Global stability of distributions of higher corank

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Studying a manifold with a certain geometric structure becomes an interesting topic in differential topology when such a structure has some stability. The local stability of symplectic and contact structures, the Darboux theorem, is a fundamental property for symplectic and contact topology. There is a well-known global stability theorem for contact structures, the Gray theorem (see [G]): a one-parameter deformation of a contact structure on a compact orientable manifold can be pursued by using a global isotopy of the manifold. We obtain a theorem of this type for distributions of higher coranks. We consider distributions of corank greater than one each of which has the tangent bundle of the underlying manifold as its first derived distribution. Furthermore, comparing the stability of distribution of corank greater than one and that of distribution of corank one, we obtain a generalization of the Gray theorem.

First we obtain the following theorem:

Theorem 1. *Let D_t, $t \in [0,1]$, be a one-parameter family of distributions of corank $k > 1$ on a compact orientable manifold M. Suppose, for any $t \in [0,1]$:*

(1) *the first derived distributions coincide with the tangent bundle of M: $D_t^2 = TM$,*

(2) *there exists a constant integrable subdistribution $K \subset D_t$ of corank one.*

Then, there exists a family $\varphi_t \colon M \to M$, $t \in [0,1]$, of global diffeomorphisms which satisfies $\varphi_0 = \mathrm{id}$ and $(\varphi_t)_* D_0 = D_t$ for any $t \in [0,1]$.

A *distribution* D on a manifold M is a subbundle of the tangent bundle TM. We use the same symbol D for the sheaf of cross-sections of $D \subset TM$. A *derived distribution* D^2 of D is a sheaf of vector fields generated by D and the Lie bracket $[X, Y]$ of vector fields $X, Y \in D$.

Further, we observe the relationship of Theorem 1 with the Gray theorem. As a result, we obtain a generalization of the Gray theorem (Theorem 3 bellow).

Before stating Theorem 3, we introduce a generalization of the Gray theorem due to R. Montgomery and M. Zhitomirskiĭ [MZh]. The Gray theorem deals with contact structures. Their generalization deals with some degeneration of non-integrability of distribution. In order to state the result, we need the following notion. The *Cauchy characteristic distribution* $L(D)$ of a distribution D is defined pointwise as follows (see [BCG3], [Y]),

$$\begin{aligned} L(D)_p &= \{X \in D_p \mid [X,Y] \in D_p, \text{ for any } Y \in D_p\}, \\ &= \{X \in D_p \mid X \lrcorner d\omega|_{D_p} = 0, \text{ for any } \omega \in \mathcal{S}(D)\}, \end{aligned}$$

where $\mathcal{S}(D)$ is a Pfaffian system annihilating D. The distribution $L(D)$ is integrable according to the Frobenius theorem. Now, a generalization, due to R. Montgomery and M. Zhitomirskiĭ, of the Gray theorem is described as follows.

Theorem 2 (Montgomery-Zhitomirskiĭ). *Let D_t, $t \in [0,1]$, be a one-parameter family of distributions of corank $k = 1$ on a compact orientable manifold M. It is assumed that D_t has the Cauchy characteristic distribution $L(D_t) \equiv L$ constant for any $t \in [0,1]$. Then, there exists a family of global diffeomorphisms $\varphi_t \colon M \to M$, $t \in [0,1]$, which satisfies $\varphi_0 = \mathrm{id}$ and $(\varphi_t)_* D_0 = D_t$ for any $t \in [0,1]$.*

We can regard Theorem 1 as a further generalization of the generalized Gray theorem due to R. Montgomery and M. Zhitomirskiĭ. For this observation, we need some notions and notation. First of all, we define a certain subdistribution $K(D)$ of a distribution D. It is defined in terms of the Pfaffian system $\mathcal{S}(D)$. We define a covariant system associated to a Pfaffian system $\mathcal{S} \subset T^*M$ according to A. Kumpera and J. L. Rubin (see [KRb]), as follows. The bundle map $\delta \colon \mathcal{S} \to \bigwedge^2(T^*M/\mathcal{S})$ defined on local sections of $\mathcal{S}$ as $\delta(\omega) = d\omega \pmod{\mathcal{S}}$ is called the *Martinet structure tensor* (see [KRz], [Ma]). We define the *polar space* $\mathrm{Pol}(\mathcal{S})_p$ of $\mathcal{S}$ at $p \in M$ as

$$\mathrm{Pol}(\mathcal{S})_p := \left\{ w \in T_p^*M/\mathcal{S}_p \mid w \wedge \delta(\omega) = 0, \text{ for any } \omega \in \mathcal{S} \right\}.$$

When the polar space $\mathrm{Pol}(\mathcal{S})_p$ has a constant rank on M, we define the *covariant system* $\widehat{\mathcal{S}}$ associated to $\mathcal{S}$ as $\widehat{\mathcal{S}} := q^{-1}(\mathrm{Pol}(\mathcal{S}))$, where $q \colon T^*M \to T^*M/\mathcal{S}$ is the quotient map. For a distribution $D \subset TM$, let $K(D)$ denote the subdistribution of D which is annihilated by the covariant system $\widehat{\mathcal{S}}(D)$ associated to the Pfaffian system $\mathcal{S}(D)$.

Example. We give an example of the polar space and the covariant system. Let $D_0 = \{\omega_1 = 0, \ldots, \omega_k = 0\}$, where $\omega_i := dx_{2i-1} + x_{2i}dt$, be a distribution on $\mathbb{R}^{2k+1}$ with the standard coordinates $(x_1, \ldots, x_{2k}, t)$. When $k > 1$, the distribution of polar spaces of $\mathcal{S}(D_0)$ is obtained as follows:

$$\begin{aligned} \mathrm{Pol}(\mathcal{S}(D_0)) &= \{w \in T^*M/\mathcal{S}(D_0) \mid w \wedge dx_{2i} \wedge dt \equiv 0 \pmod{\mathcal{S}(D_0)},\ i = 1, 2, \ldots, k\} \\ &= \{dt\}. \end{aligned}$$

Then the covariant system is obtained as follows:

$$\widehat{\mathcal{S}}(D_0) = \{\omega_1, \ldots, \omega_k, dt\} = \{dx_1, dx_3, \ldots, dx_{2k-1}, dt\}.$$

Then we have $K(D_0) = \langle \partial/\partial x_2, \partial/\partial x_4, \ldots, \partial/\partial x_{2k} \rangle$. They are clearly integrable. When $k = 1$, $D_0 = \{dx_1 - x_2 dt = 0\}$ is the standard contact structure on $\mathbb{R}^3$. Then we have $\mathrm{Pol}(\mathcal{S}(D_0)) = \{dx_2, dt\}$, $\widehat{\mathcal{S}}(D_0) = \{dx_1, dx_2, dt\}$, and $K(D_0) = \langle 0 \rangle$.

In terms of the covariant distribution $K(D)$, we obtain the following theorem. It should be noted here that the following theorem includes the case where $k = 1$, although Theorem 1 deals with the case where $k > 1$.

Theorem 3. *Let D_t, $t \in [0, 1]$, be a one-parameter family of distributions of corank $k \geq 1$ on a compact orientable manifold M. Suppose, for any $t \in [0, 1]$:*

(1) *the first derived distributions coincide with the tangent bundle of M: $D_t^2 = TM$,*

(2) *each D_t has the constant covariant distribution $K(D_t) \equiv K \subset TM$, which is integrable.*

Then, there exists a family of global diffeomorphisms $\varphi_t \colon M \to M$, $t \in [0, 1]$, which satisfies $\varphi_0 = \mathrm{id}$ and $(\varphi_t)_* D_0 = D_t$ for any $t \in [0, 1]$.

We describe a rough sketch of the proof of Theorem 3.

When $k = 1$, we obtain the result from the generalization above of the Gray theorem (Theorem 2). In fact, for each $t \in [0, 1]$, the covariant distribution $K(D_t)$ coincides with the Cauchy characteristic distribution $L(D_t)$ if $k = 1$. We remark that the Cauchy characteristic distributions are integrable from the definition and the Frobenius theorem. Then a one-parameter family D_t of distributions of corank 1 which satisfies the assumption of Theorem 3 satisfies automatically the assumption of Theorem 2. Therefore, Theorem 3 can be regarded as a generalization of a result of R. Montgomery and M. Zhitomirskiĭ, and further, as a generalization of the Gray theorem.

In what follows, we consider the case where $k > 1$. A. Kumpera and J. L. Rubin studied in [KRb] distributions whose first derived distributions coincide with the tangent bundles of the underlying manifolds. By using their results, we can induce from the assumption of Theorem 3 that of Theorem 1. In other words, for such distributions D_t as in Theorem 3, $K(D_t) \subset D$ is a subdistribution of corank 1 if it is integrable. Thus we obtain Theorem 3. We remark that the assumptions of Theorems 1 and 3 turn out to be equivalent if $k > 1$.

Examples of distributions

In the following part, we give examples of distributions whose first derived distributions are the tangent bundles of the underlying manifolds: the standard distribution on the jet bundle $J^1(1, k)$, and the generalized E. Cartan prolongation.

Example 1. Let $D_0 = \{\omega_1 = 0, \ldots, \omega_k = 0\}$, where $\omega_i := dx_{2i-1} + x_{2i}dt$, be a distribution on $\mathbb{R}^{2k+1}$ with coordinates $(x_1, \ldots, x_{2k}, t)$, $k > 1$. In other words, D_0 is the standard distribution on the jet bundle $J^1(1, k)$. This form also appears in the study of distributions whose first derived distributions coincide with the tangent bundle of the underlying manifolds by A. Kumpera and J. L. Rubin. They show in [KRb] the following.

Proposition 4 (Kumpera-Rubin). *Let D be a distribution of corank $k > 1$ on a manifold M whose derived distribution coincides with the tangent bundle: $D^2 = TM$. If the distribution $K(D)$ is integrable, then at each point $p \in M$ the distribution D admits the following local normal form: $D = \{\omega_1 = 0, \ldots, \omega_k = 0\}$,*

$$\omega_1 = dx_1 + x_2dt,\ \omega_2 = dx_3 + x_4dt,\ \ldots,\ \omega_k = dx_{2k-1} + x_{2k}dt,$$

where the coordinates x_i, t vanish at $p \in M$.

Example 2. An important example of distributions of the type discussed in this talk is constructed by the method of the generalized Cartan prolongation (see [Y], [Mor]). We belive that this construction is one of starting points of some applications of the results in this talk. Let us introduce the method. Let M be an n-dimensional manifold. We consider the Grassmannian bundle $J(M, 1)$ on M, which consists of 1-dimensional contact elements of M. We set

$$J(M, 1) := \bigcup_{p \in M} Gr(T_pM, 1), \tag{1}$$

where $Gr(T_pM, 1)$ denotes the Grassmannian manifold consisting of all the lines through the origin in T_pM. The total space $J(M, 1)$ is a $(2n - 1)$-dimensional manifold. $J(M, 1)$ has a canonical distribution D of rank n constructed as follows. Let $\pi\colon J(M, 1) \to M$ be the projection of the bundle. A point $q \in J(M, 1)$ can be regarded as a line ℓ in T_pM, where $p = \pi(q)$. We set $D_q := (d\pi)^{-1}\ell \subset T_q(J(M, 1))$. D_q is described as the standard distribution on the jet bundle $J^1(1, n-1)$ introduced in Example 1, by using certain local coordinates, as follows. Let $(x_1, \ldots, x_{n-1}, t)$ be local coordinates at a point of M. A line ℓ in T_pM is characterized by slopes $z_i := dx_i/dt$, $i = 1, 2, \ldots, n-1$, if it is not parallel to x_i-axis. Therefore, $(z_1, \ldots, z_{n-1})$ can be considered as local coordinates of fibres. The distribution D is described by these local coordinates $(x_1, \ldots, x_{n-1}, t, z_1, \ldots, z_{n-1})$ as $\{dx_1 - z_1 dt = 0, \ldots, dx_{n-1} - z_{n-1} dt = 0\}$. It is the standard distribution on the jet bundle $J^1(1, n-1)$. Last of all, we remark that when we take S^3 as the manifold M, we obtain a distribution of this type on a closed manifold $J(S^3, 1) \cong S^3 \times \mathbb{R}P^2$ or $S^3 \times S^2$.

References

[BCG3] R. L. Bryant, S. S. Chern, R. B. Gardner, H. L. Goldschmidt, P. A. Griffiths, *Exterior differential systems*, Mathematical Sciences Research Institute Publications, **18**, Springer-Verlag, New York, 1991.

[G] J. W. Gray, *Some global properties of contact structures*, Ann. of Math. (2) **69** (1959), 421–450.

[KRb] A. Kumpera, J. L. Rubin, *Multi-flag systems and ordinary differential equations*, Nagoya Math. J. **166** (2002), 1–27.

[KRz] A. Kumpera, C. Ruiz, *Sur l'équivalence locale des systèmes de Pfaff en drapeau*, Monge-Ampère equations and related topics (Florence, 1980), Ist. Naz. Alta Mat. Francesco Severi, Rome, 1982, 201–248.

[Ma] J. Martinet, *Classes caractéristiques des systèmes de Pfaff*, Lecture Notes in Math., Vol. **392**, Springer, Berlin, 1974, 30–36.

[MZh] R. Montgomery, M. Zhitomirskiĭ, *Geometric approach to Goursat flags*, Ann. Inst. H. Poincaré Anal. Non Linaire **18** (2001), no. 4, 459–493.

[Mor] P. Mormul, *Multi-dimensional Cartan prolongation and special k-flags*, preprint, Warsaw University 2002.

[Y] K. Yamaguchi, *Contact geometry of higher order*, Japan. J. Math. (N.S.) **8** (1982), no. 1, 109–176.

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