JOURNAL OF SCIENCE& ENGINEERING, MAY, 1967 VOL IV PP.1~24

THE FIXED-POINT THEOREM AT VARIOUS DIMENSIONAL SURFACES

by

Harvey H. S. Tung

I INTRODUCTION

A large portion of topological theory with important applications in other branches of mathematics, physics, astronomy, or philosophy, is concerned with fixed points of mappings. There are many so called Fixed-Point Theorems (in fact, there altogether is a Fixed-Point Theorem at various dimensional surfaces) closely related to the continuity of function, homeomorphism, and connectedness. In order to facililate the working out of this article in detail, let X and Y be two topological spaces which will be kept in use throughout the whole article.

Definition 1. A function $f: A \to B$ is called one-one (1—1) if whenever $f(a)=f(a')$ for $a, a' \in A$, then $a=a'$, or equivalently, $a \neq a'$ implies $f(a) \neq f(a')$.

Definition 2. A function $f: A \to B$ is called onto if $B=f(A)$, that is, if every member of B appears as the image of at least one element of A. If $f: A \to B$ is onto, we say "f is a function of A onto B", or "f maps A onto B", or f is an onto function".

Definition 3. Let $f: A \to B$ and $g: B \to A$ be given. The function $f: A \to B$ is called the inverse function of $g: B \to A$ and the function $g: B \to A$ is called the inverse function of $f: A \to B$ if $g(f(a))=a$ for each $a \in A$ and $f(g(b))=b$ for each $b \in B$. It is evident that the equations $f(a)=b$ and $g(b)=a$ are equivalent to the two equations above. So we can use $f(a)=b$ and $g(b)=a$ instead of $g(f(a))=a$ and $f(g(b))=b$.

In this event we shall also say that $f: A \to B$ and $g: B \to A$ are inverse functions and that each of them is invertible.

The function $f: X \to Y$ is said to be continuous at the point $a \in X$, if given $\epsilon > 0$, there is a $\delta > 0$, such that $|f(x)-f(a)| < \epsilon$ as $|x-a| < \delta$.

This relationship can be shown graphically as in Fig. 1, thus we have

Definition 4. A function $f: X \to Y$ is said to be continuous at a point $a \in X$ if for each neighborhood N of $f(a)$, there is a neighborhood $f^{-1}(N)$ of a. f is said to be

* Professor of Applied Mathematics, Provincial Chung Hsing University.

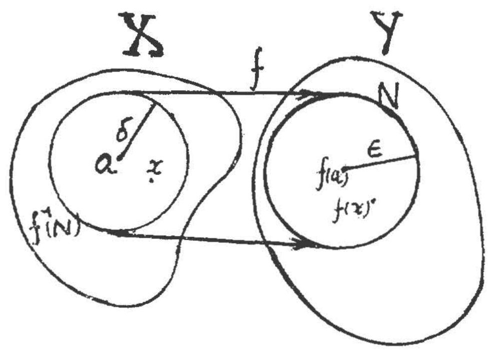


Fig. 1

continuous if f is continuous at each point of X.

Definition 5. X and Y are called ***homeomorphic*** if there exist inverse functions $f: X \to Y$ and $g: Y \to X$ such that f and g are continuous. In this case the function f and g are said to be ***homeomorphisms*** and we say that f and g define a homeomorphism between X and Y.

A subspace of a topological space is "connected" if it is all of one piece; i.e., if it is impossible to decompose the subspace into two disjoint non-empty open sets. Thus, we have

Definition 6. A subset A of a topological space X is said to be ***connected*** if the only two subsets of A that are simultaneously relatively open and relatively closed (Definition 8) in A are A and ϕ.

Definition 7. A fixed point of f is defined as a point $z \in A$ which coincides with its image; that is

$$z = f(z), \text{ clearly } z \in (A \cap f(A)).$$

Definition 8. Suppose Y is a subspace of a topological space X. A subset O that is open in Y is often called ***relatively open*** in Y or simply relatively open. The closed subsets of Y are called ***relatively closed*** in Y or simply relatively closed.

Lemma 9. Let X be a topological space, a subspace $A \subset X$. Then A is not connected *iff* (***if and only if***) there exist 2 open sets P and Q of X such that

$$A \subset P \cup Q, \quad P \cap Q \subset C(A)$$

and $$P \cap A \neq \phi, \quad Q \cap A \neq \phi$$

Proof. Suppose that A is not connected. Then, by Definition 6, there is a subset P' of A such that $P' \neq \phi$ and $P' \neq A$, and P' is both relatively open and relatively closed. This implies that $C_A(P') \neq \phi$ and $C_A(P') \neq A$, and it is relatively open. Thus $P' = P \cap A \neq \phi$, and $C_A(P') = Q \cap A \neq \phi$, where P and Q are open sets of X. We, then, have that $A = P' \cup C_A(P') \subset P \cup Q$, since $P' \subset P$ and $C_A(P') \subset Q$, and also $(P \cap Q) \cap A = (P \cap A) \cap (Q \cap A) = P' \cap C_A(P') = \phi$, so that $P \cap Q \subset C(A)$.

Conversely, given open sets P and Q satisfying the stated conditions, let

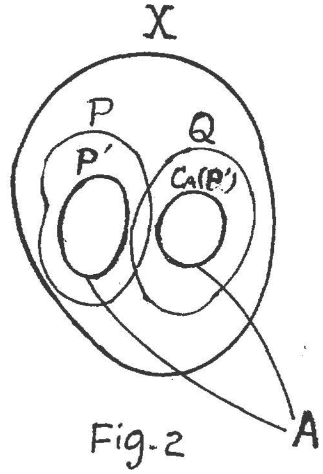


Fig-2

$P' = P \cap A$ and $Q' = Q \cap A$.

Since $A \subset P \cup Q$, $A = A \cap (P \cup Q) = (A \cap P) \cup (A \cap Q) = P' \cup Q'$; and $P' \cap Q' = (A \cap P) \cap (A \cap Q) = (P \cap Q) \cap A = \phi$, since $P \cap Q \subset C(A)$.

Thus $P' = C_A(Q')$, and P' is both relatively open and relatively closed in A. For $P' = P \cap A \neq \phi$ and $Q' = Q \cap A \neq \phi$; $P' \neq A$, since $A = P' \cup Q'$.

Hence, by Definition 6, A is not connected.

We have a corresponding result for closed sets, such as

Lemma 10. Let A be a subspace of X. Then A is not connected iff there exist 2 closed subsets $F, G \subset X$ such that

$$A \subset F \cup G, \quad F \cap G \subset C(A)$$

and $$F \cap A \neq \phi, \quad G \cap A \neq \phi.$$

Definition 11. A subset O of X is said to be open if O is a neighborhood of each of its points (namely, interior points).

Theorem 12. A function $f: X \to Y$ is continuous iff for each open subset O of Y, $f^{-1}(O)$ is an open subset of X.

Proof.

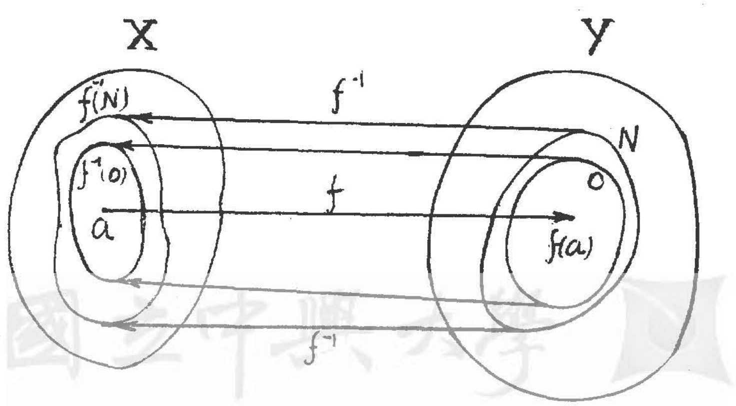


Fig.3

Suppose that f is continuous and that O is an open subset of Y. For each $a \in f^{-1}(O)$, O is a neighborhood of $f(a)$, by Definition 4, $f^{-1}(O)$ is a neighborhood of a. Since point a was arbitrary, $f^{-1}(O)$ is a neighborhood of each of its points, and $f^{-1}(O)$, by Definition 11, is an open subset of X.

Conversely, suppose that for each open subset O of Y, $f^{-1}(O)$ is an open subset of X. Let $a \in X$ and a neighborhood N of $f(a)$ be given; N contains the open set O containing $f(a)$. So by our hypotheses, $f^{-1}(N)$ contains the open set $f^{-1}(O)$ containing a. Thus, $f^{-1}(N)$ is a neighborhood of a, by Definition 4, f is continuous at a. Since a was arbitrary, f is continuous.

Now, we shall show that connectedness is preserved under continuous mappings.

Theorem 13. Let the function $f: X \to Y$ be continuous. If A is a connected subset of X, then $f(A)$ is a connected subset of Y.

Proof. Suppose $f(A)$ is not connected. Then, by Lemma 9, there are two open subsets P′ and Q′ of Y such that $f(A) \subset P' \cup Q'$, $P' \cap Q' \subset C(f(A))$, and $P' \cap f(A) \neq \phi$, $Q' \cap f(A) \neq \phi$. Since f is continuous, by Theorem 12, $P = f^{-1}(P')$ and $Q = f^{-1}(Q')$ are open sets of X. But $A \subset f^{-1}(f(A)) \subset f^{-1}(P' \cup Q') = P \cup Q$. Also $P \cap Q = f^{-1}(P' \cap Q') \subset f^{-1}(C(f(A))) = C(f^{-1}(f(A))) \subset C(A)$. Finally, $P \cap A \neq \phi$, $Q \cap A \neq \phi$, since $P' \cap f(A) \neq \phi$, $Q' \cap f(A) \neq \phi$. Thus A is not connected. It follows that if A is connected then $f(A)$ must also be connected.

Definition 14. A subset A of a set R of all real numbers is called an ***interval*** if A contains at least 2 distinct points, say a, b with $a < b$, then for each point x such that $a < x < b$, it follows that $x \in A$.

Thus, an interval contains all points between any two of its points. $[a, b]$ is a ***closed interval*** equivalent to $\{a \leq x \leq b\}$, and (a, b) is an ***open interval*** equivalent to $\{a < x < b\}$. In addition to these subsets of R that are intervals are defined in the following definition.

Definition 15. Let a and b be two real numbers with $a < b$. The subset A of R consisting of real numbers x such that

$a < x$, then $A = (a, +\infty)$;
$a \leq x$, then $A = [a, +\infty)$;
$a > x$, then $A = (-\infty, a)$;
$a \geq x$, then $A = (-\infty, a]$;
$a < x \leq b$, then $A = (a, b]$;
$a \leq x < b$, then $A = [a, b)$;
$R = (-\infty, +\infty)$.

Theorem 16. A subset A of R is an interval iff it is one of the nine forms: (a, b); $[a, b)$; $(a, b]$; $[a, b]$; $(-\infty, a)$; $(-\infty, a]$; $(a, +\infty)$; $[a, +\infty)$; $(-\infty, +\infty)$.

Proof. By Definitions 14 and 15, the given nine forms are intervals. For the "only if" part of the theorem, suppose A is an interval. We first note that if a point $x \notin A$, then either x is a lower bound of A or an upper bound of A, for otherwise there would be points a, $b\in$A with $a<x<b$ and we would obtain the contradiction $x\in$A. We shall, consequently, distinguish four cases.

Case 1. A has neither an upper bound nor a lower bound. In this case C(A) $=\phi$ so that A$=(-\infty, +\infty)$.

Case 2. A has an upper bound but no lower bound. Since an interval is non-empty, A has a least upper bound a. We claim that if $x<a$, then $x\in$A. For, suppose $x<a$, then there is a point $a'\in$A with $x<a'\leq a$ (for otherwise a would not be a least upper bound). Since, by hypothesis, x cannot be a lower bound of A, there is a point (arbitrary) $b\in$A with $b<x$. Hence we have $b<x<a'\leq a$, and a', $b\in$A imply that $x\in$A. We have thus shown that $(-\infty, a)\subset$A. On the other hand, for $x>a$, $x\notin$A. It follows that A is either of the form $(-\infty, a]$ or $(-\infty, a)$, depending on whether $a\in$A or $a\notin$A (i.e. for $x\leq a$ or $x<a$).

Case 3. A has a lower bound but no upper bound. By reasoning similar to that of Case 2, we can show that A is either of the form $[a, +\infty)$ or $(a, +\infty)$, where a is the greatest lower bound of A.

Case 4. A has a lower bound and an upper bound. Let a be the greatest lower bound of A and b the least upper bound of A. Since A contains at least two distinct points, $a<b$. If $x\in$A, then x must $\in[a, b]$, so that A$\subset[a, b]$. We claim that $a<x<b$ implies that $x\in$A. This implication follows from the fact that for any such point x, there must be points $a'\in$A and $b'\in$A, and $a\leq a'<x<b'\leq b$. Hence, $(a, b)\subset$A$\subset[a, b]$. Consequently, A must be one of the four forms (a, b), $[a, b)$, $(a, b]$, or $[a, b]$, depending on which, if any, of the two points a, b belongs to A.

Theorem 17. A subset A$\subset$R, and A containing at least two distinct points is connected iff it is an interval.

Proof. If A is not an interval, then there are points a, b, c with $a<c<b$ and a, $b\in$A, whereas $c\notin$A. Let P$=(-\infty, c)$, Q$=(c, +\infty)$. P and Q are open in the set of all real numbers R that satisfy the conditions of Lemma 9, namely, A$\subset$P$\cup$Q, P$\cap$Q$\subset$C(A), P$\cap$A$\neq\phi$, Q$\cap$A$\neq\phi$; thus A is not connected. Therefore if A is connected, then A is an interval.

Conversely, if A is not connected, by Lemma 10, there are closed subsets F and G of R such that A$\subset$F$\cup$G, F$\cap$G$\subset$C(A) and F$\cap$A$\neq\phi$, G$\cap$A$\neq\phi$. Assume that the notation is such that there is a point $a\in$A$\cap$F and a point $b\in$A$\cap$G with $a<b$. At first, let G$'=$G$\cap$[a, b]. Then G$'$ is a closed non-empty subset of R, and, consequently, contains its greatest lower bound c. We cannot have $a=c$, for $c\in$G if $a=c$ then (A$\cap$F)$\cap$G$=$A$\cap$(F$\cap$G)$\neq\phi$ contradicting F$\cap$G$\subset$C(A). Thus, $a<c$. Next, let F$'=$ F$\cap[a, c]$. F$'$ is also a closed non-empty subset of R and therefore contains its least upper bound d. (1) If $c=d$, then $c\in$F$\cap$G, F$\cap$G$\subset$C(A) implies $c\in$C(A), hence $c\notin$A and A is not an interval. (2) Otherwise $d<c$ and $(d, c)\cap$(F$\cup$G)$=\phi$. Since $d=$l.u.b. of F$'=$F$\cap[a, c]$, then $d=x\notin$F, i.e. $(d, c)\cap$F$=\phi$; $c=$g.l.b. of G$'=$G$\cap[a, b]$, then $c>x\notin$G, i.e. $(d, c)\cap$G$=\phi$. Thus $(d, c)\cap$(F$\cup$G)$=\phi$, and A$\subset$F$\cup$G, $(d, c)\cap$A$\neq\phi$.

Again, A does not contain a point e between d and c, and d, c, are between a and b, which means that A does not contain a point e between a and b, and it is therefore not an interval. That is, if A is an interval, then A is connected.

II BROUWER FIXED-POINT THEOREM, ONE DIMENSIONAL VERSION

The fixed-point theorem, first proved by the Dutch mathematician L. E. J. Brouwer, is called Brouwer fixed-point theorem.

Theorem 18. (Intermediate-Value Theorem). Let $f:[a,b]\to R$ be a continuous function and let $f(a)\neq f(b)$, where $[a,b]$ is a closed interval and R a set of all real numbers. Then for each number C between $f(a)$ and $f(b)$ there is a point $c\in[a,b]$ such that $f(c)=C$.

Proof. By Theorem 17, $[a,b]$ is connected, and f is continuous, hence $f([a,b])$, by Theorem 13, is connected. Now, $f(a), f(b)\in f([a,b])$. Thus $f([a,b])$ is an interval (Theorem 17), and if C is between $f(a)$ and $f(b)$, then $C\in f([a,b])$; that is, there is a point $c\in[a,b]$ such that $f(c)=C$.

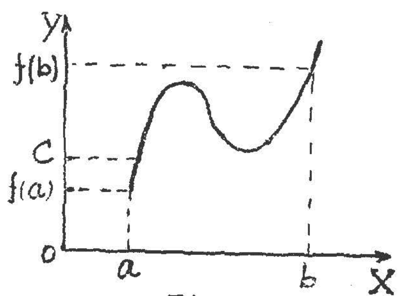


Fig. 4

Theorem 18 states that for each C between $f(a)$ and $f(b)$, the horizontal line $y=C$ intersects the curve of $y=f(x)$ at least at one point (c, C) with $a<c<b$, as shown in the figure.

As a special case of Theorem 18, let $C=0$, then we have

Corollary 19. Let $f:[a,b]\to R$ be continuous. If $f(a)f(b)<0$, i.e. $f(a)$ and $f(b)$ have different signs, then there is a $c\in[a,b]$ such that $f(c)=0$.

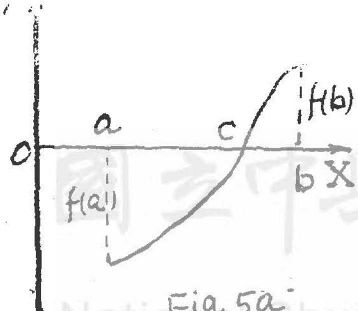


Fig. 5a

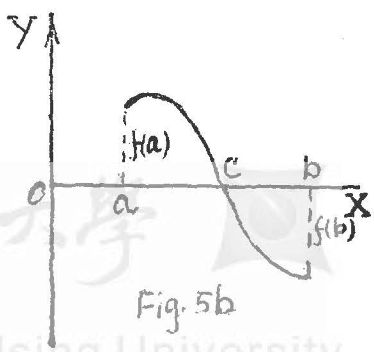


Fig. 5b

Corollary 20. (The one-dimensional version of Brouwer Fixed-Point theorem). Let $f:[0,1]\to[0,1]$ be a continuous function. Then there is a $z\in[0,1]$ such that $f(z)=z$.

Proof. For $f(0)=0$ or $f(1)=1$, the theorem is certainly true. Thus, it suffices to consider the case in which $f(0)>0$ and $f(1)<1$. Let $g:[0,1]\to R$ be a function defined by $g(z)=z-f(z)$, then, if $g(z)=0$, $f(z)=z$. g is continuous and $g(0)=-f(0)<0$, whereas $g(1)=1-f(1)>0$, i.e. $g(0)g(1)<0$. Hence, by Corollary 19, there is a $z\in[0,1]$ such that $g(z)=0$, or $z-f(z)=0$, therefore $f(z)=z$.

We may illustrate this corollary by a paper clip. Suppose we stretch a piece of wire to its full length to form a straight line. Let this length of the wire be 1. Next we fold the wire to make a paper clip. Now, it can be shown that a point

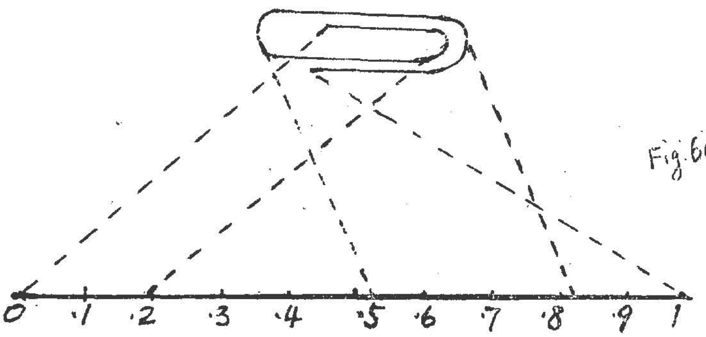


Fig. 6a

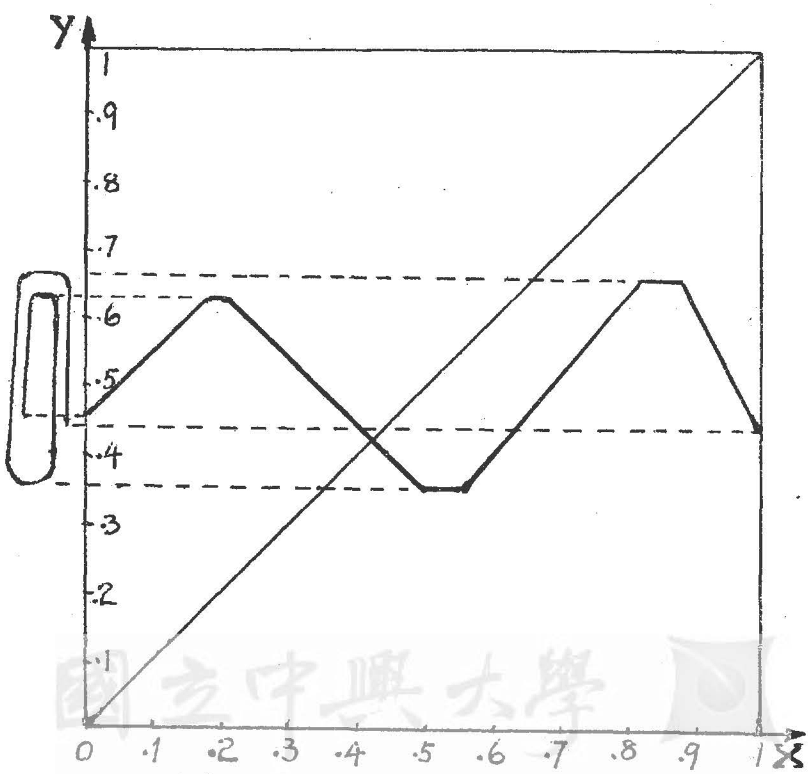


Fig. 6b

on the wire has returned to the exact spot it occupied before the manipulation and is therefore a fixed point.

The theorem is illustraled by representing both the original straight wire and the paper clip as curves on a graph, comparing the curves and demonstrating that they intersect at some point (Fig. 6b). To start, we divide the original straight wire of length 1 into 10 equal parts. Let the left end be the origin and specify each point of partition on the wire by its distance from the origin. By the same token the position of each point on the clip can be specified by its new distance from the origin. Thus, each point on the straight wire has a new position on the clip. The points on the straight wire constitute the domain $[0, 1]$ of the function $f:[0, 1]\to[0, 1]$. The new positions to which these points have been moved constitute the range $[0, 1]$ of f. Since there is no break in the clip, f is continuous.

Let $x\in[0, 1]$, and $y=f(x)$. The curve of $y=f(x)$ lies entirely within the unit square defined by $0\leq x\leq 1$, $0\leq y\leq 1$, since $f:[0, 1]\to[0, 1]$. It is represented by the jagged curve in Fig. 6b.

Suppose we have picked up the straight wire and returned it vertically with left end at origin. Then each point on the original straight wire has a new position. If we plot the new distances from the left end against the old, we get a straight line that is equidistant from the two axes, and whose equation is $y=x$. This is the diagonal of the unit square. We can consider that each point on the line $y=x$ is a fixed one. Thus, by Fig. 6b, the jagged curve of the paper clip that intersects the line has a fixed point, point of intersection. If we move the paper clip up and down within $[0, 1]$, in Fig. 6b, the new curves of the paper clip will be constructed. Each of the new curves, however, evidently still intersect the diagonal of the unit square in Fig. 6b. If z is the abscissa of any one of the points of intersection, its ordinate will be $f(z)$, and $f(z)=z$.

In the next theorem we will prove that if X and Y are homeomorphic topological spaces, Definition 5, and the fixed-point theorem is true for X, then it is also true for Y.

Theorem 21. Let X and Y be homeomorphic topological spaces. Then each continuous function $h:X\to X$ possesses a fixed point iff each continuous function $k:Y\to Y$ possesses a fixed point.

Proof. Let $f:X\to Y$ and $g:Y\to X$ be continuous inverse functions. Let $h:X\to X$, and $k:Y\to Y$ be both continuous functions, and suppose that $h:X\to X$ possesses a fixed point. Thus, we have the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "k"] \\ X & Y \arrow[l, "g"'] \end{tikzcd}$$

Then the function $h=gkf: X \to X$ is continuous and there is a $z \in X$ sach that $h(z)=z$. Let $y=f(z)$. We have

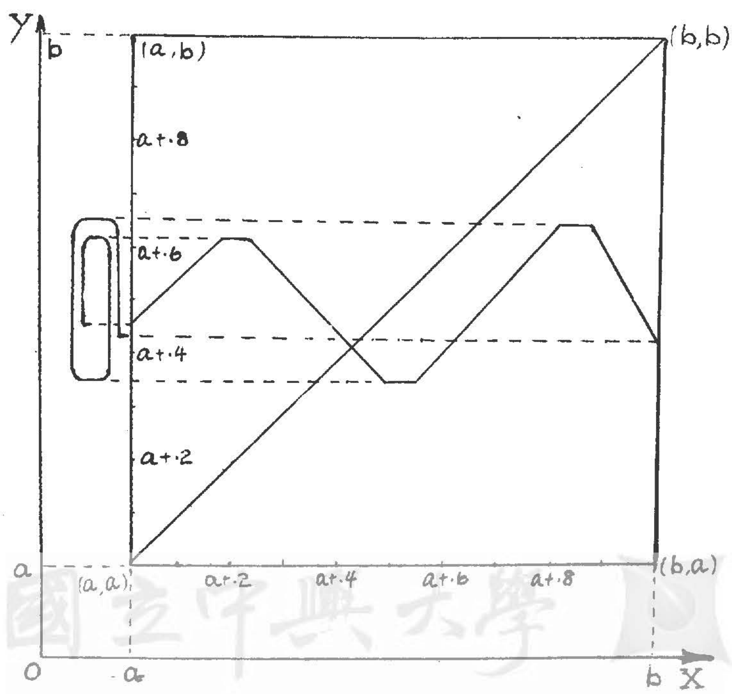
$$k(y)=k(f(z))=fg(k(f(z)))=f(gkf(z))=f(h(z))=f(z)=y.$$

Thus, y is a fixed point of k. Since the hypotheses are symmetric with regard to X and Y; so, conversely, if each continuous function $k: Y \to Y$ has a fixed point then so does each continuous function $h: X \to X$.

Any two closed intervals are homeomorphic. By Corollary 20, we get

Corollary 22. Suppose that $f: [a, b] \to [a, b]$ is continuous. Then there is a z such that $f(z)=z$. (The another one-dimensional version of Brouwer Fixed-Point Theorem).

This corollary can also be illustrated by a paper clip as shown in Fig. 7. This is very similar to that we did in the illustration for Corollary 20. The only



Fig·7

difference is that the original coordinate system was moved to a new position and "b−a" considered as of length 1.

In addition to the above illustrations, we use an clastic material such as a rubber string instead of a paper clip. Then we can have a further interesting demonstration for Corollary 20.

Suppose that there is a piece of rubber string whose length is 1 in natural condition, i.e. there is no tension in the rubber string. Furthermore, suppose that the rubber string can be made a straight line without tension in it. Divide it into 10 parts as we did for the straight wire in the preceding illustration for Corollary 20. Then let this straight rubber string be taken as the coordinate axes (namely, X- and Y- axis) in a Cartesian plane in the same manner as the straight wire stood in the preceding illustration.

However, in natural condition, the straight rubber string is likely to change its shape into a curved form as shown in Fig. 8. In this figure every new position of each point of the original straight string is shown by the original point with a prime added to the right shoulder.

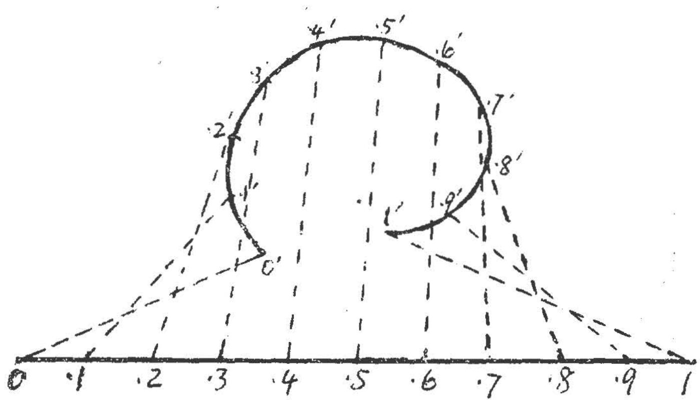


Fig. 8

It has been understood that the diagonal of the square in Fig. 9. is constituted by the set of fixed points. The curve in the square corresponding to the deformed rubber string intersects the diagonal at a point. Thus, there is still a fixed point on the rubber string after deformation. If $y=f(x)$ is the equation of the curve within the square, then $x\in[0, 1]$, $y\in[0, 1]$, and there must be an $x=z$ such that $f(z)=z$. It is evident that if we move the corved form of rubber string any where within the range [0, 1] the same conclusion will be produced.

In a similar manner, we can demonstrate Corollary 22. (comparing Fig. 9 with

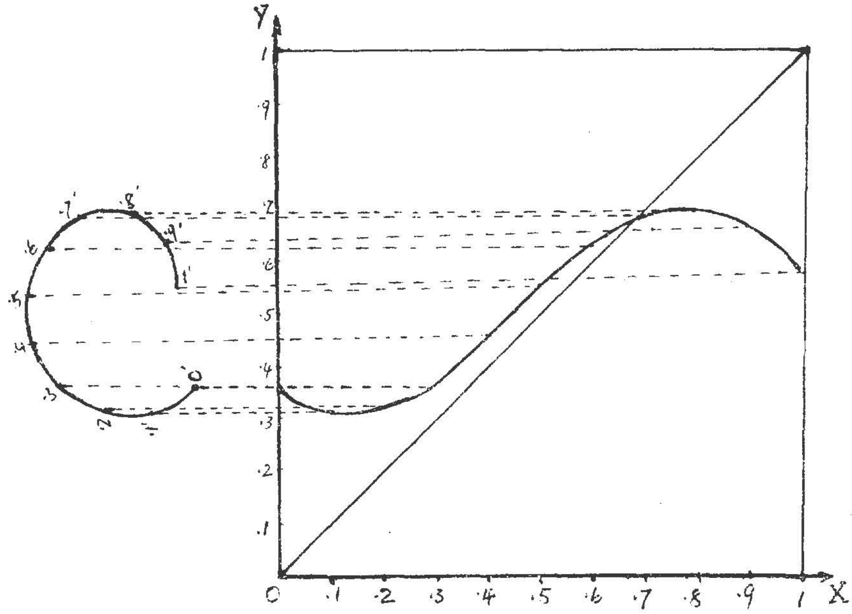


Fig. 9

Fig. 6b and 7).

Instead of the natural condition of the rubber string, we pull it from some point, say point .6, upward to a position, say .6′, with its two ends fixed at two points whose distance ≤ 1. Then, each point of the straight rubber string has a new position as shown in Fig. 10. The continuous curve in Fig. 11 intersecting the diagonal of the square tells us that there is still a fixed point in the deformed rubber string. In this manipulation of pulling, the rubber string was stretched not broken. That is, the function $f:[0,a]\rightarrow[0,1]$ defined by the equation of the curve $y=f(x)$ is continuous, where $a\geq 1$, $a=1$ if the rubber string is in natural condition, $a>1$ if there is any pulling applied on the rubber string to make it's length longer.

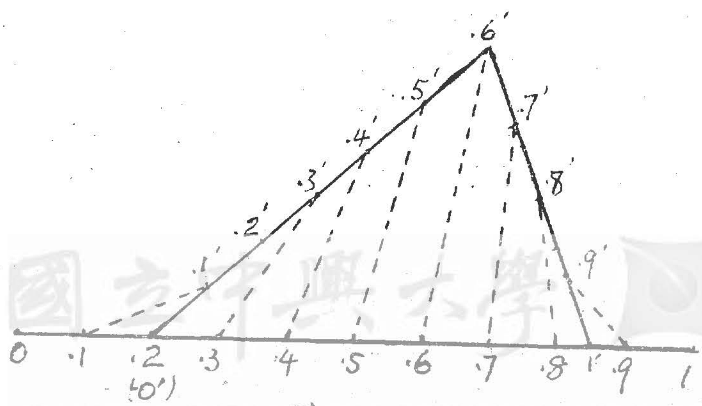


Fig. 10

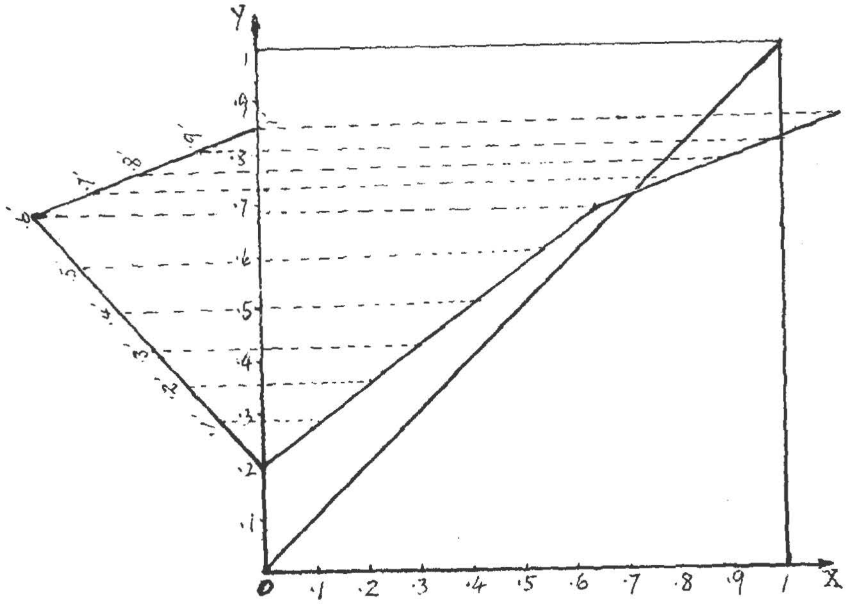


Fig. 11

Ⅲ THE HOMEOMORPHISM

According to Theorem 21, the homeomorphism between two topological spaces is closely related to the property of fixed point of mapping. So it is worth for saying something about its nature, in order to know when a homeomorphism between two topological spaces is defined.

Theorem 23. Let X, Y, and Z be topological spaces. Let $f: X \to Y$ be continuous at the point $a \in X$ and let $g: Y \to Z$ be continuous at the point $f(a) \in Y$. Then $gf: X \to Z$ is continuous at the point $a \in X$.

Proof. Let $\epsilon > 0$ be given. We must find a $\delta > 0$ such that whenever $x \in X$ and $d(x, a) = |x-a| < \delta$, then $d''(g(f(x)), g(f(a))) = |g(f(x)) - g(f(a))| < \epsilon$. Since g is continuous at $f(a)$, there is an $\eta > 0$, such that whenever $y \in Y$ and $d'(y, f(a)) = |y - f(a)| < \eta$, then $d''(g(y), g(f(a))) = |g(y) - g(f(a))| < \epsilon$. f is continuous at a, then we know that giveu $\eta > 0$, there is a $\delta > 0$, such that $x \in X$ and $d(x, a) < \delta$ imply that $d'(f(x), f(a)) < \eta$ and hence $d''(g(f(x)), g(f(a))) < \epsilon$. Therefore $gf: X \to Z$ is continuous at the point $a \in X$.

Corollary 24. Let X, Y and Z be three topological spaces. Let $f: X \to Y$ and $g: Y \to Z$ be continuous. Then $gf: X \to Z$ is continuous.

Theoaem 25. If X and Y are homeomorphic, X and any other topological space Z are homeomorphic, then Y and Z are homeomorphic.

Proof. Let $f: X \to Y$ and $g: Y \to X$ be inverse functions; Z be a topological space which combined with X forms a homeomorphism. Thus, let $h: X \to Z$ and $k: Z \to X$ be inverse functions.

By Definition 3, $g(f(a))=a$ and $f(g(b))=b$, or equivalently $f(a)=b$ and $g(b)=a$; $k(h(a))=a$ and $h(k(c))=c$, or equivalently $h(a)=c$ and $k(c)=a$; where $a \in X$, $b \in Y$, and $c \in Z$.

If we want to show that $hg: Y \to Z$ and $fk: Z \to Y$ are inverse functions, it suffices to show that $fk(hg(b))=b$, $hg(fk(c))=c$.

But $$fk(hg(b))=fk(h(a))=fk(c)=f(a)=b,$$
$$hg(fk(c))=hg(f(a))=hg(b)=h(a)=c.$$

Hence hg and fk are inverse functions. Since k, f are continuous, by Corollary 24, fk is continuous. Similarly for g and h are continuous, hg is continuous.

Therefore, by Definition 5, Y and Z are homeomorphic.

Lemma 26. The open interval $(-\pi/2, \pi/2)$, considered as a subspace of the real number system, and the real number system are homeomorphic. Any two open intervals, considered as subspaces of the real number syetem, are homeomorphic. Any open interval, considered as a subspace of the real number system, and the real number system are homeomrphic.

Proof. Let us denote the real number system by R and consider the inverse functions defined by

$$f: (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow R$$

$$f^{-1}: R \longrightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$$

where $-\pi/2 < f^{-1}(x) < \pi/2$ for $x \in R$, and f is some trigonometric function, e.g., $f(x)=\tan x$. The continuity of these functions has been shown in calculus. Therefore $(-\pi/2, \pi/2)$ and R are homeomorphic.

Now consider any two open intervals (a, b) and (c, d) in R. An invertible function f which maps (a, b) into (c, d) can simply be taken as the one representing translation and magnification (or contraction) defined by

$$f(x)=(x-a)\frac{d-c}{b-a}+c, \quad x \in (a, b) .$$

The inverse of f can readily shown to be the function $g: (c, d) \to (a, b)$ defined by

$$g(y)=(y-c)\frac{b-a}{d-c}+a, \quad y \in (c, d)$$

The continuity of these inverse functions is evident. Thus, we have shown that any two open intervals considered as subspaces of R are homeomorphic.

Now, by Theorem 25, we see that any open interval in R and R are homeomorphic, since each of them and $(-\pi/2, \pi/2)$ are homeomorphic.

Lemma 27. Let X_i, Y_i, $i=1, 2, \cdots, n$, be topological spaces (metrizable). Let $f_i: X_i \to Y_i$, $i=1, 2, \cdots, n$, be continuous functions. Let

$$X = \prod_{i=1}^{n} X_i \quad \text{and} \quad Y = \prod_{i=1}^{n} Y_i.$$

Define the function $F: X \to Y$ by

$$F(x_1, x_2, \cdots, x_n) = (f_1(x_1), f_2(x_2), \cdots, f_n(x_n)),$$

then F is continuous.

Proof. Let $a=(a_1, a_2, \cdots, a_n)\in X$, and $\epsilon>0$ be given. Choose $\delta>0$ such that

$$d'(f(x_i), f(y_i))=|f(x_i)-f(y_i)|<\frac{\epsilon}{\sqrt{n}}$$

Suppose $x=(x_1, x_2, \cdots, x_n)$ is a point such that $d(x, a)=|x-a|<\delta$, that is,

$$d'(f(x_i)), f(y_i))<\epsilon \quad \text{as} \quad d(x, a)=\max_{1\leq i\leq n}.\{|x_i-a_i|\}<\delta,$$

and $|x_i-a_i|<\delta$ for $1\leq i\leq n$.

Hence $$d'(F(x), F(a))=\sqrt{\sum_{i=1}^{n}(f_i(x_i)-f_i(y_i))^2}<\sqrt{n\epsilon^2/n}=\epsilon \quad \text{as} \quad d(x, a)<\delta.$$

Therefore F is continuous.

Lemma 28. For $i=\overline{1, 2, \cdots, n}$, let the metrizable topological spaces X_i and Y_i are homeomorphic. Then the topological spaces

$$X=\pi_{i=1}^{n} X_i \quad \text{and} \quad Y=\pi_{i=1}^{n} Y_i$$

are homeomorphic.

Proof. Since X_i and Y_i are homeomorphic, $f_i: X_i\to Y_i$ and $g_i: Y_i\to X_i$ are inverse functions, i.e.

$$g_i(f_i(x_i))=x_i, \quad f_i(g_i(y_i))=y_i,$$

where $x_i\in X_i$, $y_i\in Y_i$.

Let $x=(x_1, x_2, \cdots, x_n)\in X$, $y=(y_1, y_2, \cdots, y_n)\in Y$.

Then $g(f(x))=g(f(x_1, x_2, \cdots, x_n))=(x_1, x_2, \cdots, x_n)=x$,

and $f(g(y))=f(g(y_1, y_2, \cdots, y_n))\in(y_1, y_2, \cdots, y_n)=y$.

Hence $f: X\to Y$ and $g: Y\to X$ are inverse functions.

By hypothesis, f_i and g_i are continuous, by Lemma 27, f is continuous, and g is also continuous, since the hypothesis imposes conditions symmetrical with regard to the two functions.

Therefore X and Y are homeomorphic.

Theorem 29. The open n-cube is the set of all points $x=(x_1, x_2, \cdots, x_n)\in R^n$ such that $0<x_i<1$ for $i=1, 2, \cdots, n$. Then the n-cube, considered as a subspace of R^n, and R^n are homeomorphic.

Proof. By Lemma 26, for $i=1, 2, \cdots, n$, $0<x_i<1$ and R are homeomorphic. In addition to this, by Lemma 28, the open n-cube (R^n for $0<x_i<1$), considered as a subspace of R^n, and R^n are homeomorphic.

Theorem 30. For each pair of points $a, b\in R^n$, there is a homeomorphism between R^n and itself defined by inverse functions $f: R^n\to R^n$ and $g: R^n\to R^n$ such that $f(a)=b$.

Proof. Let $a=(a_1, a_2, \cdots, a_n)$, $b=(b_1, b_2, \cdots, b_n)$

and let $f(x_1, x_2, \cdots, x_n)=(x_1+b_1-a_1, x_2+b_2-a_2, \cdots\cdots, x_n+b_n-a_n)$.

i.e. $f(x)=f(x_1, x_2, \cdots, x_n)=x+b-a$.

Let $\epsilon>0$ be given, and choose $\delta=\epsilon$.

Then $d(f(x), f(y))=|x+b-a-(y+b-a)|=|x-y|<\delta=\epsilon$,

whenever $d(x,y)<\delta$, for each $x, y \in R^n$. Hence f is continuous.
Let $g(x)=g(x_1, x_2,\cdots, x_n)=(x_1-b_1+a_1,\ x_2-b_2+a_2, \cdots\cdots,\ x_n-b_n+a_n)=x-b+a.$
Let $\epsilon>0$ be given and choose $\delta=\epsilon$, then

$$d(g(x), g(y))=|x-b+a-(y-b+a)|=|x-y|<\delta=\epsilon \quad \text{as} \quad d(x,y)<\delta.$$

Hence g is continuous. Given $f(a)=b$, $g(b)=b-b+a=a$, and, f and g are inverse functions.

Therefore R^n and itself are homeomorphic.

In R^n, let the unit n-cube I^n be defined as the set of points $(x_1, x_2,\cdots, x_n)$ whose coordinates satisfy the inequalities $0 \leq x_i \leq 1$ for $i=1, 2,\cdots, n$.

Comparing this with the n-cube in Theorem 29, we see that I^n is equal to the open n-cube plus the condition $0=x_i=1$, for $i=1, 2,\cdots, n$.

By Theorem 29, in comparison with Theorem 30, it is evident that I^n and R^n are hemeomorphic.

IV THE TWO DIMENSIONAL VERSION OF BROUWER FIXED-POINT THEOREM

In this section we are going to have a description of the fixed-point theorem as it applies to a plane, which is of course a two-dimensional surface.

Theorem 31. (the no-retraction theorem). There is no retraction (transformation) of a plane area onto it boundary. In other words, it is impossible to shift the interior points of a disc all to the circumference by a continuous transformation without breaking the disc.

Proof. Let the n-cell be the set of points in R^n satisfying the inequality

$$x_1^2+x_2^2+\cdots+x_n^2 \leq 1$$

and S^{n-1} (the $n-1$ sphere) the set of all n-tuples $(x_0, x_1,\cdots, x_n) \in R^n$ satisfying

$$x_1^2+x_2^2+\cdots+x_n^2=1$$

In a plane, $n=2$, $2-$cell is the set of points in R^2 satisfying $x_1^2+x_2^2 \leq 1$ which constitutes a disc whose circumference is $(S^{2-1}=)S^1$ satisfying $x_1^2+x_2^2=1$.

Let $d(x,y)=|x-y|$ be the distance between the points x and y of the disc. Let $\overset{\frown}{xy}$ be the length of the short arc of the circumference S^1 when x, y are on S^1. It is known that there are two arcs for each pair of points x, y on the circumference one of which is shorter the other is longer. The one which is shorter is called short arc. According to the direction of x to y being counterclockwise or clockwise, the sign of $\overset{\frown}{xy}$ is positive or negative respectively. We know that the short arc $\overset{\frown}{xy}$ is alway less than the twice of its chord, $d(x,y)$. i.e. (1) $|\overset{\frown}{xy}|<2d(x,y)$.

Since the radius of S^1 is 1, the polar coordinate of any point, say x, is $(1,\theta)$. It is understood that each polar coordinate contains 1. In order to simplify the procedure of proof, let θ be used as the coordinate of x instead of $(1,\theta)$.

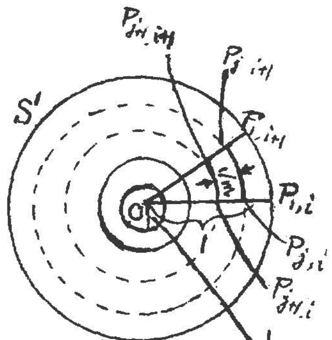


Fig. 12

If $x, y \in S^1$ and $d(x, y) < 2$, which means that x and y are not the end points of a diameter of S^1. Let the angle between ox and oy be $\Delta\theta$, then $\Delta\theta = \frac{\widehat{xy}}{r} = \widehat{xy}$, since $r = 1$. Thus the polar coordinate of y is

(2) $\theta + \Delta\theta = \theta + \widehat{xy}$

Let all $x_1, x_2, \cdots, x_n, x_i \in S^1$, and $d(x_i, x_{i+1}) < 2$, for $i = 1, 2, \cdots, n$. Then,

(3) $\widehat{x_1x_2} + \widehat{x_2x_3} + \cdots\cdots + \widehat{x_{n-1}x_n} + \widehat{x_nx_1} = 2p\pi$

where p is an integer. Since, by (1), if θ_1 is a polar coordinate of x_1, then $\theta_1 + \widehat{x_1x_2} + \widehat{x_2x_3} + \cdots\cdots + \widehat{x_{n-1}x_n} + \widehat{x_nx_1}$ will be also a polar coordinate of x_1 the difference of them, $\widehat{x_1x_2} + \widehat{x_2x_3} + \cdots\cdots + \widehat{x_{n-1}x_n} + \widehat{x_nx_1}$, is $2p\pi$.

Let C^2 be the subset of R^2 such that $x_1^2 + x_2^2 < 1$.

Suppose that there is a continuous retraction g of this n-cell onto S^1. By the definition of continuity of a function, for a $\pi/8$ there is a δ such that,

$$d(g(x), g(y)) = |g(x) - g(y)| < \pi/8$$

whenever $d(x, y) < \delta$, for $x, y \in C^2$. (4) For $x \in S^1$, $g(x) = x$.

For $x, y \in C^2$, by (1), (5) $|\widehat{g(x)g(y)}| < 2d(g(x), g(y)) < \pi/4$ as $d(x, y) < \delta$.

Dividing S^1 into n equal parts by n points of partition $P_{1,0}, P_{1,1}, P_{1,2}, \cdots, P_{1,n}$, let $P_{1,n}$ be coincided with $P_{1,0}$. Let n so large such that

(6) $d(P_{1,i}, P_{1,i+1}) > \delta$.

Let 0 be the center of the disc, drawing m concentric circles, abont 0, we get the radius of the j+1 th circle $= 1 - j/m$.

Let $P_{j,i}$, $P_{j,i+1}$ $P_{j+1,i}$, and $P_{j+1,i+1}$ be four points of intersection of the j th, j+1 th circles and the two arbitrary adjacent radii as shown in Fig. 12.

The difference of the radii of any two adjacent circles is $1/m$. We have, thus,

$$d(P_{j,i}, P_{j+1,i}) = d(P_{j,i+1}, P_{j+1,i+1}) = 1/m.$$

By the continuity of g in C^2, we can choose $m > M$ and $n > N$ such that, for $1 \leq j \leq m$, $1 \leq i \leq n$,

(7) $|\widehat{g(P_{j,i})\, g(P_{j+1,i})}| < \pi/2n$, $|\widehat{g(P_{j,i+1})\, g(P_{j+1,i+1})}| < \pi/2n$.

By Fig. 12 and (6),

$$d(P_{j,i}, P_{j+1,i}) < d(P_{1,i}, P_{1,i+1}) < \delta$$
$$d(P_{j,i+1}, P_{j+1,i+1}) < d(P_{1,i}, P_{1,i+1}) < \delta.$$

Again, by (5),

$$(8) \quad |\widehat{g(P_{j,i})\, g(P_{j,i+1})}| < \pi/4, \quad \text{and} \quad |\widehat{g(P_{j+1,i})\, g(P_{j+1,i+1})}| < \pi/4.$$

$g(P_{j,i})$, $g(P_{j+1,i})$, $g(P_{j+1,i+1})$, and $g(P_{j,i+1})$ are respectively the four imagines of $P_{j,i}$, $P_{j+1,i}$, $P_{j+1,i+1}$ and $P_{j,i+1}$ on S^1, by (3),

$$(9) \quad \widehat{g(P_{j,i})g(P_{j+1,i})} + \widehat{g(P_{j+1,i})g(P_{j+1,i+1})} + \widehat{g(P_{j+1,i+1})g(P_{j,i+1})}$$
$$+ \widehat{g(P_{j,i+1})g(P_{j,i})} = 2p\pi$$

where p is an integer.

According to the inequalities (7) and (8), the absolute value of each term on the left side of (9) is less than $\pi/4$, the sum of them must be less than π. Thus, the integer p of (9) must be 0. i.e., by changing signs,

$$-\widehat{g(P_{j+1,i})g(P_{j,i})} + \widehat{g(P_{j+1,i})g(P_{j+1,i+1})} - \widehat{g(P_{j,i+1})g(P_{j+1,i+1})}$$
$$-\widehat{g(P_{j,i})g(P_{j,i+1})} = 0$$

or by (7),

$$(10) \quad |\widehat{g(P_{j+1,i})g(P_{j+1,i+1})} - \widehat{g(P_{j,i})g(P_{j,i+1})}|$$
$$= |\widehat{g(P_{j+1,i})g(P_{j,i})} + \widehat{g(P_{j,i+1})g(P_{j+1,i+1})}|$$
$$\leq \frac{\pi}{2n} + \frac{\pi}{2n} = \frac{\pi}{n}$$

Let $S_{j+1} = \sum_{i=1}^{n} \widehat{g(P_{j+1,i})g(P_{j+1,i+1})}$, where $j = 1, 2, \cdots, m$.

By (3), $S_{j+1} = 2q\pi$, p is an integer, in the same manner, $S_j = 2q\pi$, q is an integer, and $q-p$ is also an integer.

Thus, by (10), $S_j - S_{j+1} = 2(q-p)\pi = n \times \frac{\pi}{n} = \pi$.

Hence $q-p=0$, cr $q=p$ i.e. $S_{j+1} = S_j$ for $j = 1, 2, \cdots, m$.

$\therefore$ (11) $S_1 = S_2 = \cdots\cdots = S_m$.

From Fig. 12, evidently, $S_1 = 2\pi$. However, we can choose $m > M_1$ to make that the radius of the mth circle (smallest circle), $1/m$, less than an infinitesimal η, such that the distance between each two adjacent points on the mth circle

$$d(P_{m,i}, P_{m,i+1}) < \delta_1$$

and under this condition, we have

$$\widehat{g(P_{m,i})g(P_{m,i+1})} < \frac{\pi}{n}$$

for $i = 1, 2, \cdots, n$. Thus

$$S_m < n \times \frac{\pi}{n} = \pi.$$

But $S_m = 2p\pi$, p is an integer, p must be equal to 0, i.e. $S_m = 0$. On the other hand,

$$S_m = S_1 = 2\pi.$$

This contradictory fact tells us that the no-retraction theorem is true.

Definition 32. If, following a certain transformation (or mapping) $f: A \to B$, the distance between the two points $f(x)$ and $f(y)$ is always strictly less than the distance between the original points x and y then the transformation is called a contraction, where $x, y \in A$.

According to the discussion after the proof of Theorem 30, I^n and R^n are homeomorphic; and by Theorem 21, there is no loss of generality in replacing R^n by I^n in the description of Fixed-Point Theorem. Thus, for $n=2$, we have

Theorem 33. (The two dimensional version of Brouwer FIXED-POINT THEOREM). Let $f: I^2 \to I^2$ be continuous. Then there is a point $z \in I^2$ such that $f(z)=z$.

Suppose we think of I^2 as being a surface (plane) constructed of elastic material such as rubber, we may conceive of a deformation or stretching by which we obtain a surface that is a disc; that is, the set of points (x_1, x_2) in the plane whose coordinates satisfy the inequality $x_1^2+x_2^2 \leq 1$. Thus, the disc (2-cell, Th. 31) is homeomorphic with I^2, and we may argue the validity of the fixed point theorem with regard to the disc.
Proof (1) in concerning a continuous contraction f.

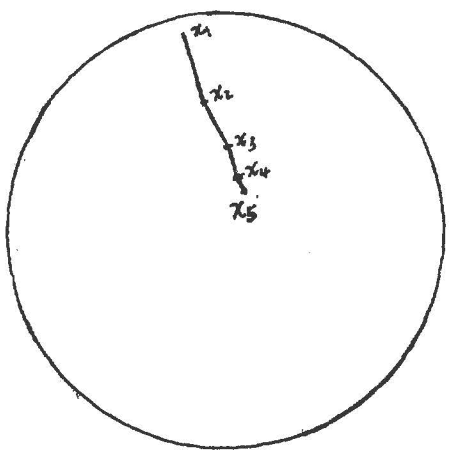


Fig. 13

When a contraction takes place, any point x_1 on the original plane assumes a new position x_2. The point we just designated x_2 occupies the position originally occupied by a point that we say has moved to x_3. This point in turn now occupies the position originally occupied by a point we say has moved to x_4; and so on. So far as we know that the transformation f under consideration is a contraction, the distance between x_2 and x_3 must be less than the distance between x_2 and x_1 (see Definition 32). In the same manner, the distance between x_4 and x_3 is less than the distance between x_3 and x_2, and so on. Thus, we obtain a sequence of points, $x_1, x_2, x_3, \cdots$, which get closer and closer together (see Fig. 13). This implies that the sequence must have a limit, which means only that all these points get closer and closer to some one point on the plane. It is evident that this limiting point is a fixed point for the transformation f. In addition to this, this fixed point is unique. Suppose x and y were two different fixed points of the contract f. If this were the case, we should have $x=f(x)$ and $y=f(y)$. Since these are fixed points, the distance between them should be the same as the distance between $f(x)$ and $f(y)$. But the distance between $f(x)$ and $f(y)$ must, by Definition 32, be strictly less than the distance between x and y. This contradiction, calling for the distance between x and y to be less than itself, shows that the original assumption that x and y are two different fixed point is untenable and thus proves that the fixed point is unique.

Proof (2) in general.

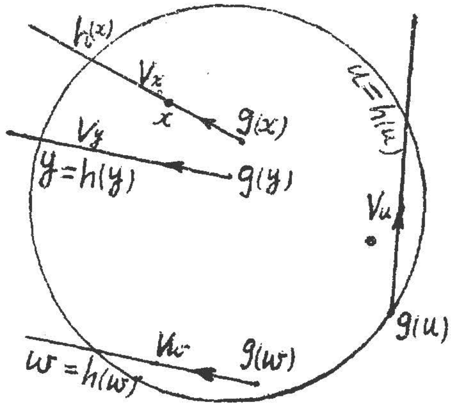


Fig. 14

Let g be a continuous transformation of the disc (shown in Fig. 14) into itself. Suppose that it were possible for each point x of the disc, we had $g(x)\neq x$. Then for each arrow-line (called transformation vector) V_x emanating from $g(x)$ and passing through x (see Fig. 14). The arrow V_x will contain a point, say $h(x)$, on the boundary of the disc other than $g(x)$. In particular, if y is a boundary point of the disc, then $h(y)=y$. (see (4), the proof of Th. 31); if even w is on the boundary, then $h(w)=w$. This is true even if $g(y)$ itself is a boundary point, as may be seen by the case $u=h(u)$ depicted in Fig. 14. Using the given transformation g we have thus constructed a new transformation h, which has the property that it carries each point of the disc into boundary point and leaves each boundary point fixed (h is called a "retraction" as shown in Th. 31).

We next assert that the transformation h is continuous, since the image $h(x)$ will vary by a small amount if we suitably restrict the variation of x. By theorem 31, there is no function such as the retraction h, we have obtained a contradiction, and therefore the supposition that g did not have a fixed point is untenable.

Ⅴ BROUWER FIXED-POINT THEOREM AT *n* DIMENSIONAL SURFACES

The Homotopy. The set of points on and between the two concentric circles, as shown in Fig. 15, is called an annulus. It is easy to see that this annulus is arcwise connected. For example, given two points $p_0(x_0, y_0)$ and $p_1(x_1, y_1)$, one may construct a path from p_0 to p_1 by first traversing the radius on which p_0 lies until we reach a point whose distance from the origin is the same as that of p_1 and then traversing in a clockwise direction the circlar arc from this point to p_1 (see Fig. 15). Let this path be F_0. In addition to this, one may construct a second path, say F_1, from p_0 to p_1, by first traversing in a clockwise direction a circular arc from p_0 to the radius on which p_1 lies and then traversing this radius until p_1 is reached.

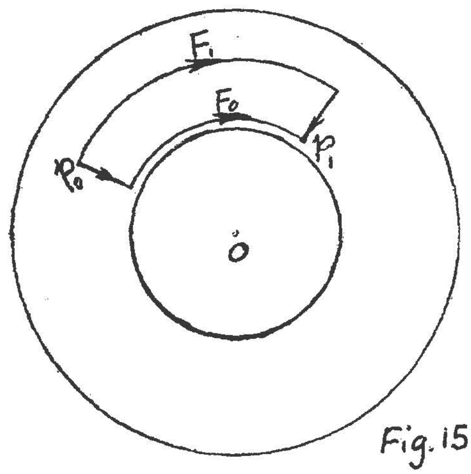


Fig. 15

If, in the meanwhile, we think of each of these two paths F_0 and F_1 as being represented by elastic strings with initial point p_0 and terminal point p_1, it is clear that in a given unit of time it would be possible to smoothly deform the path F_0 into the path F_1 (Keeping p_0 and p_1 fixed throughout the deformation). This deformation might be carried out so that at time $t=\frac{1}{4}$ the string lies over $F_{1/4}$, and at time $t=\frac{1}{2}$ the string lies over $F_{1/2}$, and at time $t=\frac{3}{4}$ the string lies over $F_{3/4}$ (see Fig. 16). We may thus think of the deformation of the path F_0 into

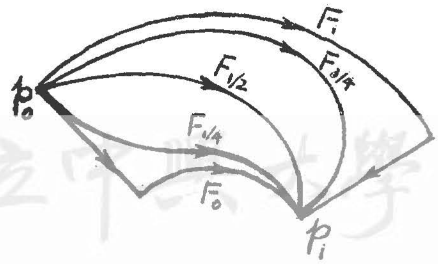


Fig. 16

the path F_1 as being accomplished by constructing an entire family of paths F_t for $0 \leq t \leq 1$, such that if t and t′ are close then the paths F_t and $F_t{}'$ are close also.

The concept of regarding two paths as being "close" implies the introduction of some sort of topology in this set of paths. Although this topology might be introduced directly by defining open set or neighborhood in the set of paths, an easier procedure is first to regard the unit of time as a unit interval on a line. Instead of viewing the two original paths F_0 and F_1 as being defined on the same unit interval, let us view F_0 as temporarily being defined on the homeomorphic image of the unit interval I_0, where I_0 is the set of points $(x, 0)$ in the plane with $0 \leq x \leq 1$ (see Fig. 17). Similarly, let us view F_1 as being defined on I_1, where I_1 is the set of points $(x, 1)$,$0 \leq x \leq 1$. For each value of t, $0 \leq t \leq 1$, we may view the path F_t as being defined on the homeomorphic image of the unit interval I_t, where I_t is the set of points (x, t), $0 \leq x \leq 1$. If we have such a situation, we may define a function $H: I^2 \rightarrow X$, where I^2 is the unit square and X is a part of our annulus (see Fig. 17), by setting

$$H(x, t) = F_t(x, t).$$

Equivalently, if we insist on viewing each path F_t as being defined on the same interval I, we may still obtain the same function H by setting

$$H(x, t) = F_t(x).$$

We now introduce the concept of closeness amongst paths by requiring that the funcion $H: I^2 \rightarrow X$ be continuous.

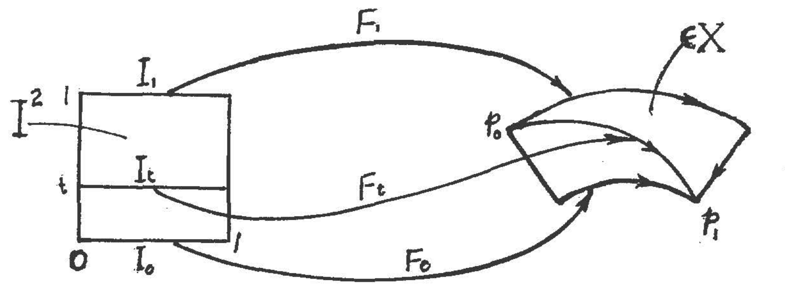


Fig. 17

Definition 34. Let F_0, F_1 be two paths in a topological space X with the initial point $p_0 = F_0(0) = F_1(0)$ and the same terminal point $p_1 = F_0(1) = F_1(1)$. F_0 is said to be homotopic to F_1 if there is a continuous function $H: I^2 \rightarrow X$ such that

$$H(0, t) = p_0 \quad (= F_0(0) = F_1(0)), \quad 0 \leq t \leq 1,$$
$$H(1, t) = p_1 \quad (= F_0(1) = F_1(1)), \quad 0 \leq t \leq 1,$$
$$H(x, 0) = F_0(x), \quad 0 \leq x \leq 1,$$
$$H(x, 1) = F_1(x), \quad 0 \leq x \leq 1.$$

The function H is called a homotopy from F_0 to F_1.

In this event we say that the path F_0 is deformable into the path F_1 with fixed end points. One may illustrate the fact a path F_0 is homotopic to F_1 by indicating that I^2 is the domain of the homotopy H, where the boundary of I^2 is mapped in agreement with conditions of Definition 34 as shown in Fig. 18.

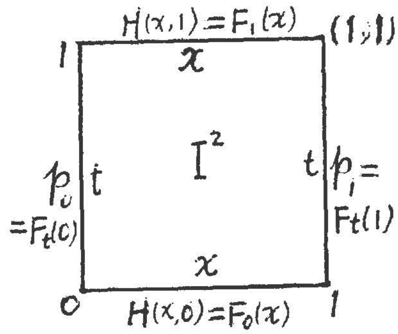


Fig. 18

The meaning of I^n defined after the proof of Theorem 30 indicates that $0 \leq x_i \leq 1$ for $i=1, 2, \cdots, n$. By this property, we see that I^n is a set of n-tuples $(x_1, x_2, \cdots, x_n)$ whose coordinates satisfy $x_1^2+x_1^2+\cdots+x_n^2 \leq 1$. Thus, I^n is homeomorphism with n-cell which was defined in the proof of Theorem 31. According to the discussion after the proof of Theorem 30, I^n is homeomorphic to R^n. Hence, by Theorem 25, an n-cell is homeomorphism with R^n. That is,

Lemma 35. n-cell and R^n are homeomorphic.

Definition 36. Consider two $(n-1)$-spheres S^{n-1} and S_1^{n-1} and a continuous mapping $f: S^{n-1} \to S_1^{n-1}$. With every such mapping f we associate an integer $\rho(f)$, called the degree of f (deg(f)). Intuitively, the degree $\rho(f)$ is the algebraic number of times that the image $f(S^{n-1})$ wraps around S_1^{n-1}, where S^{n-1} is defined as in the proof of Theorem 31.

Theorem 37. The degree (Def. 36) of a continuous mapping f of an $(n-1)$-sphere S^{n-1} into an $(n-1)$-sphere S_1^{n-1} depends only upon the homotopy class(see p.p. 152 ~153, Topology, John G. Hocking and Gall S. Young, 1961) of f.

This means that any two homotopic mappings f and g of S^{n-1} into S_1^{n-1} have the same degree. The converse theorem was proved by Hopf (see p.p. 119~122, Combinatorial Topology, Volume 3, P.S. Aleksandrov, 1960), i.e. if f and g are two mappings of S^{n-1} into S_1^{n-1} and if deg(f)=deg(g), then f and g are homotopic.

The no-retraction theorem of two dimensional surface has been shown in Theorem 31. In addition to this, we are going to have the same theorem of n-cell.

Theorem 38. There is no retraction of an n-cell onto its boundary for $n>0$.

Proof. According to Lemma 35, there is no loss of generality in taking the n-cell to be the set of points in R^n satisfying the inequality $x_1^2+x_2^2+\cdots\cdots+x_n^2\leq 1$ whose boundary is the sphere S^{n-1}, these have been defined in the proof of Theorem 31. Suppose that there is a retraction g of this n-cell onto S^{n-1}. Define the mapping

$$H(x,t)=g((1-t)\cdot x),\quad x\in S^{n-1},$$

where $0\leq x\leq 1$. Let $t=1$, we have $H(x,1)=g(0,x)=g(0)$ for each point $x\in S^{n-1}$, so $h(x,1)$ is a constant maping of S^{n-1} onto the point $g(0)$. But, let $t=0$, we have $H(x,0)=g(x)=x$ is the identity mapping of S^{n-1} cnto itself. Thus $H(x,t)$ is a homotopy between a constant mapping which has degree zero and the identity mapping which has degree 1. By Theorem 37, this is impossible, so the retraction g cannot exist.

Theorem 39. Given any continuous mapping f of an n-cell into itself, there is at least one point $z\in$ (n-cell) such that $f(z)=z$.

Proof. Let the n-cell be defined as in the proof of Theorem 31. Now suppose there is a mapping f of this n-cell into itself which has no fixed point. For each point x in this n-cell, let V_x be the arrow-line (transformation vector) from $f(x)$ to x. Since there is no fixed point, there is a unique arrow-line V_x for each point x in the n-cell, and hence a unique point in $V_x\cap S^{n-1}$. Let g be the mapping defined by $g(x)=V_x\cap S^{n-1}$. That is, we map x onto $f(x)$ and then back along V_x until we meet S^{n-1}. It is easy to see that g is continuous, and clearly $g(x)=x$ for each point x in S^{n-1}. This means that g is a retraction of the n-cell onto its boundary, which contradicts Theorem 38. Thus the unique arrow line V_x cannot exist as claimed and there must be a fixed point.

By Lemma 35, n-cell and R^n are homeomorphic. Thus, from Theorem 39, by Theorem 21, we have finally

Theorem 40. Let $f: R^n\to R^n$ be continuous. Then there is a point $z\in R^n$ such that $f(z)=z$.

VI CONCLUSION

There are two main points included in this conclusion. At first, we gave some emphasis to the fixed point theorem at cne and two dimensional surfaces. The popular type of representation was more frequent in use than usual to describe this theorem in detail. In addition to this, illustrations were offered in order to make the theorem more interesting in application. At last, we studied the fixed point thecrem at n-dimensional surface. To this end, we laid strong emphsis upon the description of homeomorphism, in order to generalize the proof of the fixed point theorem concerning n-cell to the fixed point theorem concerning R^n (real unmber system of n-dimensional Euclidean space), since the proof of the former is much easier to carry out than that of the latter.

REFERENCES

1. Samuel Eilenberg and Norman Steenrod, Foundations of Algebraic Topology, p.p. 44, 54, 301; 1951.
2. Bert Mendelson, Introduction to Topology, p.p. 125～134; 1962.
3. P. S. Aleksandrov, Combinatorial Topology, Volume 3, p.p. 115～122, 128～145; 1960.
4. John G. Hocking and Gall S. Young, Topology, p.p. 152～153, 271～273; 1961.
5. Stewart Scott Cairns, Introductory Topology, p.p. 61～62, 71, 74, 136～139; 1961.
6. Scientific American, January 1966, p.p. 105～110.
7. 中國自然科學促進會，科學教育，第12卷，第5,6期，定點定理，劉世超，p.p. 17～19; 民國五十五年六月三十日出版。

各維空間之定點定理

董憲舒

提　　要

定點定理首由荷蘭數學家 L.E.J. Brouwer 先生所證明，爲一代數拓撲學中具有濃厚趣味之重要定理，該定理經四十餘年來不斷發展與應用，已成爲數學、物理、天文，甚至哲學等研究工作中之一重要依據。

該定理牽涉範圍既屬廣泛，理解方法及所作有關之敍述或所用符號自不一致。本文於緒論中闡述各有關定義定理以構成本文之基礎。然後就一維、二維空間用最易於一般化之方式研討此定理並例證之，以充分表現其所含之意義，而更見吾人對之發生濃厚興趣之原因。

進一步在n 維空間內研討定點定理，特別着重拓撲同等 (Homemorphism) 意義之闡述，以備應用於定理之推廣，因在定理證明方面，希望用較易之方法並儘量避免採用高深而牽涉太廣之理論，循此方式獲得n 維局部拓撲空間，如 n-cell, 定點定理之證明後，再用拓撲同等之作用，將證明推廣到全部拓撲空間，如 R^n, 之定點定理，本文所採用之此種觀念及方法對於高深研究或有裨益。

* 臺灣省立中興大學理工學院應用數學系教授.