Toward an Alternative Optimal Control Theory for Replacement and Recyclable Assets

Wan-Tran Huang*
黃萬傳

I 、 Introduction

The dynamic optimizing problem is to allocate scarce mneans among competing ends from an initial time to a terminal time (Intrilligator). There are many theories about optimal asset replacement (Faustman ; Samuelson ; Prienreich; Bellman; Jorgenson; Hirschleifer ; Reid and Bradford 1983, 1987). Among those theories, the Faustman-Samuelson (F-S) gave the replacement criterion that states the maximizing the pressnt discounted value of net returns with respect to constant outpnt prices. In 1985, Chavas, Kliebenstein , and Crershaw developed and applied the techniques of optimal control within the theoretical replacement framework. In 1989 , McClelland , Wetzstein, and Noles developed a dynamic modlel (M-W-N model) describing the optimal replacement strategy for a recyclable asset.

The M-W-N model is essentially a calculus of variation problem where the state of the asset or its productivity is a only function of time or the age of the asset. The purpose of this article is to expand the generality of the M-W-N model by allowing the state of the asset and in productivity to be a function of one or more inputs or control variables. Essentially this transforms the M-W-N model from one using the calculus of variations to one using the maximun principle.

In the following sections we identify the recycle and replace problem, and then presset the model with an extension to the M-W-N model. The paper ends with conclusions.

* Associate Professor, National Ping-Tung Institute of Agriculture , Taiwan, R.O.C.

Ⅱ、The Model

The objective is to develop a model that describes the optimal recycling or replacement strategy for a profit maximizing agent, participating in a competitive market who uses a recyclable asset. The output of the asset faces a cyclical price structure. All other prices such as input costs, the cost of new assets and the price received for a retired asset are held constant. As the productivity of an asset declines the agent has the option of either recycling the asset or replacing the asset . We assume that when an asset is retired it is immediately replaced with another asset. Each asset is in exactly the same condition or state when initially acquired. The agent, therefore, faces the problem not of maximizing returns to a single asset but of maximizing the total present value of an infinitely repeating stream of assets.

The output of the asset is described by the following functions:

(1) $f=f(z(t),u(t),t)$

(2) $fR=fR(x(t),u(t),t)$

where f is the production function for an asset that has not been recycled, f^R is the production function for an asset that has been recycled, x(t) is the state of the asset at time t and u(t) is a vector of input or control variable levels at time t and f^R are assumed to be continuous, differentiable, and concave.

Output is thus defined as being a function of the current state of the asset, the current input or control variable level and time. This model could be generalized to allow the asset to be recyled more than once by adding additional production functions.

The change in the state of the asset through time is described by the following equations of motion:

(3) $$\frac{dx(t)}{dt} = x' = g(x(t),u(t),t)$$

$$(4)\quad \frac{dx^R(t)}{1\ dt} = x^{R\prime} = g^R(x(t),u(t),t)$$

where g is the continuous and differentiable equation of motion that applies to an unrecycled asset and g^R is the continuous and differentiable equation of motion that applies to a recycled asset. The salvage valve of the asset is represented by

$$(5)\quad S = S(X_{Ti})$$

This states that the value of an asset is a function of its state when it is retired. This represents a small departure from the M-W-N model where the salvage value was held to be constant. This salvage value can be either positive or negative with a negative value representing a cost of disposal. In determining the optimal control trajectory, a profit maximizing agent must consider both the effect of the control variables on the output during the asset's life and on the salvage value of the asset. We assume the same cyclical price structure as in the M-W-N model.

$$(6)\quad P(t) = P(Zi,t)$$

where Zi[Zi =Zi(Ti, Ti′)] is the point in the price cycle an asset is initially acquired, Ti, or the point that the asset is brought back into production after being recycled, Ti′. Each price level in the cycle is represented by a point on a unit circle. The circumference of the circle is the length of one price cycle. The price received per unit of output at any point in time can be determined given Zi and the time elapsed since the asset was brought into production. Since all assets are assumed to be in the same state when acquired, we are letting "F" represent the constant cost of a new asset.

As stated previously, the objective of our profit maximizing agent is to maximize the total present value of the infinitely repeating stream of assets. Using the functions defined above, the objective functional is

$$(6)\quad \max TPV(u_1,u_2,u_3,\ldots,T_1',T_2',T_3',\ldots) = -F$$

$$+ \int_0^{T_1'} (pf - ru)e^{-kt}dt - w_1\left[\int_{T_1'}^{T_1'+L} ce^{-kt}dt - \int_{T_1'+L}^{T_1} (pf^R - ru)e^{-kt}dt\right]$$

$$+ (S(x_{T_1}) - F)e^{-kT_1} + \int_{T_1}^{T_2'} (pf - ru)e^{-kt}dt - w_2\left[\int_{T_2}^{T_2'+L} ce^{-kt}dt\right.$$

$$\left. - \int_{T_2'+L}^{T_2} (pf^R - ru)e^{-kt}dt\right] + (S(x_{T_2})-F)e^{-kT_2} + \ldots$$

subject to : $x' = g(x,u,t)$

$x^{R'} = g^R(x,u,t)$

where Wi= 1 if the asset is to be recycled
　　　　0 if the asset is not to be recycl

r= the constant cost per level of the control variables.

c= the constant incremental cost incurred while an asset is being recycled.

L= the length of time required to recycle an asset.

The solution to this problem consists of finding the optimal control trajectories for each asset $(u_1, u_2, u_3, .\ )$ the optimal time to recycle an asset $(T_1', T_2', T_3', \ldots)$ and the optimal time to retire an asset $(T_1\ , T_2\ , T_3\ , \ldots)$ By arbitrarily choosing Ti and Ti′ we can determine the optimal control trajectory for an individual asset by using optimal control theory. Once the optimal control trajectories are determined we can use the transveraslity conditions

$\frac{\partial TPV_i}{\partial T_i'} = 0$, and $\frac{\partial TPV_i}{\partial T_i} = 0$, to determine the optimal opints in time to recycle and/or replace an asset. The present value of the i^{th} asset is

$$(7)\ PVi = -Fe^{-k(Ti-1)} + \int_{T_{i-1}}^{T_i'} (pf - ru)e^{-kt}dt - w_i \left[\int_{T_i}^{T_i'+L} ce^{-kt}dt - \int_{T_i'+L}^{T_i} (pf^R - ru)e^{-kt}dt \right] + S(x_{T_i})e^{-kT_i}$$

subject to : $x' = g(x,u,t)$

$x^{R'} = g^R(x,u,t)$

We can determine the optimal control trajectory that will maximize the returns to the asset before it has been recycled and the control trajectory that will maximize returns to the asset after it has been recycled. The first order conditions for both situations are similar The only difference is in the production functions and equations of motion applied in the problem (f or f^R, g or g^R). The Hamiltonian function(Kamien and Schwartze) is

$$(8)\ H = (pf-ru)e^{-kt}dt + \Phi\, g_u$$

The first order conditions for a maximum are:

$$(9)\ \frac{\partial H}{\partial u} = (pf_u - r)e^{-kt} + \Phi g_u = 0$$

$$\Phi = \frac{-e^{-kt}}{g_u}(pf_u - r)e^{-kt}$$

$$(10)\quad \frac{\partial H}{\partial x} = -\Phi' = pf_x e^{-kt} + \Phi g_x$$

Φ is the marginal valuation of the asset at time t (Kamien and Schwartz). pfur is the net marginal return from output resulting from a unit change in the control variable. g_u is the marginal change in the state of the asset resulting from a change in the level of the control variable. Thus, as Chavas, Kliebenstein, and Crenshaw point out, condition(9) states that the optimum level of the control variable causes the net marginal returns from current output to be exactly counterbalanced by the value of the marginal change of the asset at any given time.

Chavas, Kliebenstein, and Crenshaw point out that Φ' is the rate , the marginal value of the asset is changing through time. From condition (2) one can see that at the optimum control levels a change in the value of the marginal output due to a change in the state of the asset ($pf_x e^{-kt}$) must be exactly offset by the change in the value of the asset ($\Phi' g_x$)

These two conditions provide us with two unknowns, Φ (t) and u(t) , and two equations. Further insights into the nature of these first order conditions and the optimal control trajectory can be obtained be simultaneously solving the two equations representing the first order conditions for the two unknowns. Taking the total differential of(9) with respect to time we find

$$(11)\quad \Phi' = \frac{e^{-kt}}{g_u}\left([k(pf_u - r)] - [p'f_u + pf_u' - \frac{g_u'(pf_u - r)}{g_u}]\right)$$

Substituting this result into condition(10) and solving for Φ we find

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$$(12)\quad \Phi = -e^{-kt}\left(pf_x + \frac{1}{g_u}\left([k(pf_u - r)] - [p'f_u + pf_u' - \frac{g_u'(pf_u - r)}{g_u}]\right)\right)\frac{1}{g_x}$$

Equating this result with condition(9) we find after same manipulation

$$(13)\quad (pf_u - r)\left(g_x - k - \frac{g_u'}{g_u}\right) = p(g_u f_x - f_u') - p' f_u$$

The left side of this equality is the net value of the marginal output of the control variable $(pf_u - r)$ multiplied by the effect the current state of the asset has on the change of the asset , g_x ,less the discount rate less the ratio of the time rate of change of the marginal effect of the control variable on the asset to the marginal effect of the control variable on the asset, g_u'/g_u . The right side is the value of the marginal effect of the control variable on the asset multiplied by the marginal effect of the current state of the asset on output, $p\,g_u f_x$, minus the value of the time rate of change of the marginal effect of the control variable on output, pf_u', minus the time rate of change of the price of the ooutput multiplied by the marginal effect of the control variable on output, $p' f_u$.

The optimum level of the control variable for any given point in time can be determined by solving the above equality for u(t). Equation(13) gives the optimal value of u(t) regardless of whether or not the optimal control trajectory has been followed up to the present time. Given the equations of motion describing the change in the state of the asset through time, the production function, the current state of the asset, and the current prices, the optimal level of the control variable at a given time can be determined.

The optimun points in time to recycle the i^{th} asset(T_i') and to retire the i^{th} asset (T_i) can be determined with the aid of the following transversality conditions.

$$(14)\quad \frac{\partial TPV_i}{\partial T_i'} = (pf - ru)T_i' e^{-kT_i} - ce^{-kT_i+L'} + ce^{-kT_i'} + \int_{T_i'+L}^{T_i} \frac{\partial p}{\partial T_i'}(f^R)dt = 0$$

$$(15)\quad \frac{\partial TPV_i}{\partial T_i} = (pf^R - ru)]_{T_i} e^{-kT_i} + \frac{\partial S(x_{T_i})}{\partial T_i}(e^{-kT_i}) + ke^{-kT_i}(F - S(x_{T_i})) + \int_{T_i}^{T_i'+l} \frac{\partial p}{\partial T_i}(f)dt = 0$$

These are essentially the same transversality conditions that apply to the M-W-N model. Equation(14) states that an asset should be recycled when the net value of its output at T_i', is equal to the marginal cost of recycling the asset less the when the returns from the last point of production plus the marginal change in the salvage value due to a change in the time of its retirement equal the marginal cost of disposal of the asset less the marginal returns resulting from a change in the time the next asset into production.

Note that if an asset is not recycled $T_i' = T_i$. There is no change in the first order conditions for the optimal control trajectory, The transversality condition, however, is slightly different if an asset is not recycled.

$$(16)\quad \frac{\partial TPV_i}{\partial T_i} = (pf - ru)_{Ti}e^{-kTi} + \frac{\partial S(x_{Ti})}{\partial T_i} e^{-kTi} + ke^{-kTi}(F - S(x_{Ti})) + \int_{T_i}^{T_{i+1}} \frac{\partial p}{\partial T_i} f\, dt = 0$$

The interpretation of this condition is the same as the interpretation given for (15) with the exception that the production function for an unrecycled asset is used in the first term rather than the procuction function for an asset that has been recycled.

Ⅲ、Integrating the Two Sets of Conditions

Determining the optimal control trajectories and durations of the various productions phases given the first order and transversality conditions developed earlier follows essentially the same procedure as used by M-W-N model. Each possible price level is represented by a point on a unit circle. Because all of the assets in the replacement stream are exactly alike when aquired the only thing that can cause differences in the optimal control decisions for individual assets is the price cycle. Assets acquired at different points in the price cycle face different series of prices for their output. Therefore, the optimal control trajectory, duration of the production period before the asset is recycled, and the duration of the prodction period from the time the asset is recycled until it is retired can be determined for an asset that is brought into production at each possible beginning point in the price cycle.

By equation(15) the only succeeding asset that affects the duration of the productive life of an asset is the one that immediately succeeds it. A possible solution method, therefore, would be to estimate the durations of the two phases of an asset's productive life (before and after it is recycled)that was brought into production at a given Zi in the price cycle. The durations of the two phases of the productive life of the succeeding asset could also be esimated. The optimal control trajectories could then be determined using the first order conditions derived earlier. To determine whether or not the estimated T_i' and T_i are optimal, the transversality conditions could be applied. If $\frac{\partial TPV_i}{\partial T_i'} > 0$ it implies that the present value could be increased by increasing the duration of the production period before the asset is recycled. If $\frac{\partial TPV_i}{\partial T_i} > 0$ it implies that the present value could be increased by increasing the duration of the production period after the asset has been recycled it is retired. Of course, if the above derivatives are negative it implies that the durations of the respective production periods should be shortened. Using this information new estimates of the beginning and ending times can be made and the entire process repeated until no new estimates of the beginning and ending times are indicated. This process can be repeated for an asset beginning at every possible Zi in the price cycle.

The above procedure could be repeated assuming the asset is not recycled. By comparing the two sets of results it can be determined whether or not it is optimal to recycle an individual asset.

IV、 Conclusions

An extension to the theory on the replacement and recyclable asset is presented in this paper. That is to say, this paper is to derive the generality of the M-W-N model by allowing the state of the asset and its productivity to be a function of one or more inputs or control variables. The model developed by this paper describes the optimal recycling or replacement strategy for a profit maxinixing agent, participating in a competitive market, who uses a recyclable asset.

The theoretical differences between M-W-N model and this study are as followings : (a) M-W-N model employed the calculus of variations, here used the maximum principle. (b) The production functions and equations of motion are different. However, caution is required in applying the model described in previous sections to the uncontrolled and stochastic environment.

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置換性與循環性資產之最適控制理論

黃萬傳*

摘　　要

本文旨在修正及擴展由McClelland、Wetzstein和Noles(M-W-N)等所發展之置換性與循環性資產之最適理論，即修正資產狀態及其生產力是一個或以上控制變數之函數。本文所誘導之模式和結果，可用在一競爭市場情境下，追求最大利潤的循環性資產使用者之最適置換策略。

本文模式與M-W-N模式的主要差異爲：(1)M-W-N模式係應用變微分析法，然本文則引用最大原則法；(2)本文所設立的生產函數和移動方程式是不同於M-W-N 模式。

＊作者爲國立屛東農專副教授