is some positive constant. We denote  $\langle x \rangle := (1 + |x|^2)^{1/2}$ . We identify any set with its characteristic function. Thus for any set  $A$ ,  $A(x) = 1$  if  $x \in A$  and  $A(x) = 0$  otherwise.

## 2. FRACTIONAL SOBOLEV SPACES ON THE SPHERE

In this section, we recall some basic facts that we need on the fractional Sobolev spaces on the unit sphere  $S^2$ . See [14, 17] for more general information. We denote the polar coordinates  $x = r\theta$ ,  $r = |x|$  and  $\theta \in S^2$ . Let  $\Delta_\theta$  denote the Laplace-Beltrami operator on  $S^2$ . For any function  $f(r\theta)$ , we have

$$\Delta_\theta f(x) = |x \times \nabla|^2 f(x). \quad (2.1)$$

The Lebesgue and Sobolev spaces on  $S^2$  are defined by the norms

$$\|f\|_{L_\theta^p} = \left( \int_{S^2} |f(\theta)|^p d\theta \right)^{1/p}, \quad \|f\|_{H_\theta^{s,p}} = \|(1 - \Delta_\theta)^{s/2} f\|_{L_\theta^p}. \quad (2.2)$$

Throughout this paper, we will use these norms in the mixed form:

$$\|f(x)\|_{L_r^p(X_\theta)} = \left( \int \|f(r\theta)\|_{X_\theta}^p r^2 dr \right)^{1/p}. \quad (2.3)$$

The fractional power of  $\Delta_\theta$  can be written explicitly by introducing the spherical harmonics. Let  $F_\nu^k(x)$  be a homogeneous polynomial of degree  $\nu$  satisfying  $\Delta F_\nu^k(x) = 0$ , such that  $\{F_\nu^k(\theta)\}_{\nu,k}$  makes a complete orthonormal basis of  $L^2(S^2)$ . Then any function  $f(r\theta)$  can be decomposed as

$$f(r\theta) = \sum_{\nu=0}^{\infty} \sum_{k=1}^{N(\nu)} a_\nu^k(r) F_\nu^k(\theta), \quad (2.4)$$

where  $a_\nu^k(r)$  are determined by  $f$ , and

$$(1 - \Delta_\theta)^{s/2} f = \sum_{\nu,k} (1 + \nu(\nu + 1))^{s/2} a_\nu^k(r) F_\nu^k(\theta), \quad (2.5)$$

where we used  $\Delta_\theta F_\nu^k(\theta) = -\nu(\nu + 1)F_\nu^k(\theta)$ . In the case  $p = 2$ , we may use the orthogonality to deduce that

$$\|f\|_{L_r^2(H_\theta^{s,2})}^2 \sim \sum_{\nu,k} \langle \nu \rangle^s \|a_\nu^k\|_{L_r^2}^2. \quad (2.6)$$

For nonlinear estimates, we use the equivalent norms defined through local coordinates. Let  $\{(O_j, \Psi_j)\}_{j=1}^N$  be a system of coordinate neighborhoods, and  $\{\lambda_j\}$  be a smooth partition of unity subordinate to  $\{O_j\}$ . Let  $\{\chi_j\} \subset C_0^\infty(\mathbb{R}^2)$  satisfy  $\chi_j = 1$  on  $\Psi_j(\text{supp } \lambda_j)$  and  $\text{supp } \chi_j \subset \Psi_j(O_j)$ . Then, for any functions  $f : S^2 \rightarrow \mathbb{C}$  and  $h : (\mathbb{R}^2)^N \rightarrow \mathbb{C}$ , we define  $Sf : (\mathbb{R}^2)^N \rightarrow \mathbb{C}$  and  $Rh : S^2 \rightarrow \mathbb{C}$  by

$$(Sf)_j(x) := (\lambda_j f)(\Psi_j^{-1}(x)), \quad Rh(y) := \sum_{j=1}^N (\chi_j h)(\Psi_j(y)). \quad (2.7)$$

Then we can define the Sobolev norms by

$$\|f\|_{H^{s,p}(S^2)} = \|Sf\|_{(H^{s,p}(\mathbb{R}^2))^N}. \quad (2.8)$$

This gives an equivalent norm of  $H_\theta^{s,p}$  for  $1 < p < \infty$  (see [17]). We do not deal with the cases  $p = 1$  or  $\infty$  in this paper.

It is easily seen that  $RSf = f$  and  $SR$  is bounded from  $(H^{s,p}(\mathbb{R}^2))^N$  into itself, and so,  $R$  is a retraction from  $(H^{s,p}(\mathbb{R}^2))^N$  to  $H^{s,p}(S^2)$  with a coretraction  $S$ . Therefore we have the same embeddings and interpolations for  $H^{s,p}(S^2)$  as on  $\mathbb{R}^2$ . We may introduce another equivalent norm

$$(S'f)_j(x) := \chi_j(x)f(\Psi_j^{-1}(x)), \quad \|S'f\|_{(H^{s,p}(\mathbb{R}^2))^N} \sim \|Sf\|_{(H^{s,p}(\mathbb{R}^2))^N}. \quad (2.9)$$

Then the Hölder inequality and the Leibniz rule easily transfers from the Euclidean case as follows. Let  $s \geq 0$  and  $1/p = 1/q_1 + 1/r_1 = 1/q_2 + 1/r_2, 1 < p < \infty, q_1 \neq \infty, r_2 \neq \infty$ . We have

$$\begin{aligned} \|fg\|_{H^{s,p}(S^2)} &\sim \sum_j \|(Sf)_j(S'g)_j\|_{H^{s,p}(\mathbb{R}^2)} \\ &\lesssim \sum_j (\|(Sf)_j\|_{H^{s,q_1}(\mathbb{R}^2)}\|(S'g)_j\|_{L^{r_1}(\mathbb{R}^2)} + \|(Sf)_j\|_{L^{q_2}(\mathbb{R}^2)}\|(S'g)_j\|_{H^{s,r_2}(\mathbb{R}^2)}) \\ &\lesssim \|f\|_{H^{s,q_1}(S^2)}\|g\|_{L^{r_1}(S^2)} + \|f\|_{L^{q_2}(S^2)}\|g\|_{H^{s,r_2}(S^2)}, \end{aligned} \quad (2.10)$$

where we used the standard estimate on pointwise multiplication on  $\mathbb{R}^2$  on the second line.

Finally we check the equivalence of the following norms,

$$\|(1 - \Delta_\theta)^{s/2}f\|_{H^1} \sim \|f\|_{H^1(H_\theta^s)}, \quad (2.11)$$

where the right hand side was introduced in (1.12). Note that  $\nabla$  and  $\Delta_\theta$  are not commutative. Since (2.11) is obvious if we replace  $H^1$  by  $L^2$ , it suffices to prove the homogeneous version, i.e., for  $\dot{H}_x^1$ . Since  $|\nabla| = \sqrt{-\Delta}$  commutes with  $\Delta_\theta$ , the above equivalence (2.11) reduces to the following one:

$$\|\nabla|f|\|_{L_r^2(H_\theta^s)} \sim \|\nabla f\|_{L_r^2(H_\theta^s)}, \quad (2.12)$$

which is equivalent to the boundedness of the Riesz operators:

$$\nabla/|\nabla| : L_r^2(H_\theta^s) \rightarrow L_r^2(H_\theta^s) \quad \text{bounded}. \quad (2.13)$$

This is easily checked when  $s$  is an (even) integer by computing the commutators of  $x \times \nabla$  and  $\nabla$ . Then the remaining case is covered by interpolation.

### 3. ENDPOINT STRICHARTZ ESTIMATES

In this section, we consider the endpoint Strichartz estimate. Although one might expect that the estimates in Theorem 1.1 were easier for the Klein-Gordon ( $m > 0$ ) because of the faster decay ( $t^{-3/2}$ ), the estimate for the Klein-Gordon actually implies that for the wave. In fact, suppose that we have an estimate of the form:

$$\|u\|_{L_t^2 L_r^\infty X_\theta} \leq CE(u)^{1/2} \quad (3.1)$$

for a fixed  $m = m_0 > 0$ . Then we obtain the same estimate for all  $m > 0$  just by rescaling  $u \mapsto u(tm/m_0, xm/m_0)$ . Taking the limit  $m \rightarrow 0$ , we obtain the same estimate for  $m = 0$  as well. On the other hand, it is not trivial to extend such an estimate from  $m = 0$  to  $m > 0$ .

The rest of this section is devoted to the proof.

**3.1. Sharpness of  $\sqrt{p}$ .** First we prove the optimality of  $\sqrt{p}$  in (1.5). Let  $m = 0$ . We consider the function given by

$$\begin{aligned} g(x) &= \chi_A(x)g_0(x), \\ g_0(x) &= |x|^{-2}(1 + |\log |x||)^{-\alpha}, \end{aligned} \tag{3.2}$$

where  $\alpha \in (1/2, 1)$ ,  $\chi_A$  is the characteristic function of  $A = K \setminus B$ ,  $K$  is a sufficiently large cube, and  $B$  is a ball tangent to the boundary of  $K$  from its inside at the origin:

$$\begin{aligned} K &:= [0, 10] \times [-5, 5] \times [-5, 5], \\ B &:= \{x \in \mathbb{R}^3 \mid |x - e_1| < 1\}, \end{aligned} \tag{3.3}$$

where  $e_1 = (1, 0, 0)$ . This function is a slight modification of that given by T. Tao [16] as a counterexample for the endpoint Strichartz estimate

$$\|u\|_{L_t^2 L_x^\infty} \lesssim \|g\|_{L^2}$$

for free solutions with data  $u(0) = 0$  and  $\partial_t u(0) = g$ . In fact, by a simple calculation we know that the above function  $g$  satisfies  $g \in L^2$  but the free solution, which is given by

$$u(t, x) = \frac{t}{4\pi} \int_{S^2} g(x + t\theta') d\theta', \tag{3.4}$$

satisfies  $u(t, te_1) = \infty$  for all  $1 < t < 2$ . This function also shows sharpness of our  $L_\theta^p$  estimate as we see in the following. Let  $0 < t - 1 \ll 1$  and  $x = t\theta$  with  $\ell := |\theta - e_1| < \varepsilon \ll 1$ . We want to estimate  $u(t, x)$  given by (3.4) from below.

First we consider the restriction on the integral region of  $\theta'$  due to the cut-off  $\chi_A$ . Let  $y = x + t\theta'$  and we denote the region for  $y$  by  $S := \{x + t\theta' \mid \theta' \in S^2\}$ . It is easily seen that  $y \in S$  is contained in the cube  $K$  when  $|y| \gtrsim \ell$ . Since the radius of  $S$  is greater than that of  $B$ , it is clear that  $\mathbb{R}^3 \setminus B$  contains at least one of the hemispheres of  $S$  divided by a plane containing 0 and  $x$ . Thus we can estimate, by taking the integral with respect to  $\rho = |y|$ ,

$$u(t, x) \gtrsim \int_{C_\ell}^1 g_0(\rho) \rho d\rho \gtrsim \int_0^{|\log \ell|/2} s^{-\alpha} ds \gtrsim |\log \ell|^{1-\alpha}, \tag{3.5}$$

where we changed the integral variable as  $s = -\log \rho$ . Then we estimate the  $L_\theta^p$  norm for sufficiently large  $p$  with  $x = t\theta$ ,  $\ell = |e_1 - \theta|$  as

$$\begin{aligned} \|u(t, t\theta)\|_{L_\theta^p}^p &\gtrsim \int_0^\varepsilon |\log \ell|^{(1-\alpha)p} \ell d\ell \gtrsim \int_{-\log \varepsilon}^\infty s^{(1-\alpha)p} e^{-2s} ds \\ &\gtrsim p^{(1-\alpha)p} \int_p^\infty e^{-2s} ds = (p^{(1-\alpha)} e^{-2})^p / 2, \end{aligned} \quad (3.6)$$

where we changed the integral variable as  $s = -\log \ell$  and assumed that  $p > -\log \varepsilon$ . Therefore we have

$$\|u(t, t\theta)\|_{L_\theta^p} \gtrsim p^{1-\alpha} \quad (3.7)$$

for any  $\alpha > 1/2$  and large  $p$ . This completes the proof of (1.5) for  $m = 0$ .

Next we consider the Klein-Gordon case  $m > 0$ . Fix  $m > 0$ ,  $\varepsilon > 0$  and suppose that we have the estimate of the form

$$\sup_{p>1} \|u\|_{L_t^2 L_r^\infty L_\theta^p} / p^{1/2-\varepsilon} \leq CE(u)^{1/2}. \quad (3.8)$$

Then by the rescaling argument at the beginning of this section, we have the same estimate for  $m = 0$ , which we have just disproved. Therefore (3.8) is false, which means that there exists a finite energy solution for which the left hand side is infinite.

**3.2.  $TT^*$  argument.** Now we start to prove the main Strichartz estimates. First of all, we convert them into the  $TT^*$  versions. Our desired estimates can be rewritten as

$$\|\omega_m^{-1} e^{\pm i\omega_m t} \varphi\|_{L_t^2 L_r^\infty X_\theta} \lesssim \|\varphi\|_{L_x^2}, \quad \omega_m := \sqrt{m^2 - \Delta}, \quad (3.9)$$

where  $X_\theta$  denotes some Banach space ( $L_\theta^p$  for (I) and  $H_\theta^s$  for (III)). We apply the  $TT^*$  argument to the operators  $T_\pm := \omega_m^{-1}(e^{i\omega_m t} \pm e^{-i\omega_m t})$ . We have

$$T_\pm T_\pm^* u = 2 \int_{\mathbb{R}} \omega_m^{-2} \{\cos(\omega_m(t-s)) \pm \cos(\omega_m(t+s))\} u(s) ds. \quad (3.10)$$

Hence, by time reversibility, it suffices to prove

$$\left\| \int_{\mathbb{R}} \omega_m^{-2} \cos \omega_m(t-s) u(s) ds \right\|_{L_t^2 L_r^\infty X_\theta} \lesssim \|u\|_{L_t^2 L_r^1 X_\theta^*}, \quad (3.11)$$

where  $X_\theta^*$  denotes the  $L^2$  dual of  $X_\theta$ . It is important for our later argument that we do not have ‘sin’ but ‘cos’ above. We denote the operator in (3.11) by  $L_m(t) := \omega_m^{-2} \cos(\omega_m t)$  and its kernel function by

$$\mathcal{L}_m(t, x) = \mathcal{F}^{-1} \langle \xi \rangle_m^{-2} \cos \langle \xi \rangle_m t, \quad \langle \xi \rangle_m := \sqrt{|\xi|^2 + m^2}. \quad (3.12)$$

We use the following  $TT^*$  version of the Hardy–Littlewood maximal operator as the key estimate on  $(t, r)$ . In the lemma below, we forget about the polar coordinates and so  $L_r^p$  denotes the standard  $L^p((0, \infty); dr)$  without weights.

**Lemma 3.1.** *Let  $g(r)$  be a nonnegative nonincreasing integrable function on  $(0, \infty)$ . Then the following estimate holds*

$$\left\| \int_0^\infty \int_{\mathbb{R}} \frac{1}{r \vee l} g\left(\frac{|t-s|}{r \vee l}\right) h(s, l) ds dl \right\|_{L_t^2 L_r^\infty} \lesssim \|g\|_{L_r^1} \|h\|_{L_t^2 L_r^1}. \quad (3.13)$$

where  $r \vee l = \max(r, l)$ .

*Proof.* The Hardy-Littlewood maximal function theorem shows the boundedness of the operator

$$M\varphi(t, r) = \frac{1}{r} \int_{|t-s|<r} \varphi(s) ds : L_t^2 \rightarrow L_t^2 L_r^\infty. \quad (3.14)$$

So  $MM^*$  is bounded

$$MM^* : L_t^2 L_r^1 \rightarrow L_t^2 L_r^\infty, \quad (3.15)$$

and it is written explicitly by

$$MM^* h(t, r) = \int_0^\infty \int_{\mathbb{R}} \frac{1}{rl} I(|t-s|, r, l) h(s, l) ds dl, \quad (3.16)$$

where

$$I(t, r, l) = \begin{cases} 2 \min(r, l), & (t < |r-l|), \\ r+l-t, & (|r-l| < t < r+l), \\ 0, & (r+l < t). \end{cases} \quad (3.17)$$

Denote the operator in (3.13) by  $\mathcal{M}(g, h)$ . Since

$$\frac{1}{rl} I(t, r, l) \geq \frac{1}{r \vee l} \{0 < t < r \vee l\}, \quad (3.18)$$

the boundedness of  $MM^*$  implies the desired estimate for  $\mathcal{M}([0, 1], h)$ , and by rescaling, for any interval  $\mathcal{M}([0, a], h)$ . (Remember that we identify any set with its characteristic function.) Then the general case follows by slicing  $g$  into intervals:

$$\begin{aligned} \|\mathcal{M}(g, h)\|_{L_t^2 L_r^\infty} &= \left\| \int_0^\infty -g'(a) \mathcal{M}([0, a], h) da \right\|_{L_t^2 L_r^\infty} \\ &\lesssim \int_0^\infty -g'(a) a \|h\|_{L_t^2 L_r^1} da = \|g\|_{L_r^1} \|h\|_{L_t^2 L_r^1}. \end{aligned} \quad (3.19)$$

□



**3.3.  $L_\theta^p$  estimate (3.9) for the wave.** We fix  $t$  and estimate  $L_0(t)$  pointwise. By symmetry, we may assume that  $t > 0$ . Using the well known formula for the fundamental solution, we obtain

$$\begin{aligned} L_0(t) &= \int_t^\infty \omega_0^{-1} \sin \omega_0 s ds, \\ \mathcal{L}_0(t, x) &= \int_t^\infty \frac{1}{4\pi s} \delta(s - r) ds = \frac{1}{4\pi r} \{t < r\}. \end{aligned} \quad (3.20)$$

Here again we identify the set with its characteristic function. Using the polar coordinates we may write it as

$$L_0(t)\varphi = \int_0^\infty \Omega[\varphi(l\theta)l^2] dl, \quad (3.21)$$

where  $\Omega$  is an operator on  $S^2$  defined by

$$\Omega\varphi(\theta) = \int_{S^2} F(|r\theta - l\alpha|)\varphi(\alpha) d\alpha, \quad F(r) = (4\pi r)^{-1} \{t < r\}. \quad (3.22)$$

We estimate the  $L_\theta^p$  norm of  $\Omega$  as follows. First we have the trivial  $L_\theta^\infty$  bound:

$$\|\Omega\varphi\|_{L_\theta^\infty} \leq \|F(|r\theta - l\alpha|)\|_{L_\alpha^\infty} \|\varphi\|_{L_\theta^1} \lesssim t^{-1} \{t < r + l\} \|\varphi\|_{L_\theta^1}. \quad (3.23)$$

For the  $L_\theta^2$  estimate, we apply the Young inequality for the convolution on  $SO(3)$ . Using the identity

$$\int_{S^2} f(\theta) d\theta = C \int_{SO(3)} f(Ae) dA, \quad e \in S^2, \quad (3.24)$$

we estimate

$$\begin{aligned} \|\Omega\varphi\|_{L_\theta^2} &\sim \left\| \int_{SO(3)} F(|re - lBe|)\varphi(ABe) dB \right\|_{L_A^2} \\ &\lesssim \|\varphi(Ae)\|_{L_A^2} \int_{SO(3)} F(|re - lBe|) dB \\ &\sim \|\varphi\|_{L_\theta^2} \int_{S^2} F(|re - l\theta|) d\theta, \end{aligned} \quad (3.25)$$

where we changed the variables as  $\theta \mapsto Ae$  and  $\alpha \mapsto ABe$ . The last integral of  $F$  is dominated by

$$\{t < r + l\} \int_{S^2} |re - l\theta|^{-1} d\theta \lesssim \{t < r + l\} (r \vee l)^{-1}. \quad (3.26)$$

Interpolating these estimates, we obtain

$$\|\Omega\varphi\|_{L_\theta^p} \lesssim t^{2/p-1} (r \vee l)^{-2/p} \{t < r + l\} \|\varphi\|_{L_\theta^{p'}}, \quad (3.27)$$

for  $2 \leq p \leq \infty$ , where  $p' = p/(p-1)$  is the dual exponent. Plugging this estimate into  $L_0(t)$ , we obtain

$$\|\mathcal{L}_0 * f(t, r\theta)\|_{L_\theta^p} \lesssim \int_{\mathbb{R}} \int_0^\infty \frac{1}{r \vee l} g_p \left( \frac{|t-s|}{r \vee l} \right) \|f(s, l\theta)l^2\|_{L_\theta^{p'}} dlds, \quad (3.28)$$

where

$$g_p(t) = t^{2/p-1}\{0 < t < 2\}. \quad (3.29)$$

Then the desired  $L_\theta^p$  estimate (1.4) for  $m = 0$  follows from Lemma 3.1 together with the estimate  $\|g_p\|_{L^1} \lesssim p$ . The case  $p < 2$  is covered by the embedding  $L_\theta^2 \hookrightarrow L_\theta^p$ .

**3.4.  $L_\theta^p$  estimate (3.9) for the Klein-Gordon.** Next we extend the above result to the Klein-Gordon  $m > 0$ . Since our estimate is global in time and the large time behavior is essentially different between the wave and the Klein-Gordon, it seems meaningless to approximate the latter by the former. Nevertheless, we will show that the  $TT^*$  operator  $L_m(t)$  for the Klein-Gordon can be dominated by the wave correspondence and a “dispersive” part, which is smooth and decays fast in time.

By the rescaling argument, it suffices to prove the estimate for  $m = 1$ . We may assume  $t > 0$  by symmetry. We calculate the kernel  $\mathcal{L}_m$  by writing the Fourier transform in the polar coordinates as

$$\begin{aligned} \mathcal{L}_m(t, x) &= C \int_0^\infty \int_{S^2} \langle \rho \rangle_m^{-2} \cos(t\langle \rho \rangle_m) e^{ir\theta \cdot \rho \alpha} \rho^2 d\alpha d\rho \\ &= C \int_0^\infty \int_0^1 \langle \rho \rangle_m^{-2} \cos(t\langle \rho \rangle_m) \cos(r\rho\lambda) \rho^2 d\lambda d\rho \\ &= C \int_0^\infty \cos(r\nu) \int_\infty^{t\langle \nu \rangle_m} \frac{\cos l}{l} dl d\nu, \end{aligned} \quad (3.30)$$

where we changed the variables as  $\lambda = \cos(\theta \cdot \alpha)$ ,  $\nu = \rho\lambda$  and  $l = t\langle \rho \rangle_m$ . Then we obtain a uniform bound

$$|\mathcal{L}_1(t, x) - \mathcal{L}_0(t, x)| \lesssim \int_0^\infty \int_{t\nu}^{t\langle \nu \rangle_m} \frac{dl}{l} d\nu \lesssim 1. \quad (3.31)$$

Integrating by parts after changing the variable  $l \mapsto l/\langle \nu \rangle_m$ , we further rewrite (3.30) as

$$\mathcal{L}_m(t, x) = Ct^{-1}\mathcal{K}_m(t, x) + C \int_\infty^t \mathcal{K}_m(l, x) l^{-2} dl, \quad (3.32)$$

where  $\mathcal{K}_m(t)$  denotes the one-dimensional fundamental solution of the Klein-Gordon. When  $m = 1$ , we have

$$\begin{aligned} \mathcal{K}_1(t, r) &= C \int_0^\infty \langle \nu \rangle^{-1} \sin(t\langle \nu \rangle) \cos(r\nu) d\nu = C J_0(\sqrt{t^2 - r^2}) \{r < t\} \\ &\lesssim \langle \sqrt{t^2 - r^2} \rangle^{-1/2} \{r < t\}, \end{aligned} \quad (3.33)$$

where  $J_0$  is the Bessel function of order 0 and we used the estimate  $|J_0(s)| \lesssim \langle s \rangle^{-1/2}$  [12, p. 98]. Hence we have for  $t < r$ ,

$$|\mathcal{L}_1(t, x)| \lesssim \int_r^\infty (l^2 - r^2)^{-1/4} l^{-2} dl \lesssim r^{-3/2}. \quad (3.34)$$

When  $t/2 < r < t$ , we estimate  $|\mathcal{K}_1(t, r)| \lesssim 1$  and

$$|\mathcal{L}_1(t, x)| \lesssim t^{-1} + \int_t^\infty l^{-2} dl \lesssim t^{-1} \lesssim r^{-1}. \quad (3.35)$$

When  $r < t/2$ , we have  $\sqrt{t^2 - r^2} \gtrsim t$  and so

$$|\mathcal{L}_1(t, x)| \lesssim t^{-3/2} + t^{-1/2} \int_t^\infty l^{-2} dl \lesssim t^{-3/2}. \quad (3.36)$$

Gathering the estimates (3.20), (3.31), (3.35) and (3.36), we conclude

$$|\mathcal{L}_1(t, x)| \lesssim \mathcal{L}_0(t/2, x) + \langle t \rangle^{-3/2}. \quad (3.37)$$

Thus we have reduced the desired estimate for  $m = 1$  to that for  $m = 0$  and the  $L_t^2 L_x^\infty$  estimate for the dispersive part  $\langle t \rangle^{-3/2}$ , which follows simply from the Young inequality.

**3.5.  $H_\theta^{3/4}$  estimate (1.6) for  $m = 0$ .** First we derive an expression of  $L_0(t)$  restricted to each spherical harmonic (2.4), using the identities (3.20). Since we have

$$\Delta(a(r)H_\nu(\theta)) = (\Delta_\nu a(r))H_\nu(\theta), \quad \Delta_\nu = \Delta - \nu(\nu + 1)r^{-2} \quad (3.38)$$

for any spherical harmonic  $H_\nu(\theta)$  of order  $\nu$ , we have the same relation for any function of  $\Delta$ , and in particular

$$\omega_0^{-1} \sin \omega_0 t (a(r)H_\nu(\theta)) = (K_{0\nu}(t)a(r))H_\nu(\theta) \quad (3.39)$$

with a certain operator  $K_{0\nu}(t)$  on radial functions. Choosing  $H_\nu(\theta) = P_\nu(e \cdot \theta)$  [9, Theorem 3], where  $P_\nu(s) = (2^\nu \nu!)^{-1} [(d/ds)^\nu (s^2 - 1)^\nu]$  is the Legendre polynomial, and then letting  $\theta = e$  and using that  $P_\nu(1) = 1$ , we obtain

$$\begin{aligned} K_{0\nu}(t)a(r) &= \frac{1}{4\pi t} \delta(t - r) * (a(r)P_\nu(e \cdot \theta))(re) \\ &= \frac{1}{2r} \int_{\Delta(l,t,r)} P_\nu(\cos \beta) a(l) l dl, \end{aligned} \quad (3.40)$$

where  $\Delta(l, t, r)$  denotes restriction to the region where a triangle holds with side lengths  $l, t$  and  $r$ , i.e.  $2 \max(l, t, r) \leq l + t + r$ , and the respective opposite angles are denoted by  $\alpha, \beta$  and  $\gamma$ .

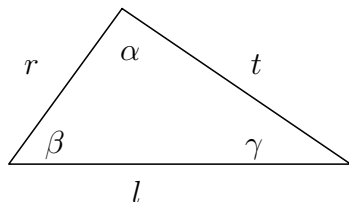


Figure.  $\Delta(l, t, r)$

Hence we have by (3.20)

$$\begin{aligned} L_0(t)(a(r)H_\nu(\theta)) &= (L_{0\nu}(t)a(r))H_\nu(\theta), \\ L_{0\nu}(t)a(r) &:= \frac{1}{2} \int_t^\infty \int_{\Delta(l,s,r)} P_\nu(\cos \beta) a(l) l^2 \frac{s}{rl} dl \frac{ds}{s}. \end{aligned} \quad (3.41)$$

The  $TT^*$  argument (3.11) and the orthogonal decomposition (2.4) reduce our desired estimate to

$$\left\| \int_{\mathbb{R}} L_{0\nu}(|t-s|)v(s)ds \right\|_{L_t^2 L_r^\infty} \lesssim \langle \nu \rangle^{-3/2} \|v(t,r)\|_{L_t^2 L_r^1}, \quad (3.42)$$

where we used the time symmetry of  $L_{0\nu}(t)$ . Since the estimate for  $\nu = 0$  follows from the endpoint estimate (1.4) with  $p = 2$ , we assume that  $\nu \geq 1$  as well as  $t > 0$  in the following.

In order to derive the decay in  $\nu$ , we exploit the oscillatory property of the Legendre polynomial in (3.41). We integrate by parts for the variable  $s$ . Using the identity  $\nu(\nu+1)P_\nu(x) = ((x^2-1)P'_\nu(x))'$  and the relation  $s^2 = r^2 + l^2 - 2rl \cos \beta$  by the triangle, we obtain

$$\int_{\Delta(l,s,r)} P_\nu(\cos \beta) h(s) \frac{s ds}{rl} = \int_{\Delta(l,s,r)} \frac{\sin^2 \beta}{\nu(\nu+1)} P'_\nu(\cos \beta) h'(s) ds, \quad (3.43)$$

where we put  $h(s) = 1/s \{s > t\}$ , hence  $h'(s) = \delta(s-t)/t - 1/s^2 \{s > t\}$ . Applying the classical estimate (see [6])

$$|\sin \beta|^{3/2} |P'_\nu(\cos \beta)| \lesssim \nu^{1/2} \quad (3.44)$$

and the sine theorem

$$\left| \frac{\sin \beta}{s} \right| = \left| \frac{\sin \alpha}{l} \right| = \left| \frac{\sin \gamma}{r} \right| \leq \frac{1}{r \vee l}, \quad (3.45)$$

we dominate (3.43) by

$$\nu^{-3/2} \int_{\Delta(l,s,r)} \sqrt{\frac{s}{r \vee l}} |h'(s)| ds \lesssim \nu^{-3/2} \frac{\{t < r+l\}}{\sqrt{t(r \vee l)}}. \quad (3.46)$$

In conclusion we have

$$|L_{0\nu}(t)a(r)| \lesssim \langle \nu \rangle^{-3/2} \int_0^\infty \frac{\{t < r+l\}}{\sqrt{t(r \vee l)}} a(l) l^2 dl \quad (3.47)$$

and the desired (3.42) follows from Lemma 3.1 with  $g(t) = t^{-1/2} \{0 < t < 2\}$ .

#### 4. GLOBAL SOLUTIONS FOR THE NONLINEAR DIRAC EQUATION

In this section, we prove Theorem 1.2. We rewrite the equation (1.8) as the following integral equation:

$$u = U_m(t)\varphi + \int_0^t U_m(t-s)F(u(s))ds, \quad (4.1)$$

where  $F(u) = -i\lambda\gamma^0(\gamma^0 u, u)u$  and  $U_m(t)$  denotes the propagator of the free Dirac equation given by

$$U_m(t) = \cos(\omega_m t) - \gamma^0 \left( \sum_{j=1}^3 \gamma^j \partial_j + im \right) \omega_m^{-1} \sin(\omega_m t), \quad (4.2)$$

where  $\omega_m = \sqrt{m^2 - \Delta}$ . We set  $\Phi u = \text{R.H.S of (4.1)}$  and apply the contraction mapping theorem.

For the linear term, we use the Strichartz estimates (1.4). We see from (4.2) that  $\omega_m^{-1} U_m(t)$  is a linear combination of  $\omega_m^{-1} e^{\pm i\omega_m t}$  with bounded Fourier multipliers. So we have estimates for  $m \geq 0, 1 \leq p < \infty$  as

$$\|U_m(t)\varphi\|_{L_t^2 L_r^\infty L_\theta^p} \lesssim \|\varphi\|_{H^1}. \quad (4.3)$$

Moreover, from the fact that  $\Delta$  is commutative with  $\Delta_\theta$ , it follows that

$$\|U_m(t)\varphi\|_{L_t^2 L_r^\infty H_\theta^{s,p}} \lesssim \|(1 - \Delta_\theta)^{s/2} \varphi\|_{H^1} \sim \|\varphi\|_{H^1(H_\theta^s)}. \quad (4.4)$$

Therefore putting  $X = L_t^\infty H^1(H_\theta^s) \cap L_t^2 L_r^\infty H_\theta^{s,p}$  with  $p$  sufficiently large as  $p > 2/s$ , we have

$$\begin{aligned} \|\Phi u\|_X &\lesssim \|\varphi\|_{H^1(H_\theta^s)} + \int_0^\infty \|U_m(t-s)F(u(s))\|_X ds \\ &\lesssim \|\varphi\|_{H^1(H_\theta^s)} + \|F(u)\|_{L_t^1 H^1(H_\theta^s)}. \end{aligned} \quad (4.5)$$

By (2.10), we estimate the nonlinear term  $F(u)$  as

$$\begin{aligned} \|F(u)\|_{H_\theta^s} &\lesssim \|u\|_{L_\theta^\infty}^2 \|u\|_{H_\theta^s}, \\ \|\nabla F(u)\|_{H_\theta^s} &\lesssim \|u\|_{H_\theta^{s,p}} \|u\|_{L_\theta^\infty} \|\nabla u\|_{L_\theta^q} + \|u\|_{L_\theta^\infty}^2 \|\nabla u\|_{H_\theta^s} \end{aligned} \quad (4.6)$$

with  $1/p + 1/q = 1/2$ . By the embeddings  $H_\theta^{s,p} \hookrightarrow L_\theta^\infty$  for  $s > 2/p$ ,  $H_\theta^s \hookrightarrow L_\theta^q$  for  $s \geq 2/p$ , and the Hölder inequality for variables  $t$  and  $r$ , we have

$$\|F(u)\|_{L_t^1 H^1(H_\theta^s)} \lesssim \|u\|_{L_t^2 L_r^\infty H_\theta^{s,p}}^2 \|u\|_{L_t^\infty H^1(H_\theta^s)}. \quad (4.7)$$

Analogously we have

$$\|\Phi u - \Phi v\|_X \lesssim (\|u\|_X^2 + \|v\|_X^2) \|u - v\|_X. \quad (4.8)$$

Therefore  $\Phi$  is a contraction map on a small closed ball in  $X$ .

For the uniqueness of solutions in the class of (1.13), we consider the  $L_t^\infty L_x^2$  metric. By the  $L^2$  invariance of  $U(t)$ , we have

$$\|u - v\|_{L_t^\infty L_x^2} \lesssim (\|u\|_{L_t^2 L_x^\infty}^2 + \|v\|_{L_t^2 L_x^\infty}^2) \|u - v\|_{L_t^\infty L_x^2}. \quad (4.9)$$

We can conclude  $u = v$  time locally, so that for the entire time interval by the repetition.

## 5. DISCUSSION

Theorem 1.2 implies that if the initial data is spherically symmetric and small in  $H^1$ , then the solution is global. For another examples, we can find some of studies on the following form of solutions for Dirac equations in [1], [2], [13] etc.,

$$u(t, x) = \begin{bmatrix} f(t, r) \\ 0 \\ g(t, r) \cos \omega \\ g(t, r) \sin \omega e^{i\phi} \end{bmatrix}, \quad \begin{cases} x_1 = r \sin \omega \cos \phi, \\ x_2 = r \sin \omega \sin \phi, \\ x_3 = r \cos \omega. \end{cases} \quad (5.1)$$

We can apply Theorem 1.2 to this type solution, namely, if the initial data takes the form  $\varphi = (f_0(r), 0, g_0(r) \cos \omega, g_0(r) \sin \omega e^{i\phi})$  and  $\|f_0\|_{H^1(\mathbb{R}^3)}, \|g_0\|_{H^1(\mathbb{R}^3)}$  are sufficiently small, then there exists a global solution of the form (5.1). Indeed, since  $U_m(t)$  and the nonlinear term  $(\gamma^0 u, u)u$  preserve the form of (5.1), the functions given by the iteration argument which starts from the free solution  $U_m(t)\varphi$  have the form and the limiting function which is the solution of (1.8) also has the form.

Moreover, our argument also applies to nonlinear Klein-Gordon equations of the form

$$u_{tt} - \Delta u + m^2 u + F(\partial u, m u) = 0, \quad (5.2)$$

where  $\partial$  denotes the space-time derivatives. We can deduce local wellposedness in  $H^1(H_\theta^s)$  for  $F = u\partial u$ , global wellposedness for small data in  $H^1(H_\theta^s)$  for  $F = u^2\partial u$ , local wellposedness in  $H^2(H_\theta^s)$  for  $F = (\partial u)^2$ , global wellposedness for small data in  $H^2(H_\theta^s)$  for  $F = (\partial u)^3$ , etc. Compare with [5],[11]. Notice that systems of nonlinear wave equations in most cases do not possess radial symmetric solutions but have a certain class of solutions with the Lorentz covariance, just as in the above case of the nonlinear Dirac. The radial endpoint Strichartz estimate does not simply apply to such classes, since the reduced equations for the radial part of solutions would have terms of the form  $u/r^2$ . But one can apply our argument directly to such classes to have wellposedness, say in  $H^1$ , without even knowing algebraic properties of the symmetry.

Finally we give upper bounds for  $s$  in the  $L_t^2 L_r^\infty H_\theta^s$  estimate for both the Klein-Gordon (on  $\mathbb{R}^3$ ) and the Schrödinger (on  $\mathbb{R}^2$ ). This implies that we can not recover the  $L_\theta^p$  estimate for all  $p < \infty$  from  $H_\theta^s$  estimate and the Sobolev embedding.

**Theorem 5.1.** (i) *Let  $m \geq 0$ ,  $s \in \mathbb{R}$  and suppose that we have the estimate of the form*

$$\|u\|_{L_t^2 L_r^\infty H_\theta^s} \lesssim E(u)^{1/2}, \quad (5.3)$$

*for any finite energy solution  $u$  of the Klein-Gordon equation (1.1). Then we have  $s \leq 5/6$ .*

(ii) Let  $s \in \mathbb{R}$  and suppose that we have the estimate for the Schrödinger equation on  $\mathbb{R}^2$  of the form

$$\|e^{it\Delta}\varphi\|_{L_t^2 L_r^\infty H_\theta^s} \lesssim \|\varphi\|_{L^2(\mathbb{R}^2)}. \quad (5.4)$$

Then  $s \leq 1/3$ .

*Proof.* First we consider the Klein-Gordon case. By the scaling argument, we may assume  $m = 0$  without loss of generality. Then by the Strichartz estimate and the duality we have

$$|(\psi(x)f(t), \omega_0^{-1}e^{-it\omega_0}\varphi(x))_{t,x}| \lesssim \|\psi\|_{L_r^1 H_\theta^{-s}} \|f\|_{L_t^2} \|\varphi\|_{L_x^2}, \quad (5.5)$$

where  $(\cdot, \cdot)_{t,x}$  denotes the  $L^2$  inner product on  $\mathbb{R}^{1+3}$ . We can rewrite the inner product by using the Plancherel for  $(t, x)$

$$(\tilde{\psi}(x)\tilde{f}(|x|), |x|^{-1}\tilde{\varphi}(x))_x, \quad (5.6)$$

where  $\tilde{\psi}$  denotes the Fourier transform of  $\psi$ , and  $(\cdot, \cdot)_x$  denotes the inner product on  $\mathbb{R}^3$ . Thus we obtain

$$|(\tilde{\psi}(x), r^{-1}f(r)\varphi(x))_x| \lesssim \|\psi\|_{L_r^1 H_\theta^{-s}} \|f\|_{L_t^2} \|\varphi\|_{L_x^2}. \quad (5.7)$$

We can decompose any  $g \in L_r^1 L_\theta^2$  as

$$g(r\theta) = r^{-1}f(r)\varphi(x), \quad f(r) = \|g(r\theta)\|_{L_\theta^2}^{1/2} r, \quad (5.8)$$

then we have

$$\|g\|_{L_r^1 L_\theta^2} = \|f\|_{L_t^2} \|\varphi\|_{L_x^2}. \quad (5.9)$$

Plugging this into (5.7), we obtain

$$|(\tilde{\psi}(x), g(x))_x| \lesssim \|\psi\|_{L_r^1 H_\theta^{-s}} \|g\|_{L_r^1 L_\theta^2}. \quad (5.10)$$

Hence the Plancherel and the duality imply that

$$\|\tilde{g}\|_{L_r^\infty H_\theta^s} \lesssim \|g\|_{L_r^1 L_\theta^2}. \quad (5.11)$$

Now let  $g(r\theta) = a(r)H_\nu(\theta)$  where  $H_\nu(\theta)$  is a spherical harmonic of order  $\nu$ . Then we have

$$\tilde{g}(r\theta) = CH_\nu(\theta)i^\nu \int_0^\infty a(\rho)\mathcal{J}_\nu(r\rho)\rho^2 d\rho, \quad (5.12)$$

where

$$\mathcal{J}_\nu(r) = r^{-1/2}J_{\nu+1/2}(r), \quad (5.13)$$

and  $J_\nu(r)$  denotes the Bessel function (see [10, p. 164]). Then (5.11) implies that

$$\left\| \int_0^\infty a(\rho)\mathcal{J}_\nu(r\rho)\rho^2 d\rho \right\|_{L_r^\infty} \lesssim \nu^{-s} \|a\|_{L_r^1}, \quad (5.14)$$

which is equivalent by duality to

$$\|\mathcal{J}_\nu(r)\|_{L_r^\infty} \lesssim \nu^{-s}. \quad (5.15)$$

By choosing  $r = \nu + 1/2$  and using the asymptotic behavior of the Bessel function  $J_\nu(\nu) \sim \nu^{-1/3}$  for  $\nu \rightarrow \infty$  [19, p. 231], we conclude that  $s \leq 5/6$ .

The proof in the Schrödinger case is almost the same. By the same argument we obtain instead of (5.7)

$$|(\tilde{\psi}(x)f(r^2), \varphi(x))_x| \lesssim \|\psi\|_{L_r^1 H_\theta^{-s}} \|f\|_{L_r^2} \|\varphi\|_{L_r^2}, \quad (5.16)$$

on  $\mathbb{R}^2$ . By using the following decomposition for any  $g \in L_r^1 L_\theta^2$ :

$$g(r\theta) = f(r^2)\varphi(r\theta), \quad f(r^2) = \|g(r\theta)\|_{L_\theta^2}^{1/2}, \quad (5.17)$$

and the duality, we arrive at the same estimate as above

$$\|\tilde{g}\|_{L_r^\infty H_\theta^s} \lesssim \|g\|_{L_r^1 L_\theta^2}. \quad (5.18)$$

Again we assume  $g$  is a spherical harmonic  $g(r\theta) = a(r)e^{i\nu\theta}$ . In this case we have

$$\tilde{g}(r\theta) = C e^{i\nu\theta} i^\nu \int_0^\infty a(\rho) J_\nu(r\rho) \rho d\rho, \quad (5.19)$$

and (5.18) implies that

$$\|J_\nu(r)\|_{L_r^\infty} \lesssim \nu^{-s}. \quad (5.20)$$

Then we obtain  $s \leq 1/3$  by the asymptotic of  $J_\nu(\nu)$ .  $\square$

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