Random Schrödinger Operators of Anderson Type with Generalized Laplacians and Sparse Potentials

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DÉDICACE

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Chapter 2 of the present thesis was submitted for publication in the *Proceedings of the MolchanovFest (CRM)* [36]. Chapter 3 will be submitted for publication as a joint paper with Vojkan Jakšić [21].

ABSTRACT

The first part of the thesis concerns Green's functions of discrete Laplacians on lattices. In the continuous case, it is well known that the corresponding Green's functions decay polynomially. However, an identical proof of this fact fails in the discrete case, since the constant energy surfaces of the discrete Laplacian are not convex. Two approaches are presented to turn around this problem. One consists of adapting the stationary phase method in order to treat non convex surfaces admitting $\kappa > 0$ non vanishing principal curvatures at each point, as suggested by Littman [27]. The other consists of changing the discretization of the Laplacian, as suggested by Molchanov and Vainberg [30].

The second part of the thesis concerns random Schrödinger operators of type Anderson on the d-dimensional lattice. Sufficient conditions are presented for such operators, $H = \Delta + V$, to satisfy almost surely the following, remarkable spectral and scattering properties:

1. Outside spec(Δ), the spectrum of H is pure point with exponentially decaying eigenfunctions (so-called *Anderson localization*). Examples where the spectrum of H is equal to the whole real line are also exhibited, in which case the eigenvalues of H are in addition *dense* in $\mathbb{R} \setminus$ spec(Δ);
2. Inside spec(Δ), the spectrum of H is purely absolutely continuous (so-called *delocalization*);
3. Inside spec(Δ), the wave operators between H and Δ exist and are complete.

Such Anderson operators are exhibited for the first time in the literature. Using the estimate of the first part of the thesis, the mentioned sufficient conditions appear to be sparseness conditions on the support of the potential.

ABRÉGÉ

La première partie de cette thèse traite des fonctions de Green des laplaciens discrets sur $\mathbb{Z}^d$. Rappelons que, dans le cas continu, les fonctions de Green correspondantes décroissent polynomialement. Toutefois, la preuve de ce résultat ne peut être reproduite pour les laplaciens discrets, puisque les surfaces d'énergie constante de ces derniers ne sont pas convexes. Deux solutions à ce problème sont proposées. La première, suivant Littman, consiste à adapter la méthode des phases stationnaires pour qu'elle s'applique aux surfaces non convexes dont en chaque point au moins $\kappa > 0$ courbures principales ne s'annulent pas. La seconde, suivant Molchanov et Vainberg, consiste à modifier adéquatement la discrétisation du laplacien.

La seconde partie de cette thèse traite des opérateurs aléatoires de Schrödinger de type Anderson sur des réseaux. Des conditions suffisantes pour que de tels opérateurs, $H = \Delta + V$, vérifient presque sûrement les propriétés remarquables suivantes sont présentées:

1. En dehors de spec(Δ), le spectre d'H est purement ponctuel et ses fonctions propres décroissent exponentiellement (*localisation d'Anderson*); nous montrons en plus que, pour certains exemples, le spectre d'H est égal à $\mathbb{R}$, dans lequel cas sa partie purement ponctuelle est *dense* dans $\mathbb{R} \setminus \mathrm{spec}(\Delta)$.
2. À l'intérieur de spec(Δ), le spectre d'H est purement absolument continu (*délocalisation*);
3. À l'intérieur de spec(Δ), les opérateurs d'ondes entre H et Δ existent et sont complets.

De tels opérateurs d'Anderson sont présentés pour la première fois dans la littérature. Au moyen de la borne polynomiale établie dans la première partie de cette thèse, nous montrons que les conditions suffisantes en question reviennent à ce que le potentiel soit clairsemé (*sparse*).

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[1] Precise statements of the main theorems proven in this thesis are found in its conclusion (section 5.1).

[2] For a detailed description of results, terminology, and notations used in this chapter, the reader is invited to consult Appendix 4.2.

CHAPTER 1
Introduction[1]

1.1 Historical Background

Quantum physics is governed by the Schrödinger equation, following which observables (like the position, momentum, and energy of a particle) are captured by selfadjoint operators on a Hilbert space. In particular, the energy of a single particle is given by a Schrödinger operator, $\Delta + V$, acting on a certain Hilbert space (made of square summable functions). Here, the kinetic energy, Δ, is an extension of -1 times the usual Laplacian, while the potential energy, V, is the operator of multiplication by a certain function which depends on the physical context. The possible values of the energy are then given by the spectrum of $\Delta + V$, that is, the numbers, e, such that $\Delta + V - e$ is not invertible. To develop spectral theory (which studies spectra of selfadjoint operators on Hilbert spaces) has thus been a major concern for understanding quantum phenomena since the beginning of the last century.

About fifty years ago, new developments in quantum mechanics arised after Anderson introduced his model, in which the potential, V, is affected by a random parameter [4]. This last model was designed for studying solid state physics (for instance, the evolution of electrons submitted to a potential coming from impurities;

[1] Precise statements of the main theorems proven in this thesis are found in its conclusion (section 5.1).

since the exact nature of the impurities is not known, the best description of the induced potential is given by a probability distribution). The Anderson model provides a new insight in quantum mechanics, since, being interested in results that happen with probability one, pathological, unobserved counterexamples are discarded. The introduction of this model contributed to Anderson's Nobel prize in physics. His main conjecture, about the spectral nature of $\Delta + cV$ for c small, is still unsolved and attracts great scientists around the world (like the Fields medalist Jean Bourgain).

In order to find results motivating the Anderson conjecture, other models were suggested. For instance, in the discrete framework the underlying Hilbert space consists of square summable sequences over the d-dimensional lattice, and the Laplacian becomes the adjacency operator of this grid (up to an additive constant). Moreover, scientists have been interested in the case where the potential is sparse, *i.e.*, has non zero values on more and more distant sites only. One then investigates the spectral nature of the operator and asks which parts of its spectrum are absolutely continuous (so-called delocalization), pure point with exponentially decaying eigenfunctions (so-called Anderson localization), singular continuous, which parts admit possibly complete wave operators (scattering theory), etc.

1.2 Objectives

The present thesis concerns discrete, random Schrödinger operators of Anderson type with sparse potentials. Its objective is to exhibit for the first time in the literature a family of random Schrödinger operators, $H = \Delta + V$, acting on $l^2(\mathbb{Z}^d)$, satisfying almost surely the following, remarkable spectral and scattering properties:

1. The spectrum of H is dense pure point outside spec(Δ) with exponentially decaying eigenfunctions (so-called *Anderson localization*);
2. The spectrum of H is purely absolutely continuous on spec(Δ) (so-called *delocalization*);
3. The wave operators between Δ and H exist and are complete on spec(Δ).

Our main result, developed in Chapter 3, states that the above properties may hold under a suitable sparseness condition on the sites of the random potential, V. This result is an application of famous theorems in random perturbation theory (Simon–Wolff Theorem, Jakšić–Last Theorem), and more specifically of the Jakšić–Last criteria of existence and completeness of wave operators for Schrödinger operators on graphs [17]. These last criteria apply under the following, main condition: suppose the random potential, V, is supported on $\Gamma \subseteq \mathbb{Z}^d$; let us denote by Γ_1 the set consisting of all sites in Γ and their immediate neighbors; denoting by $\mathbf{1}_1$ the projection on $l^2(\Gamma_1)$, it is required that for all $n \in \Gamma$

$$\|\mathbf{1}_1(H - e - \mathrm{i}0)^{-1}\delta_n\| < \infty$$

on the considered interval of energy (say, for $e \in]a, b[$), where δ_n denotes the Kronecker delta.

In the present thesis two approaches are used to verify the previous condition: one is deterministic, the other is probabilistic.

In the deterministic case we investigate the free, restricted resolvent

$$\mathbf{1}_1(\Delta - e - \mathrm{i}0)^{-1}\mathbf{1}_1 \tag{1.1}$$

and explain how it is affected by a sparseness condition on Γ. Our key observation is the following: if Γ is sufficiently sparse, then (1.1) is the sum of a superdiagonal operator and a compact operator. Using this last decomposition, the sparseness of Γ also permits to control $\|\mathbf{1}_1(\Delta - e - \mathrm{i}0)^{-1}\mathbf{1}_1\|$. Then, one passes from Δ to H using the resolvent identity in conjunction with Fredholm's analytic theory.

In the probabilistic case, (1.1) is estimated by means of the Aizenman–Molchanov theory. The key observation is the following: if Γ is sufficiently sparse, then there exists a finite set, $\mathcal{F} \subset \Gamma$, such that the Aizenman–Molchanov method applies to $H + \mathbf{1}_{\Gamma\setminus\mathcal{F}}V$, where $\mathbf{1}_{\Gamma\setminus\mathcal{F}}$ is the characteristic function of $\Gamma \setminus \mathcal{F}$. One then goes from this last operator to H by the resolvent identity.

A preliminary prolem occured, which is discussed in the first part of the thesis. The conditions found in our main theorems are expressed in terms of the Green's functions of Δ, more precisely, in terms of

$$G(n, e + \mathrm{i}0) = \langle \delta_0 \,|\, (\Delta - e - \mathrm{i}0)^{-1}\delta_n \rangle.$$

They constitute a sparseness condition on Γ only if an *a priori* estimate on $G(n, e+\mathrm{i}0)$ (when $|n| \to \infty$) is known. In the continuous case such an *a priori* estimate is easily established using the stationary phase method, due to the fact that constant energy surfaces[2] of the continuous Laplacian are strictly convex (indeed, they are spheres). However, strict convexity of constant energy surfaces fails in the discrete case.

Two approaches may be used to turn around this problem. One consists of generalizing the stationary phase method in order to treat non convex surfaces without planar point [27], which gives a weak, but satisfying estimate. The other consists of changing the discretization of the usual Laplacian, as suggested by Molchanov and Vainberg [30].

This last idea is simple. At the first glance, periodicity of $\widehat{\Delta}$ forbids convexity of its level surfaces, since, when lifting the level surfaces to the usual covering of the torus (*i.e.*, to Euclidean space), one may obtain *unbounded connected components* which decompose in patterns reproduced at every $(2\pi, \ldots, 2\pi)$, creating an oscillation. However, convexity is still possible if $\widehat{\Delta}$ is factorized, in which case the level surfaces consist of bounded connected components enclosed in a system of hyperplanes.

One thus seeks for a discretization of the Laplacian whose symbol is factorized, which is easily found when considering the associated random walk. For instance, if *each single step* of a random walk is determined by several independent trials, one per axis, each trial determining the direction of the walk along its corresponding axis, then the resulting stochastic process is the product of 1-dimensional, independent

[2] Given an operator, A, on $L^2(\mathbb{R}^d)$ or $l^2(\mathbb{Z}^d)$, its *symbol*, $\widehat{A}$, is its lifting via the Fourier transform. If the symbol of A is (the multiplication by) a function, the *constant energy surfaces* of A are the level surfaces of $\widehat{A}$.

processes, so the resulting symbol is factorized (each factor corresponding to the symbol of a 1-dimensional random walk). Notice that the random walk just described then goes along full diagonals; hence, the construction of the proposed Laplacian is based on full diagonal neighbors instead of immediate neighbors. It is not difficult to verify that the constant energy surfaces of the resulting operator are convex [35].

The above, preliminary problem and its solutions were the occasion for the author, firstly, to write a chapter reviewing the stationary phase method and its applications to Green's functions of discrete Laplacians—this text, based on [45, 46, 42], constitute the first part of the present thesis and will appear in [36]; [3] secondly, to promote the use of the diagonal Laplacian in the context of scattering theory of the Anderson model. In this context, the operator in question has been named the *Molchanov–Vainberg Laplacian.*

1.3 Pre-requisites

Since Chapter 2 is a review of the stationary phase method and its application to Green's functions, only a small knowledge of differential geometry of surfaces

[3] In certain respects the mentioned text goes beyond a simple review. For instance, results and proofs coming from [45] are adapted in order treat a parameter, thanks to which proofs are simplified (*e.g.*, the corollary of Theorem 3 is deduced by induction). This treatment of a parameter also permits to deduce a decay for Fourier transforms of smooth functions over non convex surfaces (in particular, technical details omitted in [27] are complemented here). Then, the strategy presented in [42] for estimating Green's function of the standard discrete Laplacian on a certain interval of energy becomes available for other intervals of energy, or for other discrete Laplacians *without assuming that their constant energy surfaces are convex on the considered intervals.*

in $\mathbb{R}^d$ is assumed [6]. However, Chapter 3 uses pre-requisites in measure theory, harmonic analysis, functional analysis, probability theory, and especially in random perturbation theory.

The interested reader may consult [41] for a standard exposition of measure theory and harmonic analysis.

In functional analysis, a strong knowledge of the spectral theorem for unbounded selfadjoint operators is required; a good reference is [40]. As a complement, we strongly recommend [13]. In this last reference, the proof of the spectral theorem follows an interesting outline which is described in the first part of Appendix 4.2; this appendix may be read before Chapter 3, since it gives an accurate overview of results, notations, and terminology used in this last chapter.

In random perturbation theory the Simon–Wolff theorem [44], the Aizenman–Molchanov theory [3], and the Jakšić–Last theorem [16] are assumed. These specialized results are described in the second part of Appendix 4.2.

Finally, for a general treatment of random Schrödinger operators, the monographies [7, 8] are recommended.

CHAPTER 2
Stationary Phase Method and Applications to Green's Functions

An *oscillatory integral* is an integral of the form

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\varphi(x)} f(x)\,\mathrm{d}x,$$

where $r \in \mathbb{R}$ and $\varphi(x)$ is real valued. Its *amplitude* and *phase* are $f(x)$ and $\varphi(x)$, respectively. The phase is *stationary at* x_0 if $\nabla\varphi(x_0) = 0$. Such a stationary phase point, x_0, is *non degenerate* if in addition $\det \mathrm{D}_x^2\varphi(x_0) \neq 0$.

In this chapter we establish *1)* the rapid decay of an oscillatory integral whose phase is nowhere stationary, for a given compactly supported amplitude; *2)* the polynomial decay of an oscillatory integral whose stationary phase points are not degenerate; *3)* similar results for Cauchy principal values of oscillatory integrals. Given an analytic function, $\Phi(x)$, on $\mathbb{T}^d$, we then consider Fourier transforms of functions over level surfaces of the form

$$\Gamma(e) = \{x \in \mathbb{T}^d\ ;\ \Phi(x) = e\}$$

and investigate their decay when the $\Gamma(e)$'s are regular, compact, and admit at every point at least $\kappa > 0$ non vanishing principal curvatures. Then, we do a similar study for Cauchy principal values of such Fourier transforms. Finally, we deduce the decay of Green's functions of generalized Laplacians for energies inside their associated spectra.

In all our results a parameter $t \in \mathbb{R}^m$ is considered. It permits to deduce multidimensional results from 1-dimensional ones using a simple induction. When studying Fourier transforms over $\Gamma(e)$, it also permits to deduce uniform estimates in e and, furthermore, to generalize our results to surfaces whose Gaussian curvature may vanish, but which admit at least $\kappa > 0$ non vanishing principal curvatures. It permits to study the decay of Cauchy principal values of such Fourier transforms and hence, to calculate the decay of Green's functions of a general class of operators. Finally, it permits to show that this last decay is uniform in e, where e is the level of energy.

We adopt the following conventions: most of our theorems establish the existence of a neighborhood on which a certain phenomenon occurs. For sake of simplicity (and without loss of generality), we consider only non empty, bounded, open, cubic neighborhoods and we call them *cubes.*[1]

Given a real valued phase, $\varphi(x,t)$, we define

$$I_f(r,t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\varphi(x,t)} f(x,t)\,\mathrm{d}x,$$

where $r > 0$, $t \in \mathbb{R}^m$, and $f(x,t)$ is a complex valued function—provided that this integral makes sense.

In the present text, *smooth* is used for *infinitely differentiable.* The vector space of all (complex valued) smooth functions on $\mathbb{R}^d$ is denoted by $C^\infty(\mathbb{R}^d)$. It contains

[1] Recall that a *cubic neighborhood* in $\mathbb{R}^d$ is a subset of the form $I_1 \times \cdots \times I_d$, where each I_d is an interval.

two important subspaces: $C^{\omega}(\mathbb{R}^d)$, consisting of all analytic functions on $\mathbb{R}^d$, and $C_c^{\infty}(\mathbb{R}^d)$, consisting of all compactly supported smooth functions on $\mathbb{R}^d$.

The *transpose* of a linear transformation on $C_c^{\infty}(\mathbb{R}^d)$ is defined by duality with respect to the following bracket:

$$(f(x)|g(x)) = \int_{\mathbb{R}^d} f(x)g(x)\,\mathrm{d}x.$$

For instance, denoting by $\partial_{x^{(j)}}$ the differentiation with respect to $x^{(j)}$, integration by parts gives ${\partial_{x^{(j)}}}^{\mathrm{t}} = -\partial_{x^{(j)}}$. Moreover, denoting by $F(x)$ the multiplication by a smooth function of the same name, $F(x)^{\mathrm{t}} = F(x)$.

Our notations regarding asymptotic behavior are standard: for instance,

$$f(r) = O(r^{-\alpha}) \text{ when } r \to \infty$$

means the existence of a positive constant, C, such that $|f(r)| \leqslant Cr^{-\alpha}$ when r is sufficiently large. In the weaker circumstance where $f(r) = O(r^{-\alpha+\varepsilon})$ for all $\varepsilon > 0$ (where the constant C depends on ε), one writes

$$f(r) = O(r^{-\alpha^+}) \text{ when } r \to \infty.$$

In the stronger circumstance where $f(r) = O(r^{-\alpha})$ for all $\alpha \geqslant 0$ (where the constant C depends on α), one writes

$$f(r) = O(r^{-\infty}) \text{ when } r \to \infty.$$

Finally,

$$f(r) \sim \sum_{j=0}^{\infty} a_j r^{-j} \text{ when } r \to \infty$$

means that for any $N \geqslant 0$

$$f(r) - \sum_{j=0}^{N} a_j r^{-j} = O(r^{-N-1}).$$

If the function f also depends on a parameter, t, we say that the previous estimates/asymptotics are *uniform* in t if the constants C may be chosen independently of t.

2.1 Oscillatory Integral without Stationary Phase Point

The following theorems are stated in the way they are used when studying Fourier transforms over surfaces. They establish the existence of neighborhoods on which a certain phenomenon occurs, given a fixed phase. The given phase, $\varphi(x, t)$, is supposed to be smooth in $(x, t) \in \mathbb{R}^d \times \mathbb{R}^m$.

Lemma *Let $d = 1$. Suppose $\partial_x \varphi \neq 0$ at a given $(x_0, t_0) \in \mathbb{R} \times \mathbb{R}^m$. Then, there exists an arbitrarily small cube, $U \times B$, containing (x_0, t_0) such that the following holds: if the amplitude, $f(x, t)$, is smooth in the neighborhood of $\mathbb{R} \times \overline{B}$ and vanishes on $U^c \times \overline{B}$, then*

$$|I_f(r, t)| = O(r^{-\infty})$$

uniformly in $t \in \overline{B}$.[2]

[2] U^c denotes the complementary of U with respect to a set determined by the context; here, $U^c = \mathbb{R} \setminus U$.

Proof: By continuity, $\partial_x\varphi(x,t) \neq 0$ on a certain cube $U' \times B'$ containing (x_0, t_0). The operator $D = \frac{1}{\partial_x\varphi(x,t)} \circ \partial_x$ is thus well defined on this cube, where it satisfies $D\mathrm{e}^{\mathrm{i}r\varphi(x,t)} = \mathrm{i}r\mathrm{e}^{\mathrm{i}r\varphi(x,t)}$ and $D^{\mathrm{t}} = -\partial_x \circ \frac{1}{\partial_x\varphi(x,t)}$. Let $U \times B \ni (x_0, t_0)$ be a cube whose closure is in $U' \times B'$. If $f(x,t)$ satisfies the asserted properties, then for any $N \geqslant 0$ and $t \in \overline{B}$

$$\begin{aligned} |I_f(r,t)| &= \frac{1}{r^N}\left|\int_U (D^N \mathrm{e}^{\mathrm{i}r\varphi(x,t)}) f(x,t)\,\mathrm{d}x\right| \\ &= \frac{1}{r^N}\left|\int_U \mathrm{e}^{\mathrm{i}r\varphi(x,t)} (D^{\mathrm{t}})^N f(x,t)\,\mathrm{d}x\right| \\ &\leqslant \frac{C_N}{r^N}, \end{aligned}$$

where the constant C_N does not depend on $t \in \overline{B}$. □

The multidimensional analogue follows:

Theorem 1 *Suppose $\nabla_x\varphi(x_0,t_0) \neq 0$ for a given $(x_0,t_0) \in \mathbb{R}^d \times \mathbb{R}^m$. Then, there exists an arbitrarily small cube, $U \times B$, containing (x_0,t_0) such that the following holds: if the amplitude, $f(x,t)$, is smooth in the neighborhood of $\mathbb{R}^d \times \overline{B}$ and vanishes on $U^{\mathrm{c}} \times \overline{B}$, then*

$$|I_f(r,t)| = O(r^{-\infty})$$

uniformly in $t \in \overline{B}$.

Proof: By assumption, $\partial_{x^{(k)}}\varphi(x_0,t_0) \neq 0$ for a certain $1 \leqslant k \leqslant d$, say, for $k = 1$. Interpreting $(x^{(2)}, \ldots, x^{(d)}, t)$ as a parameter, there exists an arbitrarily small cube, $U_1 \times (U_2 \times \cdots \times U_d \times B)$, containing $(x_0^{(1)}; x_0^{(2)}, \ldots, x_0^{(d)}, t_0)$ on which the previous lemma applies. Hence, if $f(x,t)$ satisfies the asserted properties, then

$$\left|\int_{U_1} \mathrm{e}^{\mathrm{i}r\varphi(x,t)} f(x,t)\,\mathrm{d}x^{(1)}\right| \leqslant \frac{C_N}{r^N}$$

uniformly in $(x^{(2)}, \ldots, x^{(d)}, t) \in \overline{U_2} \times \ldots \times \overline{U_d} \times \overline{B}$. In particular,

$$\begin{aligned} |I_f(r,t)| &= \left|\int_{U_2\times\ldots\times U_d} \int_{U_1} \mathrm{e}^{\mathrm{i}r\varphi(x,t)} f(x,t)\,\mathrm{d}x^{(1)}\ \mathrm{d}x^{(2)} \ldots\ \mathrm{d}x^{(d)}\right| \\ &\leqslant |U_2 \times \ldots \times U_d| \frac{C_N}{r^N} \end{aligned}$$

uniformly in $t \in \overline{B}$. □

Scholium *A similar result holds when considering oscillatory integrals over the torus,*

$$\int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}r\varphi(x,t)} f(x,t)\,\mathrm{d}x,$$

without stationary phase point. One then assumes $f(\cdot,t)$ is periodic for every t (instead of being compactly supported). The same proof works, due to the absence of boundary term when integrating by parts a smooth periodic function.

2.1.1 Cauchy Principal Value

We now turn our attention to Cauchy principal values of oscillatory integrals without stationary phase point. To this end, we now consider a parameter $(e,t) \in \mathbb{R} \times \mathbb{R}^m$ and a phase, $\varphi(x;e,t)$, smooth in $(x;e,t) \in \mathbb{R}^d \times (\mathbb{R} \times \mathbb{R}^m)$. We thus let

$$I_f(r;e,t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\varphi(x;e,t)} f(x;e,t)\,\mathrm{d}x$$

and study

$$\text{p.v.} \int_{|\eta - e| < \delta} \frac{1}{\eta - e} I_f(r; \eta, t) \, \mathrm{d}\eta = \lim_{\varepsilon \downarrow 0} \int_{\varepsilon < |\eta - e| < \delta} \frac{1}{\eta - e} I_f(r; \eta, t) \, \mathrm{d}\eta$$

for a given $\delta > 0$. Our results are based on the following elementary estimate:

Lemma *Suppose $f(h)$ is a continuously differentiable function in the neighborhood of $[-\delta, \delta]$, where $\delta > 0$ is given. Then, for any $\varepsilon \in]0, \delta[$*

$$\left| \int_{\varepsilon \leqslant |h| \leqslant \delta} \frac{f(h)}{h} \, \mathrm{d}h \right| \leqslant 2\delta \max_{|h| \leqslant \delta} |f'(h)|.$$

In particular,

$$\left| \text{p.v.} \int_{-\delta}^{\delta} \frac{f(h)}{h} \, \mathrm{d}h \right| \leqslant 2\delta \max_{|h| \leqslant \delta} |f'(h)|.$$

Proof: By the mean value theorem, there exist numbers $|\xi_h| < \delta$ such that

$$\begin{aligned} \left| \int_{\varepsilon \leqslant |h| \leqslant \delta} \frac{f(h)}{h} \, \mathrm{d}h \right| &= \left| \int_{\varepsilon}^{\delta} \frac{f(h) - f(-h)}{h} \, \mathrm{d}h \right| \\ &= \left| \int_{\varepsilon}^{\delta} 2f'(\xi_h) \, \mathrm{d}h \right| \\ &\leqslant 2\delta \max_{|h| \leqslant \delta} |f'(h)|. \end{aligned}$$

□

Theorem 2 *Suppose $\nabla_x \varphi(x_0, 0, t_0) \neq 0$ for a given $(x_0, t_0) \in \mathbb{R}^d \times \mathbb{R}^m$. Then, there exists an arbitrarily small cube, $U \times B$, containing (x_0, t_0) and an arbitrarily small*

$\delta > 0$ *such that the following holds: if the amplitude,* $f(x, h, t)$, *is smooth in the neighborhood of* $\mathbb{R}^d \times [-\delta, \delta] \times \overline{B}$ *and vanishes on* $U^c \times [-\delta, \delta] \times \overline{B}$, *then*

$$\left| \text{p.v.} \int_{-\delta}^{\delta} \frac{I_f(r, h, t)}{h} \, \mathrm{d}h \right| = O(r^{-\infty})$$

uniformly in $t \in \overline{B}$.

Proof: By the lemma it suffices to estimate $\partial_h I_f(r, h, t)$. In fact, the dominated convergence theorem implies

$$\partial_h I_f(r, h, t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\varphi(x,h,t)} (\partial_h f(x, h, t) + \mathrm{i}r f(x, h, t) \partial_h \varphi(x, h, t)) \, \mathrm{d}x.$$

Hence, by Theorem 1 there exist an arbitrarily small cube, $U \times B$, containing (x_0, t_0) and an arbitrarily small $\delta > 0$ such that for $f(x, h, t)$ as stipulated

$$|\partial_h I_f(r, h, t)| \leqslant \frac{C_N}{r^N}$$

uniformly in $(h, t) \in [-\delta, \delta] \times \overline{B}$. The result follows. □

Applying the above to the phase $\tilde{\varphi}(x, h; e, t) = \varphi(x, h + e, t)$ and the amplitude $\tilde{f}(x, h; e, t) = f(x, h + e, t)$, one obtains:

Corollary *Suppose* $\nabla_x \varphi(x_0, e_0, t_0) \neq 0$ *for a given* $(x_0, e_0, t_0) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^m$. *Then, there exist an arbitrarily small cube,* $U \times B$, *containing* (x_0, t_0) *and an arbitrarily small* $\delta > 0$ *such that the following holds: if the amplitude,* $f(x, e, t)$, *is smooth in the neighborhood of* $\mathbb{R}^d \times [e_0 - \delta, e_0 + \delta] \times \overline{B}$ *and vanishes on* $U^c \times [e_0 - \delta, e_0 + \delta] \times \overline{B}$, *then*

$$\left|\mathrm{p.v.}\int_{|\eta-e|<\delta}\frac{I_f(r,\eta,t)}{\eta-e}\,\mathrm{d}\eta\right| = O(r^{-\infty})$$

uniformly in $(e,t)\in[e_0-\delta, e_0+\delta]\times\overline{B}$.

2.2 Oscillatory Integral with Non Degenerate Stationary Phase Points

2.2.1 Quadratic Phase

We now investigate the decay of oscillatory integrals admitting non degenerate stationary phase points. Via the *Morse lemma*, which is proven below, our study reduces to oscillatory integrals whose phases are canonical, not degenerate quadratic forms:

$$Q(x)=\sum_{j=1}^{s}(x^{(j)})^2-\sum_{k=s+1}^{d}(x^{(k)})^2,$$

where $0\leqslant s\leqslant d$.

Exceptionally, in the next three lemmas the variable $r\in\mathbb{R}$ is allowed to be negative. Moreover $d=1$ (so x varies in $\mathbb{R}$).

Lemma *Suppose* $|r|>1$. *For all* $l\in\mathbb{N}$ *there exist constants* $c_j^{(l)}\in\mathbb{C}$ *independent of* $|r|$ *such that*

$$\int_{-\infty}^{\infty}\mathrm{e}^{-(1-\mathrm{i}r)x^2}x^l\,\mathrm{d}x=|r|^{-\frac{l+1}{2}}\sum_{j=0}^{\infty}c_j^{(l)}|r|^{-j}.$$

Proof: Let us denote by $(\cdot)^{\frac{1}{2}}$ the branch of the square root whose singular cut is the positive imaginary axis. Let $z = (1 - \mathrm{i}r)^{\frac{1}{2}}x$. Then,

$$\int_{\mathbb{R}} \mathrm{e}^{-(1-\mathrm{i}r)x^2} x^l \,\mathrm{d}x = \left(\int_\gamma \mathrm{e}^{-z^2} z^l \,\mathrm{d}z\right) [(1-\mathrm{i}r)^{\frac{1}{2}}]^{-l-1}, \tag{2.1}$$

where γ is the oriented path $(1 - \mathrm{i}r)^{\frac{1}{2}}\mathbb{R}$. Observe that

$$(1-\mathrm{i}r)^{\frac{1}{2}} = \sigma_r |r|^{\frac{1}{2}} \left(\frac{1}{r} - \mathrm{i}\right)^{\frac{1}{2}}, \tag{2.2}$$

where

$$\sigma_r = \begin{cases} 1 & \text{if} \quad r > 1 \\ \mathrm{i} & \text{if} \quad r < -1 \end{cases}$$

(so σ_r does not depend on $|r|$). Moreover the Taylor expansion

$$[(w-\mathrm{i})^{\frac{1}{2}}]^{-l-1} = \sum_{j=0}^{\infty} a_j^{(l)} w^j \tag{2.3}$$

is valid for all $|w| < 1$, by choice of the branch of $z^{\frac{1}{2}}$. Thus, substituting $w = \mathrm{sgn}(r)\frac{1}{|r|}$ in the above, the result follows from (2.1), (2.2), and (2.3). $\square$

Lemma *Suppose $f(x,t)$ is smooth in the neighborhood of $\mathbb{R} \times \overline{B}$ and vanishes on $U^{\mathrm{c}} \times \overline{B}$, where $U \subset \mathbb{R}$ and $B \subset \mathbb{R}^m$ are cubes. For any $l \in \mathbb{N}$ there exists a $C_l > 0$ independent of $|r|$ satisfying*

$$\left| \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}rx^2} x^l f(x,t) \,\mathrm{d}x \right| \leqslant C_l |r|^{-\frac{l+1}{2}}$$

uniformly in $t \in \overline{B}$.

Proof: Let $\chi(x)$ be a compactly supported smooth function on $\mathbb{R}$ such that $0 \leqslant \chi(x) \leqslant 1$ everywhere and

$$\chi(x) = \begin{cases} 0 & \text{if} \quad |x| \geqslant 2 \\ 1 & \text{if} \quad |x| \leqslant 1. \end{cases}$$

For an arbitrarily fixed $\varepsilon > 0$, $\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}rx^2} x^l f(x,t)\,\mathrm{d}x = I + II$, where

$$\begin{aligned} I &= \int_{|x|<2\varepsilon} \mathrm{e}^{\mathrm{i}rx^2} x^l f(x,t) \chi\left(\frac{x}{\varepsilon}\right)\,\mathrm{d}x, \\ II &= \int_{|x|>\varepsilon} \mathrm{e}^{\mathrm{i}rx^2} x^l f(x,t) \left(1 - \chi\left(\frac{x}{\varepsilon}\right)\right)\,\mathrm{d}x. \end{aligned}$$

Concerning I, there exist constants (generically denoted by *Const*) independent of $t \in \overline{B}$, but depending on l, satisfying

$$|I| \leqslant Const \int_{-2\varepsilon}^{2\varepsilon} |x|^l\,\mathrm{d}x = Const\,\varepsilon^{l+1}.$$

Concerning II, let $D = \frac{1}{x} \circ \partial_x$, which is well defined on the support of the integrand. Since $\frac{1}{2\mathrm{i}r} D$ fixes $\mathrm{e}^{\mathrm{i}rx^2}$,

$$II = \left(\frac{1}{2\mathrm{i}r}\right)^N \int_{|x|>\varepsilon} \mathrm{e}^{\mathrm{i}rx^2} (D^{\mathrm{t}})^N x^l f(x,t) \left(1 - \chi\left(\frac{x}{\varepsilon}\right)\right)\,\mathrm{d}x.$$

Letting $F(x,y,t) = f(x,t)(1 - \chi(y))$, notice that

$$D^{\mathrm{t}} x^l F(x, x/\varepsilon, t) = x^{l-2} F_0(x, x/\varepsilon, t) + x^{l-1} \varepsilon^{-1} F_1(x, x/\varepsilon, t),$$

where $F_0(x,y,t)$ and $F_1(x,y,t)$ are bounded on $\mathbb{R}\times\mathbb{R}\times\overline{B}$. More generally, notice that $(D^{\mathrm{t}})^N F(x,x/\varepsilon,t)$ is of the form

$$x^{l-2N}F_0(x,x/\varepsilon,t)+x^{l-2N+1}\varepsilon^{-1}F_1(x,x/\varepsilon,t)+\cdots+x^{l-N}\varepsilon^{-N}F_N(x,x/\varepsilon,t),$$

where $F_0(x,y,t),\ldots,F_N(x,y,t)$ are bounded on $\mathbb{R}\times\mathbb{R}\times\overline{B}$. Hence,

$$\begin{aligned}
|(D^{\mathrm{t}})^N F(x,x/\varepsilon,t)| \;&\leqslant \\
&\leqslant\; \mathit{Const}\,(|x|^{l-2N}+|x|^{l-2N+1}\varepsilon^{-1}+|x|^{l-2N+2}\varepsilon^{-2}+\cdots+|x|^{l-N}\varepsilon^{-N}) \\
&\leqslant\; \mathit{Const}\,(|x|^{l-2N}+|x|^{l-N}\varepsilon^{-N}),
\end{aligned}$$

where $N>l+1$ is fixed. Therefore,

$$\begin{aligned}
|II| \;&\leqslant\; \mathit{Const}\,|r|^{-N}\int_{|x|>\varepsilon}(|x|^{l-2N}+|x|^{l-N}\varepsilon^{-N})\,\mathrm{d}x \\
&=\; \mathit{Const}\,|r|^{-N}\varepsilon^{l-2N+1}.
\end{aligned}$$

In total,

$$|I+II|\leqslant \mathit{Const}\,(\varepsilon^{l+1}+|r|^{-N}\varepsilon^{l-2N+1}),$$

where $N>l+1$ is fixed. Choosing $\varepsilon=|r|^{-\frac{1}{2}}$ then completes the proof. □

Lemma *Consider a function, $f(x,t)$, smooth in the neighborhood of $\mathbb{R}\times\overline{B}$ (where $B\subset\mathbb{R}^m$ is a cube) and vanishing when $|x|<\varepsilon$ (where $\varepsilon>0$). Suppose that for all*

$N \geqslant 0$ there exist a $D_N > 0$ and an $\alpha_N > 0$ such that

$$|\partial_x^N f(x,t)| \leqslant D_N(1 + |x|^{\alpha_N})$$

uniformly in $t \in \overline{B}$. Then,

$$\int_{-\infty}^{\infty} \mathrm{e}^{(\mathrm{i}r-1)x^2} f(x,t)\,\mathrm{d}x = O(r^{-\infty})$$

uniformly in $t \in \overline{B}$.

Proof: The derivation $D = \frac{1}{x} \circ \partial_x$ is well defined on the support of the integrand. Notice that $\frac{1}{2\mathrm{i}r} D$ fixes $\mathrm{e}^{\mathrm{i}rx^2}$ and $D^{\mathrm{t}} = -\partial_x \circ \frac{1}{x}$. Then,

$$\left| \int_{\mathbb{R}} \mathrm{e}^{(\mathrm{i}r-1)x^2} f(x,t)\,\mathrm{d}x \right| = \left(\frac{1}{2|r|}\right)^N \left| \int_{|x| \geqslant \varepsilon} \mathrm{e}^{\mathrm{i}rx^2} (D^{\mathrm{t}})^N (\mathrm{e}^{-x^2} f(x,t))\,\mathrm{d}x \right|.$$

Our assumption on the derivatives of $f(x,t)$ makes the integral on the right side of the previous equation uniformly bounded in $t \in \overline{B}$, where N is arbitrarily fixed. The result follows. $\square$

In dimension one we are interested in the phase x^2, so let

$$I_f(r,t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}rx^2} f(x,t)\,\mathrm{d}x.$$

According to our convention $r > 0$. However, since we are also interested in the phase $-x^2$, we allow r to be negative. We are ready to compute the asymptotic expansion of $I_f(r,t)$ (resp. $I_f(-r,t)$) when $|r| \to \infty$:

Theorem 3 *Consider an amplitude, $f(x,t)$, smooth in the neighborhood of $\mathbb{R} \times \overline{B}$ and vanishing on $U^c \times \overline{B}$, where $U \subset \mathbb{R}$ and $B \subset \mathbb{R}^m$ are cubes. There exist constants, $a_j(t)$, depending smoothly on $t \in \overline{B}$ such that*

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}rx^2} f(x,t)\,\mathrm{d}x \sim |r|^{-\frac{1}{2}} \sum_{j=0}^{\infty} a_j(t)|r|^{-\frac{j}{2}}$$

uniformly in $t \in \overline{B}$ when $r \to \infty$. The same result holds when $r \to -\infty$, with different constants $a_j(t)$.

Proof: Let $\chi(x)$ be a smooth, bounded, compactly supported function such that $\chi(x) = 1$ on an interval containing $\overline{U} \cup \{0\}$. Then, for $f(x,t)$ as stated

$$I_f(r,t) = \int_{\mathbb{R}} \mathrm{e}^{(\mathrm{i}r-1)x^2} (\,\mathrm{e}^{x^2} f(x,t)\,) \chi(x)\,\mathrm{d}x.$$

By Taylor's theorem (in dimension 1), for any fixed N there exists a polynomial,

$$P_t(x) = \sum_{l=0}^{N} b_l(t) x^l,$$

and a smooth remainder, $R_t(x)$, both depending smoothly on $(x,t) \in \mathbb{R} \times \overline{B}$, such that

$$\mathrm{e}^{x^2} f(x,t) = P_t(x) + x^{N+1} R_t(x).$$

Thus, $I_f(r,t)$ decomposes into $I + II + III$, where

$$\begin{aligned} I &= \sum_{l=0}^{N} b_l(t) \int_{\mathbb{R}} \mathrm{e}^{(\mathrm{i}r-1)x^2} x^l\,\mathrm{d}x, \\ II &= \int_{\mathbb{R}} \mathrm{e}^{(\mathrm{i}r-1)x^2} x^{N+1} R_t(x)\chi(x)\,\mathrm{d}x, \\ III &= \int_{\mathbb{R}} \mathrm{e}^{(\mathrm{i}r-1)x^2} P_t(x)(\chi(x) - 1)\,\mathrm{d}x. \end{aligned}$$

Concerning I the first lemma gives the existence of $c_j^{(l)} \in \mathbb{C}$ such that for any $|r| > 1$

$$\begin{aligned} I &= \sum_{l=0}^{N} b_l(t)|r|^{-\frac{l+1}{2}} \sum_{j=0}^{\infty} c_j^{(l)}|r|^{-j} \\ &= |r|^{-\frac{1}{2}} \sum_{l=0}^{N} b_l(t)|r|^{-\frac{l}{2}} \sum_{j\leqslant\frac{N-l}{2}} c_j^{(l)}|r|^{-j} + \sum_{l=0}^{N} b_l(t)|r|^{-\frac{l+1}{2}} \sum_{j>\frac{N-l}{2}} c_j^{(l)}|r|^{-j}. \end{aligned}$$

Notice that

$$\sum_{l=0}^{N} b_l(t)|r|^{-\frac{l+1}{2}} \sum_{j>\frac{N-l}{2}} c_j^{(l)}|r|^{-j} = O(|r|^{-\frac{N}{2}-1})$$

uniformly in $t \in \overline{B}$ when $r \to \infty$ (resp. $r \to -\infty$). Consequently, we have found coefficients, $a_k(t)$, smooth in $t \in \overline{B}$, satisfying

$$I = |r|^{-\frac{1}{2}} \sum_{k=0}^{N} a_k(t)|r|^{-\frac{k}{2}} + O(|r|^{-\frac{N}{2}-1})$$

uniformly in $t \in \overline{B}$.

Moreover, the last two lemmas give respectively

$$II = O(|r|^{-\frac{N}{2}-1}) \quad \text{and} \quad III = O(|r|^{-\infty})$$

uniformly in $t \in \overline{B}$. In total,

$$I_f(r,t) = |r|^{-\frac{1}{2}} \sum_{k=0}^{N} a_k(t)|r|^{-\frac{k}{2}} + O(|r|^{-\frac{N}{2}-1})$$

uniformly in $t \in \overline{B}$ when $r \to \infty$ (resp. $r \to -\infty$), as desired. □

Our treatment of a parameter $t \in \mathbb{R}^m$ permits to deduce from the previous theorem its multidimensional analogue by induction. Indeed, let us consider the phase

$$Q(x) = \sum_{j=1}^{s} (x^{(j)})^2 - \sum_{k=s+1}^{d} (x^{(k)})^2,$$

where $x \in \mathbb{R}^d$ and $0 \leqslant s \leqslant d$. Returning to the convention that $r > 0$, we let

$$I_f(r,t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}rQ(x)} f(x,t)\,\mathrm{d}x.$$

Then,

Corollary *Consider an amplitude, $f(x,t)$, smooth in the neighborhood of $\mathbb{R}^d \times \overline{B}$ and vanishing on $U^{\mathrm{c}} \times \overline{B}$, where $U \subset \mathbb{R}^d$ and $B \subset \mathbb{R}^m$ are cubes. There exist constants $a_j(t)$ depending smoothly on $t \in \overline{B}$ such that*

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}rQ(x)} f(x,t)\,\mathrm{d}x \sim r^{-\frac{d}{2}} \sum_{j=0}^{\infty} a_j(t) r^{-\frac{j}{2}}$$

uniformly in $t \in \overline{B}$, when $r \longrightarrow \infty$.

Proof: Suppose the result holds for a certain $d-1$. Let $x_1 = (x^{(1)}, \ldots, x^{(d-1)})$ and $Q_1(x_1)$ be defined by

$$Q(x) = Q_1(x_1) \pm (x^{(d)})^2.$$

Recall that U may be written $U_1 \times \cdots \times U_d$. Then, by the inductive hypothesis, for an arbitrary $N \geqslant 0$

$$\begin{aligned} I_f(r,t) &= \int_{U_d} \mathrm{e}^{\pm \mathrm{i}r(x^{(d)})^2} \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}rQ_1(x_1)} f(x_1; x^{(d)}, t)\, \mathrm{d}x_1\, \mathrm{d}x^{(d)} \\ &= \int_{U_d} \mathrm{e}^{\pm \mathrm{i}r(x^{(d)})^2} \left(r^{-\frac{d-1}{2}} \sum_{k=0}^{N+1} b_k(x^{(d)}, t) r^{-\frac{k}{2}} + O(r^{-\frac{d+N+1}{2}}) \right) \mathrm{d}x^{(d)} \end{aligned}$$

uniformly in $(x^{(d)}, t) \in \overline{U_d} \times \overline{B}$. The following estimates then hold when $r \to \infty$, uniformly in $t \in \overline{B}$:

$$\begin{aligned} I_f(r,t) &= \sum_{k=0}^{N+1} r^{-\frac{d+k-1}{2}} \int_{U_d} \mathrm{e}^{\pm \mathrm{i}r(x^{(d)})^2} b_k(x^{(d)}, t)\, \mathrm{d}x^{(d)} + O(r^{-\frac{d+N+1}{2}}) \\ &= \sum_{k=0}^{N+1} r^{-\frac{d+k-1}{2}} \left(r^{-\frac{1}{2}} \sum_{l=0}^{N} c_{k,l}(t) r^{-\frac{l}{2}} \right) + O(r^{-\frac{d+N+1}{2}}) \\ &= r^{-\frac{d}{2}} \sum_{j=0}^{N} a_j(t) r^{-\frac{j}{2}} + O(r^{-\frac{d+N+1}{2}}). \end{aligned}$$

Since N is arbitrary, this completes the proof. $\square$

2.2.2 Morse Lemma

The investigation of oscillatory integrals with non degenerate stationary phase points reduces to the above case by means of *Morse's lemma*, which we now prove.

Let the anticipated phase, $\phi(h,t)$, be a smooth function in $(h,t) \in \mathbb{R}^d \times \mathbb{R}^m$ satisfying

$$\phi(0,t) = 0, \quad \nabla_h \phi(0,t) = 0 \quad \text{and} \quad \det \mathrm{D}_h^2 \phi(0,t) \neq 0$$

for *all* $t \in B$, where $B \subset \mathbb{R}^m$ is a given cube containing a given $t_0 \in \mathbb{R}^m$. Such a function can be expressed *like* a quadratic form, but with coefficients varying smoothly in (h, t):

Lemma *In the above circumstances there exist functions, $\phi_{jk}(h,t)$, smooth in $(h,t) \in \mathbb{R}^d \times \mathbb{R}^m$ and satisfying*

$$\phi(h,t) = \sum_{j=1}^{d} \sum_{k=1}^{d} \phi_{jk}(h,t) h^{(j)} h^{(k)},$$

where $\phi_{jk}(h,t) = \phi_{kj}(h,t)$.

Proof: Using our hypotheses on $\phi(h,t)$, the fundamental theorem of calculus and integration by parts give for any $t \in B$

$$\phi(h,t) = \int_0^1 \partial_s(\phi(sh,t))\,\mathrm{d}s = \int_0^1 (1-s)\partial_s^2(\phi(sh,t))\,\mathrm{d}s.$$

Expanding $\partial_s^2(\phi(sh,t))$ in the above gives the result. $\square$

The next step consists of applying *Lagrange's algorithm*, which is better understood using matrices. Using the standard basis on $\mathbb{R}^d$, h is represented by a column (also denoted by h), while the "quadratic form" given by the previous lemma is represented by a $d \times d$ matrix, denoted by $\Phi(h,t)$. The (j,k)-th element of $\Phi(h,t)$ is then given by the function $\phi_{jk}(h,t)$, so the previous lemma gives

$$\phi(h,t) = h^{\mathrm{t}} \Phi(h,t) h.$$

For any given $t \in B$ the rank of $\Phi(0,t)$ is read through the Hessian of $\phi(h,t)$ using the following, straightforward relation:

Lemma *Under the above circumstances, for* any $t \in B$, $\mathrm{D}_h^2\phi(0,t) = 2\Phi(0,t)$.

In order to perform Lagrange's algorithm one uses the following elementary line/column operations:

- Given a scalar $c \neq 0$, to multiply the j-th row and then the j-th column by c, which is denoted by $CL_j(c)$;
- Given a scalar $c \in \mathbb{C}$, to add c times the k-th row to the j-th row and then c times the k-th column to the j-th column, which is denoted by $CL_{jk}(c)$;
- To interchange the j-th row with the k-th row and then the j-th column with the k-th column, which is denoted by CL_{jk}.

By the previous lemma, since $\det \mathrm{D}_h^2\phi(0,t) \neq 0$ and since $\Phi(0,t)$ consists of smooth elements, there exists a cube $V_0 \times B_0 \subset \mathbb{R}^d \times B$ containing $(0,t_0)$ such that $\Phi(h,t)$ is invertible for all $(h,t) \in V_0 \times B_0$.

Without loss of generality $\phi_{11}(h,t) \neq 0$ on a certain cube $V_1 \times B_1 \subseteq V_0 \times B_0$ containing $(0,t_0)$. Otherwise, $\phi_{11}(h,t)$ vanishes at $(0,t_0)$. However, considering the Laplace expansion of the above determinant along the first row, there exist a cube, $V_1' \times B_1' \subseteq V_0 \times B_0$, containing $(0,t_0)$ and an index $1 \leqslant k \leqslant d$ such that $\phi_{1k}(h,t) \neq 0$ for all $(h,t) \in V_1' \times B_1'$. Applying $CL_{1k}(1)$ to $\Phi(h,t)$, the resulting upper left element does not vanish on $V_1' \times B_1'$ (which then replaces $V_1 \times B_1$), as desired.

Then, one may reduce the upper left element to ± 1 (depending on the sign of $\phi_{11}(h,t)$, which does not change on $V_1 \times B_1$) by applying $CL_1(|\phi_{11}(h,t)|^{-\frac{1}{2}})$ to (the possibly refreshed) $\Phi(h,t)$. Finally, this resulting constant on the upper left corner permits to cancel the rest of the first line and column, by applying $CL_{k1}(f(h,t))$ for $k = 2, \ldots, d$ successively—where $f(h,t)$ is equal to the element to cancel (up to the sign). All these operations are represented by matrices having smooth elements in $(h,t) \in V_1 \times B_1$. They transform $\Phi(h,t)$ in a block diagonal matrix, having ± 1 as its first block and a square $(d-1) \times (d-1)$ matrix as its second block.

Repeating this procedure for the second block, $\Phi(h,t)$ is transformed in a block diagonal matrix having ± 1 as its first two blocks and a square $(d-2) \times (d-2)$ matrix as its third block. All the operations used for this second step are represented by matrices having smooth elements in $(h,t) \in V_2 \times B_2$, where

$$(0, t_0) \in V_2 \times B_2 \subseteq V_1 \times B_1 \subseteq V_0 \times B_0.$$

So on and so forth one transforms $\Phi(h,t)$ into a diagonal matrix having elements ± 1 only. All the required operations are smooth (in the previous sense) for (h,t) varying in a cube $V_d \times B_d$ containing $(0, t_0)$. One then applies permutations CL_{jk}, so the resulting matrix becomes $\operatorname{diag}(1, \ldots, 1, -1, \ldots, -1)$, where the element 1 is repeated, say, s times. Since s does not depend on $(h,t) \in V_d \times B_d$, the previous lemma and the Sylvester Inertia Theorem permit to recover s from the signature of $\mathrm{D}_h^2\phi(0, t_0)$, which is then $(s, d-s, 0)$.

In summary, we have proven:

Lemma *In the above circumstances there exist a cube,* $V_d \times B_d$*, containing* $(0, t_0)$ *and a nonsingular linear map,* $Q(h,t)$*, whose matrix elements are smooth in* $(h,t) \in V_d \times B_d$ *such that*

$$Q(h,t)^{\mathrm{t}}\Phi(h,t)Q(h,t) = \mathrm{diag}(1,\ldots,1,-1,\ldots,-1).$$

In the above the element 1 *is repeated* s *times, where* $(s, d-s, 0)$ *is the signature of* $\mathrm{D}_h^2\phi(0,t_0)$.

It is thus tempting to consider the *non linear* mapping $h \mapsto Q(h,t)^{-1}h$ defined on V_d as a potential change of variables given a fixed $t \in B_d$. Considering $t \in B_d$ as a parameter, $\{Q(h,t)^{-1}h\}_{t\in B_d}$ is indeed a family of smooth mappings depending smoothly on t in the strong sense that $(h,t) \mapsto Q(h,t)^{-1}h$ is jointly smooth on $V_d \times B_d$. Moreover, all these mappings map 0 to 0. They really consist of invertible changes of variables when restricting suitably the ranges of h and t, as shown below:

Lemma *In the above circumstances there exists a cube,* $V' \times B'$*, such that*

$$(0,t_0) \in V' \times B' \subset V_d \times B_d,$$

on which

$$(h,t) \mapsto (Q(h,t)^{-1}h, t)$$

is a smooth diffeomorphism. In addition, this diffeomorphism maps $V' \times B'$ onto an open bounded set, hence contained in a certain cube $\tilde{V}' \times B'$.

Proof: Let $F(h,t) = (Q(h,t)^{-1}h, t)$ for $(h,t) \in V_d \times B_d$. A direct computation shows that

$$\det \mathrm{D}_{(h,t)} F(0,t_0) = \det Q(0,t_0)^{-1},$$

which is not zero. Thus, $F(h,t)$ is a local diffeomorphism in a neighborhood of $(0,t_0)$. Choosing a cube, $V' \times B' \ni (0,t_0)$, whose closure is included in this last neighborhood (and in $V_d \times B_d$) then yields the result. □

The previous diffeomorphism maps $V' \times B'$ onto a bounded open set $\tilde{\mathcal{D}} \subseteq \tilde{V}' \times B'$. We want to fix $t \in B'$, so let $\tilde{\mathcal{D}}_t = \{h \in \mathbb{R}^d \ ; \ (h,t) \in \tilde{\mathcal{D}}\}$. Then, $\tilde{\mathcal{D}}_t \subseteq \tilde{V}'$ is an open set in $\mathbb{R}^d$. Let us consider the restriction

$$V' \longrightarrow \tilde{\mathcal{D}}_t, \ h \mapsto Q(h,t)^{-1}h,$$

which is also a smooth diffeomorphism. It may be used as a smooth invertible change of variables by setting

$$\tilde{h} = Q(h,t)^{-1}h$$

for $h \in V'$. Notice that 0 is then mapped to 0. Let P_t be the inverse change of variables, so

$$h = P_t(\tilde{h})$$

for $\tilde{h} \in \tilde{\mathcal{D}}_t$. Then, $\tilde{h} = P_t^{-1}(h) = Q(h,t)^{-1}h$, which implies $h = Q(h,t)\tilde{h}$ and hence

$$\begin{aligned}
\phi(P_t(\tilde{h}),t) &= \phi(h,t) \\
&= \phi(Q(h,t)\tilde{h},t) \\
&= \tilde{h}^{\mathrm{t}}Q(h,t)^{\mathrm{t}}\Phi(h,t)Q(h,t)\tilde{h} \\
&= \sum_{j=1}^{s}(\tilde{h}^{(j)})^2 - \sum_{k=s+1}^{d}(\tilde{h}^{(k)})^2.
\end{aligned}$$

We have thus proven the *Morse lemma* with special care of the parameter t:

Theorem 4 *Given a cube $B \subset \mathbb{R}^m$ containing a fixed t_0, suppose $\phi(h,t)$ is smooth in $(h,t) \in \mathbb{R}^d \times B$ and satisfies*

$$\phi(0,t) = 0, \quad \nabla_h\phi(0,t) = 0, \quad \textit{and} \quad \det \mathrm{D}_h^2\phi(0,t) \neq 0$$

for all $t \in B$. *Then, there exists a cube $V' \times B' \subset \mathbb{R}^d \times B$ containing $(0,t_0)$ such that the following holds: for all $t \in B'$ there exists an invertible change of variables*

$$\tilde{h} = P_t^{-1}(h)$$

on V', smooth and with smooth inverse, mapping 0 to 0, which satisfies

$$\phi(P_t(\tilde{h}),t) = \sum_{j=1}^{s}(\tilde{h}^{(j)})^2 - \sum_{k=s+1}^{d}(\tilde{h}^{(k)})^2.$$

The resulting family of changes of variables, $\{P_t(\tilde{h})\}_{t\in B'}$, depends diffeomorphically on $t \in B'$ in the following sense: setting $\tilde{\mathcal{D}}_t = P_t^{-1}(V')$,

$$\tilde{\mathcal{D}} = \bigcup_{t\in B'} \tilde{\mathcal{D}}_t \times \{t\}$$

is an open set in $\mathbb{R}^d \times \mathbb{R}^m$ *(contained in a cube* $\tilde{V}' \times B'$*) on which*

$$\tilde{\mathcal{D}} \to V' \times B', \quad (\tilde{h}, t) \mapsto (P_t(\tilde{h}), t)$$

is a diffeomorphism.

2.2.3 Continuation

The corollary of Theorem 3 joined with the Morse lemma finally yield:

Theorem 5 *Suppose* $\varphi(x,t)$ *is smooth in* $(x,t) \in \mathbb{R}^d \times \mathbb{R}^m$ *and satisfies*

$$\nabla_x \varphi(x_0, t_0) = 0 \quad \textit{and} \quad \det \mathrm{D}_x^2 \varphi(x_0, t_0) \neq 0.$$

Then, there exists an arbitrarily small cube, $U \times B$*, containing* (x_0, t_0) *such that the following holds: if* $f(x,t)$ *is smooth in the neighborhood of* $\mathbb{R}^d \times \overline{B}$ *and vanishes on* $U^{\mathrm{c}} \times \overline{B}$*, then*

$$I_f(r,t) \sim \mathrm{e}^{\mathrm{i}r\theta(t)} r^{-\frac{d}{2}} \sum_{j=0}^{\infty} a_j(t) r^{-\frac{j}{2}},$$

where $\theta(t)$ *is real valued, the* $a_j(t)$*'s are complex valued, and all these functions are smooth in* $t \in \overline{B}$*. Moreover, these estimates are uniform in* $t \in \overline{B}$*.*

Proof: By the Implicit Function Theorem there exists a smooth function

$$B_0 \subset \mathbb{R}^m \to \mathbb{R}^d, \quad t \mapsto \mathrm{x}(t)$$

defined on a cube $B_0 \ni t_0$ such that $(\nabla_x \varphi)(\mathrm{x}(t), t) = 0$ for all $t \in B_0$ and $\mathrm{x}(t_0) = x_0$. Without loss of generality we also suppose

$$\det \mathrm{D}_x^2 \varphi(\mathrm{x}(t), t) \neq 0$$

for all $t \in B_0$. Let

$$\phi(h,t) = \varphi(\mathrm{x}(t)+h,t) - \varphi(\mathrm{x}(t),t),$$

which is smooth in $(h,t) \in \mathbb{R}^d \times B_0$. Then, $\phi(h,t)$ satisfies the hypotheses of Morse's lemma. Hence, there exists a cube, $V' \times B'$, containing $(0,t_0)$ and whose closure is in $\mathbb{R}^d \times B_0$, and a family of diffeomorphisms

$$P_t^{-1}\colon V' \to \tilde{\mathcal{D}}_t \subseteq \tilde{V}' \quad \text{(where } t \in B'\text{)}$$

such that, letting $\tilde{h} = P_t^{-1}(h)$, one obtains

$$\phi(P_t(\tilde{h}),t) = \sum_{j=1}^{s} (\tilde{h}^{(j)})^2 - \sum_{k=s+1}^{d} (\tilde{h}^{(k)})^2.$$

Notice that $0 = x_0 - \mathrm{x}(t_0) \in V'$. Consequently, there exists an arbitrarily small cube, $U \times B \ni (x_0,t_0)$, such that

$$(x - \mathrm{x}(t), t) \in V' \times B'$$

for all $(x,t) \in \overline{U} \times \overline{B}$. In other words, $U \times B$ is mapped onto a region whose closure is in $V' \times B'$ via the change of variables $(h,t) = (x - \mathrm{x}(t), t)$.

Let us consider an amplitude, $f(x,t)$, satisfying the asserted properties. Since for a fixed $t \in \overline{B}$ the integrand in

$$I_f(r,t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\varphi(x,t)} f(x,t)\,\mathrm{d}x$$

is supported in U, it follows that the right-hand side in

$$I_f(r,t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\varphi(h+\mathrm{x}(t),t)} f(h+\mathrm{x}(t),t)\,\mathrm{d}h = \mathrm{e}^{\mathrm{i}r\varphi(\mathrm{x}(t),t)} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\phi(h,t)} f(h+\mathrm{x}(t),t)\,\mathrm{d}h$$

is supported in V'. Then, by Morse's lemma the change of variables $\tilde{h} = P_t^{-1}(h)$ is available, yielding

$$I_f(r,t) = \mathrm{e}^{\mathrm{i}r\varphi(\mathrm{x}(t),t)} \int_{\tilde{\mathcal{D}}_t} \mathrm{e}^{\mathrm{i}rQ(\tilde{h})} f(P_t(\tilde{h}) + \mathrm{x}(t), t) J_t(\tilde{h})\,\mathrm{d}\tilde{h},$$

where $J_t(\tilde{h})$ is the Jacobian and

$$Q(\tilde{h}) = \sum_{j=1}^{s} (\tilde{h}^{(j)})^2 - \sum_{j=s+1}^{d} (\tilde{h}^{(j)})^2.$$

Since the amplitude in the above extends smoothly on $\mathbb{R}^d \times \overline{B}$ and vanishes for $\tilde{h} \notin \tilde{V}'$, the corollary of Theorem 3 then completes the proof. $\square$

Finally, the following result is an interesting application of our treatment of a parameter:

Theorem 6 *Suppose $\varphi(x,t)$ is smooth in $(x,t) \in \mathbb{R}^d \times \mathbb{R}^m$ and satisfies*

$$\nabla_x \varphi(x_0, t_0) = 0 \quad \textit{and} \quad \operatorname{rank} \mathrm{D}_x^2 \varphi(x_0, t_0) = \kappa,$$

where $\kappa \geqslant 1$. Then, there exists an arbitrarily small cube, $U \times B$, containing (x_0, t_0) such that the following holds: if $f(x,t)$ is smooth in a neighborhood of $\mathbb{R}^d \times \overline{B}$ and vanishes on $U^{\mathrm{c}} \times \overline{B}$, then

$$I_f(r,t) = O(r^{-\frac{\kappa}{2}})$$

uniformly in $t \in \overline{B}$.

Proof: Since the rank of $\mathrm{D}_x^2\varphi(x_0, t_0)$ is equal to the order of its largest non zero principal minor, there exist indices, $j_1, \ldots, j_\kappa$, such that, letting

$$\xi = (x^{(j_1)}, \ldots, x^{(j_\kappa)}),$$

$\det \mathrm{D}_\xi^2(x_0, t_0) \neq 0$. After permuting the variables we write

$$x = (\xi, \chi), \quad \text{and} \quad x_0 = (\xi_0, \chi_0),$$

with the obvious definitions of χ, ξ_0, and χ_0. Interpreting (χ, t) as a parameter the previous theorem gives the result. □

Remark *Using the previous decomposition one cannot derive the complete asymptotic expansion of the considered oscillatory integral, since the resulting coefficients would be oscillatory integrals themselves! Their decay is not known* a priori.

2.2.4 Cauchy Principal Value

We now derive similar results for Cauchy principal values of oscillatory integrals with non degenerate stationary phase points. To this end let us consider first

$$\text{p.v.} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}rh} \frac{f(h)}{h}\, \mathrm{d}h = \lim_{\varepsilon \downarrow 0} \int_{|h|>\varepsilon} \mathrm{e}^{\mathrm{i}rh} \frac{f(h)}{h}\, \mathrm{d}h,$$

where the amplitude, $f(h)$, is smooth in $h \in \mathbb{R}$ and compactly supported.

Notice that the lemma of Theorem 1 generalizes to a complex valued phase provided that the path of integration remains in $\overline{\mathbb{C}_+}$, explicitly: *Given a smooth* *regular*[3] *path, $\gamma(t)$, lying in the closure of $\mathbb{C}_+$, and an amplitude, $f(z) = f(x,y)$, smooth in x and compactly supported along this path,*

$$\int_\gamma \mathrm{e}^{\mathrm{i}rz} f(z)\,\mathrm{d}z = O(r^{-\infty})$$

when $r \longrightarrow \infty$.

Lemma *Suppose $f(h)$ is smooth in $h \in \mathbb{R}$, compactly supported, and analytic at 0. Then,*

$$\text{p.v.} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}rh} \frac{f(h)}{h}\,\mathrm{d}h = \pi \mathrm{i} f(0) + O(r^{-\infty})$$

when $r \longrightarrow \infty$.

Proof: For $\varepsilon > 0$, let C_ε be a smooth regular path starting at -2ε, going through $[-2\varepsilon, -\varepsilon]$, then avoiding the origin, but staying in

$$\{z \in \mathbb{C}\,;\ |\mathrm{Re}\,(z)| < \varepsilon \text{ and } 0 < \mathrm{Im}\, z < \varepsilon\},$$

and finally going through $[\varepsilon, 2\varepsilon]$. Since

$$\mathrm{e}^{\mathrm{i}rz} \frac{f(z)}{z} = \frac{f(0)}{z} + \ \textit{analytic}$$

in a punctured neighborhood of the origin,

$$\lim_{\varepsilon \downarrow 0} \int_{C_\varepsilon} \mathrm{e}^{\mathrm{i}rz} \frac{f(z)}{z}\,\mathrm{d}z = -\pi \mathrm{i} f(0).$$

[3] A smooth path, $\gamma(t)$, is *regular* if $\gamma'(t) \neq 0$ for any t.

Thus, letting $\gamma_\varepsilon =]-\infty, -2\varepsilon] * C_\varepsilon * [2\varepsilon, \infty[$ (where $*$ denotes the concatenation), which is smooth and regular,

$$\begin{aligned}
\text{p.v.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}rh} \frac{f(h)}{h} \, \mathrm{d}h - \pi \mathrm{i} f(0) &= \lim_{\varepsilon \downarrow 0} \int_{\gamma_\varepsilon} \mathrm{e}^{\mathrm{i}rz} \frac{f(z)}{z} \, \mathrm{d}z \\
&= \int_{\gamma} \mathrm{e}^{\mathrm{i}rz} \frac{f(z)}{z} \, \mathrm{d}z,
\end{aligned}$$

where $\gamma = \gamma_{\varepsilon_0}$ for a fixed, but small enough $\varepsilon_0 > 0$. The result follows from the statement preceding the lemma. $\square$

The analyticity assumption may be removed in the following way—where a parameter $t \in \mathbb{R}^m$ is also introduced for later purpose:

Theorem 7 *Given $\delta > 0$ and a cube $B \subset \mathbb{R}^m$, consider a function, $f(h, t)$, smooth in a neighborhood of $\mathbb{R} \times \overline{B}$ and vanishing on $]-\delta, \delta[^{\mathrm{c}} \times \overline{B}$. Then,*

$$\text{p.v.} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}rh} \frac{f(h,t)}{h} \, \mathrm{d}h = \pi \mathrm{i} f(0,t) + O(r^{-\infty})$$

uniformly in $t \in \overline{B}$, when $r \to \infty$.

Proof: Let $\chi(x)$ be a smooth function in $x \in \mathbb{R}$, compactly supported, such that $0 \leqslant \chi(x) \leqslant 1$ on $\mathbb{R}$ and $\chi(x) \equiv 1$ in a neighborhood of 0. The considered principal value then decomposes into $I + II + III$, where

$$\begin{aligned}
I &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}rh} \frac{f(h,t)}{h} (1 - \chi(h)) \, \mathrm{d}h, \\
II &= \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}rh} \frac{f(h,t) - f(0,t)}{h} \chi(h) \, \mathrm{d}h, \quad \text{and} \\
III &= f(0,t) \text{p.v.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}rh} \frac{\chi(h)}{h} \, \mathrm{d}h.
\end{aligned}$$

The lemma of Theorem 1 implies $I = O(r^{-\infty})$ and $II = O(r^{-\infty})$ uniformly in $t \in \overline{B}$.[4] Since $\chi(x)$ is analytic at 0 and $\chi(0) = 1$, the result follows from the lemma. $\square$

We now turn our attention to a phase, $\varphi(h,t)$, smooth in $(h,t) \in \mathbb{R} \times \mathbb{R}^m$, such that $\partial_h \varphi(0, t_0) \neq 0$. Our study reduces to the previous theorem by a change of variables, regarding which the following elementary result is helpful: *Given a cube $B \subset \mathbb{R}^m$, suppose $f(h,t)$ is smooth in a neighborhood of $\{0\} \times \overline{B}$, where it satisfies*

$$f(0,t) = 0 \quad \textit{and} \quad \partial_h f(0,t) \neq 0.$$

Then, $\frac{h \partial_h f(h,t)}{f(h,t)}$ extends smoothly to a neighborhood of $\{0\} \times \overline{B}$, the extension being equal to 1 when $h = 0$.

Lemma *Suppose $\partial_h \varphi(0, t_0) \neq 0$ for a certain $t_0 \in \mathbb{R}^m$. There exist a $\delta > 0$ and an arbitrarily small cube, B, containing t_0 such that the following holds: if $f(h,t)$ is smooth in a neighborhood of $\mathbb{R} \times \overline{B}$ and vanishes on $]-\delta, \delta[^{\mathrm{c}} \times \overline{B}$, then*

$$\mathrm{p.v.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i} r \varphi(h,t)} \frac{f(h,t)}{h} \, \mathrm{d}h = \sigma \pi \mathrm{i} f(0,t) \mathrm{e}^{\mathrm{i} r \varphi(0,t)} + O(r^{-\infty})$$

uniformly in $t \in \overline{B}$, where $\sigma = \operatorname{sgn}(\partial_h \varphi(0, t_0))$.

[4] More precisely, the result follows from the proof of this lemma, by noticing that $U' \times B'$ may be chosen to be equal to $\mathbb{R} \times \mathbb{R}^m$—since the derivative of the phase is $1 \neq 0$ everywhere.

Proof: By continuity there exists a cube, $]-\delta, \delta[\times B' \ni (0, t_0)$, on which $\partial_h \varphi(h,t) \neq 0$. Let $\varphi_t(h) = \varphi(h,t) - \varphi(0,t)$ and $F(h,t) = (\varphi_t(h), t)$, which is smooth in $(h,t) \in \mathbb{R} \times \mathbb{R}^m$. Then,

$$\det \mathrm{D}_{(h,t)} F(h,t) = \partial_h \varphi(h,t) \neq 0$$

on $]-\delta, \delta[\times B'$. Hence, $F(h,t)$ admits a smooth inverse, $G(\tilde{h}, t) = (\psi_t(\tilde{h}), t)$, defined on $F(]-\delta, \delta[\times B')$. Notice that $\varphi_t(0) = 0 = \psi_t(0)$ and

$$\psi_t'(0) = \frac{1}{\varphi_t'(0)} \neq 0.$$

In particular, $\sigma = \operatorname{sgn}(\psi_t'(0))$. Moreover, by the statement preceding the lemma $\tilde{h}\psi_t'(\tilde{h})/\psi_t(\tilde{h})$ has a smooth extension to a neighborhood of $\{0\} \times \overline{B}$ (where $\overline{B} \subset B'$ is arbitrarily chosen), which is equal to 1 when $\tilde{h} = 0$. For $f(h,t)$ of the stipulated form and $t \in \overline{B}$, the change of variables $\tilde{h} = \varphi_t(h)$ then gives

$$\begin{aligned} \text{p.v.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}r\varphi(h,t)} \frac{f(h,t)}{h} \, \mathrm{d}h &= \text{p.v.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}r\tilde{h}} \mathrm{e}^{\mathrm{i}r\varphi(0,t)} \frac{f(\psi_t(\tilde{h}), t)}{\psi_t(\tilde{h})} |\psi_t'(\tilde{h})| \, \mathrm{d}\tilde{h} \\ &= \sigma \mathrm{e}^{\mathrm{i}r\varphi(0,t)} \text{p.v.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}r\tilde{h}} \frac{g(\tilde{h}, t)}{\tilde{h}} \, \mathrm{d}\tilde{h}, \end{aligned}$$

where

$$g(\tilde{h}, t) = f(\psi_t(\tilde{h}), t) \frac{\tilde{h}\psi_t'(\tilde{h})}{\psi_t(\tilde{h})}.$$

Notice that $g(0,t) = f(0,t)$. Moreover, $g(\tilde{h}, t)$ vanishes when $t \in \overline{B}$ and $\tilde{h} \notin]-\tilde{\delta}, \tilde{\delta}[$ for a certain $\tilde{\delta} > 0$. The previous theorem thus applies, which completes the proof. □

Let us now consider a phase, $\varphi(x,h,t)$, smooth in $(x,h,t) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^m$, and

$$I_f(r,h,t) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\varphi(x,h,t)} f(x,h,t)\,\mathrm{d}x.$$

Theorem 8 *Suppose* $\nabla_x \varphi(x_0,0,t_0) = 0$, $\det \mathrm{D}_x^2 \varphi(x_0,0,t_0) \neq 0$, *and*

$$\partial_h \varphi(x_0,0,t_0) \neq 0.$$

Then, there exist a $\delta > 0$ *and an arbitrarily small cube,* $U \times B$, *containing* (x_0,t_0) *such that the following holds: if* $f(x,h,t)$ *is smooth in the neighborhood of* $\mathbb{R}^d \times \mathbb{R} \times \overline{B}$ *and vanishes on* $(U \times]-\delta,\delta[)^{\mathrm{c}} \times \overline{B}$, *then*

$$\mathrm{p.v.} \int_{\mathbb{R}} \frac{1}{h} I_f(r,h,t)\,\mathrm{d}h \sim \mathrm{e}^{\mathrm{i}r\theta(t)} r^{-\frac{d}{2}} \sum_{j=0}^{\infty} a_j(t) r^{-\frac{j}{2}}$$

uniformly in $t \in \overline{B}$, *where* $\theta(t)$ *and* $a_j(t)$ *are smooth in the neighborhood of* $\overline{B}$.

Proof: Interpreting $(x,t) \in \mathbb{R}^d \times \mathbb{R}^m$ as a parameter, the previous lemma holds, since $\partial_h \varphi(x_0,0,t_0) \neq 0$. Moreover, interpreting $(h,t) \in \mathbb{R} \times \mathbb{R}^m$ as a parameter, the theorem 5 also holds, since $\nabla_x \varphi(x_0,0,t_0) = 0$ and $\det \mathrm{D}_x^2 \varphi(x_0,0,t_0) \neq 0$. Consequently, there exist a $\delta > 0$ and an arbitrarily small cube, $U \times B$, containing (x_0,t_0) such that for $f(x,h,t)$ of the stipulated form

$$\mathrm{p.v.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}r\varphi(x,h,t)} \frac{f(x,h,t)}{h}\,\mathrm{d}h = \sigma\pi\mathrm{i} f(x,0,t) \mathrm{e}^{\mathrm{i}r\varphi(x,0,t)} + O(r^{-\infty}) \tag{2.4}$$

uniformly in $(x,t) \in \overline{U} \times \overline{B}$ and

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}r\varphi(x,0,t)} f(x,0,t)\,\mathrm{d}x \sim \mathrm{e}^{\mathrm{i}r\theta(t)} r^{-\frac{d}{2}} \sum_{j=0}^{\infty} a_j(t) r^{-\frac{j}{2}} \tag{2.5}$$

uniformly and smoothly in $t \in \overline{B}$, both when $r \to \infty$.

By Fubini's and the dominated convergence theorems

$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{h} I_f(r,h,t)\, \mathrm{d}h = \int_{\mathbb{R}^d} \text{p.v.} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}r\varphi(x,h,t)} \frac{f(x,h,t)}{h}\, \mathrm{d}h\, \mathrm{d}x,$$

where the dominator is given by the lemma of Theorem 2. By the equation (2.4), the above is equal to

$$\int_{\mathbb{R}^d} \sigma\pi \mathrm{i} f(x,0,t) \mathrm{e}^{\mathrm{i}r\varphi(x,0,t)}\, \mathrm{d}x + O(r^{-\infty})$$

uniformly in $t \in \overline{B}$. The equation (2.5) then yields the result. □

In the same way as we derived from Theorem 2 its corollary,

Corollary *Suppose* $\nabla_x \varphi(x_0, e_0, t_0) = 0$, $\det \mathrm{D}_x^2 \varphi(x_0, e_0, t_0) \neq 0$, *and*

$$\partial_e \varphi(x_0, e_0, t_0) \neq 0.$$

Then, there exist a $\delta > 0$ *and an arbitrarily small cube* $U \times B$ *containing* (x_0, t_0) *such that the following holds: if* $f(x,e,t)$ *is smooth in the neighborhood of* $\mathbb{R}^d \times \mathbb{R} \times \overline{B}$ *and vanishes on* $(U \times]e_0 - \delta, e_0 + \delta[)^{\mathrm{c}} \times \overline{B}$, *then*

$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} I_f(r,\eta,t)\, \mathrm{d}\eta \sim \mathrm{e}^{\mathrm{i}r\theta(e,t)} r^{-\frac{d}{2}} \sum_{j=0}^{\infty} a_j(e,t) r^{-\frac{j}{2}}$$

uniformly in $(e,t) \in [e_0 - \delta, e_0 + \delta] \times \overline{B}$, *where* $\theta(e,t)$ *and* $a_j(e,t)$ *are smooth in the neighborhood of* $[e_0 - \delta, e_0 + \delta] \times \overline{B}$.

More generally, by the argument used in Theorem 6 (which we repeat!),

Theorem 9 *Suppose* $\nabla_x \varphi(x_0, e_0, t_0) = 0$, $\operatorname{rank} \mathrm{D}_x^2 \varphi(x_0, e_0, t_0) = \kappa$, *and*

$$\partial_e \varphi(x_0, e_0, t_0) \neq 0.$$

Then, there exist a $\delta > 0$ *and an arbitrarily small cube,* $U \times B$, *containing* (x_0, t_0) *such that the following holds: if* $f(x, e, t)$ *is smooth in the neighborhood of* $\mathbb{R}^d \times \mathbb{R} \times \overline{B}$ *and vanishes on* $(U \times]e_0 - \delta, e_0 + \delta[)^c \times \overline{B}$, *then*

$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} I_f(r, \eta, t)\, \mathrm{d}\eta = O(r^{-\frac{\kappa}{2}})$$

uniformly in $(e, t) \in [e_0 - \delta, e_0 + \delta] \times \overline{B}$.

Proof: There exist indices, $j_1, \cdots, j_\kappa$, such that, for

$$\xi = (x^{(j_1)}, \cdots, x^{(j_\kappa)}),$$

$\det \mathrm{D}_\xi^2(x_0, t_0) \neq 0$. After permuting the variables we write

$$x = (\xi, \chi) \quad \text{and } x_0 = (\xi_0, \chi_0)$$

with the obvious definitions of χ, ξ_0, and χ_0. Interpreting (χ, t) as a parameter, the above corollary gives the existence of a $\delta > 0$ and an arbitrarily small cube, $U' \times (U'' \times B) \ni (\xi_0; \chi_0, t_0)$, such that, letting $U = U' \times U'' \ni x_0$, for $f(x, t)$ of the stipulated form

$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} \int_{\mathbb{R}^\kappa} \mathrm{e}^{\mathrm{i}r\varphi(\xi;\chi,\eta,t)} f(\xi; \chi, \eta, t)\, \mathrm{d}\xi\, \mathrm{d}\eta = O(r^{-\frac{\kappa}{2}})$$

uniformly in $(\chi, e, t) \in \overline{U''} \times [e_0 - \delta, e_0 + \delta] \times \overline{B}$. Therefore,

$$\int_{\mathbb{R}^{d-\kappa}} \text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} \int_{\mathbb{R}^\kappa} \mathrm{e}^{\mathrm{i}r\varphi(\xi;\chi,\eta,t)} f(\xi; \chi, \eta, t) \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\chi = O(r^{-\frac{\kappa}{2}})$$

uniformly in $(e, t) \in [e_0 - \delta, e_0 + \delta] \times \overline{B}$. By Fubini's and the dominated convergence theorems this last integral is equal to

$$\int_{\mathbb{R}^d} \text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} \mathrm{e}^{\mathrm{i}r\varphi(x;\eta,t)} f(x; \eta, t) \, \mathrm{d}\eta \, \mathrm{d}x,$$

and hence (for the same reasons) to

$$\text{p.v.} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{1}{\eta - e} \mathrm{e}^{\mathrm{i}r\varphi(x;\eta,t)} f(x; \eta, t) \, \mathrm{d}x \, \mathrm{d}\eta,$$

where both times the dominator is given by the lemma of Theorem 2. The proof is thus complete. $\square$

2.3 Fourier Transforms over Level Surfaces of Analytic Functions

We now consider a real valued function, $\Phi(x)$, analytic in $x \in \mathbb{R}^d$. Usually, the level surfaces of $\Phi(x)$ consist of several connected components; let us focus on some of them, say, the connected components whose reunion is given by

$$\Gamma(e) = \{x \in \mathcal{R} \, ; \, \Phi(x) = e\}$$

for an appropriate domain $\mathcal{R} \subseteq \mathbb{R}^d$.

Given a function, $f(x)$, summable on $\Gamma(e)$, its Fourier transform over $\Gamma(e)$ is defined as

$$\mathcal{F}(\Gamma(e), f)(n) = \int_{\Gamma(e)} \mathrm{e}^{\mathrm{i}n\cdot x} f(x) \, \mathrm{dS}(x)$$

for $n \in \mathbb{Z}^d$, where $\mathrm{dS}(x)$ denotes the element of surface on $\Gamma(e)$. In this section we derive the decay of $\mathcal{F}(\Gamma(e), f)(n)$ when $|n| \to \infty$ and show its uniformity when e varies on an appropriate interval. We then derive an analogous result for the Cauchy principal value of such a Fourier transform.

Let $a' < b'$ and $\mathcal{S}' = \bigcup_{e \in]a',b'[} \Gamma(e)$ be given. We assume:

Assumption A

- $\Phi(x)$ *is real valued and analytic in* $\mathbb{R}^d$*;*
- $\nabla\Phi(x) \neq 0$ *for all* $x \in \mathcal{S}'$*;*
- $\Gamma(e)$ *is compact for all* $e \in]a', b'[$.

The second statement in Assumption A and the Implicit Function Theorem ensure that $\Gamma(e)$ is a *regular smooth surface* for any $e \in]a', b'[$. In particular, $\Gamma(e)$ may be covered by real-analytic local parameterizations, $U \xrightarrow{\sigma} \Gamma(e)$, where each $U \subset \mathbb{R}^{d-1}$ is open, each $\sigma(u)$ is a smooth homeomorphism between U and $\sigma(U)$ in the topology of $\Gamma(e)$, and

$$\bigcup_{(U,\sigma)} \sigma(U) = \Gamma(e).$$

Moreover, if two of the previous local parameterizations, (U, σ) and (V, τ), have a non empty overlap, $\mathcal{D} = \sigma(U) \cap \tau(V) \subseteq \Gamma(e)$, then the change of parameterizations $\tau^{-1} \circ \sigma(u)$ is a real-analytic diffeomorphism from $\sigma^{-1}(\mathcal{D})$ to $\tau^{-1}(\mathcal{D})$.

Given a local parameterization, (U, σ), the restriction of the Fourier transform to $\sigma(U)$ gives

$$\begin{aligned}\mathcal{F}(\sigma(U), f)(n) &= \int_{\sigma(U)} \mathrm{e}^{\mathrm{i}n\cdot x} f(x)\,\mathrm{dS}(x) \\ &= \int_U \mathrm{e}^{\mathrm{i}n\cdot\sigma(u)} f(\sigma(u)) J(u)\,\mathrm{d}u\end{aligned}$$

for any $f(x)$ summable on $\sigma(U)$, where $J(u)\,\mathrm{d}u$ is the element of surface. Of course the decay of such an integral when $|n| \to \infty$ is studied by means of the stationary phase method.

Let $n = r\omega$ be the polar form of $n \in \mathbb{Z}^d$, so $r = |n|$ and $\omega \in \mathrm{S}^{d-1}$. The *phase* in the previous integral then becomes $\varphi(u, \omega) = \omega \cdot \sigma(u)$. Let us consider a point, $x_0 = \sigma(u_0)$, in $\sigma(U)$. Remarkably, the fact that u_0 is or is not a stationary phase point depends on *intrinsic properties of* $\Gamma(e)$ *only*; if u_0 is stationary, the rank of $\mathrm{D}_u^2\varphi(u_0, \omega)$ is also intrinsic. Indeed,

Theorem 10 *In the above circumstances u_0 is stationary if, and only if ω is perpendicular to $\Gamma(e)$ at $x_0 = \sigma(u_0)$. Then, the rank of the Hessian of $\varphi(u, \omega)$ at u_0 is equal to the number of non vanishing principal curvatures of $\Gamma(e)$ at x_0.*

Proof: Notice that u_0 is stationary iff

$$\nabla_u \omega \cdot \sigma(u_0) = (\omega \cdot \mathrm{d}_{u^{(1)}}\sigma(u_0), \ldots, \omega \cdot \mathrm{d}_{u^{(d-1)}}\sigma(u_0)) = 0.$$

Since the tangent plane of $\Gamma(e)$ at x_0 is generated by

$$\{\mathrm{d}_{u^{(j)}}\sigma(u_0) \;;\; j = 1, \ldots, d-1\},$$

the first statement follows.

Suppose u_0 is a stationary phase point and consider any other local parameterization, (V, τ), of a neighborhood of $x_0 = \sigma(u_0)$. Let $\mathcal{D} = \sigma(U) \cap \tau(V)$ and $F(v) = \sigma^{-1} \circ \tau(v)$. Then, $F(v)$ is a smooth diffeomorphism from $\tau^{-1}(\mathcal{D})$ to $\sigma^{-1}(\mathcal{D})$ satisfying

$$\omega \cdot \tau(v) = \omega \cdot \sigma(F(v)). \tag{2.6}$$

Let $v_0 = \tau^{-1}(x_0)$, so $(\nabla_u(\omega \cdot \sigma))(F(v_0)) = \nabla_u(\omega \cdot \sigma)(u_0) = 0$. Then, the chain rule applied to (2.6) gives

$$\mathrm{D}_v^2(\omega \cdot \tau)(v_0) = \mathrm{D}_u^2(\omega \cdot \sigma)(u_0)(\mathrm{D}_v F(v_0))^2.$$

Since $F(v)$ is a diffeomorphism, $\mathrm{D}_v F(v_0)$ is invertible and hence

$$\operatorname{rank} \mathrm{D}_v^2(\omega \cdot \tau)(v_0) = \operatorname{rank} \mathrm{D}_u^2(\omega \cdot \sigma)(u_0),$$

which shows that this last rank is intrinsic.

Finally, since u_0 is a stationary phase point, ω is perpendicular to $\Gamma(e)$ at x_0, so

$$\omega = \frac{\pm\nabla_x \Phi(x_0)}{\|\nabla_x \Phi(x_0)\|}.$$

Moreover, there exists a $j \in \{1, \ldots, d\}$ such that $\omega^{(j)} \neq 0$. Suppose without loss of generality $\omega^{(d)} \neq 0$, and hence $\partial_{x^{(d)}} \Phi(x_0) \neq 0$. Let $w = (x^{(1)}, \ldots, x^{(d-1)})$ and $w_0 = (x_0^{(1)}, \ldots, x_0^{(d-1)})$. By the Implicit Function Theorem there exists a function, $h(w)$, smooth in the neighborhood of w_0, such that $\Phi(w, h(w)) = e$ and $h(w_0) = x_0^{(d)}$. Hence, $\gamma(w) = (w, h(w))$ gives a smooth local parameterization of a neighborhood of x_0 as a graph of a smooth function. Differential geometry then shows that $\operatorname{rank} \mathrm{D}_w^2 h(w)$ is equal to the number of non vanishing principal curvatures at $x_0 \in \Gamma(e)$. Since

$$\mathrm{D}_w^2(\omega \cdot \gamma)(w) = \omega^{(d)} \mathrm{D}_w^2 h(w),$$

the proof is complete. $\square$

2.3.1 Joint system of parameterizations

Given a fixed $e_0 \in]a', b'[$, we now construct a system of parameterizations for $\Gamma(e_0)$ compatible with all $\Gamma(e)$'s for e varying in a small neighborhood of e_0 (so the derived estimate for Fourier transforms will be uniform in e). We make the following hypothesis:

Assumption B *For every $e \in]a', b'[$, $\Gamma(e)$ admits at least κ non vanishing principal curvatures at any point, where $\kappa \geqslant 1$ is a fixed integer.*

The plan is the following: starting from an arbitrary system of real-analytic parameterizations for $\Gamma(e_0)$, we will parametrize $\Gamma(e)$ using the local coordinates of $\Gamma(e_0)$, by lifting them orthogonally to $\Gamma(e_0)$ (for e very close to e_0).

Let $\{(\mathcal{U}_\beta, \gamma_\beta)\}_{\beta=1}^{M}$ be a system of real-analytic parameterizations covering $\Gamma(e_0)$. Since $\Gamma(e_0)$ is compact, we assume M to be finite. Without loss of generality, we also assume $\gamma_\beta(u)$ is analytic in a neighborhood of $\overline{\mathcal{U}_\beta}$, so the expression $\gamma_\beta(u)$ for $u \in \partial\mathcal{U}_\beta$ makes sense.

Let $x_0 \in \Gamma(e_0)$, say, $x_0 = \gamma_\beta(u_0)$ for a given $u_0 \in \mathcal{U}_\beta$ and a given $1 \leqslant \beta \leqslant M$. Since $\nabla_x \Phi(x)$ is perpendicular to $\Gamma(e_0)$ at any $x \in \Gamma(e_0)$, we need to solve

$$\Phi(\,\gamma_\beta(u) + \lambda \nabla_x \Phi(\gamma_\beta(u))\,) - e = 0 \tag{2.7}$$

in the neighborhood of $(u, e, \lambda) = (u_0, e_0, 0)$. The derivative at 0 with respect to λ of the left-hand side in (2.7) is $\|\nabla_x \Phi(\gamma_\beta(u))\|^2$, which is strictly positive. Hence, there exists an analytic function, $\lambda(u, e)$, defined on an arbitrarily small cube, $\mathcal{U}' \times\,]e_0 - \Delta', e_0 + \Delta'[\, \ni (u_0, e_0)$, satisfying

$$\gamma_\beta(u) + \lambda(u, e)\nabla_x \Phi(\gamma_\beta(u)) \in \Gamma(e), \;\; \lambda(u_0, e_0) = 0. \tag{2.8}$$

Remarkably, $\lambda(u, e)$ depends on $\gamma_\beta(u)$ only, not on its local coordinates u:

Lemma *Suppose $\gamma_{\beta'}(u'_0) = x_0$ and let $u' = \gamma_{\beta'}^{-1} \circ \gamma_\beta(u)$, where u varies in*

$$\gamma_\beta^{-1}(\gamma_\beta(\mathcal{U}_\beta) \cap \gamma_{\beta'}(\mathcal{U}_{\beta'})).$$

Define $\lambda'(u', e)$ as above with respect to $\gamma_{\beta'}(u')$—while $\lambda(u, e)$ was defined with respect to $\gamma_\beta(u)$. Then, $\lambda(u, e) = \lambda'(u', e)$ when e is close enough to e_0.

Proof: $\lambda(u, e)$ is the unique solution of

$$\Phi(\gamma_\beta(u) + \lambda(u, e)\nabla_x \Phi(\gamma_\beta(u))) = e, \;\; \lambda(u_0, e_0) = 0$$

in the neighborhood of (u_0, e_0), similarly for $\lambda'(u', e)$ with respect to $\gamma_{\beta'}(u')$. Hence, letting $F = \gamma_\beta^{-1} \circ \gamma_{\beta'}$ on $\gamma_{\beta'}^{-1}(\gamma_\beta(\mathcal{U}_\beta) \cap \gamma_{\beta'}(\mathcal{U}_{\beta'}))$, one obtains

$$\Phi(\gamma_{\beta'}(u') + \lambda(F(u'), e)\nabla_x \Phi(\gamma_{\beta'}(u'))) = e, \;\; \lambda(F(u'_0), e_0) = 0$$

in a neighborhood of (u'_0, e_0). By uniqueness of $\lambda'(u', e)$, it follows that

$$\lambda'(u', e) = \lambda(F(u'), e),$$

in other words, that $\lambda'(u', e) = \lambda(u, e)$ for any e sufficiently close to e_0. $\square$

Since the considered surfaces are compact, this last lemma ensures *the existence of an analytic function,* $\Lambda(x, e)$*, defined on* $\Gamma(e_0) \times]e_0 - \Delta', e_0 + \Delta'[$ *(where* $\Gamma(e_0)$ *is endowed with its surface structure), such that*

$$x + \Lambda(x, e)\nabla_x \Phi(x) \in \Gamma(e) \quad \textit{and} \quad \Lambda(x_0, e_0) = 0. \tag{2.9}$$

Incidentally, the function $\lambda(u, e)$ has the following, interesting properties:

Lemma *For every* $u \in \mathcal{U}'$*,* $\lambda(u, e_0) = 0$*. In particular,* $\partial_{u(j)}\lambda(u, e_0) = 0$ *for any* $j = 1, \ldots, d-1$.

Proof: Let $v_0 \in \mathcal{U}'$ and consider the equation

$$\Phi(\gamma_\beta(u) - \tilde{\lambda}\nabla_x \Phi(\gamma_\beta(u))) - e_0 = 0$$

in the neighborhood of v_0. Since the derivative of its left-hand side at $\tilde{\lambda} = 0$ (with respect to $\tilde{\lambda}$) is $\|\nabla_x \Phi(\gamma_\beta(u))\|^2 > 0$, there exists a unique implicit function, $\tilde{\lambda}(u)$, satisfying

$$\Phi(\gamma_\beta(u) - \tilde{\lambda}(u)\nabla_x\Phi(\gamma_\beta(u))) = e_0, \quad \tilde{\lambda}(v_0) = e_0 \tag{2.10}$$

in the neighborhood of v_0. This implicit function is thus identically zero. Indeed, letting v_0 vary in $\mathcal{U}'$, the family of equations (2.10) defines piecewise a unique solution to

$$\Phi(\gamma_\beta(u) - \tilde{\lambda}(u)\nabla_x\Phi(\gamma_\beta(u))) = e_0, \quad \tilde{\lambda}(u_0) = e_0$$

on the whole $\mathcal{U}'$. Since $\lambda(u, e_0)$ is such a solution, it identically vanishes. □

Lemma $\partial_e \lambda(u_0, e_0) > 0$.

Proof: In fact, since $\Phi(x_0 + \lambda(u_0, e)\nabla_x\Phi(x_0)) = e$, the chain rule gives

$$\partial_e \lambda(u_0, e_0) = \frac{1}{\|\nabla_x \Phi(x_0)\|^2}.$$

□

The equation (2.8) permits to define the following local parameterization of $\Gamma(e)$:

$$\sigma(u, e) = \gamma_\beta(u) + \lambda(u, e)\nabla_x\Phi(\gamma_\beta(u)),$$

where $u \in \mathcal{U}'$ and $|e - e_0| < \Delta'$. Notice that $\sigma(u, e)$ is real-analytic. Indeed,

Lemma *$\sigma(u,e)$ is a real-analytic diffeomorphism in a neighborhood of (u_0, e_0).*

Proof: Notice that the ambient space, $\mathbb{R}^d$, is generated by the tangent vectors of $\Gamma(e_0)$ at x_0 (namely, $\mathrm{d}_{u^{(j)}}\gamma_\beta(u_0)$ for $1 \leqslant j \leqslant d-1$) and $\nabla_x \Phi(x_0)$. The penultimate lemma and a direct computation show that the columns of the matrix $\mathrm{D}_{(u,e)}\sigma(u_0, e_0)$ in canonical basis are

$$\begin{aligned} \mathrm{d}_{u^{(j)}}\gamma_\beta(u_0) \quad &\text{for} \quad 1 \leqslant j \leqslant d-1 \\ \partial_e \lambda(u_0, e_0)\nabla_x \Phi(x_0) \quad &\text{for} \quad j = d, \end{aligned}$$

which are linearly independent by the last lemma. The Inverse Mapping Theorem then completes the proof. $\square$

We thus select an arbitrarily small cube, $\mathcal{U} \times]e_0 - \Delta, e_0 + \Delta[\ni (u_0, e_0)$, contained in $\mathcal{U}' \times]e_0 - \Delta', e_0 + \Delta'[$, such that $\sigma(u,e)$ is a real-analytic diffeomorphism in the neighborhood of $\overline{\mathcal{U}} \times [e_0 - \Delta, e_0 + \Delta]$.

Let $\varphi(u; e, \omega) = \omega \cdot \sigma(u, e)$ be the anticipated phase, where $u \in \mathcal{U}$, $|e - e_0| < \Delta$, and $\omega \in \mathrm{S}^{d-1}$. By Theorem 10, the associated stationary phase points and $\operatorname{rank} \mathrm{D}_u^2 \varphi$ at these points are intrinsic properties of $\Gamma(e)$. Let $\omega_0 \in \mathrm{S}^{d-1}$ be arbitrarily fixed. It appears that if x_0 is a stationary phase point, then $\partial_e \varphi(u_0; e_0, \omega_0)$ is also "intrinsic" in the following sense:

Theorem 11 *Suppose $x_0 = \sigma(u_0, e_0)$ is a stationary phase point. Then,*

$$\partial_e \varphi(u_0; e_0, \omega_0) = \frac{\pm 1}{\|\nabla_x \Phi(x_0)\|}.$$

Proof: By Theorem 10, since x_0 is stationary, $\omega_0 = \pm \frac{\nabla_x \Phi(x_0)}{\|\nabla_x \Phi(x_0)\|}$. In particular,

$$\begin{aligned} \partial_e \varphi(u_0; e_0, \omega_0) &= \omega_0 \cdot \partial_e \sigma(u_0, e_0) \\ &= \frac{\pm 1}{\|\nabla_x \Phi(x_0)\|} \nabla_x \Phi(x_0) \cdot \partial_e \sigma(u_0, e_0). \end{aligned}$$

Since $\Phi(\sigma(u, e)) = e$, the chain rule gives $\nabla_x \Phi(x_0) \cdot \partial_e \sigma(u_0, e_0) = 1$, which completes the proof. $\square$

Consequently, $\partial_e \varphi(u_0; e_0, \omega_0) \neq 0$ if x_0 is a stationary phase point. Hence, whether $\nabla_u \varphi(u_0; e_0, \omega_0) \neq 0$ or

$$\begin{aligned} \nabla_u \varphi(u_0; e_0, \omega_0) &= 0, \\ \operatorname{rank} \mathrm{D}_u^2 \varphi(u_0; e_0, \omega_0) &\geqslant \kappa, \\ \partial_e \varphi(u_0; e_0, \omega_0) &\neq 0. \end{aligned}$$

By Theorems 1, 6, the corollary of Theorem 2, and Theorem 9 there exists an arbitrarily small cube, $U \times B \subset \mathcal{U} \times \mathbb{R}^d$, containing (u_0, ω_0) and an arbitrarily small $0 < \delta < \Delta$ such that the following holds:

If $f(u; e, \omega)$ is smooth in the neighborhood of $\mathbb{R}^{d-1} \times \mathbb{R} \times \overline{B}$ and vanishes on $(U \times]e_0 - \delta, e_0 + \delta[)^c \times \overline{B}$, then

$$I_f(r; e, \omega) = O(r^{-\frac{\kappa}{2}}) \text{ and}$$

$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} I_f(r; \eta, \omega) \, d\eta = O(r^{-\frac{\kappa}{2}}),$$

where both estimates are uniform in $(e, \omega) \in [e_0 - \delta, e_0 + \delta] \times \overline{B}$.

Of course, the previous cube and associated constructions depend on the fixed e_0, ω_0, and $x_0 \in \Gamma(e_0)$. Repeating this procedure for all $x \in \Gamma(e_0)$, where e_0 and ω_0 are still fixed, one obtains a system of real-analytic diffeomorphisms,

$$\{(U_x \times]e_0 - \delta_x, e_0 + \delta_x[\, , \sigma_x)\}_{x \in \Gamma(e_0)},$$

and a family of cubes, $\{B_\alpha\}_{\alpha=1}^N$, satisfying the above properties, from which one extracts a finite subsystem, $\{(U_\alpha \times]e_0 - \delta_\alpha, e_0 + \delta_\alpha[\, , \sigma_\alpha)\}_{\alpha=1}^N$, which covers $\Gamma(e_0)$:

$$\Gamma(e_0) = \bigcup_{\alpha=1}^{N} \sigma_\alpha(U_\alpha, e_0).$$

We limit our considerations to an arbitrarily small cube, B, whose closure is inside $\bigcap_{\alpha=1}^N B_\alpha$, and to $\{(U_\alpha \times]e_0 - \delta', e_0 + \delta'[\, , \sigma_\alpha)\}_{\alpha=1}^N$, where $0 < \delta' < \min_{\alpha=1}^N \delta_\alpha$ is arbitrarily small.

By the equation (2.9) the local parameterizations

$$\sigma_\alpha(u, e) = \gamma_{\beta_\alpha}(u) + \lambda_\alpha(u, e) \nabla_x \Phi(\gamma_{\beta_\alpha}(u))$$

yield the following, arguably global parameterization of $\Gamma(e)$:

$$\Sigma(x, e) = x + \Lambda(x, e) \nabla_x \Phi(x), \tag{2.11}$$

where $x \in \Gamma(e_0)$ and $|e - e_0| < \delta'$. In particular, when x varies over the entire $\Gamma(e_0)$, $\Sigma(x, e)$ describes a whole "closed" surface included in $\Gamma(e)$ (which should therefore corresponds to $\Gamma(e)$, as long as $\Gamma(e_0)$ and $\Gamma(e)$ have the same number of connected components). In other words, one expects

$$\Gamma(e) = \bigcup_{\alpha=1}^{N} \sigma_\alpha(U_\alpha, e). \tag{2.12}$$

However, it is easier to shorten δ' in order to *prove* the previous relation.

This may be done in the following way: suppose by contradiction there does not exist a δ such that the relation (2.12) holds for all $e \in]e_0 - \delta, e_0 + \delta[$, where $0 < \delta \leqslant \delta'$. Then there exist a sequence, $\{e_n\}$, converging to e_0, and points, x_n, on $\Gamma(e_n)$, such that $x_n \notin \bigcup_{\alpha=1}^N \sigma_\alpha(U_\alpha, e_n)$. Letting $\mathcal{S} = \bigcup_{|e-e_0|<\delta} \Gamma(e)$, the x_n's lie in the compact set $\overline{\mathcal{S}}$, so they accumulate towards a certain $x^* \in \overline{\mathcal{S}}$. Going to a subsequence, again denoted by $x_n \to x^*$, one finds

$$\Phi(x^*) = \Phi(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} \Phi(x_n) = \lim_{n\to\infty} e_n = e.$$

Therefore, x^* lies in $\bigcup_{\alpha=1}^N \sigma_\alpha(U_\alpha \times]e_0 - \delta', e_0 + \delta'[)$, while the x_n's do not, contradicting the fact that this last region is open in $\mathbb{R}^d$.

Indeed,

Theorem 12 *There exists a $\delta \in]0, \delta'[$ such that for any $e \in]e_0 - \delta, e_0 + \delta[$, $\Sigma(x, e)$ is a real-analytic diffeomorphism between $\Gamma(e_0)$ and $\Gamma(e)$ (endowed with their surface structures).*

Proof: The previous paragraph shows that $\Sigma(x, e)$ is surjective for e close enough to e_0. By a similar argument suppose there does not exist a $\delta > 0$ such that $\Sigma(x, e)$ is injective for every $e \in]e_0 - \delta, e_0 + \delta[$. Then, there exist a sequence, $\{e_n\}$, converging to e_0, and points, x_n, on $\Gamma(e_n)$, such that

$$x_n = \Sigma(y_n, e_n) = \Sigma(z_n, e_n)$$

for distinct points y_n and z_n on $\Gamma(e_0)$. Since the x_n's accumulate towards a certain $x^* \in \Gamma(e_0)$, y_n and z_n are eventually in the same coordinates neighborhood, contradicting the fact that each σ_α is a diffeomorphism. $\square$

We limit our considerations to the system $\{(U_\alpha \times]e_0 - \delta, e_0 + \delta[\, , \sigma_\alpha)\}_{\alpha=1}^N$, where δ is specified by the previous theorem. We have proven:

Theorem 13 *Let $\omega_0 \in S^{d-1}$, $e_0 \in]a', b'[$, and $\varepsilon > 0$ be arbitrarily fixed. Under the assumptions A and B, there exists a finite family of cubes of diameters less than ε,*

$$\{U_\alpha \times]e_0 - \delta, e_0 + \delta[\, \times B\}_{\alpha=1}^N,$$

where $[e_0 - \delta, e_0 + \delta] \subset]a', b'[$ and $\omega_0 \in B \subset \mathbb{R}^{d-1}$, and functions,

$$\sigma_\alpha \colon U_\alpha \times]e_0 - \delta, e_0 + \delta[\to \bigcup_{|e-e_0|<\delta} \Gamma(e)$$

such that the following holds:

1. *For every α, $\sigma_\alpha(u, e)$ is a real-analytic diffeomorphism from a neighborhood of $\overline{U_\alpha} \times [e_0 - \delta, e_0 + \delta]$ to its image.*
2. *For all $e \in]e_0 - \delta, e_0 + \delta[$, $\Gamma(e) = \bigcup_{\alpha=1}^N \sigma_\alpha(U_\alpha, e)$.* [5]
3. *Let us denote by $I_f^{(\alpha)}(r; e, \omega)$ the oscillatory integral of amplitude $f(u; e, \omega)$ with respect to the phase $\varphi_\alpha(u; e, \omega) = \omega \cdot \sigma_\alpha(u, e)$. If $f(u; e, \omega)$ is smooth in the neighborhood of $\mathbb{R}^{d-1} \times \mathbb{R} \times \overline{B}$ and vanishes on $(U_\alpha \times]e_0 - \delta, e_0 + \delta[)^c \times \overline{B}$,*

[5] In other words, $\{(U_\alpha, \sigma_\alpha(\cdot, e))\}_{\alpha=1}^N$ is a system of real-analytic parameterizations of $\Gamma(e)$ for *any* $e \in]e_0 - \delta, e_0 + \delta[$.

then

$$I_f^{(\alpha)}(r;e,\omega) = O(r^{-\frac{\kappa}{2}}) \quad \textit{and}$$
$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} I_f(r;\eta,\omega)\,\mathrm{d}\eta = O(r^{-\frac{\kappa}{2}}),$$

where both estimates are uniform in $(e,\omega) \in [e_0 - \delta, e_0 + \delta] \times \overline{B}$.

We now construct a smooth, joint "partition of unity" subordinated to the system of neighborhoods $\{\sigma_\alpha(U_\alpha, e)\}_{\alpha=1}^N$.

Theorem 14 *Let* $0 < \tilde{\delta} < \delta$ *and* $\mathcal{S} = \bigcup_{|e-e_0|<\tilde{\delta}} \Gamma(e)$. *There exists a family of functions,* $\{\chi_\alpha(x)\}_{\alpha=1}^N$, *smooth in* $x \in \mathbb{R}^d$, *satisfying:*

- $0 \leqslant \chi_\alpha \leqslant 1$,
- $\operatorname{supp} \chi_\alpha \subset \sigma_\alpha(U_\alpha \times]e_0 - \delta, e_0 + \delta[)$,
- $\sum_{\alpha=1}^N \chi_\alpha(x) = 1$ *for all* $x \in \overline{\mathcal{S}}$.

Proof: Notice that $\overline{\mathcal{S}} \subset \bigcup_{\alpha=1}^N \sigma_\alpha(U_\alpha \times]e_0 - \delta, e_0 + \delta[)$, so in particular

$$\{\sigma_1(U_1 \times]e_0 - \delta, e_0 + \delta[), \ldots, \sigma_N(U_N \times]e_0 - \delta, e_0 + \delta[), \mathbb{R}^d \setminus \overline{\mathcal{S}}\}$$

is a finite open covering of $\mathbb{R}^d$. After discarding $\chi_{N+1}(x)$, any partition of unity, $\{\chi_\alpha(x)\}_{\alpha=1}^{N+1}$, subordinated to the previous covering satisfies the stated properties. □

2.3.2 Fourier Transforms

We are ready to compute (uniform!) decays of Fourier transforms, $\mathcal{F}(\Gamma(e), f)$, for suitable amplitudes and derive an analogous result for Cauchy principal values.

Theorem 15 *Let* $n = r\omega$ *be the polar form of* $n \in \mathbb{Z}^d$, *where* $n \neq 0$, *and let the amplitude,* $f(x)$, *be smooth in* $x \in \mathbb{R}^d$. *Consider any interval* $[a,b] \subset]a', b'[$. *Under*

Assumptions A and B,

$$\mathcal{F}(\Gamma(e), f)(n) = O(r^{-\frac{\kappa}{2}})$$

uniformly in $(e, \omega) \in [a, b] \times \mathrm{S}^{d-1}$.

Proof: Let $e_0 \in [a, b]$ and $\omega_0 \in \mathrm{S}^{d-1}$ be fixed. By Theorems 13 and 14, there exist a $\delta > 0$, a cube, $B \ni \omega_0$, a joint system of parameterizations,

$$\{(U_\alpha \times]e_0 - 2\delta, e_0 + 2\delta[\,, \sigma_\alpha)\}_{\alpha=1}^{N},$$

and a joint partition of unity, $\{\chi_\alpha\}_{\alpha=1}^{N}$, summing at 1 on $\bigcup_{|e-e_0|<\delta} \Gamma(e)$, such that the following holds: Let

$$I^{(\alpha)}(r; e, \omega) = \int_{U_\alpha} \mathrm{e}^{\mathrm{i}r\omega\cdot\sigma_\alpha(u,e)} f(\sigma_\alpha(u, e))\chi_\alpha(\sigma_\alpha(u, e))J_\alpha(u, e)\, \mathrm{d}u,$$

where $J_\alpha(u, e)\, \mathrm{d}u$ is the element of surface of $\sigma_\alpha(U_\alpha, e)$. Then,

$$\mathcal{F}(\Gamma(e), f)(r\omega) = \sum_{\alpha=1}^{N} I^{(\alpha)}(r; e, \omega)$$

for every $(e, \omega) \in [e_0 - \delta, e_0 + \delta] \times \overline{B}$. Since $\chi_\alpha \circ \sigma_\alpha$, and hence the integrand of $I^{(\alpha)}(r; e, \omega)$ vanish outside $U_\alpha \times]e_0 - 2\delta, e_0 + 2\delta[$, by Theorem 13, $I^{(\alpha)}(r; e, \omega) = O(r^{-\frac{\kappa}{2}})$ uniformly in $(e, \omega) \in [e_0 - \delta, e_0 + \delta] \times \overline{B}$. Thus,

$$\mathcal{F}(\Gamma(e), f)(r\omega) = O(r^{-\frac{\kappa}{2}})$$

uniformly in $[e_0 - \delta, e_0 + \delta] \times \overline{B}$. Since $e_0 \in [a, b]$ and $\omega_0 \in \mathrm{S}^{d-1}$ are arbitrary, the result follows from the compactness of $[a, b] \times \mathrm{S}^{d-1}$. $\square$

2.3.3 Cauchy Principal Value

An analogous result for principal values may be derived when affecting the amplitude by a cutoff function.

We establish first that the element of surface on any local part of $\Gamma(e)$ varies smoothly in (u, e). Having in hand a common system of parameterizations, $\sigma_\alpha(u, e)$, for $u \in U_\alpha$ and $|e - e_0| < \delta$ (where $e_0 \in]a', b'[$ is arbitrarily fixed), let us compute this element of surface. To this end, one considers the determinants, $M_\alpha^{(l)}$, of the submatrices of format $(d-1) \times (d-1)$ obtained from $[\partial_{u^{(k)}} \sigma_\alpha^{(l)}(u, e)]_{k,l}$ by removing its l^{th} column, where $\sigma_\alpha = (\sigma_\alpha^{(1)}, \ldots, \sigma_\alpha^{(d)})$. Let

$$J_\alpha(u, e) = (M_\alpha^{(1)}(u, e), -M_\alpha^{(2)}(u, e), \cdots, (-1)^{d-1} M_\alpha^{(d)}(u, e)).$$

By definition the element of surface is $\|J_\alpha(u, e)\|\, du$.

Theorem 16 $\|J_\alpha(u, e)\| = \|\nabla_x \Phi(\sigma_\alpha(u, e))\|\, |\det \mathrm{D}_{(u,e)} \sigma_\alpha(u, e)|$.

Proof: The chain rule applied to $\Phi(\sigma_\alpha(u, e)) = e$ gives

$$\nabla_x \Phi(\sigma_\alpha(u, e)) \mathrm{D}_{(u,e)} \sigma_\alpha(u, e) = [0 \cdots 0\, 1],$$

which we abbreviate $\nabla \Phi \mathrm{D} \sigma_\alpha = {e_d}^{\mathrm{t}}$. Since $\sigma_\alpha(u, e)$ is a diffeomorphism, $\mathrm{D}\sigma_\alpha$ is invertible. Thus,

$$\nabla \Phi = {e_d}^{\mathrm{t}} (\mathrm{D}\sigma_\alpha)^{-1} = \frac{1}{\det \mathrm{D}\sigma_\alpha} {e_d}^{\mathrm{t}} (\mathrm{adj} \mathrm{D}\sigma_\alpha),$$

where adj stands for the classical adjoint. The result follows from

$${e_d}^{\mathrm{t}} (\mathrm{adj} \mathrm{D}\sigma_\alpha) = (-1)^{d-1} J_\alpha(u, e).$$

□

Corollary $\|J_\alpha(u,e)\|$ *is smooth in* $(u,e) \in U_\alpha \times]e_0-\delta, e_0+\delta[$.

For fixed $\omega_0 \in \mathrm{S}^{d-1}$ and $e_0 \in]a', b'[$, let $\{(U_\alpha \times]e_0 - 3\delta, e_0+3\delta[\,, \sigma_\alpha)\}_{\alpha=1}^N$, $B \ni \omega_0$, and $\{\chi_\alpha\}_{\alpha=1}^N$, summing at 1 on $\bigcup_{|e-e_0|<2\delta} \Gamma(e)$, be given by Theorems 13 and 14. Let us define a cutoff function, $0 \leqslant \chi_{e_0}(x) \leqslant 1$, smooth in $x \in \mathbb{R}^d$, such that[6]

$$\chi_{e_0}(x) = \begin{cases} 1 & \text{if} \quad |\Phi(x) - e_0| < \delta, \\ 0 & \text{if} \quad |\Phi(x) - e_0| > 2\delta. \end{cases}$$

Notice that $\chi_{e_0}(x)$ is analytic in the neighborhood of $\Gamma(e_0)$—which will become important later. Under Assumptions A and B,

Theorem 17 *For $f(x)$ smooth in $x \in \mathbb{R}^d$ and $r > 0$*

$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} \mathcal{F}(\Gamma(\eta), \chi_{e_0} f)(r\omega)\, \mathrm{d}\eta = O(r^{-\frac{\kappa}{2}})$$

uniformly in $(e, \omega) \in [e_0 - \delta, e_0 + \delta] \times \mathrm{S}^{d-1}$.

Proof: The considered principal value is equal to

$$\text{p.v.} \int_{\mathbb{R}} \frac{1}{\eta - e} \sum_{\alpha=1}^{N} I_{F_\alpha}^{(\alpha)}(r; \eta, \omega)\, \mathrm{d}\eta,$$

[6] Since the amplitude we will use does not depend on ω, the following cutoff function does not depend on ω_0; in other circumstances if it does, the following argument still holds.

where $F_\alpha(u,\eta) = (\chi_{e_0} f \chi_\alpha)(\sigma_\alpha(u,\eta))\|J_\alpha(u,\eta)\|$. Notice that

$$\chi_{e_0}(\sigma_\alpha(u,\eta)) = \begin{cases} 1 & \text{if} \quad |\eta - e_0| < \delta, \\ 0 & \text{if} \quad |\eta - e_0| > 2\delta. \end{cases}$$

Hence, $F_\alpha(u,\eta)$ is smooth on $\mathbb{R}^{d-1} \times \mathbb{R}$ and vanishes when

$$(u,\eta) \notin U_\alpha \times \left]e_0 - 2\delta, e_0 + 2\delta\right[.$$

By Theorem 13 the principal value under consideration is thus $O(r^{-\frac{\kappa}{2}})$ uniformly in $(e,\omega) \in [e_0 - \delta, e_0 + \delta] \times \overline{B}$. Since $\omega_0 \in \mathrm{S}^{d-1}$ is arbitrarily fixed, the result follows from compactness of S^{d-1}. $\square$

2.3.4 Analyticity of Fourier Transform

We close this section by showing that for any fixed $n \in \mathbb{Z}^d$ the following Fourier transform, $\mathcal{F}(\Gamma(e), f)(n)$, is analytic at e_0 when the amplitude, $f(x)$, is analytic in the neighborhood of $\Gamma(e_0) \subset \mathbb{R}^d$ (where $e_0 \in \left]a', b'\right[$ is arbitrarily fixed). To this end the diffeomorphism between $\Gamma(e_0)$ and $\Gamma(e)$, $\Sigma(x,e)$, defined in the relation (2.11) is helpful, because, given a system of *disjoint* open neighborhoods on $\Gamma(e_0)$ covering all $\Gamma(e_0)$ except a set of area zero, its lifting to $\Gamma(e)$ via $\Sigma(x,e)$ also covers the whole $\Gamma(e)$ except a set of area zero.

In details, let $\{(U_\alpha \times \left]e_0 - \delta, e_0 + \delta\right[, \sigma_\alpha)\}_{\alpha=1}^N$ be given by Theorem 13.

Theorem 18 *There exists a finite, joint system of local parameterizations,*

$$\{(V_\beta \times \left]e_0 - \delta, e_0 + \delta\right[, \sigma_\beta)\}_{\beta=1}^M,$$

such that:

- *For all* $\beta \in \{1, \ldots, M\}$ *there exists an* $\alpha_\beta \in \{1, \ldots, N\}$ *such that* $V_\beta \subseteq U_{\alpha_\beta}$ *and* $\sigma_\beta = \sigma_{\alpha_\beta} \lceil V_\beta$*;*
- *For all* $e \in]e_0 - \delta, e_0 + \delta[$ *the coordinates neighborhoods* $\{\sigma_\beta(V_\beta, e)\}_{\beta=1}^{M}$ *are mutually disjoints;*
- *For all* $e \in]e_0 - \delta, e_0 + \delta[$ *the area of* $\Gamma(e) \setminus \bigcup_{\beta=1}^{M} \sigma_\beta(V_\beta, e)$ *is zero.*[7]

Proof: One may construct a system of disjoint, open neighborhoods of full area on $\Gamma(e_0)$ by considering all non empty cells of the form[8]

$$\sigma_{\alpha_1}(U_{\alpha_1}, e_0) \cap \cdots \cap \sigma_{\alpha_l}(U_{\alpha_l}, e_0) \cap \sigma_{\alpha_{l+1}}(\overline{U_{\alpha_{l+1}}}, e_0)^c \cap \cdots \cap \sigma_{\alpha_N}(\overline{U_{\alpha_N}}, e_0)^c,$$

where $\{\alpha_1, \cdots, \alpha_N\} = \{1, \cdots, N\}$. These cells are open, disjoint, and cover all $\Gamma(e_0)$ except the set $\sigma_1(\partial U_1, e_0) \cup \cdots \cup \sigma_N(\partial U_N, e_0)$, whose area is zero. The diffeomorphism $\Sigma(x, e)$ lifts this exceptional set onto an exceptional set in $\Gamma(e)$ (which is not a surprise). The peculiarity of $\Sigma(x, e)$ is that *all points on* $\Gamma(e)$ *which are not the image of an exceptional* $x \in \Gamma(e_0)$ *are covered.* The result follows. □

Theorem 19 *Suppose* $f(x)$ *is analytic in a neighborhood of* $\Gamma(e_0) \subset \mathbb{R}^d$*. Then, for any fixed* $n \in \mathbb{Z}^d$ *the Fourier transform* $\mathcal{F}(\Gamma(e), f)(n)$ *is analytic at* $e_0 \in]a', b'[$.

---

[7] In other words, the integral of 1 over this surface is zero.

[8] The complements are taken with respect to $\Gamma(e_0)$.

Proof: By the previous theorem, for any $e \in]e_0 - \delta, e_0 + \delta[$

$$\mathcal{F}(\Gamma(e), f)(n) = \sum_{\beta=1}^{M} \int_{V_\beta} e^{in\cdot\sigma_\beta(u,e)} f(\sigma_\beta(u,e)) J_\beta(u,e)\, \mathrm{d}u, \tag{2.13}$$

where $J_\beta(u,e)\,\mathrm{d}u$ is the element of surface of $\sigma_\beta(V_\beta, e)$. Since the integrand in this last expression is analytic in the neighborhood of $\overline{V_\beta} \times [e_0 - \delta/2, e_0 + \delta/2]$ (that is, converges to its Taylor's series), the result follows. □

2.4 Green's Functions

We turn our attention to

$$G(n,z) = \int_{\mathbb{T}^d} \frac{e^{in\cdot x}}{\Phi(x) - z}\, \mathrm{d}x, \tag{2.14}$$

where $z \in \mathbb{C}_+$, $n \in \mathbb{Z}^d$, and $\Phi(x)$ is real valued, analytic, and *periodic* on $\mathbb{T}^d$. Here, $\mathbb{T}^d$ denotes the *torus* of dimension d, that is, the set $[-\pi,\pi]^d$ endowed with the quotient topology induced by congruence modulo 2π. In particular,

$$\{x \in \mathbb{T}^d\ ;\ \Phi(x) = e\} = \Gamma(e)$$

is a compact manifold. It may be covered by a finite system of coordinates neighborhoods admitting a subordinated partition of unity. Since each coordinate neighborhood is embedded in $\mathbb{R}^d$, it is clear that all results in section 2.3 apply under appropriate hypotheses, namely: Let $a' < b'$ and $\mathcal{S}' = \bigcup_{e\in]a',b'[} \Gamma(e)$. We assume:

Assumption C

- $\Phi(x)$ *is real valued, analytic, and periodic on* $\mathbb{T}^d$*;*

- $\nabla\Phi(x) \neq 0$ *for all* $x \in \mathcal{S}'$;
- *For every* $e \in\,]a', b'[$, $\Gamma(e)$ *admits at least* κ *non vanishing principal curvatures at any point, where* $\kappa \geqslant 1$ *is a fixed integer.*

For $e \in \mathbb{R}$ let

$$G(n, e) = \lim_{\substack{z \to e \\ z \in \mathbb{C}_+}} G(n, z),$$

where of course the existence of such a limit has to be established. In this section we compute the decay of $G(n, e)$ as $|n| \to \infty$. We are especially interested in the case where e is in the range of $\Phi(x)$; otherwise, since by assumption $\Phi(x)$ admits a holomorphic extension, one may slightly change the domain of integration in (2.14) and deduce that $G(n, e)$ decays exponentially.

The following decomposition theorem is interesting in its own and will be used in the next chapter. Before, we need this elementary lemma:

Lemma *Consider a function,* $f(\eta)$*, continuous in* $\eta \in [a, b]$ *and analytic at* $e \in\,]a, b[$*. Then,*

$$\lim_{\substack{z \to e \\ z \in \mathbb{C}_+}} \int_a^b \frac{f(\eta)}{\eta - z}\, \mathrm{d}\eta = \pi \mathrm{i} f(e) + \text{p.v.} \int_a^b \frac{f(\eta)}{\eta - e}\, \mathrm{d}\eta.$$

Proof: Given an $\varepsilon \in\,]0, (b-a)/2[$, let C_ε be a path joining $e - \varepsilon$ and $e + \varepsilon$, and lying inside the lower half-plane. Let $\gamma_\varepsilon = [a, e - \varepsilon] * C_\varepsilon * [e + \varepsilon, b]$, where $*$ denotes the concatenation. By Cauchy's theorem

$$\int_a^b \frac{f(\eta)}{\eta - z}\,\mathrm{d}\eta = \int_{\gamma_\varepsilon} \frac{f(\eta)}{\eta - z}\,\mathrm{d}\eta$$

for any $z \in \mathbb{C}_+$ if ε is sufficiently small. Thus, by the dominated convergence theorem

$$\begin{aligned} \lim_{\substack{z \to e \\ z \in \mathbb{C}_+}} \int_a^b \frac{f(\eta)}{\eta - z}\,\mathrm{d}\eta &= \int_{\gamma_\varepsilon} \frac{f(\eta)}{\eta - e}\,\mathrm{d}\eta \\ &= \lim_{\varepsilon \downarrow 0} \int_{C_\varepsilon} \frac{f(\eta)}{\eta - e}\,\mathrm{d}\eta + \lim_{\varepsilon \downarrow 0} \int_{|\eta - e| > \varepsilon} \frac{f(\eta)}{\eta - e}\,\mathrm{d}\eta \end{aligned}$$

(by Cauchy's theorem again). The result follows. □

For fixed $\omega_0 \in \mathrm{S}^{d-1}$ and $e_0 \in]a', b'[$, let $\{(U_\alpha \times]e_0 - 3\delta, e_0 + 3\delta[\,, \sigma_\alpha)\}_{\alpha=1}^N$, $B \ni \omega_0$, $\{\chi_\alpha\}_{\alpha=1}^N$, summing to 1 on $\bigcup_{|e-e_0|<2\delta} \Gamma(e)$, and χ_{e_0} be given by Theorems 13, 14, and 17.

Theorem 20 *Suppose* $|e - e_0| < \delta$ *and* $\omega \in B$. *Then,* $G(n, e)$ *exists and is equal to*

$$\pi \mathrm{i} \mathcal{F}(\Gamma(e), P)(n) + \mathrm{p.v.} \int_{-\infty}^{\infty} \frac{1}{\eta - e} \mathcal{F}(\Gamma(\eta), \chi_{e_0} P)(n)\,\mathrm{d}\eta + \int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}n \cdot x} \frac{1 - \chi_{e_0}(x)}{\Phi(x) - e}\,\mathrm{d}x,$$

where $P(x) = \frac{1}{\|\nabla_x \Phi(x)\|}$.

Proof: Notice that

$$\int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}n \cdot x} \frac{\chi_{e_0}(x)}{\Phi(x) - z}\,\mathrm{d}x = \sum_{\alpha=1}^N \int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}n \cdot x} \frac{\chi_{e_0}(x) \chi_\alpha(x)}{\Phi(x) - z}\,\mathrm{d}x.$$

The change of variables $x = \sigma_\alpha(u, \eta)$ applied to the above gives

$$\sum_{\alpha=1}^N \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \mathrm{e}^{\mathrm{i}n \cdot \sigma_\alpha(u,\eta)} \frac{\chi_{e_0}(\sigma_\alpha(u, \eta)) \chi_\alpha(\sigma_\alpha(u, \eta))}{\eta - z} |\det \mathrm{D}_{(u,\eta)} \sigma_\alpha(u, \eta)|\,\mathrm{d}u\,\mathrm{d}\eta,$$

which, by Theorem 16, is equal to

$$\sum_{\alpha=1}^{N} \int_{\mathbb{R}} \frac{1}{\eta - z} \int_{\Gamma(\eta)} \mathrm{e}^{\mathrm{i}n\cdot x} \chi_{e_0}(x) \chi_\alpha(x) P(x)\, \mathrm{dS}(x)\, \mathrm{d}\eta,$$

where $\mathrm{dS}(x)$ denotes the element of surface. Hence,

$$\int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}n\cdot x} \frac{\chi_{e_0}(x)}{\Phi(x) - z}\, \mathrm{d}x = \int_{\mathbb{R}} \frac{1}{\eta - z} \mathcal{F}(\Gamma(\eta), \chi_{e_0} P)(n)\, \mathrm{d}\eta. \tag{2.15}$$

Notice that $\chi_{e_0} P$ is analytic in the neighborhood of $\Gamma(e)$ when $|e - e_0| < \delta$. Hence, by Theorem 19, $\mathcal{F}(\Gamma(\eta), \chi_{e_0} P)$ is analytic at $\eta = e$. The result follows from the lemma and the dominated convergence theorem. □

The desired decay follows:

Theorem 21 *Let $n = r\omega$ be the polar form of $n \in \mathbb{Z}^d$, where $n \neq 0$, and consider any interval $[a, b] \subset]a', b'[$. Under Assumption C*

$$G(n, e) = O(r^{-\frac{\kappa}{2}})$$

uniformly in $(e, \omega) \in [a, b] \times \mathrm{S}^{d-1}$.

Proof: For ω_0, e_0, etc. as above, by Theorems 15 and 17

$$\pi\mathrm{i}\mathcal{F}(\Gamma(e), P)(n) = O(r^{-\frac{\kappa}{2}}) \quad \text{and}$$

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{1}{\eta - e} \mathcal{F}(\Gamma(\eta), \chi_{e_0} P)(n)\, \mathrm{d}\eta = O(r^{-\frac{\kappa}{2}})$$

uniformly on $[e_0 - \delta, e_0 + \delta] \times \overline{B}$. Moreover, by the scholium of Theorem 1

$$\int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}n\cdot x} \frac{1 - \chi_{e_0}(x)}{\Phi(x) - e}\, \mathrm{d}x = O(r^{-\infty})$$

uniformly on $[e_0 - \delta, e_0 + \delta] \times \overline{B}$. Since (e_0, ω_0) is arbitrary in $[a, b] \times \mathrm{S}^{d-1}$, which is compact, the result follows. $\square$

In the second part of the present thesis uniformity is needed in the complex plane:

Theorem 22 *In the above circumstances, let*

$$\mathcal{S} = \{e + \mathrm{i}y \; ; \; a < e < b, \, 0 < y < 1\}.$$

Then, $G(n, z) = O(r^{-\frac{\kappa}{2}} \ln r)$ *uniformly in* $(z, \omega) \in \overline{\mathcal{S}} \times \mathrm{S}^{d-1}$.

Proof: Since $G(n, z) = O(|n|^{-\infty})$ uniformly in $\{e + \mathrm{i}y \; ; \; a \leqslant e \leqslant b \text{ and } c < y \leqslant 1\}$ for any $c > 0$, we assume w.l.o.g. $y \in [0, c]$ for an arbitrarily small c. Moreover, we restrict our attention to $e \in [e_0 - \delta, e_0 + \delta]$ for an arbitrarily chosen $e_0 \in [a, b]$ and a sufficiently small $\delta > 0$, which is done w.l.o.g. since $[a, b]$ is compact. Then, the cutoff function $\chi_{e_0}(x)$ restricts $\Phi(x)$ to the interval $[e_0 - 2\delta, e_0 + 2\delta]$ and the decomposition used in Theorem 20 gives (by the equation 2.15)

$$G(n, z) = \int_{e_0 - 2\delta}^{e_0 + 2\delta} \frac{1}{\eta - z} \mathcal{F}(\Gamma(\eta), \chi_{e_0} P)(n) \, \mathrm{d}\eta + \int_{\mathbb{T}^d} \mathrm{e}^{\mathrm{i}n \cdot x} \frac{1 - \chi_{e_0}(x)}{\Phi(x) - e} \, \mathrm{d}x.$$

Integration by parts shows that the second integral in the above is $O(|n|^{-\infty})$ uniformly on the considered strip. Hence, it suffices to analyze the first term, which we denote $F(n, z)$.

Letting $\mathcal{F}(n, e) = \mathcal{F}(\Gamma(e), \chi_{e_0} P)(n)$, we thus consider

$$F(n, z) = \int_{e_0 - 2\delta}^{e_0 + 2\delta} \frac{1}{\eta - z} \mathcal{F}(n, \eta) \, \mathrm{d}\eta$$

for z varying over the strip $\{e+\mathrm{i}y\ ;\ |e-e_0|\leqslant\delta \text{ and } 0\leqslant y\leqslant c\}$. Our strategy consists of estimating $F(n,e+\mathrm{i}y)$ and $\partial_{\mathrm{i}y}F(n,e+\mathrm{i}y)$;[9] applying the fundamental theorem of calculus to the latter; joining the former and the resulting estimate in order to obtain the desired decay.

If c is sufficiently small,

$$\begin{aligned}
|F(n,e+\mathrm{i}y)| &\leqslant \sup_{|\eta-e_0|<2\delta}|\mathcal{F}(n,\eta)|\int_{e_0-2\delta}^{e_0+2\delta}\frac{1}{\sqrt{(\eta-e)^2+y^2}}\,\mathrm{d}\eta\\
&\leqslant \sup_{|\eta-e_0|<2\delta}|\mathcal{F}(n,\eta)|\int_{e-3\delta}^{e+3\delta}\frac{1}{\sqrt{(\eta-e)^2+y^2}}\,\mathrm{d}\eta\\
&= 2\sup_{|\eta-e_0|<2\delta}|\mathcal{F}(n,\eta)|\ln\left(\frac{3\delta+\sqrt{9\delta^2+y^2}}{y}\right)\\
&\leqslant \mathit{Const}\,|n|^{-\frac{\kappa}{2}}(1+\ln(1/y)\,),
\end{aligned}$$

so in total

$$|F(n,e+\mathrm{i}y)|\leqslant \mathit{Const}\,|n|^{-\frac{\kappa}{2}}\ln(1/y) \tag{2.16}$$

uniformly on the considered strip.

---

[9] For a fixed n we denote by $\partial_{\mathrm{i}y}F(n,z)$ the derivative of $F(n,z)$ along a line parallel to the imaginary axis. Since $F(n,z)$ is holomorphic, this last derivative is indeed equal to the complex derivative.

On the other hand, by the dominated convergence theorem

$$\begin{aligned}
\partial_{\mathrm{i}y} F(n, e+\mathrm{i}y) &= \int_{e_0-2\delta}^{e_0+2\delta} \frac{1}{(\eta-e-\mathrm{i}y)^2} \mathcal{F}(n,\eta)\,\mathrm{d}\eta \\
&= \int_{e_0-2\delta}^{e_0+2\delta} \frac{1}{(\eta-e-\mathrm{i}y)^2} (\mathcal{F}(n,\eta)-\mathcal{F}(n,e))\,\mathrm{d}\eta \; + \\
&\quad + \mathcal{F}(n,e) \int_{e_0-2\delta}^{e_0+2\delta} \frac{1}{(\eta-e-\mathrm{i}y)^2}\,\mathrm{d}\eta \\
&= I + II.
\end{aligned}$$

Notice that $\left|\int_{e_0-2\delta}^{e_0+2\delta} \frac{1}{(\eta-e-\mathrm{i}y)^2}\,\mathrm{d}\eta\right| \leqslant Const$ uniformly when $e+\mathrm{i}y$ varies on the considered strip, which may be seen by changing the integration path. Hence, $|II| \leqslant Const\,|n|^{-\frac{\kappa}{2}}$ uniformly on this last strip. Moreover, the mean value theorem implies

$$|I| \leqslant \sup_{|\eta-e_0|<2\delta} |\partial_\eta \mathcal{F}(n,\eta)| \int_{e_0-2\delta}^{e_0+2\delta} \frac{1}{(\eta-e)^2+y^2} |\eta-e|\,\mathrm{d}\eta.$$

Observe that the dominated convergence theorem applied to the explicit decomposition of $\mathcal{F}(n,\eta)$ given in the equation (2.13) yields $\partial_\eta \mathcal{F}(n,\eta) = O(|n|^{-\frac{\kappa}{2}+1})$ uniformly in $\eta \in [e_0-2\delta, e_0+2\delta]$. Therefore,

$$\begin{aligned}
|I| &\leqslant Const\,|n|^{-\frac{\kappa}{2}+1} \int_{e_0-2\delta}^{e_0+2\delta} \frac{|\eta-e|}{(\eta-e)^2+y^2}\,\mathrm{d}\eta \\
&\leqslant Const\,|n|^{-\frac{\kappa}{2}+1} \int_{e-3\delta}^{e+3\delta} \frac{|\eta-e|}{(\eta-e)^2+y^2}\,\mathrm{d}\eta \\
&= Const\,|n|^{-\frac{\kappa}{2}+1} \ln\left(\frac{y^2+9\delta^2}{y^2}\right) \\
&\leqslant Const\,|n|^{-\frac{\kappa}{2}+1}(1+\ln(1/y)),
\end{aligned}$$

so in total

$$|\partial_{\mathrm{i}y} F(n, e+\mathrm{i}y)| \leqslant Const\,|n|^{-\frac{\kappa}{2}+1} \ln(1/y)$$

uniformly on the considered strip.

By the fundamental theorem of calculus

$$\begin{aligned}
|F(n,e+\mathrm{i}y)-F(n,e)| &\leqslant \int_0^y |\partial_s F(n,e+\mathrm{i}s)|\,\mathrm{d}s \\
&= \int_0^y |\partial_{\mathrm{i}y} F(n,e+\mathrm{i}s)|\,\mathrm{d}s \\
&\leqslant \mathit{Const}\,|n|^{-\frac{\kappa}{2}+1}\int_0^y \ln(1/s)\,\mathrm{d}s \\
&= \mathit{Const}\,|n|^{-\frac{\kappa}{2}+1}(y\ln(1/y)+y) \\
&= \mathit{Const}\,|n|^{-\frac{\kappa}{2}+1}y\ln(1/y).
\end{aligned}$$

Hence, since $F(n,e)=O(|n|^{-\frac{\kappa}{2}})$ uniformly in $e\in[e_0-\delta,e_0+\delta]$,

$$|F(n,e+\mathrm{i}y)|\leqslant \mathit{Const}\,|n|^{-\frac{\kappa}{2}+1}y\ln(1/y)$$

on the considered strip. Notice that $\sqrt{y}\ln(1/y)$ goes to 0 when $y\downarrow 0$. *A fortiori*, this last expression is bounded. Consequently,

$$|F(n,e+\mathrm{i}y)|\leqslant \mathit{Const}\,|n|^{-\frac{\kappa}{2}+1}\sqrt{y}.$$

In particular, for $y<\frac{1}{|n|^2}$,

$$|F(n,e+\mathrm{i}y)|\leqslant \mathit{Const}\,|n|^{-\frac{\kappa}{2}}.$$

Otherwise, $y>\frac{1}{|n|^2}$, so $\ln(1/y)<2\ln|n|$ and the equation (2.16) implies

$$|F(n,e+\mathrm{i}y)|\leqslant \mathit{Const}\,|n|^{-\frac{\kappa}{2}}\ln|n|$$

uniformly on the considered strip. The proof is thus complete. $\square$

As an immediate corollary notice that

$$G(n, z) = O(r^{-\frac{\kappa}{2}^{+}}) \text{ when } |n| \to \infty$$

uniformly in $(\omega, z) \in \mathrm{S}^{d-1} \times \overline{S}$, *i.e.*, for all $\varepsilon > 0$ there exists a C_ε independent of ω and z such that

$$|G(n, z)| \leqslant C_\varepsilon r^{-\frac{\kappa}{2}+\varepsilon}$$

for all $r > 0$. The above theorem will be used in this last form in the applications.

2.5 Application to Generalized Laplacians

In this section we apply the previous results to Green's functions of concrete Laplacians on $l^2(\mathbb{Z}^d)$. We focus on two specific examples: the *standard Laplacian* and the *Molchanov–Vainberg Laplacian.*

At a higher level of generality let us consider a simple graph without loop, whose set of vertices is denoted by X (where X is allowed to be infinite). For $m, n \in X$, $\mathrm{d}(m, n)$ denotes the *graph distance* between m and n, that is, the length of the minimal chain joining m and n in the graph (∞ if m and n lie in two different connected components). Of course, d is a metric distance with values in $\mathbb{N} \cup \{\infty\}$. Notice also that (X, d) determines the graph completely, since $\{m, n\}$ is an edge if and only if $\mathrm{d}(m, n) = 1$.

We are interested in the Hilbert space, $l^2(X)$, consisting of square summable sequences indexed by X. Its usual basis is $\{\delta_n\}_{n \in X}$, where δ denotes the Kronecker delta:

$$\delta_n(m) = \mathbf{1}_{\{n\}}(m) = \begin{cases} 1 & \text{if} \quad m = n, \\ 0 & \text{if} \quad m \neq n. \end{cases}$$

The *adjacency operator* on $l^2(X)$ with respect to (X, d), sometimes called *Laplacian*, is defined as

$$(\Delta\psi)(n) = \sum_{\mathrm{d}(m,n)=1} \psi(m),$$

where $\psi \in l^2(X)$. In particular, $(\Delta\delta_n)(m) = \mathbf{1}_{\mathcal{V}_n}(m)$, where

$$\mathcal{V}_n = \{m \in X \; ; \; \mathrm{d}(m,n) = 1\}.$$

In the sequel we suppose that the degrees of the vertices of the considered graph are bounded, in other words,

$$\sup_n \#\mathcal{V}_n \leqslant B$$

for a certain $B < \infty$.[10] Then,

Theorem 23 *The adjacency operator, Δ, is a bounded, selfadjoint operator on $l^2(X)$.*

[10] In general $\#A$ denotes the cardinality of A.

Proof: For any $\psi(n) \in l^2(X)$

$$\begin{aligned}
\|\Delta\psi\|^2 &= \sum_{n\in X} |(\Delta\psi)(n)|^2 \\
&= \sum_{n\in X} \left| \sum_{m\in\mathcal{V}_n} \psi(m) \right|^2 \\
&\leqslant B \sum_{n\in X} \sum_{m\in\mathcal{V}_n} |\psi(m)|^2 \\
&\leqslant B^2 \|\psi\|^2,
\end{aligned}$$

which shows that the adjacency operator is bounded. Moreover,

$$\langle \delta_m \,|\, \Delta\delta_n \rangle = \mathbf{1}_{\mathcal{V}_n}(m) = \mathbf{1}_{\mathcal{V}_m}(n) = \langle \Delta\delta_m \,|\, \delta_n \rangle,$$

which completes the proof. □

We are now interested in the case where $X = \mathbb{Z}^d$ and the graph distance, $\mathrm{d}(m,n)$, is translational invariant. In these circumstances we call Δ a *generalized Laplacian.* Then, $\mathrm{d}(m,n)$ is a function of $m-n$ only. Hence, letting $\mathcal{V} = \mathcal{V}_0$ (where $0 \in \mathbb{Z}^d$ denotes the origin), the considered graph is clearly determined by $(\mathbb{Z}^d, \mathcal{V})$. In particular,

$$\Delta\psi(n) = \sum_{v\in\mathcal{V}} \psi(n+v),$$

where $\#\mathcal{V} \leqslant B$ is still assumed to be finite.

Recall that the Fourier transform of $\psi \in l^2(\mathbb{Z}^d)$ is the following function, defined for $x \in \mathbb{T}^d$:

$$\widehat{\psi}(x) = (\mathcal{F}\psi)(x) = (2\pi)^{-\frac{d}{2}} \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}n\cdot x}\psi(n).$$

The *symbol* of Δ, denoted by $\widehat{\Delta}$, is the lifting of Δ via the Fourier transform:

$$\widehat{\Delta} = \mathcal{F}\Delta\mathcal{F}^{-1}.$$

Theorem 24 *Let Δ be the generalized Laplacian associated with a given $\mathcal{V} \subset \mathbb{Z}^d$. Then, its symbol is the multiplication by*

$$\Phi(x) = \sum_{v\in\mathcal{V}} \mathrm{e}^{\mathrm{i}v\cdot x} = \sum_{v\in\mathcal{V}} \cos(v\cdot x),$$

where $x \in \mathbb{T}^d$.

Proof: $\widehat{\Delta}$ maps $\widehat{\psi}(x)$ to the following function:

$$\begin{aligned}(2\pi)^{-\frac{d}{2}} \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}n\cdot x} \sum_{v\in\mathcal{V}} \psi(n+v) &= (2\pi)^{-\frac{d}{2}} \sum_{v\in\mathcal{V}} \sum_{n\in\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}(n-v)\cdot x}\psi(n) \\ &= \sum_{v\in\mathcal{V}} \mathrm{e}^{-\mathrm{i}v\cdot x}\widehat{\psi}(x).\end{aligned}$$

Notice that

$$v \in \mathcal{V} \iff \mathrm{d}(v,0) = 1 \iff \mathrm{d}(0,-v) = 1 \iff -v \in \mathcal{V}.$$

Hence, denoting by $\mathcal{V}^+$ the set of $v \in \mathcal{V}$ whose first non zero coordinate is positive,

$$\sum_{v\in\mathcal{V}} \mathrm{e}^{-\mathrm{i}v\cdot x} = \sum_{v\in\mathcal{V}} \mathrm{e}^{\mathrm{i}v\cdot x} = \sum_{v\in\mathcal{V}^+} 2\cos(v\cdot x) = \sum_{v\in\mathcal{V}} \cos v\cdot x,$$

which completes the proof. □

Since the range of $\Phi(x)$ is equal to the spectrum of Δ, one obtains:

Corollary $\operatorname{spec}(\Delta) = [\min \Phi, \#\mathcal{V}]$.

Notice that $\Phi(x)$ is analytic at each $x \in \mathbb{T}^d$. Hence,

$$E = \{x \in \mathbb{T}^d \; ; \; \nabla_x \Phi(x) = 0\}$$

has Lebesgue measure zero. In particular, $\operatorname{spec}(\Delta)$ is purely absolutely continuous. [11]

The *Green's function* of Δ is defined as

$$G(m, n; z) = \langle \delta_m \,|\, (\Delta - z)^{-1} \delta_n \rangle$$

for $m, n \in \mathbb{Z}^d$ and $z \in \mathbb{C}_+$. .Since Δ is translational invariant, this last function depends on $m - n$ and z only. In fact, $G(m, n; z) = G(0, n - m; z)$ so we denote this latter simply by $G(n - m, z)$. Since the Fourier transform is unitary, for any $n \in \mathbb{Z}^d$ and $z \in \mathbb{C}_+$

$$\begin{aligned} G(n, z) &= \langle \widehat{\delta_0}(x) \,|\, (\widehat{\Delta} - z)^{-1} \widehat{\delta_n}(x) \rangle_2 \\ &= (2\pi)^{-d} \int_{\mathbb{T}^d} \frac{e^{in \cdot x}}{\Phi(x) - z} \, dx. \end{aligned}$$

[11] This may be seen in the following way: $L^2(\mathbb{T}^d) = L^2(\mathbb{T}^d \setminus E)$. Moreover, by the Inverse Function Theorem $\Phi(x)$ is invertible on $\mathbb{T}^d \setminus E$. Let $\Psi(y)$ be its inverse, so the change of variables $y = \Phi(x)$ gives a unitary equivalence between $L^2(\mathbb{T}^d)$ with the Lebesgue measure and $L^2(\Phi(\mathbb{T}^d \setminus E))$ with the measure $\|\nabla_y \Psi(y)\| \, dy$. Via this unitary equivalence the operator of multiplication by $\Phi(x)$ is lifted to the operator of multiplication by y, so by definition $\|\nabla_y \Psi(y)\| \, dy \restriction \Phi(\mathbb{T}^d \setminus E)$ is a spectral measure for Δ—and is clearly absolutely continuous.

By Theorem 20 the limit

$$G(n,e) = \lim_{\substack{z \to e \\ z \in \mathbb{C}_+}} G(n,z)$$

exists for any $e \notin E$. We will compute its decay when $|n| \to \infty$ in two particular examples: the *standard discrete Laplacian* and the *Molchanov–Vainberg Laplacian*, which are described in the rest of the section. Both are important in mathematical physics, since both are discretizations of the continuous Laplacian on $\mathbb{R}^d$ (up to an additive constant).

2.5.1 Standard Laplacian

The *standard Laplacian* is the adjacency operator of the usual grid on $\mathbb{Z}^d$; it is specified by the graph distance

$$\mathrm{d}(m,n) = |m-n|_1 = \sum_{j=1}^{d} |m^{(j)} - n^{(j)}|,$$

so the set of immediate neighbors of the origin is

$$\mathcal{V} = \{(\pm 1, 0, \ldots, 0), (0, \pm 1, \ldots, 0), \ldots, (0, 0, \ldots, \pm 1)\}.$$

Hence, by Theorem 24 the symbol of the standard Laplacian is the operator of multiplication by

$$\Phi(x) = 2\sum_{j=1}^{d} \cos x^{(j)}.$$

Thus, the spectrum of Δ is purely absolutely continuous and equal to $[-2d, 2d]$. Notice that

$$\nabla_x \Phi(x) = (-2\sin x^{(1)}, \ldots, -2\sin x^{(d)}),$$

which vanishes only if $\cos x^{(j)} = \pm 1$ for all $j = 1, \ldots, d$, in which case $\Phi(x) \in \{-2d, -2d+4, \ldots, 2d-4, 2d\}$. In particular, the level surfaces of $\Phi(x)$,

$$\Gamma(e) = \{x \in \mathbb{T}^d \, ; \; \Phi(x) = e\},$$

are regular for all $e \notin \{-2d, -2d+4, \ldots, 2d-4, 2d\}$.

Let us show that for such e's the level surfaces are exempt of planarity, except for $e = 0$:

Theorem 25 *Let*

$$E = \{-2d, -2d+4, \ldots, 2d-4, 2d\} \cup \{0\}$$

and suppose $e \in [-2d, 2d] \setminus E$. *Then,* $\Gamma(e)$ *admits at least one non vanishing principal curvature at any point.*

Proof: Let $x_0 \in \Gamma(e)$ be fixed. By choice of e, $\Gamma(e)$ is regular, so there exists a j such that $\sin x_0^{(j)} \neq 0$. After renaming the variables, the equation defining $\Gamma(e)$ in a neighborhood of x_0 thus becomes

$$2\cos u^{(1)} + \cdots + 2\cos u^{(d-1)} + 2\cos h = e, \tag{2.17}$$

where $\sin h \neq 0$ on a certain interval. In particular, the derivative with respect to h of the left-hand side in (2.17) is not zero. Therefore, writing $u = (u^{(1)}, \ldots, u^{(d-1)})$, there exists an implicit function, $\mathrm{h}(u)$, such that

$$2\cos u^{(1)} + \cdots + 2\cos u^{(d-1)} + 2\cos \mathrm{h}(u) = e$$

for all u in a neighborhood of u_0, where $(u_0, \mathrm{h}(u_0))$ is the permuted x_0. The number of non vanishing principal curvatures at x_0 is then given by the rank of $\mathrm{D}^2_u\mathrm{h}(u_0)$. Indeed, for any $j = 1, \ldots, d-1$, differentiating the previous equation with respect to $u^{(j)}$ gives

$$-\sin u^{(j)} - \sin h\, \partial_{u^{(j)}} h = 0, \tag{2.18}$$

where $h = \mathrm{h}(u)$. Consequently,

$$-\sin h\, \mathrm{D}^2_u \mathrm{h}(u) = \mathrm{diag}(\cos u^{(1)}, \ldots, \cos u^{(d-1)}) + \cos h\, [\partial_{u^{(j)}}\mathrm{h}(u)\partial_{u^{(k)}}\mathrm{h}(u)]_{j,k=1}^{d-1}.$$

Since $\sin h \neq 0$, it suffices to show that the right side in the previous equation does not vanish in a neighborhood of u_0. By the equation (2.18) this matrix is equal to

$$\mathrm{diag}(\cos u^{(1)}, \ldots, \cos u^{(d-1)}) + \frac{\cos h}{\sin^2 h}[\sin u^{(j)} \sin u^{(k)}]_{j,k=1}^{d-1}. \tag{2.19}$$

Suppose by contradiction there exists an u in the considered neighborhood such that the above vanishes. Then, for all $j = 1, \ldots, d-1$

$$\cos u^{(j)} \sin^2 h + \cos h \sin^2 u^{(j)} = 0,$$

which is equivalent to

$$(1 - \cos h \cos u^{(j)})(\cos u^{(j)} + \cos h) = 0.$$

Moreover, since $\sin h \neq 0$, it follows that $1 - \cos h \cos u^{(j)} \neq 0$, so

$$\cos u^{(j)} = -\cos h \quad \text{for } j = 1, \ldots, d-1$$

at such an u. Hence, by the equation (2.17), $2(2-d)\cos h = e$. If $d=2$, then $e=0$, contrary to our assumption. Thus, the previous situation would occur only when $d>2$, in which case $\cos h = -\frac{e}{2d-4}$ and $\cos u^{(j)} = \frac{e}{2d-4}$ for $j=1,\dots,d-1$. Then, $\cos^2 h = \cos^2 u^{(j)}$ for any j, so $\sin u^{(j)} \sin u^{(k)} = \pm \sin^2 h$. Consequently, the (j,k)-th element of the considered matrix, (2.19), is 0 when $j=k$, but $\pm\frac{e}{2d-4}$ otherwise. This last quantity differs from zero (since $e \notin E$), which provides a contradiction.

In conclusion, at any point in a neighborhood of x_0, $\Gamma(e)$ admits at least one non vanishing principal curvature. Since x_0 is arbitrary, this completes the proof. □

Theorems 21 and 22 thus give a polynomial decay for the Green's function, $G(n,e)$, associated with the standard Laplacian. Without asserting that this decay is (or is not) optimal, it suffices for our applications in the second part of this thesis.

Corollary *Let $E = \{-2d, -2d+4, \dots, 2d-4, 2d\} \cup \{0\}$ and suppose $e \in [-2d, 2d] \setminus E$. Then,*

$$G(n,e) = \lim_{\substack{z \to e \\ z \in \mathbb{C}_+}} G(n,z) = O(|n|^{-\frac{1}{2}})$$

when $|n| \longrightarrow \infty$, uniformly in e on each compact and uniformly in $\omega \in \mathrm{S}^{d-1}$, where $n = |n|\omega$ is the polar form of $n \neq 0$.

Corollary *Suppose $[a,b] \subset [-2d, 2d] \setminus E$ and let*

$$\mathcal{S} = \{e + \mathrm{i}y \; ; \; a < e < b, \, 0 < y < 1\}.$$

Then,

$$G(n,z) = O(|n|^{-\frac{1}{2}^{+}})$$

uniformly in $(z,\omega) \in \overline{\mathcal{S}} \times \mathrm{S}^{d-1}$.

2.5.2 Molchanov–Vainberg Laplacian

In order to avoid convexity problems, Molchanov and Vainberg have suggested to change the discretization of the Laplacian. They have based their construction on the 2^d full-diagonal neighbors of elements in $\mathbb{Z}^d$, instead of their $2d$ immediate neighbors. The constant energy surfaces of the resulting operator are strictly convex in any dimension, as shown below.

Explicitly, the *Molchanov–Vainberg Laplacian* (or *diagonal Laplacian*) is the adjacency operator of the translational invariant graph specified by the following set of points adjacent to the origin:

$$\mathcal{V} = \{(v^{(1)}, \ldots, v^{(d)}) \ ; \ v^{(j)} \in \{1, -1\} \ \text{ for } j = 1, \ldots, d\}.$$

By an elementary combinatorial argument, $n \in \mathbb{Z}^d$ is in the component of the origin if and only if the $n^{(j)}$'s are all even or all odd. Indeed, the considered graph consists of 2^{d-1} connected components with set of representatives

$$\{(0, n^{(2)}, \ldots, n^{(d)}) \ ; \ n^{(j)} \in \{0, 1\} \ \text{ for } j = 2, \ldots, d\}.$$

The graph is also specified by the following metric:

$$\mathrm{d}(m,n) = \begin{cases} |m-n|_\infty & \text{if the components of } m-n \text{ have the same parity,} \\ \infty & \text{otherwise,} \end{cases}$$

where $|n|_\infty = \max_{j=1}^{d} |n^{(j)}|$.

Remarkably, the symbol of the Molchanov–Vainberg Laplacian factorizes:

Theorem 26 $\widehat{\Delta}$ *is the operator of multiplication by* $\Phi(x) = 2^d \cos x^{(1)} \dots \cos x^{(d)}$, *where* $x = (x^{(1)}, \dots, x^{(d)}) \in \mathbb{T}^d$.

Proof: Let us denote by $\{e_1, \dots, e_d\}$ the standard basis of $\mathbb{Z}^d$. By Theorem 24 the symbol of Δ is the multiplication by

$$\begin{aligned} \Phi(x) &= \sum_{v\in\mathcal{V}} \mathrm{e}^{\mathrm{i}x\cdot v} = \sum_{v\in\mathcal{V}} \mathrm{e}^{\mathrm{i}x\cdot\sum_{j=1}^{d} v^{(j)} e_j} = \sum_{v\in\mathcal{V}} \prod_{j=1}^{d} \mathrm{e}^{\mathrm{i}x\cdot v^{(j)} e_j} \\ &= \prod_{j=1}^{d} (\mathrm{e}^{\mathrm{i}x\cdot e_j} + \mathrm{e}^{-\mathrm{i}x\cdot e_j}) = 2^d \prod_{j=1}^{d} \cos x^{(j)}, \end{aligned}$$

as claimed. □

Consequently, the spectrum of Δ is purely absolutely continuous and equal to $[-2^d, 2^d]$. Moreover,

Lemma *Suppose* $0 < |e| < 2^d$. *Then, for all* $x \in \Gamma(e)$, $\nabla_x \Phi(x) \neq 0$. *In particular,* $\Gamma(e)$ *defines a regular surface for such an* e.

Proof: If $x \in \Gamma(e)$, then $\cos x^{(j)} \neq 0$ for $j = 1, \ldots, d$, so $\partial_{x^{(j)}}\Phi(x) = -2^d e \tan x^{(j)}$. Thus, $\|\nabla_x \Phi(x)\|^2 = 4^d e^2 \sum_{j=1}^{d} \tan^2 x^{(j)}$, which differs from 0, since $e \neq \pm 2^d$. $\square$

Let us investigate the constant energy surfaces associated with $\Phi(x)$. Firstly, let us consider the covering $\tilde{\Gamma}(e) = \{x \in \mathbb{R}^d \; ; \; \Phi(x) = e\}$.

If $e = 0$, $\tilde{\Gamma}(e)$ consists of the hyperplanes of equation $x^{(j)} = (2k+1)\frac{\pi}{2}$ for $k \in \mathbb{Z}$ and $j = 1, \ldots, d$. These hyperplanes divide $\mathbb{R}^d$ into open hypercubes, which we call *cells*. The cells admit a good bicoloration in the following sense: starting from a set of two colors, say, red and blue, it is possible to paint each cell in such a way that the $2d$ neighbors of any red cell are blue and *vice versa*. Let us accomplish this, the cell containing the origin being painted in red.

If $e = 2^d$, then $\tilde{\Gamma}(e)$ is a discrete set consisting of the centers of the red cells. On the other hand, if $e = -2^d$, then $\tilde{\Gamma}(e)$ consists of the centers of the blue cells.

When x varies continuously, $\Phi(x)$ changes sign each time one of the previous hyperplanes is crossed. It follows that the connected components of $\tilde{\Gamma}(e)$ are enclosed in the red cells when $e > 0$, each red cell containing one component. Moreover, these components are all congruent. The situation is the same when $e < 0$, but replacing the red cells with the blue ones.

Finally, $\Gamma(e)$ is obtained from the previous surface by restricting $\tilde{\Gamma}(e)$ to the torus, where $e \in \left]-2^d, 2^d\right[\setminus \{0\}$ is fixed. It follows that $\Gamma(e)$ consists of 2^{d-1} identical connected components.

As Molchanov and Vainberg conjectured,

Theorem 27 *For $0 < |e| < 2^d$ any component of $\Gamma(e)$ is strictly convex.*

Proof: Suppose $0 < e < 2^d$, the other case being similar. Then, without loss of generality the considered component is

$$\{x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[^d ;\ 2^d \prod_{j=1}^{d} \cos x^{(j)} = e\}.$$

Let $m = d-1$ and $h = x^{(d)}$. The equation defining the previous component becomes

$$2^d \cos x^{(1)} \ldots \cos x^{(m)} \cos h = e.$$

Since each factor in the above is positive,

$$d \ln 2 + \ln \cos x^{(1)} + \cdots + \ln \cos x^{(m)} + \ln \cos h - \ln e = 0. \tag{2.20}$$

Since the considered component is symmetric with respect to the hyperplanes $x^{(j)} = 0$ and $x^{(j)} = x^{(l)}$, where $j,\, l \in \{1, \ldots, d\}$ are distinct, it suffices to show the result on the fundamental domain $h \leqslant x^{(1)} \leqslant \cdots \leqslant x^{(m)} \leqslant 0$. There, $h \neq 0$ since $e \neq 2^d$.

The derivative with respect to h of the left side in (2.20) is $-\tan h$, which does not vanish. Consequently, an implicit function, $h = \mathrm{h}(x, e)$, satisfying (2.20) in a neighborhood of an arbitrarily fixed point in the fundamental domain exists, is analytic, and induces a local parameterization of the previous component.

It thus suffices to show that $\mathrm{D}_x^2 \mathrm{h}(x, e)$ is positive definite. Differentiating (2.20) with respect to $x^{(j)}$ gives

$$-\tan x^{(j)} - \tan h\, \partial_{x^{(j)}} h = 0,$$

where $h = \mathrm{h}(x, e)$. Differentiating this last equation with respect to $x^{(l)}$ gives

$$\begin{aligned} -\sec^2 x^{(j)} - \sec^2 h\,(\partial_{x^{(j)}} h)^2 - \tan h\,\partial^2_{x^{(j)}} h &= 0 \quad \text{if } l = j, \\ -\sec^2 h\,\partial_{x^{(j)}} h \partial_{x^{(l)}} h - \tan h\,\partial_{x^{(j)}} \partial_{x^{(l)}} h &= 0 \quad \text{otherwise.} \end{aligned}$$

Let $a_{jl} = -\tan h\,\partial_{x^{(j)}}\partial_{x^{(l)}} h$. Since $-\tan h > 0$, it suffices to show $[a_{jl}] > 0$. By the above,

$$[a_{jl}] = \mathrm{diag}(\sec^2 x^{(1)}, \ldots, \sec^2 x^{(d-1)}) + \sec^2 h\,[\partial_{x^{(j)}} h \partial_{x^{(l)}} h].$$

Clearly, the first term of the right-hand side in the previous equation is strictly positive. Moreover, the second term is non negative, since all its principal minors are zero except the first, which is a square. Hence, $[a_{jl}] > 0$, which completes the proof.

$\square$

Theorems 21 and 22 then give an optimal decay for the Green's function, $G(n, e)$, associated with the Molchanov–Vainberg Laplacian. Explicitly,

Corollary *Let $E = \{-2^d, 0, 2^d\}$ and suppose $e \in \left[-2^d, 2^d\right] \setminus E$. Then,*

$$G(n, e) = \lim_{\substack{z \to e \\ z \in \mathbb{C}_+}} G(n, z) = O(|n|^{-\frac{d-1}{2}})$$

when $|n| \to \infty$, uniformly in e on each compact and uniformly in $\omega \in \mathrm{S}^{d-1}$, where $n = |n|\omega$ is the polar form of $n \neq 0$.

Corollary *Suppose* $[a,b] \subset \left[-2^d, 2^d\right]$ *and let*

$$\mathcal{S} = \{x + \mathrm{i}y \; ; \; a < x < b \quad and \quad 0 < y < 1\}.$$

Then,

$$G(n,z) = O(|n|^{-\frac{d-1}{2}^{+}})$$

uniformly in $(z, \omega) \in \overline{\mathcal{S}} \times \mathrm{S}^{d-1}$.

CHAPTER 3
Scattering From Sparse Potentials[1]

3.1 Basics in Scattering Theory

Let $\mathcal{H}$ be a Hilbert space.[2] Given two selfadjoint operators, $A, B \in \mathcal{L}(\mathcal{H})$, and a Borel set, $S \subseteq \mathbb{R}$, the *wave operators* on S are defined as the strong limits

$$\Omega^{\pm}(B, A) = \lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tB} \mathrm{e}^{-\mathrm{i}tA} \mathbf{1}_S(A).$$

In the sequel we suppose that $\Omega \in \{\Omega^+(B, A), \Omega^-(B, A)\}$ exists.

Proposition 28 *Ω is a partial isometry whose initial space is $\mathbf{1}_S(A)\mathcal{H}$.*

Proof: If $\varphi \perp \mathbf{1}_S(A)\mathcal{H}$, then $\mathbf{1}_S(A)\varphi = 0$ and hence $\Omega\varphi = 0$. Since $\mathrm{e}^{\mathrm{i}tB}$ and $\mathrm{e}^{-\mathrm{i}tA}$ are unitaries, the result follows. □

Proposition 29 $\mathrm{e}^{\mathrm{i}sB}\Omega = \Omega \mathrm{e}^{\mathrm{i}sA}$.

Proof: $\mathrm{e}^{\mathrm{i}sB}\Omega \mathrm{e}^{-\mathrm{i}sA} = \lim_t \mathrm{e}^{\mathrm{i}(s+t)B} \mathrm{e}^{-\mathrm{i}(s+t)A} \mathbf{1}_S(A) = \Omega$. □

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[1] For a detailed description of results, terminology, and notations used in this chapter, the reader is invited to consult Appendix 4.2.

[2] In this thesis all Hilbert spaces under consideration are separable. Also, given a Hilbert space, $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ denotes the set of bounded linear operators on $\mathcal{H}$, while $\mathcal{L}(\mathcal{H})$ denotes the set of all linear operators on $\mathcal{H}$, bounded or unbounded.

Indeed, using Stone's theorem one obtains:

Proposition 30 $B\Omega = \Omega A$. *In other words,* $\Omega\psi \in \operatorname{dom} B$ *iff* $\psi \in \operatorname{dom} A$ *and for any such* ψ, $B\Omega\psi = \Omega A\psi$.

Proof: Let $\psi \in \operatorname{dom} A$, so $\lim_{s\to 0} \frac{e^{isA}\psi - \psi}{s} = iA\psi$. Since Ω is bounded,

$$\begin{aligned} i\Omega A\psi &= \lim_{s\to 0} \frac{\Omega e^{isA}\psi - \Omega\psi}{s} \\ &= \lim_{s\to 0} \frac{e^{isB}\Omega\psi - \Omega\psi}{s} \\ &= iB\Omega\psi \end{aligned}$$

These last relations also hold for $\Omega\psi \in \operatorname{dom} B$; thus, $\Omega A = B\Omega$. □

In particular, the isometry $\mathbf{1}_S(A)\mathcal{H} \xrightarrow{\Omega} \operatorname{ran}\Omega$ provides an identification between the restrictions $\mathbf{1}_S(A)\mathcal{H} \xrightarrow{A\restriction} \mathbf{1}_S(A)\mathcal{H}$ and $\operatorname{ran}\Omega \xrightarrow{B\restriction} \operatorname{ran}\Omega$:[3]

$$\begin{array}{ccc} \mathbf{1}_S(A)\mathcal{H} & \stackrel{A\restriction}{\rightarrow} & \mathbf{1}_S(A)\mathcal{H} \\ \Omega\downarrow & & \downarrow\Omega \\ \operatorname{ran}\Omega & \stackrel{B\restriction}{\rightarrow} & \operatorname{ran}\Omega \end{array}$$

Notice that $\operatorname{ran}\Omega$ is closed, since $\mathbf{1}_S(A)\mathcal{H}$ is.

Proposition 31 $\operatorname{ran}\Omega \subseteq \mathbf{1}_S(B)\mathcal{H}$.

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[3] Except if $S \subseteq \mathbb{R}$ is bounded, we do not pretend by the following picture that the domains of the identified restrictions are full, but only that they coincide via Ω.

Proof: Trivially $\mathbf{1}_S(A\lceil)(A\lceil -z)^{-1} = (A\lceil -z)^{-1}$ for any $z \notin \mathbb{R}$. Since $A\lceil$ and $B\lceil$ are unitarily equivalent, the functional calculus gives

$$\mathbf{1}_S(B\lceil)(B\lceil -z)^{-1} = (B\lceil -z)^{-1}.$$

In other words, the spectral measure of any $\varphi \in \operatorname{ran}\Omega$ with respect to $B\lceil$ is concentrated on S. The result follows. □

The following proposition is known as the *chain rule*:

Proposition 32 *Let $A, B, C \in \mathcal{L}(\mathcal{H})$ be selfadjoints operators. Then,*

$$\Omega^{\pm}(C,B)\Omega^{\pm}(B,A) = \Omega^{\pm}(C,A),$$

provided that these wave operators exist.

Proof: By the previous lemma, for all $\varphi \in \mathcal{H}$

$$0 = (1 - \mathbf{1}_S(B))\Omega^{\pm}(B,A)\varphi = \lim_{t\to\pm\infty}(1 - \mathbf{1}_S(B))\mathrm{e}^{\mathrm{i}tB}\mathrm{e}^{-\mathrm{i}tA}\mathbf{1}_S(A)\varphi$$

strongly. Writing $\mathrm{e}^{\mathrm{i}tC}\mathrm{e}^{-\mathrm{i}tA}\mathbf{1}_S(A)\varphi$ as

$$\mathrm{e}^{\mathrm{i}tC}\mathrm{e}^{-\mathrm{i}tB}(1 - \mathbf{1}_S(B))\mathrm{e}^{\mathrm{i}tB}\mathrm{e}^{-\mathrm{i}tA}\mathbf{1}_S(A)\varphi + \mathrm{e}^{\mathrm{i}tC}\mathrm{e}^{-\mathrm{i}tB}\mathbf{1}_S(B)\mathrm{e}^{\mathrm{i}tB}\mathrm{e}^{-\mathrm{i}tA}\mathbf{1}_S(A)\varphi$$

and taking the strong limit of both expressions, the previous relation yields

$$\Omega^{\pm}(C,A)\varphi = \Omega^{\pm}(C,B)\Omega^{\pm}(B,A)\varphi.$$

□

In connection with the penultimate proposition the wave operator Ω is said to be *complete* if $\operatorname{ran}\Omega = \mathbf{1}_S(B)\mathcal{H}$. Scattering theory is concerned with existence and completeness of wave operators. The following proposition shows that these problems are *formally* equivalent.

Proposition 33 *$\Omega = \Omega^{\pm}(B, A)$ is complete iff $\Omega^{\pm}(A, B)$ exists.*

Proof: Suppose Ω is complete and let $\varphi \in \mathcal{H}$. By hypothesis there exists a $\psi \in \mathcal{H}$ such that $\mathbf{1}_S(B)\varphi = \Omega\psi$. Hence,

$$0 = \lim_t \|\mathbf{1}_S(B)\varphi - \mathrm{e}^{\mathrm{i}tB}\mathrm{e}^{-\mathrm{i}tA}\mathbf{1}_S(A)\psi\| = \lim_t \|\mathrm{e}^{\mathrm{i}tA}\mathrm{e}^{-\mathrm{i}tB}\mathbf{1}_S(B)\varphi - \mathbf{1}_S(A)\psi\|,$$

so $\Omega^{\pm}(A, B)$ exists.

Conversely, suppose $\Omega^{\pm}(A, B)$ exists. It suffices to show that $\operatorname{ran}\mathbf{1}_S(B) \subseteq \operatorname{ran}\Omega$. By the chain rule $\mathbf{1}_S(B) = \Omega^{\pm}(B, B) = \Omega\,\Omega^{\pm}(A, B)$, from which the result follows. □

Let us indicate explicitly that Ω depends on S:

$$\Omega_S^{\pm} = \Omega_S^{\pm}(B, A) = \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tB}\mathrm{e}^{-\mathrm{i}tA}\mathbf{1}_S(A) \quad \text{(strongly)}.$$

Proposition 34 *Suppose $S_N \nearrow S$, i.e., $S_1 \subseteq S_2 \subseteq \cdots$ and $\bigcup_N S_N = S$. If every $\Omega_{S_N}^{\pm}$ exists, then $\Omega_S^{\pm}$ also exists.*

Proof: Let $\varphi \in \mathcal{H}$ be arbitrarily fixed. From $S \setminus S_N \searrow \emptyset$, it follows that $\mathbf{1}_{S_N}(A)\varphi \to \mathbf{1}_S(A)\varphi$ and hence

$$\mathrm{e}^{\mathrm{i}tB}\mathrm{e}^{-\mathrm{i}tA}\mathbf{1}_{S_N}(A)\varphi \to \mathrm{e}^{\mathrm{i}tB}\mathrm{e}^{-\mathrm{i}tA}\mathbf{1}_S(A)\varphi \tag{3.1}$$

uniformly in t. Hence, since the left-hand side of (3.1) is Cauchy in t, the right-hand side is also Cauchy. The result follows. $\square$

Let S, $T \subset \mathbb{R}$ be Borel sets. Suppose $T \subseteq S$. If $\Omega_S^\pm$ exists, then $\Omega_T^\pm$ also exists, for $\Omega_T^\pm = \Omega_S^\pm \mathbf{1}_T(A)$. Suppose instead $T \cap S = \emptyset$. If $\Omega_S^\pm$ and $\Omega_T^\pm$ both exist, then $\Omega_{S \sqcup T}^\pm$ also exists, for $\Omega_{S \sqcup T}^\pm = \Omega_S^\pm + \Omega_T^\pm$. Finally, suppose S and T are arbitrary. If $\Omega_S^\pm$ and $\Omega_T^\pm$ both exist, then $\Omega_{S \cup T}^\pm$ also exists; this follows from our previous considerations, writing

$$S \cup T = (S \cap (\mathbb{R} \setminus T)) \sqcup (S \cap T) \sqcup (T \cap (\mathbb{R} \setminus S)).$$

Of course, the previous result may be generalized to finite unions by induction. Hence,

Corollary *Suppose $S = \bigcup_{n=1}^{\infty} S_n$. If every $\Omega_{S_n}^\pm$ exists, then $\Omega_S^\pm$ also exists.*

Proof: We have just seen that the wave operators exist on $\bigcup_{n=1}^{N} S_n$ for every $N \geqslant 1$. The result follows, since $\bigcup_{n=1}^{N} S_n \nearrow S$ when $N \to \infty$. $\square$

3.2 First Criterion of Completeness of Wave Operators

In this section we establish a sufficient condition for the existence and completeness of wave operators coming from Schrödinger operators on graphs. This criterion, due to Jakšić and Last, is based on Kato's smooth perturbation theory.

The setting is the following (for more details see Section 2.5): we consider a simple graph, (X, d), having countably many vertices, whose degrees are assumed to be bounded. Here, X denotes the set of vertices of the graph, while $\mathrm{d}(m, n)$ denotes the distance between m, $n \in X$, that is, the length of the shortest path connecting them in X (∞ if m and n lie on two different components). The *adjacency operator*, Δ, on X is an operator acting on $\mathcal{H} = l^2(X)$ as follows: for $\varphi \in l^2(X)$

$$\Delta\varphi(n) = \sum_{\mathrm{d}(m,n)=1} \varphi(m).$$

Let $\Gamma \subseteq X$ and $V \colon \Gamma \to \mathbb{R}$ be given. We interpret V as a potential supported on Γ and study Hamiltonians of the form

$$H = \Delta + \sum_{n\in\Gamma} V(n)\langle\delta_n \,|\, \cdot\rangle\delta_n,$$

where $\{\delta_n\}_{n\in\mathbb{Z}^d}$ is the standard orthonormal basis of $\mathcal{H} = l^2(X)$.

We set $H_0 = \Delta$ and $V = H - H_0$, so $H = H_0 + V$. *However, the conscientious reader will notice that H_0 and H may be reversed in the present section without affecting the results.* This important notice will be used in the corollary of the main theorem.

We denote by Γ_R the *R-fattening* of Γ:

$$\Gamma_R = \{n \in X \ ; \ \mathrm{d}(n, \Gamma) \leqslant R\}.$$

The projection on $l^2(\Gamma_R)$ is denoted by $\mathbf{1}_R$, while the projection on its orthogonal complement is denoted by $\mathbf{1}_{\bar{R}}$.

We denote by $\mathcal{K}$ be the Hilbert subspace cyclically generated by $\{\delta_n \ ; \ n \in \Gamma\}$ with respect to H_0, which is clearly invariant for H.[4] Since H is selfadjoint, $\mathcal{K}^\perp$ is also H-invariant. In particular, for any bounded Borel function, f, the projections on $\mathcal{K}$ and $\mathcal{K}^\perp$ commute with the calculus:

$$f(H \restriction \mathcal{K}) = f(H) \restriction \mathcal{K}, \quad f(H \restriction \mathcal{K}^\perp) = f(H) \restriction \mathcal{K}^\perp.$$

Notice that $\mathcal{K}^\perp$ is included in the Hilbert space generated by $\{\delta_n \ ; \ n \notin \Gamma\}$, which implies

$$H \restriction \mathcal{K}^\perp = H_0 \restriction \mathcal{K}^\perp.$$

In particular, for any $t \in \mathbb{R}$

$$\mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH} = \mathrm{e}^{\mathrm{i}tH_0 \restriction \mathcal{K}}\mathrm{e}^{-\mathrm{i}tH \restriction \mathcal{K}} \oplus 1$$

with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$. Thus, when proving the existence or completeness of $\Omega^\pm(H, H_0)$, it suffices to restrict our considerations to $\mathcal{K}$. The same argument yields a similar conclusion when replacing $\mathcal{K}$ with $\mathcal{K}_1$, where the latter denotes the Hilbert subspace cyclically generated by $\{\delta_n \ ; \ n \in \Gamma_1\}$ with respect to H_0.

In order to state our main result we need the following, abstract definition. Given a Hilbert space, $\mathcal{H}$, a selfadjoint operator, $A \in \mathcal{L}(\mathcal{H})$, a bounded operator, $B \in \mathcal{B}(\mathcal{H})$, and a Borel set, $U \subseteq \mathbb{R}$, B is *A-smooth* on U if there exists a $C > 0$ such

[4] In fact, using perturbation theory, Jakšić and Last [19] observed that this space is equal to the subspace cyclically generated by $\{\delta_n \ ; \ n \in \Gamma\}$, but with respect to H.

that for all $\varphi \in \mathbf{1}_U(A)\mathcal{H}$

$$\int_{\mathbb{R}} \|B\mathrm{e}^{-\mathrm{i}tA}\varphi\|^2 \,\mathrm{d}t \leqslant C\|\varphi\|^2.$$

The following characterization is also used in the sequel (see [38]):

Theorem 35 *In the above circumstances, B is A-smooth on U iff*

$$\sup_{\substack{0<\varepsilon<1 \\ e\in U}} \|B(A-e-\mathrm{i}\varepsilon)^{-1}B^*\| < \infty.$$

Remark *In fact, by continuity of the resolvent B is then A-smooth on $\overline{U}$.*

In the sequel $U \subseteq \mathbb{R}$ denotes a fixed *open* set. Let

$$\mathcal{D} = \left\{\varphi \in \operatorname{ran} \mathbf{1}_U(H) \ ; \ \int_{\mathbb{R}} \|\mathbf{1}_1\mathrm{e}^{-\mathrm{i}tH}\varphi\|^2 \,\mathrm{d}t < \infty\right\},$$

so $\mathcal{D}$ is a vector space, not necessarily closed, but satisfying the following invariance properties: clearly, $\mathcal{D}$ is invariant for $\mathrm{e}^{-\mathrm{i}sH}$ for any $s \in \mathbb{R}$. Moreover,

Lemma *Given an $f \in L^1(\mathbb{R})$, let us denote its Fourier transform by*

$$\widehat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x) \,\mathrm{d}x.$$

Let $\varphi \in \mathcal{D}$ and $C = \int_{\mathbb{R}} \|\mathbf{1}_1\mathrm{e}^{-\mathrm{i}tH}\varphi\|^2 \,\mathrm{d}t$, which is thus finite. Then,

$$\int_{\mathbb{R}} \|\mathbf{1}_1\mathrm{e}^{-\mathrm{i}tH}\widehat{f}(H)\varphi\|^2 \,\mathrm{d}t \leqslant \frac{C}{2\pi}\|f\|_1.$$

In particular, $\mathcal{D}$ is invariant for $\widehat{f}(H)$.

Proof: By Jensen's inequality

$$\begin{aligned}
\|\mathbf{1}_1 e^{-itH}\widehat{f}(H)\varphi\|^2 &= \frac{1}{2\pi}\left\|\int_{\mathbb{R}} \mathbf{1}_1 e^{-itH} e^{-ixH} f(x)\varphi \, dx\right\|^2 \\
&\leqslant \frac{1}{2\pi}\|f\|_1^2 \left(\int_{\mathbb{R}} \|\mathbf{1}_1 e^{-itH} e^{-ixH}\varphi\| \frac{|f(x)|\, dx}{\|f\|_1}\right)^2 \\
&\leqslant \frac{1}{2\pi}\|f\|_1^2 \int_{\mathbb{R}} \|\mathbf{1}_1 e^{-i(t+x)H}\varphi\|^2 \frac{|f(x)|\, dx}{\|f\|_1}.
\end{aligned}$$

Hence, by Tonelli's theorem

$$\int_{\mathbb{R}} \|\mathbf{1}_1 e^{-itH}\widehat{f}(H)\varphi\|^2 \, dt \leqslant \frac{1}{2\pi}\|f\|_1^2 \int_{\mathbb{R}} \|\mathbf{1}_1 e^{-itH}\varphi\|^2 \, dt,$$

as claimed. □

Corollary *With φ and C as above, let $z = e + iy \notin \mathbb{R}$. Then,*

$$\int_{\mathbb{R}} \|\mathbf{1}_1 e^{-itH}(H-z)^{-1}\varphi\|^2 \, dt \leqslant \frac{C}{y^2}.$$

In particular, $\mathcal{D}$ is invariant for $(H-z)^{-1}$.

Proof: By Kato's formula, for a fixed $z \in \mathbb{C}_+$

$$\begin{aligned}
(H-z)^{-1} &= i\int_0^\infty e^{-ix(H-z)} \, dx \\
&= \sqrt{2\pi} i \, \mathcal{F}(\mathbf{1}_{]0,\infty[}(x) e^{ixz})(H).
\end{aligned}$$

Since $\|\mathbf{1}_{]0,\infty[}(x)\mathrm{e}^{\mathrm{i}xz}\|_1 = \frac{1}{y}$, the result follows from the lemma. Similarly, for a fixed $z \in \mathbb{C}_-$

$$\begin{aligned}(H-z)^{-1} &= -\mathrm{i}\int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}x(H-z)}\,\mathrm{d}x \\ &= -\sqrt{2\pi}\mathrm{i}\,\mathcal{F}(\mathrm{e}^{\mathrm{i}xz}\mathbf{1}_{]-\infty,0[}(x))(H),\end{aligned}$$

and again the result follows. □

Scholium *The previous corollary has an interesting consequence: if $\varphi \in \mathcal{D}$, then for any $y > 0$*

$$\int_{\mathbb{R}} \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}tH} y(H - e - \mathrm{i}y)^{-1}\varphi\|^2 \leqslant C.$$

Letting $y \downarrow 0$, Fatou's lemma yields

$$\int_{\mathbb{R}} \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}tH}\mathrm{i}\mathbf{1}_{\{e\}}(H)\varphi\|^2\,\mathrm{d}t \leqslant C,$$

that is, $\int_{\mathbb{R}} \|\mathbf{1}_1\mathbf{1}_{\{e\}}(H)\varphi\|^2\,\mathrm{d}t < \infty$. Hence, $\mathbf{1}_1\mathbf{1}_{\{e\}}(H)\varphi = 0$ for all $\varphi \in \mathcal{D}$. In particular, restricting our attention to the Hilbert subspace $\mathcal{K}$,

$$\mathbf{1}_1\mathbf{1}_{\{e\}}(H \restriction \mathcal{K})\varphi = 0$$

for all $\varphi \in \mathcal{D} \cap \mathcal{K}$.

Let $n \in \Gamma$ and $z \notin \mathbb{R}$, so $(H-z)^{-1}\delta_n$ is a typical generator of $\mathcal{K}$. Then, for any $\varphi \in \mathcal{D} \cap \mathcal{K}$ the previous relation yields

$$\langle (H-z)^{-1}\delta_n \mid \mathbf{1}_{\{e\}}(H)\varphi\rangle = \frac{1}{e-\bar{z}}\langle \mathbf{1}_1\delta_n \mid \mathbf{1}_{\{e\}}(H)\varphi\rangle = 0,$$

which implies $\mathbf{1}_{\{e\}}(H \restriction \mathcal{K})\varphi = 0$ *for any* $\varphi \in \mathcal{D} \cap \mathcal{K}$. *In particular,* if $\mathcal{D} \cap \mathcal{K}$ is dense in $\mathcal{K}$, then the spectrum of $H \restriction \mathcal{K}$ is purely continuous. *The same argument yields a similar conclusion for* $H \restriction \mathcal{K}_1$.

Our main theorem states that in the present setting *if* $\mathbf{1}_1$ *is* H_0*-smooth on* U *and* $\mathcal{D}$ *is dense in* $\operatorname{ran} \mathbf{1}_U(H)$, *then the wave operators* $\Omega^{\pm}(H_0, H)$ *exist on* U. Since in this context the usual wave operators are $\Omega^{\pm}(H, H_0)$, we establish their completeness, but without assuming their existence. The proof is preceded by several lemmas, which are shown under the same assumptions as the main theorem, namely:

Assumption D

- $\mathbf{1}_1$ *is* H_0*-smooth on* U,
- $\mathcal{D}$ *is dense in* $\operatorname{ran} \mathbf{1}_U(H)$.

Let $T = [H_0, \mathbf{1}_{\bar{0}}]$. The following, trivial properties are useful. Observe how H and H_0 may be interchanged in their statement:

Lemma

- $\mathbf{1}_{\bar{0}}H = \mathbf{1}_{\bar{0}}H_0$, $H\mathbf{1}_{\bar{0}} = H_0\mathbf{1}_{\bar{0}}$, *and hence* $[H, \mathbf{1}_{\bar{0}}] = T = [H_0, \mathbf{1}_{\bar{0}}]$.
- *Therefore,* $[H, \mathbf{1}_0] = [H_0, \mathbf{1}_0]$.
- $T = \mathbf{1}_1 T = T\mathbf{1}_1 = \mathbf{1}_1 T \mathbf{1}_1$ *and* $\|T\| \leqslant 2\|H_0\|$.

Proof: By direct computations. $\square$

Finally, the following computational tool is frequently invoked:

Proposition 36 *Let $\mathcal{B}$ be a Banach space and $\mathbb{R} \xrightarrow{f} \mathcal{B}$ be a continuous, strongly differentiable function whose derivative is uniformly bounded:*

$$\exists M : \forall t \in \mathbb{R} : \|f'(t)\| < M.$$

Suppose moreover that $f \in L^p(\mathbb{R})$, that is

$$\int_{\mathbb{R}} \|f(t)\|^p \, \mathrm{d}t < \infty$$

for a given $0 < p < \infty$. Then, $\lim_{t\to\pm\infty} f(t) = 0$.

Proof: Suppose by contradiction $\limsup_{t\to\infty} \|f(t)\| > 0$ (the case where $t \to -\infty$ being similar). Then, there exists an $\varepsilon \in]0, M[$ and an increasing sequence, $t_n \to \infty$, satisfying $t_{n+1} - t_n > 1$ and $\|f(t_n)\| \geqslant 2\varepsilon$ for all n. Let $s_n = t_n + \frac{\varepsilon}{M}$, so the intervals $]t_n, s_n[$ do not intersect. Notice that the assumption $\|f'(t)\| < M$ and the mean value theorem imply $\|f(t) - f(s)\| \leqslant M|t - s|$ for all $s, t \in \mathbb{R}$. In particular, for $t \in]t_n, s_n[$

$$\begin{aligned} \|f(t)\| &\geqslant \|f(t_n)\| - \|f(t_n) - f(t)\| \\ &\geqslant 2\varepsilon - M|t - t_n| \\ &\geqslant \varepsilon. \end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}} \|f(t)\|^p \,\mathrm{d}t &\geqslant \sum_n \int_{t_n}^{s_n} \|f(t)\|^p \,\mathrm{d}t \\ &\geqslant \varepsilon^p \sum_n (s_n - t_n) \\ &= \sum_n \frac{\varepsilon^{p+1}}{M} \\ &= \infty,\end{aligned}$$

a contradiction. The result follows. $\square$

We now come to our succession of lemmas:

Lemma *For all* $\varphi \in \mathbf{1}_U(H)\mathcal{H}$, $\lim_{t\to\pm\infty} \mathbf{1}_0 \mathrm{e}^{-\mathrm{i}tH}\varphi = 0$.

Proof: Let $f(t) = \mathrm{e}^{\mathrm{i}tH}\mathbf{1}_0\mathrm{e}^{-\mathrm{i}tH}\varphi$. $f(t)$ is continuous, since $\|f(t+h) - f(t)\| \leqslant \|\mathrm{e}^{\mathrm{i}hH}\mathbf{1}_0\mathrm{e}^{-\mathrm{i}hH} - \mathbf{1}_0\| \, \|\varphi\| \xrightarrow{h\to 0} 0$. Furthermore, $f(t)$ is square integrable, since by Assumption D

$$\int_{\mathbb{R}} \|f(t)\|^2 \,\mathrm{d}t = \int_{\mathbb{R}} \|\mathbf{1}_0 \mathrm{e}^{-\mathrm{i}tH}\varphi\|^2 \,\mathrm{d}t \leqslant \int_{\mathbb{R}} \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}tH}\varphi\|^2 \,\mathrm{d}t < \mathit{Const}\, \|\varphi\|^2.$$

Finally, $f'(t) = \mathrm{i}\mathrm{e}^{\mathrm{i}tH}[H, \mathbf{1}_0]\mathrm{e}^{-\mathrm{i}tH}\varphi = \mathrm{i}\mathrm{e}^{\mathrm{i}tH}[H_0, \mathbf{1}_0]\mathrm{e}^{-\mathrm{i}tH}\varphi$, which implies that $\|f'(t)\| \leqslant 2\|H_0\|$. The previous proposition then completes the proof. $\square$

The proof of the following lemma uses Dunford's calculus, which specializes as follows. Let γ be positively oriented Jordan curve in $\mathbb{C}$, whose interior is denoted by

Θ. Consider any closed operator, $F \in \mathcal{L}(\mathcal{H})$, whose spectrum does not intersect γ. Then,

$$\mathbf{1}_\Theta(F) = -\frac{1}{2\pi\mathrm{i}}\oint_\gamma (F - z)^{-1}\,\mathrm{d}z.$$

In particular, for *any* selfadjoint operator $A \in \mathcal{L}(\mathcal{H})$ and Borel set B whose closure is in Θ the trivial inclusion $\mathrm{spec}(\mathbf{1}_B(A)A) \subset \Theta$ implies

$$\begin{aligned}
\mathbf{1}_B(A) &= \mathbf{1}_B(A)\mathbf{1}_\Theta(\mathbf{1}_B(A)A) \\
&= \mathbf{1}_B(A)\left(-\frac{1}{2\pi\mathrm{i}}\oint_\gamma (\mathbf{1}_B(A)A - z)^{-1}\,\mathrm{d}z\right) \\
&= -\frac{1}{2\pi\mathrm{i}}\oint_\gamma \mathbf{1}_B(A)(\mathbf{1}_B(A)A - z)^{-1}\,\mathrm{d}z \\
&= -\frac{1}{2\pi\mathrm{i}}\oint_\gamma (A - z)^{-1}\mathbf{1}_B(A)\,\mathrm{d}z.
\end{aligned}$$

Thus, if $\varphi \in \mathbf{1}_B(A)\mathcal{H}$, then $\varphi = -\frac{1}{2\pi\mathrm{i}}\oint_\gamma (A - z)^{-1}\varphi\,\mathrm{d}z$.

On the other hand, let us denote the exterior of γ by Θ'. For any Borel set B' whose closure is inside Θ', the trivial inclusion $\mathrm{spec}(\mathbf{1}_{B'}(A)A) \subset \Theta'$ implies

$$\begin{aligned}
0 &= \mathbf{1}_{B'}(A)\mathbf{1}_\Theta(\mathbf{1}_{B'}(A)A) \\
&= \mathbf{1}_{B'}(A)\left(-\frac{1}{2\pi\mathrm{i}}\oint_\gamma (\mathbf{1}_{B'}(A)A - z)^{-1}\,\mathrm{d}z\right) \\
&= -\frac{1}{2\pi\mathrm{i}}\oint_\gamma \mathbf{1}_{B'}(A)(\mathbf{1}_{B'}(A)A - z)^{-1}\,\mathrm{d}z \\
&= -\frac{1}{2\pi\mathrm{i}}\oint_\gamma (A - z)^{-1}\mathbf{1}_{B'}(A)\,\mathrm{d}z.
\end{aligned}$$

Thus, if $\varphi' \in \mathbf{1}_{B'}(A)\mathcal{H}$, then $-\frac{1}{2\pi\mathrm{i}}\oint_\gamma (A - z)^{-1}\varphi'\,\mathrm{d}z = 0$.

In the following lemma we denote $U^{\mathrm{c}} = \mathbb{R} \setminus U$.

Lemma $\lim_{t\to\pm\infty} \mathbf{1}_{U^c}(H_0)\mathrm{e}^{\mathrm{i}tH_0}\mathbf{1}_{\bar{0}}\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_U(H) = 0$ *strongly.*

Proof: Let $I \subset U$ be a finite closed interval whose endpoints are not eigenvalues of H and whose opening is not empty. Let γ be a positively oriented Jordan curve in $\mathbb{C}$ separating I from U^c (in $\mathbb{R}$). Given a vector $\psi \in \mathcal{D}$, let us consider $\varphi(t) = \mathbf{1}_{U^c}(H_0)\mathrm{e}^{\mathrm{i}tH_0}\mathbf{1}_{\bar{0}}\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_I(H)\psi$. We first show $\lim_{t\to\pm\infty}\varphi(t) = 0$ and then conclude the proof by a limiting argument.

Since w.l.o.g. U^c is outside γ and $\varphi(t) \in \mathbf{1}_{U^c}(H_0)\mathcal{H}$, the previous discussion gives $0 = -\frac{1}{2\pi\mathrm{i}}\oint_\gamma (H_0 - z)^{-1}\varphi(t)\,\mathrm{d}z$, so

$$0 = -\frac{1}{2\pi\mathrm{i}}\oint_\gamma \mathbf{1}_{U^c}(H_0)\mathrm{e}^{\mathrm{i}tH_0}(H_0 - z)^{-1}\mathbf{1}_{\bar{0}}\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_I(H)\psi\,\mathrm{d}z.$$

Furthermore, since I is inside γ and $\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_I(H)\psi \in \mathbf{1}_I(H)\mathcal{H}$, the same discussion gives $\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_I(H)\psi = -\frac{1}{2\pi\mathrm{i}}\oint_\gamma (H - z)^{-1}\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_I(H)\psi\,\mathrm{d}z$, so

$$\varphi(t) = -\frac{1}{2\pi\mathrm{i}}\oint_\gamma \mathbf{1}_{U^c}(H_0)\mathrm{e}^{\mathrm{i}tH_0}\mathbf{1}_{\bar{0}}(H - z)^{-1}\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_I(H)\psi\,\mathrm{d}z.$$

Subtracting both equations,

$$\varphi(t) = \frac{1}{2\pi\mathrm{i}}\oint_\gamma \mathbf{1}_{U^c}(H_0)\mathrm{e}^{\mathrm{i}tH_0}((H_0 - z)^{-1}\mathbf{1}_{\bar{0}} - \mathbf{1}_{\bar{0}}(H - z)^{-1})\mathrm{e}^{-\mathrm{i}tH}\mathbf{1}_I(H)\psi\,\mathrm{d}z.$$

By the resolvent identity

$$\begin{aligned}(H_0 - z)^{-1}\mathbf{1}_{\bar{0}} - \mathbf{1}_{\bar{0}}(H - z)^{-1} &= (H_0 - z)^{-1}(\mathbf{1}_{\bar{0}}H - H_0\mathbf{1}_{\bar{0}})(H - z)^{-1} \\ &= -(H_0 - z)^{-1}T(H - z)^{-1}.\end{aligned}$$

Thus,

$$\varphi(t) = -\frac{1}{2\pi \mathrm{i}} \oint_\gamma \mathbf{1}_{U^c}(H_0) \mathrm{e}^{\mathrm{i}tH_0}(H_0 - z)^{-1} T (H - z)^{-1} \mathrm{e}^{-\mathrm{i}tH} \mathbf{1}_I(H)\psi \, \mathrm{d}z.$$

Since $\mathbf{1}_{U^c}(H_0)\mathrm{e}^{\mathrm{i}tH_0}(H_0 - z)^{-1}$ is bounded uniformly in t, $\|T\| \leqslant 2\|H_0\|$, and $T = T\mathbf{1}_1$, it follows that

$$\|\varphi(t)\| \leqslant Const \oint_\gamma \|\mathbf{1}_1(H - z)^{-1} \mathrm{e}^{-\mathrm{i}tH} \mathbf{1}_I(H)\psi\| \, \mathrm{d}z.$$

By the dominated convergence theorem, in order to establish $\lim_t \varphi(t) = 0$ it suffices to show that the integrand in this last expression tends to zero.

For a fixed $z \in \gamma$ let

$$B_z = \|\mathbf{1}_1(H - z)^{-1} \mathrm{e}^{-\mathrm{i}tH} \mathbf{1}_I(H)\|,$$

which is finite. There exists a bounded sequence of C^∞ functions supported in I that converges pointwise to $\mathbf{1}_{I^\circ}(x)$, where I° denotes the opening of I. Notice that $\mathbf{1}_{I^\circ}(H) = \mathbf{1}_I(H)$, since the endpoints of I are not eigenvalues of H. Hence, for any $\varepsilon > 0$ there exists a $g(x)$ in the previous sequence satisfying

$$\|(\mathbf{1}_I(H) - g(H))\psi\| < \frac{\varepsilon}{B_z},$$

where $z \in \gamma$ is still fixed. Since $g(H) = \mathbf{1}_I(H)g(H)$, it follows that

$$\|\mathbf{1}_1(H - z)^{-1} \mathrm{e}^{-\mathrm{i}tH}(\mathbf{1}_I(H) - g(H))\psi\| < \varepsilon.$$

Consequently, the integrand under consideration satisfies the following relation:

$$\|\mathbf{1}_1(H - z)^{-1} \mathrm{e}^{-\mathrm{i}tH} \mathbf{1}_I(H)\psi\| < \varepsilon + \|F_z(t)\|,$$

where $F_z(t) = \mathbf{1}_1(H - z)^{-1}\mathrm{e}^{-\mathrm{i}tH}g(H)\psi$. Since ε is arbitrarily small, the problem reduces to show $\lim_t F_z(t) = 0$.

This last fact follows from the preliminary proposition: $F_z(t)$ is clearly continuous. Furthermore, it is square integrable, since $\psi \in \mathcal{D}$ and g is a smooth, compactly supported function, so the invariance properties of $\mathcal{D}$ imply $(H-z)^{-1}g(H)\psi \in \mathcal{D}$.[5] Finally, $F_z'(t) = \mathbf{1}_1(H - z)^{-1}(-\mathrm{i}H)\mathrm{e}^{-\mathrm{i}tH}g(H)\psi$, which is clearly bounded, since $Hg(H)$ is bounded. Thus, $F_z(t)$, and hence $\varphi(t)$ tend to zero.

Let $B(t) = \mathbf{1}_{U^c}(H_0)\mathrm{e}^{\mathrm{i}tH_0}\mathbf{1}_{\tilde{0}}\mathrm{e}^{-\mathrm{i}tH}$, which is uniformly bounded. We then have to prove that $B(t) \xrightarrow{t} 0$ strongly. Since ψ is arbitrarily fixed in $\mathcal{D}$, which is dense in $\operatorname{ran}\mathbf{1}_U(H)$, the above shows that for every finite, closed interval $I \subset U$ whose endpoints are not eigenvalues of H, $B(t)\mathbf{1}_I(H) \xrightarrow{t} 0$ strongly. As an immediate consequence, if $F \subset U$ is a finite union of disjoint closed intervals whose endpoints are not eigenvalues of H, $B(t)\mathbf{1}_F(H) \xrightarrow{t} 0$ strongly. Since U is open and the set of eigenvalues of H is countable, U is approachable by such F's, say $F_1 \subset F_2 \subset \cdots$ with $\bigcup_n F_n = U$. Then, $\mathbf{1}_{F_n}(H) \xrightarrow{n} \mathbf{1}_U(H)$ strongly, so $B(t)\mathbf{1}_{F_n}(H) \xrightarrow{n} B(t)\mathbf{1}_U(H)$ strongly, uniformly in t. These facts imply that $B(t)\mathbf{1}_U(H) \xrightarrow{t} 0$ strongly, which completes the proof. $\square$

We now come to the announced result, asserting in some sense the "completeness" of the usual wave operators, $\Omega^{\pm}(H, H_0)$, but without assuming their existence.

[5] Recall that a smooth, compactly supported function is the Fourier transform of an L^1 function (indeed, of a Schwartz' rapidly decreasing function).

Theorem 37 (Jakšić–Last) *Under Assumption D, $\Omega^{\pm}(H_0, H)$ exist on U.*

Proof: Let $\varphi \in \mathcal{D}$. Since $\mathcal{D}$ is dense and $\varphi \in \mathcal{D}$ is arbitrary, it suffices to establish the existence of $\lim_t \mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH}\varphi$. By the last two lemmas

$$\lim_t \mathrm{e}^{\mathrm{i}tH_0}\mathbf{1}_0\mathrm{e}^{-\mathrm{i}tH}\varphi = 0 \quad \text{and}$$

$$\lim_t \mathbf{1}_{U^c}(H_0)\mathrm{e}^{\mathrm{i}tH_0}\mathbf{1}_{\bar{0}}\mathrm{e}^{-\mathrm{i}tH}\varphi = 0.$$

Letting $\zeta(t) = \mathbf{1}_U(H_0)\mathrm{e}^{\mathrm{i}tH_0}\mathbf{1}_{\bar{0}}\mathrm{e}^{-\mathrm{i}tH}\varphi$, it thus suffices to show that $\lim_{t\to\pm\infty}\zeta(t)$ both exist.

For $\psi \in \mathcal{H}$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi\,|\,\zeta(t)\rangle &= \mathrm{i}\langle\psi\,|\,\mathbf{1}_U(H_0)\mathrm{e}^{\mathrm{i}tH_0}(H_0\mathbf{1}_{\bar{0}} - \mathbf{1}_{\bar{0}}H)\mathrm{e}^{-\mathrm{i}tH}\varphi\rangle \\
&= \mathrm{i}\langle\mathrm{e}^{-\mathrm{i}tH_0}\mathbf{1}_U(H_0)\psi\,|\,T\mathrm{e}^{-\mathrm{i}tH}\varphi\rangle.
\end{aligned}$$

By the fundamental theorem of calculus, for $s < t$

$$\begin{aligned}
\langle\psi\,|\,\zeta(t) - \zeta(s)\rangle &= \mathrm{i}\int_s^t \langle\mathrm{e}^{-\mathrm{i}\tau H_0}\mathbf{1}_U(H_0)\psi\,|\,T\mathrm{e}^{-\mathrm{i}\tau H}\varphi\rangle\,\mathrm{d}\tau \\
&= \mathrm{i}\int_s^t \langle\mathbf{1}_1\mathrm{e}^{-\mathrm{i}\tau H_0}\mathbf{1}_U(H_0)\psi\,|\,T\mathbf{1}_1\mathrm{e}^{-\mathrm{i}\tau H}\varphi\rangle\,\mathrm{d}\tau,
\end{aligned}$$

since $T = \mathbf{1}_1 T\mathbf{1}_1$. Thus, by the Cauchy–Schwarz inequality

$$\begin{aligned}
|\langle\psi\,|\,\zeta(t) - \zeta(s)\rangle| &\leqslant \|T\|\int_s^t \|\mathbf{1}_1\mathrm{e}^{-\mathrm{i}\tau H_0}\mathbf{1}_U(H_0)\psi\|\,\|\mathbf{1}_1\mathrm{e}^{-\mathrm{i}\tau H}\varphi\|\,\mathrm{d}\tau \\
&\leqslant \|T\|\left(\int_s^t \|\mathbf{1}_1\mathrm{e}^{-\mathrm{i}\tau H_0}\mathbf{1}_U(H_0)\psi\|^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}\left(\int_s^t \|\mathbf{1}_1\mathrm{e}^{-\mathrm{i}\tau H}\varphi\|^2\,\mathrm{d}\tau\right)^{\frac{1}{2}} \\
&\leqslant \mathit{Const}\left(\int_s^t \|\mathbf{1}_1\mathrm{e}^{-\mathrm{i}\tau H}\varphi\|^2\,\mathrm{d}\tau\right)^{\frac{1}{2}}\|\psi\|,
\end{aligned}$$

since $\mathbf{1}_1$ is H_0-smooth on U. Furthermore, the integral $\int_{\mathbb{R}} \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}\tau H}\varphi\|^2 \,\mathrm{d}\tau$ is finite by the definition of $\mathcal{D}$. Hence,

$$\begin{aligned} \|\zeta(t)-\zeta(s)\| &= \sup_{\|\psi\|=1} |\langle \psi \,|\, \zeta(t)-\zeta(s)\rangle| \\ &\leqslant Const \left(\int_s^t \|\mathbf{1}_1 \mathrm{e}^{-\mathrm{i}\tau H}\varphi\|^2 \,\mathrm{d}\tau \right)^{\frac{1}{2}} \quad \xrightarrow{s,t} \quad 0. \end{aligned}$$

Thus, both $\lim_{t\to\pm\infty}\zeta(t)$ exist and the result follows. □

Scholium *The beginning of the proof asserts the following: it suffices to establish the existence of* $\lim_t \mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}tH}\varphi$ *for* $\varphi \in \mathcal{D}$, *because* $\mathcal{D}$ *is dense in* $\mathbf{1}_U(H)\mathcal{H}$. *However, as discussed at the beginning of the section the result is trivial for* $\varphi \in \mathcal{K}^\perp$. *Since* $\mathcal{D}\cap\mathcal{K}$ *is dense in* $\mathbf{1}_U(H)\mathcal{K}$, *it then suffices to prove the existence of the limit for* $\varphi \in \mathcal{D}\cap\mathcal{K}$. *In other words, Assumption D may be weakened in the following way:*

- $\mathbf{1}_1$ *is* H_0*-smooth on* U,
- $\mathcal{D}\cap\mathcal{K}$ *is dense in* $\mathbf{1}_U(H)\mathcal{K}$.

Notice that a similar conclusion holds by replacing $\mathcal{K}$ *with* $\mathcal{K}_1$ *(see the proof of Theorem 43).*

Applying twice the previous theorem,

Corollary *If* $\mathbf{1}_1$ *is both* H*-smooth and* H_0*-smooth on* U, *then the wave operators* $\Omega^\pm(H,H_0)$ *exist and are complete on* U.

3.3 Deterministic Approach

Using the definitions and conventions of the previous section, we now turn our attention to the scattering properties of $H = H_0 + V$. In the present section V is assumed to be bounded.

Let $a < b$. The existence and completeness of wave operators on $]a, b[$ except on a closed set of Lebesgue measure zero will follow from various hypotheses regarding the following quantities:

$$\begin{aligned} I &= \inf_{n\in\Gamma} \inf_{\substack{a<e<b \\ 0<\varepsilon<1}} \operatorname{Im} \langle \delta_n \,|\, (H_0 - e - \mathrm{i}\varepsilon)^{-1} \delta_n \rangle, \\ \tau(n,m) &= \sup_{\substack{a<e<b \\ 0<\varepsilon<1}} |\langle \delta_n \,|\, (H_0 - e - \mathrm{i}\varepsilon)^{-1} \delta_m \rangle|, \text{ where } n, m \in X, \\ l(n) &= \sum_{\substack{m\in\Gamma \\ m\neq n}} \tau(n,m), \text{ where } n \in \Gamma. \end{aligned}$$

We adopt the following convention: except mentioned explicitly, z varies in the rectangle

$$\mathcal{S} = \{e + \mathrm{i}y \;;\; a < e < b \text{ and } 0 < y < 1\},$$

while N, M vary in X, $\underline{N}, \underline{M}$ in Γ_1, and n, m in Γ. Thus, the previous quantities may be abbreviated

$$
\begin{aligned}
I &= \inf_{n,z} \operatorname{Im} \langle \delta_n \,|\, (H_0 - z)^{-1} \delta_n \rangle, \\
\tau(N, M) &= \sup_z |\langle \delta_N \,|\, (H_0 - z)^{-1} \delta_M \rangle|, \\
l(n) &= \sum_{m \neq n} \tau(n, m).
\end{aligned}
$$

Theorem 38 *The spectral measure associated with H_0, δ_N, and δ_M is real valued.*

Proof: Let us denote this measure by μ_{NM}. It is characterized by the relation

$$
\langle \delta_N \,|\, f(H_0) \delta_M \rangle = \int_{\mathbb{R}} f(t) \, \mathrm{d}\mu_{NM}(t),
$$

which holds for any bounded measurable f. Since $\langle \delta_N \,|\, H_0^l \delta_M \rangle$ represents the number of paths from N to M of length l in the graph (X, d), it is a positive integer, *a fortiori* a real number. Thus, $\langle \delta_N \,|\, p(H_0) \delta_M \rangle$ is real for any polynomial p on $\mathbb{R}$. By density, it follows that $\langle \delta_N \,|\, \mathbf{1}_B(H_0) \delta_M \rangle = \mu_{NM}(B)$ is real for any Borel set B. $\square$

As a consequence, the matrix elements of $(H_0 - z)^{-1}$ are "real symmetric" in the sense that $\langle \delta_N \,|\, (H_0 - z)^{-1} \delta_M \rangle = \langle \delta_M \,|\, (H_0 - z)^{-1} \delta_N \rangle$, which implies $\tau(N, M) = \tau(M, N)$.

In this section we make the following assumptions, *which involve H_0 only*:

Assumption E *$l(n) \to 0$ as $n \to \infty$, i.e., for all $\varepsilon > 0$ there exists a finite set $\mathcal{F} \subset \Gamma$ such that $l(n) < \varepsilon$ for every $n \notin \mathcal{F}$.*

Assumption F $l(n) < \infty$ *for all* $n \in \Gamma$.

Assumption G $I > 0$.

Assumption H *The function* $\mathbb{C}_+ \longrightarrow \mathcal{B}(\mathcal{H})$, $z \mapsto \mathbf{1}_1(H_0 - z)^{-1}\mathbf{1}_1$ *extends continuously to* $\mathbb{C}_+ \cup [a, b]$, *where* $\mathcal{B}(\mathcal{H})$ *is endowed with the uniform topology.*

Notice that the property E is vacuously true when Γ is finite. Similarly, the case where the potential lies on a single site is not really discarded by the definition of $l(n)$, interpreting a sum over an empty set of indices as zero. However, we are mainly concerned in the case where Γ is infinite.

Concretely E, F, and H may come from sparseness of Γ and an *a priori* estimate on the Green's function of H_0. At the level of operators theory the condition G implies that the diagonal part of $(H_0 - z)^{-1} \restriction l^2(\Gamma)$ is invertible, while E and F imply that its remaining part is compact, as we shall see later.

Finally, Assumptions G and H imply the following:

Theorem 39 *The spectrum of* H_0 *restricted to* $\mathcal{K}_1$ *is purely absolutely continuous on* $[a, b]$. *Moreover, its essential support contains* $[a, b]$.

Proof: Let $\mu_{\underline{N}}$ be the spectral measure of $\delta_{\underline{N}}$ with respect to H_0, and $P_{\underline{N}}$ be its Poisson transform. Then, for $z = x + \mathrm{i}y \in \mathbb{C}_+$

$$\begin{aligned} P_{\underline{N}}(z) &= y \int_{\mathbb{R}} \frac{1}{(t-x)^2 + y^2} \, \mathrm{d}\mu_{\underline{N}}(t) \\ &= \operatorname{Im} \langle \delta_{\underline{N}} \,|\, (H_0 - x - \mathrm{i}y)^{-1} \delta_{\underline{N}} \rangle. \end{aligned}$$

Assumption H ensures that $P_{\underline{N}}(z)$ extends continuously to a function $P_{\underline{N}}(x)$ on $[a,b]$. Moreover, by classical Harmonic Analysis the measure $\frac{1}{\pi}P_{\underline{N}}(x+iy)\,\mathrm{d}x$ converges vaguely to $\mathrm{d}\mu_{\underline{N}}(x)$ when $y \downarrow 0$. It follows that $\mathrm{d}\mu_{\underline{N}}(x)$ is equal to $\frac{1}{\pi}P_{\underline{N}}(x)\,\mathrm{d}x$ on $[a,b]$, *i.e.*, for all $\underline{N}$, $\mu_{\underline{N}}$ is purely absolutely continuous on $[a,b]$. Limiting our considerations to the subspace cyclically generated by $\{\delta_{\underline{N}}\}_{\underline{N}\in\Gamma_1}$,

$$\mu = \sum_{\underline{N}} a_{\underline{N}}\mu_{\underline{N}}$$

then defines a spectral measure for $H_0 \restriction \mathcal{K}_1$, where $a_{\underline{N}} > 0$ and $\sum_{\underline{N}} a_{\underline{N}} = 1$. Since each $\mu_{\underline{N}}$ is purely absolutely continuous on $[a,b]$, the first conclusion follows. Finally, since the density of μ with respect to the Lebesgue measure is given by

$$\frac{1}{\pi}\sum_{\underline{N}} a_{\underline{N}}\mathrm{Im}\,\langle\delta_{\underline{N}}\,|\,(H_0 - x - \mathrm{i}0)^{-1}\delta_{\underline{N}}\rangle,$$

which exists almost everywhere (by classical Harmonic Analysis) and is positive for $x \in [a,b]$ (by Assumption G), the second conclusion follows. $\square$

Let us focus on the Hilbert subspace $l^2(\Gamma)$ and consider

$$W_0\colon \mathcal{S} \to \mathcal{B}(l^2(\Gamma)),\ W_0(z) = \mathbf{1}_0(H_0 - z)^{-1}\mathbf{1}_0.$$

Clearly, $W_0(z)$ is analytic on $\mathcal{S}$ and extends to a continuous function

$$W_0\colon \overline{\mathcal{S}} \to \mathcal{B}(l^2(\Gamma))$$

by Assumption H. Moreover,

Theorem 40 *$W_0(z)$ is invertible for all $z \in \mathcal{S}$.*

Proof: Let φ vary in $l^2(\Gamma)$ and let us denote its spectral measure with respect to H_0 by μ_φ. For a fixed $z = x + \mathrm{i}y \in \mathcal{S}$,

$$\begin{aligned}\operatorname{Im}\langle\varphi\,|\,W_0(z)\varphi\rangle &= \operatorname{Im}\langle\varphi\,|\,(H_0-z)^{-1}\varphi\rangle \\ &= y\int_{\mathbb{R}}\frac{1}{(t-x)^2+y^2}\,\mathrm{d}\mu_\varphi(t).\end{aligned}$$

This last quantity is strictly positive; indeed, it is bounded away from zero when φ varies in $l^2(\Gamma)$, since $|t| \leqslant \|H_0\|$ on the support of the integrand. Thus, the closure of the numerical range[6] of $W_0(z)$ is included in $\mathbb{C}_+$. In particular, $0 \notin \operatorname{spec}(W_0(z))$, from which the result follows. □

Let us define the following operators, which act on the underlying Hilbert space $l^2(\Gamma)$:

$$\begin{aligned}D_0(z) &= \operatorname{diag} W_0(z) \\ &= \sum_n \langle\delta_n\,|\,W_0(z)\delta_n\rangle\langle\delta_n\,|\,\cdot\rangle\delta_n, \\ K_0(z) &= W_0(z) - D_0(z).\end{aligned}$$

Notice that $\|\operatorname{diag}A\| \leqslant \|A\|$ for any bounded operator A, since

$$\|\operatorname{diag}A\| = \sup_n |\langle\delta_n\,|\,A\delta_n\rangle| \leqslant \|A\|.$$

[6] Recall that the *numerical range* of a bounded, selfadjoint operator, A, is defined as $\{\langle\varphi\,|\,A\varphi\rangle\ ;\ \varphi\in\mathcal{H},\ \|\varphi\|=1\}$; it is well known that its closure contains $\operatorname{spec}(A)$.

In particular, $D_0(z)$ is bounded for any $z \in \mathcal{S}$. Moreover, D_0 is clearly analytic on $\mathcal{S}$. It extends continuously to $\overline{\mathcal{S}}$, letting

$$D_0(x) = \operatorname{diag} W_0(x)$$

for $x \in [a, b]$. Hence, K_0 inherits the same properties, namely: $K_0(z)$ is bounded for any fixed $z \in \mathcal{S}$, K_0 is analytic, and K_0 extends continuously to $\overline{\mathcal{S}}$.

Lemma *$K_0(z)$ is compact for any $z \in \overline{\mathcal{S}}$.*

Proof: Let $\varepsilon > 0$. By Assumption E there exists a finite set, $\mathcal{F}_1 \subseteq \Gamma$, such that $l(n) \leqslant \frac{\varepsilon}{2}$ for any $n \in \Gamma \setminus \mathcal{F}_1$. In other words, $\sup_{n \notin \mathcal{F}_1} \sum_{m \neq n} \tau(n, m) \leqslant \frac{\varepsilon}{2}$, that is, for an arbitrarily fixed $z \in \overline{\mathcal{S}}$

$$\sup_{n \notin \mathcal{F}_1} \sum_m |\langle \delta_n \mid K_0(z) \delta_m \rangle| \leqslant \frac{\varepsilon}{2}.$$

Moreover, by Assumption F there exists a finite set $\mathcal{F}_2 \subseteq \Gamma$ such that

$$\sup_{n \in \mathcal{F}_1} \sum_{m \notin \mathcal{F}_2} |\langle \delta_n \mid K_0(z) \delta_m \rangle| \leqslant \frac{\varepsilon}{2}.$$

Letting $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$, one finally finds

$$\sup_{n \notin \mathcal{F}} \sum_m |\langle \delta_n \mid K_0(z) \delta_m \rangle| + \sup_{n \in \mathcal{F}} \sum_{m \notin \mathcal{F}} |\langle \delta_n \mid K_0(z) \delta_m \rangle| \leqslant \varepsilon. \tag{3.2}$$

Let F be the finite rank operator defined as follows:

$$\langle \delta_n \mid F \delta_m \rangle = \begin{cases} \langle \delta_n \mid K_0(z) \delta_m \rangle & \text{if } n, m \in \mathcal{F}, \\ 0 & \text{otherwise.} \end{cases}$$

As already noticed the matrix elements of $K_0(z)$ are "symmetric in the real sense". Hence, so are the matrix elements of $K_0(z) - F$, *i.e.*,

$$\langle \delta_n \,|\, (K_0(z) - F)\delta_m \rangle = \langle \delta_m \,|\, (K_0(z) - F)\delta_n \rangle.$$

In particular, by the Riesz–Thorin Interpolation Theorem (*cf.* Appendix 4.3)

$$\|K_0(z) - F\|_1 = \|K_0(z) - F\|_\infty \geqslant \|K_0(z) - F\|.$$

Since the infinity norm of $K_0(z) - F$ is precisely the left-hand side in (3.2), one concludes

$$\|K_0(z) - F\| \leqslant \varepsilon.$$

Since ε is arbitrary, $K_0(z)$ is compact. □

Scholium *We proved more, namely: for all $\varepsilon > 0$ there exists a finite dimensional projection, P, such that for* any $z \in \overline{\mathcal{S}}$

$$\|K_0(z) - PK_0(z)P\| \leqslant \varepsilon.$$

This comes from the fact that our choice of $\mathcal{F}$ did not depend on $z \in \overline{\mathcal{S}}$—defining P as the projection onto the vector space generated by $\{\delta_n \,;\, n \in \mathcal{F}\}$.

Lemma *The diagonal operator $1 + D_0(z)V$ acting on $l^2(\Gamma)$ is invertible for any*

$z \in \overline{\mathcal{S}}$.

Proof: By Assumption G, $D_0(z)$ consists of invertible diagonal elements. If [7]

$$|V(n)| \leqslant \frac{1}{2\|D_0(z)\|},$$

then

$$\begin{aligned} |(1 + D_0(z)V)(n)| &\geqslant 1 - |D_0(z)(n)|\,|V(n)| \\ &\geqslant 1 - \|D_0(z)\|\frac{1}{2\|D_0(z)\|} \\ &= \frac{1}{2}, \end{aligned}$$

so $|(1 + D_0(z)V)^{-1}(n)| \leqslant 2$. Otherwise, $|V(n)| > (2\|D_0(z)\|)^{-1}$, and then

$$\begin{aligned} |(1 + D_0(z)V)(n)| &\geqslant |\mathrm{Im}\,(1 + D_0(z)V)(n)| \\ &= |V(n)|\,|\mathrm{Im}\, D_0(z)(n)| \\ &\geqslant \frac{I}{2\|D_0(z)\|}, \end{aligned}$$

so $|(1 + D_0(z)V)^{-1}(n)| \leqslant \frac{2\|D_0(z)\|}{I}$. In total,

$$\|(1 + D_0(z)V)^{-1}\| \leqslant \max\left\{2, \frac{2\|D_0(z)\|}{I}\right\},$$

[7] In the sequel we denote by $A(n)$ the n^{th} diagonal element of a diagonal operator A.

which ensures that the inverse of $1 + D_0(z)V$, whose existence follows from Assumption G, is really in $\mathcal{B}(l^2(\Gamma))$. $\square$

Let us define

$$W\colon \mathcal{S} \longrightarrow \mathcal{B}(l^2(\Gamma)),\ W(z) = \mathbf{1}_0(H-z)^{-1}\mathbf{1}_0.$$

From the fact that V is bounded, the argument used in Theorem 40 shows that $W(z)$ is invertible for any $z \in \mathcal{S}$. Moreover,

Theorem 41 *There exists a closed set of Lebesgue measure zero, $\mathcal{R} \subset [a,b]$, such that W extends continuously to a function $\overline{\mathcal{S}} \setminus \mathcal{R} \longrightarrow \mathcal{B}(l^2(\Gamma))$.*

Proof: For the moment, let z vary in $\mathcal{S}$. By the resolvent identity

$$\begin{aligned} W(z) - W_0(z) &= \mathbf{1}_0((H-z)^{-1} - (H_0-z)^{-1})\mathbf{1}_0 \\ &= -\mathbf{1}_0(H_0-z)^{-1}V(H-z)^{-1}\mathbf{1}_0 \\ &= -W_0(z)VW(z), \end{aligned}$$

since $V = \mathbf{1}_0 V \mathbf{1}_0$. Thus,

$$(1 + W_0(z)V)W(z) = W_0(z).$$

Notice that $1 + W_0(z)V$ is invertible for any fixed $z \in \mathcal{S}$, since $W(z)$ and $W_0(z)$ are. Thus,

$$W(z) = (1 + W_0(z)V)^{-1}W_0(z), \tag{3.3}$$

where $z \in \mathcal{S}$. We wonder to which extent $(1 + W_0(z)V)^{-1}$ is still invertible for $z \in \partial\mathcal{S}$. Indeed, for any $z \in \overline{\mathcal{S}}$,

$$\begin{aligned} 1 + W_0(z)V &= 1 + D_0(z)V + K_0(z)V \\ &= (1 - K(z))(1 + D_0(z)V). \end{aligned}$$

where $K(z) = -K_0(z)V(1 + D_0(z)V)^{-1}$ is compact. Since for $z \in \mathcal{S}$ both $1+D_0(z)V$ and $1 + W_0(z)V$ are invertible, $1 - K(z)$ is. By a variant of the Fredholm analytic theorem (*cf.* Appendix 4.4), $1 - K(z)$ is thus invertible in $\mathcal{B}(l^2(\Gamma))$ for all $z \in [a, b] \setminus \mathcal{R}$, where $\mathcal{R} \subset [a, b]$ is a closed set of Lebesgue measure zero. Since $\overline{\mathcal{S}} \setminus \mathcal{R} \longrightarrow \mathcal{B}(l^2(\Gamma))$, $z \mapsto (1 - K(z))^{-1}$ is still continuous (*cf.* Appendix 4.4), so is $z \mapsto (1 + W_0(z)V)^{-1}$. Hence, the right-hand side of (3.3) extends continuously to $\overline{\mathcal{S}} \setminus \mathcal{R}$, as desired. $\square$

Since the natural embedding $\mathcal{B}(l^2(\Gamma)) \hookrightarrow \mathcal{B}(\mathcal{H})$ is an isometry,

Corollary *There exists a closed set of Lebesgue measure zero, $\mathcal{R} \subset [a, b]$, such that W extends continuously to a function $\overline{\mathcal{S}} \setminus \mathcal{R} \longrightarrow \mathcal{B}(\mathcal{H})$.*

The main theorem in this section is deduced from this last corollary, by working out

$$W_1 \colon \mathcal{S} \to \mathcal{B}(\mathcal{H}), \; W_1(z) = \mathbf{1}_1(H - z)^{-1}\mathbf{1}_1.$$

Lemma *There exists a closed set of Lebesgue measure zero, $\mathcal{R} \subset [a, b]$, such that*

W_1 *extends continuously to a function* $\overline{\mathcal{S}} \setminus \mathcal{R} \longrightarrow \mathcal{B}(\mathcal{H})$.

Proof: We first prove the existence of a similar extension for $\mathbf{1}_1(H-z)^{-1}\mathbf{1}_0$. By the resolvent identity

$$\begin{aligned}\mathbf{1}_1(H_0-z)^{-1}\mathbf{1}_0 - \mathbf{1}_1(H-z)^{-1}\mathbf{1}_0 &= \mathbf{1}_1((H_0-z)^{-1} - (H-z)^{-1})\mathbf{1}_0 \\ &= \mathbf{1}_1(H_0-z)^{-1}V(H-z)^{-1}\mathbf{1}_0 \\ &= \mathbf{1}_1(H_0-z)^{-1}\mathbf{1}_0 V W(z),\end{aligned}$$

since $V = \mathbf{1}_0 V \mathbf{1}_0$. By Assumption H, $\mathbf{1}_1(H_0-z)^{-1}\mathbf{1}_0 = \mathbf{1}_1(H_0-z)^{-1}\mathbf{1}_1\mathbf{1}_0$ extends to $\overline{\mathcal{S}}$, while by the previous corollary $W(z)$ extends to $\overline{\mathcal{S}} \setminus \mathcal{R}$. Thus, $\mathbf{1}_1(H-z)^{-1}\mathbf{1}_0$ extends on $\overline{\mathcal{S}} \setminus \mathcal{R}$, as claimed. Again, by the resolvent identity,

$$\begin{aligned}\mathbf{1}_1(H_0-z)^{-1}\mathbf{1}_1 - W_1(z) &= \mathbf{1}_1((H_0-z)^{-1} - (H-z)^{-1})\mathbf{1}_1 \\ &= \mathbf{1}_1(H-z)^{-1}V(H_0-z)^{-1}\mathbf{1}_1 \\ &= \mathbf{1}_1(H-z)^{-1}\mathbf{1}_0 V \mathbf{1}_1(H_0-z)^{-1}\mathbf{1}_1.\end{aligned}$$

Both $\mathbf{1}_1(H_0-z)^{-1}\mathbf{1}_1$ and $\mathbf{1}_1(H-z)^{-1}\mathbf{1}_0$ extend appropriately by Assumption H and the above. The result follows. $\square$

Using the criterion in Section 3.2 (corollary of Theorem 37), we have proven:

Theorem 42 *Under Assumptions E, F, G, and H, there exists a closed set of Lebesgue measure zero,* $\mathcal{R} \subset \,]a,b[$*, such that the wave operators,* $\Omega^{\pm}(H, H_0)$*, exist and are complete on* $]a,b[\, \setminus \mathcal{R}$.

More generally,

Corollary *Let $\Theta \subseteq \mathbb{R}$ be an open set. Suppose Assumptions E, F, G, and H hold for any $[a, b] \subset \Theta$. Then, there exists a set of Lebesgue measure zero, $\mathcal{R} \subset \Theta$, such that the wave operators, $\Omega^{\pm}(H, H_0)$, exist and are complete on $\Theta \setminus \mathcal{R}$.*

Proof: Since Θ is open,

$$\Theta = \bigcup_{\substack{[a,b]\subset\Theta \\ a,b\in\mathbb{Q}}} [a, b] \,.$$

By the theorem, for all $[a, b] \subset \Theta$ there exists a closed set of Lebesgue measure zero, $\mathcal{R}_{a,b}$, such that the wave operators exist and are complete on $[a, b] \setminus \mathcal{R}_{a,b}$. Letting

$$\mathcal{R} = \bigcup_{\substack{[a,b]\subset\Theta \\ a,b\in\mathbb{Q}}} \mathcal{R}_{a,b},$$

which is not necessarily closed but has measure zero, it follows from the discussion after Proposition 34 that the wave operators exist and are complete on each $[a, b] \setminus \mathcal{R}$, where $[a, b] \subset \Theta$ has rational endpoints. Hence, by the corollary of Proposition 34 the wave operators exist on $\Theta \setminus \mathcal{R}$. □

3.3.1 Conclusion in Random Frame

It is possible to remove the exceptional set, $\mathcal{R}$, in the above theorem by working in the random frame. Then,

$$H = H_0 + V,$$

where $\{V(n)\}_{n\in\Gamma}$ is a family of independent, identically distributed random variables of law μ, where μ is a probability measure on $\mathbb{R}$ (*cf.* the beginning of Section 3.5). Assume the support of μ is bounded, so V is bounded almost surely. Then,

Theorem 43 *Let $\Theta \subseteq \mathbb{R}$ be an open set. Suppose Assumptions E, F, G, and H hold for any $[a, b] \subset \Theta$. Then, the wave operators $\Omega^{\pm}(H, H_0)$ exist and are complete on Θ almost surely.*

Proof: It suffices to establish the existence and completeness of the wave operators $\Omega^{\pm}(H \restriction \mathcal{K}_1, H_0 \restriction \mathcal{K}_1)$ on Θ.

For all V the previous theorem ensures the existence of a random set of Lebesgue measure zero, $\mathcal{R}_V$, such that $\Omega^{\pm}(H, H_0)$ exist and are complete on $\Theta \setminus \mathcal{R}_V$, a conclusion that persists when restricting the operators to $\mathcal{K}_1$. By Theorem 39 the spectrum of $H_0 \restriction \mathcal{K}_1$ is purely absolutely continuous on Θ. Thus, $\mathbf{1}_{\Theta\setminus\mathcal{R}_V}(H_0 \restriction \mathcal{K}_1) = \mathbf{1}_{\Theta}(H_0 \restriction \mathcal{K}_1)$, so the wave operators $\Omega^{\pm}(H \restriction \mathcal{K}_1, H_0 \restriction \mathcal{K}_1)$ exist on Θ for all V.

By Theorem 39 the essential support of $H_0 \restriction \mathcal{K}_1$ on Θ is full. Since the random variables $\{V(n)\}_{n\in\Gamma}$ are i.i.d., the essential support of $H \restriction \mathcal{K}_1$ is almost surely full by random perturbation theory. By the Jakšić–Last theorem, it follows that $\mathrm{spec}(H \restriction \mathcal{K}_1)$ is purely absolutely continuous on Θ (a.s.), so the wave operators $\Omega^{\pm}(H_0 \restriction \mathcal{K}_1, H \restriction \mathcal{K}_1)$ almost surely exist on Θ. $\square$

3.3.2 Application to Generalized Laplacians

In this section we show that the previous, abstract theorem applies to random Schrödinger operators with generalized Laplacians and sparse potentials. Our sparseness conditions will come from the *a priori* estimates on free generalized Laplacians derived in Chapter 2.

Let $H_0 = \Delta$ be a generalized Laplacian. Then, $H_0 = \Delta$ comes from a translational invariant graph on $\mathbb{Z}^d$, whose distance is denoted by $\mathrm{d}(m, n)$. In order to apply Theorem 43 to the random operator

$$H = \Delta + V,$$

where V is assumed to be bounded almost surely, we show that Δ satisfies Assumptions E, F, G, and H.

On specific examples condition G comes from Theorem 20. Let $\Theta \subset \mathrm{spec}(\Delta)$ be an open set on which Theorem 20 applies. For instance, suppose Δ is the standard Laplacian and let

$$E = \{-2d, -2d + 4, \ldots, 2d - 4, 2d\} \cup \{0\};$$

alternatively, suppose Δ is the Molchanov–Vainberg Laplacian and let

$$E = \{-2^d, 0, 2^d\}.$$

We then set $\Theta = \mathrm{spec}(\Delta) \setminus E$. As in the previous section we focus on an interval $[a, b] \subset \Theta$ and define $\mathcal{S}$ with respect to this interval.

Firstly, by translational invariance

$$\begin{aligned} I &= \inf_{z} \operatorname{Im} \langle \delta_0 \,|\, (H_0 - z)^{-1} \delta_0 \rangle \\ &= \inf_{z} \operatorname{Im} G(0, z), \end{aligned}$$

where following our convention $z \in \mathcal{S}$. Secondly, by Theorem 20, for any fixed $e \in [a, b]$

$$\operatorname{Im} G(0, e) = \pi \int_{\Gamma(e)} \frac{1}{\|\nabla_x \Phi(x)\|} \, \mathrm{d}x > 0.$$

On the other hand, if $e + \mathrm{i}y \in \overline{\mathcal{S}}$ is not real, then

$$\operatorname{Im} G(0, e + \mathrm{i}y) = (2\pi)^{-d} \int_{\mathbb{T}^d} \frac{y}{(\Phi(x) - e)^2 + y^2} \, \mathrm{d}x > 0.$$

Since $G(n, z)$ exists and is continuous on $\overline{\mathcal{S}}$, which is compact, it follows that $I > 0$. Hence, all generalized Laplacians (in particular, both the standard and the Molchanov–Vainberg ones) satisfy Assumption G for $[a, b]$ included in their respective Θ.

The main work consists of verifying Assumption H. To this end, we use the following, stronger versions of Assumptions G and E, which yield our sparseness assumption.

Before stating these assumptions, let us partition Γ_1 in the following way: for all $n \in \Gamma$, we select a non empty neighborhood

$$\mathcal{B}(n) \subseteq \{\underline{N} \in \Gamma_1 \;;\; \mathrm{d}(\underline{N}, n) \leqslant 1\} \tag{3.4}$$

in such a way that $\bigcup_n \mathcal{B}(n) = \Gamma_1$ and $\mathcal{B}(m) \cap \mathcal{B}(n) \neq \emptyset$ if $m \neq n$. Of course, equality is necessary in (3.4) if $\mathrm{d}(m, n) > 2$ for all $m \neq n$ (where according to our convention

$m, n \in \Gamma$). Moreover, for all $\underline{N} \in \Gamma_1$ there exists exactly one $n \in \Gamma$ such that $\underline{N} \in \mathcal{B}(n)$. We thus set $\mathcal{B}(\underline{N}) = \mathcal{B}(n)$, and then define

$$L(\underline{N}) = \sum_{\underline{M} \notin \mathcal{B}(\underline{N})} \tau(\underline{M}, \underline{N}).$$

This last sum is an analogue of $l(n)$ for $\underline{M}, \underline{N}$ varying Γ_1 instead of Γ. Moreover, instead of removing only the diagonal element ($\underline{M} = \underline{N}$) from this summation, the whole $\mathcal{B}(\underline{N})$ is removed.

We suppose:

Assumption I $L(\underline{N}) \to 0$ *when* $\underline{N} \to \infty$. *In other words, for all* $\varepsilon > 0$ *there exists a finite set,* $\mathcal{F} \subset \Gamma_1$, *such that* $L(\underline{N}) < \varepsilon$ *for all* $\underline{N} \notin \mathcal{F}$.

Assumption J $L(\underline{N}) < \infty$ *for all* $\underline{N} \in \Gamma_1$.

We then decompose $W_1(z) = \mathbf{1}_1(H_0 - z)^{-1}\mathbf{1}_1$ into two summands: a superdiagonal,

$$D_1(z) = \sum_{\underline{N}} \sum_{\underline{M} \in \mathcal{B}(\underline{N})} \langle \delta_{\underline{N}} \,|\, (H_0 - z)^{-1} \delta_{\underline{M}} \rangle \langle \delta_{\underline{M}} \,|\, \cdot \rangle \delta_{\underline{N}},$$

and the other part,

$$K_1(z) = W_1(z) - D_1(z).$$

By Theorem 20, $\langle \delta_N \,|\, (H_0 - e - \mathrm{i}0)^{-1} \delta_M \rangle$ exists for any $e \in [a, b]$ and $M, N \in \mathbb{Z}^d$. In particular, *defining*

$$W_1(e) = \sum_{\underline{M},\underline{N}} \langle \delta_N \,|\, (H_0 - e - \mathrm{i}0)^{-1} \delta_M \rangle \langle \delta_{\underline{M}} \,|\, \cdot \rangle \delta_{\underline{N}},$$

it follows that

$$\lim_{\substack{z \to e \\ z \in \mathbb{C}_+}} W_1(z) = W_1(e) \quad \text{weakly.}$$

In this situation $W_1(e)$ is bounded, which is a well known application of the Steinhaus–Banach uniform boundedness principle (see [40]); here, this fact will be deduced from further computations.

Similarly, let us define $D_1(e)$ and $K_1(e)$ in the obvious way, so these last operators are weak limits of $D_1(z)$ and $K_1(z)$ respectively. We want to show that under Assumptions I and J

$$\lim_{\substack{z \to e \\ z \in \mathbb{C}_+}} W_1(z) = W_1(e) \quad \textit{uniformly,} \tag{3.5}$$

the two mentioned assumptions joint with the *a priori* estimate in Chapter 2 then yielding our sparseness assumption.

Observe that $W_1(z) = D_1(z) + K_1(z)$ for all $z \in \overline{S}$ (including values in $\mathbb{R}$). In order to prove (3.5) we first show:

Lemma *For any $e \in [a, b]$, $\lim_{\substack{z \to e \\ z \in \mathbb{C}_+}} D_1(z) = D_1(e)$ uniformly.*

Proof: Let $\{\mathcal{C}_j\}_{j=1}^{L}$ be an indexation of all subsets of $\{N \in \mathbb{Z}^d \ ; \ \mathrm{d}(N, 0) \leqslant 1\}$ containing 0. Then, for all $n \in \Gamma$ there exists one and only one j, which we denote j(n), such that $\mathcal{B}(n) - n = \mathcal{C}_j$. Thus, one may rearrange $D_1(z)$ in the following way:

$$D_1(z) = \sum_{j=1}^{L} \sum_{\{n\,;\,\mathrm{j}(n)=j\}} \sum_{J,K\in\mathcal{C}_j} \langle \delta_{n+K} \,|\, (H_0 - z)^{-1}\delta_{n+J}\rangle \langle \delta_{n+J} \,|\, \cdot \rangle \delta_{n+K}.$$

By translational invariance this last expression is equal to

$$\sum_{j=1}^{L} \sum_{J,K\in\mathcal{C}_j} G(J-K,z) \sum_{\{n\,;\,\mathrm{j}(n)=j\}} \langle \delta_{n+J} \,|\, \cdot \rangle \delta_{n+K},$$

from which the result follows. □

It remains to show that the convergence of $K_1(z)$ to $K_1(e)$ is also uniform. Exactly as we did for $K_0(z)$,

Lemma (of the lemma) *Let $\varepsilon > 0$ be arbitrarily fixed. Then, there exists a finite rank operator, $F_\varepsilon(z)$, such that*

$$\|K_1(z) - F_\varepsilon(z)\| \leqslant \varepsilon$$

for any $z \in \overline{\mathcal{S}}$ (including values in $\mathbb{R}$). Moreover, the function $F_\varepsilon : \overline{\mathcal{S}} \to \mathcal{B}(\mathcal{H})$ is continuous at $z \in \overline{\mathcal{S}}$, where $\mathcal{B}(\mathcal{H})$ is endowed with the uniform topology.

Proof: By Assumptions I and J, for all $\varepsilon > 0$ there exists a finite set, $\mathcal{F} \subset \Gamma_1$, such that

$$\sup_{\underline{N}\notin\mathcal{F}} \sum_{\underline{M}} |\langle \delta_{\underline{N}} \,|\, K_1(z)\delta_{\underline{M}}\rangle| + \sup_{\underline{N}\in\mathcal{F}} \sum_{\underline{M}\notin\mathcal{F}} |\langle \delta_{\underline{N}} \,|\, K_1(z)\delta_{\underline{M}}\rangle| \leqslant \varepsilon. \tag{3.6}$$

Let $\mathcal{V}$ be the vector space generated by $\{\delta_{\underline{N}}\ ;\ \underline{N} \in \mathcal{F}\}$ and

$$F_\varepsilon(z) = K_1(z) \restriction \mathcal{V}.$$

Notice that $F_\varepsilon(z)$ is weakly continuous and hence uniformly continuous at $z \in \overline{\mathcal{S}}$, since $\mathcal{V}$ is finite dimensional. Moreover, the matrix elements of $K_1(z) - F_\varepsilon(z)$ are "symmetric in the real sense", so the equation (3.6) is equivalent to

$$\|K_1(z) - F_\varepsilon(z)\|_1 = \|K_1(z) - F_\varepsilon(z)\|_\infty \leqslant \varepsilon.$$

The Riesz–Thorin Interpolation Theorem then completes the proof (*cf.* Appendix 4.3). □

Lemma *For any* $e \in [a, b]$, $\lim\limits_{\substack{z \to e \\ z \in \mathbb{C}_+}} K_1(z) = K_1(e)$ *uniformly.*

Proof: Let $\varepsilon > 0$ be arbitrarily fixed. Then, for a given $e \in [a, b]$ and z varying in $\overline{\mathcal{S}}$

$$\begin{aligned} \|K_1(z) - K_1(e)\| &\leqslant \|K_1(z) - F_\varepsilon(z)\| + \|K_1(e) - F_\varepsilon(e)\| + \|F_\varepsilon(z) - F_\varepsilon(e)\| \\ &\leqslant \|F_\varepsilon(z) - F_\varepsilon(e)\| + 2\varepsilon. \end{aligned}$$

Since $\lim\limits_{\substack{z \to e \\ z \in \mathbb{C}_+}} F_\varepsilon(z) = F_\varepsilon(e)$ uniformly and ε is arbitrarily small, this completes the proof. □

In conclusion, by the two previous lemmas $W_1(z)$ admits a uniformly continuous extension to $\overline{\mathcal{S}}$ (*i.e.*, Assumption H holds) and this, under Assumptions I and J. Since I and J are stronger versions of E and F, and since G is already shown, we have proven:

Theorem 44 *Consider a random Schrödinger operator acting on* $l^2(\mathbb{Z}^d)$,

$$H = \Delta + V,$$

where Δ *is a generalized Laplacian and* V *is a random potential bounded almost surely. More precisely, assume* $\{V(n)\}_{n\in\Gamma}$ *are i.i.d. random variables of law* μ, *where* μ *is compactly supported. Let* $\Theta \subset \mathrm{spec}(\Delta)$ *be an open region of validity of Theorem 20. If Assumptions I and J hold for any* $[a,b] \subset \Theta$, *then Theorem 43 applies, i.e., the wave operators* $\Omega^{\pm}(H,\Delta)$ *exist and are complete on* Θ *almost surely.*

Assumptions I and J with the estimate derived in Chapter 2 then yield our sparseness assumption. On specific examples,

Theorem 45 *Consider* H *as above, where* Δ *is the standard Laplacian and* V *is bounded almost surely. Suppose* Γ *is* sparse *in the following sense: there exists an* $\epsilon > 0$ *such that*

$$\sum_{m\neq n} |n-m|^{-\frac{1}{2}+\epsilon} < \infty \quad \textit{for all } n \in \Gamma \textit{ and} \tag{3.7}$$

$$\lim_{|n|\to\infty} \sum_{m\neq n} |n-m|^{-\frac{1}{2}+\epsilon} = 0, \tag{3.8}$$

where m *and* n *vary in* Γ. *Then, the wave operators* $\Omega^{\pm}(H,\Delta)$ *exist and are complete on* $\mathrm{spec}(\Delta) = [-2d, 2d]$ *almost surely.*

Remark *It is perhaps possible to remove the condition (3.7) using (3.8) and the finiteness of d, but from our point of view* le jeu n'en vaut pas la chandelle, *i.e., to verify (3.7) on concrete examples is so easy that seeking for a general argument does not seem appropriate.*

Proof: Let $\Theta = [-2d, 2d] \setminus E$, where $E = \{-2d, -2d+4, \ldots, 2d-4, 2d\} \cup \{0\}$. Since the spectrum of Δ is purely absolutely continuous,

$$\Omega_{\Theta}^{\pm}(H, \Delta) = \Omega_{[-2d,2d]}^{\pm}(H, \Delta)$$

if these wave operators exist. Moreover, by a theorem of Jakšić and Last [16] based on spectral averaging, the deterministic and finite set E does not contain any eigenvalue of $H \restriction \mathcal{K}$ almost surely. Therefore,

$$\Omega_{\Theta}^{\pm}(\Delta, H) = \Omega_{[-2d,2d]}^{\pm}(\Delta, H)$$

almost surely if these last operators exist. Hence, it suffices to prove existence and completeness (a.s.) of the wave operators on Θ.

Equation (3.8) implies $|n - m| \to \infty$ when m and n are distinct and go to infinity. For this reason, in order to verify I and J it suffices to show that

$$\sum_{\mathrm{d}(\underline{M},\underline{N})>2} \tau(\underline{M}, \underline{N})$$

is finite for every $\underline{N} \in \Gamma_1$ and tends to 0 when $|\underline{N}| \to \infty$. Indeed, for $\underline{M}, \underline{N} \in \Gamma_1$, since

$$\#\{N \in \mathbb{Z}^d \; ; \; \mathrm{d}(N, 0) \leqslant 1\} = 2d + 1 < \infty,$$

the second corollary of Theorem 25 gives for any fixed $\underline{N} \in \Gamma_1$

$$\begin{aligned}
\sum_{\mathrm{d}(\underline{M},\underline{N})>2} \tau(\underline{M}, \underline{N}) &\leqslant \sum_{\mathrm{d}(\underline{M},\underline{N})>2} |\underline{N} - \underline{M}|^{-\frac{1}{2}+\epsilon} \\
&\leqslant (2d+1) \sum_{m \neq n} (|n - m| - 2)^{-\frac{1}{2}+\epsilon} \\
&\leqslant Const \sum_{m \neq n} |n - m|^{-\frac{1}{2}+\epsilon},
\end{aligned}$$

where $n \in \Gamma$ is adjacent or equal to the given $\underline{N} \in \Gamma_1$. Notice that n is unique eventually and $|n| \to \infty$ when $|\underline{N}| \to \infty$. Hence, Assumptions I and J are satisfied. The result follows. □

By exactly the same argument and with the same remark, which we will not repeat, a weaker sparseness assumption is obtained when Δ is the Molchanov–Vainberg Laplacian:

Theorem 46 *Consider H as above, where Δ is the Molchanov–Vainberg Laplacian and V is bounded almost surely. Suppose Γ is* sparse *in the following sense: there exists an $\epsilon > 0$ such that*

$$\sum_{m \neq n} |n - m|^{-\frac{d-1}{2}+\epsilon} < \infty \quad \textit{for all } n \in \Gamma \textit{ and} \tag{3.9}$$

$$\lim_{|n| \to \infty} \sum_{m \neq n} |n - m|^{-\frac{d-1}{2}+\epsilon} = 0, \tag{3.10}$$

where m and n vary in Γ. Then, the wave operators $\Omega^{\pm}(H,\Delta)$ exist and are complete on $\mathrm{spec}(\Delta) = \left[-2^d, 2^d\right]$ *almost surely.*

More generally,

Theorem 47 *Consider H as above, where Δ is a generalized Laplacian and V is bounded almost surely. Let Θ be an open region of validity of Theorems 20 and 21; in particular, we assume that the constant energy surfaces of the Green's function associated with Δ have at least $\kappa > 0$ non vanishing principal curvatures at any point when the energy lies in Θ. Suppose Γ is* sparse *in the following sense: there exists an $\epsilon > 0$ such that*

$$\sum_{m\neq n} |n-m|^{-\frac{\kappa}{2}+\epsilon} < \infty \quad \textit{for all } n \in \Gamma \textit{ and} \tag{3.11}$$

$$\lim_{|n|\to\infty} \sum_{m\neq n} |n-m|^{-\frac{\kappa}{2}+\epsilon} = 0, \tag{3.12}$$

where m and n vary in Γ. Then, the wave operators $\Omega^{\pm}(H,\Delta)$ exist and are complete on Θ almost surely.

To provide numerical, explicit examples is easy. For instance, if Δ is the standard or the Molchanov–Vainberg Laplacian, one may consider the set of sites

$$\Gamma = \{(j^4, 0, \ldots, 0) \in \mathbb{Z}^d \; ; \; j \in \mathbb{Z}\}.$$

Since there exists a constant such that $|j^4 - k^4| \geqslant \mathit{Const}\, |j|^3$ for any distinct $j,\ k \in \mathbb{Z}$, it follows that

$$\sum_{j\neq k} |j^4 - k^4|^{-\frac{1}{2}+\epsilon} \leqslant \mathit{Const} \sum_{j\neq 0} |j|^{-\frac{3}{2}+\epsilon} < \infty$$

for any fixed $k \in \mathbb{Z}$ (by choosing $\epsilon < \frac{1}{2}$). Then, not only our first sparseness assumption is satisfied, but the dominated convergence theorem applies and yields

$$\lim_{|k|\to\infty} \sum_{j\neq k} |j^4 - k^4|^{-\frac{1}{2}+\epsilon} = 0.$$

3.3.3 Remark About Clusters

It is possible to relax the geometric constraint imposed by our sparseness condition by considering *clusters*. Let $D \in \mathbb{N}^*$ and $\mathcal{C} \subseteq \Gamma$, $\mathcal{C} \neq \emptyset$. Let us denote the elements of $\mathcal{C}$ by underlined small letters, *e.g.*, $\underline{n} \in \mathcal{C}$. Interpreting them as the centers of the clusters, and D as the maximal radius of the clusters, we suppose

$$\Gamma \subseteq \bigcup_{\underline{n}} \{N \in \mathbb{Z}^d \; ; \; \mathrm{d}(\underline{n}, N) \leqslant D\}.$$

Then, the sparseness conditions found for n varying in Γ may be replaced with similar conditions for $\underline{n}$ varying in $\mathcal{C}$ (without affecting the exponents).

This can be seen in the following way. Imitating the argument used in the previous case, one partitions Γ_1 into classes

$$\mathcal{B}(\underline{n}) \subseteq \{\underline{N} \in \Gamma_1 \; ; \; \mathrm{d}(\underline{N}, \underline{n}) \leqslant D+1\},$$

so for all $\underline{N} \in \Gamma_1$ there exists one and only one $\underline{n} \in \mathcal{C}$ such that $\mathcal{B}(\underline{N}) = \mathcal{B}(\underline{n})$. Then, one defines

$$L(\underline{N}) = \sum_{\underline{M}\notin\mathcal{B}(\underline{N})} \tau(\underline{M}, \underline{N})$$

and make the usual sparseness hypotheses: $L(\underline{N}) < \infty$ and $L(\underline{N}) \to 0$ when $|\underline{N}| \to \infty$. The operator $W_1(z)$ then decomposes into $D_1(z) + K_1(z)$, where

$$D_1(z) = \sum_{\underline{N}} \sum_{\underline{M} \in \mathcal{B}(\underline{N})} \langle \delta_{\underline{N}} \,|\, (H_0 - z)^{-1} \delta_{\underline{M}} \rangle \langle \delta_{\underline{M}} \,|\, \cdot \rangle \delta_{\underline{N}}.$$

Let $\{\mathcal{C}_j\}_{j=1}^N$ be an indexation of all subsets of $\{N \in \mathbb{Z}^d \,;\, \mathrm{d}(N, 0) \leqslant D + 1\}$ containing 0. Hence, there exists a unique $j = \mathrm{j}(\underline{n})$ such that $\mathcal{B}(\underline{n}) - \underline{n} = \mathcal{C}_j$, so

$$D_1(z) = \sum_{j=1}^{N} \sum_{J,K \in \mathcal{C}_j} G(J - K, z) \sum_{\{\underline{n}\,;\,\mathrm{j}(\underline{n}) = j\}} \langle \delta_{\underline{n}+J} \,|\, \cdot \rangle \delta_{\underline{n}+K}.$$

The rest of the proof is identical. We thus have:

Theorem 48 *Consider a random Schrödinger operator acting on $l^2(\mathbb{Z}^d)$,*

$$H = \Delta + V,$$

where Δ is a generalized Laplacian and V is a random potential almost surely bounded. More precisely, assume $\{V(n)\}_{n \in \Gamma}$ are i.i.d. random variables of law μ, where μ is compactly supported. Let $\Theta \subset \mathrm{spec}(\Delta)$ be an open region of validity of Theorem 20. Let $\underline{n} \in \mathcal{C}$ be centers of clusters of radius (at most) $D < \infty$ in Γ. If $L(\underline{N}) < \infty$ and $L(\underline{N}) \to 0$ when $|\underline{N}| \to \infty$ (with respect to any $[a, b] \subset \Theta$), then the wave operators $\Omega^{\pm}(H, \Delta)$ exist and are complete on Θ almost surely.

The resulting conditions for the standard and the Molchanov–Vainberg Laplacians are respectively similar, replacing $n \in \Gamma$ with $\underline{n} \in \mathcal{C}$. However, when proving this fact one shows that

$$\sum_{\mathrm{d}(\underline{M}, \underline{N}) > 2D} \tau(\underline{M}, \underline{N})$$

is finite and tends to 0 when $|\underline{N}| \to \infty$, using a similar argument.

3.4 Second Criterion of Completeness of Wave Operators

This section is a continuation of Section 3.2. Using the same setting (and similar notations), we present a sufficient criterion for Theorem 37 to apply. This provides a second criterion of "completeness" for the usual wave operators, $\Omega^{\pm}(H, H_0)$, in the sense that $\Omega^{\pm}(H_0, H)$ exist, but without assuming that the formers do. The derived criterion will be used in Section 3.5 for unbounded potentials.

Recall that $\mathcal{K}$ and $\mathcal{K}_1$ are the Hilbert subspaces cyclically generated by $\{\delta_n \,;\, n \in \Gamma\}$ and $\{\delta_{\underline{N}} \,;\, \underline{N} \in \Gamma_1\}$ respectively (with respect to H_0). Therefore, $\mathcal{K}$, $\mathcal{K}_1$, and their orthogonal complements are H_0- and H-invariant, so the restrictions to $\mathcal{K}$, $\mathcal{K}_1$, $\mathcal{K}^{\perp}$, or ${\mathcal{K}_1}^{\perp}$ commute with the calculus for H_0 and for H.

Recall that $U \subseteq \mathbb{R}$ is a given, non empty, open set and

$$\mathcal{D} \cap \mathcal{K} = \{\varphi \in \mathbf{1}_U(H)\mathcal{K} \,;\, \sum_{\underline{M}} \int_{\mathbb{R}} |\langle \delta_{\underline{M}} \,|\, \mathrm{e}^{-\mathrm{i}tH}\varphi\rangle|^2 \, \mathrm{d}t < \infty\}.$$

In this section we establish a sufficient condition for $\mathcal{D} \cap \mathcal{K}$ to be dense in $\mathbf{1}_U(H)\mathcal{K}$ (*cf.* Scholium of Theorem 37). Explicitly,

Theorem 49 *Suppose that the spectrum of $H \restriction \mathbf{1}_U(H)\mathcal{K}_1$ is purely absolutely continuous. If for all $n \in \Gamma$ and almost all $e \in U$*

$$\sum_{\underline{M}} |\mathrm{Im}\, \langle \delta_{\underline{M}} \,|\, (H - e - \mathrm{i}0)^{-1}\delta_n\rangle|^2 < \infty, \tag{3.13}$$

then $\mathcal{D} \cap \mathcal{K}$ is dense in $\mathbf{1}_U(H)\mathcal{K}$.

Proof: Let $\mathcal{K}_n$ be the cyclic subspace generated by δ_n with respect to H_0 and let $\mathcal{D}_n = \mathcal{D} \cap \mathbf{1}_U(H)\mathcal{K}_n$, where $n \in \Gamma$ is arbitrarily fixed.[8] Since $\mathcal{D}$ is a vector space, it suffices to prove that $\mathcal{D}_n$ is dense in $\mathbf{1}_U(H)\mathcal{K}_n$, in which case $\sum_n \mathcal{D}_n$ is dense in $\mathbf{1}_U(H)\sum_n \mathcal{K}_n$, so $\mathcal{D}$ is dense in $\mathbf{1}_U(H)\mathcal{K}$.[9]

Let

$$U_j = \{e \in U \cap [-j,j] \; ; \; \sum_{\underline{M}} |\mathrm{Im}\, \langle \delta_{\underline{M}} \,|\, (H - e - \mathrm{i}0)^{-1}\delta_n \rangle|^2 < j\}.$$

By assumption, U differs from $\bigcup_j U_j$ by a set of Lebesgue measure zero. Moreover,

$$U_1 \subseteq U_2 \subseteq \cdots \nearrow \bigcup_j U_j,$$

so $\mathbf{1}_U(x) = \lim_j \mathbf{1}_{U_j}(x)$ almost everywhere. Since the spectrum of $H \restriction \mathcal{K}_1$ is purely absolutely continuous on U, it follows that $\mathbf{1}_U(H \restriction \mathcal{K}_1) = \lim_j \mathbf{1}_{U_j}(H \restriction \mathcal{K}_1)$ strongly. Since $\mathcal{K}_1$ and $\mathcal{K}_1{}^{\perp}$ are H-invariant, this last equation is equivalent to

$$\mathbf{1}_U(H) \restriction \mathcal{K}_1 = \lim_j \mathbf{1}_{U_j}(H) \restriction \mathcal{K}_1 \quad \text{strongly.}$$

Consequently, $\{\mathbf{1}_{U_j}(H)\mathcal{K}_n \; ; \; j \geqslant 1\}$ is dense in $\mathbf{1}_U(H)\mathcal{K}_n$, so in particular

$$\mathcal{D}_n' = \{f(H)\mathbf{1}_{U_j}(H)\delta_n \; ; \; f \in L^\infty(\mathbb{R}), j \geqslant 1\}$$

---

[8] We declare w.l.o.g. $1 \notin X$ in order to avoid a conflictual notation with $\mathcal{K}_1$.

[9] Here, $\sum_n \mathcal{D}_n$ denotes the linear, not necessarily closed space generated by $\cup_n \mathcal{D}_n$, and similarly for $\sum_n \mathcal{K}_n$.

is dense in $\mathbf{1}_U(H)\mathcal{K}_n$. It thus suffices to show

$$\sum_{\underline{M}} \int_{\mathbb{R}} |\langle \delta_{\underline{M}} \,|\, \mathrm{e}^{-\mathrm{i}tH}\varphi \rangle|^2 \,\mathrm{d}t < \infty$$

for any $\varphi \in \mathcal{D}'_n$, say, for $\varphi = f(H)\mathbf{1}_{U_j}(H)\delta_n$, where $f \in L^\infty(\mathbb{R})$, $j \geqslant 1$, and $n \in \Gamma$ are arbitrarily fixed.

By assumption the spectral measure of $\delta_{\underline{M}}$ and δ_n with respect to $H \restriction \mathcal{K}_1$ is purely absolutely continuous on U_j. In particular,

$$\begin{aligned} \langle \delta_{\underline{M}} \,|\, \mathrm{e}^{-\mathrm{i}tH}\varphi \rangle &= \int_{U_j} \mathrm{e}^{-\mathrm{i}te} f(e) \frac{1}{\pi} \mathrm{Im}\, \langle \delta_{\underline{M}} \,|\, (H - e - \mathrm{i}0)^{-1}\delta_n \rangle \,\mathrm{d}e \\ &= \sqrt{\frac{2}{\pi}} \mathcal{F}[\mathbf{1}_{U_j}(e) f(e) \mathrm{Im}\, \langle \delta_{\underline{M}} \,|\, (H - e - \mathrm{i}0)^{-1}\delta_n \rangle](t), \end{aligned}$$

where $\mathcal{F}$ denotes the Fourier transform. By Plancherel's theorem,

$$\int_{\mathbb{R}} |\langle \delta_{\underline{M}} \,|\, \mathrm{e}^{-\mathrm{i}tH}\varphi \rangle|^2 \,\mathrm{d}t = \frac{2}{\pi} \int_{U_j} |f(e)|^2 |\mathrm{Im}\, \langle \delta_{\underline{M}} \,|\, (H - e - \mathrm{i}0)^{-1}\delta_n \rangle|^2 \,\mathrm{d}e.$$

It follows that

$$\begin{aligned} \sum_{\underline{M}} \int_{\mathbb{R}} |\langle \delta_{\underline{M}} \,|\, \mathrm{e}^{-\mathrm{i}tH}\varphi \rangle|^2 \,\mathrm{d}t &= \\ &= \sum_{\underline{M}} \frac{2}{\pi} \int_{U_j} |f(e)|^2 |\mathrm{Im}\, \langle \delta_{\underline{M}} \,|\, (H - e - \mathrm{i}0)^{-1}\delta_n \rangle|^2 \,\mathrm{d}e \\ &\leqslant \frac{2}{\pi} \|f\|_\infty^2 \, j |U_j| \\ &\leqslant \frac{4}{\pi} \|f\|_\infty^2 \, j^2 \\ &< \infty, \end{aligned}$$

which completes the proof. $\square$

The scholium of Theorem 37 and the above immediately give another criterion of "completeness" in the sense that:

Corollary (Jakšić–Last) *Under the hypotheses of the previous theorem, if in addition* $\mathbf{1}_1$ *is* H_0*-smooth on* U*, then the wave operators* $\Omega^{\pm}(H_0, H)$ *exist on* U*.*

3.5 Random Result Inside $\mathrm{spec}(H_0)$

We now turn to the study of the Anderson type Hamiltonian $H = H_0 + V$, where $\{V(n)\}_{n\in\Gamma}$ is a family of independent, identically distributed random variables of law μ.[10] This time, we do not assume that μ is compactly supported, *i.e.*, that V is bounded almost surely; some cases where μ has full support are in fact of special interest (*cf.* Scholium of Theorem 61).

In this section we establish a sufficient condition for the following, stronger version of (3.13) to hold:

$$\forall n \in \Gamma : \|\mathbf{1}_1(H - e - \mathrm{i}0)^{-1}\delta_n\| < \infty. \tag{3.14}$$

[10] Explicitly, the probability space $\Omega = \mathbb{R}^\Gamma$ is endowed with its Borel σ-algebra and the probability measure $\mathrm{d}\mathbb{P} = \prod_{n\in\Gamma} \mathrm{d}\mu$, where $\mathrm{d}\mu$ is a given probability measure on $\mathbb{R}$. The n^{th} random variable in our family is then the projection $\Omega \longrightarrow \mathbb{R}$, $V \mapsto V(n)$.

Our computations are based on the Aizenman–Molchanov theory and apply under these conditions on μ:

1. μ is absolutely continuous,
2. For a given $0 < s < 1$ the decoupling constants

$$\begin{aligned} k_s &= \inf_{\alpha,\beta\in\mathbb{C}} \frac{\int_{\mathbb{R}} |x-\alpha|^s |x-\beta|^{-s}\, \mathrm{d}\mu(x)}{\int_{\mathbb{R}} |x-\beta|^{-s}\, \mathrm{d}\mu(x)} \quad \text{and} \\ K_s &= \sup_{\beta\in\mathbb{C}} \frac{\int_{\mathbb{R}} |x|^s |x-\beta|^{-s}\, \mathrm{d}\mu(x)}{\int_{\mathbb{R}} |x-\beta|^{-s}\, \mathrm{d}\mu(x)} \end{aligned}$$

   satisfy $k_s > 0$ and $K_s < \infty$.

Given a non empty, open interval $]a, b[$, let us define $\tau(M, N)$ as in Section 3.3 and let

$$\mathfrak{I} = \inf_{n,z} |\langle \delta_n \,|\, (H_0 - z)^{-1} \delta_n \rangle|,$$

where following our convention $n \in \Gamma$ and $z \in \{e + iy \,;\; a < e < b \text{ and } 0 < y < 1\}$.[11] We now state our assumptions on H_0, which involve these last quantities and

$$l^{(s)}(m) = \sum_{n\neq m} \tau(n, m)^s.$$

Assumption K *$l^{(s)}(m) \to 0$ when $m \to \infty$, i.e., for all $\varepsilon > 0$ there exists a finite set $\mathcal{F} \subseteq \Gamma$ such that $l^{(s)}(m) < \varepsilon$ for all $m \notin \mathcal{F}$.*

[11] This replacement of I with $\mathfrak{I}$ becomes essential in Section 3.6 only.

Assumption L $\sup_n \sum_{\underline{M}} \tau(n, \underline{M})^s < \infty$.

Assumption M $\mathfrak{I} > 0$.

On concrete examples Assumptions K and L come from sparseness of Γ and an *a priori* estimate on the Green's function associated with H_0. At the level of random operators theory Assumption K permits to control the l^1 and l^∞ norms of the operator whose matrix elements are $\tau(m,n)^s$; this is essential for the Aizenman–Molchanov method to apply. Explicitly, there exists a finite set, $\mathcal{F} \subset \Gamma$ (*which we now fix until the end of this section*), such that

$$l^{(s)}(m) < \frac{\mathfrak{I}^s k_s}{2K_s}$$

for all $m \notin \mathcal{F}$. *Making the convention that $\underline{n}$, $\underline{m}$, ...* $\in \Gamma \setminus \mathcal{F}$, the previous relation is equivalent to

$$\sup_{\underline{m}} \sum_{n \neq \underline{m}} \tau(n, \underline{m})^s < \frac{\mathfrak{I}^s k_s}{2K_s}. \tag{3.15}$$

The Aizenman–Molchanov method will then apply and provide an estimate on the resolvents when restricting the potential to $l^2(\Gamma \setminus \mathcal{F})$. Then, Assumption L will permit to recover from this last estimate the full strength of the equation (3.14).

Remark *Let $0 \leqslant R \leqslant \infty$ be a fixed integer. Remarkably, all computations derived in the present section still work when replacing Γ_1 with Γ_R. Hence, for later reference* *the conscientious reader will adopt the convention*

$$\underline{N},\ \underline{M},\ \ldots \in \Gamma_R$$

for an arbitrarily fixed $R \in \mathbb{N} \cup \{\infty\}$. *The interesting case in the present section is still* $R = 1$, *but results are used in Section 3.6 with* $R = \infty$.

Let us define

$$\widehat{H} = H_0 + \sum_{\underline{n}} V(\underline{n})\langle \delta_{\underline{n}} \,|\, \cdot \rangle \delta_{\underline{n}}$$

and use the abbreviations

$$\begin{aligned} R_0(N, M, z) &= \langle \delta_N \,|\, (H_0 - z)^{-1} \delta_M \rangle, \\ R(N, M, z) &= \langle \delta_N \,|\, (H - z)^{-1} \delta_M \rangle, \\ \widehat{R}(N, M, z) &= \langle \delta_N \,|\, (\widehat{H} - z)^{-1} \delta_M \rangle. \end{aligned}$$

The following lemma implies that $R(N, M, z)$ is "symmetric in the real sense" for any fixed $z \in \mathcal{S}$, *i.e.*,

$$R(N, M, z) = R(M, N, z),$$

which is repeatedly used in the sequel without explicit mention. Of course similar conclusions hold for R_0 and $\widehat{R}$.

Lemma *The spectral measure of* δ_M *and* δ_N *with respect to* H *is real valued.*

Proof: Since the characteristic function of any Borel set is approachable in the L^2 sense by bounded, continuous, real valued functions, it suffices to show

$$\langle \delta_M \mid f(H)\delta_N \rangle \in \mathbb{R}$$

for any such f. Since $\langle \delta_M \mid (H_0 + V)^j \delta_N \rangle \in \mathbb{R}$ for any $j \geqslant 0$,

$$\langle \delta_M \mid p(H)\delta_N \rangle \in \mathbb{R}$$

for any real valued polynomial, p. Assuming first that V is bounded, the Weierstrass Theorem (applied to real valued functions on the interval $[-\|H\|, \|H\|]$) implies

$$\langle \delta_M \mid f(H)\delta_N \rangle \in \mathbb{R}$$

for any bounded, continuous, real valued f.

Suppose now that V is not bounded. Let

$$V_L(n) = \begin{cases} V(n) & \text{if } |V(n)| \leqslant L, \\ 0 & \text{otherwise} \end{cases}$$

and $H_L = H_0 + V_L$.

We claim that $\lim_{L\to\infty} H_L = H$ in the strong resolvent sense. Indeed, by the resolvent identity

$$(H - z)^{-1} - (H_L - z)^{-1} = -(H_L - z)^{-1}(V - V_L)(H - z)^{-1}.$$

Notice that $(H - z)^{-1}\varphi \in \operatorname{dom} V$ for any $\varphi \in \mathcal{H}$. Hence, it suffices to show

$$\lim_{L\to\infty} (V - V_L)\psi = 0$$

for any $\psi \in \operatorname{dom} V$. In fact, by the dominated convergence theorem (with dominator $|V(n)\psi(n)|^2$)

$$\|(V - V_L)\psi\|^2 = \sum_n \mathbf{1}_{\{|V(n)|>L\}} |V(n)|^2 |\psi(n)|^2 \overset{L\to\infty}{\longrightarrow} 0,$$

which proves our claim.

Consequently (see [40]), for any bounded Borel f

$$\lim_{L\to\infty} f(H_0 + V_L) = f(H) \quad \text{strongly,}$$

a fortiori

$$\langle \delta_M \mid f(H)\delta_N \rangle = \lim_{L\to\infty} \langle \delta_M \mid f(H_0 + V_L)\delta_N \rangle,$$

which is real (for f as stipulated) by the first part of the proof. □

In the sequel we use repeatedly the Aizenman–Molchanov decoupling lemma in conjunction with the resolvent identity; this latter implies

$$\widehat{R}(N, M, z) = R_0(N, M, z) - \sum_{\underline{p}} R_0(N, \underline{p}, z) V(\underline{p}) \widehat{R}(\underline{p}, M, z) \tag{3.16}$$

for any $M, N \in X$. As a first instance,

Lemma [12]

$$\mathbb{E}\,|\widehat{R}(\underline{n},\underline{m},z)|^s \leqslant \frac{1}{k_s\mathfrak{J}^s}\tau(\underline{n},\underline{m})^s + \frac{K_s}{k_s\mathfrak{J}^s}\sum_{\underline{p}\neq\underline{n}}\tau(\underline{n},\underline{p})^s\mathbb{E}\,|\widehat{R}(\underline{p},\underline{m},z)|^s.$$

Proof: By the equation (3.16)

$$\widehat{R}(\underline{n},\underline{m},z)(1+R_0(\underline{n},\underline{n},z)V(\underline{n})) = R_0(\underline{n},\underline{m},z) - \sum_{\underline{p}\neq\underline{n}} R_0(\underline{n},\underline{p},z)V(\underline{p})\widehat{R}(\underline{p},\underline{m},z).$$

Using the triangle inequality for $|\cdot|^s$ and then taking the expectation,

$$\mathbb{E}\,|1+R_0(\underline{n},\underline{n},z)V(\underline{n})|^s|\widehat{R}(\underline{n},\underline{m},z)|^s \leqslant$$
$$\leqslant |R_0(\underline{n},\underline{m},z)|^s + \sum_{\underline{p}\neq\underline{n}}|R_0(\underline{n},\underline{p},z)|^s\mathbb{E}\,|V(\underline{p})|^s|\widehat{R}(\underline{p},\underline{m},z)|^s.$$

The decoupling lemmas then give

$$k_s|R_0(\underline{n},\underline{n},z)|^s\mathbb{E}\,|\widehat{R}(\underline{n},\underline{m},z)|^s \leqslant$$
$$\leqslant |R_0(\underline{n},\underline{m},z)|^s + K_s\sum_{\underline{p}\neq\underline{n}}|R_0(\underline{n},\underline{p},z)|^s\mathbb{E}\,|\widehat{R}(\underline{p},\underline{m},z)|^s$$

[12] Our convention consists of using parentheses with $\mathbb{E}$ in analogy to Σ. For instance, we write $\mathbb{E}\,X^s$ for $\mathbb{E}\,(X^s)$ as opposed to $(\mathbb{E}\,X)^s$, and $\mathbb{E}\,XY$ for $\mathbb{E}\,(XY)$ as opposed to $(\mathbb{E}\,X)Y$.

and the result follows from the definitions of $\mathfrak{I}$ and τ. $\square$

Let us fix $\underline{m}$ and z, $\underline{n} \in \Gamma \setminus \mathcal{F}$ being thought as the only variable. We define the following vectors on $l^\infty(\Gamma \setminus \mathcal{F})$:

$$\begin{aligned} X(\underline{n}) &= \mathbb{E}\,|\widehat{R}(\underline{n}, \underline{m}, z)|^s, \\ B(\underline{n}) &= \frac{1}{k_s \mathfrak{I}^s} \tau(\underline{n}, \underline{m})^s. \end{aligned}$$

They are well defined, since

$$\|X\|_\infty \leqslant |\mathrm{Im}\, z|^{-s}$$

and $\|B\|_\infty < \infty$, the latter by Assumption L . Indeed, this last assumption also ensures that $\|B\|_1 < \infty$. Let us define the operator

$$(A\psi)(\underline{n}) = \frac{K_s}{k_s \mathfrak{I}^s} \sum_{\underline{p} \neq \underline{n}} \tau(\underline{n}, \underline{p})^s \psi(\underline{p}),$$

which acts on both $l^\infty(\Gamma \setminus \mathcal{F})$ and $l^1(\Gamma \setminus \mathcal{F})$, where it is bounded. Indeed, by the equation (3.15)

$$\|A\|_\infty = \|A\|_1 = \frac{K_s}{k_s \mathfrak{I}^s} \sup_{\underline{n}} \sum_{\underline{p} \neq \underline{n}} \tau(\underline{n}, \underline{p})^s < \frac{1}{2},$$

where we have used $\tau(\underline{n}, \underline{p})^s = \tau(\underline{p}, \underline{n})^s$; see also Appendix 4.3. Hence, the previous lemma may be restated as

$$X \leqslant AX + B \quad \text{(pointwise)}.$$

From this fact and since we have controlled the l^1 and l^∞ norms of the operator A, we obtain:

Theorem 50 $\sup_z \sup_{\underline{m}} \sum_{\underline{n}} \mathbb{E}\,|\widehat{R}(\underline{n},\underline{m},z)|^s < \infty.$

Proof: By the lemma, $(1-A)X \leqslant B$ (pointwise). Furthermore, by our choice of $\mathcal{F}$, $\|A\|_1 = \|A\|_\infty < \frac{1}{2}$, so the geometric series gives

$$(1-A)^{-1} = \sum_{j=0}^{\infty} A^j \quad \text{and} \quad \|(1-A)^{-1}\|_1 \leqslant 2.$$

Since all matrix elements of A are positive, the matrix elements of $(1-A)^{-1}$ are also positive. Thus, this last operator preserves pointwise positivity of vectors. In particular,

$$X \leqslant (1-A)^{-1}B \quad \text{(pointwise)}, \tag{3.17}$$

so

$$\|X\|_1 \leqslant \|(1-A)^{-1}\|_1 \|B\|_1 \leqslant 2\|B\|_1.$$

Explicitly,

$$\sum_{\underline{n}} \mathbb{E}\,|\widehat{R}(\underline{n},\underline{m},z)|^s \leqslant \frac{2}{k_s \mathfrak{J}^s} \sum_{\underline{n}} \tau(\underline{n},\underline{m})^s.$$

Since $\underline{m}$ and z are arbitrary, Assumption L finally yields

$$\sup_z \sup_{\underline{m}} \sum_{\underline{n}} \mathbb{E}\,|\widehat{R}(\underline{n},\underline{m},z)|^s \leqslant \frac{2}{k_s \mathfrak{J}^s} \sup_{\underline{m}} \sum_{\underline{n}} \tau(\underline{n},\underline{m})^s < \infty.$$

$\square$

We now use the full strength of Assumption L in order to improve the last theorem. Before stating the resulting theorem, we need:

Lemma

$$\mathbb{E}\,|\widehat{R}(N,M,z)|^s \leqslant \tau(N,M)^s + K_s \sum_{\underline{p}} \tau(N,\underline{p})^s \mathbb{E}\,|\widehat{R}(\underline{p},M,z)|^s. \tag{3.18}$$

Proof: The triangle inequality for $|\cdot|^s$ applied to (3.16) gives

$$|\widehat{R}(N,M,z)|^s \leqslant |R_0(N,M,z)|^s + \sum_{\underline{p}} |R_0(N,\underline{p},z)|^s |V(\underline{p})|^s |\widehat{R}(\underline{p},M,z)|^s.$$

Taking the expectation,

$$\begin{aligned}
\mathbb{E}\,|\widehat{R}(N,M,z)|^s &\leqslant |R_0(N,M,z)|^s + \sum_{\underline{p}} |R_0(N,\underline{p},z)|^s \mathbb{E}\,|V(\underline{p})|^s |\widehat{R}(\underline{p},M,z)|^s \\
&\leqslant \tau(N,M)^s + \sum_{\underline{p}} \tau(N,\underline{p})^s \mathbb{E}\,|V(\underline{p})|^s |\widehat{R}(\underline{p},M,z)|^s,
\end{aligned}$$

so the decoupling lemma yields the result. □

Theorem 51 $\sup_z \sup_n \sum_{\underline{M}} \mathbb{E}\,|\widehat{R}(n,\underline{M},z)|^s < \infty.$

Proof: Let

$$\begin{aligned}
C &= \sup_n \sum_{\underline{M}} \tau(n,\underline{M})^s, \\
D &= \sup_z \sup_{\underline{m}} \sum_{\underline{n}} \mathbb{E}\,|\widehat{R}(\underline{n},\underline{m},z)|^s,
\end{aligned}$$

which are finite by Assumption L and the previous theorem respectively. By the lemma,

$$\mathbb{E}\,|\widehat{R}(\underline{N},\underline{m},z)|^s \leqslant \tau(\underline{N},\underline{m})^s + K_s \sum_{\underline{p}} \tau(\underline{N},\underline{p})^s \mathbb{E}\,|\widehat{R}(\underline{p},\underline{m},z)|^s,$$

so

$$\sum_{\underline{N}} \mathbb{E}\,|\widehat{R}(\underline{N},\underline{m},z)|^s \leqslant C + K_s C \sum_{\underline{p}} \mathbb{E}\,|\widehat{R}(\underline{p},\underline{m},z)|^s$$

and hence

$$\sup_z \sup_{\underline{m}} \sum_{\underline{N}} \mathbb{E}\,|(\widehat{R}(\underline{N},\underline{m},z)|^s \leqslant C + K_s CD.$$

By the lemma,

$$\mathbb{E}\,|\widehat{R}(n,\underline{M},z)|^s \leqslant \tau(n,\underline{M})^s + K_s \sum_{\underline{p}} \tau(n,\underline{p})^s \mathbb{E}\,|\widehat{R}(\underline{p},\underline{M},z)|^s,$$

so by the above

$$\begin{aligned}\sum_{\underline{M}} \mathbb{E}\,|\widehat{R}(n,\underline{M},z)|^s &\leqslant C + K_s C \sup_{\underline{q}} \sum_{\underline{M}} \mathbb{E}\,|\widehat{R}(\underline{q},\underline{M},z)|^s \\ &\leqslant C + K_s C(C + K_s CD) < \infty\end{aligned}$$

uniformly in n and z, as desired. □

The previous theorem is used in the following weaker form only:

$$\forall n : \sup_z \sum_{\underline{M}} \mathbb{E}\,|\widehat{R}(n,\underline{M},z)|^s < \infty.$$

We want to abstract from this last relation information about $\widehat{R}(n,\underline{M},e+\mathrm{i}0)$. By classical Harmonic Analysis, for a given potential V the previous limit exists almost everywhere (a.e.) on $]a, b[$. In the case where V is random the limit in question exists almost everywhere and almost surely (a.e. & a.s.) on $]a, b[\times \Omega$ (by Fubini's theorem). We obtain:

Lemma *For all $n \in \Gamma$*

$$\sum_{\underline{M}} |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^s < \infty \text{ a.e. \& a.s. on }]a, b[\times \Omega.$$

Proof: Let n be fixed. Since $\widehat{R}(n, \underline{M}, e + \mathrm{i}0)$ exists a.e. & a.s. on $]a, b[\times \Omega$,

$$\begin{aligned} \int_a^b \mathbb{E} \sum_{\underline{M}} |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^s \, \mathrm{d}e &\leqslant (b - a) \operatorname*{ess\,sup}_{a<e<b} \mathbb{E} \sum_{\underline{M}} |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^s \\ &= (b - a) \operatorname*{ess\,sup}_{a<e<b} \sum_{\underline{M}} \mathbb{E} \, |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^s, \end{aligned}$$

where ess sup denotes the essential supremum with respect to the Lebesgue measure. By Fatou's lemma, it follows that

$$\begin{aligned} \int_a^b \mathbb{E} \sum_{\underline{M}} |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^s \, \mathrm{d}e &\leqslant \\ &\leqslant (b - a) \operatorname*{ess\,sup}_{a<e<b} \liminf_{\varepsilon \downarrow 0} \sum_{\underline{M}} \mathbb{E} \, |\widehat{R}(n, \underline{M}, e + \mathrm{i}\varepsilon)|^s \\ &\leqslant (b - a) \sup_z \sum_{\underline{M}} \mathbb{E} \, |\widehat{R}(n, \underline{M}, z)|^s, \end{aligned}$$

which is finite by the previous theorem. The result follows. □

The triangle inequality for $|\cdot|^{\frac{s}{2}}$ immediately yields:

Corollary *For all $n \in \Gamma$*

$$\sum_{\underline{M}} |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^2 < \infty \text{ a.e. \& a.s. on }]a, b[\times \Omega.$$

It remains to go from $\widehat{R}$ to R to conclude the argument:

Theorem 52 *Under the assumptions of the present section (with respect to a given $0 \leqslant R \leqslant \infty$), for all $n \in \Gamma$*

$$\|\mathbf{1}_R(H - e - \mathrm{i}0)^{-1}\delta_n\| < \infty \text{ a.e. \& a.s. on }]a, b[\times \Omega.$$

Proof: Let $n \in \Gamma$ be fixed. By the resolvent identity

$$R(n, \underline{M}, z) = \widehat{R}(n, \underline{M}, z) - \sum_{p \in \mathcal{F}} V(p)\widehat{R}(p, \underline{M}, z)R(n, p, z)$$

for any $V \in \Omega$ and $z \in \mathcal{S}$. Hence, by classical Harmonic Analysis

$$R(n, \underline{M}, e + \mathrm{i}0) = \widehat{R}(n, \underline{M}, e + \mathrm{i}0) - \sum_{p \in \mathcal{F}} V(p)\widehat{R}(p, \underline{M}, e + \mathrm{i}0)R(n, p, e + \mathrm{i}0)$$

a.e. & a.s. on $]a, b[\times \Omega$. Consequently, by Schwarz' inequality

$$|R(n, \underline{M}, e + \mathrm{i}0)|^2 \leqslant$$

$$\leqslant A\,(|\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^2 + \sum_{p \in \mathcal{F}} |V(p)|^2|\widehat{R}(p, \underline{M}, e + \mathrm{i}0)|^2|R(n, p, e + \mathrm{i}0)|^2)$$

a.e. & a.s., where A is the number of elements of $\mathcal{F}$ plus one. Consequently,

$$\sum_{\underline{M}} |R(n, \underline{M}, e + \mathrm{i}0)|^2 \leqslant$$
$$\leqslant A\,(\sum_{\underline{M}} |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^2 + M(e) \sum_{p \in \mathcal{F}} |V(p)|^2 |R(n, p, e + \mathrm{i}0)|^2)$$

a.e. & a.s., where $M(e) = \max_{p\in\mathcal{F}} \sum_{\underline{M}} |\widehat{R}(p, \underline{M}, e + \mathrm{i}0)|^2$. Notice that the finiteness of $\mathcal{F}$ and the previous corollary give $M(e) < \infty$ a.e. & a.s.; the latter also gives $\sum_{\underline{M}} |\widehat{R}(n, \underline{M}, e+\mathrm{i}0)|^2 < \infty$ a.e. & a.s. for our fixed n, while the former and classical Harmonic Analysis yield

$$\sum_{p\in\mathcal{F}} |V(p)|^2 |R(n, p, e + \mathrm{i}0)|^2 < \infty \text{ a.e. \& a.s.}$$

Hence, $\sum_{\underline{M}} |R(n, \underline{M}, e + \mathrm{i}0)|^2 < \infty$ almost everywhere and almost surely on $]a, b[\times\Omega$. In other words,

$$\|\mathbf{1}_R(H - e - \mathrm{i}0)^{-1}\delta_n\|^2 < \infty \text{ a.e. \& a.s.,}$$

which concludes the proof. □

3.5.1 Conclusion

As already noticed, the previous theorem is a technical requirement for the second criterion of completeness to apply. We are especially interested in the following context, which may be realized inside the spectrum of H_0 only (see the remark below):

Lemma *Suppose*

$$\sum_{\underline{N}} \mathrm{Im}\,\langle \delta_{\underline{N}} \,|\, (H_0 - e - \mathrm{i}0)^{-1} \delta_{\underline{N}} \rangle \;>\; 0$$

for almost all $e \in \,]a, b[$, *and that the wave operator* $\Omega^+(H, H_0)$ *exists almost surely on* $]a, b[$. *Then, on* $]a, b[$, *the spectrum of* $H \restriction \mathcal{K}_1$ *is purely absolutely continuous and its essential support is full, almost surely.*

Proof: Let $\mu_{\underline{N}}$ be the spectral measure of $\delta_{\underline{N}}$ with respect to H_0. Then, for an arbitrary choice of $a_{\underline{N}} > 0$ such that $\sum_{\underline{N}} a_{\underline{N}} = 1$,

$$\mu_{H_0} = \sum_{\underline{N}} a_{\underline{N}} \mu_{\underline{N}}$$

is a spectral measure for $H_0 \restriction \mathcal{K}_1$, whose Radon–Nicodym derivative with respect to the Lebesgue measure is equal to

$$\frac{1}{\pi} \sum_{\underline{N}} a_{\underline{N}} \mathrm{Im}\,\langle \delta_{\underline{N}} \,|\, (H_0 - e - \mathrm{i}0)^{-1} \delta_{\underline{N}} \rangle.$$

Thus, our first assumption asserts that the absolutely continuous spectrum of $H_0 \restriction \mathcal{K}_1$ on $]a, b[$ has full essential support.

By assumption, $\Omega^+_{]a,b[}(H, H_0)$, equivalently $\Omega^+_{]a,b[}(H \restriction \mathcal{K}_1, H_0 \restriction \mathcal{K}_1)$, exists almost surely. Let V be a potential for which this last property holds. Then, the restrictions $H_0 \restriction \mathbf{1}_{]a,b[}(H_0)\mathcal{K}_1$ and $H \restriction \Omega^+(H, H_0)\mathcal{K}_1$ are unitarily equivalent (see Section 3.1). In particular, their spectral measures, which we denote by $\mu_{H_0 \restriction \mathbf{1}_{]a,b[}(H_0)\mathcal{K}_1}$ and $\mu_{H \restriction \Omega^+(H,H_0)\mathcal{K}_1}$ respectively, are equivalent. It follows that the Radon–Nicodym derivative of $\mu_{H \restriction \Omega^+(H,H_0)\mathcal{K}_1}$ with respect to the Lebesgue measure is strictly positive on $]a, b[$.

Moreover, $\Omega^+(H,H_0)\mathcal{K}_1 \subseteq \mathbf{1}_{]a,b[}(H)\mathcal{K}_1$ by Proposition 31 in Section 3.1. Denoting by $\mathcal{O}$ the orthogonal complement of $\Omega^+(H,H_0)\mathcal{K}_1$ in $\mathbf{1}_{]a,b[}(H)\mathcal{K}_1$,

$$\mu_{H\restriction \mathbf{1}_{]a,b[}(H)\mathcal{K}_1} = \mu_{H\restriction \Omega^+(H,H_0)\mathcal{K}_1} + \mu_{H\restriction \mathcal{O}}$$

is a spectral measure for $H \restriction \mathbf{1}_{]a,b[}(H)\mathcal{K}_1$, since $\Omega^+(H,H_0)\mathcal{K}_1$ is H-invariant. Thus, the Radon–Nicodym derivative of $\mu_{H\restriction \mathbf{1}_{]a,b[}(H)\mathcal{K}_1}$ with respect to the Lebesgue measure is also strictly positive. In particular, the essential support of $\mathrm{spec}_{\mathrm{ac}}(H \restriction \mathcal{K}_1)$ on $]a,b[$ is full for any such V, *i.e.*, almost surely. The Jakšić–Last theorem finally yields that $\mathrm{spec}(H \restriction \mathcal{K}_1)$ is purely absolutely continuous on $]a,b[$, almost surely. $\square$

Remark *The beginning of the previous argument shows that under the circumstances of the lemma,* $\mathrm{spec}(H_0)$ *is purely absolutely continuous on* $]a,b[$ *and has full support on this last interval, which justifies the title of the present section.*

Assumption L may be strengthened for the following lemma to apply:

Lemma *Assume* $\sup_{\underline{N}} \sum_{\underline{M}} \tau(\underline{N},\underline{M})^s < \infty$ *with respect to a given* $R \in \mathbb{N} \cup \{\infty\}$. *Then,* $\mathbf{1}_R$ *is* H_0*-smooth on* $]a,b[$.

Proof: By assumption

$$\begin{aligned} \infty &> \sup_{\underline{N}} \sum_{\underline{M}} \sup_{z} |\langle \delta_{\underline{N}} \,|\, (H_0 - z)^{-1} \delta_{\underline{M}} \rangle|^s \\ &\geqslant \sup_{z} \sup_{\underline{N}} \sum_{\underline{M}} |\langle \delta_{\underline{N}} \,|\, (H_0 - z)^{-1} \delta_{\underline{M}} \rangle|^s. \end{aligned}$$

By the triangle inequality for $|\cdot|^s$ it follows that

$$\sup_{z} \sup_{\underline{N}} \sum_{\underline{M}} |\langle \delta_{\underline{N}} \,|\, (H_0 - z)^{-1} \delta_{\underline{M}} \rangle| < \infty.$$

Interpreting $\mathbf{1}_R(H_0 - z)^{-1}\mathbf{1}_R$ as an operator on $l^2(\Gamma_R)$, its l^1 and l^∞ norms are then given by the above quantity. The Riesz–Thorin Theorem then implies (see Appendix 4.3)

$$\sup_{z} \|\mathbf{1}_R(H_0 - z)^{-1}\mathbf{1}_R\| < \infty.$$

Thus, $\mathbf{1}_R$ is H_0-smooth on $]a, b[$, as desired. □

Finally, this general, abstract conclusion may be drawn:

Theorem 53 *Suppose*

1. *The wave operators $\Omega^\pm(H, H_0)$ exist on $]a, b[$ almost surely,*
2. *$l^{(s)}(m) \to 0$ when $m \to \infty$,*
3. *$\mathfrak{I} > 0$,*
4. *For $R = 1$, $\sup_{\underline{N}} \sum_{\underline{M}} \tau(\underline{N}, \underline{M})^s < \infty$.*

Then, spec(H) *is purely absolutely continuous on $]a, b[$ and the wave operators $\Omega^\pm(H, H_0)$ are complete on $]a, b[$, almost surely.*

Proof: By the above lemmas, the theorem 52, and the second criterion of completeness (corollary of Theorem 49). □

3.5.2 Application to Generalized Laplacians

We now consider the case where $H_0 = \Delta$ is a generalized Laplacian. In the sequel Θ denotes an open region of validity of Theorems 20 and 21; in particular, we assume that the constant energy surfaces of the Green's function associated with H_0 at any level of energy inside Θ have at least $\kappa > 0$ non vanishing principal curvatures. For instance, if Δ is the standard or the Molchanov–Vainberg Laplacian, we let

$$\Theta = \mathrm{spec}(\Delta) \setminus E,$$

where $E = \{-2d, -2d+4, \ldots, 2d-4, 2d\} \cup \{0\}$ when Δ is the standard Laplacian and $E = \{-2^d, 0, 2^d\}$ when it is the Molchanov–Vainberg one.[13]

In Section 3.3.2 we established that for any $[a, b] \subset \Theta$

$$\inf_{n,z} \mathrm{Im}\, \langle \delta_n \,|\, (\Delta - z)^{-1} \delta_n \rangle > 0.$$

A fortiori, the condition 3 of the previous theorem holds. By the *a priori* estimates calculated in the first part of the present thesis, the conditions 2 and 4 reduce to a sparseness assumption on the sites of the potential:

[13] Recall that $\kappa = 1$ for the standard Laplacian in any dimension (without pretending that this result is optimal) and $\kappa = d - 1$ for the Molchanov–Vainberg Laplacian in dimension d.

Theorem 54 *In the present circumstances suppose the sites of the potential are* sparse *in the following sense: there exists an $\epsilon > 0$ such that*

$$\sum_{n \neq m} |n-m|^{-\frac{s\kappa}{2}+\epsilon} < \infty \quad \textit{for any } m \in \Gamma \textit{ and} \tag{3.19}$$

$$\lim_{|m|\to\infty} \sum_{n \neq m} |n-m|^{-\frac{s\kappa}{2}+\epsilon} = 0, \tag{3.20}$$

where m and n vary in Γ. If the wave operators $\Omega^{\pm}_{\Theta}(H, H_0)$ exist a.e., then they are complete on Θ and the spectrum of H is purely absolutely continuous on Θ, almost surely.

Proof: It suffices to show that the conditions 2 and 4 of the previous theorem apply for any $[a, b] \subset \Theta$. The former is an immediate consequence of the equation (3.20) and the estimate

$$\tau(n, m) = O(|n-m|^{-\frac{\kappa}{2}^{+}})$$

given by Theorem 22. We now prove the latter.

In many details, let $(\mathbb{Z}^d, \mathrm{d})$ be the graph from which the considered generalized Laplacian is defined. Since this graph is translational invariant and since the degrees of its vertices are bounded, there exists a constant, α, such that

$$|N-M| \leqslant \alpha \quad \text{when} \quad \mathrm{d}(N, M) \leqslant 1.$$

The condition 4 is weaker than the following,

$$\sup_{\underline{N}} \sum_{|\underline{M}-\underline{N}|>4\alpha} \tau(\underline{N}, \underline{M})^s + \sup_{\underline{N}} \sum_{|K|\leqslant 4\alpha} \tau(\underline{N}, \underline{N}+K) < \infty, \tag{3.21}$$

which we now verify. Indeed, translational invariance and the first part of the thesis yield that for any fixed K

$$\tau(\underline{N}, \underline{N} + K) = \sup_z |G(K, z)| < \infty.$$

In particular, the second sum in the equation (3.21) is finite and independent of $\underline{N}$. Hence, the problem reduces to show $\sup_{\underline{N}} \sum_{|\underline{M}-\underline{N}|>4\alpha} \tau(\underline{N}, \underline{M})^s < \infty$. Indeed, by Theorem 22, $\tau(\underline{N}, \underline{M}) = O(|\underline{N} - \underline{M}|^{-\frac{\kappa}{2}^+})$, so it suffices to show

$$\sup_{\underline{N}} \sum_{|\underline{M}-\underline{N}|>4\alpha} |\underline{N} - \underline{M}|^{-\frac{s\kappa}{2}+\epsilon} < \infty.$$

By the definition of Γ_1, each $\underline{M} \in \Gamma_1$ is adjacent to or equal with an $m \in \Gamma$. Since $|\underline{M} - m| \leqslant \alpha$ and since the degrees of the m's are bounded, it suffices to show $\sup_{\underline{N}} \sum_{|m-\underline{N}|>3\alpha} (|\underline{N} - m| - \alpha)^{-\frac{s\kappa}{2}+\epsilon} < \infty$. For the same reasons, but regarding $\underline{N}$ instead of $\underline{M}$, it suffices to show $\sup_n \sum_{|m-n|>2\alpha} (|n - m| - 2\alpha)^{-\frac{s\kappa}{2}+\epsilon} < \infty$, which is clearly equivalent to

$$\sup_n \sum_{|m-n|>2\alpha} |n - m|^{-\frac{s\kappa}{2}+\epsilon} < \infty.$$

Finally, this last relation is implied by (3.19), provided that (3.20) holds. The proof is now complete. $\square$

Remark *As explained in the proof of Theorem 45, if Δ is the standard Laplacian or the Molchanov–Vainberg one, existence and completeness of the wave operators on $\Theta = \mathrm{spec}(\Delta) \setminus E$ are equivalent to their existence and completeness on* $\mathrm{spec}(\Delta)$ *(with the suggested definition of E). This last equivalence holds for the two following* *reasons:* spec(Δ) *is absolutely continuous and E does almost surely not contain any eigenvalue of $H \restriction \mathcal{K}$—by a theorem of Jakšić and Last.*

Supplemental conditions may be imposed to the geometry of Γ in order to assure the existence of the wave operators. These include other sparseness conditions discussed in the literature—for instance, see [32]. It is thus natural to wonder if the conditions (3.20) and (3.19) indeed suffice. However, the present thesis does not answer this legitimate question.

Example Consider the Anderson type Hamiltonian $H = \Delta + V$, where Δ is the standard (or the Molchanov–Vainberg) Laplacian. Suppose the potential, V, consists of independent random variables, lying on

$$\Gamma = \{(j^4, 0, \ldots, 0) \in \mathbb{Z}^d \; ; \; j \in \mathbb{Z}\}$$

and whose common distribution is Cauchy (alternatively, normal).[14] Observe that Γ is sparse in the sense of the previous theorem (with s sufficiently close to 1). Moreover, the wave operators $\Omega^{\pm}(H, \Delta)$ exist on spec(Δ): since Γ is included in the hyperplane $\mathbb{Z}^{d-1} \subset \mathbb{Z}^d$, their existence follows from a deterministic result of Jakšić

[14] Notice that V is not bounded a.s., so the deterministic approach presented in Section 3.3 does not apply.

and Last [15].[15] Hence, by the previous theorem not only $\mathrm{spec}(H)$ is purely absolutely continuous on $\mathrm{spec}(\Delta)$, but the wave operators are also complete on this last region (almost surely).

3.6 Random Result Outside $\mathrm{spec}(H_0)$

In this section we apply techniques and results of the previous one for values of energy outside $\mathrm{spec}(H_0)$. We show that eigenfunctions associated with these values of energy decay exponentially. The setting and notations of the previous section (especially our convention regarding indices) are maintained in the sequel. In particular (without restating our hypotheses exhaustively) $H = H_0 + V$, where H_0 is the adjacency operator of a graph, (X, d), and V is random. Also, m, n vary in Γ, z varies in a set $\{x + \mathrm{i}y\ ;\ x \in\ ]a,b[\ \text{ and }\ 0 < y < 1\}$, where this time $[a,b] \subset \mathbb{R} \setminus \mathrm{spec}(H_0)$,

$$\mathfrak{I} = \inf_{n,z} |\langle \delta_n \mid (H_0 - z)^{-1} \delta_n \rangle|,$$

etc.

At the highest level of generality the desired decay of eigenfunctions will be expressed in terms of a given *weight*, γ, on X, that is, a function

$$\gamma\colon X \times X \to [0, \infty]$$

[15] The model considered in this last work is the half-space model (for which the Laplacian is not translational invariant) with a random potential at the boundary; however, according to V. Jakšić the argument in the mentioned work may be slightly modified in order to include the above situation.

satisfying all axioms of metric distances except positive definiteness.[16] Hence, in the sequel (X,γ) is a given weighted graph sharing its vertices, but not necessarily its edges with (X,d).[17]

Our results hold under the following hypotheses:

Assumption N *For any $k>0$, $\sup_N \sum_M \mathrm{e}^{-k\gamma(N,M)} < \infty$.*

Assumption O *There exist constants D and β such that*

$$\tau(n,\underline{M})^s \leqslant D\mathrm{e}^{-\beta\gamma(n,\underline{M})}$$

for all $n \in \Gamma$ and $\underline{M} \in \Gamma_R$.

Assumption P $\inf_{n\neq m}\gamma(n,m) \to \infty$ *when $m \to \infty$, i.e., for all $L>0$ there exists a finite set, $\mathcal{E} \subseteq \Gamma$, such that for every $m \notin \mathcal{E}$*

$$\inf_{n\neq m} \gamma(n,m) \geqslant L.$$

[16] More precisely, $\gamma(M,N)=0$ is allowed when $M \neq N$; however, $\gamma(N,N)=0$ for all $N \in X$.

[17] On concrete examples, $X = \mathbb{Z}^d$, d is translational invariant, and γ is the Pythagorean distance: $\gamma(M,N) = |M-N|$. Then, (X,d) is a translational invariant simple graph, while (X,γ) is a complete graph on $\mathbb{Z}^d$ whose vertices, $\{M,N\}$, are weighted by $|M-N|$ for any distinct $M,N \in \mathbb{Z}^d$.

Assumption Q $\mathfrak{I} > 0$.

Notice that Assumption O is realistic outside $\mathrm{spec}(H_0)$—which justifies the title of the present section. Moreover, Assumption Q is trivially verified in our applications (where $H_0 = \Delta$ comes from a translational invariant graph—see Section 2.5). Finally, Assumption N extends by induction:

Theorem 55 *For any k and α such that $0 < \alpha < k$ there exists a $C_{k,\alpha} > 0$ satisfying*

$$\sum_{P_1,\cdots,P_l} e^{-k(\gamma(N,P_1)+\gamma(P_1,P_2)+\cdots+\gamma(P_l,M))} \leqslant C_{k,\alpha}^l e^{-\alpha\gamma(N,M)} \tag{3.22}$$

for every $N, M \in X$ and $l \in \mathbb{N}$.

Proof: Since $0 < \alpha < k$, there exists an $s \in]0, 1[$ such that $\alpha = sk$. By Assumption N,

$$B_k = \sup_N \sum_M e^{-k\gamma(N,M)} < \infty$$

for any $k > 0$. Let us show that $C_{k,\alpha} = B_{tk}$ satisfies the desired property, where $t = 1 - s$.

The triangle inequality for γ implies that the left-hand side in (3.22) is bounded above by

$$\sum_{P_1,\ldots,P_l} e^{-tk(\gamma(M,P_1)+\cdots+\gamma(P_l,N))} e^{-\alpha\gamma(M,N)}$$

for any fixed $l \geqslant 0$. It thus suffices to show

$$\sum_{P_1,\ldots,P_l} e^{-tk(\gamma(M,P_1)+\cdots+\gamma(P_l,N))} \leqslant B_{tk}^l$$

for any $l \geqslant 0$, which we do by induction on l.

The result is trivial for $l = 0$, so suppose it holds for $l-1$. Then,

$$\begin{aligned}
\sum_{P_1,\ldots,P_l} \mathrm{e}^{-tk(\gamma(M,P_1)+\cdots+\gamma(P_l,N))} &= \\
&= \sum_{P_1} \mathrm{e}^{-tk\gamma(M,P_1)} \sum_{P_2,\ldots,P_l} \mathrm{e}^{-tk(\gamma(P_1,P_2)+\cdots+\gamma(P_l,N))} \\
&\leqslant B_{tk} B_{tk}^{l-1} = B_{tk}^l,
\end{aligned}$$

as desired. □

As a final preliminary remark,

Theorem 56 *All assumptions of Section 3.5 are satisfied.*

Proof: Assumption L follows from Assumptions N and O. More interestingly, Assumption K is satisfied, since

$$\begin{aligned}
l^{(s)}(n) &= \sum_{m\neq n} \tau(n,m)^s \\
&\leqslant D \sum_{m\neq n} \mathrm{e}^{-\beta\gamma(n,m)} \\
&\leqslant D \sup_{m\neq n} \mathrm{e}^{-\frac{\beta}{2}\gamma(n,m)} \sum_{q\neq n} \mathrm{e}^{-\frac{\beta}{2}\gamma(n,q)} \\
&\leqslant \left(D \sup_p \sum_{q\neq p} \mathrm{e}^{-\frac{\beta}{2}\gamma(p,q)} \right) \sup_{m\neq n} \mathrm{e}^{-\frac{\beta}{2}\gamma(n,m)},
\end{aligned}$$

which goes to zero when $n \to \infty$ by Assumptions N and P. Finally, Assumption M is satisfied by fiat. □

We are thus free to use results and computations of the previous section in order to establish our main theorem: *there exists a universal constant $k > 0$ such that the following assertion holds for almost all $e \in \left]a, b\right[$ and almost all $V \in \Omega$:*

For all $n \in \Gamma$ there exists a $K > 0$ such that

$$|\langle \delta_n \,|\, (H - e - \mathrm{i}0)^{-1} \delta_{\underline{M}} \rangle| \leqslant K \mathrm{e}^{-k\gamma(n, \underline{M})}$$

for all $\underline{M} \in \Gamma_R$.

The main part of the present section is devoted to proving the above, from which the exponential decay of the eigenfunctions will be deduced using Simon–Wolff's theorem.

Recall that $\mathcal{F} \subset \Gamma$ is a finite set, chosen in such a way that (3.15) holds (where $\underline{m}, \underline{n} \in \Gamma \setminus \mathcal{F}$). From now, by enlarging $\mathcal{F}$ if necessary, we also require[18]

$$\mathrm{e}^{-\frac{\beta}{2}\widehat{d}} < \frac{\mathfrak{I}^s k_s}{K_s C_{\frac{\beta}{2}, \frac{\beta}{3}} D}, \tag{3.23}$$

where $\widehat{d} = \inf_{\underline{m}} \inf_{\underline{n} \neq \underline{m}} \gamma(\underline{n}, \underline{m})$, which is possible by Assumption P.

Let $\underline{m}$ and z be fixed, $\underline{n}$ being thought as the only variable. Then, with the notation explained immediately before Theorem 50 the inequation (3.17) applies, namely

$$X \leqslant (1 - A)^{-1} B \quad \text{(pointwise)}.$$

Consequently,

[18] In the following expression, β, D, and $C_{\frac{\beta}{2}, \frac{\beta}{3}}$ refer to Assumption O and Theorem 55.

Lemma *$X \leqslant Const\,(1-A)^{-1}\delta_{\underline{m}}$ (pointwise), where the constant is universal.*

Proof: Since

$$\begin{aligned}(A\delta_{\underline{m}})(\underline{n}) &= \frac{K_s}{k_s\mathfrak{J}^s}(1-\delta_{\underline{n}})(\underline{m})\tau(\underline{n},\underline{m})^s \\ &= K_s B(\underline{n}) - \frac{K_s}{k_s\mathfrak{J}^s}\tau(\underline{m},\underline{m})^s\delta_{\underline{m}}(\underline{n}),\end{aligned}$$

it follows that $B = \frac{1}{K_s}A\delta_{\underline{m}} + \frac{1}{k_s\mathfrak{J}^s}\tau(\underline{m},\underline{m})^s\delta_{\underline{m}}$. The inequation (3.17) thus becomes

$$\begin{aligned}X &\leqslant \frac{1}{K_s}(1-A)^{-1}A\delta_{\underline{m}} + \frac{\tau(\underline{m},\underline{m})^s}{k_s\mathfrak{J}^s}(1-A)^{-1}\delta_{\underline{m}} \\ &= -\frac{1}{K_s}\delta_{\underline{m}} + \left(\frac{1}{K_s} + \frac{\tau(\underline{m},\underline{m})^s}{k_s\mathfrak{J}^s}\right)(1-A)^{-1}\delta_{\underline{m}} \\ &\leqslant \left(\frac{1}{K_s} + \frac{\tau(\underline{m},\underline{m})^s}{k_s\mathfrak{J}^s}\right)(1-A)^{-1}\delta_{\underline{m}} \quad \text{(pointwise)}.\end{aligned}$$

The result follows, where $Const$ is explicitly equal to $\frac{1}{K_s} + \frac{1}{k_s\mathfrak{J}^s}\sup_{\underline{p}}\tau(\underline{p},\underline{p})^s$, which is finite by Assumption L. □

Theorem 57 *There exist universal constants Const and k such that*

$$\mathbb{E}\,|\widehat{R}(\underline{n},\underline{m},z)|^s \leqslant Const\ \mathrm{e}^{-k\gamma(\underline{n},\underline{m})}$$

for all $\underline{n},\underline{m} \in \Gamma\setminus\mathcal{F}$ and $z \in \{x+\mathrm{i}y\ ;\ x\in\,]a,b[\,, 0<y<1\}$.

Proof: The lemma and the geometric series give

$$\mathbb{E}\,|\widehat{R}(\underline{n},\underline{m},z)|^s \leqslant Const\sum_{j=0}^{\infty}\langle\delta_{\underline{n}}\,|\,A^j\delta_{\underline{m}}\rangle. \tag{3.24}$$

Moreover,

$$A^j(\underline{n},\underline{m}) = \left(\frac{K_s}{k_s\mathfrak{J}^s}\right)^j \sum_{\underline{p}_1,\cdots,\underline{p}_{j-1}} \mathbf{1}_{\underline{n}\neq\underline{p}_1}\tau(\underline{n},\underline{p}_1)^s\cdots\mathbf{1}_{\underline{p}_{j-1}\neq\underline{m}}\tau(\underline{p}_{j-1},\underline{m})^s,$$

where $\mathbf{1}_{\underline{p}\neq\underline{q}}$ abbreviates $1-\delta_{\underline{p}}(\underline{q})$. Since by Assumption O

$$\begin{aligned}\mathbf{1}_{\underline{p}\neq\underline{q}}\tau(\underline{p},\underline{q})^s &\leqslant D\mathbf{1}_{\underline{p}\neq\underline{q}}\mathrm{e}^{-\beta\gamma(\underline{p},\underline{q})}\\ &\leqslant D\mathrm{e}^{-\frac{\beta\hat{d}}{2}}\mathrm{e}^{-\frac{\beta}{2}\gamma(\underline{p},\underline{q})},\end{aligned}$$

Theorem 55 yields

$$\begin{aligned}A^j(\underline{n},\underline{m}) &\leqslant \left(\frac{K_sD\mathrm{e}^{-\frac{\beta\hat{d}}{2}}}{k_s\mathfrak{J}^s}\right)^j \sum_{\underline{p}_1,\cdots,\underline{p}_{j-1}} \mathrm{e}^{-\frac{\beta}{2}\gamma(\underline{n},\underline{p}_1)}\cdots\mathrm{e}^{-\frac{\beta}{2}\gamma(\underline{p}_{j-1},\underline{m})}\\ &\leqslant \frac{1}{C_{\frac{\beta}{2},\frac{\beta}{3}}}\left(\frac{K_sC_{\frac{\beta}{2},\frac{\beta}{3}}D\mathrm{e}^{-\frac{\beta\hat{d}}{2}}}{k_s\mathfrak{J}^s}\right)^j \mathrm{e}^{-\frac{\beta}{3}\gamma(\underline{n},\underline{m})}.\end{aligned}$$

By choice of $\mathcal{F}$ the equation (3.23) holds, so there exist constants *Const* and k such that

$$\sum_{j=0}^{\infty} A^j(\underline{n},\underline{m}) \leqslant \mathit{Const}\,\mathrm{e}^{-k\gamma(\underline{n},\underline{m})}.$$

The equation (3.24) then completes the proof. □

We now use the full strength of Assumption O in order to improve the previous result:

Corollary *There exist universal constants Const and k such that*

$$\mathbb{E}\,|\widehat{R}(n,\underline{M},z)|^s \leqslant \mathit{Const}\ \mathrm{e}^{-k\gamma(n,\underline{M})}.$$

Proof: The equation (3.18), Assumption O, and the previous theorem yield

$$\begin{aligned}\mathbb{E}\,|\widehat{R}(\underline{N},\underline{m},z)|^s &\leqslant \tau(\underline{N},\underline{m})^s + K_s \sum_{\underline{p}} \tau(\underline{N},\underline{p})^s \mathbb{E}\,|\widehat{R}(\underline{p},\underline{m},z)|^s \\ &\leqslant \mathit{Const}\ \mathrm{e}^{-k\gamma(\underline{N},\underline{m})} + K_s \sum_{\underline{p}} \mathit{Const}\ \mathrm{e}^{-k\gamma(\underline{N},\underline{p})}\mathrm{e}^{-k\gamma(\underline{p},\underline{m})}\end{aligned}$$

for some constants generically denoted by *Const* and k. It follows from Theorem 55 that $\mathbb{E}\,|\widehat{R}(\underline{N},\underline{m},z)|^s \leqslant \mathit{Const}\,\mathrm{e}^{-k\gamma(\underline{N},\underline{m})}$. Using this last relation and the equation (3.18) again, a similar computation gives

$$\begin{aligned}\mathbb{E}\,|\widehat{R}(n,\underline{M},z)|^s &\leqslant \tau(n,\underline{M})^s + K_s \sum_{\underline{p}} \tau(n,\underline{p})^s \mathbb{E}\,|\widehat{R}(\underline{p},\underline{M},z)|^s \\ &\leqslant \mathit{Const}\ \mathrm{e}^{-k\gamma(n,\underline{M})},\end{aligned}$$

as desired. □

We now obtain the announced result for $\widehat{R}$ instead of R:

Theorem 58 *For all $n \in \Gamma$ and almost all $(e,V) \in \,]a,b[\, \times \Omega$ there exist constants, Const and k, the latter universal, satisfying*

$$|\widehat{R}(n,\underline{M},e+\mathrm{i}0)| \leqslant \mathit{Const}\ \mathrm{e}^{-k\gamma(n,\underline{M})}$$

for all $\underline{M} \in \Gamma_R$.

Proof: By classical Harmonic Analysis $\widehat{R}(n, \underline{M}, e + \mathrm{i}0)$ exists for almost all $(e, V) \in$ $]a, b[\times \Omega$. Thus, the previous corollary and Fatou's lemma give

$$\begin{aligned} \mathbb{E} \int_a^b |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^s \, \mathrm{d}e &\leqslant (b-a)\, Const\, \mathrm{e}^{-k\gamma(n,\underline{M})} \\ &= Const\, \mathrm{e}^{-k\gamma(n,\underline{M})}. \end{aligned}$$

For a fixed $n \in \Gamma$ let us define

$$A_{\underline{M}} = \{(e, V) \in]a, b[\times \Omega \,;\, |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)| > \mathrm{e}^{-\frac{k}{2s}\gamma(n,\underline{M})}\},$$

where k is determined by the previous inequality. Then, denoting the Lebesgue measure by d,

$$\begin{aligned} \sum_{\underline{M}} (\,\mathrm{d} \times \,\mathrm{d}\mathbb{P})(A_{\underline{M}}) &\leqslant \sum_{\underline{M}} \mathbb{E} \int_a^b \mathrm{e}^{\frac{k}{2}\gamma(n,\underline{M})} |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)|^s \, \mathrm{d}e \\ &\leqslant Const \sum_{\underline{M}} \mathrm{e}^{-\frac{k}{2}\gamma(n,\underline{M})}, \end{aligned}$$

which is finite by Assumption N. The Cantelli lemma then implies

$$(\,\mathrm{d} \times \,\mathrm{d}\mathbb{P})\Big(\bigcap_{\substack{\mathcal{E} \subseteq \Gamma_R \\ \mathcal{E} \text{ finite}}} \bigcup_{\underline{M} \notin \mathcal{E}} A_{\underline{M}} \Big) = 0.$$

In other words, for an arbitrarily fixed $n \in \Gamma$ there exists a finite set $\mathcal{E} \subseteq \Gamma_R$ such that for all $\underline{M} \notin \mathcal{E}$

$$|\widehat{R}(n, \underline{M}, e + \mathrm{i}0)| \leqslant \mathrm{e}^{-\frac{k}{2s}\gamma(n,\underline{M})} \quad \text{a.e. \& a.s.}$$

Since $\mathcal{E}$ is finite, one concludes the existence a.e. & a.s. of a constant, *Const*, depending on e, V, and n, but not on $\underline{M}$, satisfying

$$|\widehat{R}(n, \underline{M}, e + \mathrm{i}0)| \leqslant \mathit{Const}\ \mathrm{e}^{-\frac{k}{2s}\gamma(n,\underline{M})},$$

which completes the proof. □

The above theorem will be used in the following special form:

Corollary *Let $\mathcal{E} \subset \Gamma$ be a finite set. For all $n \in \Gamma$ and almost all $(e, V) \in \,]a, b[\, \times \Omega$ there exist constants, K and k, the latter universal, satisfying*

$$|\widehat{R}(q, \underline{M}, e + \mathrm{i}0)| \leqslant K\mathrm{e}^{-k\gamma(n,\underline{M})}$$

for every $\underline{M} \in \Gamma_R$ and $q \in \mathcal{E}$.

Proof: Since $\mathcal{E}$ is finite, the theorem ensures for almost all (e, V) the existence of constants satisfying

$$|\widehat{R}(q, \underline{M}, e + \mathrm{i}0)| \leqslant \mathit{Const}\ \mathrm{e}^{-k\gamma(q,\underline{M})}$$

for all $\underline{M} \in \Gamma_R$ and $q \in \mathcal{E}$. Since

$$-\gamma(q, \underline{M}) \leqslant \gamma(n, q) - \gamma(n, \underline{M}),$$

one obtains

$$\mathit{Const}\ \mathrm{e}^{-k\gamma(q,\underline{M})} \leqslant \mathit{Const}\ \mathrm{e}^{k\gamma(n,q)}\mathrm{e}^{-k\gamma(n,\underline{M})}.$$

The result follows, letting $K = Const\ \mathrm{e}^{k \sup_{q \in \mathcal{E}} \gamma(n,q)}$. $\square$

We now prove the announced, main result:

Theorem 59 *There exists a universal constant $k > 0$ such that the following proposition holds for almost all $e \in]a, b[$ and almost all $V \in \Omega$:*

For all $n \in \Gamma$ there exists a $K > 0$ such that

$$|\langle \delta_n \,|\, (H - e - \mathrm{i}0)^{-1} \delta_{\underline{M}} \rangle| \leqslant K \mathrm{e}^{-k\gamma(n,\underline{M})}$$

for all $\underline{M} \in \Gamma_R$.

Proof: By the resolvent identity

$$R(n, \underline{M}, z) = \widehat{R}(n, \underline{M}, z) - \sum_{p \in \mathcal{F}} R(n, p, z) V(p) \widehat{R}(p, \underline{M}, z).$$

Moreover, for any $M, N \in X$ both $R(N, M, e + \mathrm{i}0)$ and $\widehat{R}(N, M, e + \mathrm{i}0)$ exist almost everywhere and almost surely. Thus,

$$R(n, \underline{M}, e + \mathrm{i}0) = \widehat{R}(n, \underline{M}, e + \mathrm{i}0) - \sum_{p \in \mathcal{F}} R(n, p, e + \mathrm{i}0) V(p) \widehat{R}(p, \underline{M}, e + \mathrm{i}0) \text{ a.e. \& a.s.}$$

In particular, for almost all (e, V) and for all $n \in \Gamma$ there exists a constant, $L = \sup_{p \in \mathcal{F}} |R(n, p, e + \mathrm{i}0) V(p)|$, which depends on n, e, and V, but not on $\underline{M}$, and satisfies

$$|R(n, \underline{M}, e + \mathrm{i}0)| \leqslant |\widehat{R}(n, \underline{M}, e + \mathrm{i}0)| + L \sum_{p \in \mathcal{F}} |\widehat{R}(p, \underline{M}, e + \mathrm{i}0)|. \tag{3.25}$$

The result follows from the previous corollary applied to $\mathcal{E} = \mathcal{F} \cup \{n\}$. $\square$

3.6.1 Conclusion

We now draw a general conclusion under the following hypothesis,

$$\sup_N \sum_M \tau(N, M)^s < \infty,$$

which is a reinforcement of Assumption L with the supremum taken over Γ and with $R = \infty$.[19] For this choice of R and under the previous assumptions, the last lemma of Theorem 53 and the definition of relative smoothness then give

$$\sup_z \|(H_0 - z)^{-1}\| < \infty,$$

where z varies in $\{x + \mathrm{i}y \ ; \ x \in]a, b[\, , y > 0\}$. Then, $H_0 - x$ is invertible for all $x \in]a, b[$, which justifies the title of the present section.[20]

Moreover, under the conditions of Section 3.5 with $R = \infty$ the first part of Simon–Wolff's theorem and Theorem 52 yield that the spectrum of $H \restriction \mathcal{K}$ is almost surely pure point on $]a, b[$. Since $H = H_0$ on $\mathcal{K}^\perp$ and $]a, b[$ is in the resolvent of H_0, and since $\mathcal{K}$ and $\mathcal{K}^\perp$ are H-invariant, it follows that the spectrum of H is almost surely pure point on $]a, b[$, where it is equal to the spectrum of $H \restriction \mathcal{K}$. Finally, under the assumptions of the present section Theorem 59 holds with $R = \infty$. Hence, by the second part of Simon–Wolff's theorem the eigenfunctions decay exponentially. In summary,

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[19] Notice that for this choice of R, $\underline{M}$ may be denoted by M, since $\Gamma_R = \Gamma$.

[20] We use the fact that $\limsup_{z \to x, z \notin \mathbb{R}} \|(H_0 - z)^{-1}\| = \infty$ if and only if $x \in \mathrm{spec}(H_0)$, which is easily seen using Weyl's sequences.

Theorem 60 *Suppose*

1. *For any* $k > 0$, $\sup_N \sum_M \mathrm{e}^{-k\gamma(N,M)} < \infty$,
2. $\tau(n, M)^s \leqslant D\mathrm{e}^{-\beta\gamma(n,M)}$ *for all* $n \in \Gamma$ *and* $M \in X$,
3. $\mathfrak{I} > 0$,
4. $\sup_N \sum_M \tau(N, M)^s < \infty$,
5. $\inf_{n \neq m} \gamma(n, m) \xrightarrow{m} \infty$.

Then, the spectrum of H *on* $]a, b[$ *is almost surely pure point with simple eigenvalues obeying the following exponential decay: for an eigenfunction,* $\psi \in L^2(X)$, *associated with an eigenvalue,* e, *there exists a fixed site,* $n_0 \in X$, *and a coefficient, Const, both depending on* V *and* e, *and a universal exponent,* $k > 0$, *such that*

$$|\psi(N)| \leqslant \mathit{Const}\, \mathrm{e}^{-k\gamma(N,n_0)}$$

for all $N \in X$.

3.6.2 Application to Generalized Laplacians

Assume $H_0 = \Delta$ is a generalized Laplacian on $X = \mathbb{Z}^d$. Let $\Theta = \mathbb{R} \setminus \mathrm{spec}(H_0)$ and suppose $[a, b] \subset \Theta$. Let γ be the Pythagorean distance on $\mathbb{Z}^d$:

$$\gamma(M, N) = |M - N|.$$

The condition 1 of the previous theorem is then trivially satisfied. If in addition there exist a $D > 0$ and a $\beta > 0$ such that

$$\tau(N, M)^s \leqslant D\mathrm{e}^{-\beta|N-M|} \tag{3.26}$$

for all $N, M \in X$, then the condition 4 is deduced from 1, while the condition 2 holds by fiat. Since $z = e + \mathrm{i}y$ is bounded away from $\operatorname{spec}(H_0)$ and $|z|$ is bounded, the condition 3 is also satisfied, for, denoting by μ_n the spectral measure of δ_n with respect to H_0,

$$\mathfrak{I} \geqslant \inf_{n,z} \left| \int_{\mathbb{R}} \frac{t-e}{(t-e)^2+y^2} \, \mathrm{d}\mu_n(t) \right| = \inf_{n,z} \int_{\mathbb{R}} \frac{|t-e|}{(t-e)^2+y^2} \, \mathrm{d}\mu_n(t) > 0.$$

Therefore, the theorem applies under the sparseness condition 5.

In fact, the relation (3.26) is an immediate consequence of the equation

$$G(N,z) = \int_{\mathbb{T}^d} \frac{\mathrm{e}^{\mathrm{i}N\cdot x}}{\Phi(x)-z} \, \mathrm{d}x,$$

since $\overline{\mathcal{S}}$ is at a positive distance of the range of Φ. More precisely, since $\Phi(x)$ is analytic, one may replace each $x^{(j)}$ by $x^{(j)} + \mathrm{i}\beta$ in the previous integral without affecting its value, where $\beta > 0$ is so small that $\Phi(\,x + \mathrm{i}(\beta, \ldots, \beta)\,)$ remains bounded away from $\overline{\mathcal{S}}$, and deduce $G(N,z) = O(\mathrm{e}^{-\beta|N|})$ uniformly in $z \in \overline{\mathcal{S}}$ when $|N| \to \infty$. In other words,

$$\tau(N,M) = \sup_{z\in\overline{\mathcal{S}}} G(M-N,z) = O(\mathrm{e}^{-\beta|N-M|})$$

when $|N-M| \to \infty$.

We have proven:

Theorem 61 *Consider a random Schrödinger operator acting on* $l^2(\mathbb{Z}^d)$,

$$H = \Delta + V,$$

where Δ *is a generalized Laplacian and* V *is a random potential supported on* $\Gamma \subseteq \mathbb{Z}^d$*; we assume that the random variables* $\{V(n)\}_{n\in\Gamma}$ *are i.i.d. and absolutely continuous.*

Suppose V *is* sparse *in the sense that*

$$\lim_{\substack{|n|\to\infty \\ n\in\Gamma}} \inf_{\substack{m\neq n \\ m\in\Gamma}} |n-m| = \infty.$$

Then, almost surely the spectrum of $H = \Delta + V$ *outside* $\mathrm{spec}(\Delta)$ *is pure point with simple, exponentially decaying eigenfunctions. More precisely, given such an eigenfunction,* $\psi \in l^2(\mathbb{Z}^d)$*, almost surely there exist constants, Const and* k*, both depending on* ψ*, such that*

$$\psi(N) = Const\ \mathrm{e}^{-k|N|}$$

for any $N \in \mathbb{Z}^d$.

Scholium *Since in the present model the* $V(n)$*'s are i.i.d., it is well known that the essential spectrum of* $H = \Delta + V$ *is almost surely equal to a certain deterministic set [33]. This last set was characterized by Molchanov and Vainberg [30, 32].*[21] *Using their result, one may construct examples in which the spectrum of* H *covers the whole real line. This happens for instance when the random potential on a single site has a Cauchy or a normal distribution. Then, the spectrum of* H *is* dense pure point *in* $\mathbb{R} \setminus \mathrm{spec}(\Delta)$.

[21] In their original proof Molchanov and Vainberg considered only the case where Δ is the standard Laplacian. However, their proof may easily be adapted in order to treat any generalized Laplacian; in particular, the spectrum of Δ does not have to be centered.

Example Consider the example following Theorem 54, in which H consists of the standard (or Molchanov–Vainberg) Laplacian, Δ, added to a random potential, V, lying on

$$\Gamma = \{(j^4, 0, \ldots, 0) \in \mathbb{Z}^d \, ; \, j \in \mathbb{Z}\}.$$

Assume again that $\{V(n)\}_{n\in\Gamma}$ is a family of independent random variables whose common distribution is Cauchy (alternatively, normal). As we have seen, the spectrum of H is then purely absolutely continuous spec(Δ), and the wave operators exist and are complete on this last region (almost surely). Moreover, since Γ is sparse, the previous theorem implies that the spectrum of H on $\mathbb{R} \setminus \mathrm{spec}(\Delta)$ is pure point with exponentially decaying eigenfunctions (almost surely). Finally, as pointed out in the previous scholium, the spectrum of H in this situation covers the whole line (almost surely), which implies in particular that the eigenfunctions of H are dense in $\mathbb{R} \setminus \mathrm{spec}(\Delta)$.

CHAPTER 4
Appendices

4.1 Estimate Using Bessel Function

Let Δ be the *standard* discrete Laplacian in dimension $d \geqslant 4$. Its Green's function is denoted by

$$G(n-m,z) = \langle \delta_m \,|\, (\Delta - z)^{-1}\delta_n \rangle,$$

where $m, n \in \mathbb{Z}^d$, $z \in \mathbb{C}_+$, and δ is the Kronecker delta. By Kato's formula

$$G(n,z) = \mathrm{i}\int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}tz} \langle \delta_0 \,|\, \mathrm{e}^{\mathrm{i}t\Delta}\delta_n \rangle \,\mathrm{d}t. \tag{4.1}$$

Recall that the symbol of Δ is the multiplication by $\Phi(x) = 2\sum_{j=1}^{d} \cos x^{(j)}$, where $x = (x^{(1)}, \ldots, x^{(d)}) \in \mathbb{T}^d$. Hence,

$$\begin{aligned}
\langle \delta_0 \,|\, \mathrm{e}^{\mathrm{i}t\Delta}\delta_n \rangle &= (2\pi)^{-d} \prod_{j=1}^{d} \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}n^{(j)}k} \mathrm{e}^{2\mathrm{i}t\cos k} \,\mathrm{d}k \\
&= \prod_{j=1}^{d} \mathrm{i}^{(n^{(j)})} J_{n^{(j)}}(2t)
\end{aligned}$$

at any $n = (n^{(1)}, \ldots, n^{(d)})$ and $z \in \mathbb{C}_+$, where $J_m(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}mk}\mathrm{e}^{-\mathrm{i}t\sin k}\,\mathrm{d}k$ is the Bessel function.

It is well known that there exist a universal constant, C, such that

$$|J_m(t)| \leqslant \frac{C}{|m|^{\frac{1}{3}}} \quad \text{and} \quad |J_m(t)| \leqslant \frac{C}{|t|^{\frac{1}{3}}}$$

for any $m \in \mathbb{Z}$ and $t \in \mathbb{R}$ (see [26]). Consequently, since $d \geqslant 4$, for an arbitrarily fixed $\varepsilon > 0$

$$|\langle \delta_0 \,|\, \mathrm{e}^{\mathrm{i}t\Delta} \delta_n \rangle| \leqslant C\, |t|^{-\frac{1}{3}} |t|^{-\frac{1}{3}} |t|^{-\frac{1}{3}} (|t|^{-\frac{\varepsilon}{3}} |n^{(j_4)}|^{-\frac{1-\varepsilon}{3}}) |n^{(j_5)}|^{-\frac{1}{3}} \ldots |n^{(j_d)}|^{-\frac{1}{3}},$$

where $(n^{(j_1)}, \ldots, n^{(j_d)})$ is a permutation of $(n^{(1)}, \ldots, n^{(d)})$. This last estimate, the equation (4.1), and the dominated convergence theorem then give

$$|G(n, e + \mathrm{i}0)| \leqslant \mathit{Const}\, |n^{(j_4)}|^{-\frac{1-\varepsilon}{3}} |n^{(j_5)}|^{-\frac{1}{3}} \ldots |n^{(j_d)}|^{-\frac{1}{3}}.$$

Since ε and the permutation of $(n^{(1)}, \ldots, n^{(d)})$ are arbitrary, it follows *a fortiori* that

$$G(n, e + \mathrm{i}0) = O(|n|^{-\frac{1}{3}})$$

in dimension at least 4.

4.2 Overview of Spectral Theorem and Random Perturbation Theory

In the first part of this appendix we sketch a proof of the spectral theorem (following [13] and [14]), which is based on elementary harmonic analysis. The second part discusses the Simon–Wolff theorem, the Aizenman–Molchanov theory, and the Jakšić–Last theorem in random perturbation theory. Aside, the reader will find results, notation and terminology used in Chapter 3.

For the purpose of spectral theory, harmonic analysis, which studies relationship between harmonic functions and their boundary values, is better realized in the upper half-plane, which we denote $\mathbb{C}_+$. As an instance of a harmonic function on $\mathbb{C}_+$, which is positive, one may start from a Borel positive measure on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{1+t^2} < \infty$ and constitutes its *Poisson transform*:

$$P_\mu(x + \mathrm{i}y) = y \int_{\mathbb{R}} \frac{\mathrm{d}\mu(t)}{(x-t)^2 + y^2},$$

where $y > 0$. The relationship between $P_\mu(z)$ and its boundary values is the following: denoting by $\frac{\mathrm{d}\mu}{\mathrm{d}x}(x)$ the Radon-Nikodym derivative of μ with respect to the Lebesgue measure,

$$\frac{1}{\pi} P_\mu(x + \mathrm{i}y)\,\mathrm{d}x \xrightarrow{y\downarrow 0} \mathrm{d}\mu(x) \quad \text{vaguely},$$

while

$$\frac{1}{\pi} P_\mu(x + \mathrm{i}y) \xrightarrow{y\downarrow 0} \frac{\mathrm{d}\mu}{\mathrm{d}x}(x) \quad \text{a.e.}^{1}$$

[1] Almost everywhere.

Indeed, given a positive harmonic function, $f(z)$, on $\mathbb{C}_+$, the *Poisson representation theorem* ensures the existence of a unique Borel positive measure, μ, and a unique constant, c, such that $f(x+\mathrm{i}y) = cy + P_\mu(x+\mathrm{i}y)$. In particular, there exists a μ such that

$$\begin{aligned} \tfrac{1}{\pi}V(x+\mathrm{i}y)\,\mathrm{d}x &\xrightarrow{y\downarrow 0} \mathrm{d}\mu(x) \quad \text{vaguely,} \\ \tfrac{1}{\pi}V(x+\mathrm{i}y) &\xrightarrow{y\downarrow 0} \tfrac{\mathrm{d}\mu}{\mathrm{d}x}(x) \quad \text{a.e.} \end{aligned} \tag{4.2}$$

As an immediate corollary,[2] if $F(z) \in H^\infty(\mathbb{C}_+)$, then $F(x+\mathrm{i}0)$ exists almost everywhere.

Let $\mathcal{H}$ be a separable Hilbert space and consider a selfadjoint operator, H, acting on $\mathcal{H}$. By the Fredholm analytic theorem, for an arbitrarily fixed $\varphi \in \mathcal{H}$

$$z \mapsto \operatorname{Im}\langle\varphi \,|\, (H-z)^{-1}\varphi\rangle$$

is harmonic. Moreover, this last function is strictly positive on $\mathbb{C}_+$, where it is equal to $\operatorname{Im} z\|(H-z)^{-1}\varphi\|^2$. Hence, by the Poisson representation theorem there exists a positive Borel measure, μ_φ, satisfying

$$\operatorname{Im}\langle\varphi \,|\, (H-z)^{-1}\varphi\rangle = \operatorname{Im}\int_{\mathbb{R}} \frac{\mathrm{d}\mu_\varphi(t)}{t-z}$$

for $z \in \mathbb{C}_+$. Since the holomorphic functions $\langle\varphi \,|\, (H-z)^{-1}\varphi\rangle$ and $\int_{\mathbb{R}} \frac{\mathrm{d}\mu_\varphi(t)}{t-z}$ have the same imaginary parts and their limits when $|z| \to \infty$ are both equal to 0, they are indeed equal for any $z \in \mathbb{C}_+$. It is not hard to deduce the same relation for $z \in \mathbb{C}_-$,

[2] Recall that the Hardy class, $H^\infty(\mathbb{C}_+)$, consists of all bounded analytic functions on $\mathbb{C}_+$.

so for all $z \notin \mathbb{R}$

$$\langle \varphi \,|\, (H-z)^{-1}\varphi \rangle = \int_{\mathbb{R}} \frac{\mathrm{d}\mu_\varphi(t)}{t-z}. \tag{4.3}$$

The positive measure μ_φ, which is characterized by the previous equation, is called the *spectral measure of φ with respect to H.*

Let

$$F(z) = \frac{1}{\mathrm{i} + \langle \varphi \,|\, (H-z)^{-1}\varphi \rangle},$$

where $z \in \mathbb{C}_+$. Clearly, $F(z)$, which never vanishes, is in the Hardy class $H^\infty(\mathbb{C}_+)$; $1 - \mathrm{i}F(z)$ is also in this last class. Since

$$\langle \varphi \,|\, (H-z)^{-1}\varphi \rangle = \frac{1 - \mathrm{i}F(z)}{F(z)},$$

it follows that $\langle \varphi \,|\, (H-e-\mathrm{i}0)^{-1}\varphi \rangle$ exists for almost every $e \in \mathbb{R}$. By polarization $\langle \psi \,|\, (H-e-\mathrm{i}0)^{-1}\varphi \rangle$ also exists a.e. for any given $\varphi, \psi \in \mathcal{H}$.

The idea is to use (4.3) in conjunction with the *resolvent identity*: for $u, v \notin \mathbb{R}$

$$(H-u)^{-1} - (H-v)^{-1} = (H-u)^{-1}(u-v)(H-v)^{-1}.$$

Doing so, the equation (4.3) yields that for every u, $v \notin \mathbb{R}$

$$\langle (H-u)^{-1}\varphi \,|\, (H-v)^{-1}\varphi \rangle = \langle (t-u)^{-1} \,|\, (t-v)^{-1} \rangle_2,$$

where $\langle \cdot \,|\, \cdot \rangle_2$ denotes the scalar product on $L^2(\mathbb{R},\, \mathrm{d}\mu_\varphi(t))$ (which we abbreviate $L^2(\mu_\varphi)$). Since $\{\frac{1}{t-u}\ ;\ u \notin \mathbb{R}\}$ is total in $L^2(\mu_\varphi)$, this last relation suggests that the closed vector space generated by $\{(H-z)^{-1}\varphi\ ;\ z \notin \mathbb{R}\}$, which we denote by $\mathcal{K}_\varphi$, is unitarily equivalent to $L^2(\mu_\varphi)$. Moreover, since

$$\langle (H-u)^{-1}\varphi \,|\, (H-z)^{-1}(H-v)^{-1}\varphi\rangle = \langle (t-u)^{-1} \,|\, (t-z)^{-1}(t-v)^{-1}\rangle_2,$$

it suggests that the operator $(H-z)^{-1}$ is lifted to the multiplication by $(t-z)^{-1}$ via this last unitary equivalence.

Indeed, not only the previous considerations are right, but H itself is lifted to the multiplication by t. More precisely, denoting by U the unitary equivalence in question, the following diagram commute: [3]

$$\begin{array}{ccc} \mathcal{K}_\varphi & \xrightarrow{H} & \mathcal{K}_\varphi \\ U \downarrow & & \downarrow U \\ L^2(\mu_\varphi) & \xrightarrow{t} & L^2(\mu_\varphi) \end{array}$$

The subspace $\mathcal{K}_\varphi$ is called the *cyclic space* generated by φ with respect to H; as one may expect, if H is bounded it corresponds to the smallest H-invariant, closed subspace containing φ.[4]

[3] By this, we do not pretend that H and the multiplication by t are bounded, but that their respective domains coincide via U.

[4] Given a subset, $\mathcal{F} \subseteq \mathcal{H}$, the *cyclic space* generated by $\mathcal{F}$ with respect to H is then defined as the closure of the linear span of $\{(H-z)^{-1}\phi \ ; \ \phi \in \mathcal{F},\, z \notin \mathbb{R}\}$. However, a closed subspace is said to be *cyclic* only if it may be generated by a single element.

More generally, $\mathcal{H}$ decomposes into a direct sum, $\oplus_n \mathcal{H}_n$, of cyclic spaces. Hence, there exists cyclic generators, φ_n, and a unitary equivalence between $\mathcal{H}$ and $\oplus_n L^2(\mu_{\varphi_n})$ such that H is lifted to the operator of multiplication by t on each summand. Then, H is said to be *diagonalized* in the representation $\oplus_n L^2(\mu_{\varphi_n})$. As we have just sketched, all selfadjoint operators on a separable Hilbert space are diagonalizable. This last statement constitutes the *spectral theorem.*

The main application of the spectral theorem is the following: since H is identified with t on each cyclic summand, $f(t)$ being known, $f(H)$ is also known! More precisely, suppose $\mathcal{H}$ is identified with $\oplus_n L^2(\mu_{\varphi_n})$. Then, denoting by $\oplus_n f(t)$ the operator of multiplication by $f(t)$ on each summand of $\oplus_n L^2(\mu_{\varphi_n})$, $f(H)$ is defined as the lifting of $\oplus_n f(t)$ via the given identification.

Doing this for any bounded Borel f, one obtains a *functional calculus* for H, that is, a morphism of $*$-algebras between Borel bounded functions on $\mathbb{R}$ and $\mathcal{B}(\mathcal{H})$, the set of all bounded linear applications on $\mathcal{H}$. Indeed, the calculus for H is unique, being characterized by several properties it satisfies.[5]

Since H is identified with t on each cyclic summand, the set of *values* of H, say, on the n-th cyclic summand, is commonly defined as the support of μ_{φ_n}; in total, the set of values of H is thus equal to its spectrum. However, one may seek for more precise information and try to identify on which Borel sets values of H (identified with t) are relevant, *i.e.*, on which Borel sets not all spectral measures, μ_{φ_n}, vanish.

[5] Among these properties, the following is frequently used: if $f_n(t) \xrightarrow{n} f(t)$ for *all* $t \in \mathbb{R}$, where the $f_n(t)$ are *uniformly bounded*, then $f_n(H) \xrightarrow{n} f(H)$ *strongly.*

In other words, from our point of view a complete knowledge of the values of H consists of identifying every Borel set, E, such that $\mathbf{1}_E(H) \neq 0$, where $\mathbf{1}_E$ denotes the characteristic function of E.[6]

As a first step, one may wonder which parts of the spectrum of H are pure point, which parts are absolutely continuous, and which parts are singular continuous, with the following, obvious definitions: given a Borel set, B, the spectrum of H is *pure point* on B if there exists a countable set, $P \subseteq B$, such that $\mathbf{1}_{B\setminus P}(H) = 0$; it is *purely absolutely continuous* on B if $\mathbf{1}_S(H) = 0$ for all $S \subseteq B$ of Lebesgue measure zero; finally, it is *purely singular continuous* on B if there exists a Borel set of Lebesgue measure zero, $S \subseteq B$, such that $\mathbf{1}_{B\setminus S}(H) = 0$ and furthermore $\mathbf{1}_{\{x\}}(H) = 0$ for all $x \in B$.

Alternatively, one may define the above notions in the following way: For an arbitrary sequence of positive numbers, $\{a_n\}$, let us consider $\mu = \sum_n a_n \mu_{\varphi_n}$ and call it a *spectral measure* of H. It is easy to see that the spectral measures of H are all equivalent, *i.e.*, they induce the same sets of measure zero. The spectrum of H is *pure point*, *purely absolutely continuous*, or *purely singular continuous* on a given Borel set, B, if and only if $\mu \restriction B$ is pure point, purely absolutely continuous, or purely singular continuous on B, respectively.

Remark *The values of H are said to be* localized on B *if the spectrum of H is pure point on B, and* delocalized on B *if the spectrum of H is purely absolutely continuous*

[6] The operators $\mathbf{1}_E(H)$ are called *spectral projections* of H.

on B.

* * *

The scientific community is interested in localization and delocalization of the operator of energy, $H = \Delta + V$, in the Anderson model, discussed in the introduction of the present thesis. Here, the kinetic energy, Δ, is a discrete Laplacian on $l^2(\mathbb{Z}^d)$, while the potential, V, is random; V is supported on a given set of sites, $\Gamma \subseteq \mathbb{Z}^d$. A general criterion of localization applying to this model was given by Simon and Wolff [44]. It is used in Chapter 3 of the present thesis in the following special form.

Let $\{V(n)\}_{n\in\Gamma}$ be a family of i.i.d. random variables of law ν, where ν is absolutely continuous.[7] Let $\{\delta_n\}_{n\in\mathbb{Z}^d}$ be the usual basis of $l^2(\mathbb{Z}^d)$, where δ denotes the Kronecker delta. Given a subset of sites, $\Gamma \subseteq \mathbb{Z}^d$, we focus on the subspace cyclically generated by $\{\delta_n \; ; \; n \in \Gamma\}$ with respect to Δ, which we denote by $\mathcal{K}$.

Theorem 62 (Simon–Wolff) *Consider an arbitrary Borel set, $B \subseteq \mathbb{R}$. If with probability one $\|(H - e - \mathrm{i}0)^{-1}\delta_n\| < \infty$ for all $n \in \Gamma$ and almost all $e \in B$, then the spectrum of $H \restriction \mathcal{K}$ on B is almost surely pure point with simple eigenvalues.*

[7] The underlying probability space is given by $\mathbb{R}^\Gamma$, *i.e.*, by the set of functions from Γ to $\mathbb{R}$, which is endowed with its Borel σ-algebra and the probability measure $\mathbb{P} = \prod_{n\in\Gamma} \nu$. The random variables in question are then the projections $V \longrightarrow V(n)$ for $n \in \Gamma$, where $V \in \mathbb{R}^\Gamma$ is the random parameter.

Suppose in addition that for almost all V, almost all $e \in B$, and all $n \in \Gamma$, there exists constants, $C > 0$ and $k > 0$, such that

$$|\langle \delta_n \,|\, (H - e - \mathrm{i}0)^{-1} \delta_m \rangle| \leqslant C\mathrm{e}^{-k|m|}$$

uniformly in $m \in \mathbb{Z}^d$. Then, the eigenfunctions of $H \restriction \mathcal{K}$ are almost surely exponentially bounded, which means: if φ is such an eigenfunction, for almost all V, there exists constants, $D > 0$ and $l > 0$, both depending on φ and V, satisfying

$$|\varphi(n)| \leqslant D\mathrm{e}^{-l|n|}$$

for every $n \in \Gamma$.

The proof of this last theorem is based on spectral averaging (see [43]), conditional Fubini's theorem, and rank one perturbation theory (especially, the Aronszajn–Donoghue theorem).

In order to apply the Simon–Wolff theorem, it is convenient to estimate quantities of the form

$$R(m, n, z) = \langle \delta_m \,|\, (H - z)^{-1} \delta_n \rangle;$$

the *Aizenman–Molchanov theory* is designed to this end [3]. More precisely, it is designed to estimate $\mathbb{E}\,|R(m, n, z)|^s$, where $s \in \,]0, 1[$ is a structural constant. One then removes the expectation by using Cantelli's lemma.[8]

[8] The Cantelli lemma asserts: *if $\{A_n\}_{n \in \mathbb{N}}$ is a family of events satisfying $\sum_n \mathbb{P}(A_n) < \infty$, then $\mathbb{P}(\bigcap_N \bigcup_{n \geqslant N} A_n) = 0$.*

The Aizenman–Molchanov method is based on the *resolvent identity*,

$$(H-u)^{-1}-(H-v)^{-1}=(H-u)^{-1}(u-v)(H-v)^{-1},$$

where u and v are appropriate numbers or operators, in conjunction with the *decoupling lemmas*, which we now state.

Lemma *Suppose there exists an* $s\in\,]0,1[$ *such that*

$$k_s=\inf_{\alpha,\beta\in\mathbb{C}}\frac{\int_{\mathbb{R}}|x-\alpha|^s|x-\beta|^{-s}\,\mathrm{d}\nu(x)}{\int_{\mathbb{R}}|x-\beta|^{-s}\,\mathrm{d}\nu(x)}>0.$$

Then, for any deterministic function, $F(n,m,z)$,

$$\mathbb{E}\,|V(m)-F(m,n,z)|^s|R(m,n,z)|^s\geqslant k_s\mathbb{E}\,|R(m,n,z)|^s.$$

Suppose instead there exists an $s\in\,]0,1[$ *such that*

$$K_s=\sup_{\beta\in\mathbb{C}}\frac{\int_{\mathbb{R}}|x|^s|x-\beta|^{-s}\,\mathrm{d}\nu(x)}{\int_{\mathbb{R}}|x-\beta|^{-s}\,\mathrm{d}\nu(x)}<\infty.$$

Then, $\mathbb{E}\,|V(m)|^s|R(m,n,z)|^s\leqslant K_s\mathbb{E}\,|R(m,n,z)|^s$.

Notice that both hypotheses in the previous lemma are satisfied for large classes of probability measures, which include Gaussians, Cauchy distributions and uniform distributions [1, 2, 3, 11, 20, 28].

Finally, the celebrated *Jakšić–Last theorem* gives a criterion of delocalization applying to the Anderson model [16]. It involves the notion of *essential support*,

$\Sigma(H)$, of the absolutely continuous spectrum of H, which is defined as follows: denoting by μ a spectral measure of H and by $\frac{\mathrm{d}\mu}{\mathrm{d}x}(x)$ its Radon–Nikodym derivative with respect to the Lebesgue measure,

$$\Sigma(H) = \left\{ x \in \mathbb{R}\,;\ \frac{\mathrm{d}\mu}{\mathrm{d}x}(x) > 0 \right\}.$$

Notice that $\Sigma(H)$ is defined "up to a set of Lebesgue measure zero".[9] Notice also that by the first part of this appendix

$$\frac{\mathrm{d}\mu}{\mathrm{d}x}(x) = \frac{1}{\pi} \sum_n \langle \delta_n \,|\, (H - e - \mathrm{i}0)^{-1} \delta_n \rangle \quad \text{a.e.},$$

where $\{\delta_n\}$ is any set generating cyclically $l^2(\mathbb{Z}^d)$ with respect to Δ.

Before stating the Jakšić–Last theorem, let us mention the following property of the essential support: in the present setting—where the random variables $V(n)$ are independent—there exists a deterministic set, $\Sigma \subseteq \mathbb{R}$, such that $\Sigma(H) = \Sigma$ almost surely. This last property, which is a consequence of the Kolmogorov 0-1 law and random perturbation arguments, is used in Chapter 3.

In the present setting,

Theorem 63 (Jakšić–Last) *Consider an arbitrary Borel set, $B \subseteq \mathbb{R}$. If with probability one $B \subseteq \Sigma(H)$, then almost surely the spectrum of H on B is purely absolutely continuous.*

[9] Rigorously, the essential support is thus defined as the equivalence class of the given $\Sigma(H)$, where two Borel sets are said to be *equivalent* iff the Lebesgue measure of their symmetric difference is zero.

This last result is a non trivial consequence of Poltoratskii's theorem (see [18]).

4.3 l^1, l^∞, and l^2 Norms of Operators

In this appendix we present a special instance of the Riesz–Thorin theorem. More precisely, from the l^1 and the l^∞ norms of an operator we derive a bound on its l^2 norm.

Given an index set, Γ, let A be a linear operator acting on the vector space generated by $\{\delta_n \,;\, n \in \Gamma\}$, where δ denotes the Kronecker delta:

$$\delta_n(m) = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $A(m,n) = \langle \delta_m \mid A\delta_n \rangle$ the (m,n)-th matrix element of A with respect to the previous basis.

Lemma *If* $\sup_{n\in\Gamma} \sum_{m\in\Gamma} |A(m,n)| < \infty$, *then A extends continuously to a bounded operator from $l^1(\Gamma)$ to $l^1(\Gamma)$, whose norm is given by*

$$\|A\|_1 = \sup_{n\in\Gamma} \sum_{m\in\Gamma} |A(m,n)|.$$

Proof: Let $\varphi \in l^1(\Gamma)$. The inequation

$$\sum_m \sum_n |A(m,n)|\,|\varphi(n)| \leqslant \sup_p \sum_m |A(m,p)| \sum_n |\varphi(n)|$$

shows that $\sum_n A(m,n)\varphi(n)$ is absolutely convergent for any $m \in \Gamma$, is in $l^1(\Gamma)$, and that

$$\|A\|_1 \leqslant \sup_p \sum_m |A(m,p)|. \tag{4.4}$$

Moreover, since for any $p \in \Gamma$

$$\begin{aligned} \|A\|_1 &= \sup_{\|\varphi\|_1=1} \sum_m \left| \sum_n A(m,n)\varphi(n) \right| \\ &\geqslant \sum_m \left| \sum_n A(m,n)\delta_p(n) \right| \\ &= \sum_m |A(m,p)|, \end{aligned}$$

the equality is attained in (4.4). □

Lemma *If* $\sup_{m\in\Gamma} \sum_{n\in\Gamma} |A(m,n)| < \infty$, *then A extends continuously to a bounded operator from $l^\infty(\Gamma)$ to $l^\infty(\Gamma)$, whose norm is given by*

$$\|A\|_\infty = \sup_{m\in\Gamma} \sum_{n\in\Gamma} |A(m,n)|.$$

Proof: Let $\varphi \in l^\infty(\Gamma)$. The inequation

$$\sup_m \sum_n |A(m,n)|\,|\varphi(n)| \leqslant \sup_m \sum_n |A(m,n)| \sup_p |\varphi(p)|$$

shows that $\sum_n A(m,n)\varphi(n)$ is absolutely convergent for any $m \in \Gamma$, is in $l^\infty(\Gamma)$, and that

$$\|A\|_\infty \leqslant \sup_m \sum_n |A(m,n)|.$$

Moreover, for a fixed $m \in \Gamma$, let $\varphi_m(n) = \mathrm{e}^{-\mathrm{i}\arg A(m,n)}$. Then, $\|\varphi_m\|_\infty = 1$ and $A(m,n)\varphi_m(n) = |A(m,n)|$, so

$$\begin{aligned}\|A\|_\infty &= \sup_m \sup_{\|\varphi\|_\infty=1} \left|\sum_n A(m,n)\varphi(n)\right| \\ &\geqslant \sup_m \sum_n |A(m,n)|.\end{aligned}$$

□

Scholium *One may deduce* $\|A\|_1$ *from* $\|A\|_\infty$ *and* vice versa, *using*

$$\|A\|_1 = \|A^*\|_\infty, \tag{4.5}$$

where $A^*(m,n) = \overline{A(n,m)}$. *The equation (4.5) follows from the Hölder inequality,*

$$|\langle\varphi \mid \psi\rangle| \leqslant \|\varphi\|_1 \|\psi\|_\infty,$$

where $\varphi \in l^1(\Gamma)$, $\psi \in l^\infty(\Gamma)$, and $\langle \cdot \mid \cdot \rangle$ denotes the canonical pairing between $l^1(\Gamma)$ and $l^\infty(\Gamma)$. In particular, since in this last expression the equality is attained,

$$\begin{aligned}
\|A\|_1 &= \sup_{\substack{\|\varphi\|_1=1 \\ \|\psi\|_\infty=1}} |\langle A\varphi \mid \psi \rangle| \\
&= \sup_{\substack{\|\varphi\|_1=1 \\ \|\psi\|_\infty=1}} |\langle \varphi \mid A^*\psi \rangle| \\
&= \|A^*\|_\infty.
\end{aligned}$$

The knowledge of the l^1 and l^∞ norms of an operator provides a bound on its l^2 norm:[10]

Theorem 64 $\|A\|_2^2 \leqslant \|A\|_1 \|A\|_\infty$.

Proof:

$$\begin{aligned}
\|A\|_2^2 &= \sup_{\|\varphi\|_2=1} \|A\varphi\|_2^2 \\
&= \sup_{\|\varphi\|_2=1} \sum_n |(A\varphi)(n)|^2 \\
&= \sup_{\|\varphi\|_2=1} \sum_n \left| \sum_k A(n,k)\varphi(k) \right|^2 \\
&\leqslant \sup_{\|\varphi\|_2=1} \sum_n \sum_k |A(n,k)\varphi(k)| \sum_l |A(n,l)\varphi(l)|.
\end{aligned}$$

[10] Notice that a relation similar to the following one is trivially derived for sequences instead of operators.

For any fixed φ such that $\|\varphi\|_2 = 1$, the triple sum in this last expression is equal to the following, which is bounded by applying twice Schwarz' inequality:

$$\begin{aligned}
&\sum_k |\varphi(k)| \sum_l |\varphi(l)| \sum_n |A(n,k)A(n,l)| \leqslant \\
&\qquad\leqslant \sum_k |\varphi(k)| \left(\sum_l (\sum_n |A(n,k)A(n,l)|)^2 \right)^{\frac{1}{2}} \\
&\qquad\leqslant \left(\sum_k \sum_l (\sum_n |A(n,k)A(n,l)|)^2 \right)^{\frac{1}{2}} .
\end{aligned}$$

Consequently, expanding the square in the last expression,

$$\begin{aligned}
\|A\|_2^2 &\leqslant \left(\sum_k \sum_l \sum_m \sum_n |A(m,k)|\,|A(m,l)|\,|A(n,k)|\,|A(n,l)| \right)^{\frac{1}{2}} \\
&= \left(\sum_k \{ \sum_{m,l} |A(m,k)|\,|A(m,l)| \} \sum_n |A(n,k)|\,|A(n,l)| \right)^{\frac{1}{2}} \\
&\leqslant \left(\{ \sup_p \sum_{m,l} |A(m,p)|\,|A(m,l)| \} \sum_{k,n} |A(n,k)|\,|A(n,l)| \right)^{\frac{1}{2}}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\sup_p \sum_{k,l,m,n} |A(m,p)|\,|A(m,l)|\,|A(n,k)|\,|A(n,l)|\right)^{\frac{1}{2}} \\
&= \left(\sup_p \sum_l \{\sum_{n,k} |A(n,l)|\,|A(n,k)|\} \sum_m |A(m,p)|\,|A(m,l)|\right)^{\frac{1}{2}} \\
&\leqslant \left(\sup_p \{\sup_q \sum_{n,k} |A(n,q)|\,|A(n,k)|\} \sum_{m,l} |A(m,p)|\,|A(m,l)|\right)^{\frac{1}{2}} \\
&= \sup_q \sum_{n,k} |A(n,q)|\,|A(n,k)| \\
&= \sup_q \sum_n \{\sum_k |A(n,k)|\} |A(n,q)| \\
&\leqslant \sup_q \{\sup_p \sum_k |A(p,k)|\} \sum_n |A(n,q)| \\
&= \|A\|_\infty \|A\|_1.
\end{aligned}
$$

$\square$

4.4 Variant of Analytic Fredholm Theorem

In this appendix we present a natural extension of the analytic Fredholm theorem, due to B. Simon, which gives us a sufficient criterion for a set to be closed and of Lebesgue measure zero.

Given a domain, $\mathcal{D} \subseteq \mathbb{C}$, let us consider an analytic function[11]

$$f\colon \mathcal{D} \to \mathcal{B}(\mathcal{H}).$$

For later purpose, if f admits a continuous extension $\overline{\mathcal{D}} \to \mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is endowed with the uniform topology, we set $\mathcal{E} = \overline{\mathcal{D}}$; otherwise, we set $\mathcal{E} = \mathcal{D}$. Hence, $f\colon \mathcal{E} \to \mathcal{B}(\mathcal{H})$ is continuous on $\mathcal{E}$ and analytic in the interior of $\mathcal{E}$.

We will use the following elementary lemmas:

Lemma *Suppose $f(z)$ is invertible for any $z \in \mathcal{E}$. Then, $z \mapsto f(z)^{-1}$ is continuous on $\mathcal{E}$.*

Proof: Given a fixed $z \in \mathcal{E}$, consider any h such that $z + h \in \mathcal{E}$. Then,

$$f(z+h)^{-1} - f(z)^{-1} = f(z+h)^{-1}(f(z) - f(z+h))f(z)^{-1}. \tag{4.6}$$

[11] Recall that f is *analytic* at z iff the strong limit $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists. It is well known that this last requirement is equivalent to the following, *a priori* weaker one: for all $l \in \mathcal{B}(\mathcal{H})^*$, the complex valued function $l \circ f$ is analytic.

Therefore,

$$f(z+h)^{-1}(1-(f(z)-f(z+h))f(z)^{-1}) = f(z)^{-1},$$

where $1 \in \mathcal{B}(\mathcal{H})$ denotes the identity. Since f is continuous, given our fixed $z \in \mathcal{E}$, there exists a $\delta > 0$ such that, if $|h| < \delta$ and $z + h \in \mathcal{E}$, then

$$\|(f(z)-f(z+h))f(z)^{-1}\| < \frac{1}{2}.$$

Then, by the geometric series $1-(f(z)-f(z+h))f(z)^{-1}$ admits an inverse whose norm is less than 2 uniformly in h. Thus, the norm of

$$f(z+h)^{-1} = (\,1-(f(z)-f(z+h))f(z)^{-1}\,)^{-1}f(z)^{-1}$$

is bounded for such h's. The result then follows from the continuity of f and the equation (4.6). $\square$

Lemma *Suppose $f(z)$ is invertible for any $z \in \mathcal{D}$. Then, $f^{-1}(z) = f(z)^{-1}$ is analytic on $\mathcal{D}$, where it satisfies*

$$(f^{-1})'(z) = -f(z)^{-1}f'(z)f(z)^{-1}.$$

Proof: Given a fixed $z \in \mathcal{D}$, if h is sufficiently small, then

$$\frac{f(z+h)^{-1}-f(z)^{-1}}{h} + f(z)^{-1}f'(z)f(z)^{-1} \tag{4.7}$$

is well defined and equal to

$$(f(z+h)^{-1} - f(z)^{-1})\frac{f(z) - f(z+h)}{h}f(z)^{-1} + \\ + f(z)^{-1}\left(\frac{f(z) - f(z+h)}{h} + f'(z)\right)f(z)^{-1}.$$

Hence, the norm of (4.7) is bounded by

$$\|f(z+h)^{-1} - f(z)^{-1}\| \left\|\frac{f(z) - f(z+h)}{h}\right\| \|f(z)^{-1}\| + \\ + \|f(z)^{-1}\| \left\|\frac{f(z+h) - f(z)}{h} - f'(z)\right\| \|f(z)^{-1}\|,$$

which clearly tends to zero when $h \to 0$. □

We now prove the Fredholm analytic theorem. Let $z_0 \in \mathcal{E}$ be fixed. Then, there exists a $\delta > 0$ such that $\|f(z) - f(z_0)\| < \frac{1}{2}$ when $z \in \mathcal{B}(z_0, \delta) \cap \mathcal{E}$. Moreover, since $f(z_0)$ is compact, there exists a finite rank operator, F, such that

$$\|f(z_0) - F\| < \frac{1}{2}.$$

Thus, $\|f(z) - F\| < 1$ on $\mathcal{B}(z_0, \delta) \cap \mathcal{E}$, which implies (by the geometric series) that $1 - f(z) + F$ is invertible. One then studies the injectivity of

$$1 - f(z) = (1 - F(1 - f(z) + F)^{-1})(1 - f(z) + F),$$

which is realized iff $1 - F(1 - f(z) + F)^{-1}$ is injective.

Let us denote by π the projection of $\mathcal{H}$ onto $\operatorname{ran} F$.

Lemma *In the above circumstances, $1-f(z)$ is not injective iff the secular equation,*

$$\det(1 - \pi F(1 - f(z) + F)^{-1}\pi) = 0, \tag{4.8}$$

is satisfied.

Proof: Let φ be in the kernel of $1 - F(1 - f(z) + F)^{-1}$, so

$$\varphi = F(1 - f(z) + F)^{-1}\varphi.$$

In particular, φ belongs to the range of F, so

$$\varphi = \pi F(1 - f(z) + F)^{-1}\pi\varphi.$$

Since $\operatorname{ran} F$ is finite dimensional, the result follows. $\square$

Lemma *In the above circumstances, if $1-f(z)$ is injective, then it is also surjective.*

Proof: Again, $1 - f(z)$ is surjective iff $1 - F(1 - f(z) + F)^{-1}$ is surjective. For an arbitrarily fixed $\psi \in \mathcal{H}$, one wonders if there exists a $\phi \in \mathcal{H}$ satisfying

$$(1 - F(1 - f(z) + F)^{-1})\phi = \psi.$$

If so, ψ is of the form $\phi - \varphi$ for a $\varphi \in \operatorname{ran} F$; thus, one may assume $\phi = \psi + \varphi$ and seek for a $\varphi \in \operatorname{ran} F$ satisfying

$$(1 - F(1 - f(z) + F)^{-1})(\psi + \varphi) = \psi.$$

This last equation is equivalent to

$$(1 - \pi F(1 - f(z) + F)^{-1}\pi)\varphi = F(1 - f(z) + F)^{-1}\psi,$$

which admits a solution when $\det(1 - \pi F(1 - f(z) + F)^{-1}\pi) \neq 0$. The result follows from the previous lemma. □

Therefore, by the Inverse Mapping Theorem:

Corollary *In the above circumstances, if $1 - f(z)$ is injective, then it admits a bounded inverse.*

Proving now the classical Fredholm analytic theorem, let us consider the case where $z_0 \in \mathcal{D}$. Without loss of generality, we assume $\mathcal{B}(z_0, \delta) \subseteq \mathcal{D}$. Then,

$$\det(1 - \pi F(1 - f(z) + F)^{-1}\pi)$$

is analytic in $\mathcal{B}(z_0, \delta)$. In particular, either this last function is identically zero on $\mathcal{B}(z_0, \delta)$, either its zeroes on $\mathcal{B}(z_0, \delta)$ are isolated. In conclusion,

Theorem 65 (Fredholm) *Given a domain $\mathcal{D} \subseteq \mathbb{C}$, consider an analytic function, $f : \mathcal{D} \to \mathcal{B}(\mathcal{H})$, whose values are compact operators. Then, either the operators*

$1 - f(z)$ *are not invertible for* $z \in \mathcal{D}$, *either they are invertible for all* $z \in \mathcal{D}$ *except isolated points.*

Let $\mathbb{D} \subset \mathbb{C}$ denote the open unit disk. Our variant of the previous theorem is based on the following classical result, whose proof is given, for instance, in [23]: [12]

Theorem 66 *If* $f : \overline{\mathbb{D}} \to \mathbb{C}$ *is continuous on* $\overline{\mathbb{D}}$ *and analytic in* $\mathbb{D}$, *then either* $\{z \in \overline{\mathbb{D}}\,;\, f(z) = 0\}$ *is the whole* $\overline{\mathbb{D}}$, *either it is a closed set of Lebesgue measure zero whose intersection with* $\mathbb{D}$ *consists of isolated points.*

We now consider the case where $\mathcal{D} = \mathbb{C}_+$ and where $f\colon\, \mathbb{C}_+ \to \mathcal{B}(\mathcal{H})$ admits a continuous extension,

$$f\colon\, \overline{\mathbb{C}_+} \to \mathcal{B}(\mathcal{H})\,;$$

in this case $\mathcal{E} = \overline{\mathbb{C}_+}$.

If the secular equation, (4.8), identically holds on $\mathbb{C}_+$, then by continuity it holds on $\overline{\mathbb{C}_+}$, so $1 - f(z)$ is never invertible.

Otherwise, assume $z_0 \in \mathbb{R}$, the case where $z_0 \in \mathbb{C}_+$ being covered by the classical theorem. By Riemann's Conformal Mapping Theorem (or by an explicit construction) there exists a conformal equivalence from $\mathcal{B}(z_0, \delta) \cap \mathbb{C}_+$ to $\mathbb{D}$, which extends to a homeomorphism from $\overline{\mathcal{B}(z_0, \delta) \cap \mathbb{C}_+}$ to $\overline{\mathbb{D}}$ (since the boundary of the former region is regular). Thus, by Theorem 66 the secular equation for $z \in \overline{\mathcal{B}(z_0, \delta) \cap \mathbb{C}_+}$ is never satisfied, except on a closed set of Lebesgue measure zero whose intersection with $\mathcal{B}(z_0, \delta) \cap \mathbb{C}_+$ consists of isolated points. Thus, a connectedness argument yields:

[12] Indeed, the following result holds under weaker assumptions, for instance, for f bounded in $\mathbb{D}$ or, which is even better, for f of exponential type in $\mathbb{D}$.

Theorem 67 (Simon) *Suppose $f\colon \overline{\mathbb{C}_+} \to \mathcal{B}(\mathcal{H})$ is continuous on $\overline{\mathbb{C}_+}$ and analytic in $\mathbb{C}_+$. Then, either $1 - f(z)$ is never invertible, either it is invertible except on a closed set of Lebesgue measure zero whose intersection with $\mathbb{C}_+$ consists of isolated points.*

CHAPTER 5
Conclusion

5.1 Main Results

After an extensive review of the stationary phase method, Chapter 2 is devoted to Green's functions of discrete Laplacians on $\mathbb{Z}^d$. Here, *generalized Laplacians* are defined as adjacency operators of translational invariant graphs on $\mathbb{Z}^d$. Explicitly, given such a graph, whose distance is denoted by d, the associated Laplacian is defined as

$$\Delta\varphi(n) = \sum_{\mathrm{d}(m,n)=1} \varphi(m),$$

where $\varphi \in l^2(\mathbb{Z}^d)$ and $n \in \mathbb{Z}^d$.

Let $\mathcal{V} = \{n \in \mathbb{Z}^d \ ; \ \mathrm{d}(n,0) = 1\}$. Then, the symbol of Δ is (the multiplication by) $\Phi(x) = \sum_{v \in \mathcal{V}} \mathrm{e}^{\mathrm{i}v\cdot x}$, so its Green's function is

$$\langle \delta_m \,|\, (H - z)^{-1}\delta_n \rangle = G(n - m, z) = (2\pi)^{-d} \int_{\mathbb{T}^d} \frac{\mathrm{e}^{\mathrm{i}(n-m)\cdot x}}{\Phi(x) - z} \,\mathrm{d}x$$

for any $m, n \in \mathbb{Z}^d$ and $z \in \mathbb{C}_+$, where δ denotes the Kronecker delta. We are interested in the decay of $G(n,e) = \lim_{\substack{z\to e \\ z\in\mathbb{C}_+}} G(n,z)$ when $|n| \to \infty$ for spectral values of energy, $e \in \operatorname{spec}(\Delta) = \operatorname{ran}\Phi$. The stationary phase method yields:

Theorem *Consider an open set,* $\Theta \subset \operatorname{spec}(\Delta)$, *such that* $\nabla\Phi(x) \neq 0$ *on* $\Phi^{-1}(\Theta)$.

Notice that $\Phi^{-1}(\{e\})$ is then a real-analytic regular surface for any $e \in \Theta$; suppose that for any $e \in \Theta$ this last surface admits at least κ non vanishing principal curvatures at any point, where $\kappa \geqslant 1$. Then, for any compact $K \subset \Theta$

$$G(n, e) = O(|n|^{-\frac{\kappa}{2}})$$

uniformly in $(e, \omega) \in K \times \mathrm{S}^d$, where $n = |n|\omega$.

Two concrete examples are emphasized: the *standard Laplacian*, whose graph is determined by $\mathcal{V} = \{(\pm 1, 0, \cdots, 0), (0, \pm 1, \cdots, 0), \cdots, (0, 0, \cdots, \pm 1)\}$, and the *Molchanov–Vainberg Laplacian*, whose graph is determined by

$$\mathcal{V} = \{(\sigma_1, \cdots, \sigma_d) \in \mathbb{Z}^d \, ; \, \sigma_j \in \{-1, 1\} \text{ for any } j\}.$$

Notice that the spectrum of the former is $[-2d, 2d]$, while the spectrum of the latter is $[-2^d, 2^d]$. In the former case an elementary argument shows that the previous theorem applies on $\Theta = [-2d, 2d] \setminus (\{-2d, -2d + 4, \cdots, 2d - 4, 2d\} \cup \{0\})$ for $\kappa = 1$ (without pretending that this result is optimal). In the latter case the theorem applies on $\Theta = [-2^d, 2^d] \setminus \{-2^d, 0, 2^d\}$ for $\kappa = d - 1$, which is optimal.

After a revision of basic scattering theory, Chapter 3 is devoted to random Schrödinger operators of the form $H = \Delta + V$, where Δ is a generalized Laplacian and V is a random potential. We are interested in scattering and spectral properties of H that hold almost surely when the sites of the potential are sparse. Our work is a continuation of [17], where abstract criterions of existence of the wave operators

$\Omega_S^{\pm}(H,\Delta) = \lim_{t\to\pm\infty} e^{itH} e^{-it\Delta} \mathbf{1}_S(\Delta)$ and $\Omega_S^{\pm}(\Delta, H) = \lim_{t\to\pm\infty} e^{it\Delta} e^{-itH} \mathbf{1}_S(H)$ are presented in a more general framework (in which Δ is the adjacency operator of *any* simple, countable graph). Two approaches are used in our study: one, based on Fredholm's analytic theory, is *deterministic*, while the other, based on the Aizenman–Molchanov theory, is *probabilistic.*

Let us denote by $\Gamma \subseteq \mathbb{Z}^d$ the sites of the random potential, V.

In the deterministic approach $\{V(n)\}_{n\in\Gamma}$ consists of independent random variables of law μ, where μ is a compactly supported probability measure on $\mathbb{R}$ (so V is almost surely *bounded*). In these circumstances,

Theorem *Let Θ be a region of validity of the previous theorem with respect to a certain $\kappa \geqslant 1$. If Γ is* sparse *in the sense that for a certain $\epsilon > 0$*

$$\sum_{m\in\Gamma\setminus\{n\}} |n-m|^{-\frac{\kappa}{2}+\epsilon}$$

is finite for all $n \in \Gamma$ and tends to 0 when $|n| \to \infty$ in Γ, then the wave operators $\Omega_\Theta^{\pm}(H,\Delta)$ and $\Omega_\Theta^{\pm}(\Delta, H)$ exist almost surely.

More generally, if the sites of the potential are partitioned in clusters whose diameters are bounded, the previous theorem still holds if one replaces Γ with the set of centers of the clusters.

In the probabilistic approach $\{V(n)\}_{n\in\Gamma}$ consists of independent random variables of law μ, where μ is an absolutely continuous probability measure, not necessarily compactly supported, and satisfying the *decoupling hypotheses* for a given

$s \in]0,1[$:

$$\inf_{\alpha,\beta\in\mathbb{C}} \frac{\int_{\mathbb{R}} |x-\alpha|^s|x-\beta|^{-s}\,\mathrm{d}\mu(x)}{\int_{\mathbb{R}} |x-\beta|^{-s}\,\mathrm{d}\mu(x)} > 0 \quad\text{and}\quad \sup_{\beta\in\mathbb{C}} \frac{\int_{\mathbb{R}} |x|^s|x-\beta|^{-s}\,\mathrm{d}\mu(x)}{\int_{\mathbb{R}} |x-\beta|^{-s}\,\mathrm{d}\mu(x)} < \infty.$$

In these circumstances,

Theorem *Suppose* Γ *is* sparse *in the sense that for* Θ *and* κ *as above and for a certain* $\epsilon > 0$

$$\sum_{m\in\Gamma\setminus\{n\}} |n-m|^{-\frac{s\kappa}{2}+\epsilon}$$

is finite for all $n \in \Gamma$ *and tends to* 0 *when* $|n| \to \infty$ *in* Γ. *If the wave operators* $\Omega^{\pm}_{\Theta}(H,\Delta)$ *exist almost surely, then the wave operators* $\Omega^{\pm}_{\Theta}(\Delta,H)$ *also exist and the spectrum of* H *is purely absolutely continuous on* Θ, *almost surely.*

Notice that the existence of $\Omega^{\pm}_{\Theta}(\Delta,H)$ may come from other sparseness conditions on Γ found in the literature; see for instance [32].

In the present circumstances *similar calculations* outside $\mathrm{spec}(\Delta)$ yield:

Theorem *The spectrum of* H *is pure point outside* $\mathrm{spec}(\Delta)$ *with exponentially decaying eigenfunctions, almost surely. More precisely, for almost all* V, *if* φ *is such an eigenfunction, then there exist positive constants,* C *and* α, *both depending on* V *and* φ, *such that* $|\varphi(n)| \leqslant C\mathrm{e}^{-\alpha|n|}$ *for all* $n \in \mathbb{Z}^d$.

If Δ is the standard or the Molchanov–Vainberg Laplacian, then the spectrum of Δ is purely absolutely continuous, while Θ is equal to $S = \mathrm{spec}(\Delta)$ minus a finite set. Hence, $\Omega_{\Theta}^{\pm}(H,\Delta) = \Omega_{S}^{\pm}(H,\Delta)$ (if they exist). For the same reason, under the conditions of the penultimate theorem $\Omega_{\Theta}^{\pm}(\Delta,H) = \Omega_{S}^{\pm}(\Delta,H)$ almost surely, by a theorem of Jakšić and Last [19].

In summary, if Γ is sufficiently sparse, $H = \Delta + V$ then satisfies almost surely the following, remarkable properties:

1. Outside spec(Δ) the spectrum of H is (possibly dense) pure point with exponentially decaying eigenfunctions;
2. Inside spec(Δ) the spectrum of H is purely absolutely continuous;
3. Inside spec(Δ) the wave operators $\Omega^{\pm}(H,\Delta)$ and $\Omega^{\pm}(\Delta,H)$ exist.

The existence of a family of random Schrödinger operators satisfying these last properties is thus established (at our knowledge for the first time in the literature).

* * *

Historically, sparse potentials were introduced by Pearson [34] in order to exhibit examples of Schrödinger operators whose spectra present singular continuous parts. Since this time, several models of sparse potentials have been suggested in the literature, having in common that the number of sites of the potential included in a cube of length L centered at the origin decreases with L. Both continuous and discrete cases have been investigated.

The idea to construct wave operators for showing that Schrödinger operators with sparse potentials possess an absolutely continuous part is due to Krishna [24], who used a deterministic model. Krishna *et al.* [25] then exhibited *mixed* spectra (*i.e.*, spectra containing both an absolutely continuous part and a singular one) for Schrödinger operators submitted to random, sparse potentials in high disorder regime.[1]

Sparse potentials were also investigated by Kirsch *et al.* [5, 12, 22] and Molchanov *et al.* [28, 29, 30, 31, 32]. These teams established that the spectrum of a random Schrödinger operator, $H = \Delta + \lambda V$, with sparse potential is mixed: almost surely, its absolutely continuous part covers $\mathrm{spec}(\Delta)$, where in addition the wave operators exist; almost surely, its singular part lies outside this last region and is pure point. Using potentials almost surely bounded, they exhibited examples where the pure point spectrum is discrete outside $\mathrm{spec}(\Delta)$ (with accumulation points at the edges) *if and only if* the disorder is small. Moreover, they characterized the essential spectrum, which was already known to be fixed almost surely.

In summary, examples of random Schrödinger operators with sparse potentials satisfying the properties 1 and 2 in the above enumeration, whose discrete pure point spectrum outside $\mathrm{spec}(\Delta)$ has dense parts (alternatively, is discrete), and for which wave operators exist were constructed in the past.[2] Therefore, the main novelty of the present dissertation, which is a continuation of [17], is the second part of the property 3: the wave operators are *complete* on $\mathrm{spec}(\Delta)$. Our result thus provides a complete description of the absolutely continuous spectrum for a class of random Schrödinger operators with sparse potentials.

[1] Given a Schrödinger operator, $H = \Delta + \lambda V$, the *disorder* is defined as λ.

[2] Notice however that our results use new techniques and yield substantially different sparseness criterions.

5.2 Perspectives

Several little projects may be built from the present thesis.

Regarding our model, the potentials of the considered Schrödinger operators are random, but supported on deterministic sets of sites; it may be physically relevant to randomize these last sets. Moreover, we studied generalized Laplacians coming from translational invariant graphs only. Other discretizations of the Laplacian may be investigated, for instance, coming from non invariant, weighted graphs or, in the hardest case, from non invariant, weighted, oriented graphs.

Regarding Chapter 2 one may wonder on which subintervals of energy the constant energy surfaces of the standard Laplacian admit $2, 3, \ldots$ non vanishing principal curvatures at every point. One may also calculate the complete asymptotic expansion of the Green's function of the standard Laplacian in concrete dimensions, say, $2, 3, \ldots$[3] We also presented the Molchanov–Vainberg Laplacian, whose symbol (with respect to the Fourier transform over $\mathbb{Z}^d$) has strictly convex level surfaces. One may try to generalize this result using a different lattice.

Regarding Chapter 3, in the unbounded case our sparseness condition, which ensures the existence of $\Omega^{\pm}(\Delta, H)$ inside $\mathrm{spec}(\Delta)$, may be compared with sparseness conditions found in the literature which ensure the existence of $\Omega^{\pm}(H, \Delta)$ instead. In particular, one may wonder if our sparseness condition indeed suffices to ensure

[3] The complete asymptotic expansion of the Molchanov–Vainberg Laplacian may easily be derived in dimension d using techniques shown in the present thesis.

the existence of all these wave operators. Finally, other forms of sparseness may be investigated (especially, sparse clusters in the unbounded case).

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