Efficient Determination of Four-Point Form-Closure Optimal Constraints of Polygonal Objects

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***Abstract*—This paper proposes a new and more efficient solution to the problem of determining optimal form-closure constraints of polygonal objects using four contacts. New grasp parameters are determined based only on the directions of the applied forces, which are then used to determine the optimal grasp. Given a set of contact edges, using an analytical procedure a solution that is either the optimal one or is very close to it is obtained (only in this second case an iterative procedure is needed to find a root of a nonlinear equation). This procedure is used for an efficient determination of the optimal grasp on the whole object. The algorithms have been implemented and numerical examples are shown.**

***Note to Practitioners*—This paper presents an algorithm that improves previous approaches in terms of efficiency in the determination of the optimal object constraint maximizing the minimum wrench that the object can support in any direction. The problem can always be solved using numerical optimization techniques but when time is relevant an efficient algorithm becomes of interest. Practical applications include optimal determination of fixtures and object grasps.**

***Index Terms*—Fixture design, form closure, grasp synthesis, intrinsic grasp parameters, optimal constraint.**

I. Introduction

The obtention of grasps or fixtures capable of ensuring the immobility of the object despite external disturbances has been a topic of extensive study and has been characterized by one of the following properties: form-closure (the position of the fingers ensures the object immobility) or force-closure (the forces applied by the fingers ensure the object immobility) [1]. Relevant works on this topic have determined necessary and sufficient conditions for the existence of a form-closure grasp [2]: the necessary conditions that four frictionless contacts and two frictional contacts must satisfy to obtain force-closure grasps of 2-D polygonal objects [3], a sufficient condition for force-closure grasps with three fingers on 2-D polygonal objects [4] and for four finger grasps on 3-D polyhedral objects [5], and necessary and sufficient conditions for three finger force-closure grasps of 2-D and 3-D objects [6]. Considering any number of contact points, a qualitative test was presented to determine whether they allow a force-closure grasp of a 3-D object [7] and an algorithm to determine the set of all the force-closure grasps of polygonal objects [8], [9].

Finding the optimal force/form-closure grasp was the next problem in grasp and fixture planning and several criteria were proposed for the grasp quality evaluation [10]. Some criteria consider only geometrical aspects of the grasp, for instance the minimization of the distance between the object's center of mass and the geometric center of the grasping points [4], used in several grasp synthesis algorithms (e.g., [11]). Other criteria determine the optimal object constraint considering constraints on the fingers forces [12]; the most used criterion in this line, known as the criterion of the maximum ball, evaluates the maximum wrench that the grasp can safely resist in any direction considering limited finger forces [13], [14]. This criterion was used to evaluate force-closure grasps generated with different strategies [15]–[17], but although these approaches provide good grasps they do not generate the optimum. The synthesis of optimal grasps considering this criterion and with a low computational cost is a problem that remains unsolved. The main drawback of the general approaches developed until now is their computational cost, thus they must be simplified to be applied in systems with time constraints [18]. Variations of this criterion were also used to obtain general procedures, for instance using linear programming [19], or using a different norm to compute the module of the wrenches [20] even when the convergence to the optimal grasp is not guaranteed. Specifically, in the field of fixture design, it is common to use heuristics and exhaustive search procedures to obtain the final fixture design [21], [22]. Other criteria decouple forces and torques [23] or define an invariant metric [24]. A comparison of several criteria was done in [25]. Good overviews of the related problems in this field were done by Shimoga [26] and Bicchi [27].

After this introduction, Section II details the contributions and scope of this paper and Section III describes the constraint on the finger forces and the grasp quality measure used in this paper. Section IV presents the algorithm to obtain the optimal grasp on the whole object and Section V presents an efficient procedure to obtain the optimal grasp on a given set of edges, which is the main contribution of this paper. Numerical examples are included in Section VI. Section VII presents some concluding remarks and future research lines. There is also an Appendix describing a geometrical substitution to reduce the computational cost of the proposed approach.

II. Contributions of This Paper

This paper presents a new and efficient procedure to determine the optimal form-closure grasp (hereafter FC grasp) of 2-D

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polygonal objects using four frictionless contacts and the quality measure of the maximum ball (four is the minimum number of frictionless contacts that allow an FC grasp [28], and it is a conservative solution with respect to friction, whose existence increases the robustness of the grasp). The obtention of a procedure to solve this specific problem with a reasonable computational cost was presented as an open problem in the literature [12] and, to our knowledge, it has not been solved yet. The approach developed here follows a previous work [29], where the optimal position of a fourth finger given the positions of the other three was solved in a fully analytical way. In this paper, the determination of the optimal position of the four fingers is deeply analyzed and a procedure to determine the optimal grasp without involving hard iterative searches is presented. Specifically, the main contributions of this paper are as follows.

1) *Grasp analysis:* Determination of a new set of intrinsic grasp parameters function of the object shape, which are used to identify different cases for the optimal grasp determination.
2) *Grasp synthesis:* Development of an efficient procedure to determine the optimal grasp in each case considering one of the most popular quality measures. The procedure analytically obtains a solution that is either the optimal one or is very close to it. In this second case, an iterative procedure is needed to find a root of a nonlinear equation.

The authors are not aware of any previous work that analytically determines the optimal grasp of 2-D objects using the quality measure of the maximum ball. The approach presented by [30] identified equivalent cases for the optimal grasp although not all of them were solved. Here, a faster identification of each case is presented as well as the methodology to solve all of them. The proposed approach to determine FC grasps with four frictionless contacts is of practical interest in the design of fixtures for 2-D polygonal objects and some particular cases of 3-D polyhedral objects [31]–[33].

In this paper, it is assumed that the contacts between the object and the fingertips are punctual and that the forces applied by the fingers act only against the object boundary (positivity constraint). The vertices of the object are not considered as possible contact points even when concave vertices may be actually considered for grasping purpose. There is no constraint regarding the number of fingers per edge, thus, in this approach, it is possible to consider two fingers on the same edge (for polygonal objects a minimum of three edges must be contacted to allow an FC grasp).

III. Grasp Quality Measure

A. Constraint on Finger Forces

Let $\boldsymbol{p}_i$ be a contact point on the object boundary described with respect to the object center of mass, and let $\boldsymbol{f}_i = \alpha_i \hat{\boldsymbol{f}}_i$, with $\alpha_i \geq 0$ and $\|\hat{\boldsymbol{f}}_i\| = 1$, be the force exerted by the finger i at $\boldsymbol{p}_i$. In the absence of friction, $\hat{\boldsymbol{f}}_i = (\cos\theta_i \sin\theta_i)^T$, where θ_i indicates the inward direction normal to the contact edge (Fig. 1). The force exerted by each finger produces a torque with respect to the object center of mass $\tau_i = \boldsymbol{p}_i \times \boldsymbol{f}_i$, and the components of $\boldsymbol{f}_i$ and τ_i form the wrench vector $\boldsymbol{\omega}_i = (\boldsymbol{f}_i^T \lambda\tau_i)^T$, where λ is the constant that adjusts the metric of the wrench space. The proposed approach is independent of λ, thus, for simplicity, from now on it is considered $\lambda = 1$ and therefore it is removed from the equations.

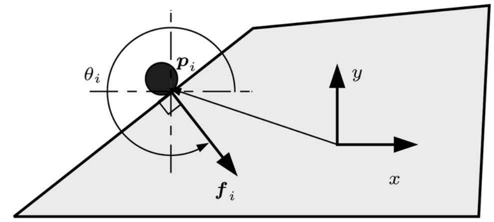


Fig. 1. Force $\boldsymbol{f}_i$ applied by finger i at contact point $\boldsymbol{p}_i$.

The forces applied by the fingers can be subject to different constraints [12]). Here, it is considered that the total force exerted by the fingers is limited to $\alpha_{\max}$ (for instance, due to a maximum available power for all the finger actuators); then, the resultant force $\boldsymbol{f}$ on the object is

$$\boldsymbol{f} = \sum_{i=1}^{4} \alpha_i \hat{\boldsymbol{f}}_i = \alpha \hat{\boldsymbol{f}} \quad \text{with} \sum_{i=1}^{4} \alpha_i \leq \alpha_{\max}. \tag{1}$$

Geometrically, this implies that $\boldsymbol{f}$ lies inside the polygon $\mathcal{P}_f$ defined in the force space as [Fig. 2(a)]

$$\mathcal{P}_f = \text{Convex Hull}\left(\bigcup_{i=1}^{4}\{\boldsymbol{f}_i\}\right) \quad \text{with } \boldsymbol{f}_i = \alpha_{\max}\hat{\boldsymbol{f}}_i. \tag{2}$$

Analogously, the resultant wrench on the object lies inside the polyhedron $\mathcal{P}_\omega$ defined in the wrench space as [Fig. 2(b)]

$$\mathcal{P}_\omega = \text{Convex Hull}\left(\bigcup_{i=1}^{4}\{\boldsymbol{\omega}_i\}\right) \quad \text{for } \boldsymbol{f}_i = \alpha_{\max}\hat{\boldsymbol{f}}_i. \tag{3}$$

In an FC grasp, $\mathcal{P}_f$ and $\mathcal{P}_\omega$ must contain the origin of the force and wrench space, respectively [2].

In the rest of this paper, for simplicity and without loss of generality, we consider $\alpha_{\max} = 1$ and therefore $\boldsymbol{f}_i$ will refer always to the maximum (unitary) possible applied force.

B. Quality Measure Definition

The quality Q of an FC grasp is given by the maximum wrench that the finger forces can generate in any direction of the wrench space [13], i.e.,

$$Q = \min_{\boldsymbol{\omega} \in \text{Boundary}(\mathcal{P}_\omega)} \|\boldsymbol{\omega}\|. \tag{4}$$

Geometrically, Q is the radius of the maximum ball centered at the origin of the wrench space and fully contained inside $\mathcal{P}_\omega$. Let D_{ijk} be the distance from the origin of the wrench space to the plane defined by $\boldsymbol{\omega}_i, \boldsymbol{\omega}_j$, and $\boldsymbol{\omega}_k$ (the wrenches produced by fingers i, j and k). Then, Q can be expressed as

$$Q = \min_{i,j,k \in \{1,\ldots,4\}, i \neq j \neq k} \{D_{ijk}\}. \tag{5}$$

The same concept can be applied to define a quality measure considering only the force space [23] as

$$Q_f = \min_{\boldsymbol{f} \in \text{Boundary}(\mathcal{P}_f)} \|\boldsymbol{f}\|. \tag{6}$$

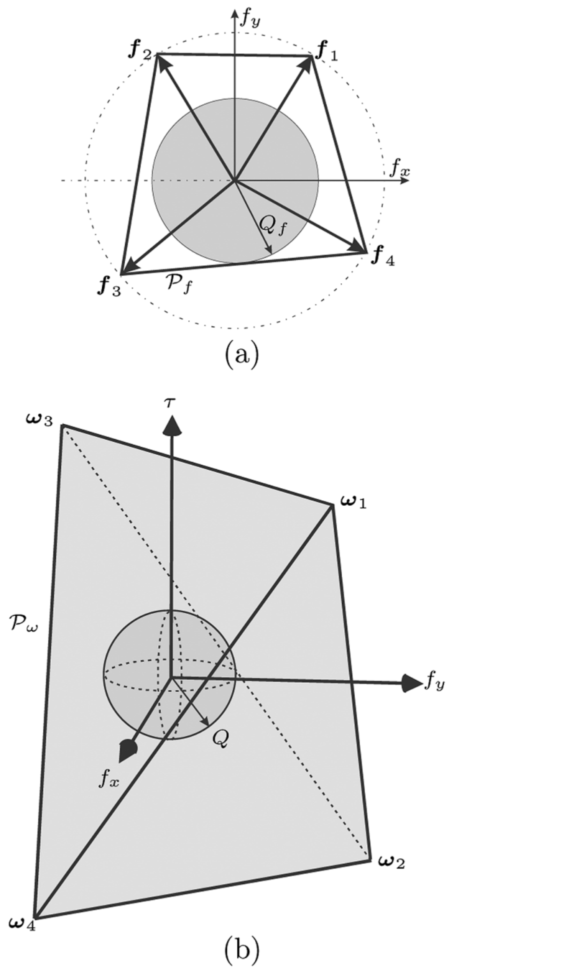


Fig. 2. Constraint and quality measure on: (a) force space (polygon $\mathcal{P}_f$ and circumference of radius Q_f) and (b) wrench space (polyhedron $\mathcal{P}_\omega$ and sphere of radius Q).

Fig. 2 shows the geometrical interpretation of the constraints $\mathcal{P}_f$ and $\mathcal{P}_\omega$ and the quality measures Q_f and Q, respectively. Note that $\mathcal{P}_f$ is the projection of $\mathcal{P}_\omega$ on the force space and it is not possible to obtain a sphere fully contained in $\mathcal{P}_\omega$ with radius larger than Q_f. Therefore, Q_f is an *upper bound* for Q.

IV. Optimal Grasp Over Whole Object

This section presents the procedure to obtain the optimal grasp over the whole object. The following terms will be used.

Definition 1: The *edge-optimal grasp* G_e is the set of four contact points that generates the optimal grasp on a given set of three or four contact edges.

Definition 2: The *object-optimal grasp* G_o is the set of four contact points that generates the optimal grasp over the whole object (i.e., G_o is the best G_e).

Given a combination of three or four edges for the finger contacts, the direction $\theta_i, i = 1, \ldots, 4$, of the force applied by each finger, is known, and from them $\boldsymbol{f}_i = (\cos\ \theta_i\ \sin\ \theta_i)^T$ and then $\mathcal{P}_f$ are directly obtained. Therefore, Q_f can be easily computed from (6) once the contact edge of each finger is given.

The object-optimal grasp G_o over the whole object is obtained with the following algorithm, which uses Q_f of a set of contact edges as an upper bound for the quality Q of any grasp produced on those edges.

Algorithm 1(Computation of G_o)

Let C be the set of possible different combinations of three and four edges:
1) Initialize $Q = 0$
2) Determine the subset C' of C with the combinations of edges that satisfy $\mathbf{0} \in \mathcal{P}_f$.
3) Compute Q_f for each combination of edges in C'.
4) Order C' from better to worse Q_f.
5) For each combination of edges in C' and following the order established in step 3, do:
    5.1) Determine G_e and its quality Q'.
    5.2) If $Q < Q'$ then $G_o = G_e$ and $Q = Q'$.
    5.3) If Q is greater that the value of Q_f of the next combination of edges then exit the loop.
6) Return G_o and its quality Q.

The determination of G_e in step 5.1 is the critical operation in terms of computational cost. The rest of this paper deals with an efficient procedure to solve this problem, which is the key contribution of this paper.

V. Optimal Grasp Over a Set of Contact Edges

Given the set of three or four contact edges, the direction θ_i and, therefore, the component $\boldsymbol{f}_i$ of each wrench $\boldsymbol{\omega}_i = (\boldsymbol{f}_i^T \tau_i)^T$, is known; then, determining G_e is equivalent to determine the values of τ_i, $i = 1, \ldots, 4$ that maximize Q (refer again to Fig. 2). Since $\boldsymbol{f}_i$ is known, the value of τ_i determines the position of the contact point $\boldsymbol{p}_i$ on the corresponding edge; for this reason, from now on we will frequently refer to the problem of finding the optimal contact points $\boldsymbol{p}_i$ as the problem of finding the optimal values of τ_i.

A. Ranges of Torque That Allow Form-Closure Grasp

Based on the univocal relation between the contact point $\boldsymbol{p}_i$ and the torque τ_i, the following concepts are defined.

Definition 3: The *real range* of τ_i, R_i is the set of values of τ_i produced by the contact force $\boldsymbol{f}_i$ applied at any point $\boldsymbol{p}_i$ on the contact edge E_i, i.e., $R_i = \{\tau_i = \boldsymbol{p}_i \times \boldsymbol{f}_i / \boldsymbol{p}_i \in E_i\}$.

Definition 4: The *directional range* of τ_i, R_{d_i} is the set of values of τ_i produced by the contact force $\boldsymbol{f}_i$ at any point $\boldsymbol{p}_i$ on the supporting line e_i of the contact edge E_i, that allows an FC grasp given any other three wrenches $\boldsymbol{\omega}_h, \boldsymbol{\omega}_j$, and $\boldsymbol{\omega}_k$ applied on the object, i.e.,

$$R_{d_i} = \{\tau_i = \boldsymbol{p}_i \times \boldsymbol{f}_i / \boldsymbol{p}_i \in e_i \quad \text{and } \mathbf{0} \in \mathcal{P}_\omega\}.$$

Note that R_{d_i} may include values of τ_i that are not physically possible due to the actual finite length of the edge E_i. Then, four contact points $\boldsymbol{p}_i$, $i = 1, \ldots, 4$ allow an FC grasp if $\tau_i \in R_i \cap R_{d_i}$. Fig. 3 shows an example of an FC grasp and the directional ranges for each contact.

R_{d_i} is defined for three other fixed wrenches, thus it is a continuous set bounded by values τ_{i_m} of τ_i that produce a grasp with $Q = 0$; geometrically, τ_{i_m} implies that the origin of the wrench space belongs to a face of $\mathcal{P}_\omega$ whose vertices are $\boldsymbol{\omega}_i$ and

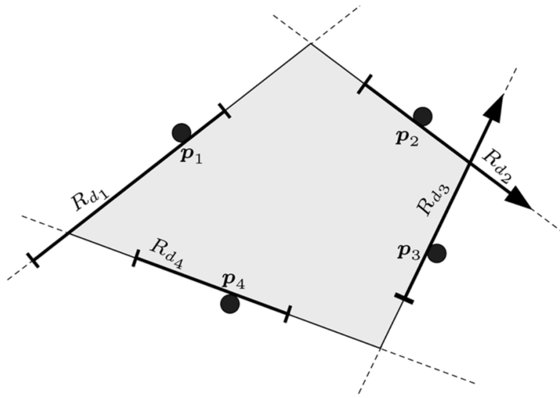


Fig. 3. FC grasp and directional ranges R_{d_i}.

two other wrenches $\boldsymbol{\omega}_j$ and $\boldsymbol{\omega}_k$, i.e., $\mathbf{0} = \alpha_i\boldsymbol{\omega}_i + \alpha_j\boldsymbol{\omega}_j + \alpha_k\boldsymbol{\omega}_k$ with $\alpha_i > 0, \alpha_j, \alpha_k \geq 0$ and $\alpha_i + \alpha_j + \alpha_k = 1$. Solving this equation for $\boldsymbol{\omega}_i$ and expanding its components results in the following:

$$\cos\theta_i = \beta_{i,jk}\cos\theta_j + \beta_{i,kj}\cos\theta_k \tag{7}$$

$$\sin\theta_i = \beta_{i,jk}\sin\theta_j + \beta_{i,kj}\sin\theta_k \tag{8}$$

$$\tau_{i_m} = \beta_{i,jk}\tau_j + \beta_{i,kj}\tau_k \tag{9}$$

where $\beta_{i,jk} = -(\alpha_j/\alpha_i) \leq 0$ and $\beta_{i,kj} = -(\alpha_k/\alpha_i) \leq 0$, but $\beta_{i,jk}$ and $\beta_{i,kj}$ cannot be simultaneously null because $\cos(\theta_i)$ and $\sin(\theta_i)$ cannot be simultaneously null.

Given two known wrenches $\boldsymbol{\omega}_j$ and $\boldsymbol{\omega}_k$, the corresponding extreme of R_{d_i} can be determined as follows.

1) Solving $\beta_{i,jk}$ and $\beta_{i,kj}$ from (7) and (8) is

$$\beta_{i,jk} = \frac{\sin(\theta_i - \theta_k)}{\sin(\theta_j - \theta_k)} \quad \beta_{i,kj} = \frac{\sin(\theta_j - \theta_i)}{\sin(\theta_j - \theta_k)}. \tag{10}$$

2) If $\beta_{i,jk} \leq 0$ and $\beta_{i,kj} \leq 0$, then the τ_{i_m} resulting from (9) is an extreme of R_{d_i} that produces $Q = 0$ (if either $\beta_{i,jk} > 0$ or $\beta_{i,kj} > 0$, the resulting τ_{i_m} from (10) makes that the plane defined by $\boldsymbol{\omega}_i, \boldsymbol{\omega}_j$, and $\boldsymbol{\omega}_k$ contains the origin, but with the origin outside the face of $\mathcal{P}_\omega$).

The exact determination of the extremes of R_{d_i} is only possible when the other three applied wrenches are known (i.e., the positions of the other three contact points are given), because τ_j and τ_k are necessary in (9), but knowing whether an extreme exists or not can be determined by knowing how many pairs $\beta_{i,jk}$ and $\beta_{i,kj}$ [from (10)] have nonpositive values for $i, j, k \in \{1, \ldots, 4\}$, and this depends only on the directions of the applied forces. According to the number of existing finite extremes for a directional range R_{d_i} it is classified as follows.

1) *Limited:* $R_{d_i} = [\tau_{i_1}, \tau_{i_2}], \tau_{i_1}$ and τ_{i_2} are two finite extremes where $Q = 0$ (e.g., R_{d_1} and R_{d_4} in Fig. 3).
2) *Infinite:* $R_{d_i} = (-\infty, \tau_{i_1}]$ or $R_{d_i} = [\tau_{i_1}, \infty), \tau_{i_1}$ are the unique finite extremes where $Q = 0$ while the quality for $\tau_i \to \pm\infty$ is a finite value L (e.g., R_{d_2} and R_{d_3} in Fig. 3). (L is the distance from the origin of the force space to the segment $\overline{\boldsymbol{f}_j\boldsymbol{f}_k}$).

Given the contact edges, the directions $\theta_i, i = 1, \ldots, 4$ of the applied forces are known, and analyzing the combinations of signs in (10) for the possible relative directions of forces the following cases were identified [34].

1) *General case*: If the angles between the applied forces are different from π, there are two infinite directional ranges corresponding to the torques generated by the two forces lying between the negated of the other two [Fig. 4(a) and (b)].
2) *Particular cases*: If the angle between two forces is π, there are three infinite directional ranges corresponding to the torques generated by the other two forces and the force that lies between them [Fig. 4(c)], and if the angles between two pairs of forces are π, the four directional ranges are infinite [Fig. 4(d)].

Then, in an FC grasp there always exist two wrenches whose force components define two consecutive vertices of $\mathcal{P}_f$ and whose torque components have infinite directional ranges.

Let $\boldsymbol{f}_i$ and $\boldsymbol{f}_j$ be two consecutive vertices of $\mathcal{P}_f, \tau_i$ and τ_j be torques with infinite directional ranges, and, considering the general case, let $R_{d_k} = [\tau_{k_1}, \tau_{k_2}]$ be one of the two limited directional ranges, i.e., $\tau_{k_1} \leq \tau_k \leq \tau_{k_2}$. Substituting τ_{k_1} and τ_{k_2} by the expressions derived from (9), we obtain (note that the subscripts i and j could be swapped)

$$\beta_{k,hj}\tau_h + \beta_{k,jh}\tau_j \leq \tau_k \leq \beta_{k,ih}\tau_i + \beta_{k,hi}\tau_h \tag{11}$$

and solving τ_i and τ_j from (11)

$$\tau_i \leq \frac{1}{\beta_{k,ih}}(\tau_k - \beta_{k,hi}\tau_h) \tag{12}$$

$$\tau_j \geq \frac{1}{\beta_{k,jh}}(\tau_k - \beta_{k,hj}\tau_h). \tag{13}$$

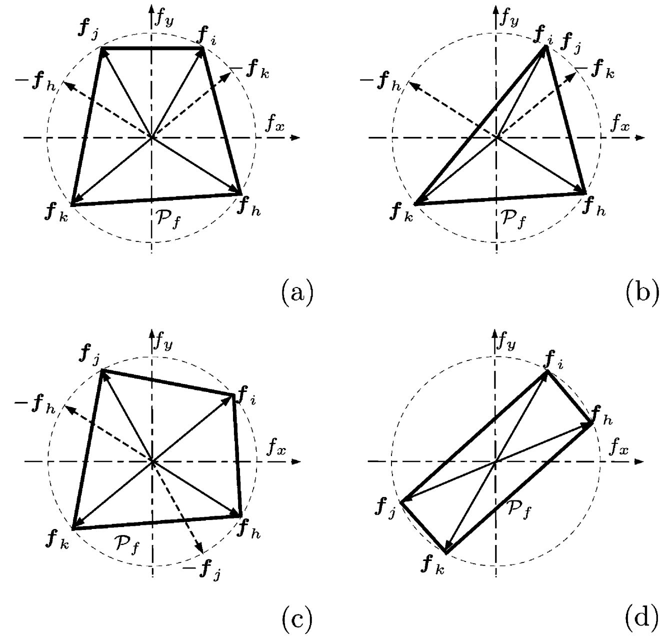


Fig. 4. Examples of determination of types of directional ranges from applied forces. (a) General case: R_{d_i} and R_{d_j} are infinite and R_{d_h} and R_{d_k} are limited. (b) General case (with two fingers on the same edge): R_{d_i} and R_{d_j} are infinite and R_{d_h} and R_{d_k} are limited. (c) Particular case (with two opposite forces): R_{d_k} is limited and R_{d_h}, R_{d_i}, and R_{d_j} are infinite. (d) Particular case (with two pairs of opposite forces): all directional ranges are infinite.

Therefore, τ_i has an upper bound while τ_j has a bottom bound, implying that $R_{d_i} = (-\infty, \tau_{i_1}]$ and $R_{d_j} = [\tau_{j_1}, \infty)$ (or *vice versa* if the subscripts i and j were swapped).

The two particular cases are limits of the general case. Adding $\delta\theta$ arbitrarily small to the direction of one of the aligned forces, the particular cases are transformed into the general case, and the same result is obtained when $\delta\theta \to 0$.

B. Optimal Grasp Cases

Jia [30] established a relation between the number of faces of $\mathcal{P}_\omega$ at a distance Q of the origin and the number of contact points lying on an extreme of an edge, obtaining the following.

Case 1: If Q is the distance to one face of $\mathcal{P}_\omega$, then the four contact points lie on the extremes of the edges.

Case 2: If Q is the distance to two faces of $\mathcal{P}_\omega$, then at least two contact points lie on the extremes of the edges.

Case 3: If Q is the distance to three faces of $\mathcal{P}_\omega$, then at least one contact point lies on an extreme of the edge.

Case 4: If Q is the distance to the four faces of $\mathcal{P}_\omega$, then there may be no contact point lying on an extreme of the edge.

This determines the number of contact points that lie on an extreme of an edge in the optimal grasp; nevertheless, given a set of contact edges, the approach does not determine which cases can actually exist and which contact points lie on extremes of the edges. Therefore, all the possible combinations have to be checked. Moreover, only the first and second cases were solved.

The approach presented here completely solves this problem and identifies (with reduced computation) which cases are possible and which contact points lie on extremes of the edges. For this purpose, the following concepts are defined, based only on the directions of the given contact edges.

Definition 5: The *internal bounds*, C_{hj} and C_{ik}, of an FC grasp are the distances from the origin of the force space to each one of the segments determined by two nonconsecutive vertices of $\mathcal{P}_f$ (e.g., $\overline{\boldsymbol{f}_h\boldsymbol{f}_j}$ and $\overline{\boldsymbol{f}_i\boldsymbol{f}_k}$ in Fig. 5).

Definition 6: The *directional-optimal grasp* G_d is the set of four points that generates the optimal grasp on the supporting lines of the given grasping edges (i.e., the lengths of the edges are not considered and only their directions are relevant).

Note that the points that determine G_d may not lie on the actual object boundary, thus G_d could actually be unreachable.

Consider now the following propositions whose detailed proofs can be found in [34].

Proposition 1: Let $\boldsymbol{\omega}_h, \boldsymbol{\omega}_i$, and $\boldsymbol{\omega}_j$ be three known wrenches (i.e., three wrenches produced at three known contact points on the object) and let $\boldsymbol{\omega}_k$ be a wrench whose torque component τ_k is unknown (i.e., the contact edge is known but the contact point is unknown). The optimal value τ_{d_k} that produces the optimal grasp without considering the real range R_k can be analytically determined knowing the upper bound Q_f, the internal bounds C_{hj}, and C_{ik}, and the type of the directional range R_{d_k}, according to the following cases.

1) If R_{d_k} is infinite and $C_{hj} \geq Q_f$, then $\tau_{d_k} \to \pm\infty$ according to R_{d_k}.
2) Else (i.e., R_{d_k} is limited or $C_{hj} < Q_f$):
   a) If $C_{ik} \geq Q_f$, then τ_{d_k} is the solution of

$$D_{hik} = D_{hjk} \tag{14}$$

   where D_{hik} and D_{hjk} are the distances from the origin of the wrench space to the faces of $\mathcal{P}_\omega$ defined by $\{\boldsymbol{\omega}_h, \boldsymbol{\omega}_i, \boldsymbol{\omega}_k\}$ and $\{\boldsymbol{\omega}_h, \boldsymbol{\omega}_j, \boldsymbol{\omega}_k\}$, such that the triangles defined by $\{\boldsymbol{f}_h, \boldsymbol{f}_i, \boldsymbol{f}_k\}$ and $\{\boldsymbol{f}_h, \boldsymbol{f}_j, \boldsymbol{f}_k\}$ intersect with the circumference of radius Q_f in the force space.
   b) Else (i.e., $C_{hj} < Q_f$) τ_{d_k} is one of the solutions of

$$D_{hik} = D_{hjk} \tag{15}$$
$$D_{hik} = D_{ijk} \tag{16}$$
$$D_{hjk} = D_{ijk} \tag{17}$$

   where D_{hik}, D_{hjk} and D_{ijk} are the distances from the origin of the wrench space to the faces of $\mathcal{P}_\omega$ defined by $\{\boldsymbol{\omega}_h, \boldsymbol{\omega}_i, \boldsymbol{\omega}_k\}, \{\boldsymbol{\omega}_h, \boldsymbol{\omega}_j, \boldsymbol{\omega}_k\}$, and $\{\boldsymbol{\omega}_i, \boldsymbol{\omega}_j, \boldsymbol{\omega}_k\}$ (i.e., the three faces of $\mathcal{P}_\omega$ that contain $\boldsymbol{\omega}_k$).

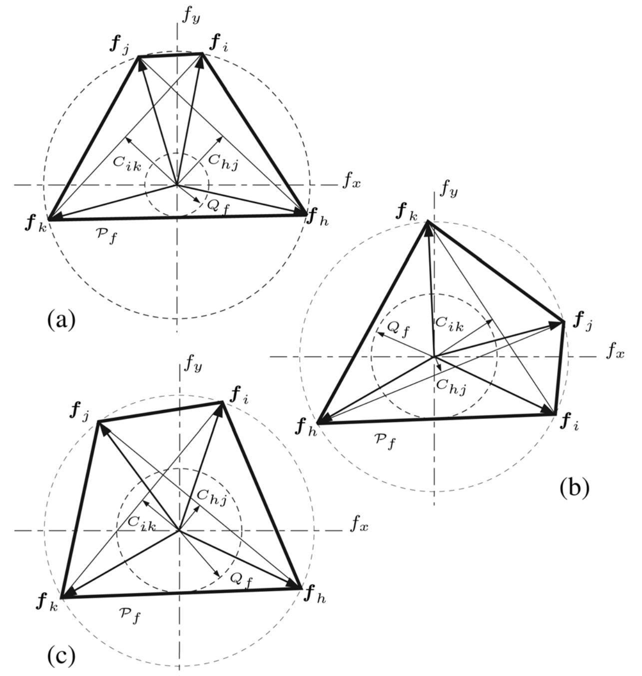


Fig. 5. Examples of internal bounds and of three cases in determination of directional-optimal grasp of Proposition 2: (a) $C_{hj} \geq Q_f$ and $C_{ik} \geq Q_f$; (b) $C_{hj} < Q_f$ and $C_{ik} \geq Q_f$; and (c) $C_{hj} < Q_f$ and $C_{ik} < Q_f$.

This proposition refines the results presented in [29] and, in order to obtain G_d, it is applied considering all the possible relations between the upper bound, the internal bounds, and the types of directional ranges, obtaining the following result.

Proposition 2: Let $\boldsymbol{\omega}_h$ and $\boldsymbol{\omega}_i$ be the wrenches whose force components determine the upper bound Q_f, and let $\boldsymbol{\omega}_j$ and $\boldsymbol{\omega}_k$ be the other two wrenches. The directional-optimal grasp, G_d can be determined according to the values of Q_f and the internal bounds C_{hj} and C_{ik} as follows:

1) If $C_{hj} \geq Q_f$ and $C_{ik} \geq Q_f$, then $\tau_{d_j} \to \pm\infty, \tau_{d_k} \to \mp\infty, \tau_{d_h}$ and τ_{d_i} are determined from

$$\text{Max} \quad D_{hij} \tag{18}$$
$$\text{subject to} \quad D_{hij} = D_{hik}. \tag{19}$$

2) If $C_{hj} < Q_f$ and $C_{ik} \geq Q_f$, then $\tau_{d_j} \rightarrow \pm\infty, \tau_{d_h}, \tau_{d_i}$ and τ_{d_k} are determined from

$$\text{Max} \quad D_{hij} \tag{20}$$
$$\text{subject to} \quad D_{hij} = D_{hik} = D_{hjk}. \tag{21}$$

3) If $C_{hj} < Q_f$ and $C_{ik} < Q_f$, then $\tau_{d_h}, \tau_{d_i}, \tau_{d_j}$ and τ_{d_k} are determined from

$$\text{Max} \quad D_{hij} \tag{22}$$
$$\text{subject to} \quad D_{hij} = D_{hik} = D_{hij} = D_{ijk}. \tag{23}$$

Fig. 5 shows examples of the force directions that produce each case in Proposition 2.

Note that G_d has always at least one torque tending to infinite (since the optimization problem is unbounded, it also happens when $C_{hj} < Q_f$ and $C_{ik} < Q_f$, but in this case it is not possible to determine which one tends to infinite). Then, in order to obtain the reachable optimal grasp, some optimal contact points lie on extremes of the edges, according to the following proposition.

Proposition 3: If $\tau_{d_i} \rightarrow \pm\infty$, then the optimal reachable torque is the extreme of R_i closest to τ_{d_i}.

From Propositions 2 and 3, the use of the upper bound and the internal bounds (parameters that depend only on the directions of the applied forces) allows us an easy identification of the optimal contact points that for sure lie on the extremes of the edges and of the faces of $\mathcal{P}_\omega$ at a distance Q from the origin in the optimal case. Then, the cases presented by [30] are easily identified.

Since the real range of all the contact points has not been considered yet, the current optimal grasp may actually be unreachable; in this case, the reachable optimal solution will have additional contact points lying on edge extremes and the possible combinations of edge extremes for this contact points must be considered in the search of G_e.

C. Computation of Edge-Optimal Grasp G_e

In the wrench space, determining G_e is equivalent to determining four reachable wrenches $\boldsymbol{\omega}_{e_i} = (\boldsymbol{f}_i^T \ \tau_{e_i})^T, i = 1, \ldots, 4$ (i.e., with $\tau_{e_i} \in R_i$) that fix the vertices of the polyhedron $\mathcal{P}_\omega$ to contain the largest possible sphere centered at the origin.

From Propositions 2 and 3, the optimal positions of some points lie on extremes of the edges while the optimal positions of the others are the solution of a optimization problem that can be expressed in a generic form as

$$\text{Max} \quad D_{hij} \tag{24}$$
$$\text{subject to} \quad \boldsymbol{\mathcal{C}}_s = \mathbf{0} \tag{25}$$

where $\boldsymbol{\mathcal{C}}_s$ includes $s = 1, \ldots, 3$ constraints depending on the considered optimization problem (note that the number of unknown torques is always $s + 1$).

Since the constraints of the optimization problem defined by (24) and (25) are equalities, this problem can be translated into a system of equations using the Lagrange theorem [35]. Letting $\boldsymbol{\mathcal{L}} = [\mathcal{L}_1 \cdots \mathcal{L}_s]^T$ be the Lagrange multipliers vector, the solution of the optimization problem can be determined by solving the following system of equations:

$$\nabla D_{hij} + \boldsymbol{\mathcal{L}}^T \nabla \boldsymbol{\mathcal{C}}_s = \mathbf{0} \tag{26}$$
$$\boldsymbol{\mathcal{C}}_s = \mathbf{0} \tag{27}$$

where ∇ is the gradient operator. Since there are s constraints and $s + 1$ unknown torques, (26) and (27) represent a system of $2s + 1$ equations with $2s + 1$ unknowns (including the torques and the Lagrange multipliers).

Equation (26) represents $s + 1$ linear equations with respect to the Lagrange multipliers. Since the determination of the Lagrange multipliers is not necessary, an evaluation function $\mathcal{F}$ can be obtained eliminating them from (26). For instance, for the optimization problem described by (18) and (19) with two unknown torques, $\mathcal{F}$, is

$$\mathcal{F} = \frac{\frac{\partial D_{hij}}{\partial \tau_h}}{\frac{\partial (D_{hij} - D_{hik})}{\partial \tau_h}} - \frac{\frac{\partial D_{hij}}{\partial \tau_i}}{\frac{\partial (D_{hij} - D_{hik})}{\partial \tau_i}}. \tag{28}$$

Equation (27) represents s constraints (nonlinear equations) with respect to $s + 1$ unknown torques. Analytically, it is possible to solve a maximum of two constraints with two unknowns. Considering this fact and the evaluation function $\mathcal{F}$, the following algorithm allows to efficiently determine the edge-optimal grasp G_e on a given set of edges.

Algorithm 2 (Computation of G_e)

Given a set of contact edges, G_e is determined as:

1) Determine Q_f, C_{hj} and C_{ik}.
2) Obtain the s constraints that form $\boldsymbol{\mathcal{C}}_s$ and the contact points whose optimal positions lie on extremes of the edges (Propositions 2 and 3).
3) Depending on s, do:
    a) If $s = 1$ or $s = 2$, solve $\boldsymbol{\mathcal{C}}_s = \mathbf{0}$ from (27) for, respectively, each of the four or eight combinations resulting from fixing the position of each unknown torque on each extreme of the corresponding edge.
    b) If $s = 3$, solve the four resulting subsystems of two constraints of $\boldsymbol{\mathcal{C}}_s$ resulting from fixing the positions of each pair of unknown torques on two extremes of the corresponding edges.
4) As a result of step 3:
    a) If at least one of the computed sets of torques is reachable, take as initial reachable solution the one with largest Q.
    b) If none of the obtained sets of torques is reachable:
        i) If there is only one unknown torque, its optimal value is on the edge extreme closest to the value computed in step 3. Then, G_e has all the contact points lying on extremes of the edges. **Return G_e.**
        ii) Else fix the position of each unknown torque on an extreme of an edge and obtain new constraints $\boldsymbol{\mathcal{C}}_s$ applying Proposition 1 for the

remaining unknown torques (note that $\mathcal{C}_s$ is independent of the selected extremes). Go to step 3.

5) Obtain the evaluation function $\mathcal{F}$ and evaluate the initial reachable solution.
6) If $\mathcal{F} \neq 0$, determine in which direction the contact points fixed on extremes of the edges in step 3 have to be moved in order to make $\mathcal{F} \to 0$:
   a) If the points have to be moved inside the edge, apply an iterative numerical procedure to obtain the solution that satisfies $\mathcal{C}_s = \mathbf{0}$ and $\mathcal{F} = 0$.
   b) Else the initial reachable solution cannot be improved.
7) If $s = 3$, determine in which direction the contact points fixed on extremes of the edges in step 3 have to be moved in order to make the distances from the origin to the four faces of $\mathcal{P}_\omega$ be the same.
   a) If the points have to be moved inside the edges, apply an iterative numerical procedure to obtain a new solution.
   b) Else the initial reachable solution cannot be improved
8) If Steps **Return** as G_e the best of the solutions computed in steps 4a, 6a, or 7a.

As a difference from the approach proposed by [30], where cases 3 and 4 were not solved, Algorithm 2 is complete, since it always finds the optimal grasp taking into account all the possible cases. Moreover, this algorithm is also really efficient, since in many cases the initial reachable solution obtained in step 4 either is G_e or is very close to it, completely avoiding or at least decreasing the number of iterations in steps 6a or 7a, which, in any case, are not hard iterative procedures since they are function of only one torque and they can be easily solved using the Bolzano theorem.

VI. Examples

Numerical examples of the proposed methodology are presented here using the object shown in Fig. 6. Since the optimal grasp has always at least one contact point on an edge extreme, in order to avoid placing a contact point on a vertex of the object, the real ranges were slightly reduced considering a security distance from the object vertices. The object has eight edges, so the total number of possible sets of three and four edges is 238 (95 of them with $\mathcal{P}_f$ containing the origin). Considering the upper bounds Q_f, only 26 of them have been evaluated by Algorithm 1 to obtain G_o.

The examples show the determination of G_o and the determination of G_e for other three combinations of edges. The optimal grasp was also computed using the brute force method taking 50 sample points per edge and evaluating all the possible contact combinations. The results were always coincident (up to the sample resolution).

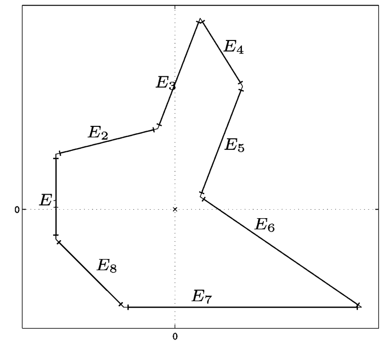
In all the examples the contact points are numbered such that the normal forces define consecutive vertices of $\mathcal{P}_f$ and the upper bound is determined by $\overline{\boldsymbol{f}_1\boldsymbol{f}_2}$.

Example 1 (Edges $E_1, E_4, E_6,$ and E_7): The edge-optimal grasp G_e on this set of edges is the object-optimal grasp G_o. This is the 11th evaluated set of edges considering the order based on the upper bounds. According to the numbering convention of the contact points: $\boldsymbol{p}_1 \in E_6, \boldsymbol{p}_2 \in E_1, \boldsymbol{p}_3 \in E_7$, and $\boldsymbol{p}_4 \in E_4$. Following Algorithm 2 to obtain G_e:

1) Determine $Q_f = 0.4665, C_{13} = 0.2956, C_{24} = 0.2756$.
2) $C_{13} < Q_f$ and $C_{24} < Q_f$. Then from (23) $s = 3$, $\mathcal{C}_s = (D_{123} - D_{124}\ D_{123} - D_{134}\ D_{123} - D_{234})^T$, and it is not possible to determine which contact points lie on an extreme of the edge.
3) $s = 3$, then the four subsystems of two constraints of $\mathcal{C}_s$ are solved fixing the positions of two unknown contacts on extremes of the edges.
4) Step 3 produced reachable solutions, with the best one, produced when $\tau_2 = \tau_{\max_2}$ and $\tau_3 = \tau_{\min_3}$, being: $\tau_1 = -1.0819, \tau_2 = 0.3021, \tau_3 = -0.5539$, $\tau_4 = 1.5742, Q = D_{124} = D_{134} = D_{234} = 0.4040$, and $D_{123} = 0.4309$.
5) Evaluating this initial reachable grasp in $\mathcal{F}$:
   If τ_2 is fixed, then $\mathcal{F} = 0.0062$.
   If τ_3 is fixed, then $\mathcal{F} = -0.0349$.
6) $\mathcal{F} \to 0$ for values greater than $\tau_{\max_2}$ or smaller than $\tau_{\min_3}$. Then, the solution of step 4 cannot be improved.
7) The distances to the four faces of $\mathcal{P}_\omega$ tend to be equal for values greater than $\tau_{\max_2}$. Then, the solution of step 4 cannot be improved.
8) Steps 6 and 7 do not improve the solution, then G_e is the grasp obtained in step 4.

Fig. 7(a) shows the resulting contact points.

Example 2 (Edges $E_2, E_5, E_6,$ and E_8): The quality of G_e on this set of edges is close to the quality of G_o. This is the fourth evaluated set of edges considering the order based on the upper bounds. According to the numbering convention of the contact points: $\boldsymbol{p}_1 \in E_2, \boldsymbol{p}_2 \in E_8, \boldsymbol{p}_3 \in E_5$, and $\boldsymbol{p}_4 \in E_6$. Following Algorithm 2, G_e is obtained when $\tau_1 = \tau_{\max_1}$ and $\tau_4 = \tau_{\min_4}$, being: $\tau_1 = 1.1548, \tau_2 = -0.4940, \tau_3 = 1.0218$,



	Normal direction (θ_i)	Minimum torque (τ_{min_i})	Maximum torque (τ_{max_i})
E_1	0	-0.5979	0.3021
E_2	4.9574	0.0179	1.1548
E_3	5.9160	-2.1544	-0.8615
E_4	3.7002	0.8559	1.6993
E_5	2.7744	0.2943	1.5872
E_6	4.1123	-2.4130	-0.2108
E_7	1.5708	-0.5539	2.1461
E_8	0.7854	-0.6937	0.3376

Fig. 6. Object used in examples and initial data (directions normal to edges and real ranges R_i of possible actual torques).

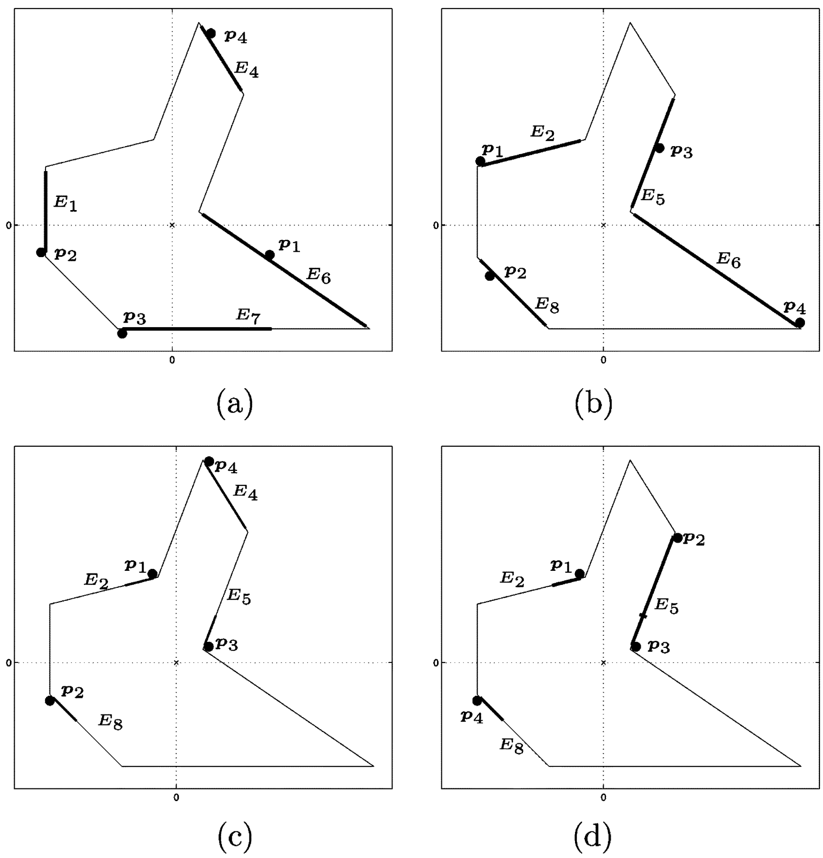


Fig. 7. Edge-optimal grasps and intersection of directional range with real range for each contact point (bold segments). Case (a) is object-optimal grasp G_e.

$\tau_4 = -2.4130, Q = D_{123} = D_{124} = D_{234} = 0.3999$ and $D_{134} = 0.6120$. Fig. 7(b) shows the resulting contact points.

Example 3 (Edges E_2, E_4, E_5, and E_8): This set of edges produces one of the worst cases in the determination of the optimal grasp because none of the constraints of the optimization algorithms can be satisfied. This is the third evaluated set of edges considering the order based on the upper bounds. According to the numbering convention of the contact points: $\boldsymbol{p}_1 \in E_2, \boldsymbol{p}_2 \in E_8, \boldsymbol{p}_3 \in E_5$, and $\boldsymbol{p}_4 \in E_4$. Following Algorithm 2, G_e is obtained for $\tau_1 = 0.0179, \tau_2 = -0.6937, \tau_3 = 0.2943, \tau_4 = 1.6993$, with $Q = D_{123} = 0.0986, D_{124} = 0.2145, D_{134} = 0.3842$, and $D_{234} = 0.3281$. Fig. 7(c) shows the resulting contact points.

Example 4 (Edges E_2, E_5 with two contact points, and E_8): This is the 17th evaluated set of edges considering the order based on the upper bounds. According to the numbering convention of the contact points: $\boldsymbol{p}_1 \in E_2, \boldsymbol{p}_2 \in E_5, \boldsymbol{p}_3 \in E_5$, and $\boldsymbol{p}_4 \in E_8$. Following Algorithm 2 to obtain G_e:

1) Determine $Q_f = 0.4612, C_{13} = 0.4612, C_{24} = 0.5449$.
2) $C_{13} \geq Q_f$ and $C_{24} \geq Q_f$. Then, from (19) $s = 1$, $\boldsymbol{C}_s = (D_{124} - D_{134})$ and the optimal positions of $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$ lie on an extreme of the edge.
3) $s = 1$, then the constraint of $\boldsymbol{C}_s$ is solved for the four combinations resulting from fixing the position of each unknown contact point on each extreme of an edge.
4) None of the results computed in step 3 are reachable. Then, the following best solution is obtained when all the contacts lie on extremes: $\tau_1 = 0.0179, \tau_2 = 0.2943, \tau_3 = 1.5872, \tau_4 = -0.6937, Q = D_{124} = 0.0986, D_{123} = 0.4611, D_{134} = 0.1832$, and $D_{234} = 0.5449$. As a result, this grasp is G_e and the algorithm ends.

Fig. 7(d) shows the resulting contact points (note the mark on E_5 limiting the intersection of the directional range with the real range for each of the two contact points on this edge).

VII. Conclusion and Future Work

This paper provides a new efficient approach to determine the optimal form-closure grasp on polygonal objects using the quality measure of the maximum ball. The upper bound, the internal bounds, and the type of directional range have being identified as intrinsic grasp parameters that can be easily determined since they depend only on the directions of the applied forces and are useful in the optimal grasp search. These parameters are used to identify and solve the different cases in the determination of the object-optimal grasp. The upper bound is also used as a bound in the search of the object-optimal grasp. One advantage of the proposed approach is that there are cases in the computation of an edge-optimal grasp where some contact points can be analytically determined; in addition, in the search of the object-optimal grasp it is not necessary to obtain the edge-optimal grasp for all the sets of edges.

This paper introduces a new concept: the directional-optimal grasp, defined considering virtual edges with infinity lengths. Although the directional-optimal grasp may actually be unreachable, it is useful to determine the feasible cases of optimal grasp and which optimal contact points lie on extremes of the

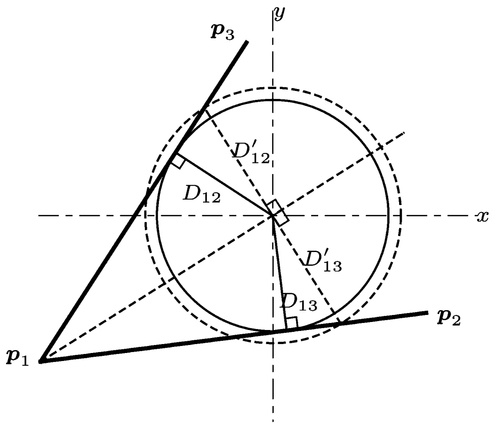


Fig. 8. Two-dimensional qualitative example showing that $D'_{12} = D'_{13}$ when $D_{12} = D_{13}$.

edges. With this information, an initial solution is obtained in a fully analytical way, that is either the edge-optimal grasp or very close to it; in this second case an iterative numerical procedure, function of only one unknown, is used to obtain the edge-optimal grasp.

The main concepts used in this approach depend only on the directions of the applied forces and can be easily determined. This encourages us to extend this work considering frictional contacts, nonpolygonal objects, and 3-D objects in future work. As a step in this direction, a new necessary and sufficient condition for the existence of form-closure grasps has been defined both for frictionless and frictional contacts [36] and extended to nonpolygonal objects [37]. The extension of the methodology presented here to the determination of the optimal grasp in these cases is still a problem under development.

APPENDIX
COMPUTATIONAL ASPECTS

The constraints included in (27) are four-order equations when the distances $D_{ijk}, i, j, k \in \{1, \ldots, 4\}$ are considered, but the order of these constraints can be reduced using the following geometrical substitution.

Consider the constraint $D_{hij} = D_{hik}$ (the same reasoning can be applied to the other constraints) and let Π_{hij}, Π_{hik}, and Π_{hi0} be the planes defined in the wrench space by $\{\boldsymbol{\omega}_h, \boldsymbol{\omega}_i, \boldsymbol{\omega}_j\}, \{\boldsymbol{\omega}_h, \boldsymbol{\omega}_i, \boldsymbol{\omega}_k\}$ and $\{\boldsymbol{\omega}_h, \boldsymbol{\omega}_i, \mathbf{0}\}$, expressed as

$$\Pi_{hij}: \quad \boldsymbol{n}_{hij}\boldsymbol{\omega} + d_{hij} = 0 \tag{29}$$

$$\Pi_{hik}: \quad \boldsymbol{n}_{hik}\boldsymbol{\omega} + d_{hik} = 0 \tag{30}$$

$$\Pi_{hi0}: \quad \boldsymbol{n}_{hi0}\boldsymbol{\omega} = 0 \tag{31}$$

where $\boldsymbol{n}_{hij}, \boldsymbol{n}_{hik}$ and $\boldsymbol{n}_{hi0}$ are the vectors normal to the planes and d_{hij} and d_{hik} are the independent terms, all of them being linear functions of τ_h, τ_i, τ_j and τ_k.

The constraint $D_{hij} = D_{hik}$ implies that Π_{hi0} is a bisector plane of Π_{hij} and Π_{hik}, and any vector normal to Π_{hi0} intersects Π_{hij} and Π_{hik} at points located at the same distance from Π_{hi0}. Then, selecting the normal vector that passes through the origin, the distances D'_{hij} and D'_{hik} from the origin to Π_{hij} and Π_{hik}, respectively, satisfy $D'_{hij} = -D'_{hik}$ and can be used instead of $D_{hij} = D_{hik}$ (see Fig. 8). Using (29)–(31), $D'_{hki} = D'_{hkj}$ can be expressed as

$$d_{hik}(\boldsymbol{n}_{hij} \cdot \boldsymbol{n}_{hi0}) = -d_{hij}(\boldsymbol{n}_{hik} \cdot \boldsymbol{n}_{hi0}). \tag{32}$$

Since $\boldsymbol{n}_{hij}, d_{hij}, \boldsymbol{n}_{hik}, d_{hik}$, and $\boldsymbol{n}_{hi0}$ are linear functions of τ_h, τ_i, τ_j, and τ_k, (32) expressed as a function of τ_h or τ_i is a three-order equation, while the same equation expressed as a function of τ_j or τ_k is a linear equation. Using (32) to represent the constraints, a system of two constraints with two unknowns torques can be solved in a fully analytical way.

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