The maximum power output and maximum efficiency of an irreversible Carnot heat engine

1994 J. Phys. D: Appl. Phys. 27 1144

(http://iopscience.iop.org/0022-3727/27/6/011)

J. Phys. D: Appl. Phys. 27 (1994) 1144–1149. Printed in the UK

The maximum power output and maximum efficiency of an irreversible Carnot heat engine

Jincan Chen

China Centre of Advanced Science and Technology (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China and Department of Physics, Xiamen University, Xiamen 361005, People's Republic of China

Received 23 November 1993, in final form 18 February 1994

Abstract. The effect of thermal resistance, heat leakage and internal irreversibility resulting from the working fluid on the performance of a Carnot heat engine is investigated using a new cyclic model. The power output and efficiency of the heat engine are adopted as objective functions for heat engine optimization. The optimal performance of the heat engine is analysed systematically. Some significant results are obtained. For example, the maximum power output and maximum efficiency are determined. The efficiency of the heat engine at maximum power output and the power output of the heat engine at maximum efficiency are also calculated. Curves of the power output varying with the efficiency of the heat engine are obtained. These curves can indicate clearly the rational regions of efficiency and power output for an irreversible Carnot heat engine. It is pointed out that all the conclusions concerning a reversible Carnot heat engine, an endoreversible Carnot heat engine only affected by thermal resistance and an irreversible Carnot heat engine with internal irreversibility and/or heat leakage can be deduced from the results in this paper.

1. Introduction

The first goal of classical thermodynamics was evaluating how well heat engines perform and how well they might perform in an idealized limit. The idealized limit became the reversible Carnot heat engine and its efficiency is

$$\eta_C = 1 - T_C/T_H \tag{1}$$

where T_H and T_C are the temperatures of the hot and cold reservoirs between which the heat engine operates. In fact, the Carnot efficiency η_C is invariably far above the efficiency of real heat engines and hence is of very limited practical value, because the reversible conditions under which heat engines would have to operate to achieve Carnot efficiency correspond to zero power output. This has resulted in the advent of a new field: 'finite-time thermodynamics' [1–3].

The main goal of finite-time thermodynamics is to seek universal limits resulting from the finite-time mode of operating thermodynamical processes [4]. Finite-time processes, besides their practical importance as more realistic models than those provided by classical thermodynamics, create a deeper understanding of how irreversibility affects the performance of thermodynamical processes. The best studied model is that of the endoreversible heat engine [1, 5–14], in which the heat engine is modelled as internally reversible, with all irreversibilities being incorporated into engine heat exchange with its reservoirs. Using the endoreversible cyclic model, Curzon and Ahlborn [1] calculated the efficiency

$$\eta_{CA} = 1 - (T_C/T_H)^{1/2} \tag{2}$$

of a Carnot heat engine at maximum power output. On the basis of the work, a series of investigations [5–14] related to endoreversible heat engines have been carried out. Some new irreversible heat engine models [15–21] have recently been proposed and many significant results have been obtained. These results have laid a foundation for studying further the optimal performance of real heat engines.

Real heat engines are complex devices. Besides the irreversibility of finite-rate heat transfer, there are also other sources of irreversibility, such as heat leaks, dissipative processes inside the working fluid, and so on. Thus, it is necessary to investigate more comprehensively the influence of finite-rate heat transfer together with other major irreversibilities on the performance of heat engines. For this aim, we will establish a new general cyclic model including three major irreversibilities, which often exist in heat engines, and use it to optimize the performance of an irreversible Carnot heat engine.

0022-3727/94/061144+06$19.50

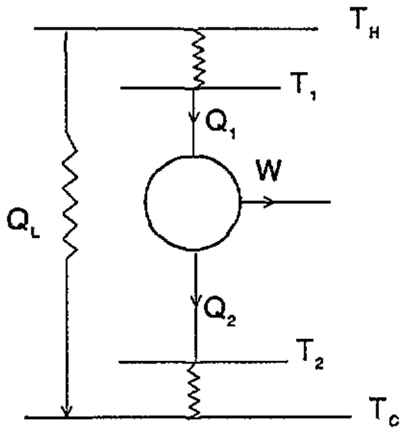

Figure 1. Schematic diagram of an irreversible Carnot heat engine.

2. The irreversible Carnot cycle

The cyclic system of a Carnot heat engine, including the irreversibilities of finite-rate heat transfer between the heat engine and its reservoirs, heat leak between the reservoirs, and internal dissipations of the working fluid, is shown schematically in figure 1. The working fluid in the system is alternately connected to a hot reservoir at constant temperature T_H and to a cold reservoir at constant temperature T_C and its temperatures are, respectively, T_1 and T_2. The heat engine operates in a cyclic fashion with fixed time t allotted for each cycle. Thus, after time t has elapsed, the working fluid returns to its initial state. All heat transfer are assumed to be linear in temperature differences, that is, Newtonian [3–6, 22–25]. The heat Q_1 transference from the hot reservoir to the heat engine occurs for a time t_1 with thermal conductance K_1 and the heat Q_2 transference from the heat engine to the cold reservoir occurs for a time t_2 with thermal conductance K_2. The heat Q_L leakage from the hot reservoir to the cold reservoir occurs during the full engine cycle time t with thermal conductance K_L and is given by

$$Q_L = K_L(T_H - T_C)t. \tag{3}$$

Then, the heat Q_H transferred from the hot reservoir is

$$Q_H = Q_1 + Q_L = K_1(T_H - T_1)t_1 + K_L(T_H - T_C)t \tag{4}$$

and the heat Q_C transferred to the cold reservoir is

$$Q_C = Q_2 + Q_L = K_2(T_2 - T_C)t_2 + K_L(T_H - T_C)t \tag{5}$$

where all the heats are defined as positive.

To obtain the simple expressions of the efficiency and power output, the connecting adiabatic branches are often assumed to proceed in negligible time [6, 9, 20], such that the cycle time t is approximately given by

$$t = t_1 + t_2. \tag{6}$$

Owing to internal dissipations of the working fluid, all branches of the cycle are irreversible. The entropy of the working fluid in two adiabatic branches increases, as shown schematically in figure 2. This is different from the T–S diagram of an endoreversible Carnot cycle [12]. In figure 2, ΔS_1 and ΔS_2 are, respectively, the entropy productions of the working fluid in two isothermal branches at temperatures T_1 and T_2, and they are defined as positive. According to the second law of thermodynamics, one has

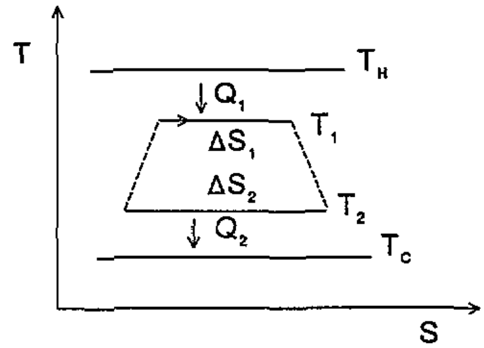

Figure 2. The T–S diagram of an irreversible Carnot heat engine.

$$\frac{Q_2}{T_2} - \frac{Q_1}{T_1} > 0. \tag{7}$$

In order to describe quantitatively the effect of the internal dissipations of the working fluid on the performance of the heat engine, we can introduce a parameter [19]

$$I_S = \frac{\Delta S_2}{\Delta S_1} \tag{8}$$

which can characterize fully the degree of internal irreversibility resulting from the working fluid, such that the inequality in equation (7) can be written as

$$\frac{Q_2}{T_2} - I_S\frac{Q_1}{T_1} = 0. \tag{9}$$

Equation (9) shows clearly that when $I_S = 1$, the heat engine is endoreversible [1, 9, 20]; when $I_S > 1$, the heat engine is internally irreversible. It is thus clear that the cyclic model adopted in this paper is more approximate to real heat engines than the endoreversible cyclic model, because all real heat engines are internally irreversible.

Using equations (3)–(6), (9) and the definitions of the power output and efficiency, we can obtain the expressions of the power output and efficiency

$$\begin{aligned} P &= \frac{Q_H - Q_C}{t} = \frac{Q_1 - Q_2}{t} \\ &= \frac{1 - I_S x}{[1/K_1(T_H - T_1)] + [I_S x/K_2(xT_1 - T_C)]} \end{aligned} \tag{10}$$

$$\begin{aligned} \eta = 1 - \frac{Q_C}{Q_H} &= (1 - I_S x)[\![1 + K_L(T_H - T_C) \\ &\times \{[1/K_1(T_H - T_1)] + [I_S x/K_2(xT_1 - T_C)]\}]\!]^{-1} \end{aligned} \tag{11}$$

respectively, where $x = T_2/T_1$.

The cyclic model is very general. Various cyclic models have often been adopted in the literature. For example, the reversible Carnot heat engine is recovered for $I_S = 1$, $K_L = 0$, $K_1 \to \infty$ and $K_2 \to \infty$; the endoreversible Carnot heat engine only affected by the irreversibility of finite-rate heat transfer is recovered for $I_S = 1$ and $K_L = 0$; and the heat engine with internal irreversibility or heat leak is recovered for $K_L = 0$ or $I_S = 1$.

3. The maximum power output and the corresponding efficiency

It can be seen from equation (10) that the power output of the heat engine is not affected by heat leak and only depends on the irreversiblity of finite-rate heat transfer and the internal irreversibility of the working fluid.

When $I_S = 1$, the heat engine is endoreversible and its maximum power output is given by [1, 9]

$$P_{CA} = \frac{K_1 K_2}{(\sqrt{K_1} + \sqrt{K_2})^2}(\sqrt{T_H} - \sqrt{T_C})^2. \qquad (12)$$

In such a case, the efficiency of the heat engine is given by equation (2). When $I_S > 1$, the power output of the heat engine decreases as I_S increases. It is thus clear that P_{CA} is an important parameter of heat engines in theory, because it determines the upper bound of the power output for all heat engines operating between the reservoirs at temperatures T_H and T_C.

For given parameters K_1, K_2, T_H, T_C and I_S, the power output of the heat engine is a function of T_1 and x. Using equation (10) and the extremal conditions

$$\partial P/\partial T_1 = 0 \qquad (13)$$

$$\partial P/\partial x = 0 \qquad (14)$$

we can find that when the power output attains maximum, T_1 and x are given by

$$T_1 = \frac{CT_H x + T_C}{(1+C)x} \qquad (15)$$

$$x = \left(\frac{T_C}{I_S T_H}\right)^{1/2} \qquad (16)$$

respectively, where $C = (I_S K_1/K_2)^{1/2}$. Substituting equation (15) into equation (10), we obtain the relationship between P and x

$$P = K(1 - I_S x)(T_H - T_C/x) \qquad (17)$$

where $K = K_1/(1+C)^2$. Using equations (17) and (12), we can easily generate the P/P_{CA} versus x curves, as shown in figure 3. Substituting equation (16) into equation (17), we obtain the maximum power output of the heat engine

$$P_{max} = K[\sqrt{T_H} - (I_S T_C)^{1/2}]^2. \qquad (18)$$

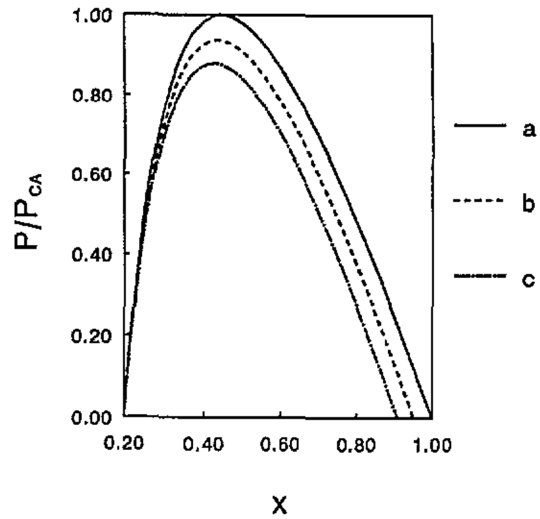

Figure 3. The P/P_{CA} versus x curves of an irreversible Carnot heat engine. Plots are presented for $T_C/T_H = 0.2$ and $K_1 = K_2$. Curves a, b and c correspond to the cases of $I_S = 1$ (endoreversible), 1.05 and 1.1, respectively.

Now, substituting equations (15) and (16) into equation (11), we obtain the efficiency

$$\eta_m = \frac{(1 - \sqrt{b_2})^2}{1 + b_1 - \sqrt{b_2}} \qquad (19)$$

of the heat engine at maximum power output, where $b_1 = (K_L/K)(1 - T_C/T_H)$ and $b_2 = I_S T_C/T_H$. When $K_L = 0$, the efficiency of the heat engine at maximum power output is given by [19]

$$\eta_m = 1 - (I_S T_C/T_H)^{1/2} \qquad (20)$$

whereas the maximum power output remains constant and is still given by equation (18), because the power output of the heat engine is not affected by heat leak.

4. The maximum efficiency and the corresponding power output

It can be seen from equation (11) that the efficiency of the heat engine is dependent on all the irreversibilities, each of which can decrease the efficiency of the heat engine.

When $K_L = 0$, we obtain the efficiency of the heat engine

$$\eta = 1 - I_S x \qquad (21)$$

from equation (11). Obviously, η increases as x decreases. When $x = T_C/T_H$, η attains its maximum

$$\eta_{max} = 1 - I_S T_C/T_H \equiv \eta_{CI}. \qquad (22)$$

In such a case, the power output of the heat engine is equal to zero. This shows clearly the fact that the efficiency of all heat engines is always smaller than η_{CI} so long as there exists the internal irreversibility of the working fluid. Only if $K_L = 0$ and $I_S = 1$ can the

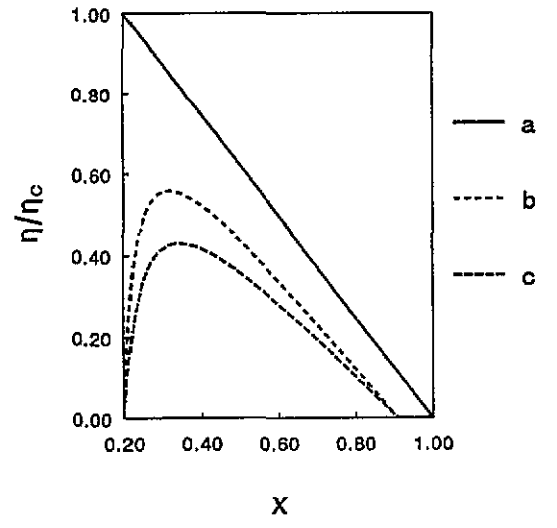

Figure 4. The η/η_C versus x curves of an irreversible Carnot heat engine. The values of T_C/T_H and K_1/K_2 are the same as in figure 3. Curves a, b and c correspond to the cases of $I_S = 1$ and $K_L = 0$, $I_S = 1.1$ and $K_L/K_1 = 0.05$, $I_S = 1.1$ and $K_L/K_1 = 0.1$, respectively.

maximum efficiency of a heat engine with zero power output be equal to the Carnot efficiency η_C.

When $K_L > 0$, substituting directly equation (15) into equation (11), we obtain the following relationship between η and x

$$\eta = \frac{(1 - I_S x)(T_H x - T_C)}{(1 + b_1)T_H x - T_C} \tag{23}$$

because there is a relation, $\partial P/\partial T_1 = \partial \eta/\partial T_1 = 0$, for equations (10) and (11). Using equations (23) and (1), we can easily generate the η/η_C versus x curves, as shown in figure 4. Using equation (23) and the extremal condition

$$\partial \eta/\partial x = 0 \tag{24}$$

we can find that when η attains its maximum

$$x = \frac{T_C}{T_H}\frac{1 + b_1[(1 + \eta_{CI}/b_1)/b_2]^{1/2}}{1 + b_1} \tag{25}$$

Substituting equation (25) into equation (23), we obtain the maximum efficiency of the heat engine

$$\eta_{\max} = \frac{b_1[(1 + \eta_{CI}/b_1)^{1/2} - \sqrt{b_2}]^2}{(1 + b_1)^2}. \tag{26}$$

Substituting equation (25) into equation (17), we obtain the power output

$$P_m = KT_H \frac{b_1^2(1 + \eta_{CI}/b_1)^{1/2}[(1 + \eta_{CI}/b_1)^{1/2} - \sqrt{b_2}]^2}{(1 + b_1)[b_1(1 + \eta_{CI}/b_1)^{1/2} + \sqrt{b_2}]} \tag{27}$$

of the heat engine at maximum efficiency.

When $I_S = 1$, the maximum efficiency of the heat engine and the corresponding power output

$$\eta_{\max} = \frac{b_1[(1 + K/K_L)^{1/2} - (T_C/T_H)^{1/2}]^2}{(1 + b_1)^2} \tag{28}$$

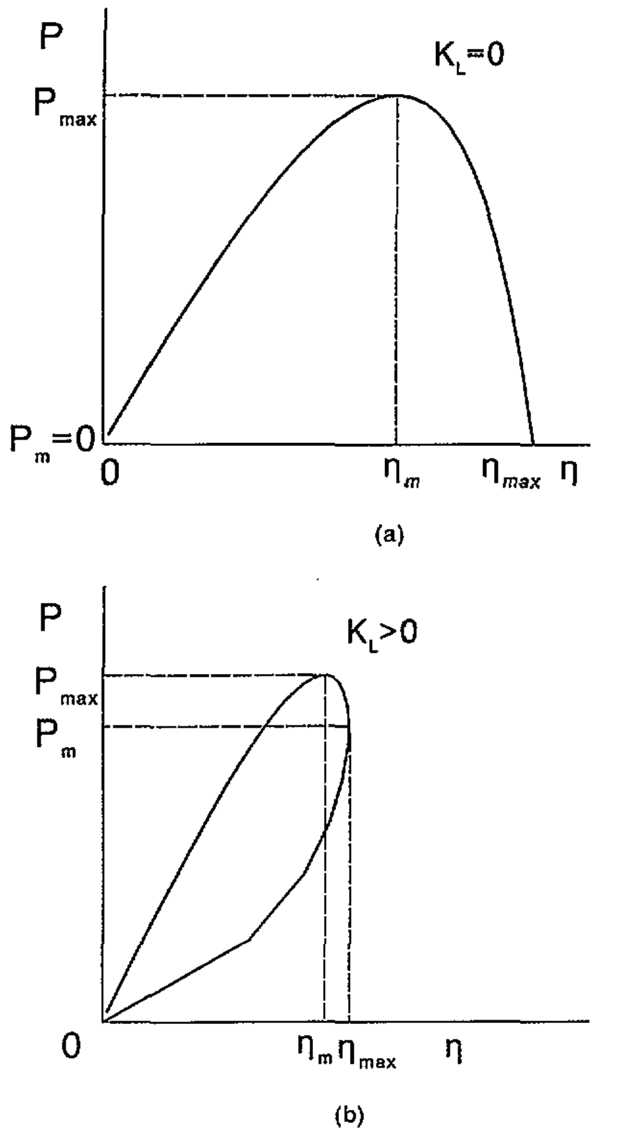

Figure 5. The power output versus efficiency curves of an irreversible Carnot heat engine: (a) $K_L = 0$ and (b) $K_L > 0$.

$$P_m = KT_H \times \frac{b_1^2(1 + K/K_L)^{1/2}[(1 + K/K_L)^{1/2} - (T_C/T_H)^{1/2}]^2}{(1 + b_1)[b_1(1 + K/K_L)^{1/2} + (T_C/T_H)]} \tag{29}$$

can be deduced from equations (26) and (27), respectively.

5. The power output versus efficiency curves

From equations (17) and (23), we obtain the relationship between the power output and the efficiency

$$P = KT_H\left(1 - \frac{1 - \eta(1 + b_1) + b_2 + \sqrt{U}}{2}\right) \times \left(1 - \frac{1 - \eta(1 + b_1) + b_2 - \sqrt{U}}{2(1 - \eta)}\right) \tag{30}$$

where $U = (\eta_{CI} - \eta)^2 - 2\eta b_1(1 - \eta + b_2) + (\eta b_1)^2$. From equation (30), we can generate the curves of the power output varying with the efficiency of the heat engine, as shown in figure 5.

Figure 5 shows clearly that, when $K_L = 0$, very large differences exist between η_m and η_{max} and between P_{max} and P_m of the heat engine; when $K_L > 0$, these differences decrease. Figure 5(*b*) shows further that, when the efficiency is less than η_m, the power output of the heat engine decreases as the efficiency decreases; when the power output is less than P_m, the efficiency of the heat engine also decreases as the power output decreases. Obviously, the working states of $P < P_m$ and $\eta < \eta_m$ are not the optimal operating states of the heat engine. In general, the heat engine should be operated between η_m and η_{max} or P_{max} and P_m. In other words, we can determine that the rational regions of the efficiency and power output of the heat engine are

$$\eta_m \leqslant \eta \leqslant \eta_{max} \tag{31}$$

$$P_{max} \geqslant P \geqslant P_m \tag{32}$$

respectively.

In addition, when heat leakage is considered, the power output versus efficiency curves become loop shapes. The qualitative behaviour of the loop-shaped power versus efficiency curves appears to be a common characteristic of some real heat engines [26–28]. However, it should be pointed out that, when it is a requirement to use different equivalent networks or different definitions of efficiency [8, 14] to analyse the performance of an irreversible Carnot heat engine, the power output versus efficiency curves may appear in different shapes.

6. Optimum criteria of other key parameters

In order to make the heat engine operate under optimal working conditions, the temperatures of the working fluids in two isothermal branches of the cycle are not chosen arbitrarily. Using equations (15) and (16), we can find that when the heat engine operates in the state of maximum power output, the temperatures of the working fluids in two isothermal branches of the cycle are determined by

$$T_{1P} = T_H \frac{C + \sqrt{b_2}}{1 + C} \tag{33}$$

$$T_{2P} = T_C \frac{1 + (1/b_2)^{1/2}}{1 + C} \tag{34}$$

respectively. Using equations (15) and (25), we can also find that when the heat engine operates in the state of the maximum efficiency, the temperatures of the working fluid in two isothermal branches of the cycle are determined by

$$T_{1\eta} = T_H \frac{C + \{(1 + b_1)/1 + b_1[(1 + \eta_{CI}/b_1)/b_2]^{1/2}\}}{1 + C} \tag{35}$$

$$T_{2\eta} = T_C \frac{1 + C + b_1 + Cb_1[(1 + \eta_{CI}/b_1)/b_2]^{1/2}}{(1 + C)(1 + b_1)} \tag{36}$$

respectively. According to equation (31) or (32), we can determine easily that the rational regions of the temperatures of the working fluid in two isothermal branches of the cycle for an irreversible Carnot heat engine are

$$T_{1\eta} \geqslant T_1 \geqslant T_{1P} \tag{37}$$

$$T_{2\eta} \leqslant T_2 \leqslant T_{2P} \tag{38}$$

respectively.

Similarly, the times spent in two isothermal branches of the cycle are also not chosen arbitrarily. Using equations (33)–(36) and (9), we can find that the optimal ratio of the times spent in two isothermal branches of the cycle is

$$t_1/t_2 = [K_2/(I_S K_1)]^{1/2} \tag{39}$$

whether the heat engine operates in the state of maximum power output or maximum efficiency. Moreover, we can also prove that, so long as equation (15) is satisfied, equation (39) is always true for arbitrary values of x. It is seen clearly from equation (39) that the optimal ratio of the times spent on two isothermal branches of the cycle for an irreversible Carnot heat engine is independent of heat leakage, but dependent on internal irreversibility resulting from the working fluid. This is different from an endoreversible Carnot heat engine in which the ratio of the times spent on two isothermal branches of the cycle is only dependent on K_2/K_1 [1, 20].

7. Conclusions

We have demonstrated how a general cyclic model, including finite-rate heat transfer, heat leakage, and internal irreversibility of the working fluid, is established. The model can capture the principal irreversibility sources of some real heat engines. It is used to discuss the optimal performance of an irreversible Carnot heat engine. The key qualitative features of the power–efficiency curves of some heat engines are revealed from theory and the bounds of some important performance parameters of heat engines such as efficiency, power output and temperatures of the working fluids in isothermal branches, and so on, are determined. These results can provide the basis for both determination of optimal operating conditions and design of real heat engines.

References

[1] Curzon F L and Ahlborn B 1975 *Am. J. Phys.* **43** 22
[2] Andresen B, Salamon P and Berry R S 1984 *Phys. Today* **37** 62
[3] Sieniutyze S and Salamon P 1991 *Finite Time Thermodynamics, Advances in Thermodynamics* vol 4 (London: Taylor and Francis)
[4] Geva E and Kosloff R 1992 *J. Chem. Phys.* **97** 4398
[5] Rubin M H and Andresen B 1981 *J. Appl. Phys.* **52** 1
[6] Ondrechen M J, Rubin M H and Band Y B 1983 *J. Chem. Phys.* **78** 4721
[7] De Vos A 1985 *Am. J. Phys.* **53** 570
[8] De Vos A 1987 *J. Phys. D: Appl. Phys.* **20** 232
[9] Chen L and Yan Z 1989 *J. Chem. Phys.* **90** 3740

[10] Gordon J M 1989 *Am. J. Phys.* **57** 1136
[11] Yan Z and Chen J 1990 *J. Chem. Phys.* **92** 1994
[12] Wu C 1991 *Energy Conversion Management* **32** 249
[13] Yan Z and Chen J 1992 *Am. J. Phys.* **60** 475
[14] Chen J 1992 *J. Appl. Phys.* **72** 3778
[15] Andresen B, Salamon P and Berry R S 1977 *J. Chem. Phys.* **66** 1571
[16] Gordon J M and Huleihil M 1990 *J. Appl. Phys.* **69** 1
[17] Orlov V N and Berry R S 1990 *Phys. Rev.* A **42** 7230
[18] Yan Z and Chen J 1992 *Int. J. Energy Environment Economics* **2** 63
[19] Wu C and Kiang R L 1992 *Energy* **17** 1173
[20] Gordon J M and Huleihil M 1992 *J. Appl. Phys.* **72** 829
[21] Orlov V N and Berry R S 1992 *Phys. Rev.* A **45** 7202
[22] Chen J and Yan Z 1988 *J. Appl. Phys.* **63** 4795
[23] Chen J and Yan Z 1989 *Phys. Rev.* A **39** 4140
[24] Yan Z and Chen J 1990 *J. Phys. D: Appl. Phys.* **23** 136
[25] Wu C 1991 *Energy Conversion Management* **31** 561
[26] Taylor C F 1977 *The Internal-Combustion Engine in Theory and Practice* vol 1 and 2 (Cambridge, MA: MIT Press)
[27] Sorensen H A 1951 *Gas Turbines* (New York: Ronald)
[28] Walker G 1973 *Stirling Cycle Machines* (London: Oxford University Press)