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The threshold dividend strategy on a class of dual model with tax payments

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Abstract: A class of dual risk model was considered in which dividends are paid under a threshold strategy and tax payments are paid according to a loss-carry forward system. For this model, the expectation of the discounted dividends until ruin was investigated and their corresponding integral equations, integro-differential equations and analytical expressions were derived. Finally, the case where profits follow an Erlang(2) distribution was solved.

Key words: compound Poisson surplus process; expected discounted dividend function; dual risk model; threshold strategy; ruin time

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一类带税的对偶模型的门槛分红策略

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摘要: 研究了一类在安全负载体系下进行赋税且按照门槛策略进行分红的对偶风险模型. 分析了此模型破产前折现分红的期望, 得到了其满足的积分方程、积分-微分方程和相关的表达式. 最后, 在特例 Erlang(2) 分布下给出了一般解.

关键词: 复合 Poisson 盈余过程; 期望折现分红函数; 对偶风险模型; 门槛策略; 破产时刻

0 Introduction

In insurance mathematics, some interesting results have been obtained on a class of model which is dual to the classical Lundberg risk model. See Refs. [1-6]. In this model, the surplus at

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time t is

$$U(t) = u - ct + S(t) \tag{1}$$

where u is the initial surplus, c the constant rate at which expenses are paid out and $\{S(t), t\geqslant 0\}$ is the aggregate positive profits process. Thus, process (1) can describe a class of companies whose inherent business involves a constant flow of expenses while profits arrive occasionally due to some contingent events.

In resent years, quite a few papers have discussed risk models with tax payments of loss-carry forward type. Albrecher et al.[7] investigated how the loss-carry forward tax payments affect the behavior of the dual process (1) with general inter-innovation times and exponential innovation sizes. In their model, the company pays tax at rate $\gamma \in [0,1)$ on the excess of each new record high of the surplus over the previous one. Obviously, a new record high can only be achieved by an innovation and hence tax payments only occur at the innovation times. More results can be found in Refs. [8-15].

Following their work, we now consider the dual model with tax payments according to a loss-carry forward system and dividends under a threshold strategy. There are three motivations for why the study of our objective model is relevant. Firstly, the tax payments of any company are necessary, but the amount of tax payments should not lead to bankruptcy, so we investigate a class of risk model with tax payments according to a loss-carry forward system. Secondly, a barrier strategy distributes all excess surplus to shareholders immediately and always caps the surplus at barrier level $b(b>0)$. Under the strategy shareholders can get big dividends, but which may not be realistic, and may cause liquidity problems in the future. In a threshold strategy, excess surplus is paid at a constant rate but not in a single "burst". Comparing the two strategies with barrier and threshold level both equal to the same b, the ruin time under the threshold strategy is longer and shareholders may prefer a threshold strategy. Finally, the dividend problem of the barrier strategy in dual risk model can be seen as a special case of our study, that is, the threshold dividend rate α equals to 0 and the tax payments rate γ equals to 0.

In our model, $\{S(t), t\geqslant 0\}$ are assumed to be a pure jump process defined as $S(t) = \sum_{n=1}^{N(t)} Y_n$ with the innovation number process $N(t)$ being a renewal process whose inter-innovation times $T_i (i=1,2,\cdots)$ have common distribution F. We also assume that the innovation sizes $\{Y_i, i\geqslant 1\}$, independent of $\{T_i, i\geqslant 1\}$, form a sequence of i.i.d. exponentially distributed random variables with exponential parameter β $(\beta>0)$. The loss-carry forward system assumes that the company pays tax at rate $\gamma\in[0,1)$ on the excess of each new record high of the after-tax surplus over the previous one. Furthermore, when the after-tax surplus is lower than a threshold level b, no dividends are paid; when the surplus is higher than b, the company pays dividends at a constant rate α, causing the surplus to decrease more quickly. The surplus process $\{R_{\gamma,b}(t), t\geqslant 0\}$ with initial principal $R_{\gamma,b}(0)=u$ can be expressed as

$$\mathrm{d}R_{\gamma,b}(t) = \begin{cases} -c\mathrm{d}t + \mathrm{d}S(t)\mathbf{1}_{\{R_{\gamma,b}(t^-)+\mathrm{d}S(t) < \max\limits_{0\leqslant s<t} R_{\gamma,b}(s)\}} + \\ (1-\gamma)(R_{\gamma,b}(t^-) + \mathrm{d}S(t) - \\ \max\limits_{0\leqslant s<t} R_{\gamma,b}(s)) \times \mathbf{1}_{\{R_{\gamma,b}(t^-)+\mathrm{d}S(t) \geqslant \max\limits_{0\leqslant s<t} R_{\gamma,b}(s)\}}, \\ \qquad R_{\gamma,b}(t) \geqslant b; \\ -(c-\alpha)\mathrm{d}t + \mathrm{d}S(t)\mathbf{1}_{\{R_{\gamma,b}(t^-)+\mathrm{d}S(t) < \max\limits_{0\leqslant s<t} R_{\gamma,b}(s)\}} + \\ (1-\gamma)(R_{\gamma,b}(t^-) + \mathrm{d}S(t) - \\ \max\limits_{0\leqslant s<t} R_{\gamma,b}(s)) \times \mathbf{1}_{\{R_{\gamma,b}(t^-)+\mathrm{d}S(t) \geqslant \max\limits_{0\leqslant s<t} R_{\gamma,b}(s)\}}, \\ \qquad R_{\gamma,b}(t) < b \end{cases} \tag{2}$$

where $\mathbf{1}_{\{A\}}$ is the indicator function of event A and $R_{\gamma,b}(t^-)$ is the surplus immediately before time t. For practical consideration, we assume that the positive safety loading condition

$$c < E(Y_1)/E(T_1) \tag{3}$$

holds all through this paper. The time of ruin is defined as $T_{\gamma,b} = \inf\{t\geqslant 0: R_{\gamma,b}(t)\leqslant 0\}$ with $T_{\gamma,b}=\infty$ if $R_{\gamma,b}(t)>0$ for all $t\geqslant 0$.

For initial surplus $u>0$, we denote by $V_\gamma(u,b)$ the present value of all dividends until ruin

$$V_\gamma(u,b)=\int_0^{T_\gamma,b} e^{-\delta t}\,dD(t) \tag{4}$$

where $D(t)$ is the aggregate dividends paid from time 0 to t, and $\delta>0$ is the discount factor. It needs to be mentioned that we shall drop the subscript γ whenever γ is zero.

The rest of this paper is organized as follows. In Section 1, analytical expressions of the expected discounted dividends are derived in terms of the corresponding quantity without tax. In Section 2, for Erlang(2) distributed inter-innovation times, explicit expressions of the expected discounted dividends are given.

1 Main results and proofs

For certain common distributions F_{T_1} (the density function f_{T_1}) of T_i ($i=1,2,\cdots$), one can derive integro-differential equations for $V(u,b)$.

Lemma 1.1 For $0<u<b$, $V(u,b)$ satisfies the following integral equation:

$$V(u,b)=\int_0^{u/(c-\alpha)} e^{-\delta t} f_{T_1}(t)\,dt\cdot \int_0^\infty V(u-(c-\alpha)t+y,b)\beta e^{-\beta y}\,dy \tag{5}$$

For $u\geqslant b$, $V(u,b)$ satisfies the following integral equation:

$$\begin{aligned}V(u,b)=&\int_0^{(u-b)/c} f_{T_1}(t)\,dt\cdot\\&\left\{\int_0^\infty e^{-\delta t}V(u-ct+y,b)\beta e^{-\beta y}\,dy+\int_0^t \alpha e^{-\delta s}\,ds\right\}+\\&\int_{(u-b)/c}^{(u-b)/c+b/(c-\alpha)} f_{T_1}(t)\,dt\cdot\\&\int_0^\infty e^{-\delta t}V(b-(c-\alpha)(t-(u-b)/c)+y,b)\beta e^{-\beta y}\,dy+\\&\int_{(u-b)/c}^\infty f_{T_1}(t)\,dt\int_0^{(u-b)/c}\alpha e^{-\delta s}\,ds\end{aligned} \tag{6}$$

Proof Consider the infinitesimal interval from 0 to dt. By conditioning on the occurrence of the first claim, one obtains that when $0<u<b$,

$$\begin{aligned}V(u,b)=&\\&e^{-\delta dt}\{P(T_1>dt)\cdot E_u[V(u-(c-\alpha)dt,b)]+\\&P(T_1\leqslant dt)\cdot E_u[V(u-(c-\alpha)dt+Y_1,b)]\}.\end{aligned}$$

Notice that the first term in the brace above equals to zero, then using

$$P(T_1>dt)=1-\lambda dt+o(dt),$$
$$P(T_1\leqslant dt)=\lambda dt+o(dt),$$

we can get Eq. (5).

Through a similar analytical approach to those used above one can derive Eq. (6). □

Furthermore, using the result that ruin is immediate and no dividend is paid when $u=0$ and the continuity of $V(u,b)$ at $u=b$, we have the boundary conditions

$$V(0^+,b)=0,\ V(b^-,b)=V(b^+,b) \tag{7}$$

Let us assume that the i.i.d innovation waiting times have a common generalized Erlang(n) distribution, i.e. the T_i's are distributed as the sum of n independent and exponentially distributed random variances, which are denoted by $S_n=\eta_1+\eta_2+\cdots+\eta_n$ with η_i having exponential parameters $\lambda_i>0$.

The following Lemma 1.2 gives the integro-differential equations for $V(u,b)$ when T_i's have a generalized Erlang(n) distribution.

Lemma 1.2 Let $\boldsymbol{I}$ and $\boldsymbol{D}$ denote the identity operator and differentiation operator respectively. Then the expected discounted dividend payments $V(u,b)$ satisfies the following integro-differential equation

$$\prod_{k=1}^n\left[\left(1+\frac{\delta}{\lambda_k}\right)\boldsymbol{I}+\frac{c-\alpha}{\lambda_k}\boldsymbol{D}\right]V(u,b)=\int_0^\infty V(u+y,b)\beta e^{-\beta y}\,dy,\ 0<u<b \tag{8}$$

and

$$\prod_{k=1}^n\left[\left(1+\frac{\delta}{\lambda_k}\right)\boldsymbol{I}+\frac{c}{\lambda_k}\boldsymbol{D}\right]V(u,b)=\int_0^\infty V(u+y,b)\beta e^{-\beta y}\,dy+B_n,\ u\geqslant b \tag{9}$$

with

$$B_k=\sum_{i=1}^k\frac{\alpha}{\lambda_i+\delta}\prod_{j=i+1}^k\left(1+\frac{\delta}{\lambda_j}\right),\ k=1,2,\cdots,n \tag{10}$$

In addition, the boundary conditions for $V(u,b)$ are as follows:

$$\left.\begin{aligned}&\prod_{i=1}^k\left[\left(1+\frac{\delta}{\lambda_i}\right)\boldsymbol{I}+\frac{c-\alpha}{\lambda_i}\boldsymbol{D}\right]V(b^-,b)=\\&\prod_{i=1}^k\left[\left(1+\frac{\delta}{\lambda_i}\right)\boldsymbol{I}+\frac{c}{\lambda_i}\boldsymbol{D}\right]V(b^+,b)-B_k,\\&k=1,2,\cdots,n\end{aligned}\right\} \tag{11}$$

$$\left.\begin{array}{c}\prod_{i=1}^{k}\left[\left(1+\frac{\delta}{\lambda_i}\right)\boldsymbol{I}+\frac{c-\alpha}{\lambda_i}\boldsymbol{D}\right]V(0^+,b)=0,\\ k=1,2,\cdots,n-1\end{array}\right\}\quad(12)$$

with Eq. (5).

Proof We follow Ref. [16] and rewrite $V(u,b)$ as $V_k(u,b)$ when $T_i\overset{d}{=}S_n-S_{k-1}$ with $S_0=0$ in the surplus process (2) with $\gamma=0$. Thus, we have $V_1(u,b)=V(u,b)$. When $0<u<b$,

$$\left.\begin{array}{c}V_k(u,b)=\\ \int_0^{u/(c-\alpha)}\lambda_k e^{-(\lambda_k+\delta)t}V_{k+1}(u-(c-\alpha)t,b)\,dt,\\ k=1,2\cdots,n-1\end{array}\right\}\quad(13)$$

and

$$V_n(u,b)=\int_0^{u/(c-\alpha)}\lambda_n e^{-(\lambda_n+\delta)t}dt\int_0^{\infty}V(u-(c-\alpha)t+y,b)\beta e^{-\beta y}dy\quad(14)$$

Let $u-(c-\alpha)t=x$, by changing variables in Eq. (13) and Eq. (14), one gets that when $0<u<b$,

$$\left.\begin{array}{c}V_k(u,b)=\int_0^u\frac{\lambda_k}{c-\alpha}e^{-(\lambda_k+\delta)\frac{u-x}{c-\alpha}}V_{k+1}(x,b)\,dx,\\ k=1,2\cdots,n-1\end{array}\right\}\quad(15)$$

and

$$V_n(u,b)=\int_0^u\frac{\lambda_n}{c-\alpha}e^{-(\lambda_n+\delta)\frac{u-x}{c-\alpha}}dx\int_0^{\infty}V(x+y,b)\beta e^{-\beta y}dy\quad(16)$$

Now differentiating both sides of Eq. (15) and Eq. (16) with respect to u and calculating carefully, one gets

$$\left.\begin{array}{c}V_{k+1}(u,b)=\left[\left(1+\frac{\delta}{\lambda_k}\right)\boldsymbol{I}+\frac{c-\alpha}{\lambda_k}\boldsymbol{D}\right]V_k(u,b),\\ k=1,2\cdots,n-1\end{array}\right\}\quad(17)$$

and

$$\left[\left(1+\frac{\delta}{\lambda_n}\right)\boldsymbol{I}+\frac{c-\alpha}{\lambda_n}\boldsymbol{D}\right]V_n(u,b)=\int_0^{\infty}V(u+y,b)\beta e^{-\beta y}dy\quad(18)$$

Combining Eq. (17) with Eq. (18), we can derive Eq. (8) for $V(u,b)$ on $(0,b)$.

Using a similar approach, we have for $u\geqslant b$

$$V_k(u,b)=\int_0^{\frac{u-b}{c}}\lambda_k e^{-(\lambda_k+\delta)t}V_{k+1}(u-ct,b)\,dt+\int_{\frac{u-b}{c}}^{\frac{u-b}{c}+\frac{b}{c-\alpha}}\lambda_k e^{-(\lambda_k+\delta)t}V_{k+1}\left(b-(c-\alpha)\left(t-\frac{u-b}{c}\right),b\right)dt\quad(19)$$

for $k=1,2,\cdots,n-1$, and

$$V_n(u,b)=\int_0^{\frac{u-b}{c}}\lambda_n e^{-(\lambda_n+\delta)t}dt\times\left\{\int_0^{\infty}V(u-ct+y,b)\beta e^{-\beta y}dy+\alpha t\right\}+\int_{\frac{u-b}{c}}^{\frac{u-b}{c}+\frac{b}{c-\alpha}}\lambda_n e^{-(\lambda_n+\delta)t}dt\times\left\{\int_0^{\infty}V\left(b-(c-\alpha)\left(t-\frac{u-b}{c}\right)+y,b\right)\beta e^{-\beta y}dy+\alpha\frac{u-b}{c}\right\}\quad(20)$$

Again, by changing variables in Eq. (19) and Eq. (20) (let $u-ct=x$) and then differentiating them with respect to u, we obtain for $u\geqslant b$

$$\left.\begin{array}{c}V_{k+1}(u,b)=\left[\left(1+\frac{\delta}{\lambda_k}\right)\boldsymbol{I}+\frac{c}{\lambda_k}\boldsymbol{D}\right]V_k(u,b),\\ k=1,2\cdots,n-1\end{array}\right\}\quad(21)$$

and

$$\left[\left(1+\frac{\delta}{\lambda_n}\right)\boldsymbol{I}+\frac{c}{\lambda_n}\boldsymbol{D}\right]V_n(u,b)=\int_0^{\infty}V(u+y,b)\beta e^{-\beta y}dy+\frac{\alpha}{\lambda_n+\delta}\quad(22)$$

Using Eq. (21) and Eq. (22), we obtain Eq. (10) for $V(u,b)$ on $[b,\infty)$. Finally, since when $u=0$ ruin is immediate and $V(u,b)$ is continuous at b, we have the boundary conditions Eq. (11) and Eq. (12). □

With the preparations made above, we can now derive analytical expressions of the expected discounted dividend payments $V_\gamma(u,b)$ for the surplus process $\{R_{\gamma,b}(t),t\geqslant0\}$. We claim that the process $\{R_{\gamma,b}(t),t\geqslant0\}$ shall up-cross the initial surplus level u at least once to avoid ruin.

Now, let

$$g_\delta(u):=E_u[e^{-\delta\tau_u}]\quad(23)$$

denote the Laplace transform of the first upper exit time τ_u, which is the time until the risk process $\{R_b(t),t>0\}$ starting with initial capital u reaching a new record high for the first time without leading to ruin before that event. In particular, $g_0(u):=\lim_{\delta\downarrow0}g_\delta(u)$ is the probability that the process $\{R_b(t),t>0\}$ reaches a new record high before ruin.

We have mentioned that $D(t)$ is the aggregate dividends paid from time 0 to t of the risk process $\{R_{\gamma,b}(t), t\geqslant 0\}$ and $V_\gamma(u,b) = E_u\left[\int_0^{T_{\gamma,b}} e^{-\delta s}\,dD(s)\right]$ denotes the expected discounted total sum of dividends until ruin. Obviously, the trivial bounds $0\leqslant V_\gamma(u,b)\leqslant\frac{\alpha}{\delta}$ always hold.

In the following theorem, we are to derive the analytical expressions of $V_\gamma(u,b)$ in terms of $V(u,b)$.

Theorem 1.1 When $0<u<b$, we have

$$V_\gamma(u,b) = \frac{V_\gamma(b,b)}{g_\delta(b)} g_\delta(u)\, e^{-\frac{\beta}{1-\gamma}\int_u^b (1-g_\delta(t))\,dt} \tag{24}$$

and when $u\geqslant b$, we have

$$V_\gamma(u,b) = g_\delta(u)\, e^{\frac{\beta}{1-\gamma}\int_b^u (1-g_\delta(t))\,dt}\cdot \int_u^\infty A(s)(g_\delta(s))^{-1} e^{-\frac{\beta}{1-\gamma}\int_b^s (1-g_\delta(t))\,dt}\,ds \tag{25}$$

where

$$A(u) = \left(\frac{g_\delta'(u)}{g_\delta(u)} + \frac{\beta}{1-\gamma}(1-g_\delta(u))\right)V(u,b) + \frac{\gamma\beta}{1-\gamma}g_\delta(u)V(u,b) - \frac{\gamma\beta}{1-\gamma}g_\delta(u)\cdot \int_u^\infty \beta e^{-\beta(x-u)}V(x,b)\,dx - V'(u,b) \tag{26}$$

Proof When $0<u<b$, considering the fact that the process $\{R_{\gamma,b}(t), t\geqslant 0\}$ shall up-cross the initial level u or else there will be no dividends paid to the shareholders. Implementing these considerations we have

$$\left.\begin{aligned} V_\gamma(u,b) &= g_\delta(u)\int_0^\infty \frac{\beta}{1-\gamma}e^{-\frac{\beta}{1-\gamma}x}V_\gamma(u+x,b)\,dx, \\ & 0<u<b \end{aligned}\right\} \tag{27}$$

Changing variables $y=u+x$ in Eq. (27) and then differentiating it with respect to u yields

$$V_\gamma'(u,b) = \frac{g_\delta'(u)}{g_\delta(u)}V_\gamma(u,b) + \frac{\beta}{1-\gamma}V_\gamma(u,b) - \frac{\beta}{1-\gamma}g_\delta(u)V_\gamma(u,b) \tag{28}$$

Solving the ordinary differential equation of first order, we arrive at Eq. (24).

When $u\geqslant b$, since the surplus process $\{R_{\gamma,b}(t), t\geqslant 0\}$ will either reach a new record high before ruin or never reach a new record high until ruin, in the latter case the sample paths of $\{R_{\gamma,b}(t), t\geqslant 0\}$ will coincide with those of $\{R_b(t), t\geqslant 0\}$. By conditioning on the two cases we obtain

$$\begin{aligned} V_\gamma(u,b) &= E_u\left[\int_0^{\tau_u} e^{-\delta t}\,dD(t)\,\mathbf{1}_{\tau_u<T_{\gamma,b}}\right] + E_u\left[\int_{\tau_u}^{T_{\gamma,b}} e^{-\delta t}\,dD(t)\,\mathbf{1}_{\tau_u<T_{\gamma,b}}\right] + \\ & E_u\left[\int_0^{T_{\gamma,b}} e^{-\delta t}\,dD(t)\,\mathbf{1}_{\tau_u>T_{\gamma,b}}\right] = \\ & E_u\left[\int_0^{\tau_u} e^{-\delta t}\,dD(t)\,\mathbf{1}_{\tau_u<T_{\gamma,b}}\right] + \\ & g_\delta(u)\int_0^\infty \frac{\beta}{1-\gamma}e^{-\frac{\beta}{1-\gamma}x}V_\gamma(u+x,b)\,dx + \\ & E_u\left[\int_0^{T_b} e^{-\delta t}\,dD(t)\,\mathbf{1}_{\tau_u>T_{\gamma,b}}\right] = \\ & E_u\left[\int_0^{\tau_u} e^{-\delta t}\,dD(t)\,\mathbf{1}_{\tau_u>T_{\gamma,b}}\right] + \\ & g_\delta(u)\int_0^\infty \frac{\beta}{1-\gamma}e^{-\frac{\beta}{1-\gamma}x}V_\gamma(u+x,b)\,dx + \\ & E_u\left[\int_0^{T_b} e^{-\delta t}\,dD(t)\right] - E_u\left[\int_0^{\tau_u} e^{-\delta t}\,dD(t)\,\mathbf{1}_{\tau_u<T_{\gamma,b}}\right] - \\ & g_\delta(u)\int_0^\infty \beta e^{-\beta x}V(u+x,b)\,dx = \\ & V(u,b) + g_\delta(u)\int_0^\infty \frac{\beta}{1-\gamma}e^{-\frac{\beta}{1-\gamma}x}V_\gamma(u+x,b)\,dx - \\ & g_\delta(u)\int_0^\infty \beta e^{-\beta x}V(u+x,b)\,dx \end{aligned} \tag{29}$$

Differentiating Eq. (29) with respect to u yields

$$\begin{aligned} V_\gamma'(u,b) &= \\ & \left(\frac{g_\delta'(u)}{g_\delta(u)} + \frac{\beta}{1-\gamma}(1-g_\delta(u))\right)V_\gamma(u,b) - \\ & \left(\frac{g_\delta'(u)}{g_\delta(u)} - \beta g_\delta(u) + \frac{\beta}{1-\gamma}\right)V(u,b) + \\ & \frac{\gamma\beta}{1-\gamma}g_\delta(u)\int_u^\infty \beta e^{-\beta(x-u)}V(x,b)\,dx + V'(u,b) \end{aligned} \tag{30}$$

The general solution of Eq. (30) can be expressed as

$$V_\gamma(u,b) = \left(C - \int_b^u A(s)(g_\delta(s))^{-1} e^{-\frac{\beta}{1-\gamma}\int_b^s (1-g_\delta(t))\,dt}\,ds\right)\cdot g_\delta(u)\, e^{\frac{\beta}{1-\gamma}\int_b^u (1-g_\delta(t))\,dt} \tag{31}$$

Furthermore, due to the trivial fact that $0<g_\delta(\infty)\leqslant E_u[e^{-\delta T_1}]<1$ and $V_\gamma(u,b)\leqslant\frac{\alpha}{\delta}$ we immediately have

$$C = \int_b^\infty A(s)(g_\delta(s))^{-1} e^{-\frac{\beta}{1-\gamma}\int_b^s (1-g_\delta(t))\,dt}\,ds \tag{32}$$

Plugging Eq. (32) into Eq. (31) we obtain Eq. (25). □

2 Explicit results for Erlang(2) innovation waiting times

In this section, we assume that T_i's are Erlang (2) distributed with parameters λ_1 and λ_2. We also assume that $\lambda_1 < \lambda_2$ (without loss of generality).

Example 2.1 Applying the operator $(\beta \boldsymbol{I} - \boldsymbol{D})$ to Eq. (8) and Eq. (9), we have

$$(\beta \boldsymbol{I} - \boldsymbol{D}) \prod_{k=1}^{2} \left[\left(1 + \frac{\delta}{\lambda_k}\right) \boldsymbol{I} + \frac{c-\alpha}{\lambda_k} \boldsymbol{D} \right] V(u,b) = \beta V(u,b),\ 0 < u < b \tag{33}$$

and

$$(\beta \boldsymbol{I} - \boldsymbol{D}) \prod_{k=1}^{2} \left[\left(1 + \frac{\delta}{\lambda_k}\right) \boldsymbol{I} + \frac{c}{\lambda_k} \boldsymbol{D} \right] V(u,b) = \beta V(u,b) + \beta B_2,\ u \geqslant b \tag{34}$$

The characteristic equation for Eq. (33) is

$$(\beta - r) \prod_{k=1}^{2} \left[\left(1 + \frac{\delta}{\lambda_k}\right) + \frac{c-\alpha}{\lambda_k} r \right] = \beta \tag{35}$$

We know that Eq. (35) has three real roots, say r_1, r_2 and r_3 which satisfies

$$\beta > r_1 > 0 > r_2 > -\frac{\lambda_1 + \delta}{c-\alpha} > -\frac{\lambda_2 + \delta}{c-\alpha} > r_3 > -\frac{\lambda_1 + \delta}{c-\alpha} - \frac{\lambda_2 + \delta}{c-\alpha}.$$

With c replacing $c-\alpha$ in Eq. (34), we get the characteristic equation of Eq. (35), whose roots are denoted by r_4, r_5 and r_6 with

$$\beta > r_4 > 0 > r_5 > -\frac{\lambda_1 + \delta}{c} > -\frac{\lambda_2 + \delta}{c} > r_6 > -\frac{\lambda_1 + \delta}{c} - \frac{\lambda_2 + \delta}{c}.$$

Note that

$$V(u,b) \equiv \left(\left(1 + \frac{\delta}{\lambda_1}\right)\left(1 + \frac{\delta}{\lambda_2}\right) - 1 \right)^{-1} B_2 = \frac{\alpha}{\delta}$$

is a special solution of Eq. (11), we have

$$V(u,b) = c_1 e^{r_1 u} + c_2 e^{r_2 u} + c_3 e^{r_3 u},\ 0 < u < b \tag{36}$$

and

$$V(u,b) = c_5 e^{r_5 u} + c_6 e^{r_6 u} + \frac{\alpha}{\delta},\ u \geqslant b \tag{37}$$

The characteristic equation for Eq. (34) is

$$(\beta - r) \prod_{k=1}^{2} \left[\left(1 + \frac{\delta}{\lambda_k}\right) + \frac{c-\alpha}{\lambda_k} r \right] = \beta \tag{38}$$

where c_i's are arbitrary constants. With the boundary conditions of $V(u,b)$, we know that c_i's are determined by

$$\begin{aligned}
& c_1 + c_2 + c_3 = 0, \\
& r_1 c_1 + r_2 c_2 + r_3 c_3 = 0, \\
& e^{r_1 b} c_1 + e^{r_2 b} c_2 + e^{r_3 b} c_3 - e^{r_5 b} c_5 - e^{r_6 b} c_6 = \frac{\alpha}{\delta}, \\
& ((c-\alpha) r_1 + \delta) e^{r_1 b} c_1 + ((c-\alpha) r_2 + \delta) e^{r_2 b} c_2 + \\
& \quad ((c-\alpha) r_3 + \delta) e^{r_3 b} c_3 - (c r_5 + \delta) e^{r_5 b} c_5 - \\
& \quad (c r_6 + \delta) e^{r_6 b} c_6 = 0, \\
& \frac{r_1 e^{r_1 b}}{\beta - r_1} c_1 + \frac{r_2 e^{r_2 b}}{\beta - r_2} c_2 + \frac{r_3 e^{r_3 b}}{\beta - r_3} c_3 - \\
& \quad \frac{r_5 e^{r_5 b}}{\beta - r_5} c_5 - \frac{r_6 e^{r_6 b}}{\beta - r_6} c_6 = 0.
\end{aligned}$$

Some calculations give

$$\left.\begin{aligned}
c_1 &= \frac{\frac{\alpha}{\delta}(r_3 - r_2)\left(\frac{(c r_5 + \delta) r_6}{\beta - r_6} - \frac{(c r_6 + \delta) r_5}{\beta - r_5}\right)}{(r_3 - r_2) e^{r_1 b} \theta_1(r_1, r_5, r_6) + (r_2 - r_1) e^{r_3 b} \theta_1(r_3, r_5, r_6) + (r_1 - r_3) e^{r_2 b} \theta_1(r_2, r_5, r_6)}, \\
c_2 &= \frac{\frac{\alpha}{\delta}(r_1 - r_3)\left(\frac{(c r_5 + \delta) r_6}{\beta - r_6} - \frac{(c r_6 + \delta) r_5}{\beta - r_5}\right)}{(r_3 - r_2) e^{r_1 b} \theta_1(r_1, r_5, r_6) + (r_2 - r_1) e^{r_3 b} \theta_1(r_3, r_5, r_6) + (r_1 - r_3) e^{r_2 b} \theta_1(r_2, r_5, r_6)}, \\
c_3 &= \frac{\frac{\alpha}{\delta}(r_2 - r_1)\left(\frac{(c r_5 + \delta) r_6}{\beta - r_6} - \frac{(c r_6 + \delta) r_5}{\beta - r_5}\right)}{(r_3 - r_2) e^{r_1 b} \theta_1(r_1, r_5, r_6) + (r_2 - r_1) e^{r_3 b} \theta_1(r_3, r_5, r_6) + (r_1 - r_3) e^{r_2 b} \theta_1(r_2, r_5, r_6)}, \\
c_5 &= \frac{-\frac{\alpha}{\delta}((r_2 - r_1) e^{r_2 b} \theta_2(r_3, r_6) + (r_3 - r_2) e^{r_1 b} \theta_2(r_1, r_6) + (r_1 - r_3) e^{r_2 b} \theta_2(r_2, r_6))}{(r_3 - r_2) e^{(r_1 + r_5) b} \theta_1(r_1, r_5, r_6) + (r_2 - r_1) e^{(r_3 + r_5) b} \theta_1(r_3, r_5, r_6) + (r_1 - r_3) e^{(r_2 + r_5) b} \theta_1(r_2, r_5, r_6)}, \\
c_6 &= \frac{\frac{\alpha}{\delta}((r_2 - r_1) e^{r_3 b} \theta_2(r_3, r_5) + (r_3 - r_2) e^{r_1 b} \theta_2(r_1, r_5) + (r_1 - r_3) e^{r_2 b} \theta_2(r_2, r_5))}{(r_3 - r_2) e^{(r_1 + r_6) b} \theta_1(r_1, r_5, r_6) + (r_2 - r_1) e^{(r_3 + r_6) b} \theta_1(r_3, r_5, r_6) + (r_1 - r_3) e^{(r_2 + r_6) b} \theta_1(r_2, r_5, r_6)}
\end{aligned}\right\} \tag{39}$$

where

$$\theta_1(r_i, r_j, r_k) = \frac{c(r_k - r_j)r_i}{\beta - r_i} + \frac{((c-\alpha)r_i - cr_k)r_j}{\beta - r_j} + \frac{(cr_j - (c-\alpha)r_i)r_k}{\beta - r_k},$$
$$1 \leqslant i < j < k \leqslant 6,$$

and

$$\theta_2(r_i, r_j) = \frac{(cr_j + \delta)r_i}{\beta - r_i} - \frac{((c-\alpha)r_i + \delta)r_j}{\beta - r_j},$$
$$1 \leqslant i < j \leqslant 6.$$

In the sequel, we write $c_i(b)$ $(i=1,\cdots,6)$ for c_i $(i=1,\cdots,6)$ to stress the dependence of c_i's on b.

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