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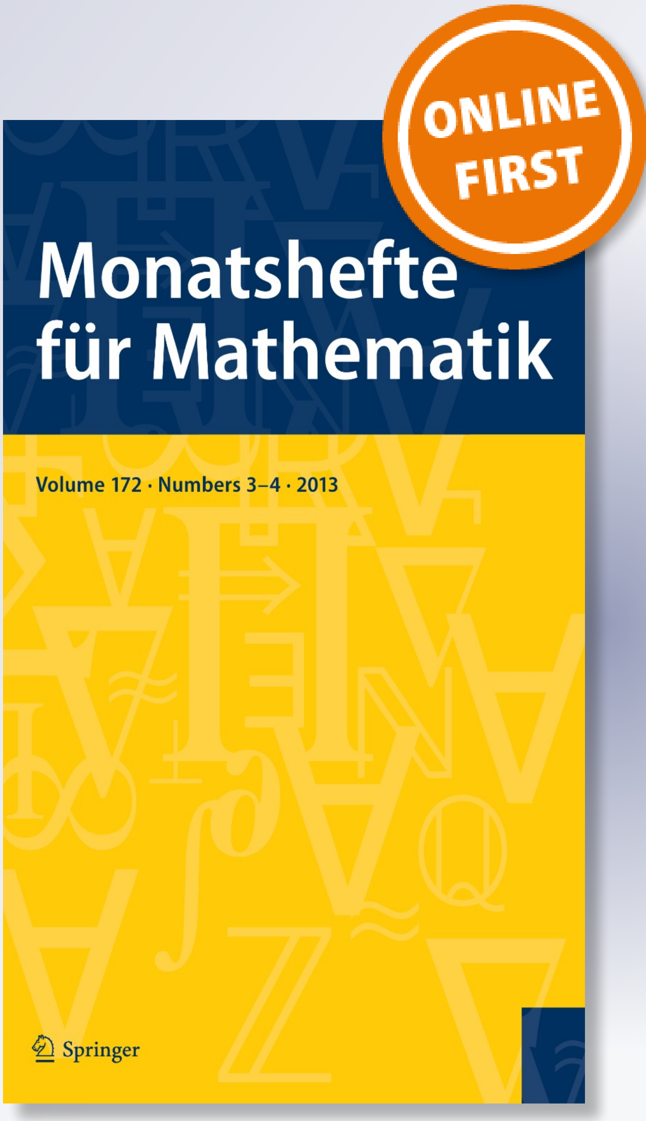

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Abstract In this paper, we present an easy way to construct partial Galois extensions; in particular, any direct sum of finitely many Galois extensions forms a partial Galois extension. The idea is inspired by the study of how Galois extensions are embedded in a partial Galois extension via minimal elements in an associated Boolean semigroup.

Keywords Partial action · Partial Galois extension · Galois extension · Boolean semigroup

Mathematics Subject Classification (2000) 13B05 · 16W22

1 Introduction

Partial actions of groups have been introduced in the theory of operator algebras as a powerful tool (see [12,13,16,19]). Partial actions on rings in a pure algebraic context were first studied in [10]. Briefly speaking, a partial action α of a group G on a unital algebra S is a collection of ideals S_g together with isomorphisms $\alpha_g : S_{g^{-1}} \to S_g$, $g \in G$ satisfying certain conditions. It was shown in [10] that a partial action on a unital algebra possesses an enveloping action if and only if every S_g is an algebra with identity.

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Further investigations of partial actions since [10] have been made in [1–3,5–7,11,14,15,18]. In particular, Dokuchaev et al. [11] introduced the notion of a partial Galois extension and generalized the results on Galois theory of commutative rings by Chase et al. [4] in the context of partial actions, assuming the existence of an enveloping action.

In this paper, we shall investigate the structure of a partial Galois extension possessing an enveloping action via minimal elements in the Boolean semigroup generated by identities of the unital ideals in the definition of a partial action. We shall show that a partial Galois extension decomposes as a direct sum of Galois extensions and possibly a partial Galois extension if there exist minimal elements invariant under the partial action. To interpret this result, we show that any direct sum of a finite number of Galois extensions is a partial Galois extension. This provides an easy way of constructing partial Galois extensions, much different from those shown in earlier papers. These results will be presented in Sect. 3. In the next section, we will recall the notions of a partial action and a partial Galois extension.

2 Preliminaries

We first recall the notion of a partial action of a group on a ring following [10]. A partial action α of a group G on a unital ring S is a collection

$$\alpha = (\{S_g\}_{g\in G}, \{\alpha_g\}_{g\in G}),$$

where for each $g \in G$, S_g is an ideal of S and $\alpha_g : S_{g^{-1}} \to S_g$ is an isomorphism of (non-necessarily unital) rings, satisfying the following conditions:

1. $S_1 = S$ and α_1 is the identity automorphism of S;
2. $\alpha_g(S_{g^{-1}} \cap S_h) = S_g \cap S_{gh}$ for $g, h \in G$;
3. $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$ for every $x \in S_{h^{-1}} \cap S_{(gh)^{-1}}$ and $g, h \in G$.

In particular, if $S_g = S$ for every $g \in G$, then α is a usual global action of G on S. By [10, Theorem 4.5], every S_g is unital if and only if α has an enveloping action, which means that there exist a ring S' and a global action of G on S' by automorphisms of S' such that S can be considered as an ideal of S' and the following conditions hold:

1. $S' = \sum_{g\in G} g(S)$;
2. $S_g = S \cap g(S)$ for every $g \in G$;
3. $\alpha_g = g|_{S_{g^{-1}}}$ for every $g \in G$.

Assume that every ideal S_g is unital generated by a central idempotent 1_g of S. Then in particular (see [11, p. 79]), for every $g \in G$ and $x \in S$,

$$1_g = 1_S g(1_S), \quad \alpha_g(x 1_{g^{-1}}) = g(x) 1_S$$

and hence for every $g, h \in G$,

$$\alpha_g(1_h 1_{g^{-1}}) = 1_g 1_{gh}. \tag{1}$$

Next following [11], the subring of invariants of S under α is defined as

$$S^\alpha = \{x \in S \mid \alpha_g(x1_{g^{-1}}) = x1_g, \quad \text{for all } g \in G\},$$

and S is called an α-partial Galois extension of S^α if there exist elements x_i, y_i of $S, i = 1, \ldots, n$ for some integer n, such that $\sum_{i=1}^n x_i \alpha_g(y_i 1_{g^{-1}}) = \delta_{1,g}$ for each $g \in G$. The finite set $\{x_i; y_i \in S \mid i = 1, \ldots n\}$ is called an α-partial Galois coordinate system for S.

3 The structure theorem

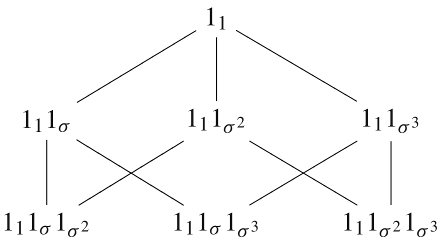
Let S be a ring with identity and α a partial action of a finite group G on S,

$$\alpha = (\{S_g\}_{g \in G}, \{\alpha_g\}_{g \in G}),$$

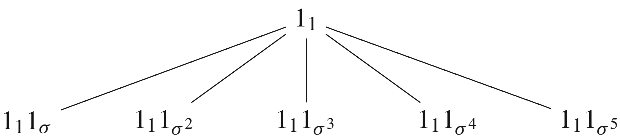
where each $S_g = S1_g$ is a unital ideal of S generated by a central idempotent 1_g. We write (S, α) for short. Let $I_G = \{1_g \mid g \in G\}$ and $(B(I_G), \cdot)$ the Boolean semigroup generated by I_G. There is a natural partial order on $B(I_G)$ and the induced tree is a rooted tree with root 1_1. For a minimal element e in $B(I_G)$, let $G(e)$ be the subset of G consisting of the elements $g \in G$ appearing in the expression of $e = \Pi_g 1_g$ with maximum number of elements g; in other words, $G(e) = \{g \in G \mid e1_g \neq 0\}$.

Example 1 For each partial action discussed in Examples 6.1 and 6.3 in [11] and Example 4.2 in [18], we sketch its associated tree.

1. The induced tree for (S, α) in [11, Example 6.1], where G is the cyclic group of order 4 generated by σ, is as follows.

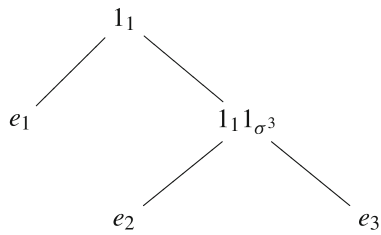


The minimal elements are $1_1 1_\sigma 1_{\sigma^2}$, $1_1 1_\sigma 1_{\sigma^3}$ and $1_1 1_{\sigma^2} 1_{\sigma^3}$.
2. The induced tree for (S, α) in [11, Example 6.3], where G is the cyclic group of order 6 generated by σ, is as follows.



For the minimal element $e = 1_1 1_{\sigma^3}$, $G(e) = \{1, \sigma^3\}$ is a subgroup of G.

3. The induced tree for (S, α) in [18, Example 4.2], where G is the cyclic group of order 6 generated by σ, is as follows.



where $e_1 = 1_1 1_\sigma = 1_1 1_{\sigma^4} = 1_1 1_\sigma 1_{\sigma^4}$, $e_2 = 1_1 1_{\sigma^2} = 1_1 1_{\sigma^2} 1_{\sigma^3}$ and $e_3 = 1_1 1_{\sigma^5} = 1_1 1_{\sigma^3} 1_{\sigma^5}$. Then $G(e_1) = \{1, \sigma, \sigma^4\}$, $G(e_2) = \{1, \sigma^2, \sigma^3\}$ and $G(e_3) = \{1, \sigma^3, \sigma^5\}$.

3.1 A one-to-one correspondence

As shown in Example 1, $G(e)$ might not be a subgroup of G. In general, suppose H is a subset of G containing 1. Let $e_H = \Pi_{g \in H} 1_g$ (e_H could be zero), $I_H = \{1_g \mid g \in H\}$ and $(B(I_H), \cdot)$ the Boolean semigroup generated by I_H. We next give some properties when H is a subgroup of G.

Lemma 2 *Suppose that H is a subgroup of G. Then for each $g \in H$, we have*

(i) $\alpha_g(e_H 1_{g^{-1}}) = e_H 1_g = e_H$.
(ii) $\alpha_g(B(I_H) 1_{g^{-1}}) \subset B(I_H)$.

Proof Let $g \in H$. Then $gh \in H$ for each $h \in H$. Thus $\alpha_g(1_h 1_{g^{-1}}) = 1_g 1_{gh}$ by Eq. (1) in Sect. 2 is in $B(I_H)$ and so (ii) follows easily. Also, $\alpha_g(e_H 1_{g^{-1}}) = \Pi_{h \in H} \alpha_g(1_h 1_{g^{-1}}) = \Pi_{h \in H} 1_g 1_{gh} = 1_g e_H = e_H$; that is, (i) holds. □

Lemma 3 *Let H be a subgroup of G. Then α induces a global action of H on Se_H. Furthermore, if S is an α-partial Galois extension of S^α and $e_H \neq 0$, then Se_H is a Galois extension of $(Se_H)^H$ with Galois group H.*

Proof Let $R = Se_H$. For each $g \in H$, $\alpha_g(R 1_{g^{-1}} e_H) = \alpha_g(Se_H 1_{g^{-1}}) = \alpha_g(S 1_{g^{-1}}) \alpha_g (e_H 1_{g^{-1}}) = S 1_g e_H = R 1_g e_H$, where Lemma 2 is applied to the second-to-last equality. Thus α induces a partial action of H on R, denoted by α', with $\alpha' = (\{R 1_g e_H\}_{g \in H}, \{\alpha'_g = \alpha_g|_{R 1_{g^{-1}}}\}_{g \in H})$. But $R 1_g e_H = R$ for each $g \in H$. Thus α' is a global action of H on R. Suppose furthermore that α on S makes it a partial Galois extension of S^α and $e_H \neq 0$. Let $\{x_i; y_i \in S \mid i = 1, \ldots, n\}$ for some integer n be an α-partial Galois coordinate system for S; that is, $\sum_{i=1}^n x_i \alpha_g(y_i 1_{g^{-1}}) = \delta_{1,g}$ for each $g \in G$. For each $i = 1, \ldots, n$, let $x'_i = x_i e_H$ and $y'_i = y_i e_H$. Then for each $g \in H$, $\sum_{i=1}^n x'_i \alpha'_g(y'_i 1_{g^{-1}} e_H) = \sum_{i=1}^n x_i e_H \alpha_g(y_i e_H 1_{g^{-1}}) = \sum_{i=1}^n x_i e_H \alpha_g(y_i 1_{g^{-1}}) \alpha_g (e_H 1_{g^{-1}}) = e_H \delta_{1,g}$, where Lemma 2 is applied again. Thus $\{x'_i; y'_i \in R \mid i = 1, \ldots, n\}$ is a Galois coordinate system for R and hence Se_H is a Galois extension of $(Se_H)^H$ with Galois group H. □

Under the assumption in the lemma above, α actually induces a partial action of H, denoted by α_H, on S such that S is an α_H-partial Galois extension of S^{α_H}, and α_H when restricted on Se_H is well-defined and makes Se_H a Galois extension of $S^{\alpha_H}e_H$ with Galois group H (see [11, Corollary 4.4 and Theorem 5.2]).

Let S be an α-partial Galois extension of S^α. Let $\mathscr{S}=\{e_H \neq 0 \mid H$ is a subgroup of $G\}$ and $\mathscr{T}$ the set of unital ideals A of S containing $(Se_H)^H$ such that A is a Galois extension of A^H with Galois group H induced by α. We shall show a one-to-one correspondence between $\mathscr{S}$ and $\mathscr{T}$. Before that we need another lemma.

Lemma 4 *Let S be an α-partial Galois extension of S^α. Let H be a subgroup of G and A a unital ideal of S containing $(Se_H)^H$. Suppose that A is a Galois extension of A^H with Galois group H induced by α. Then $e_H \neq 0$ and $A = Se_H$.*

Proof Let $1'$ denote the identity of A as a ring and α' the partial action of H on A induced by α; that is, $\alpha' = (\{A1'_g\}_{g\in H}, \{\alpha'_g\}_{g\in H})$ where $1'_g = 1'1_g$ and $\alpha'_g = \alpha_g|_{A1'_{g^{-1}}}$ for each $g \in H$. Since by assumption α' is actually a global action of H on A, then $1' = 1'_g$ for each $g \in H$. In particular, $e'_H = \Pi_{g\in H} 1'_g = 1' \neq 0$. But we also have $e'_H = \Pi_{g\in H} 1'1_g = 1'e_H$. Thus $e_H \neq 0$. Since H is a subgroup of G such that $e_H \neq 0$, then by Lemma 3, Se_H is a Galois extension of $(Se_H)^H$ with Galois group H induced by α. On the other hand, $A = A1' = Ae'_H = Ae_H \subset Se_H$ and hence $A^H \subset (Se_H)^H$. But since A contains $(Se_H)^H$, it follows that $A^H = (Se_H)^H$. Thus by assumption A is also (other than Se_H) a Galois extension of $(Se_H)^H$ with Galois group H induced by α. Now since A is contained in Se_H, we then conclude that $A = Se_H$. □

Proposition 5 *Let S be an α-partial Galois extension of S^α. Let $\mathscr{S}$ and $\mathscr{T}$ be as above. There is a one-to-one correspondence between $\mathscr{S}$ and $\mathscr{T}$.*

Proof Let $\varphi\colon \mathscr{S} \to \mathscr{T}$ be defined by sending e_H to Se_H. Then Lemma 3 implies that φ is well defined and one-to-one while Lemma 4 implies that φ is surjective. □

To compute the number of the Galois extensions given in Proposition 5, one can simply find all the subgroups H of G, determine which e_H is nonzero and then compute the number of nonzero e_H without multiplicity. The task, however, needs much effort if G is a big group with many subgroups. One can alternatively find the tree of $B(I_G)$ and hence all the minimal elements in $B(I_G)$, determine subsets of $G(e)$ which are subgroups of G for each minimal element e, and after collecting such subgroups H, compute the number of e_H without multiplicity. In this way, one does not need to determine all the subgroups of G in advance.

3.2 The main theorem

In this subsection, we will prove the structure theorem for a partial Galois extension. Before that, we make more investigations when H in the statements of Lemmas 2 and 3 is replaced by $G(e)$ for some minimal element e in $B(I_G)$. Firstly, the converse of Lemma 2 actually holds in the case when $H = G(e)$. Notice that $e_{G(e)} = e$. We characterize $G(e)$ as a subgroup of G in the following proposition.

Proposition 6 *Let e be a minimal element in $B(I_G)$. The following statements are equivalent:*

(i) $G(e)$ *is a subgroup of* G.
(ii) $\alpha_g(e1_{g^{-1}}) = e1_g$ *for each* $g \in G$*; that is,* $e \in S^\alpha$.
(iii) $\alpha_g(B(I_{G(e)})1_{g^{-1}}) \subset B(I_{G(e)})$ *for each* $g \in G(e)$.

Proof Assume that $G(e)$ is a subgroup of G. Then by Lemma 2, (iii) holds and (ii) is satisfied in the case where $g \in G(e)$. But for $g \notin G(e)$, we have $g^{-1} \notin G(e)$ and hence $e1_{g^{-1}} = e1_g = 0$. Thus (ii) follows. We next show that $G(e)$ is a subgroup of G if (ii) or (iii) holds. Assume first that (iii) holds. For any $g, h \in G(e)$, it suffices to show that $gh \in G(e)$. Because $1_g 1_{gh} = \alpha_g(1_h 1_{g^{-1}})$ by Eq. (1) is in $B(I_{G(e)})$, we have $e1_{gh} = e1_g 1_{gh} = e$. It follows that $gh \in G(e)$ by the definition of $G(e)$. Now assume that (ii) holds. For any $g, h \in G(e)$, $e = e1_g = \alpha_g(e1_{g^{-1}}) = \alpha_g(e1_h 1_{g^{-1}}) = \alpha_g(e1_{g^{-1}})\alpha_g(1_h 1_{g^{-1}}) = e1_g 1_{gh} = e1_{gh}$, where Eq. (1) is applied to the second-to-last equality. Thus as above, by the definition of $G(e)$, we see that $gh \in G(e)$. □

A combination of Lemma 3 and Proposition 6 is as follows.

Proposition 7 *Let S be an α-partial Galois extension of S^α and e a minimal element in $B(I_G)$. Suppose that e is α-invariant. Then Se is a Galois extension of $S^\alpha e$ with Galois group $G(e)$ induced by α.*

Proof Since by hypothesis e is α-invariant, then by Proposition 6, $G(e)$ is a subgroup of G. Thus, by Lemma 3, Se is a Galois extension of $(Se)^{G(e)}$ with Galois group $G(e)$ induced by α. It remains to show that $(Se)^{G(e)} = S^\alpha e$. Since $e \in S^\alpha$, then in particular we have that $S^\alpha e \subseteq (Se)^{G(e)}$. Conversely, let $x \in (Se)^{G(e)}$. Then for each $g \in G(e)$, $\alpha_g(x1_{g^{-1}}) = x1_g$. As for $g \notin G(e)$, it follows from $e1_g = 0 = e1_{g^{-1}}$ that $\alpha_g(x1_{g^{-1}}) = \alpha_g(0) = 0 = x1_g$. Therefore, $x \in S^\alpha$ and hence $x = xe \in S^\alpha e$. □

We are now ready to prove the main theorem.

Theorem 8 *Let S be an α-partial Galois extension of S^α. Suppose $\{e_1, e_2, \ldots, e_k\}$ for some integer k is a set of minimal elements in $B(I_G)$ such that each e_i is α-invariant. Then $S = \oplus_{i=1}^k Se_i \oplus Se'$, where $e' = 1_S - \sum_{i=1}^k e_i$, such that each Se_i is a Galois extension of $S^\alpha e_i$ with Galois group $G(e_i)$ induced by α and if $e' \neq 0$, Se' is a partial Galois extension of $S^\alpha e'$ under a partial action α' of G induced by α.*

Proof These e_i, $i = 1, \ldots, k$, are orthogonal idempotents of S since they are minimal elements in $B(I_G)$. Thus $S = \oplus_{i=1}^k Se_i \oplus Se'$, where $e' = 1_S - \sum_{i=1}^k e_i$. Because each e_i, $i = 1, \ldots, k$, is α-invariant, it follows that the idempotent e' is also α-invariant, and by Proposition 7, each Se_i is a Galois extension of $S^\alpha e_i$ with Galois group $G(e_i)$ induced by α. Suppose $e' \neq 0$. Then similar to the proof of Lemma 3, let $R = Se'$ for convenience. For each $g \in G$, $\alpha_g(R1_{g^{-1}}e') = \alpha_g(Se'1_{g^{-1}}) = \alpha_g(S1_{g^{-1}})\alpha_g(e'1_{g^{-1}}) = S1_g e'1_g = R1_g e'$. Thus α induces a partial action α' of G on R with $\alpha' = (\{Re'_g\}_{g\in G}, \{\alpha'_g\}_{g\in G})$ where $e'_g = 1_g e'$ and $\alpha'_g = \alpha_g|_{R1_{g^{-1}}}$ for each $g \in G$. Also, if $\{x_i; y_i \in S \mid i = 1, \ldots, n\}$ for some integer n is an α-partial Galois coordinate system for S, then $\{x_i e'; y_i e' \in R \mid i = 1, \ldots, n\}$ is an α'-partial Galois coordinate system for R. Thus Se' is an α'-partial Galois extension of $(Se')^{\alpha'}$. Finally, one can check that $(Se')^{\alpha'} = S^\alpha e'$. □

For the following, let S be an α-partial Galois extension of S^α, $\{e_1, e_2, \ldots, e_m\}$ the set of all minimal elements in $B(I_G)$, and assume that $\sum_{i=1}^m e_i = 1_S$. We will derive three results. Below we begin with the fundamental theorem provided that S is commutative.

Corollary 9 *Suppose that S is commutative and each e_i, $i = 1, \ldots, m$, is an α-invariant minimal idempotent in S. Then (S, α) satisfies the fundamental theorem; that is, there is a one-to-one correspondence between the set of subgroups of $G(e_1) \times \cdots \times G(e_m)$ and the set of separable S^α-subalgebras of S.*

Proof Notice that each Se_i, $i = 1, \ldots, m$, is a commutative ring with no idempotents other than 0 and e_i. By Proposition 7 or Theorem 8, each Se_i is a Galois extension of $S^\alpha e_i$ with Galois group $G(e_i)$. Thus the fundamental theorem holds for each Se_i ([4, Theorem 2.3]); that is, there is a one-to-one correspondence between the set of subgroups of $G(e_i)$ and the set of separable $S^\alpha e_i$-subalgebras of Se_i. Therefore it follows from $\sum_{i=1}^m e_i = 1_S$ that there is a one-to-one correspondence between the set of subgroups of $G(e_1) \times \cdots \times G(e_m)$ and the set of separable S^α-subalgebras of S. More explicitly, the subgroup $K_1 \times \cdots \times K_m$ of $G(e_1) \times \cdots \times G(e_m)$ corresponds to the separable S^α-subalgebra $\oplus_{i=1}^m (Se_i)^{K_i}$ of S. □

To present the next two results, we give two easy notions. A partial Galois extension (S, α) is called a *central partial Galois algebra* if S^α is the center of S. A partial action α of G on S is called *inner* if for each $g \in G$, there exists some unit u_g in S such that $\alpha_g(x1_{g^{-1}}) = u_g x u_g^{-1} 1_g$ for all $x \in S$. Let C denote the center of S.

Corollary 10 *Suppose that (S, α) is a central partial Galois algebra, C is a semi-local ring and each idempotent e_i, $i = 1, \ldots, m$, is minimal in S. Then $S \cong \oplus_{i=1}^m M_{n_i}(D_i)$, a direct sum of matrix rings of order n_i over Azumaya Ce_i-algebras D_i for some integers n_i.*

Proof Since $S^\alpha = C$, then $1_g \in S^\alpha$ for each $g \in G$ and hence each $e_i \in S^\alpha$, $i = 1, \ldots, m$. Thus it follows from Theorem 8 that $S = \oplus_{i=1}^m Se_i$ such that each Se_i is a central Galois algebra over Ce_i. Since by hypothesis C is semi-local and each e_i is a minimal idempotent in S, then Ce_i is semi-local with no idempotents other than 0 and e_i. Therefore it follows from the general Wedderburn theorem due to DeMeyer [9, Corollary 1] that $Se_i \cong M_{n_i}(D_i)$ for some uniquely determined integer n_i and Azumaya Ce_i-algebra D_i. □

Corollary 11 *Suppose (S, α) is a central partial Galois algebra with α an inner partial action of G on S. Then $S \cong \oplus_{i=1}^m (Ce_i)G(e_i)_{f_i}$, where $(Ce_i)G(e_i)_{f_i}$ is a projective group algebra of $G(e_i)$ over (Ce_i) with factor set $f_i : G(e_i) \times G(e_i) \to U(Ce_i)$, the multiplicative group of units of Ce_i, in the sense of DeMeyer [8].*

Proof Since α is inner, then for each $g \in G$, there exists some unit u_g in S such that $\alpha_g(x1_{g^{-1}}) = u_g x u_g^{-1} 1_g$ for all $x \in S$. In particular, for any $g' \in G$, $\alpha_g(1_{g'} 1_{g^{-1}}) = u_g 1_{g'} u_g^{-1} 1_g = 1_{g'} 1_g$; that is $1_{g'} \in S^\alpha$, and hence each $e_i \in S^\alpha$, $i = 1, \ldots, m$. Thus by Theorem 8, $S = \oplus_{i=1}^m Se_i$ such that each Se_i is a central Galois algebra over Ce_i with inner Galois group $G(e_i)$. Therefore for each $i = 1, \ldots, m$, by [8, Theorem 2], Se_i

is a projective group algebra $(Ce_i)G(e_i)_{f_i}$ for some factor set $f_i : G(e_i) \times G(e_i) \to U(Ce_i)$. □

We next construct an example of a partial Galois extension (S, α) where all minimal elements in $B(I_G)$ are α-invariant and their sum is the identity element 1_S.

Example 12 Let R be a Galois extension of R^H with Galois group H and T a Galois extension of T^K with Galois group K. We shall define a partial action α of $G = H \times K$ on $S = R \oplus T$ such that S is an α-partial Galois extension of $R^H \oplus T^K$. For $g = (h, k) \in G$, let

$$1_g = \begin{cases} (0,0) & \text{if } h \neq 1, k \neq 1, \\ (1,0) & \text{if } h \neq 1, k = 1, \\ (0,1) & \text{if } h = 1, k \neq 1, \\ (1,1) & \text{if } h = 1, k = 1. \end{cases} \quad \text{Then } S_g = S1_g = \begin{cases} 0 \oplus 0 & \text{if } h \neq 1, k \neq 1, \\ R \oplus 0 & \text{if } h \neq 1, k = 1, \\ 0 \oplus T & \text{if } h = 1, k \neq 1, \\ R \oplus T & \text{if } h = 1, k = 1. \end{cases}$$

Define $\alpha_g : S_{g^{-1}} \to S_g$ as follows

$$\begin{aligned} (0,0) &\mapsto (0,0) && \text{if } h \neq 1, k \neq 1, \\ (r,0) &\mapsto (h(r),0) && \text{if } h \neq 1, k = 1, \\ (0,t) &\mapsto (0,k(t)) && \text{if } h = 1, k \neq 1, \\ (r,t) &\mapsto (r,t) && \text{if } h = 1, k = 1. \end{aligned}$$

Obviously, each α_g is a ring isomorphism. Actually, α is a partial action of G on S. We provide below some details showing that $\alpha_g(S_{g^{-1}} \cap S_{g'}) = S_g \cap S_{gg'}$ for $g, g' \in G$. This is obvious if $g = (h, k)$ with $h = 1, k = 1$ or $h \neq 1, k \neq 1$. Suppose $g = (h, k)$ with $h \neq 1, k = 1$. Then $1_g = 1_{g^{-1}} = (1, 0)$, and α_g behaves as h on R. If $g' = (h', k')$ with $h' = 1, k' \neq 1$, then $1_{g'} = (0, 1)$ and $1_{gg'} = (0, 0)$. Thus $1_g 1_{gg'} = (0, 0) = 1_{g^{-1}} 1_{g'}$ and hence $S_{g^{-1}} \cap S_{g'} = S_g \cap S_{gg'} = 0 \oplus 0$. If $g' = (h', k')$ with $h' \neq 1, k' = 1$, then $1_{g'} = (1, 0)$ while $1_{gg'}$ is $(1, 0)$ if $h' \neq h^{-1}$ or $(1, 1)$ if $h' = h^{-1}$. In either case, $1_g 1_{gg'} = (1, 0) = 1_{g^{-1}} 1_{g'}$ and hence $S_{g^{-1}} \cap S_{g'} = S_g \cap S_{gg'} = R \oplus 0$. The remaining are similar.

Clearly, $S^\alpha = R^H \oplus T^K$. Now, let $\{r_i; r_i' \in R \mid i = 1, \ldots, m\}$ and $\{t_j; t_j' \in T \mid j = 1, \ldots, n\}$ for some integers m and n be Galois coordinate systems for R and T, respectively. Let $x_1 = (r_1, 0), \ldots, x_m = (r_m, 0), x_{m+1} = (0, t_1), \ldots, x_{m+n} = (0, t_n)$ and $x_1' = (r_1', 0), \ldots, x_m' = (r_m', 0), x_{m+1}' = (0, t_1'), \ldots, x_{m+n}' = (0, t_n')$. We have that

$$\sum_{l=1}^{m+n} x_l \alpha_{(1,1)} \left(x_l' 1_{(1,1)^{-1}}\right) = \sum_{l=1}^{m+n} x_l x_l' = \left(\sum_{i=1}^{m} r_i r_i', \sum_{j=1}^{n} t_j t_j'\right) = (1, 1).$$

Also, if $k \neq 1$ and $h \neq 1$,

$$\sum_{l=1}^{m+n} x_l \alpha_{(1,k)} \left(x_l' 1_{(1,k)^{-1}}\right) = \left(0, \sum_{j=1}^{n} t_j k\left(t_j'\right)\right) = (0, 0),$$

$$\sum_{l=1}^{m+n} x_l \alpha_{(h,1)}\left(x'_l 1_{(h,1)^{-1}}\right) = \left(\sum_{i=1}^{m} r_i h\left(r'_i\right), 0\right) = (0,0),$$
$$\sum_{l=1}^{m+n} x_l \alpha_{(h,k)}\left(x'_l 1_{(h,k)^{-1}}\right) = \sum_{l=1}^{m+n} x_l \alpha_{(h,k)}(0,0) = (0,0).$$

Thus $\{x_l; x'_l \in S \mid l = 1, \ldots, m+n\}$ is an α-partial Galois coordinate system for S. Therefore, S is an α-partial Galois extension of $R^H \oplus T^K$.

For this (S, α), $e_1 = (1,0)$ and $e_2 = (0,1)$ are the only two minimal elements in $B(I_G)$ and $e_1 + e_2 = (1,1) = 1_S$. Moreover, e_1 and e_2 are both α-invariant; indeed, $G(e_1) = H \times 1$ and $G(e_2) = 1 \times K$ are both subgroups of G. Thus by Theorem 8, $S = Se_1 \oplus Se_2$ where Se_i, $i = 1, 2$, is a Galois extension of $(Se_i)^{G(e_i)} = S^\alpha e_i$ with Galois group $G(e_i)$. This conclusion is pretty natural in this case since $Se_1 \cong R$, $G(e_1) \cong H$ and R is a Galois extension of R^H with Galois group H; similarly, $Se_2 \cong T$, $G(e_2) \cong K$ and T is a Galois extension of T^K with Galois group K.

Example 13 Let R be the quaternion algebra over the rational field $\mathbb{Q}$ generated by i, j such that $i^2 = -1 = j^2$ and $ij = -ji$ and H the inner automorphism group of R induced by $\{1, i, j, ij\}$. Then it is well-known that R is a Galois extension of $\mathbb{Q}$ with Galois group H; actually, one can easily check that $\{\frac{1}{2}, \frac{1}{2}i, \frac{1}{2}j, \frac{1}{2}ij; \frac{1}{2}, -\frac{1}{2}i, -\frac{1}{2}j, -\frac{1}{2}ij\}$ is a Galois coordinate system for R. Let T be the ring of 2×2 matrices over a ring Q in which 2 is invertible. Then as shown in [17], T is a Galois extension of Q with an inner Galois group K induced by $\left\{\begin{pmatrix}1&0\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix}, \begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\1&0\end{pmatrix}\right\}$. Now, by Example 12, $R \oplus T$ is a partial Galois extension of $\mathbb{Q} \oplus Q$ with the partial action α of $H \times K$ as defined in Example 12.

3.3 A direct sum of Galois extensions

The method used in Example 12 to construct a partial Galois extension from a direct sum of two Galois extensions can be extended to a finite set of Galois extensions. In this subsection, we shall show that any direct sum of finite number of Galois extensions is a partial Galois extension. This thus provides an easy way to construct partial Galois extensions. We first define a partial action on a direct sum of Galois extensions by the direct product of their Galois groups.

Lemma 14 *Let R_i be a Galois extension of $R_i^{H_i}$ with Galois group H_i, $i = 1, \ldots, m$ for some integer m. Let $S = R_1 \oplus \cdots \oplus R_m$ and $G = H_1 \times \cdots \times H_m$. Then there exists a partial action α of G on S.*

Proof For each $i = 1, \ldots, m$, let $G_i = 1 \times \cdots \times 1 \times H_i \times 1 \times \cdots \times 1$, $S_i = 0 \oplus \cdots \oplus 0 \oplus R_i \oplus 0 \oplus \cdots \oplus 0$ and $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in S_i$. Let

$$1_g = \begin{cases} 1_S & \text{if } g = 1_G \\ e_i & \text{if } 1_G \neq g \in G_i \\ 0 & \text{if } g \notin \cup_{i=1}^m G_i. \end{cases} \quad \text{Then } \mathrm{S_g} = \mathrm{S1_g} = \begin{cases} S & \text{if } g = 1_G \\ S_i & \text{if } 1_G \neq g \in G_i \\ 0 & \text{if } g \notin \cup_{i=1}^m G_i. \end{cases}$$

Define $\alpha_g : S_{g^{-1}} \to S_g$ to be the identity map on S if $g = 1_G$, the zero map on 0 if $g \notin \cup_{i=1}^m G_i$ and the map sending $(0, \ldots, 0, r_i, 0, \ldots, 0)$, $r_i \in R_i$, to $(0, \ldots, 0, h_i(r_i), 0, \ldots, 0)$ if $1_G \neq g = (h_1, \ldots, h_m) \in G_i$. Then one can check that α is a partial action of G on S. □

Theorem 15 *Use the notations in Lemma 14. S is an α-partial Galois extension of $\oplus_{i=1}^m R_i^{H_i}$.*

Proof Clearly $S^\alpha = R_1^{H_1} \oplus \cdots \oplus R_m^{H_m}$. For each $i = 1, \ldots, m$, since $S_i \cong R_i$ and $G_i \cong H_i$, then S_i is a Galois extension of $S_i^{G_i}$ with Galois group G_i; let $\{x_j^{(i)}; y_j^{(i)} \in S_i \mid j = 1, \ldots, l_i\}$ be a Galois coordinate system for S_i. Now for each $i = 1, \ldots, m$ and $j = 1, \ldots, l_i$, let $X_{i,j} = x_j^{(i)}$ and $Y_{i,j} = y_j^{(i)}$. Then we have that

$$\sum_{i=1}^{m}\sum_{j=1}^{l_i} X_{i,j}Y_{i,j} = \sum_{j=1}^{l_1} x_j^{(1)}y_j^{(1)} + \cdots + \sum_{j=1}^{l_m} x_j^{(m)}y_j^{(m)} = 1_{S_1} + \cdots + 1_{S_m}$$
$$= \sum_{i=1}^{m} e_i = 1_S,$$

and for $g = (h_1, \ldots, h_m) \in G_k$ for some $k \in \{1, \ldots, m\}$ but $g \neq 1_G$,

$$\sum_{i=1}^{m}\sum_{j=1}^{l_i} X_{i,j}\alpha_g(Y_{i,j}1_{g^{-1}}) = \sum_{j=1}^{l_k} x_j^{(k)} h_k(y_j^{(k)}) = 0,$$

since $h_k \neq 1$. As for $g \notin \cup_{i=1}^m G_i$, it is clear that $\sum_{i=1}^m \sum_{j=1}^{l_i} X_{i,j}\alpha_g(Y_{i,j}1_{g^{-1}}) = 0$ since $1_{g^{-1}} = 0$. Thus $\cup_{i=1}^m \{X_{i,j}; Y_{i,j} \in S \mid j = 1, \ldots, l_i\}$ is an α-partial Galois coordinate system for S. Therefore, we conclude that S is an α-partial Galois extension of $S^\alpha = R_1^{H_1} \oplus \cdots \oplus R_m^{H_m}$. □

3.4 A complete structure

To derive the complete structure of a partial Galois extension (S, α), let $\{e_1, e_2 \ldots, e_m\}$ be the set of all minimal elements in $B(I_G)$ and assume that each e_i, $i = 1, \ldots, m$, is α-invariant. By Theorem 8, $S = \oplus_{i=1}^m Se_i \oplus Se'$, where $e' = 1_S - \sum_{i=1}^m e_i$, such that each Se_i is a Galois extension of $S^\alpha e_i$ with Galois group $G(e_i)$ induced by α and if $e' \neq 0$, Se' is a partial Galois extension of $S^\alpha e'$ under a partial action α' of G induced by α. Now we use the new notations $1^{(0)}$, $e_i^{(0)}$, $S^{(1)}$, $1^{(1)}$, $\alpha^{(1)}$ for 1_S, e_i, Se', e', α', respectively. Suppose that $1^{(1)} \neq 0$. Then $\alpha^{(1)}$ is a partial action of G on $S^{(1)}$ with $\alpha^{(1)} = (\{S^{(1)}1_g^{(1)}\}_{g\in G}, \{\alpha_g^{(1)}\}_{g\in G})$ where $1_g^{(1)} = 1_g 1^{(1)}$ and $\alpha_g^{(1)} = \alpha_g|_{S^{(1)}1_{g^{-1}}}$. One can similarly define $I_G^{(1)} = \{1_g^{(1)} | g \in G\}$ and the Boolean semigroup $B\left(I_G^{(1)}\right)$ generated by $I_G^{(1)}$. Note that the tree of $B\left(I_G^{(1)}\right)$ can be derived from the tree of $B(I_G)$ by multiplying each node by $1^{(1)}$ and deleting those becoming zero.

Also, because $1^{(1)}e_i = e'e_i = 0$ for each $i = 1, \ldots, m$, we actually obtain a tree with shorter paths. Let $\{e_1^{(1)}, e_2^{(1)}, \ldots, e_{m_1}^{(1)}\}$ be the set of all minimal elements in $B(I_G^{(1)})$. Then, as before, because these $e_i^{(1)}$ are orthogonal idempotents of $S^{(1)}$, we have $S^{(1)} = \oplus_{i=1}^{m_1} S^{(1)}e_i^{(1)} \oplus S^{(1)}1^{(2)}$, where $1^{(2)} = 1^{(1)} - \sum_{i=1}^{m_1} e_i^{(1)}$. Define $G(e_i^{(1)})$ as before. Suppose that each $e_i^{(1)}$, $i = 1, \ldots, m_1$, is $\alpha^{(1)}$-invariant or equivalently that each $G(e_i^{(1)})$ is a subgroup of G. Then one can apply Theorem 8 to $S^{(1)}$ to conclude that each $S^{(1)}e_i^{(1)}$, $i = 1, \ldots, m_1$, is a Galois extension of $(S^{(1)}e_i^{(1)})^{G(e_i^{(1)})} = S^{\alpha}1^{(1)}e_i^{(1)}$ with Galois group $G(e_i^{(1)})$ and if $1^{(2)} \neq 0$, then $S^{(2)} = S^{(1)}1^{(2)}$ is a partial Galois extension of $S^{(2)\alpha^{(2)}} = S^{(1)\alpha^{(1)}}1^{(2)} = S^{\alpha}1^{(1)}1^{(2)}$ under a partial action $\alpha^{(2)}$ of G induced by $\alpha^{(1)}$ and hence by α. Notice that by the argument above, the associated tree of $S^{(2)}$ has shorter paths than that of $S^{(1)}$. One can continue this process to $S^{(2)}$ and so on if in each step j, $1^{(j)} \neq 0$ and each minimal element of $B(I_G^{(j)})$ is $\alpha^{(j)}$-invariant; equivalently, α-invariant as an element in S. But since in each stage we get a tree of shorter paths than the one in the former stage, it follows that the decomposition will eventually stop after finitely many steps. Moreover, in the final step l, $S^{(l)}$ is a *trivial* $\alpha^{(l)}$-partial Galois extension of $S^{(l)\alpha^{(l)}} = S^{\alpha}1^{(1)}1^{(2)}\ldots 1^{(l)}$; that is, $1_g^{(l)} = 0$ for each $1 \neq g \in G$. We conclude this in the following structure theorem for (S, α).

Theorem 16 *Let S be an α-partial Galois extension of S^{α}. Use the notations above. If in each step j, $1^{(j)} \neq 0$ and each minimal element of $B(I_G^{(j)})$ is α-invariant, then S is a direct sum of Galois extensions and a trivial partial Galois extension.*

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