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By Mathematical Method, To Analyze the Motion of Charged Particles in Nonlinear-dynamic-radiation Electric, Magnetic, and Various Combination of Electric & Magnetic Fields**

by

Chun-Kang Tsai*

SUMMARY

In many of the nonlinear important field problems, it is known that the time dependance of the field is dynamic-sinusoidal. This is approximately the situation with radio dynamic radiation wave phenomena, with light or optical phenomena, etc.. Also, in the more general case of complex periodic fields, the representation of the fields in a Fourier series essentially reduces these problems to one involving sinusoidal fields. The dynamic characteristics of charged particles in nonuniform fields are usually analyzed by the experimental methods. In the attached project report, mathematical method has been used to analyzed the motion of the charged particles in the most important nonlinear dynamic radiation sinusoidal electric, magnetic, and various combination of electric & magnetic fields.

The obtained exact mathematical results ore the characteristics of motion of charged particles in the following cases:

(a) A dynamic electric field.
(b) A dynamic magnetic field.
(c) A dynamic electric field with a parallel magnetic field.
(d) A dynamic electric field with a perpendicular magnetic field.
(e) A dynamic electric field accompanied by an arbitrary oriented magnetic field.

The results obtained under the influence of case (e), may also be regarded as the general results. Since, by setting the arbitrary oriented magnetic field coincides with the given electric field intensity, we may get the results as obtained under the case (c); and by setting the magnetic field equals to zero, we may get

* Associate Professor of Electronics & Network Analysis, Chung-Hsing University

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the results as shown in the case (a); etc..

1. Introduction:

In many of the nonlinear important field problems, it is known that the time dependance of the field is dynamic-sinusoidal. This is approximately the situation with radio dynamic-radiation wave phenomena, with light or optical phenomena, etc.. Also, in the more general case of complex periodic fields, the representation of the fields in a Fourier series essentially reduces these problems to one involving sinusoidal fields. The dynamic characteristics of charged particles in nonuniform fields are usually analyzed by the experimental methods. In this report, mathematical method will be used to analyze the motion of charged particles in the most important nonlinear dynamic radiation sinusoidal electric, magnetic and various combination of electric and magnetic fields.

2. General Theory

By definition, the force on a unit positive charge at any point in an electric field is the electric-field intensity E at that point. Consequently, the force on a positive charge Q in an electric field of intensity E is given by QE, the resulting force being in the direction of the electric field. If the magnitude of the charge on the electron is e, then the force on an electron in the field is eE. Since electrons are negatively charged, this force must be in the direction opposite to the field. In order to calculate the path of a charged particle in an electric field, the force, as mentioned above, must be related to the mass and the acceleration of the partiele by Newton's second law of motion. Hence, the force, F_e, on the charged particle in an electric field may be expressed as:

$$F_e = QE = mdv/dt \quad \text{Newtons} \qquad (2—1)$$

where Q is in coulombs, E in volts per meter, m in kilograms and dv/dt in meters per second per second. The solution of this equation, subject to appropriate initial conditions, will give the path of the particle resulting from the action of the electric force.

The formula that gives the force on a single charged particle moving in a magnetic field is determined by the well known "motor law". When a positive charged moving particle, carrying a charge of Q coulombs and moving with a velocity of V meters per second, is situated in a magnetic field of intensity B teslas, the force F_m acting on this particle is

$$F_m = QVB \quad \text{Newtons} \qquad (2—2)$$

This assumes that the directions of V and B are perpendicular to each other. If these quantities are not perpendicular to each other, then only the component of V perpendicular to B contributes to the force. The direction of this force is perpendicular to the plane of V and B and its sense is determined by the well-known Fleming's left hand rule. If the moving particle is negatively charged, the sense of the magnetic force as determined by the lefthand rule must be

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reversed.

In investigating the motion of charged particles which are moving in externally applied force fields of electric and magnetic origin, it will be implicity assumed that the fields alone will control the motion of the charged particles. For simplicity, unless otherwise stated, the MKS (meter-kilogram-second) rationalized system of units will be employed in the subsequent sections.

Since the motion of charged particles will take place, in general, in space, the motion will be specified mathematically with respect to the customary three mutually perpendicular Cartesian axes. The system of notation to be employed throughout the following sections is:

x=position of the particles along the x-axis, in meters.

y=position of the particles along the y-axis.

z=position of the particles along the z-axis.

v=magnitude of the velocity of the particles, in meters per second.

A=magnitude of the acceleration of the particles, in meters per second per second.

Similar subscript notation will be used for vector components. For example,

V_x=velocity component along the x-axis.

A_y=component of the acceleration of the particles along the y-axis.

F_z=component of the force along the z-axis, in Newtons.

Since the term "initial" represents the value of the specified quantity of the time t=o, the subscript o will be used to designate such initial values. For example,

X_o=initial displacement of the particle along the X-axis.

V_{oy}=initial component of the velocity in the Y direction.

3. The characteristics of motion in a dynamic electric field.

Suppose that the given dynamic sinusoidal electric ficld has a field intensity of

$$E=E_m\sin(2\pi ft) \qquad (3—1)$$

Where E_m, in volts per meter, is the maximum value of the field intensity and f, in cycles per second, is the frequency of the field. It is desired to investigate the dynamic characteristics of a charged particle, Q coulombs,injected into this field with an arbitrary initial velocity of V_o meters per second.

Let the amplitude of the given field intensity varies along the X-axis of Cartesion coordinate axes and the origin of this coordinate axes represents the initial position of the injected particle. Thus we resolve V_o into three components of V_{ox}, V_{oy} and V_{oz}, as shown in the vector diagram Fig 1. Since the given E (considered as a vector) coinsides with X-axis by Eq (3—1). We get

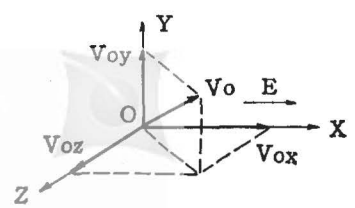

Fig 1

$$E_x=E=E_m\sin(2\pi ft) \qquad (3—2)$$

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$E_y=o$, and

$E_z=o$

These mean that there are no forces acting along Y- and Z- axes. By Newton's second law of motion, the acceleration along these axes must be zero. However, zero acceleration means constant velocity; therefore, the particle will move along these directions with constant initial velocities V_{oy} and V_{oz}, respectively.

Hence along the Y-axis, the dynamic characteristics of the charged particle t seconds after has been injected are:

$$\begin{aligned} A_y&=o \\ V_y&=V_{oy} \\ Y&=(V_y)t=(V_{oy})t \end{aligned} \tag{3—3}$$

Similiarly, the dynaimc characteristics existing along Z-axis are:

$$\begin{aligned} A_z&=o \\ V_z&=V_{oz} \\ Z&=(V_z)t=(V_{oz})t \end{aligned} \tag{3—4}$$

On the other hand, since the electric field intensity E_x existing along X-axis, by Eqs(1—1) and (3—2), we get:

$$F_e=m\frac{dV_x}{dt}=QE_x=QE_m\sin(2\pi ft)$$

Whence, the acceleration of the particle is

$$A_x=\frac{dV_x}{dt}=\frac{Q}{m}E_m\sin(2\pi ft) \tag{3—5}$$

This becomes, upon integration,

$$\int dV_x=\int\frac{Q}{m}E_m\mathrm{Sin}(2\pi ft)\,dt$$

$$\therefore\ V_x=-\frac{Q}{2\pi fm}E_m\ \mathrm{Cos}(2\pi ft)+C$$

Where C=constant of integration. C may be determined from the initial condition that $V=V_{ox}$, when $t=o$.

Thus,

$$C=V_{oy}+\frac{Q}{2\pi fm}E_m.$$

So that the velocity is given by

$$V_x=V_{ox}+\frac{Q}{2\pi fm}E_m\,[1-\mathrm{Cos}(2\pi ft)] \tag{3—6}$$

But $V_x=dx/dt$ and the initial condition is $x=o$ when $t=o$, so we get the displacement x as:

$$x=\int_{t=o}^{t=t}V_x dt=\int_o^t\{V_{ox}+\frac{Q}{2\pi fm}E_m\,[1-\mathrm{Cos}(2\pi ft)]\}dt$$

$$=(V_{ox}+\frac{Q}{2\pi fm}E_m)\,t-\frac{Q}{4\pi^2f^2m}E_m\ \mathrm{Sin}(2\pi ft) \tag{3—7}$$

Therefore, the dynamic characteristics of the particle under the given conditions are,

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Acceleration $A=iA_x+jA_y+kA_z$

$$=i\left[\frac{Q}{m}E_m \text{ Sin } (2\pi ft)\right] \qquad (3—8)$$

Velocity $V=iV_x+jV_y+kV_z$

$$=i\{V_{ox}+\frac{Q}{2\pi fm}E_m\ [1-\text{Cos } (2\pi ft)]\}+jV_{oy}+kV_{oz} \qquad (3—9)$$

Displacement $S=ix+jy+kz$

$$=i\left[(V_{ox}+\frac{Q}{2\pi fm}E_m)\ t-\frac{Q}{4\pi^2f^2m}E_m \text{ Sin } (2\pi ft)\right]$$

$$+j\ (V_{oy}t)\ +k\ (V_{oz}t) \qquad (3—10)$$

Where the resultant acceleration, velocity and displacement are all treated as vectors, and i, j & k represent the unit vector along the X-, Y-& Z-axis, respectively.

4. The characteristics of motion in a dynamic magnetic field.

Suppose that the given dynamic magneric field has a field intensity of

$$B=B_m\sin (2\pi ft) \qquad (4—1)$$

Where B_m, in teslas, is the maximum value of the field intensity and f, in cycles per second, is the frequency of the field. It is desired to investigate the dynamic characteristics of a charged particle, Q coulombs, injected into this field with an arbitrary initial velocity of V_o meters per second.

Let the amplitude of the given field intensity varies along the X axis of a Cartesian coordinate axes and the origin of this coordinate axes represents the initial position of the injected particle. Thus, we resolve V_o into components of V_{ox}, V_{oy}& V_{oz}, as shown in the vector diagram Fig. 2.

Since the given field intensity B varies along X-axis, by Eq, (4—1), we get

$$B_x=B=B_m\sin (2\pi ft)$$

$$B_y=B_z=o \qquad (4—2)$$

By Eq. (2—2) and the general theory of section 2, we get the force functions acting upon the charged particle along X-, Y- & Z-axis as:

$$F_{mx}=o$$

$$F_{my}=QB_xV_z=QV_zB_m \text{ Sin } (2\pi ft) \qquad (4—3)$$

$$F_{mz}=-QB_xV_y=-QV_y\ B_m\text{Sin } (2\pi ft)$$

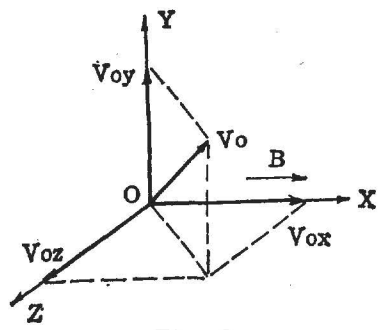

Fig. 2

Hence, along the X-axis, the dynamic characteristics of the charged particle t-second after it has been injected are:

$$A_x=o$$

$$V_x=V_{ox} \qquad (4—4)$$

$$X=V_xt=V_{ox}t$$

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On the other hand, since the magnetic force F_{my} existing along Y axis, by Eqs. (1—1) & (4—3), we get:

$$F_{my}=mdv_y/dt=QB_xV_z=QV_z\ B_m\sin(2\pi ft)$$

Whence, the acceleration of the particle along Y axis is:

$$A_y=dv_y/dt=\frac{Q}{m}V_z\ B_m\ \mathrm{Sin}\ (2\pi ft) \tag{4—5}$$

Similarly, the acceleration of the particle along Z axis is:

$$A_z=dv_z/dt=-\frac{Q}{m}V_yB_m\ \mathrm{Sin}\ (2\pi ft) \tag{4—6}$$

In order to evaluate the velocity components of V_y & V_z, Eq. (4—5) is differentiated as:

$$\frac{d^2V_y}{dt^2}=\frac{Q}{m}B_m\ \mathrm{Sin}\ (2\pi ft)\ \frac{dV_z}{dt}+\frac{Q}{m}2\pi fB_m\ V_z\ \mathrm{Cos}\ (2\pi ft)$$

Combining this Eq. with Eqs. (4—5) & (4—6), we have

$$\frac{d^2V_y}{dt^2}-2\pi f\ [\mathrm{Cot}\ (2\pi ft)]\ \frac{dV_y}{dt}+\frac{Q^2}{m^2}B_m^2\ [\mathrm{Sin}^2\ (2\pi ft)]\ V_y=o \tag{4—7}$$

This is a homogeneous, linear, second-order differential equation with nonconstant coefficient. By inspection, if we introduce a new independent variable and let

$$U=-\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft) \tag{4—8}$$

Then

$$\frac{dU}{dt}=\frac{QB_m}{m}\mathrm{Sin}\ (2\pi ft)$$

$$\frac{d^2U}{dt^2}=\frac{2\pi fQB_m}{m}\ (\mathrm{Cos}\ 2\pi ft)$$

and

$$\frac{dV_y}{dt}=\frac{dV_y}{dU}\cdot\frac{dU}{dt}=\frac{QB_m}{m}\mathrm{Sin}\ (2\pi ft)\ \frac{dV_y}{dU} \tag{4—9}$$

$$\frac{d^2V_y}{dt^2}=\frac{d}{dt}\left(\frac{dV_y}{dU}\cdot\frac{dU}{dt}\right)=\frac{dV_y}{dU}\cdot\frac{d^2U}{dt^2}+\frac{d^2V_y}{dU^2}\left(\frac{dU}{dt}\right)^2$$

$$=\frac{2\pi fQB_m}{m}\mathrm{Cos}\ (2\pi ft)\cdot\frac{dV_y}{dU}+\frac{Q^2B_m^2}{m^2}\mathrm{Sin}^2\ (2\pi ft)\cdot\frac{d^2V_y}{dU^2} \tag{4—10}$$

Substituting Eqs (4—9) & (4—10) to Eq. (4—7), we obtain

$$\frac{d^2V_y}{dU^2}+V_y=o \tag{4—11}$$

Thus, by introducing the new independent variable U, the complex non-constant coefficient differential equation of Eq. (4—7) reduces to the simple second-order differential equation with constant coefficient of Eq. (4—11). The auxiliary equation of Eq. (4—11) is

$$M^2+1=o \tag{4—12}$$

Since Eq. (4—12) has two identical equal roots of $\sqrt{-1}$, therefore, the solution of Eq. (4—11) becomes

$$V_y=K_1\mathrm{Cos}U+K_2\mathrm{Sin}U \tag{4—13}$$

Where K_1 & K_2 are arbitrary constants of integration.

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Substituting Eq. (4—8) to Eq. (4—13), then, we get the required solution of Eq. (4—7) as:

$$V_y=K_1 \text{ Cos } [-\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)] + K_2 \text{ Sin } [-\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)]$$

$$=K_1\text{Cos } [\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)] - K_2\text{Sin } [\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)] \quad (4—14)$$

Substituting Eq. (4—14) to Eq. (4—5), we get

$$V_z=\frac{m}{QB_m \text{ Sin } (2\pi ft)} \cdot \frac{dV_y}{dt}$$

$$=K_1 \text{ Sin } [\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)] + K_2\text{Cos } [\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)] \quad (4—15)$$

The integrating constants K_1 & K_2 in Eqs. (4—14) & (4—15) may be determined by the inital conditions that $V_y=V_{oy}$ & $V_z=V_{oz}$ when t=o i.e.

$$V_{oy}=K_1 \text{ Cos } (\frac{QB_m}{2\pi fm}) - K_2 \text{ Sin } (\frac{QB_m}{2\pi fm})$$

$$V_{oz}=K_1 \text{ Sin } (\frac{QB_m}{2\pi fm}) - K_2\text{Cos } (\frac{QB_m}{2\pi fm})$$

Solving these two equatons, we get:

$$K_1=V_{oy} \text{ Cos } (\frac{QB_m}{2\pi fm}) + V_{oz} \text{ Sin } (\frac{QB_m}{2\pi fm})$$

$$K_2=-V_{oy} \text{ Sin } (\frac{QB_m}{2\pi fm}) + V_{oz} \text{ Cos } (\frac{QB_m}{2\pi fm})$$

Therefore, the required general solutions for the differential equations of V_y & V_z, Eqs. (4—5) & (4—6), become:

$$V_y=K_1 \text{ Cos } [\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)] - K_2 \text{ Sin } [\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)]$$

$$= [V_{oy} \text{ Cos } (\frac{QB_m}{2\pi fm}) + V_{oz} \text{ Sin } (\frac{QB_m}{2\pi fm})] \text{ Cos } [\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)]$$

$$- [-V_{oy} \text{ Sin } (\frac{QB_m}{2\pi fm}) + V_{oz} \text{ Cos } (\frac{QB_m}{2\pi fm})] \text{ Sin } [\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)] \quad (4—16)$$

$$V_z=K_1 \text{ Sin } [\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)] + K_2 \text{ Cos } [\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)]$$

$$= [V_{oy} \text{ Cos } (\frac{QB_m}{2\pi fm}) + V_{oz} \text{ Sin } (\frac{QB_m}{2\pi fm})] \text{ Sin } [\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)]$$

$$+ [-V_{oy} \text{ Sin } (\frac{QB_m}{2\pi fm}) + V_{oz} \text{ Cos } (\frac{QB_m}{2\pi fm})] \text{ Cos } [\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)] \quad (4—17)$$

Substituting Eqs. (4—16) & (4—17) to Eqs. (4—5) & (4—6), we get the accelerations along Y-& Z-axis as:

$$A_y=\frac{Q}{m}V_z \text{ } B_m \text{ Sin } (2\pi ft)$$

$$=\frac{Q}{m}B_m \text{ Sin } (2\pi ft) \{K_1 \text{ Sin } [\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)] + K_2 \text{ Cos } [\frac{QB_m}{2\pi fm}\text{Cos}(2\pi ft)]\}$$

$$=\frac{QB_m}{m}\text{Sin } (2\pi ft)\{[V_{oy} \text{ Cos } (\frac{QB_m}{2\pi fm}) + V_{oz} \text{ Sin}(\frac{QB_m}{2\pi fm})] \text{ Sin } [\frac{QB_m}{2\pi fm}\text{Cos}(2\pi ft)]$$

$$+ [-V_{oy} \text{ Sin } (\frac{QB_m}{2\pi fm}) + V_{oz} \text{ Cos } (\frac{QB_m}{2\pi fm})] \text{ Cos } [\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)]\} \quad (4—18)$$

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$$A_z=-\frac{Q}{m}V_yB_m\ \mathrm{Sin}\ (2\pi ft)$$
$$=-\frac{Q}{m}B_m\ \mathrm{Sin}\ (2\pi ft)\ \{K_1\ \mathrm{Cos}\ [\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)]$$
$$-K_2\ \mathrm{Sin}\ [\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)]\}$$
$$=-\frac{Q}{m}B_m\ \mathrm{Sin}\ (2\pi ft)\{[V_{oy}\ \mathrm{Cos}\ (\frac{QB_m}{2\pi fm})$$
$$+V_{oz}\ \mathrm{Sin}\ (\frac{QB_m}{2\pi fm})]\ \mathrm{Cos}\ [\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)]$$
$$-[-V_{oy}\ \mathrm{Sin}\ (\frac{QB_m}{2\pi fm})+V_{oz}\ \mathrm{Cos}\ (\frac{QB_m}{2\pi fm})]\ \mathrm{Sin}\ [\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)]\}\quad(4-19)$$

Again, since the velocity is equal to the first derivative of displacement and the moving particle starts from the reference point of the coordinate origin, therefore, we get the displacement along Y-axis as:

$$Y=\int_{t=o}^{t=t}V_ydt$$
$$=\int_o^t\{K_1\ \mathrm{Cos}\ [\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)]-K_2\ \mathrm{Sin}\ [\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)]\}dt$$
$$=\int_o^t K_1\ \mathrm{Cos}\ [\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)]\ dt-\int_o^t K_2\ \mathrm{Sin}\ [\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)]\}dt\quad(4-20)$$

Let

$$\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)=U\quad(4-21)$$

Then, we get

$$t=\frac{1}{2\pi f}\mathrm{Cos}^{-1}\ (\frac{2\pi fm}{QB_m}U)$$

and

$$dt=\frac{1}{2\pi f}\cdot\frac{-\frac{2\pi fm}{QB_m}}{[1-(\frac{2\pi fm}{QB_m}U)^2]^{1/2}}dU$$
$$=-\frac{m}{QB_m}\cdot\frac{1}{[1-(\frac{2\pi fm}{QB_m}U)^2]^{1/2}}dU\quad(4-22)$$

Substituting Eqs. (4—21) & (4—22) to the first part of Eq. (4—20), we have

$$Y_1=\int_o^t K_1\ \mathrm{Cos}\ [\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)]\ dt$$
$$=\int_{U_1}^{U_2}K_1\mathrm{Cos}\ U\cdot\frac{-\frac{m}{QB_m}}{[1-(\frac{2\pi fm}{QB_m}U)^2]^{1/2}}dU$$
$$=\ (\mathrm{Set})\ \int_{U_1}^{U_2}K_3\ \mathrm{Cos}U\cdot\frac{1}{[1-(K_4U)^2]^{1/2}}dU$$
$$=K_3\int_{U_1}^{U_2}[(1-\frac{U^2}{2!}+\frac{U^4}{4!}-\frac{U^6}{6!}+\cdots\cdots)(1-\frac{1}{2}K_4^2U^2+\frac{1.3}{2.4}K_4^4U^4\cdots\cdots)]\ dU$$
$$=K_3\int_{U_1}^{U_2}[1-(\frac{1}{2!}+\frac{1}{2}K_4^2)U^2+(\frac{1}{4!}+\frac{1}{2!}\cdot\frac{1}{2}K_4^2+\frac{1.3}{2.4}K_4^4)U^4+\cdots\cdots]\ dU$$

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$$=K_3\left[(U_2-U_1)-\frac{1}{3}\left(\frac{1}{2!}+\frac{1}{2}K_4^2\right)(U_2^3-U_1^3)\right.$$

$$\left.+\frac{1}{5}\left(\frac{1}{4!}+\frac{1}{2!}\cdot\frac{1}{2}K_4^2+\frac{1.3}{2.4}K_4^4\right)(U_2^5-U_1^5)+\cdots\cdots\right]$$

Similarly, the second part of Eq. (4—20) may be determined as

$$Y_2=\int_o^t K_2\ \mathrm{Sin}\left[\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)\right]dt$$

$$=\int_{U_1}^{U_2}K_2\ \mathrm{SinU}\cdot\frac{-m/QB_m}{\left[1-\left(\frac{2\pi fm}{QB_m}U\right)^2\right]^{1/2}}dU$$

$$=(\mathrm{set})\int_{U_1}^{U_2}K_5\ \mathrm{SinU}\cdot\frac{1}{\left[1-(K_4U)^2\right]^{1/2}}dU$$

$$=K_5\int_{U_1}^{U_2}\left[\left(K_4U-\frac{1}{3!}K_4^3U^3+\frac{1}{5!}K_4^5U^5+\cdots\cdots\right)\left(1-\frac{1}{2}K_4^2U^2+\frac{1.3}{2.4}K_4^4U^4\cdots\cdots\right]dU\right.$$

$$=K_5\int_{U_1}^{U_2}\left[K_4U-\left(\frac{1}{2}+\frac{1}{3!}\right)k_4^3U^3+\left(\frac{1}{2}\cdot\frac{1}{3!}+\frac{1}{5!}+\frac{1.3}{2.4}\right)K_4^5U^5+\cdots\cdots\right]dU$$

$$=K_5\left[\frac{1}{2}K_4\ (U_2^2-U_1^2)-\frac{1}{4}\left(\frac{1}{2}+\frac{1}{3!}\right)K_4^3\ (U_2^4-U_1^4)\right.$$

$$\left.+\frac{1}{6}\left(\frac{1}{2}\cdot\frac{1}{3!}+\frac{1}{5!}+\frac{1.3}{2.4}\right)K_4^5\ (U_2^6-U_1^6)+\cdots\cdots\right]$$

Substituting the results of these two parts to Eq. (4—20), we get

$$Y=Y_1-Y_2$$

$$=K_3\ (U_2-U_1)-\frac{1}{2}K_5K_4\ (U_2^2-U_1^2)-\frac{1}{3}K_3\left(\frac{1}{2!}+\frac{1}{2}K_4^2\right)(U_2^3-U_1^3)$$

$$+\frac{1}{4}K_5K_4^3\left(\frac{1}{2}+\frac{1}{3!}\right)(U_2^4-U_1^2)+\cdots\cdots \qquad (4—23)$$

By the same way, the displacement along Z axis may be determined as:

$$Z=\int_{t=o}^{t=t}V_x dt=\int_o^t\left\{K_1\ \mathrm{Sin}\left[\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)\right]+K_2\ \mathrm{Cos}\left[\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)\right]\right\}dt$$

$$=\int_o^t K_1\ \mathrm{Sin}\left[\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)\right]dt+\int_o^t K_2\ \mathrm{Cos}\left[\frac{QB_m}{2\pi fm}\mathrm{Cos}\ (2\pi ft)\right]dt$$

$$=(\mathrm{set})\int_{U_1}^{U_2}K_6\ \mathrm{SinU}\cdot\frac{dU}{\left[1-(K_4U)^2\right]^{\frac{1}{2}}}+\int_{U_1}^{U_2}K_7\ \mathrm{CosU}\frac{dU}{\left[1-(K_4\ U)^2\right]^{\frac{1}{2}}}$$

$$=K_6\left[\frac{1}{2}K_4\ (U_2^2-U_1^2)-\frac{1}{4}K_4^3\left[\frac{1}{2}+\frac{1}{3!}\right)(U_2^4-U_1^4)\right.$$

$$+\frac{1}{6}\left(\frac{1}{2}\cdot\frac{1}{3!}+\frac{1}{5!}+\frac{1.3}{2.4}\right)K_4^5\ (U_2^6-U_1^6)+\cdots\cdots]+K_7\left[(U_2-U_1)\right.$$

$$\left.-\frac{1}{3}\left(\frac{1}{2!}+\frac{1}{2}K_4^2\right)(U_2^3-U_1^3)+\frac{1}{5}\left(\frac{1}{4!}+\frac{1}{2!}\ \frac{1}{2}K_4^2+\frac{1.3}{2.4}K_4^4\right)(U_2^5-U_1^5)+\cdots\right]$$

$$=K_7\ (U_2-U_1)+\frac{1}{2}K_6K_4\ (U_2^2-U_1^2)-\frac{1}{3}K_7\left(\frac{1}{2!}+\frac{1}{2}K_4^2\right)(U_2^3-U_1^3)$$

$$-\frac{1}{4}K_6K_4^3\left(\frac{1}{2}+\frac{1}{3!}\right)(U_2^4-U_1^4)+\cdots\cdots$$

Therefore, by combining Eqs. (4—4), (4—23), (4—24)&(4—16) through(4—19), we get the resultant dynamic characteristics of the particle under the given conditions as:

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Acceleration $A=iA_x+jA_y+kA_z$

$$=j\{\frac{Q}{m}B_m \text{ Sin } (2\pi ft)\ [K_1 \text{ Sin } (\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)+K_2 \text{ Cos } (\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)]\}$$

$$+k\{-\frac{Q}{m}B_m \text{ Sin } (2\pi ft)\ [K_1 \text{ Cos } (\frac{QB_m}{2\pi fm}\text{Cos}(2\pi ft)) - K_2 \text{ Sin}(\frac{QB_m}{2\pi fm}\text{Cos}(2\pi ft))]\} \quad (4—25)$$

Velocity $V=iV_x+jVy+kV_z$

$$=iV_{ox}+j\ [K_1 \text{ Cos } (\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)) - K_2\text{Sin } (\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft))]$$

$$+k\ [K_1 \text{ Sin } (\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft)) + K_2\text{Cos } (\frac{QB_m}{2\pi fm}\text{Cos } (2\pi ft))] \quad (4—26)$$

Displacement $S=ix+jy+kz$

$$=i[V_{ox}t]+j[K_3\ (U_2-U_1)-\frac{1}{2}K_5K_4\ (U_2^2-U_1^2)-\frac{1}{3}K_3\ (\frac{1}{2!}+\frac{1}{2}K_4^2)(U_2^3-U_1^3)+\cdots]$$

$$+k\ [K_7\ (U_2-U_1)+\frac{1}{2}K_6K_4\ (U_2^2-U_1^2)-\frac{1}{3}K_7\ (\frac{1}{2!}+\frac{1}{2}K_4^2)\ (U_2^3-U_1^3)+\cdots\cdots] \quad (4—27)$$

5. The characteristics of motion in a dynamic electric field with parallel magnetic field.

Suppose that the given dynamic clectric field has a field intensity of:

$E=E_m\sin\ (2\pi ft)$

and the amplitude of this field varies along the X axis of a Cartesian coordinate. Along the X axis, there also lies a given magnetic field B in teslas. It is desired to investigate the dynamic characteristics of a charged particle, Q coulombs, injected into these fields with an arbitrary initial velocity of V_o.

Let the origin of the chosen coordinate axes coinsides with the initial position of the injected particle, as shown in Fig.3, then, we resolve V_o into components of V_{ox}, V_{oy} and V_{oz} along X, Y and Z axies, respectively.

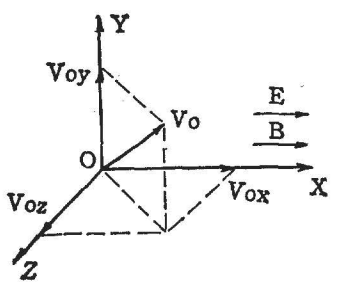

Fig 3

Since the given E and B coinside with X axis, by Eqs. (1—1) and (1—2), we get the force functions acting upon the charged particle along X, Y and Z axes, as:

$$F_x=mdV_x/dt=QE=QE_m\text{Sin } (2\pi ft) \quad (5—1)$$

$$F_y=mdV_y/dt=BQV_y \quad (5—2)$$

$$F_z=mdV_z/dt=-BQV_z \quad (5—3)$$

Hence, along the X axis, by Eq. (5—1), we get the characteristics of motion of the charged particle (t second after it has been injected) as:

$$A_x=dV_x/dt=\frac{Q}{m}E_m\text{Sin } (2\pi ft)$$

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$$V_x = V_{ox} + \int_o^t \frac{Q}{m} E_m \operatorname{Sin}(2\pi ft)\, dt$$

$$= V_{ox} + \frac{QE_m}{2\pi fm}[1 - \operatorname{Cos}(2\pi ft)]$$

$$X = \int_{t=o}^{t=t} V_x dt = \int_o^t \{V_{ox} + \frac{QE_m}{2\pi fm}[1 - \operatorname{Cos}(2\pi ft)]\} dt \qquad (5-4)$$

$$= V_{ox}t + \frac{QE_m}{2\pi fm}[t - \frac{1}{2\pi f}\operatorname{Sin}(2\pi ft)]$$

The characteristics of motion along Y- and Z- axis, may be determined by solving Eqs. (5—2) and (5—3), simultaneously. Differentiating Eq. (5—2) and substituting Eq. (5—3) to the result so obtained, we have:

$$\frac{d^2V_y}{dt^2} + \frac{Q^2B^2}{m^2}V_y = o \qquad (5-5)$$

This is a homogeneous, linear, second-order differential equation with constant coefficients. The solution of Eq. (5—5) may be assumed as:

$$V_y = K_1 \operatorname{Cos}(\frac{QB}{m}t) + K_2 \operatorname{Sin}(\frac{QB}{m}t) \qquad (5-6)$$

Where K_1 and K_2 are arbitrary integrating constants and may be determined by the initial conditions, that $V_y = V_{oy}$, $V_z = V_{oz}$ when $t = o$. Substituting the mentioned initial conditions to Eqs. (5—6) and (5—2), we obtain

$$K_1 = V_{oy}$$

$$K_2 = V_{oz}$$

Therefore, the required general solution for V_y becomes:

$$V_y = V_{oy} \operatorname{Cos}(\frac{QB}{m}t) + V_{oz} \operatorname{Sin}(\frac{QB}{m}t) \qquad (5-7)$$

Hence, subsituting Eq. (5—7) to (5—2), we have

$$V_z = -V_{oy} \operatorname{Sin}(\frac{QB}{m}t) + V_{oz} \operatorname{Cos}(\frac{QB}{m}t) \qquad (5-8)$$

Again, since the velocity is equal to the first derivative of displacement, and the moving particle starts from the reference point of the coordinate origin, therefore, we may determine the displacement along Y axis as:

$$Y = \int_{t=o}^{t=t} V_y dt = \int_o^t [V_{oy} \operatorname{Cos}(\frac{QB}{m}t) + V_{oz} \operatorname{Sin}(\frac{QB}{m}t)]\, dt$$

$$= \frac{m}{QB}V_{oz} + \frac{m}{QB}V_{oy} \operatorname{Sin}(\frac{QB}{m}t) - \frac{m}{QB}V_{oz} \operatorname{Cos}(\frac{QB}{m}t) \qquad (5-9)$$

and the displacement along Z axis as:

$$Z = \int_{t=o}^{t=t} V_z\, dt = \int_o^t [-V_{oy} \operatorname{Sin}(\frac{QB}{m}t) + V_{oz}\operatorname{Cos}(\frac{QB}{m}t)]\, dt$$

$$= -\frac{m}{QB}V_{oy} + \frac{m}{QB}V_{oy}\operatorname{Cos}(\frac{QB}{m}t) + \frac{m}{QB}V_{oz} \operatorname{Sin}(\frac{QB}{m}t) \qquad (5-10)$$

Substituting Eqs. (5—8) and (5—7) to Eqs. (5—2) and (5—3), we have the acceleration along Y axis as:

$$A_y = dV_y/dt = \frac{QB}{m}V_z$$

$$= -\frac{QB}{m}V_{oy} \operatorname{Sin}(\frac{QB}{m}t) + \frac{QB}{m}V_{oz} \operatorname{Cos}(\frac{QB}{m}t) \qquad (5-11)$$

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and the acceleration along Z axis as:

$$A_z = dV_z/dt = -\frac{QB}{m}V_y$$

$$= -\frac{QB}{m}V_{oy}\ \mathrm{Cos}\ (\frac{QB}{m}t) - \frac{QB}{m}V_{oz}\ \mathrm{Sin}\ (\frac{QB}{m}t) \qquad (5\text{—}12)$$

Therefore, by combining the corresponding results just derived, we get the resultant dynamic characteristics of the particle under the given conditions as:

Acceleration $A = iA_x + jA_y + kA_z$

$$= i\,[\frac{Q}{m}E_m\ \mathrm{Sin}\ (2\pi ft)]$$
$$+ j\,[-\frac{QB}{m}V_{oy}\ \mathrm{Sin}\ (\frac{QB}{m}t) + \frac{QB}{m}V_{oz}\ \mathrm{Cos}\ (\frac{QB}{m}t)]$$
$$+ k\,[-\frac{QB}{m}V_{oy}\ \mathrm{Cos}\ (\frac{QB}{m}t) - \frac{QB}{m}V_{oz}\ \mathrm{Sin}\ (\frac{QB}{m}t)] \qquad (5\text{—}15)$$

Velocity $V = iV_x + jV_y + kV_z$

$$= i\,[\frac{QE_m}{2\pi fm}\{1 - \mathrm{Cos}(2\pi ft)\} + V_{ox}]$$
$$+ j\,[V_{oy}\ \mathrm{Cos}\ (\frac{QB}{m}t) + V_{oz}\ \mathrm{Sin}\ (\frac{QB}{m}t)]$$
$$+ k\,[-V_{oy}\ \mathrm{Sin}\ (\frac{QB}{m}t) + V_{oz}\ \mathrm{Cos}\ (\frac{QB}{m}t)] \qquad (5\text{—}14)$$

Displacement $S = ix + jy + kz$

$$= i\,[V_{ox}t + \frac{QE_m}{2\pi fm}\{t - \frac{1}{2\pi f}\mathrm{Sin}\ (2\pi ft)\}]$$
$$+ j\,[\frac{m}{QB}V_{oz} + \frac{m}{QB}V_{oy}\ \mathrm{Sin}\ (\frac{QB}{m}t) - \frac{m}{QB}V_{oz}\ \mathrm{Cos}\ (\frac{QB}{m}t)]$$
$$+ k\,[-\frac{m}{QB}V_{oy} + \frac{m}{QB}V_{oy}\ \mathrm{Cos}\ (\frac{QB}{m}t) + \frac{m}{QB}V_{oz}\ \mathrm{Sin}\ (\frac{QB}{m}t)] \qquad (5\text{—}16)$$

6. The characteristics of motion in a dynamic electric field with perpendicular magnetic field.

Suppose that the given dynamic electric field has a field intensity of

$$E = E_m \sin (2\pi ft)$$

and the amplitude of this field varies along the X axis of the Cartesian coordinate. The given magnetic field B, in teslas, lies along the Y axis, as shown in Fig. 4. It is desired to investigate the dynamic characteristics of a charged particle, Q coulombs, injected into these fields with an arbitrary initial veloeity of V_o.

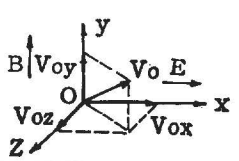

Fig 4

Let the origin of the chosen coordinate axes coinsides with the initial position of the injected particle. Then, we resolve V_o into components of V_{ox}, V_{oy} & V_{oz} along X—, Y— & Z— axis, respectively. By Eqs. (2—1) & (2—2), we get the force functions acting upon the charged particle along X—, Y— & Z—axis, as:

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$$F_x = mdV_x/dt = QE_m \text{Sin}\ (2\pi ft) - QBV_z \qquad (6-1)$$

$$F_y = mdV_y/dt = O \qquad (6-2)$$

$$F_x = mdV_z/dt = QBV_x \qquad (6-3)$$

By Eq. (6—2), we get the dynamic characteristics of the charged particle along Y axis t-second after it has been injected as:

$$A_y = dV_y/dt = O$$

$$V_y = V_{oy}$$

$$Y = V_{oy}t$$

The dynamic characteristics along X- & Z- axis may be determined by solving Eqs. (6—1) & (6—3), simultaneously. Differentiating Eq. (6—3) & substituting Eq. (6—1) to the result so obtained, we have

$$\frac{d^2V_z}{dt^2} + \frac{Q^2B^2}{m^2}V_z = \frac{Q^2}{m^2}BE_m \text{ Sin } (2\pi ft) \qquad (6-4)$$

This is a nonhomogeneous, linear, second-order differentinal epuation with constant coefficient. Thus, we get the auxiliary equation of Eq. (6—4) as:

$$M^2 + \frac{Q^2}{m^2}B^2 = O$$

Which has two identical equal roots of $\sqrt{-1}QB/m$. Therefore, the complementary function of Eq. (6—4) becomes

$$V_{zc} = K_1 \text{Sin } (\frac{Q}{m}Bt) + K_2 \text{ Ccs } (\frac{Q}{m}Bt)$$

Where K_1 & K_2 are arbitrary constants of integration. To determine the particular integral of Eq. (6—4), by the method of undetermined coefficients, we set the trial solution as:

$$V_{zp} = K_3 \text{Sin } (2\pi ft) + K_4 \text{ Cos } (2\pi ft)$$

Where K_3 & K_4 are the undermined coefficients. Substituting this trial solution into the original differential equation (6—4), we get

$$m^2 [-K_3 (2\pi f)^2 \text{ Sin } (2\pi ft) - K_4 (2\pi f)^2 \text{ Ccs } (2\pi ft)]$$
$$+ Q^2B^2 [K_3 \text{Sin } (2\pi ft) + K_4 \text{ Ccs } (2\pi ft)] = Q^2B\ E_m \text{ Sin } (2\pi ft)$$

$$\therefore K_4 = O$$

and since

$$-m^2K_3 (2\pi f)^2 + Q^2B^2K_3 = Q^2BE_m$$

$$\therefore K_3 = Q^2BE_m / [Q^2B^2 - (2\pi fm)^2]$$

Substituting these coefficients so obtained into the trial solution, the particular integral of Eq. (6—4) becomes

$$V_{zp} = K_3 \text{Sin } (2\pi ft) + K_4 \text{Cos } (2\pi ft)$$
$$= \frac{Q^2BE_m}{Q^2B^2 - (2\pi fm)^2} \text{Sin } (2\pi ft)$$

The general solution of a nonhomogeneous, linear differential equation is the sum of its complementary function and particular integral. Therefore, the general solution of Eq. (6—4) becomes:

$$V_z = V_{zp} + V_{zc} = K_3 \text{ Sin } (2\pi ft) + K_1 \text{ Sin } (\frac{QB}{m}t) + K_2 \text{Cos } (\frac{QB}{m}t)$$

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$$=\frac{Q^2BE_m}{Q^2B^2-(2\pi fm)^2}\text{Sin }(2\pi ft)+K_1\text{ Sin }(\frac{QB}{m}t)+K_2\text{ Cos }(\frac{QB}{m}t) \qquad (6-5)$$

The arbitrary constants K_1 & K_2 may be evaluated from knowledge of the initial conditions, i. e. $V_x=V_{ox}$ & $V_z=V_{oz}$ when $t=o$. Substituting these initial conditions to Eqs. (6—5) & (6—3), we get

$$K_2=V_{oz}$$

and since

$$QBV_{ox}=mdV_z/dt\Big|_{V_z=V_{oz},t=o}=m\left[\frac{2\pi fQ^2BE_m}{Q^2B^2-(2\pi fm)^2}+\frac{QB}{m}K_1\right]$$

$$\therefore K_1=V_{ox}-\frac{2\pi fmQE_m}{Q^2B^2-(2\pi fm)^2}$$

Hence, the Eq. (6—5) for V_z becomes

$$\begin{aligned}V_z&=K_3\text{ Sin }(2\pi ft)+K_1\text{ Sin }(\frac{QB}{m}t)+K_2\text{ Cos }(\frac{QB}{m}t)\\&=\frac{Q^2BE_m}{Q^2B^2-(2\pi fm)^2}\text{Sin }(2\pi ft)+(V_{ox}-\frac{2\pi fmQ\ E_m}{Q^2B^2-(2\pi fm)^2})\text{ Sin }(\frac{QB}{m}t)\\&\quad+V_{oz}\text{ Cos }(\frac{QB}{m}t)\end{aligned} \qquad (6-6)$$

Substituting Eq. (6—6) to Eq. (6—3), we get

$$\begin{aligned}V_x&=\frac{m}{QB}dV_z/dt=\frac{2\pi fmQE_m}{Q^2B^2-(2\pi fm)^2}\text{Cos }(2\pi ft)\\&\quad+(V_{ox}-\frac{2\pi fmQE_m}{Q^2B^2-(2\pi fm)^2})\text{ Cos }(\frac{QB}{m}t)-V_{oz}\text{ Sin }(\frac{QB}{m}t)\\&=(\text{set})\ K_4\text{Cos }(2\pi ft)+K_1\text{ Cos }(\frac{QB}{m}t)-K_2\text{ Sin }(\frac{QB}{m}t)\end{aligned} \qquad (6-7)$$

Integrating Eqs. (6—6) & (6—7) with respect to time t, we get the displacement along X axis as:

$$\begin{aligned}X&=\int_{t=o}^{t=t}V_xdt=\int_o^t\left[\frac{2\pi fmQE_m}{Q^2B^2-(2\pi fm)^2})\text{ Cos }(2\pi ft)\right.\\&\quad\left.+(V_{ox}-\frac{2\pi fmQE_m}{Q^2B^2-(2\pi fm)^2})\text{ Cos }(\frac{QB}{m}t)-V_{oz}\text{ Sin }(\frac{QB}{m}t)\right]dt\\&=\frac{mQE_m}{Q^2B^2-(2\pi fm)^2})\text{ Sin }(2\pi ft)+\frac{m}{QB}(V_{ox}-\frac{2\pi fmQE_m}{Q^2B^2-(2\pi fm)^2})\text{ Sin }(\frac{QB}{m}t)\\&\quad+\frac{m}{QB}V_{oz}\text{ Cos }(\frac{QB}{m}t)-\frac{m}{QB}V_{oz}\\&=(\text{set})\ X_1\text{ Sin }(2\pi ft)+X_2\text{ Sin }(\frac{QB}{m}t)+X_3\text{ Cos }(\frac{QB}{m}t)+X_4\end{aligned} \qquad (6-8)$$

the displacement along Z axis as:

$$\begin{aligned}Z&=\int_{t=o}^{t=t}V_z\ dt=\int_o^t\left[\frac{Q^2BE_m}{Q^2B^2-(2\pi fm)^2})\text{ Sin }(2\pi ft)\right.\\&\quad\left.+(V_{ox}-\frac{2\pi fmQE_m}{Q^2B^2-(2\pi fm)^2}\text{Sin }(\frac{QB}{m}t)+V_{oz}\text{Cos }(\frac{QB}{m}t)\right]dt\\&=\frac{-Q^2BE_m/2\pi f}{Q^2B^2-(2\pi fm)^2}\text{Cos }(2\pi ft)-\frac{m}{QB}\ (V_{ox}-\frac{2\pi fmQE_m}{Q^2B^2-(2\pi fm)^2})\text{ Cos }(\frac{QB}{m}t)\\&\quad+\frac{m}{QB}V_{oz}\text{ Sin }(\frac{QB}{m}t)+\frac{Q^2BE_m/2\pi f}{Q^2B^2-(2\pi fm)^2}+\frac{m}{QB}\ (V_{ox}-\frac{2\pi fmQE_m}{Q^2B^2-(2\pi fm)^2})\end{aligned}$$

$$=(\text{set})\ Z_1 \cos(2\pi ft)+Z_2 \cos\left(\frac{QB}{m}t\right)+Z_3 \sin\left(\frac{QB}{m}t\right)+Z_4 \qquad (6\text{—}9)$$

The acceleration along Z axis may be determined by substituting Eq. (6—7) to Eq. (6—3) as:

$$\begin{aligned}A_z &= \frac{dV_z}{dt} = \frac{QB}{m}V_x = \frac{QB}{m}\left[K_4 \cos(2\pi ft)+K_1 \cos\left(\frac{QB}{m}t\right)\right.\\ &\left.-K_2 \sin\left(\frac{QB}{m}t\right)\right] = \frac{2\pi fQ^2B\,E_m}{Q^2B^2-(2\pi fm)^2}\cos(2\pi ft)\\ &+\frac{QB}{m}\left(V_{ox}-\frac{2\pi fmQ\,E_m}{Q^2B^2-(2\pi fm)^2}\right)\cos\left(\frac{QB}{m}t\right)\\ &-\frac{QB}{m}V_{oz}\sin\left(\frac{QB}{m}t\right)\\ &=(\text{set})\ A_{z1}\cos(2\pi ft)+A_{z2}\cos\left(\frac{QB}{m}t\right)+A_{z3}\sin\left(\frac{QB}{m}t\right) \qquad (6\text{—}10)\end{aligned}$$

The acceleration along X axis may be determined by differentiating Eq. (6—7) directly, i.e.

$$\begin{aligned}A_x &= \frac{dV_x}{dt} = -\frac{(2\pi f)^2 mQ\,E_m}{Q^2B^2-(2\pi fm)^2}\sin(2\pi ft)\\ &-\frac{QB}{m}\left(V_{ox}-\frac{2\pi fmQ\,E_m}{Q^2B^2-(2\pi fm)^2}\right)\sin\left(\frac{QB}{m}t\right)\\ &-\frac{QB}{m}V_{oz}\,Ccs\left(\frac{QB}{m}t\right)\\ &=(\text{set})\ A_{x1}\sin(2\pi ft)+A_{x2}\sin\left(\frac{QB}{m}t\right)+A_{x3}\cos\left(\frac{QB}{m}t\right) \qquad (6\text{—}11)\end{aligned}$$

Therefore, by combining the corresponding results just derived, we get the resultant dynamic characteristics of the charged particle under the given conditions as:

Acceleration $A = iA_x + jA_y + kA_z$

$$\begin{aligned}&= i\left[A_{x1}\sin(2\pi ft)+A_{x2}\sin\left(\frac{QB}{m}t\right)+A_{x3}\cos\left(\frac{QB}{m}t\right)\right]\\ &+k\left[A_{z1}\cos(2\pi ft)+A_{z2}\cos\left(\frac{QB}{m}t\right)+A_{z3}\sin\left(\frac{QB}{m}t\right)\right] \qquad (6\text{—}12)\end{aligned}$$

Velocity $V = iV_x + jVy + kV_z$

$$\begin{aligned}&= i\left[K_4\cos(2\pi ft)+K_1\cos\left(\frac{QB}{m}t\right)-K_2\sin\left(\frac{QB}{m}t\right)\right]\\ &+j\left[V_{oy}\right]+k\left[K_3\sin(2\pi ft)+K_1\sin\left(\frac{QB}{m}t\right)+K_2\cos\left(\frac{QB}{m}t\right)\right] \qquad (6\text{—}13)\end{aligned}$$

Displacement $S = ix + jy + kz$

$$\begin{aligned}&= i\left[X_1\sin(2\pi ft)+X_2\sin\left(\frac{QB}{m}t\right)+X_3\cos\left(\frac{QB}{m}t\right)+X_4\right]\\ &+j\left[V_{oy}\ t\right]+k\left[Z_1\cos(2\pi ft)+Z_2\cos\left(\frac{QB}{m}t\right)\right.\\ &\left.+Z_3\sin\left(\frac{QB}{m}t\right)+Z_4\right] \qquad (6\text{—}14)\end{aligned}$$

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7. The characteristics of motion in a dynamic electric field accompanied by an arbitrary oriented magnetic field.

Suppose that the given dynamic electric field has a field intensity of

$$E=E_m\sin(2\pi ft)$$

and the amplitude of this field varies along the X axis of the Cartesian coordinate. The given arbitrary oriented magnetic field B lies in the XY plane and makes an angle θ with the electric field E, as shown in Fig. 5. It is desired to investigate the dynamic characteristics of a charged particle, Q coulombs, injected into these fields with an arbitrary initial velocity of V_o

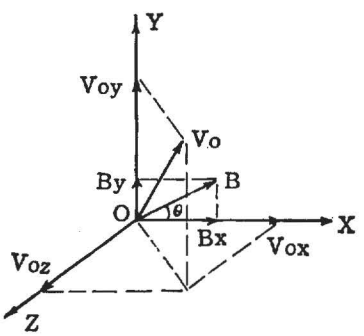

Fig 5

Let the origin of the chosen coordinate axes coinsides with the initial position of the injected particle. Then, we resolve V_o into components of V_{ox}, V_{oy} & V_{oz} along X, Y & Z axis, respectively. The magnetic field intensity B has the X component as

$$B_x=B\ \mathrm{Cos}\ \theta$$

and the Y component as

$$B_y=B\ \mathrm{Sin}\ \theta$$

By Eqs. (2—1) & (2—2), we get the force functions acting upon the charged particle along X—, Y— & Z— axis as:

$$F_x=mdV_x/dt=QE_m\mathrm{Sin}(2\pi ft)-QB_yV_z \tag{7—1}$$

$$F_y=mdV_y/dt=B_xQV_z \tag{7—2}$$

$$F_z=mdV_z/dt=QB_yV_x-QB_xV_y \tag{7—3}$$

Differentiating Eq. (7—3) with respect to t, we have

$$m\frac{d^2V_z}{dt^2}=QB_y\frac{dV_x}{dt}-QB_x\frac{dV_y}{dt} \tag{7—4}$$

Substituting Eqs (7—1) & (7—2) to Eq. (7—4), we obtain

$$\begin{aligned}m^2\frac{d^2V_z}{dt^2}&=QB_y[QE_m\mathrm{Sin}(2\pi ft)-QB_yV_z]-Q^2B_x^2V_z\\&=Q^2B_yE_m\mathrm{Sin}(2\pi ft)-Q^2(B_x^2+B_y^2)V_z\\&=Q^2B_yE_m\mathrm{Sin}(2\pi ft)-Q^2B^2V_z\end{aligned}$$

or

$$m^2\frac{d^2V_z}{dt^2}+Q^2B^2V_z=Q^2B_yE_m\mathrm{Sin}(2\pi ft) \tag{7—5}$$

This is a nonhomogeneous, linear differential equation of second order and has the similar form as Eq. (6—4). Therefore, by the similar operating procedure, we may obtain the general solution of Eq. (7—5) as:

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$$V_z = K_3\sin(2\pi ft) + K_1\sin(\frac{QB}{m}t) + K_2\cos(\frac{QB}{m}t)$$

$$= \frac{Q^2B_yE_m}{Q^2B^2-(2\pi fm)^2}\sin(2\pi ft) + K_1\sin(\frac{QB}{m}t) + K_2\cos(\frac{QB}{m}t) \qquad (7\text{—}6)$$

The arbitrary coefficients K_1 & K_2 may be evaluated from knowledge of the initial conditions that $V_x = V_{ox}$, $V_y = V_{oy}$ & $V_z = V_{oz}$ when $t = o$. Substituting these conditions to Eqs (7—6) & (7—3), we have

$$K_2 = V_{oz}$$

and since

$$QB_yV_{ox} - QB_xV_{oy} = m[\frac{2\pi fQ^2B_yE_m}{Q^2B^2-(2\pi fm)^2} + \frac{QB}{m}K_1]$$

$$\therefore \quad K_1 = \frac{1}{B}[B_yV_{ox} - B_xV_{oy} - \frac{2\pi fmQB_yE_m}{Q^2B^2-(2\pi fm)^2}]$$

Hence, the general solution of Eq. (7—6) becomes

$$V_z = K_3\sin(2\pi ft) + K_1\sin(\frac{QB}{m}t) + K_2\cos(\frac{QB}{m}t)$$

$$= \frac{Q^2B_yE_m}{Q^2B^2-(2\pi fm)^2}\sin(2\pi ft) + \frac{1}{B}[B_yV_{ox} - B_xV_{oy} - \frac{2\pi fmQB_yE_m}{Q^2B^2-(2\pi fm)^2}]\sin(\frac{QB}{m}t) + V_{oz}\cos(\frac{QB}{m}t)$$

$$= (set)V_{z1}\sin(2\tau ft) + V_{z2}\sin(\frac{QB}{m}t) + V_{z3}\cos(\frac{QB}{m}t) \qquad (7\text{—}7)$$

Substituting Eq. (7—7) to Eq. (7—2), we get

$$m\int_{V_y=V_{oy}}^{V_y=V_y} dV_y = QB_x\int_{t=o}^{t=t} V_z dt$$

$$\therefore \quad V_y = V_{oy} + \frac{QB_x}{m}\int_o^t [K_3\sin(2\pi ft) + K_1\sin(\frac{QB}{m}t) + K_2\cos(\frac{QB}{m}t)]dt$$

$$= V_{oy} + \frac{QB_x}{m}[-\frac{K_3}{2\pi f}\cos(2\pi ft) - \frac{m}{QB}K_1\cos(\frac{QB}{m}t) + \frac{m}{QB}K_2\sin(\frac{QB}{m}t)]\Big|_o^t$$

$$= -\frac{QB_xK_3}{2\pi fm}\cos(2\tau ft) - \frac{B_x}{B}K_1\cos(\frac{QB}{m}t) + \frac{B_x}{B}K_2\sin(\frac{QB}{m}t) + V_{oy} + \frac{QB_xK_3}{2\pi fm} + \frac{B_x}{B}K_1$$

$$= -\frac{Q^3B_xB_yE_m}{2\pi fm[Q^2B^2-(2\tau fm)^2]}\cos(2\tau ft) - \frac{B_x}{B^2}[B_yV_{ox} - B_xV_{oy} - \frac{2\pi fmQB_yE_m}{Q^2B^2-(2\pi fm)^2}]\cos(\frac{QB}{m}t) + \frac{B_x}{B}V_{oz}\sin(\frac{QB}{m}t) + [V_{oy} + \frac{B_x}{B^2}(B_yV_{ox} - B_xV_{oy} - \frac{2\pi fmQB_yE_m}{Q^2B^2-(2\pi fm)^2}) + \frac{Q^3B_xB_yE_m}{2\pi fm\{Q^2B^2-(2\pi fm)^2\}}]$$

$$= (set)V_{y1}\cos(2\pi ft) + V_{y2}\cos(\frac{QB}{m}t) + V_{y3}\sin(\frac{QB}{m}t) + V_{y4} \qquad (7\text{—}8)$$

Substituting Eq. (7—7) to Eq. (7—1), we obtain

$$V_x = V_{ox} + \frac{1}{m}\int_{t=o}^{t=t}[QE_m\sin(2\pi ft) - QB_yV_z]dt$$

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$$=V_{ox}+\frac{1}{m}\int_o^t QE_m\mathrm{Sin}(2\pi ft)dt$$
$$-\frac{QB_y}{m}\int_o^t[K_3\mathrm{Sin}(2\pi ft)+K_1\mathrm{Sin}(\frac{QB}{m}t)+K_2\mathrm{Cos}(\frac{QB}{m}t)]dt$$
$$=V_{ox}+\frac{QE_m}{2\pi fm}[1-\mathrm{Cos}(2\pi ft)]+\frac{QB_y}{2\pi fm}K_3\mathrm{Cos}(2\pi ft)+\frac{B_y}{B}K_1\mathrm{Cos}(\frac{QB}{m}t)$$
$$-\frac{B_y}{B}K_2\mathrm{Sin}(\frac{QB}{m}t)-\frac{QB_y}{2\pi fm}K_3-\frac{B_y}{B}K_1$$
$$=[-\frac{QE_m}{2\pi fm}+\frac{Q^3B_y^2E_m}{2\pi fm\{Q^2B^2-(2\pi fm)^2\}}]\mathrm{Cos}(2\pi ft)$$
$$+\frac{B_y}{B^2}[B_yV_{ox}-B_xV_{oy}-\frac{2\pi fmQB_yE_m}{Q^2B^2-(2\pi fm)^2}]\mathrm{Cos}(\frac{QB}{m}t)-\frac{B_y}{B}V_{oz}\mathrm{Sin}(\frac{QB}{m}t)$$
$$+[V_{ox}-\frac{Q^3B_y^2E_m}{2\pi fm\{Q^2B^2-(2\pi fm)^2\}}-\frac{B_y}{B^2}(B_yV_{ox}-B_xV_{oy}-\frac{2\pi fmQB_yE_m}{Q_2B_2-(2\pi fm)^2})$$
$$+QE_m/2\pi fm]$$
$$=(set)V_{x1}\mathrm{Cos}(2\pi ft)+V_{x2}\mathrm{Cos}(\frac{QB}{m}t)+V_{x3}\mathrm{Sin}(\frac{QB}{m}t)+V_{x4} \qquad (7-9)$$

Since the velocity is equal to the first derivative of displacement and the moving charged particle starts from the reference point of the coordinate origin, therefore, we may determine the displacement along X axis as:

$$X=\int_{t=o}^{t=t}V_xdt$$
$$=\int_o^t[V_{x1}\mathrm{Cos}(2\pi ft)+V_{x2}\mathrm{Cos}(\frac{QB}{m}t)+V_{x3}\mathrm{Sin}(\frac{QB}{m}t)+V_{x4}]dt$$
$$=\frac{V_{x1}}{2\pi f}\mathrm{Sin}(2\pi ft)+\frac{m}{QB}V_{x2}\mathrm{Sin}(\frac{QB}{m}t)-\frac{m}{QB}V_{x3}\mathrm{Cos}(\frac{QB}{m}t)+V_{x4}t+\frac{m}{QB}V_{x3}$$
$$=\frac{1}{2\pi f}[-\frac{QE_m}{2\pi fm}+\frac{Q^3B_y^2E_m}{2\pi fm\{Q^2B^2-(2\pi fm)^2\}}]\mathrm{Sin}(2\pi ft)$$
$$+\frac{mB_y}{QB^3}[B_yV_{ox}-B_yV_{oy}-\frac{2\pi fmQB_yE_m}{Q^2B^2-(2\pi fm)^2}]\mathrm{Sin}(\frac{QB}{m}t)$$
$$+\frac{mB_y}{QB^2}V_{oz}\mathrm{Cos}(\frac{QB}{m}t)+[V_{ox}-\frac{Q^3B_y^2E_m}{2\pi fm\{Q^2B^2-(2\pi fm)^2\}}$$
$$-\frac{B_y}{B^2}(B_yV_{ox}-B_xV_{oy}-\frac{2\pi fmQB_yE_m}{Q^2B^2-(2\pi fm)^2})+\frac{QE_m}{2\pi fm}]t-\frac{mB_y}{QB^2}V_{oz}$$
$$=(set)X_1\mathrm{Sin}(2\pi ft)+X_2\mathrm{Sin}(\frac{QB}{m}t)+X_3\mathrm{Cos}(\frac{QB}{m}t)+X_4t+X_5 \qquad (7-10)$$

the displacement along Y axis as:

$$Y=\int_{t=o}^{t=t}V_ydt$$
$$=\int_o^t[V_{y1}\mathrm{Cos}(2\pi ft)+V_{y2}\mathrm{Cos}(\frac{QB}{m}t)+V_{y3}\mathrm{Sin}(\frac{QB}{m}t)+V_{y4}]dt$$
$$=\frac{V_{y1}}{2\pi f}\mathrm{Sin}(2\pi ft)+\frac{m}{QB}V_{y2}\mathrm{Sin}(\frac{QB}{m}t)-\frac{m}{QB}V_{y3}\mathrm{Cos}(\frac{QB}{m}t)+V_{y4}t+\frac{m}{QB}V_{y3}$$
$$=\frac{-Q^3B_xB_yE_m}{(2\pi f)^2m[Q^2B^2-(2\pi fm)^2]}\mathrm{Sin}(2\pi ft)$$
$$-\frac{mB_x}{QB^3}[B_yV_{ox}-B_xV_{oy}-\frac{2\pi fmQB_yE_m}{Q^2B^2-(2\pi fm)^2}]\mathrm{Sin}(\frac{QB}{m}t)$$

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$$-\frac{mB_x}{QB^2}V_{oz}\cos\left(\frac{QB}{m}t\right)+\left[V_{oy}+\frac{B_x}{B^2}\left(B_yV_{ox}-B_xV_{oy}-\frac{2\pi fmQB_yE_m}{Q^2B^2-(2\pi fm)^2}\right)\right.$$
$$\left.+\frac{Q^3B_xB_yE_m}{2\pi fm\{Q^2B^2-(2\pi fm)^2\}}\right]t+\frac{mB_x}{QB^2}V_{oz}$$
$$=(set)Y_1\sin(2\pi ft)+Y_2\sin\left(\frac{QB}{m}t\right)+Y_3\cos\left(\frac{QB}{m}t\right)+Y_4t+Y_5 \qquad (7\text{—}11)$$

and the displacement along Z axis as:

$$Z=\int_{t=o}^{t=t}V_z\,dt$$
$$=\int_o^t\left[V_{z1}\sin(2\pi ft)+V_{z2}\sin\left(\frac{QB}{m}t\right)+V_{z3}\cos\left(\frac{QB}{m}t\right)\right]dt$$
$$=-\frac{V_{z1}}{2\pi f}\cos(2\pi ft)-\frac{m}{QB}V_{z2}\cos\left(\frac{QB}{m}t\right)+\frac{m}{QB}V_{z3}\sin\left(\frac{QB}{m}t\right)+\frac{V_{z1}}{2\pi f}+\frac{m}{QB}V_{z2}$$
$$=-\frac{Q^2B_yE_m}{2\pi f[Q^2B^2-(2\pi fm)^2]}\cos(2\pi ft)-\frac{m}{QB^2}\left[B_yV_{ox}-B_xV_{oy}\right.$$
$$\left.-\frac{2\pi fmQB_yE_m}{Q^2B^2-(2\pi fm)^2}\right]\cos\left(\frac{QB}{m}t\right)+\frac{m}{QB}V_{oz}\sin\left(\frac{QB}{m}t\right)$$
$$+\frac{Q^2B_yE_m}{2\pi f[Q^2B^2-(2\pi fm)^2]}+\frac{m}{QB^2}\left[B_yV_{ox}-B_xV_{oy}-\frac{2\pi fmQB_yE_m}{Q^2B^2-(2\pi fm)^2}\right]$$
$$=(set)Z_1\cos(2\pi ft)+Z_2\cos\left(\frac{QB}{m}t\right)+Z_3\sin\left(\frac{QB}{m}t\right)+Z_4 \qquad (7\text{—}12)$$

Differentiating Eq. (7—9) with respect to t, we get the acceleration along X axis as:

$$A_x=dV_x/dt$$
$$=\frac{d}{dt}\left[V_{x1}\cos(2\pi ft)+V_{x2}\cos\left(\frac{QB}{m}t\right)+V_{x3}\sin\left(\frac{QB}{m}t\right)+V_{x4}\right]$$
$$=\left[\frac{QE_m}{m}-\frac{Q^3B_y{}^2E_m}{m\{Q^2B^2-(2\pi fm)^2\}}\right]\sin(2\pi ft)$$
$$-\frac{QB_y}{mB}\left[B_yV_{ox}-B_xV_{oy}-\frac{2\pi fmQB_yE_m}{Q^2B^2-(2\pi fm)^2}\right]\sin\left(\frac{QB}{m}t\right)-\frac{QB_y}{m}V_{oz}\cos\left(\frac{QB}{m}t\right)$$
$$=(set)A_{x1}\sin(2\pi ft)+A_{x2}\sin\left(\frac{QB}{m}t\right)+A_{x3}\cos\left(\frac{QB}{m}t\right) \qquad (7\text{—}13)$$

Differentiating Eq. (7—8) with respect to t, we get the acceleration along Y axis as:

$$A_y=dV_y/dt$$
$$=\frac{d}{dt}\left[V_{y1}\cos(2\pi ft)+V_{y2}\cos\left(\frac{QB}{m}t\right)+V_{y3}\sin\left(\frac{QB}{m}t\right)+V_{y4}\right]$$
$$=\frac{Q^3B_xB_yE_m}{m[Q^2B^2-(2\pi fm)^2]}\sin(2\pi ft)$$
$$+\frac{QB_x}{mB^2}\left[B_yV_{ox}-B_xV_{oy}-\frac{2\pi fmQB_yE_m}{Q^2B^2-(2\pi fm)^2}\right]\sin\left(\frac{QB}{m}t\right)$$
$$+\frac{QB_x}{m}V_{oz}\cos\left(\frac{QB}{m}t\right)$$
$$=(set)A_{y1}\sin(2\pi ft)+A_{y2}\sin\left(\frac{QB}{m}t\right)+A_{y3}\cos\left(\frac{QB}{m}t\right) \qquad (7\text{—}14)$$

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Similarly, the acceleration along Z axis may be determined as:

$$A_z=dV_z/dt$$

$$=\frac{d}{dt}[V_{z1}\mathrm{Sin}(2\pi ft)+V_{z2}\mathrm{Sin}(\frac{QB}{m}t)+V_{z3}\mathrm{Cos}(\frac{QB}{m}t)]$$

$$=\frac{2\pi fQ^2B_yE_m}{Q^2B^2-(2\pi fm)^2}\mathrm{Cos}(2\pi ft)$$

$$+\frac{Q}{m}[B_yV_{ox}-B_xV_{oy}-\frac{2\pi fmQB_yE_m}{Q^2B^2-(2\pi fm)^2}]\mathrm{Cos}(\frac{QB}{m}t)$$

$$-\frac{QB}{m}V_{oz}\,\mathrm{Sin}(\frac{QB}{m}t)$$

$$=(\mathrm{set})A_{z1}\mathrm{Cos}(2\pi ft)+A_{z2}\mathrm{Cos}(\frac{QB}{m}t)+A_{z3}\mathrm{Sin}(\frac{QB}{m}t) \qquad (7—15)$$

Therefore, by combining the corresponding results just derived, we get the resultant dynamic characteristics of the charged particle under the given conditions as:

Acceleration $A=iA_x+jA_y+kA_z$

$$=i[A_{x1}\mathrm{Sin}(2\pi ft)+A_{x2}\mathrm{Sin}(\frac{QB}{m}t)+A_{x3}\mathrm{Cos}(\frac{QB}{m}t)]$$

$$+j[A_{y1}\mathrm{Sin}(2\pi ft)+A_{y2}\mathrm{Sin}(\frac{QB}{m}t)+A_{y3}\mathrm{Cos}(\frac{QB}{m}t)]$$

$$+k[A_{z1}\mathrm{Cos}(2\pi ft)+A_{z2}\mathrm{Cos}(\frac{QB}{m}t)+A_{z3}\mathrm{Sin}(\frac{QB}{m}t)] \qquad (7—16)$$

Velocity $V=iV_x+jV_y+kV_z$

$$=i[V_{x1}\mathrm{Cos}(2\pi ft)+V_{x2}\mathrm{Cos}(\frac{QB}{m}t)+V_{x3}\mathrm{Sin}(\frac{QB}{m}t)+V_{x4}]$$

$$+j[V_{y1}\mathrm{Cos}(2\pi ft)+V_{y2}\mathrm{Cos}(\frac{QB}{m}t)+V_{y3}\mathrm{Sin}(\frac{QB}{m}t)+V_{y4}]$$

$$+k[V_{z1}\mathrm{Sin}(2\pi ft)+V_{z2}\mathrm{Sin}(\frac{QB}{m}t)+V_{z3}\mathrm{Cos}(\frac{QB}{m}t)] \qquad (7—17)$$

Displacement $S=ix+jy+kz$

$$=i[X_1\mathrm{Sin}(2\pi ft)+X_2\mathrm{Sin}(\frac{QB}{m}t)+X_3\mathrm{Cos}(\frac{QB}{m}t)+X_4t+X_5]$$

$$+j[Y_1\mathrm{Sin}(2\pi ft)+Y_2\mathrm{Sin}(\frac{QB}{m}t)+Y_3\mathrm{Cos}(\frac{QB}{m}t)+Y_4t+Y_5]$$

$$+k[Z_1\mathrm{Cos}(2\pi ft)+Z_2\mathrm{Cos}(\frac{QB}{m}t)+Z_3\mathrm{Sin}(\frac{QB}{m}t)+Z_4] \qquad (7—18)$$

8. Discussion.

By mathematical method, we have already been derived the exact characteristics of motion of a charged particle under the influence of the following given field conditions:

(a) A dynamic electric field.

(b) A dynamic magnetic field.

(c) A dynamic electric field with a parallel magnetic field.

(d) A dynamic electric field with a perpendicular magnetic field.

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(e) A dynamic electric field accompanied by an arbitrary oriented magnetic field.

The exact mathematical results obtained under the influence of case (e), may be regarded as the general results. Since, by setting the arbitrary oriented magnetic field coinsides with the given electric field intensity, we may get the reults as obtained under the influence of case (c); by setting the magnetic field equals to zero, we may get the results as shown in the case (a); etc..

9. References.

Page: Physical Mathematics.
Reddick: Advanced Mathematics for Engineers.
Franklin: Methods of Advanced Calculus.
Cohn: Differential Equations.
Seely: Introduction to Electromagnetic Fields.
Seely: Radio Electronics.

荷電質點在非線性電磁場中運動特性之數學分析**

蔡 駿 康*

摘 要

非線性電磁場問題中之重要顯著者，當爲隨時間作正弦曲線變化之電磁場。如無線電輻射波及光波等之波動，均大體隸屬此種波形現象。事實上，大凡週期性變化之任何複雜電磁場，經運用佛里爾級數分析後，均可將其展開爲一列類率與波幅不同之正弦諧波電磁場。荷電質點在此種非線性電磁場中之運動特性，通常多用試驗方法分析之。本報告則採用數學方法，以精確分析荷電質點在非線性電場磁場及不同組合之電場與磁場中之運動特性。

本報告中所論荷電質點運動特性之數學結果，係就下列五種不同之電磁場所推求者：

一、非線性之正弦電場。

二、非線性之正弦磁場。

三、非線性之正弦電場與平行之磁場。

四、非線性之正弦電場與垂直之磁場。

五、非線性之正弦電場與任意相交之磁場。

其中第五項所獲之結果，實可視爲一最普遍通用之數學結果。蓋若令任意相交之磁場與電場相平行，吾人即可求得第三項之結果；若令此磁場爲零，則第一項之結果即可求得。

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* 臺灣省立中興大學應用數學系副教授。

** 本研究經國家長期發展科學委員會五十三年度補助，謹此深致謝忱。

National Chung Hsing University