Performance analysis of irreversible quantum Stirling cryogenic refrigeration cycles and their parametric optimum criteria

This content has been downloaded from IOPscience. Please scroll down to see the full text.

2006 Phys. Scr. 74 251
(http://iopscience.iop.org/1402-4896/74/2/019)

View the table of contents for this issue, or go to the journal homepage for more

Download details:

IP Address: 59.77.43.151
This content was downloaded on 19/05/2015 at 01:37

Please note that terms and conditions apply.
Performance analysis of irreversible quantum Stirling cryogenic refrigeration cycles and their parametric optimum criteria

Bihong Lin\textsuperscript{1,2} and Jincan Chen\textsuperscript{1}

\textsuperscript{1} Department of Physics, Xiamen University, Xiamen 361005, People’s Republic of China
\textsuperscript{2} Department of Physics, Quanzhou Normal University, Quanzhou 362000, People’s Republic of China

E-mail: jcchen@xmu.edu.cn

Received 10 January 2006
Accepted for publication 23 May 2006
Published 19 July 2006
Online at stacks.iop.org/PhysScr/74/251

Abstract
The influence of both the quantum degeneracy and the finite-rate heat transfer between the working substance and the heat reservoirs on the optimal performance of an irreversible Stirling cryogenic refrigeration cycle using an ideal Fermi or Bose gas as the working substance is investigated, based on the theory of statistical mechanics and thermodynamic properties of ideal quantum gases. The inherent regeneration losses of the cycle are analysed. Expressions for several important performance parameters such as the coefficient of performance, cooling rate and power input are derived. By using numerical solutions, the cooling rate of the cycle is optimized for a given power input. The maximum cooling rate and the corresponding parameters are calculated numerically. The optimal regions of the coefficient of performance and power input are determined. In particular, the optimal performance of the cycle in the strong and weak gas degeneracy cases and the high temperature limit are discussed in detail. The analytic expressions of some optimized parameters are derived. Some optimum criteria are given. The distinctions and connections between the Stirling refrigeration cycles working with the ideal quantum and classical gases are revealed.

PACS numbers: 05.70.–a, 07.20.Mc, 05.90.+m

1. Introduction

In recent years, the optimal analyses relative to the performance characteristics of thermodynamic cycles have been extended from classical to quantum cycles \cite{1–25}. The quantum thermodynamic cycles working with the spin systems \cite{5–10}, harmonic oscillator systems \cite{11–15} and ideal quantum gases \cite{16–25} have become one of the most interesting research subjects. These investigations facilitate the understanding of the performance of the cryogenic refrigeration cycles.

It is well known that the Stirling refrigeration cycle is one of the important cycle models of the cryogenic refrigeration systems with regeneration. The cycle has been utilized by a number of engineering firms in the construction of practical systems for the production of very low temperatures. According to the theory of classical thermodynamics, when the working substance of the cycle is regarded as an ideal gas or a van der Waals gas, the Stirling refrigeration cycle may posses the condition of perfect regeneration through the use of a reversible regenerator. However, when the temperature of the working substance is low enough or density is high enough, the working substance will deviate from the classical gas behaviour and quantum degeneracy of the gas will become important \cite{26–28}, so that the Stirling refrigeration cycle cannot process the condition of perfect regeneration \cite{29, 30}. On the other hand, the lower the temperature of the working substance is, the more obvious the influence of the finite-rate heat transfer between the working substance and the heat reservoirs. Recently, Sisman...
and Saygin [20, 22] examined the influence of the quantum degeneracy of the quantum gas on the efficiency and work output of a Stirling power cycle working with an ideal quantum gas. Several authors investigated the coefficient of performance and refrigeration load of regenerative Fermi and Bose Stirling refrigeration cycles under quantum degeneracy conditions [25, 29, 30]. It is well known that the power output of a power cycle and cooling rate of a refrigeration cycle are two of the important performance parameters of real thermodynamic cycles. The optimal relation between the power output and the efficiency or between the cooling rate and the coefficient of performance is one of the important performance characteristics of the cycles. However, these investigations mentioned above rarely dealt with the influence of both the quantum degeneracy and the finite-rate heat transfer between the working substance and the heat reservoirs on the optimal performance of an irreversible Stirling cryogenic refrigeration cycle using an ideal quantum gas as the refrigerant. Thus, it is an important task in the optimal performance analysis of the Stirling cryogenic refrigeration cycle to consider the influence of the quantum degeneracy of the gas and the irreversibility of heat transfer simultaneously.

In the present paper, we will study the optimal performance of the irreversible quantum Stirling refrigeration cycle working with an ideal Fermi or Bose gas and consisting of two isothermal and two isochoric regenerative processes. The paper is organized in the following manner. In section 2, the irreversible model of the quantum Stirling cryogenic refrigeration cycle operating between two heat reservoirs at constant temperatures \(T_H\) and \(T_L\) is established and the expressions of the amounts of heat exchange in the various processes of the cycle are given. The influence of the inherent regenerative losses on the performance of the cycle is analysed. In section 3, the general expressions of several important parameters such as the coefficient of performance, cooling rate and power input are derived. In section 4, the general performance characteristics of the cycle are revealed. The curve of the optimal relation between the cooling rate and the coefficient of performance is obtained. In section 5, the optimal performances of the cycle are discussed in detail for several interesting cases. The optimum criteria of some important parameters are obtained. Finally, some conclusions are given in section 6.

2. An irreversible cycle model

The Stirling cryogenic refrigeration cycle using an ideal Fermi or Bose gas as the working substance may be simply called the quantum (Fermi or Bose) Stirling cryogenic refrigeration cycle. It is composed of two isothermal and two isochoric processes and operated between the heat sink at temperature \(T_H\) and the cooled space at temperature \(T_L\). In order to improve the performance of the cycle, a regenerator is usually used in the isochoric processes. The temperature–entropy diagram of the cycle is shown in figure 1, where \(Q_1\) and \(Q_2\) are the amounts of heat exchange between the working substance and the heat reservoirs at temperatures \(T_H\) and \(T_L\) during the isothermal processes, \(Q_{bc}\) and \(Q_{db}\) are the amounts of heat exchange between the working substance and the regenerator during the isochoric processes, \(T_1\) and \(T_2\) are the temperatures of the working substance in the high- and low-temperature isothermal processes, and \(v_L\) and \(v_H\) are the specific volumes of the refrigerant in the two isochoric processes respectively. For the sake of convenience, all the heats \(Q_1\), \(Q_2\), \(Q_{bc}\) and \(Q_{db}\) are selected to be positive and the lowest temperature of the Stirling refrigeration cycle working with the ideal Bose gas is restricted to be higher than the temperature of Bose–Einstein condensation of the Bose gas. Due to the finite-rate heat transfer between the working substance and the heat reservoirs, the temperatures \(T_1\) and \(T_2\) of the working substance in the two isothermal processes are different from those of the heat reservoirs and there is the following relation: \(T_1 > T_H > T_L > T_2\).

According to quantum statistical mechanics and the thermodynamic properties of ideal Fermi and Bose gases, the amounts of heat exchange in the four processes mentioned above are, respectively, given by [22, 26–30]

\[
Q_1 = \int_{S_d}^{S_f} T_1 dS = \frac{3}{2} N k T_1 [F(z_a) - F(z_d)] - N k T_1 \ln(z_c/z_d),
\]

\[
Q_2 = \int_{S_d}^{S_f} T_2 dS = \frac{3}{2} N k T_2 [F(z_b) - F(z_c)] - N k T_2 \ln(z_b/z_c),
\]

\[
Q_{bc} = \int_{T_2}^{T_1} C_v(T, \nu_H) dT = \frac{3}{2} N k [T_1 F(z_c) - T_2 F(z_b)],
\]

and

\[
Q_{db} = \int_{T_2}^{T_1} C_v(T, \nu_L) dT = \frac{3}{2} N k [T_1 F(z_d) - T_2 F(z_a)],
\]

where \(S_a\), \(S_b\), \(S_c\) and \(S_d\) are the entropies of the Fermi or Bose gas at state points \(a\), \(b\), \(c\) and \(d\), \(z_a = z(T_2, \nu_L), z_b = z(T_2, \nu_H), z_c = z(T_1, \nu_H)\) and \(z_d = z(T_1, \nu_L)\) are the fugacities of the Fermi or Bose gas at state points \(a\), \(b\), \(c\) and \(d\), \(F(z_a), F(z_b), F(z_c)\) and \(F(z_d)\) are the values of the function

\[
F(z) = \left[ \frac{1}{\Gamma(5/2)} \int_0^\infty \frac{x^{5/2}}{z^{-1} e^x \pm 1} dx \right]^{-1} \left[ \frac{1}{\Gamma(3/2)} \int_0^\infty \frac{x^{3/2}}{z^{-1} e^x \pm 1} dx \right]^{-1},
\]

Figure 1. The entropy–temperature diagram of an irreversible Fermi Stirling cryogenic refrigeration cycle.
at state points a, b, c and d, \( v = V/N \) is the specific volume of the Fermi or Bose gas, \( V \) is the volume of the gas system, and \( N \) is the total number of the particles, \( \Gamma(5/2) = 1.32934, \Gamma(3/2) = 0.886227 \), and signs ‘+’ correspond to ideal Fermi and Bose gases, respectively. For small \( z \), the Fermi–Dirac integral \( f_0(z) = 1/\Gamma(n) \int_0^\infty x^n/(e^{x/z} - 1) \, dx \) and Bose–Einstein integral \( g_0(z) = 1/\Gamma(n) \int_0^\infty x^n/(e^{x/z} - 1) \, dx \) in the \( F(z) \) may be, respectively, expanded in power of \( z \), with the results \( f_0(z) = \sum_{n=0}^\infty (-1)^n/z^n/n! \) and \( g_0(z) = \sum_{n=0}^\infty z^n/n! \) [26], which are called the Polylogarithm functions [22].

The refrigerator is operated in a cyclic fashion with a fixed period time \( \tau \). In the two isothermal processes of the cycle, the working substance is, respectively, coupled to the heat reservoirs at constant temperatures \( T_H \) and \( T_L \). When heat transfer obeys a linear law [31–33], \( Q_1 \) and \( Q_2 \) can be, respectively, expressed as

\[
Q_1 = \alpha(T_1 - T_H)t_1,
\]

and

\[
Q_2 = \beta(T_L - T_2)t_2,
\]

where \( \alpha \) and \( \beta \) are, respectively, the thermal conductances between the working substance and the two heat reservoirs at temperatures \( T_H \) and \( T_L \), and \( t_1 \) and \( t_2 \) are, respectively, the times of the two isothermal processes at temperatures \( T_1 \) and \( T_2 \). The times spent on two regenerative processes are given by \( \tau = (t_1 + t_2) = (1 - \gamma/\rho)\tau \), where \( \gamma = \tau/(t_1 + t_2) \) is the ratio of the cycle period to the times of two isothermal processes. Thus, the cycle period is given by

\[
\tau = \gamma(t_1 + t_2) = \gamma \left[ \frac{Q_1}{\alpha(T_1 - T_H)} + \frac{Q_2}{\beta(T_L - T_2)} \right].
\]

Using equations (1)–(4), we obtain the work input per cycle as

\[
W = Q_1 + Q_{da} - Q_2 = Q_{hc}
\]

\[
= NK[T_1[F(z_a) - F(z_c) + \ln(z_b/z_d)] + T_2[F(z_a) - F(z_b) + \ln(z_b/z_d)]].
\]

From equations (3) and (4), one can find that the net amount of heat transfer between the working substance and the regenerator during the two regenerative processes is determined by

\[
\Delta Q = Q_{da} - Q_{hc} = \frac{1}{2}Nk[T_1[F(z_a) - F(z_c)] + T_2[F(z_a) - F(z_b)]].
\]

According to the thermodynamic properties of the Fermi and Bose gases and the results in [22], one can obtain the following relations

\[
C_v(T, v_H) > C_v(T, v_L)
\]

for the Fermi gas, and

\[
C_v(T, v_H) < C_v(T, v_L) \quad T > T_B,
\]

\[
C_v(T, v_H) > C_v(T, v_L) \quad T < T_B,
\]

where \( T_B \) is the equivalence temperature at which the heat capacities at two constant volumes are equal to each other, i.e., \( C_v(T_B, v_H) = C_v(T_B, v_L) \) [22, 30]. Consequently, it is clearly seen from equations (3), (4), (9) and (10) that there are two possible cases: (a) \( \Delta Q < 0 \), for all temperatures of a Fermi gas and the temperature regions of \( T < T_B \) of a Bose gas and (b) \( \Delta Q > 0 \), for the temperature regions of \( T > T_B \) of a Bose gas. When \( \Delta Q > 0 \), the amount of heat \( Q_{hc} \) flowing into the regenerator in the small constant-volume regenerative process is larger than that \( Q_{bc} \) flowing from the regenerator in the large constant-volume regenerative process. The redundant heat in the regenerator per cycle must be released to the cold reservoir in a timely manner. This results in the reduction of the amount of refrigeration from \( Q_2 \) to \( Q'_2 \). If not, the temperature of the regenerator would be changed such that the regenerator would not be operated normally. Similarly, when \( \Delta Q < 0 \), the amount of heat \( Q_{da} \) flowing into the regenerator in the small constant-volume regenerative process is smaller than that \( Q_{bc} \) flowing from the regenerator in the large constant-volume regenerative process. The inadequate heat in the regenerator per cycle must be compensated from the hot reservoir in a timely manner, while the amount of refrigeration \( Q_2 \) is unvarying.

According to the regenerative characteristics mentioned above, the unified expression for the amount of refrigeration per cycle can be given by

\[
Q_1 = Q_2 - \delta \Delta Q
\]

\[
= NK[T_1[F(z_b) - F(z_a) - \ln(z_b/z_d)] - \frac{1}{2}Nk[T_1[F(z_a) - F(z_c)] + T_2[F(z_a) - F(z_b)]].
\]

where \( \delta = 1 \) when \( \Delta Q > 0 \) and \( \delta = 0 \) when \( \Delta Q < 0 \).

3. Expressions for several important parameters

The coefficient of performance, cooling rate and power input are three of the important performance parameters, which are often considered in the optimal design and theoretical analysis of refrigerators. Using equations (2), (7), (8) and (11), we find that the coefficient of performance, cooling rate and power input may be, respectively, expressed as

\[
\varepsilon = \frac{Q'_2}{W}
\]

\[
= \frac{T_2[(5/2)F_{ba} - \ln(z_b/z_d)] - \frac{1}{2}(3/2)(T_2F_{ba} - T_1F_{cd})}{T_1[F_{cd} - \ln(z_c/z_a)] - T_2[F_{ba} - \ln(z_b/z_d)]},
\]

\[
R = \frac{Q'_2}{T}
\]

\[
= \frac{T_2[(5/2)F_{ba} - \ln(z_b/z_d)] - \frac{1}{2}(3/2)(T_2F_{ba} - T_1F_{cd})}{\frac{T_1[(5/2)F_{cd} - \ln(z_c/z_a)] + T_2[(5/2)F_{ba} - \ln(z_b/z_d)]}{\alpha(T_1 - T_H)} + \frac{T_1[(5/2)F_{cd} - \ln(z_c/z_a)] + T_2[(5/2)F_{ba} - \ln(z_b/z_d)]}{\beta(T_L - T_2)}},
\]

and

\[
P = \frac{W}{T}
\]

\[
= \frac{T_1[F_{cd} - \ln(z_c/z_a)] - T_2[F_{ba} - \ln(z_b/z_d)]}{\alpha(T_1 - T_H)} + \frac{T_1[F_{cd} - \ln(z_c/z_a)] - T_2[F_{ba} - \ln(z_b/z_d)]}{\beta(T_L - T_2)}.
\]
where \( F_{cd} = F(z_c) - F(z_d) \), \( F_{ba} = F(z_b) - F(z_a) \), \( z_c = z(T_1, v_1) \), \( z_d = z(T_1, v_1) \), \( z_a = z(T_2, v_1) \) and \( z_b = z(T_1, v_H) \).

Using equations (12) and (13), one can reveal the general performance characteristics of the quantum Stirling cryogenic refrigeration cycle and give the optimal criteria of some important performance parameters.

4. General optimum performance characteristics

Using a refrigerator, one always wants to obtain a cooling rate as large as possible for a given power input. For this purpose, we introduce the Lagrangian function

\[
L = R + \lambda P
\]

\[
= \frac{\left[ T_2(5/2)F_{ba} - \ln(z_b/z_a) \right] - \Phi + \lambda T_2}{T_1(5/2)F_{cd} - \ln(z_c/z_d) - \lambda T_1[F_{ba} - \ln(z_b/z_a)]} = \frac{T_2(5/2)F_{ba} - \ln(z_b/z_a) - \Phi}{T_1(5/2)F_{cd} - \ln(z_c/z_d) - \lambda T_1[F_{ba} - \ln(z_b/z_a)]},
\]

where \( \lambda \) is a Lagrange multiplier and \( \Phi = \delta(3/2)(T_2F_{ba} - T_1F_{cd}) \). From the Euler–Lagrange equation \( \partial L / \partial T_1 = 0 \) or \( \partial L / \partial T_2 = 0 \) and equation (15), we find the optimal relation between \( T_1 \) and \( T_2 \) as

\[
(D_1 - \Phi) \left[ \frac{D_1T_1}{\alpha(T_1 - T_H)} + \frac{D_2T_2}{\beta(T_L - T_2)} \right] - (D_2T_2 - \Phi) \left[ \frac{D_1(T_1 - T_2)}{\beta(T_L - T_2)} \right] = 0,
\]

where

\[
D_1 = 4F_{ba} - \ln(z_b/z_a), \quad D_2 = (5/2)F_{ba} - \ln(z_b/z_a), \quad D_3 = (5/2)F_{cd} - \ln(z_c/z_d), \quad \Phi = \delta(15/4)F_{ba} - (9/4)F_{ba}^\gamma,
\]

\[
F_{ba} = F_{ba}^3(z_b) - F_{ba}^3(z_a)
\]

\[
F_{ba}^3(z) = \frac{1}{\Gamma(3/2)}\int_0^\infty \frac{x^{3/2}}{e^x + 1} dx
\]

\[
\times \left[ \frac{1}{\Gamma(1/2)}\int_0^\infty \frac{x^{1/2}}{e^x + 1} dx \right]^{-1}.
\]

For given \( \alpha, \beta, v_1, v_H, T_1 \) and \( T_H \), the \( R^* - \varepsilon, P^* - \varepsilon \) and \( R^* - P^* \) optimum characteristic curves can be plotted by using equations (12)–(14) and (16) and choosing ¹He gas as the ideal Fermi gas, as shown by the solid curves in figures 2 and 3, where \( R^* = \gamma R/(\alpha T_1) \) and \( P^* = \gamma P/(\alpha T_1) \) are, respectively, the dimensionless cooling rate and power input. In figure 2, the parameters \( \alpha/\beta = 1, v_1 = 2 \times 10^{-29} \text{ m}^3, v_H = 10^{-28} \text{ m}^3, T_1 = 10 \text{ K} \) and \( T_H = 20 \text{ K} \) are adopted. Figure 2 shows clearly that the fundamental optimal relation between the cooling rate and the coefficient of performance is not monotonic and there exist a maximum cooling rate \( R_{\text{max}} \) and a corresponding coefficient of performance \( \varepsilon_m \) for a set of given parameters \( \alpha, \beta, v_1, v_H, T_1 \) and \( T_H \). Obviously, for different given parameters, the maximum cooling rate \( R_{\text{max}} \) and corresponding coefficient of performance \( \varepsilon_m \) will be different. It is very different from the optimal performance of a classical reversible Stirling refrigeration cycle [34, 35] using an ideal gas as the working substance, in which the cooling rate is a monotonically decreasing function of the coefficient of performance. It is also seen from figure 2 that when \( R < R_{\text{max}} \), there are two different coefficients of performance for a given cooling rate \( R \), where one is smaller than \( \varepsilon_m \) and the other is larger than \( \varepsilon_m \). When \( \varepsilon < \varepsilon_m \), the cooling rate will decrease as the coefficient of performance is decreased. It is thus clear that the region of \( \varepsilon < \varepsilon_m \) is not optimal for a Fermi Stirling cryogenic refrigeration cycle. Consequently, the optimal region of the coefficient of performance should be

\[
\varepsilon_m \leq \varepsilon < \varepsilon_{\text{max}}.
\]

For a Fermi Stirling cryogenic refrigeration cycle is operated in this region, the cooling rate will increase as the coefficient of performance is decreased and vice versa. It is of significance to note the fact that when \( \varepsilon = \varepsilon_{\text{max}} \), the cooling rate of a Fermi Stirling cryogenic refrigeration cycle is equal to zero. It shows clearly that the coefficient of performance of a real Fermi Stirling cryogenic refrigeration cycle is always smaller than \( \varepsilon_{\text{max}} \).
Using the above results and figure 3, we can further find that the optimal values of the power input should be

\[ P \leq P_m, \]  

(18)

where \( P_m \) is the power input at the maximum cooling rate. The above results show clearly that the maximum cooling rate \( R_{\text{max}} \), coefficient of performance \( \varepsilon_m \) at the maximum cooling rate, power input \( P_m \) at the maximum cooling rate and maximum coefficient of performance \( \varepsilon_{\text{max}} \) are four important performance parameters of a Fermi Stirling refrigeration cycle. \( R_{\text{max}} \) and \( \varepsilon_{\text{max}} \) determine the upper bounds of the cooling rate and coefficient of performance, and \( \varepsilon_m \) and \( P_m \) determine the allowable values of the lower and upper bounds of the optimal coefficient of performance and power input, respectively.

Similarly, choosing \(^4\)He gas as the ideal Bose gas and using the same method mentioned above, one can obtain the general optimal performance characteristics of a Bose Stirling refrigeration cycle, as shown by the dashed curves in figures 2. The values of the relevant parameters are the same as those used in the Fermi Stirling cryogenic refrigeration cycle. It can be seen from figure 2 that the fundamental optimum curve between the cooling rate and the coefficient of performance of the Bose Stirling cryogenic refrigeration cycle is similar to that of the Fermi Stirling cryogenic refrigeration cycle. Because of the influence of the inherent regenerative losses, the cooling rate of the Bose Stirling cryogenic refrigeration cycle is smaller than that of the Fermi Stirling cryogenic refrigeration cycle.

\[ R = \frac{1}{\gamma} \left( \frac{T_1^2}{\alpha(T_1 - T_H)} + \frac{T_2^2}{\beta(T_L - T_2)} \right)^{-1}, \]  

(23)

and

\[ P = \frac{1}{2\gamma} \left( T_1^2 - T_2^2 \right) \left( \frac{T_1^2}{\alpha(T_1 - T_H)} + \frac{T_2^2}{\beta(T_L - T_2)} \right)^{-1}, \]  

(24)

where \( \varepsilon_c \) is the coefficient of performance of the reversible Carnot refrigerator. In such a case, the coefficient of performance of a Fermi Stirling refrigeration cycle is only a function of temperature of the working substance and independent of other parameters. By using equations (22) and (23), the cooling rate can be further expressed as

\[ R = \frac{1}{\gamma} \left( \frac{2\varepsilon^{-1} + 1}{\alpha(\sqrt{2\varepsilon^{-1}} + T_2 - T_H)} + \frac{1}{\beta(T_L - T_2)} \right)^{-1}, \]  

(25)

Using equation (25) and the external condition \( \partial R/\partial T_2 = 0 \), we can find that the fundamental optimal relations between some important parameters and the coefficient of performance are, respectively, given by

\[ R = \frac{1}{\gamma} \frac{\alpha\beta[(2\varepsilon^{-1} + 1)T_L - T_H]}{\sqrt{\alpha(2\varepsilon + 1)T_1 + \sqrt{\beta(2\varepsilon + 1)/4}}}, \]  

(26)

\[ P = \frac{1}{\gamma} \frac{\alpha\beta[(2\varepsilon^{-1} + 1)T_L - T_H]^{\varepsilon^{-1}}}{\sqrt{\alpha(2\varepsilon + 1)T_1 + \sqrt{\beta(2\varepsilon + 1)/4}}}, \]  

(27)

\[ T_1 = \frac{\sqrt{\alpha/\beta(2\varepsilon^{-1} + 1)T_1 + (2\varepsilon^{-1} + 1)^{1/4}T_L}}{\sqrt{\alpha/\beta(2\varepsilon + 1)/4}}, \]  

(28)

and

\[ T_2 = \frac{\sqrt{\alpha/\beta T_H + (2\varepsilon^{-1} + 1)^{1/4}T_L}}{\sqrt{\alpha/\beta(2\varepsilon + 1)/4}}, \]  

(29)

It is clearly seen from equation (26) that the cooling rate \( R \) is zero when \( \varepsilon = 0 \) or \( \varepsilon = 2T_1/(T_H - T_2) = \varepsilon_{\text{max}} \). This implies the fact that when the coefficient of performance is equal to some value, there is a maximum for the cooling rate, as shown in figure 4. It can be seen from figure 4 that the larger the temperature ratio \( T_H/T_1 \), the smaller not only the maximum cooling rate \( R_{\text{max}} \) and corresponding coefficient of performance \( \varepsilon_m \) but also the maximum coefficient of performance \( \varepsilon_{\text{max}} \). In addition, it can be easily found by comparing figure 4 with figure 2 that for the same value of \( T_H/T_1 \), the stronger the quantum degeneracy of the working substance is, the smaller the maximum cooling rate \( R_{\text{max}} \) and coefficient of performance \( \varepsilon_{\text{max}} \), while the larger the coefficient of performance \( \varepsilon_m \) at the maximum cooling rate. This indicates once again that it is necessary to consider the effect of quantum degeneracy on the performance of a thermodynamic cycle at low temperatures.

Using equations (26)–(29) and the external condition \( \partial R/\partial \varepsilon = 0 \), we can find that in the strong gas degeneracy case, three important performance parameters of a Fermi Stirling cryogenic refrigeration cycle and the optimal temperatures of the working substance in the two isothermal processes can be, respectively, given by

\[ R_{\text{max}} = \frac{\alpha\beta}{\gamma} \frac{C^2T_1 - T_H}{(C\sqrt{\alpha} + C^2\sqrt{\beta})^2}. \]  

(30)
Figure 4. The dimensionless cooling rate \( R^* = \gamma R/(\alpha T_1) \) versus the coefficient of performance \( \varepsilon \) for some given values of \( \alpha/\beta \) and \( T_H/T_1 \).

\[
\varepsilon_m = \frac{2}{C^4 - 1},
\]

\[
P_m = \frac{2\alpha\beta}{\gamma} \frac{C^2 T_L - T_H}{(C^4 - 1)(C^\alpha + C^\beta)^2},
\]

\[
T_{1m} = C^2(\sqrt{\alpha/\beta}T_H + CT_L)(\sqrt{\alpha/\beta} + C)^{-1},
\]

and

\[
T_{2m} = (\sqrt{\alpha/\beta}T_H + CT_L)(\sqrt{\alpha/\beta} + C)^{-1},
\]

where

\[
C = [T_H(9\sqrt{\alpha/\beta} + \sqrt{81\alpha/\beta - 96T_H/T_L})/18T_L]^{1/3} + [16T_H^2/T_L^2/(3(9\sqrt{\alpha/\beta} + \sqrt{81\alpha/\beta - 96T_H/T_L}))]^{1/3}.
\]

Analysing equations (17), (22), (33) and (34), and figure 4, we find that the optimal ranges of the temperature of the working substance in the two isothermal processes are

\[
T_H < T_1 \leq C^2(\sqrt{\alpha/\beta}T_H + CT_L)(\sqrt{\alpha/\beta} + C)^{-1}
\]

(35)

and

\[
T_L > T_2 \geq (\sqrt{\alpha/\beta}T_H + CT_L)(\sqrt{\alpha/\beta} + C)^{-1}.
\]

Obviously, equations (17), (18), (31), (32), (35) and (36) provide some significant important criteria for the parametric optimum design of the Fermi Stirling cryogenic refrigeration cycle.

5.2. Weak gas degeneracy

Under the higher temperature or lower-density condition, i.e., the condition of weak gas degeneracy, \( 0 < z < 1 \), the Fermi–Dirac integral \( f_\alpha(z) \) and the Bose–Einstein integral \( g_\alpha(z) \) may be expanded in power of \( z \). Consequently, the function \( F(T, v) \) and fugacity \( z \) at the weak gas degeneracy cases may be expanded to the first approximation as

\[
F(T, v) = 1 \pm \frac{B}{T^{3/2}v},
\]

and

\[
\varepsilon = \frac{\sqrt{2}B}{T^{3/2}v} [1 \pm 2B/(T^{3/2}v)], \quad (38)
\]

where \( B = \hbar^3/[16g(m_k\pi)^{3/2}] \), \( g \) is a weight factor that arises from the ‘internal structure’ of the particles (such as spin), and signs ‘+’ correspond to ideal Fermi and Bose gases, respectively. By using equations (37) and (38), equations (9), (12)–(14) and (16) can be further simplified.

For a Stirling refrigeration cycle working with an ideal Fermi gases, equations (9), (12)–(14) and (16) can be, respectively, expressed as

\[
\Delta Q = (3/2)nkB\Delta v(T_1^{-1/2} - T_2^{-1/2}) < 0,
\]

(39)

\[
\varepsilon = \frac{T_2 \ln r_v - (1/2)B\Delta vT_2^{-1/2}}{(T_1 - T_2) \ln r_v + B\Delta v(T_1^{-1/2} - T_2^{-1/2})},
\]

(40)

\[
R = \frac{1}{\gamma} \frac{T_2 \ln r_v - (1/2)B\Delta vT_2^{-1/2}}{(T_1 - T_2) \ln r_v + B\Delta v(T_1^{-1/2} - T_2^{-1/2})},
\]

(41)

and

\[
\Delta Q = (3/2)nkB\Delta v(T_1^{-1/2} - T_2^{-1/2}) < 0,
\]

(39)

\[
\varepsilon = \frac{T_2 \ln r_v - (1/2)B\Delta vT_2^{-1/2}}{(T_1 - T_2) \ln r_v + B\Delta v(T_1^{-1/2} - T_2^{-1/2})},
\]

(40)

\[
R = \frac{1}{\gamma} \frac{T_2 \ln r_v - (1/2)B\Delta vT_2^{-1/2}}{(T_1 - T_2) \ln r_v + B\Delta v(T_1^{-1/2} - T_2^{-1/2})},
\]

(41)

\[
\Delta Q = (3/2)nkB\Delta v(T_1^{-1/2} - T_2^{-1/2}) < 0,
\]

(40)

\[
\varepsilon = \frac{T_2 \ln r_v - (1/2)B\Delta vT_2^{-1/2}}{(T_1 - T_2) \ln r_v + B\Delta v(T_1^{-1/2} - T_2^{-1/2})},
\]

(41)

\[
R = \frac{1}{\gamma} \frac{T_2 \ln r_v - (1/2)B\Delta vT_2^{-1/2}}{(T_1 - T_2) \ln r_v + B\Delta v(T_1^{-1/2} - T_2^{-1/2})},
\]

(42)

\[
\Delta Q = (3/2)nkB\Delta v(T_1^{-1/2} - T_2^{-1/2}) < 0,
\]

(43)
The dimensionless cooling rate $\tau = \frac{\ln n}{\ln T}$ and power input $P^* = \frac{\gamma P}{\alpha T_a}$ versus the coefficient of performance $\varepsilon$ for the parameters $\alpha/\beta = 1$, $a = 2 \times 10^{-29} \text{ m}^3$, $e = 10^{-28} \text{ m}^3$, and $T_a = 60 \text{ K}$. Curves I, II and III correspond to the cases of $\tau = T_h/T_1 = 1.5, 2.0$ and 3.0, respectively.

Using equations (44)–(48), one can also analyse the optimal performance characteristics of the Bose Stirling refrigeration cycle.

5.3. At high temperatures

When the temperature of the working substance is high enough and its density is low enough, the fugacity of the Fermi and Bose gases $z$ is much smaller than unity. In such a case, $F(z) = 1$ and $f_g(z) = z$. Equations (9), (12)–(14) and (16) can be, respectively, simplified as

$$\Delta Q = 0,$$

$$\varepsilon = \frac{T_2}{T_1 - T_2},$$

$$R = \frac{1}{\gamma} \beta \left( \frac{1}{T_1 - T_2} + \frac{1}{\alpha T_1 - T_h} \right)^{-1},$$

$$P = \frac{1}{\gamma} (T_1 - T_2) \left[ \frac{T_1}{\alpha T_1 - T_h} + \frac{T_2}{\beta T_1 - T_2} \right]^{-1},$$

and

$$T_1 = T_h T_2^2 / (T_2^2 - (\beta/\alpha) (T_1 - T_2)^2).$$

The results are just those of a Stirling refrigeration cycle working with an ideal gas [34–37]. This shows clearly that at high temperatures, the quantum Stirling refrigeration cycle becomes the classical Stirling refrigeration cycle.

From equations (50), (51) and (53), it can be obtained that the relation between the optimal cooling rate and the coefficient of performance at high temperatures is [34]

$$R = \frac{1}{\varepsilon} K [T_L - \varepsilon T_H/(1 + \varepsilon)],$$

where $K = \alpha \beta / (\sqrt{\alpha} + \sqrt{\beta})^2$. It can be seen from equation (54) that the cooling rate $R$ is a monotonically decreasing function of the coefficient of performance $\varepsilon$.

It is of interest to compare the results obtained here with those derived from a Carnot refrigeration cycle using an ideal gas as the working substance. It can be found that when the influence of finite-rate heat transfer between the working substance and the heat reservoirs on the performance of the Carnot refrigeration cycle is considered and the heat transfer is assumed to obey a linear law, the fundamental optimum relations of the quantum Stirling refrigeration cycle in the high temperature limit are the same as those of the Carnot refrigeration cycle [38]. This is just an expected result because the quantum behaviour of gas particles in this case is negligible and the refrigeration cycle may possess the condition of perfect regeneration.

6. Conclusions

On the basis of statistical mechanics, we have analysed the influence of the quantum degeneracy of the working substance and the irreversibility of the finite-rate heat transfer between the working substance and the heat reservoirs on the optimum performance characteristics of the Stirling cryogenic refrigeration cycle working with an ideal quantum gas. The concrete expressions for some important performance parameters, such as the coefficient of performance, cooling rate, power input and inherent regenerative losses, are derived. It can be clearly seen from these general expressions of important performance parameters and the corresponding curves that the performance characteristics of a Fermi or Bose Stirling cryogenic refrigeration cycle are different from those of a classical Stirling refrigeration cycle. The coefficient of performance and cooling rate are, in general, dependent not only on temperature and the thermal conductances between the working substance and the heat reservoirs but also on the volume and other parameters. By using the expressions, the effect of nonperfect regeneration is analysed and the optimal relation between the cooling rate and coefficient of performance is obtained. Several optimal performance characteristic curves are generated. The optimally operating regions of the cycle are determined and the optimum criteria of some important performance parameters are given. In general, the Fermi or Bose Stirling cryogenic refrigeration cycle does not possess the condition of perfect regeneration. The cooling rate is not a monotonic function of the coefficient of performance, and the optimal coefficient of performance always decreases as the cooling rate is increased. It is very significant to reveal the optimal performance of the Fermi or Bose Stirling cryogenic refrigeration cycle in the strong and weak gas degeneracy cases, derive the analytical expressions of some optimized parameters, and give some optimum...
criteria. Finally, it is pointed out that in the high-temperature limit, the Fermi or Bose Stirling refrigeration cycle becomes the classical Stirling refrigeration cycle, so that the optimal performance of the classical Stirling or Carnot refrigeration cycles investigated widely in literature may be directly derived from the results in the present paper.

Acknowledgments

This work has been supported by the National Natural Science Foundation (No. 10575084), People’s Republic of China and the Natural Science Foundation of Fujian Province (No. Z0512010), People’s Republic of China.

References

[29] He J, Chen J and Hua B 2002 Appl. Energy 72 541
[34] Wu C 1993 Energy Conversion Manag. 34 1249
[35] Chen J and Yan Z 1993 Cryogenics 33 863
[37] Chen J 1998 Energy Conversion Manag. 39 1255