Consecutive expansions of *k*-out-of-*n* systems

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Abstract

We introduce consecutive expansions of *k*–out–of–*n* systems, which have the property that components are totally ordered by the node criticality relation and with respect to well-known structural importance measures. We propose some formulae to easily compute these measures and study the hierarchies induced for them for large systems.

Keywords: Semicoherent structures; *k*-out-of-*n* systems; Structural importance measures

1. Introduction

An important kind of semicoherent structures is made by *k-out-of-n systems*, introduced in [1], that consist of *n* components of the same class. The entire system functions if at least *k* of its *n* components are operating. They are a very popular type of redundancy in *fault-tolerant systems*. The reliability of these systems is studied in [2] and different variations of them have been studied (see e.g. [3], [4] and [5]).

Complete or linear semicoherent structures can be described as those for which it is possible to completely arrange the components according to the *node criticality* [6] or to the *desirability relation* previously introduced in [7] in the game theory field. In this paper we focus on one of the simpler cases of complete structures, those which are *consecutive expansions of determined k-out-of-n systems*, i.e. systems which are the intersection of several*k*-out-of-*n* systems. These systems are also incomplete *k*-out-of-*n* systems [3].

The influence of the node criticality relation on the determination of component importance has been studied for different measures (see e.g. [6], [8], [9] and [10]). In this paper we study in depth the behavior of some basic structural importance measures for consecutive expansions of determined *k*-out-of-*n* systems. Although we will see that the node criticality relation gives a certain hierarchy among components to each of these systems, we will see that for some systems some non-equivalent components are almost undistinguishable.

The paper is organized as follows: In Section 2 a minimum of preliminaries is provided. In Section 3 the consecutive expansions of *k*-out-of-*n* systems are introduced and a test is provided to find whether two of these systems are equivalent is given. Moreover, it is shown that a canonical representative exists for each of these systems. Section 4 is mainly devoted to giving formulae to calculate some structural importance measures for these systems. The main advantage of these formulae is that they are useful for large systems. As an application of the results in Sections Sections 3 and 4, in Section 5 we study the behavior, for large systems, of the Barlow and Proshan and Birnbaum (normalized) structural importance measures.

2. Preliminaries

A *semicoherent structure* is a pair (N,π), where $N=\{1,2,\ldots,n\}$ is the set of components and π is an arbitrary collection of subsets of N called *path sets* such that: (i) $\emptyset \notin \pi$, (ii) $N \in \pi$ and (iii) if $S \subset T \subseteq N$ and $S \in \pi$ then $T \in \pi$. A path set is *minimal* if each proper subset of it is not a path set. The set of minimal path sets is denoted by π^m. The cardinal of a given set A is denoted hereafter by $|A|$. A component $i \in N$ is *irrelevant* in (N,π) if $i \notin S$ for every $S \in \pi^m$. A component $i \in N$ is said to be *critical* for a set $S \subseteq N \setminus \{i\}$ if $S \notin \pi$ but $S \cup \{i\} \in \pi$; we then write $S \in C(i,\pi)$. A system that is functioning if and only if at least k out of n components are functioning ($1 \le k \le n$) is called a *k-out-of-n system*.

Let p_i be the (independent) probability that a component i functions and $1-p_i$ be the probability of component i does not function. Then the probability of the semicoherent structure (N,π) to function is given by the *reliability function*

$$h(\mathbf{p}) = \sum_{S \in \pi} \prod_{i \in S} p_i \prod_{j \in N \setminus S} (1-p_j)$$

where $\mathbf{p} = (p_1, p_2, \ldots, p_n)$.

Two semicoherent structures (N,π) and (N',π') are said to be *isomorphic* if there exists a bijective map $f{:}N \to N'$ such that $S \in \pi$ if and only if $f(S) \in \pi'$.

Let (N,π) be a semicoherent structure. Set $\gamma_i=\{S \in \pi : i \in S\}$ and let $\tau_{ij}{:}N \to N$ denotes the *transposition* of components $i,j \in N$, that is, $\tau_{ij}(i)=j$, $\tau_{ij}(j)=i$ and $\tau_{ij}(k)=k$ for all $k \neq i,j$. The individual *criticality relation* between two components i and j, introduced in [6], is the binary relation $\succsim$ on N:

$$i \succsim j \quad \text{if and only if} \quad \tau_{ij}(\gamma_j) \subseteq \gamma_i ,$$

and we say that i is *at least as critical as* j. From now on we will say that $i \succsim j$ if component i has at least the same influence as component j in the structure.

It is easy to see that $\succsim$ is a preorder, with associated equivalence relation $\approx$, given by

$$i \approx j \quad \text{if and only if} \quad i \succsim j \quad \text{and} \quad j \succsim i;$$

hence $i \approx j$ means that i and j are equi-desirable or have the same influence in the structure. The relation $\succsim$ induces an ordering $\geq$ in $\{N_1,\ldots,N_l\}$, the set of equivalence classes by relation $\approx$. Thus, $N_p \geq N_q$ if and only if $i \succsim j$ for any $i \in N_p$ and any $j \in N_q$. We will use the strict notation $i \succ j$ (if and only if $i \succsim j$ but $j \not\succsim i$) for the strict criticality relation.

The criticality relation is not always complete (total). Then, if any two components are comparable by $\succsim$, (N,π) is said to be a *complete or linear semicoherent structure;* in this case, the equivalence classes are totally ordered.

3. Consecutive expansions of *k*-out-of-*n* systems

We start by introducing these structures.

Definition 3.1

Let (N,π) be a system that admits a partition $\{N_1,N_2,\ldots,N_l\}$ of N and non-negative integer numbers $m_1,m_2,\ldots,m_l$ not all of them null, such that $m_j \leq n_j$ for each $j=1,2,\ldots,l$, where $n_j=|N_j|$, in such a way that the collection of path sets of (N,π) is:

$$\{S \subseteq N : |S \cap (N_1 \cup N_2 \cup \ldots \cup N_i)| \geq m_1 + m_2 + \cdots + m_i, \quad \text{for all } i = 1,\ldots,l\}.$$

We will say that (N,π) is a *consecutive expansion of the* $(m_1+\ldots+m_i)$*-out-of-*$(n_1+\ldots+n_i)$ subsystems defined in each $N_1 \cup \ldots \cup N_i$, $i=1,\ldots,l$.
Notice that it is equivalent to say that

$$S \in \pi \Leftrightarrow S \cap (N_1 \cup N_2 \cup \ldots \cup N_i) \in \pi_i, \quad \text{for all } i = 1,\ldots,l,$$

where $(N_1 \cup \ldots \cup N_i, \pi_i)$ is a $(m_1+\ldots+m_i)$-out-of-$(n_1+\ldots+n_i)$ system, $i=1,\ldots,l$. Then, for all $S \subseteq N$ we can write

$$S \in \pi \Leftrightarrow s_1 + \cdots + s_i \geq m_1 + \cdots + m_i, \quad \text{for all } i = 1,\ldots,l,$$

where $s_i=|S \cap N_i|$.

From now on, if $S \subseteq N$ we can consider the vector of indices or *model* $\bar{s} = (s_1,\ldots,s_l)$ and we shall write $\Sigma_k(\bar{s}) = s_1 + \cdots + s_k$ for $k=1,\ldots,l$ and then, if $\bar{m} = (m_1,\ldots,m_l)$,

$$S \in \pi \Leftrightarrow \Sigma_k(\bar{s}) \geq \Sigma_k(\bar{m}) \quad \text{for all } k = 1,\ldots,l.$$

Proposition 3.2

For a consecutive expansion of k-out-of-n systems it holds:

(i) $i \approx j$ *if* $i,j \in N_p$.

(ii) $i \succsim j$ *if* $i \in N_p, j \in N_q$ *, and* $p<q$.

Proof

(i) Let $i,j \in N_p$. For any $S \subseteq N$, S and $\tau_{ij}(S)$ have the same model, so that $S \in \pi$ implies $\tau_{ij}(S) \in \pi$ and hence, $i \succsim j$. By symmetry of the argument, we conclude that $i \approx j$.

(ii) Let $j \in S \in \pi$ and $i \notin S$. We consider $\bar{s} = (s_1,\ldots,s_p,\ldots,s_q,\ldots,s_t)$, where $s_i=|S \cap N_i|$, $i=1,2,\ldots,t$. This vector satisfies $s_q \geq 1$ and $\Sigma_l(\bar{s}) \geq \Sigma_l(\bar{m})$, for $l=1,\ldots,t$.

Let $R=\tau_{ij}(S)$. Then $\bar{r} = (s_1,\ldots,s_p+1,\ldots,s_q-1,\ldots,s_t)$ and $\Sigma_k(\bar{r}) \geq \Sigma_k(\bar{s}) \geq \Sigma_k(\bar{m})$ for $k=1,\ldots,l$ and, hence $R \in \pi$. This proves $\tau_{ij}(\gamma_j) \subseteq \gamma_i$. □

Taking into account that $S \in \pi \Leftrightarrow \Sigma_k(\bar{s}) \geq \Sigma_k(\bar{m})$ for $k=1,\ldots,l$, where $\bar{s} = (s_1,\ldots,s_l)$ and $s_i=|S \cap N_i|$, for $i=1,\ldots,l$, we can check that these structures can be easily defined by using the vectors $\bar{n} = (n_1,\ldots,n_l)$ and $\bar{m} = (m_1,\ldots,m_l)$. However, for a fixed N different vectors $\bar{n} = (n_1,\ldots,n_l)$ and $\bar{m} = (m_1,\ldots,m_l)$ can define the same structure.

As a consequence of the former proposition it is clear that the expansions of k-out-of-n systems are *always* complete structures.

From now on we are interested in determining when two expansions of k-out-of-n systems, defined by a pair of integer vectors, are equivalent.

Definition 3.3

Fixed N the set of components, let $(\overline{n},\overline{m})$ and (n',m') be two consecutive expansions of k-out-of-n systems respectively representing π and π'. Then,

$(\overline{n},\overline{m}) \equiv (n',m') \Leftrightarrow \pi = \pi'$.
where $\overline{n}=(n_1,\ldots,n_l)$, $\overline{m}=(m_1,\ldots,m_l)$, $\overline{n'}=(n'_1,\ldots,n'_k)$, $\overline{m'}=(m'_1,\ldots,m'_k)$.

We denote $\mathcal{E}(\overline{n},\overline{m}) = \{(n',m') : (n',m') \equiv (\overline{n},\overline{m})\}$, the set of equivalent systems to $(\overline{n},\overline{m})$.

Proposition 3.4

For a fixed N , there is a unique representative for $\mathcal{E}(\overline{n},\overline{m})$ such that the number of components for $\overline{n}$ (and then for $\overline{m}$) is minimum. Let t be this number and $\overline{n}=(n_1,\ldots,n_t)$, $\overline{m}=(m_1,\ldots,m_t)$. Then,

$1 \le m_1 \le n_1,$
$1 \le m_k \le n_k - 1, \quad for\ 2 \le k \le t-1,\ and$
$0 \le m_t \le n_t - 1.$

Such a representation is referred hereafter as the canonical representative.

Proof

(i) (Existence) Let $(\overline{n'},\overline{m'}) \in \mathcal{E}(\overline{n},\overline{m})$, where $n'=(n'_1,\ldots,n'_l)$ and $m'=(m'_1,\ldots,m'_l)$.

If $m'_k=0$ for some $1\le k\le l-1$ we will then see that any $i \in N_k$ and any $j \in N_{k+1}$ satisfy $j \succsim i$ and, taking into account Proposition 3.2, $i\approx j$.

Let $i\in S \in\pi$ and $j\notin S$. We consider $\overline{s}=(s_1,\ldots,s_k,s_{k+1},\ldots,s_l)$, where $s_i=|S\cap N_i|$, $i=1,2,\ldots,l$. This vector satisfies $\Sigma_b(\overline{s}) \ge \Sigma_b(\overline{m'})$, for $b=1,\ldots,l$ and $s_k\ge1$.

Let $R=\tau_{ij}(S)$. Then $\overline{r}=(s_1,\ldots,s_k-1,s_{k+1}+1,\ldots,s_l)$ and it is easy to see that $\Sigma_b(\overline{r}) \ge \Sigma_b(\overline{m'})$, for all $b=1,\ldots,l$ and, hence, $R \in \pi$. This proves that $\tau_{ij}(\gamma_i) \sqsubseteq \gamma_j$.

If $m'_k = n'_k$, for some $2\le k\le l$, we will prove that $j \succsim i$, for any $j\in N_k$ and any $i\in N_{k-1}$ and, taking into account Proposition 3.2, $i\approx j$.

Let $i \in S \in \pi$ and $j\notin S$. We consider $\overline{s}=(s_1,\ldots,s_{k-1},s_k,\ldots,s_l)$, where $s_i=|S\cap N_i|$, $i=1,2,\ldots,l$. This vector satisfies $\Sigma_b(\overline{s}) \ge \Sigma_b(m')$, for $l=1,\ldots,l$; $s_k \le n'_k - 1$ and $s_{k-1}\ge1$.

Let $R=\tau_{ij}(S)$. Then $\overline{r}=(s_1,\ldots,s_{k-1}-1,s_k+1,\ldots,s_l)$ satisfies $\Sigma_k(\overline{r})=\Sigma_k(\overline{s})\ge\Sigma_k(\overline{m'})$, and, as $s_k\le n'_k-1$, it follows that $\Sigma_{k-1}(\overline{r})\ge\Sigma_k(m')$ and we conclude that $R\in\pi$. This proves that $\tau_{ij}(\gamma_i) \succsim \gamma_j$.

Let $G=\{1\le k\le l-1 : m'_k=0\}$, $H=\{2\le k\le l : m'_k=n'_k\}$ and $I=\{1\le k\le l-1 : m'_k = 0 \text{ and } m'_{k+1}=n'_{k+1}\}$. Then

$t=l-(|G|+|H|-|I|)$
and we define the minimum representative $(\overline{n},\overline{m})$ as follows:

(1) If there exists $1\le k\le l-1$ such that $m'_k=0$ then $n_k=n'_k+n'_{k+1}$ and $m_k=m'_k+m'_{k+1}$.

(2) If there exists $2 \leq k \leq l$ such that $m'_k = n'_k$ then $n_{k-1} = n'_{k-1} + n'_k$ and $m_{k-1} = m'_{k-1} + m'_k$.

(ii) (Uniqueness) Let $(\overline{n'}, \overline{m'})$ and $(\overline{n^*}, \overline{m^*})$ two minimal representatives of $\mathcal{E}(\overline{n}, \overline{m})$, where $n' = (n'_1, \ldots, n'_t)$, $m' = (m'_1, \ldots, m'_t)$, $n^* = (n^*_1, \ldots, n^*_t)$ and $m^* = (m^*_1, \ldots, m^*_t)$.

Let S a path set with model $\overline{m'}$, then $S \in \pi' = \pi^*$ and we deduce that $\Sigma_k(m') \geq \Sigma_k(\overline{m^*})$ for all $k=1,\ldots,t$. Let T a path set with model $\overline{m^*}$, then $T \in \pi^* = \pi'$ and we obtain that $\Sigma_k(\overline{m^*}) \geq \Sigma_k(m')$ for all $k=1,\ldots,t$. This proves that $m'_i = m^*_i$ for all $k=1,\ldots,t$.

If $n' \neq \overline{n^*}$, taking into account that $\Sigma_t(n'_i) = \Sigma_t(n^*_i)$ it suffices to consider that $n'_1 < n^*_1$ and check that $\pi' \neq \pi^*$.

If $n'_1 < n^*_1$, then there exists $i \in N^*_1$ such that $i \notin N'_1$ and $i \in N'_p$, for some $p>1$. Let $i \notin S$ with model $\overline{s} = (m_1 - 1, m_2, \ldots, m_t)$. It is easy to prove that $S \cup \{i\} \in \pi^*$ and $S \cup \{i\} \notin \pi'$. □

Note that if $m_t=0$, the system has n_t irrelevant components. From now on we shall write $(N, \pi) = (\overline{n}, \overline{m})$, where the pair $\overline{n} = (n_1, \ldots, n_t)$ and $\overline{m} = (m_1, \ldots, m_t)$ is the canonical representative of the system.

Proposition 3.5

Let $(\overline{n}, \overline{m})$ be the canonical representative.

Then $i \succ j$ for all $i \in N_p$, for all $j \in N_q$ and $p<q$.

Proof

From Proposition 3.2 we know that if $i \in N_p$, $j \in N_q$, $p<q$, then $i \succsim j$. Then it suffices to prove that $j \not\succsim i$. Let $i \in S \in \pi$, $j \notin S$, such that $\overline{s} = \overline{m}$. Let $R = \tau_{ij}(S)$. It is easy to prove that $R \notin \pi$ because $\overline{r} = (m_1, \ldots, m_p - 1, \ldots, m_q + 1, \ldots, m_t)$ satisfies $\Sigma_p(\overline{r}) < \Sigma_p(\overline{m})$. □

That is to say, within the set of equivalent expansions of k-out-of-n systems, the canonical representation is the *unique* which preserves the node criticality relation.

Remark 3.6

Given two vectors $\overline{n}$ and $\overline{m}$ satisfying the conditions in Proposition 3.4, there exists an expansion of tk-out-of-n systems (N,π) defined by $\overline{n}$ and $\overline{m}$, and reciprocally. We sketch here how to obtain the path sets from the two vectors $(\overline{n}, \overline{m})$ and reciprocally.

(i) Given $(\overline{n}, \overline{m})$, the structure (N,π) can be reconstructed, up to isomorphism, as follows. The cardinality of N is $n = \Sigma_t(\overline{n})$, the elements of N are denoted by $\{1,2,\ldots,n\}$. The equivalence classes of (N,π) are $N_1=\{1,\ldots,n_1\}$, $N_2=\{n_1+1,\ldots,n_1+n_2\}$, and so on.

Each $S \subseteq N$ with vector $\overline{s} = (|S \cap N_1|, |S \cap N_2|, \ldots, |S \cap N_t|)$ is a path set if $\Sigma_k(\overline{s}) \geq \Sigma_k(\overline{m})$, $k=1,\ldots,t$. Hence, the set of path sets is

$$\pi = \{S \subseteq N : \Sigma_k(s) \geq \Sigma_k(m),\ k = 1, \ldots, t\},$$

and (N,π) is a consecutive expansion of tk-out-of-n-systems. Notice that a model r is a path model if and only if each set representative R is a path set.

(ii) Conversely, if (N,π) is a semicoherent structure with equivalence classes $N_1 > N_2 > \ldots > N_t$, we will see how to obtain the vectors $(\overline{n}, \overline{m})$.

Let $\mathbb{N}$ be the set of natural numbers, if $\overline{n} \in \mathbb{N}^t$, we consider

$A(\overline{n}) = \{\overline{s} \in (\mathbb{N} \cup \{0\})^t : \overline{n} \geq \overline{s}\},$

where ≥ stands for the ordinary componentwise ordering, that is

$s \geq r \quad \text{if and only if} \quad s_k \geq r_k \quad \text{for every } k = 1, \ldots, t.$

We consider the weaker ordering δ given by comparison of partial sums, that is,

$s \, \delta \, r \quad \text{if and only if} \quad \Sigma_k(s) \geq \Sigma_k(r) \quad \text{for } k = 1, \ldots, t.$

Then, for each set S we consider the vector of *indices* or *model*

$\overline{s} = (|S \cap N_1|, |S \cap N_2|, \ldots, |S \cap N_t|),$

in $A(\overline{n})$. The vector $\overline{n}$ is $(|N_1|,|N_2|,\ldots,|N_t|)$. In particular, there are $C_{\overline{n},\overline{s}} = \binom{n_1}{s_1} \cdot \ldots \cdot \binom{n_t}{s_t}$ sets with vector $\overline{s}$. Then $\overline{m}$ is $\overline{s}$ whenever S is a path set which is δ-minimal in the lattice $(A(\overline{n}), \delta)$.

From this result we can deduce that every semicoherent structure with linearly ordered equivalence classes, $N_1 > \ldots > N_t$, that admits a minimum lattice representative can be interpreted as a linear expansion of tk-out-of-n subsystems.

Remark 3.7

These structures can be obtained from a $(m_1+\ldots+m_t)$-out-of-$(n_1+\ldots+n_t)$ system where some minimal path sets have been removed and then, they can be interpreted as incomplete $(m_1+\ldots+m_t)$-out-of-$(n_1+\ldots+n_t)$ systems (see [3] for more information about incomplete k-out-of-n systems). Taking into account this fact, their reliability function can be expressed in terms of the reliability of the $(m_1+\ldots+m_t)$-out-of-$(n_1+\ldots+n_t)$ system removing all the sets of cardinality greater than $m_1+\ldots+m_t$ which are not path sets in the structure $(\overline{n}, \overline{m})$. That is,

$$h_{\overline{n},\overline{m}}(\mathbf{p}) = h_{(m_1+\cdots+m_t):(n_1+\cdots+n_t)}(\mathbf{p}) - \sum_{\overline{s} \in \mathcal{J}} \prod_{i \in S} p_i \prod_{j \notin S} (1 - p_j),$$

where $\mathcal{J} = \{\overline{s} \in A(\overline{n}) : \overline{s} \not\delta \, \overline{m}, \sum_t(\overline{s}) \geq \sum_t(\overline{m})\}$.

Particularly, if $p_i = p$ for all $i=1,\ldots,n$ then,

$$h_{\overline{n},\overline{m}}(p) = h_{(m_1+\cdots+m_t):(n_1+\cdots+n_t)}(p) - \sum_{\overline{s} \in \mathcal{J}} C_{\overline{n},\overline{s}} \cdot p^{\sum_t(\overline{s})} \cdot (1-p)^{\sum_t(\overline{n}) - \sum_t(\overline{s})},$$

where $\sum_t(\overline{s}) = s$ and $\sum_t(\overline{n}) - \sum_t(\overline{s}) = n - s$.

A combinatorial argument counts the number of these structures, up to isomorphism.

Proposition 3.8

The number of consecutive expansions of k-out-of-n systems with n components is $\rho(n) = 2^n - 1$.

Proof

We proceed by induction on n. If $n=1$ the only semicoherent structure with minimum is given by the characteristic invariants $(\overline{n}, \overline{m}) = (1, 1)$ and hence, $\rho(1) = 2 - 1 = 1$. If $n > 1$, we first prove that $\rho(n+1) = 1 + 2\rho(n)$.

Let $\mathcal{P}_n$ be the set of descompositions of number n as a sum of natural numbers. For each $(n_1, n_2, \ldots, n_t) \in \mathcal{P}_n$ we can obtain $n_1(n_2-1)\cdot\ldots\cdot(n_{t-1}-1)n_t$ semicoherent structures with minimum information. Then, $\rho(n) = \sum_{(n_1,n_2,\ldots,n_t)\in\mathcal{P}_n} n_1(n_2-1)\cdot\ldots\cdot(n_{t-1}-1)n_t$. If the structure has n+1 components we analogously define $\mathcal{P}_{n+1}$.

Let $\mathcal{Q}$ be the subset of $\mathcal{P}_{n+1}$ formed by descompositions with first component equals to 1. Then, $|\mathcal{Q}| = |\mathcal{P}_{n+1} \setminus \mathcal{Q}| = |\mathcal{P}_n|$ and from the bijective maps $g_1 : \mathcal{P}_n \to \mathcal{Q}$ and $g_2 : \mathcal{P}_n \to \mathcal{P}_{n+1} - \mathcal{Q}$, defined by

$$g_1(n_1, n_2, \ldots, n_t) = (1, n_1, n_2, \ldots, n_t) \quad \text{and}$$
$$g_2(n_1, n_2, \ldots, n_t) = (n_1 + 1, n_2, \ldots, n_t),$$

it follows that $\rho(n+1)=1+2\rho(n)$.

Finally, by induction hypothesis $\rho(n)=2^n-1$ and, using the last formula, we can write $\rho(n+1)=1+2(2^n-1)=2^{n+1}-1$. □

4. Structural importance measures

Reliability importance was defined in [11] as the partial derivative of system reliability with respect to component reliability. Birnbaum [11] and Barlow and Proschan [12] also considered the problem of a priori quantification of relative importance of components. They referred to this as the problem of the measurement of structural importance of components which indicates the effect of a particular component's position in the system. When all components are i.i.d. with reliability 1/2 the reliability Birnbaum importance measure is called the *Birnbaum structural importance measure*. In [13], from a probabilistic point of view, different ways are introduced to evaluate importance measures for components in a given reliability system. In this case, the relative importance of a component is taken as the probability that the functioning or failure of a component makes a difference in the functioning or failure of the system. The main tools used are the *p-binomial structural importance measures*. Parameter p, whose values are between 0 and 1, defines a member of the family, and it is interpreted as the reliability of components, when all of them are i.i.d. These kinds of solutions can be computed from the reliability function of the structure as a generalization of the classical Birnbaum structural importance measure.

In this section we focus on the Birnbaum, the Barlow–Proschan and the p-binomial structural importance measures for expansion of k-out-of-n systems, providing formulae to compute them as a function of their canonical representatives.

Definition 4.1

Let (N,π) be a semicoherent structure, $i \in N$,

(a) The Birnbaum structural importance measure of component i in (N,π) is given by $IB_i(N,\pi) = \frac{|C(i,\pi)|}{2^{n-1}}$.

(b) The Barlow–Proschan structural importance measure of component i in (N,π) is given by $IBP_i(N,\pi) = \sum_{S \in C(i,\pi)} \frac{s!(n-s-1)!}{n!}$.

(c) The p-binomial structural importance measure of component i in (N,π) is given by $\psi_i^p(N,\pi) = \sum_{S \in C(i,\pi)} p^s (1-p)^{n-s-1}$, where $0 \le p \le 1$ is the (known) probability of functioning of components.

Notice that if p=1/2 we obtain the Birnbaum structural importance measure, i.e. $\psi^{1/2}=IB$.**Remark 4.2**

As it is well-known, the ranking for the node criticality relation induces the same ranking for the structural importance measures considered in Definition 4.1 (e.g. see [6], [8] and [10]). That is, $i \approx j \Rightarrow IB_i(N,\pi)=IB_j(N,\pi), IBP_i(N,\pi)=IBP_j(N,\pi)$, and $\psi_i^p(N,\pi)=\psi_j^p(N,\pi)$; and $i \succ j \Rightarrow IB_i(N,\pi)>IB_j(N,\pi), IBP_i(N,\pi)>IBP_j(N,\pi)$, and $\psi_i^p(N,\pi)>\psi_j^p(N,\pi)$. This proves that two equivalent components have the same structural importance measure and gives a complete ordering of the components with respect to their structural importance measure. In particular, for consecutive expansion of k-out-of-n systems, $(\bar{n},\bar{m})$, with classes $N_1 > \ldots > N_t$, we obtain that $IB_i > IB_j$, $IBP_i > IBP_j$ and $\psi_i^p > \psi_j^p$, for all $i \in N_a$ and $j \in N_b$, $a<b$; and $IB_i=IB_j$, $IBP_i=IBP_j$ and $\psi_i^p=\psi_j^p$ if $i,j \in N_a$.

We provide formulae to compute some structural importance measures as a function of the canonical representatives. Rather than considering the structural importance measures of components, we consider the structural importance measures of classes of equivalent components, understanding that the importance measure of a certain class is divided equally among its components.

Theorem 4.3

Let $(\bar{n},\bar{m})$ be a canonical expansion of tk-out-of-n systems. Then,

(a) $IB_{N_i}=\sum_{\bar{s}\in\Lambda_i}\frac{1}{2^{\sum_j(\bar{n})-1}}\cdot C_{\bar{n},\bar{s}}\cdot(n_i-s_i)$, $1 \leq i \leq t$;

(b) $IBP_{N_i}=\sum_{\bar{s}\in\Lambda_i}\frac{(\sum_j(\bar{s}))!\,(\sum_j(\bar{n})-\sum_j(\bar{s})-1)!}{n!}\cdot C_{\bar{n},\bar{s}}\cdot(n_i-s_i)$, $1 \leq i \leq t$;

(c) $\psi_{N_i}^p=\sum_{\bar{s}\in\Lambda_i}p^{\sum_j(\bar{s})}\cdot(1-p)^{\sum_j(\bar{n})-\sum_j(\bar{s})-1}\cdot C_{\bar{n},\bar{s}}\cdot(n_i-s_i)$, $1 \leq i \leq t$;

where $\Lambda_i=\{\bar{s}\in\Lambda(\bar{n}):\bar{s}\ngeq\bar{m},\bar{n}\geq\bar{s}+\bar{e}_i\geq\bar{m}\}$ *and* $\{\bar{e}_i\}_{i=1}^t$ *is the canonical basis of* $\mathbb{R}^t$, *i.e.,* $e_{ij}=1$ *(or 0) if* $i=j$ *(or* $i\neq j$ *).*

Proof

(a) For these kinds of structures we note that a component $j \in N_i$, $j \notin S$, satisfies S belongs to $C(j,\pi)$ when the model $\bar{s}$ of S belongs to Λ_i. For each of these sets there are $n_i - s_i$ components in $N_i \cap (N \setminus S)$ and for each model in Λ_i there are $C_{\bar{n},\bar{s}}$ sets with vector of indices $\bar{s}$. Finally, we obtain (a) adding $C_{\bar{n},\bar{s}}$ for all elements of Λ_i (notice that $\sum_j(\bar{n})=n$).

(b) and (c) are analogous to (a), taking into account that $\sum_j(\bar{s})$ is the cardinality of every path set with model $\bar{s}$. □

Example

Consider the structure defined by $\bar{n}=(20,20,20)$ and $\bar{m}=(10,15,5)$.

We can calculate the structural importance measures using the vectors $\bar{n}$ and $\bar{m}$ and it is enough to enter 6 numbers in a computer in order to get: IB_i=0.0459 for $i \in N_1$, IB_i=0.0442 for $i \in N_2$ and IB_i=0.0003 for $i \in N_3$; and IBP_i=0.025757 for $i \in N_1$, IBP_i=0.024193 for $i \in N_2$ and IBP_i=0.000050 for $i \in N_3$.

As it is well-known, both the Birnbaum and Barlow–Proschan structural measures of any semicoherent structure can be easily obtained from their reliability functions. Indeed, $IBP(N,\pi)$ is obtained by integrating the partial derivatives of the reliability function along the main diagonal $p_1=p_2=\ldots=p_n$ of the hypercube $[0,1]^N$, while $IB(N,\pi)$ can be calculated by the partial derivatives of that reliability function evaluated at point (1/2,1/2,…,1/2). This latter procedure extends well to any p-binomial structural importance measure (see [13]) by evaluating the derivatives at point $(p,p,\ldots,p)$. Next, we propose a new method to calculate these structural importance measures by means of an *auxiliary* function.

Theorem 4.4

Let $(\overline{n},\overline{m})$ *be a canonical consecutive expansion of tk-out-of-n systems. Then,*

(a) $IB_{m_i} = \frac{\partial \overline{h}}{\partial \overline{p}_i}(\overbrace{1/2,\ldots,1/2}^{t})$, $1\leq i\leq t$;

(b) $IBP_{m_i} = \int_0^1 \frac{\partial \overline{h}}{\partial \overline{p}_i}(\overbrace{p,\ldots,p}^{t})\,dp$, $1\leq i\leq t$;

(c) $\psi^p_{m_i} = \frac{\partial \overline{h}}{\partial \overline{p}_i}(\overbrace{p,\ldots,p}^{t})$, $1\leq i\leq t$;

where, $\overline{h}(\overbrace{\overline{p}_1,\ldots,\overline{p}_t}^{t}) = \sum_{\overline{s}\in\Gamma} C_{\overline{n},\overline{s}} \prod_{i=1}^{t} \overline{p}_i^{s_i}(1-\overline{p}_i)^{n_i-s_i}$, $\Gamma = \{\overline{s}\in A(\overline{n}) : \overline{s} \trianglerighteq \overline{m}\}$ *(the set of models associated to the path sets) and* $0 \leq \overline{p}_i \leq 1$ *is a value assigned to every component of* N_i , $i=1,\ldots,t$.

We omit the proof, which can be easily obtained using the fact that if $S \in \pi$ is a path set then $\overline{s}\in\Gamma$ and, for each model $\overline{s}$, there are $C_{\overline{n},\overline{s}}$ sets with vector of indices $\overline{s}$.

5. Components asymptotically equivalent

As we have seen in Section 4 the node criticality relation induces a hierarchy on any canonical consecutive expansion of k-out-of-n systems so that two arbitrary components in the same equivalence class are equally important, whereas components that belong to a certain class on the sequence $N_1,N_2,\ldots,N_t$ are more important than those that belong to a class on the right. This tells us that the structural importance measures that respect the node criticality relation, as those studied in Section 4, rank the components in exactly t different values. Let i be an arbitrary component in N_i, then any of these measures define a bijective map $\psi : \{1,\ldots,\overline{n}\} \to [0,1]^t$ which is a decreasing function, $\psi_{\overline{i}} > \psi_{\overline{j}}$ whenever $i < j$. If we normalize all these measures to satisfy $\sum_{j=1}^{t} n_j \psi_{\overline{j}} = 1$ and assume t>2, a consequence of the monotonicity of ψ can be easily derived to get upper and lower bounds:

$$\frac{1}{n} < \psi_{\overline{1}} < \frac{1}{n_1}, \quad 0 < \psi_{\overline{i}} < \frac{1}{(n_1+\cdots+n_i)} \quad \text{if } 1 < i < t$$
$$\text{and } 0 \leq \psi_{\overline{t}} < \frac{1}{n}.$$

When we consider large systems, an interesting problem is determining when these bounds are nearly attained and when two consecutive equivalence classes almost collapse, i.e. when a normalized structural importance measure ψ evaluated on components of two consecutive classes *almost* coincide.

Let all n_i be large enough numbers in the canonical system defined by vectors: $\overline{n} = (n_1, n_2, \ldots, n_t)$ and $\overline{m} = (m_1, m_2, \ldots, m_t)$. If

$$\frac{m_i}{n_i} \approx 0 \quad \text{and} \quad \frac{m_{i+1}}{n_{i+1}} \approx 1,$$

then $\psi_{\overline{i}} \approx \psi_{\overline{i+1}}$, so that asymptotically the two types of components are equally important for the system. The asymptotic results we provide in this section for large expansions of *k*-out-of-*n* systems of up to 300 components are obtained from the formulae in Section 4 to compute Birnbuaum's, Barlow and Proschan's structural importance measures. It should be noted that if we have as few as 20 components in each equivalence class and the stated condition arises, then we do not obtain any significant difference for $\psi_{\overline{i}}$ and $\psi_{\overline{i+1}}$ working with a precision of 10^{-10}; this is also true for the subsequent described cases. However, this reasoning can be extended. If

$$\frac{m_i}{n_i} \approx 0, \qquad \frac{m_{i+1}}{n_{i+1}} \approx 1 \quad \text{and} \quad \frac{m_{i+1} + m_{i+2}}{n_{i+1} + n_{i+2}} \approx 1,$$

then $\psi_{\overline{i}} \approx \psi_{\overline{i+1}} \approx \psi_{\overline{i+2}}$. Thus, components in the three consecutive equivalence classes are almost equally important. Of course, we can lead our approach to the full extent.

In order to obtain expansions of *k*-out-of-*n* systems with all components having *almost* the same importance it is sufficient to consider the condition:

$$\frac{m_1}{n_1} \approx 0, \qquad \frac{m_2}{n_2} \approx 1, \qquad \frac{m_2 + m_3}{n_2 + n_3} \approx 1, \ldots, \frac{m_2 + \cdots + m_t}{n_2 + \cdots + n_t} \approx 1.$$

Then $\psi_{\overline{i}} \approx \frac{1}{n}$ for all $i \in N$, giving examples of expansions of *k*-out-of-*n* systems which are very close to *k*-out-of-*n* systems. The most extreme case arises when the coherent system is: $\overline{n} = (n_1, n_2, \ldots, n_t)$ and $\overline{m} = (1, n_2 - 1, \ldots, n_t - 1)$ with all n_i large enough. This example illustrates that: $\psi_{\overline{1}}$ is close to its lower bound, and tends to it as long as the n_i numbers are increased, and $\psi_{\overline{t}}$ is close to its upper bound and tends to it as long as the n_i numbers are increased.

It is now simple to reproduce similar results for the other bounds. For instance, the system defined by: $\overline{n} = (n_1, n_2, \ldots, n_t)$ and $\overline{m} = (n_1, 1, \ldots, 1, 0)$ with all n_i large enough, is close to the upper bound, $1/n_1$, for $\psi_{\overline{1}}$ and to the lower bound, 0, for the rest of the components. Or, $\overline{n} = (n_1, n_2, \ldots, n_t)$ and $\overline{m} = (1, n_2 - 1, 1, \ldots, 1, 0)$ with all n_i large enough, almost tends to reach the upper bound, $1/(n_1+n_2)$, for $\psi_{\overline{2}}$; the lower bound, 0, for components $\overline{i}$ with $\overline{i} > 2$ and $1/(n_1+n_2)$ for $\psi_{\overline{1}}$.

Example

Let us consider the three following structures:

(1) $\overline{n} = (30, 30, 30, 30)$ and $\overline{m} = (1, 29, 29, 29)$. We obtain that $IB_{\overline{i}} = IBP_{\overline{i}} = 0.008\overline{3} \approx \frac{1}{120}$, for all $i \in N$.

(2) $\overline{n} = (30, 30, 30, 30)$ and $\overline{m} = (30, 1, 1, 0)$. We obtain that $IB_{\overline{1}} = IBP_{\overline{1}} = 0.0\overline{3} \approx \frac{1}{30}$ and $IB_{\overline{i}} = IBP_{\overline{i}} = 0$, for $i > 1$.

(3) $\overline{n} = (30, 30, 30, 30)$ and $\overline{m} = (1, 29, 1, 0)$. We obtain that $IB_{\overline{i}} = IBP_{\overline{i}} = 0.01\overline{6} \approx \frac{1}{60}$, for $i = 1, 2$ and $IB_{\overline{j}} = IBP_{\overline{j}} = 0$, for $j > 2$.

In summary, consecutive expansion of *k*-out-of-*n* systems are complete structures because the node criticality relation is a complete preordering for them. Any such structures admit a canonical representative with a minimum number *t* of types of distinguished. This canonical representative is easily identified because it fulfills certain algebraic conditions. Any structural measure preserves the node criticality relation on canonical representatives, so that it induces a strict hierarchy on the *t* representatives of each equivalence class. However, we have only done computations for the Birnbaum's, Barlow and Proshan's structural measures, but the results given in this section can be extended to other measures based on critical path sets. Hence, there are exactly *t* distinguished levels of importance. However, when the number of components of a certain type is high in percentage, they become more important, and conversely, if the number of components of a certain type is small in percentage then they become less important (note that this effect does not hold for an isolated *k*-out-of-*n* system, i.e., *t*=1). These effects turn out to have an equalizing effect on components belonging to consecutive equivalence classes, making them almost equally important under certain asymptotic conditions. In other words, although it is expected for these systems to have a precise hierarchy of exactly *t* levels, some of these levels can collapse to produce components that are almost equally important. If we consider this analysis to the full extent, some consecutive expansions of *k*-out-of-*n* systems can be regarded as close approximations of traditional *k*-out-of-*n* systems.

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