SIAM J. SCI. COMPUT.
Vol. 32, No. 4, pp. 1740–1760

NUMERICAL SCHEMES WITH HIGH SPATIAL ACCURACY FOR A VARIABLE-ORDER ANOMALOUS SUBDIFFUSION EQUATION*

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Abstract. In this paper, we consider a variable-order anomalous subdiffusion equation. A numerical scheme with first order temporal accuracy and fourth order spatial accuracy for the equation is proposed. The convergence, stability, and solvability of the numerical scheme are discussed via the technique of Fourier analysis. Another improved numerical scheme with second order temporal accuracy and fourth order spatial accuracy is also proposed. Some numerical examples are given, and the results demonstrate the effectiveness of theoretical analysis.

Key words. variable-order anomalous subdiffusion equation, convergence, stability, solvability, Fourier analysis, high spatial accuracy, new numerical scheme, improved numerical scheme

AMS subject classifications. 26A33, 65M06, 65M12

DOI. 10.1137/090771715

1. Introduction.

Fractional diffusion equations have been widely used in recent years in various applications in science and engineering [2], [10], [11], [12], [13], [16], [27], [28], [29], [30], [44]. Analytical solutions of most of these equations are not available. Even if these solutions can be given, their constructions by special functions make their computations difficult. Therefore, a number of authors proposed numerical methods for solving fractional diffusion equations [1], [3], [4], [5], [6], [7], [17], [20], [21], [22], [23], [24], [33], [39], [40], [41], [42], [43], [45]. It is a more difficult task to solve anomalous subdiffusion equations numerically, which are actually integro-differential equations.

A class of anomalous subdiffusion equations takes the form

$$\frac{\partial u(x,t)}{\partial t} = {}_0D_t^{1-\gamma}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) + f(x,t). \tag{1.1}$$

Yuste and Acedo [42] proposed an explicit finite difference method and a von Neumann-type stability analysis for (1.1). They pointed out the difficulty of convergence analysis when implicit methods are considered. Yuste [41] proposed weighted average finite difference methods for this equation. Langlands and Henry [17] proposed an implicit numerical scheme (L_1-approximation) and discussed its accuracy and stability. However, the global accuracy of the implicit numerical scheme has not been derived, and it seems that the unconditional stability for all γ in the range $0 < \gamma \leq 1$ has not been established. Chen et al. [7] and Chen, Liu, and Burrage [6] presented some numerical methods with first order temporal accuracy and second order spatial accuracy for the fractional diffusion equation describing subdiffusion and the fractional reaction subdiffusion equation, and they analyzed the stability, convergence, and solvability of

*Received by the editors September 19, 2009; accepted for publication (in revised form) March 24, 2010; published electronically June 18, 2010. This research was partially supported by Australian Research Council grant DP0986766, National Natural Science Foundation of China grant 10271098, and Natural Science Foundation of Fujian province grant 2009J01014.

http://www.siam.org/journals/sisc/32-4/77171.html

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these numerical methods using Fourier analysis. Liu et al. [22] considered a space-time fractional advection dispersion equation. They proposed an implicit difference method and an explicit difference method to solve this equation. Stability and convergence of these methods are discussed using mathematical induction. Zhuang et al. [45] proposed implicit numerical methods for the anomalous subdiffusion equation using the energy method. Liu, Yang, and Burrage [21] considered a modified anomalous subdiffusion equation with a nonlinear source term. An implicit difference method is constructed. Its stability and convergence are discussed using the energy method. From the continuous time random walk model, Mommer and Lebiedz [31] constructed linear reaction diffusion systems which capture the defining properties of subdiffusion.

In various applications in science and engineering, in order to more accurately describe the evolution of a system, the concept of a variable-order operator has been developed. Variable-order fractional partial differential equations have been studied [8], [9], [14], [15], [18], [25], [26], [34], [35], [36], [37]. To date, numerical methods and numerical analysis of variable-order fractional partial differential equations are still at an early stage of development. Lin et al. [19] investigated stability and convergence of an explicit finite difference approximation for the variable-order nonlinear fractional diffusion equation; Zhuang et al. [46] proposed explicit and implicit Euler approximations for the variable-order fractional advection-diffusion equation with a nonlinear source term. Stability and convergence of the methods are discussed. Moreover, they also present a fractional method of lines, a matrix transfer technique, and an extrapolation method for the equation. Sun, Chen, and Chen [38] introduced a classification of variable-order fractional diffusion models based on the possible physical origins which cause variable order.

The variable-order problem is apparently more complicated than a constant fractional order problem. We are unaware of any other published work on numerical schemes for a variable-order anomalous subdiffusion equation. This paper is our effort to remedy the situation: we develop some new numerical methods with high accuracy for a variable-order anomalous subdiffusion equation and investigate their stability and convergence.

We propose a numerical scheme with first order temporal accuracy and fourth order spatial accuracy and perform related numerical analysis for the following equation:

$$\frac{\partial u(x,t)}{\partial t} ={}_0 D_t^{1-\gamma(x,t)}\left(\frac{\partial^2 u(x,t)}{\partial x^2}\right) + f(x,t), \tag{1.2}$$

with initial and boundary conditions

$$u(x,0) = \phi(x), \qquad 0 \le x \le L, \tag{1.3}$$

$$u(0,t) = \varphi(t), \qquad 0 < t \le T, \tag{1.4}$$

$$u(L,t) = \psi(t), \qquad 0 < t \le T, \tag{1.5}$$

where $0 < \gamma_{min} \le \gamma(x,t) \le \gamma_{max} < 1$ and ${}_0D_t^{1-\gamma(x,t)} v(x,t)$ is the variable-order Riemann–Liouville fractional derivative of order $1-\gamma(x,t)$ defined as [19], [46]

$${}_0D_t^{1-\gamma(x,t)} v(x,t) = \frac{1}{\Gamma(\gamma(x,t))}\left[\frac{\partial}{\partial \xi}\int_0^{\xi} \frac{v(x,\eta)}{(\xi-\eta)^{1-\gamma(x,t)}} d\eta\right]_{\xi=t}. \tag{1.6}$$

This numerical scheme is detailed in section 2. In section 3, some lemmas are established. In sections 4 and 5, the stability and convergence of the numerical scheme are discussed. In section 6, its solvability is analyzed. In section 7, a new improved numerical scheme with second order temporal accuracy is introduced. Finally, some numerical results are given to evaluate the performance of the methods.

2. A numerical scheme for the variable-order anomalous subdiffusion equation. This section details a numerical scheme for solving (1.2) with initial and boundary conditions (1.3)–(1.5). We take an equally spaced mesh of J points for the spatial domain $0 \le x \le L$ and K constant time steps for the temporal domain $0 \le t \le T$, and we denote the spatial grid points by

$$x_j = jh, \qquad j = 0, 1, \dots, J,$$

and the temporal grid points by

$$t_k = k\tau, \qquad k = 0, 1, \dots, K,$$

where the grid spacing is simply $h = L/J$ in the spatial domain and $\tau = T/K$ in the temporal domain. We also introduce the following notation:

$$\Omega = \{(x,t) |\ 0 \le x \le L,\ 0 \le t \le T\},$$

$$\mathscr{U}(\Omega) = \left\{ u(x,t) \left| \frac{\partial^6 u(x,t)}{\partial x^6}, \frac{\partial^3 u(x,y,t)}{\partial x^2 \partial t}, \frac{\partial^2 u(x,y,t)}{\partial t^2} \in C(\Omega) \right. \right\}.$$

In this paper, we assume $u(x,t) \in \mathscr{U}(\Omega)$. Clearly, at the grid point (x_j, t_k), (1.2) becomes

$$\frac{\partial u(x_j,t_k)}{\partial t} = {}_0 D_t^{1-\gamma_j^k} \left(\frac{\partial^2 u(x_j,t_k)}{\partial x^2} \right) + f_j^k, \tag{2.1}$$

where $\gamma_j^k \equiv \gamma(x_j,t_k)$, $f_j^k \equiv f(x_j,t_k)$.

In constructing an approximation for (2.1), a key point is how to approximate the Riemann–Liouville fractional derivative. The Grünwald–Letnikov fractional derivative of order $1-\gamma_j^k$ for $g(x,t)$ is defined as

$$\lim_{\tau \to 0} \tau^{\gamma_j^k - 1} \sum_{l=0}^{[t/\tau]} (-1)^l \binom{1-\gamma_j^k}{l} g(x, t - l\tau).$$

If the function $g(x,t)$ has continuous partial derivative $\frac{\partial g(x,t)}{\partial t}$ for $t > 0$, then the Riemann–Liouville and Grünwald–Letnikov fractional derivatives of order $1-\gamma_j^k$ for $g(x,t)$ are equivalent, i.e.,

$${}_0D_t^{1-\gamma_j^k} g(x,t) = \lim_{\tau \to 0} \tau^{\gamma_j^k - 1} \sum_{l=0}^{[t/\tau]} (-1)^l \binom{1-\gamma_j^k}{l} g(x, t - l\tau). \tag{2.2}$$

From [32], we have

$$\begin{aligned} &\lim_{\tau \to 0} \tau^{\gamma_j^k - 1} \sum_{l=0}^{[t/\tau]} (-1)^l \binom{1-\gamma_j^k}{l} g(x, t - l\tau) \\ &= \tau^{\gamma_j^k - 1} \sum_{l=0}^{[t/\tau]} (-1)^l \binom{1-\gamma_j^k}{l} g(x, t - l\tau) + O(\tau). \end{aligned} \tag{2.3}$$

By (2.2) and (2.3), we get

$$ {}_0D_t^{1-\gamma_j^k} g(x,t) = \tau^{\gamma_j^k - 1} \sum_{l=0}^{[t/\tau]} (-1)^l \binom{1-\gamma_j^k}{l} g(x, t-l\tau) + O(\tau). $$

Thus,

$$ \left[{}_0D_t^{1-\gamma_j^k} g(x,t) \right]_{t=t_k} = \tau^{\gamma_j^k - 1} \sum_{l=0}^{k} \lambda_{j,k}^l g(x, t_{k-l}) + O(\tau) \tag{2.4} $$

and

$$ {}_0D_t^{1-\gamma_j^k} g(x_j,t_k) = \tau^{\gamma_j^k - 1} \sum_{l=0}^{k} \lambda_{j,k}^l g(x_j, t_{k-l}) + O(\tau), \tag{2.5} $$

where $\lambda_{j,k}^l = (-1)^l \binom{1-\gamma_j^k}{l} = (-1)^l \frac{(1-\gamma_j^k)(-\gamma_j^k)\dots(1-\gamma_j^k-l+1)}{l!}$. From (2.5) and $u(x,t) \in \mathscr{U}(\Omega)$, it follows that

$$ \begin{aligned} &{}_0D_t^{1-\gamma_j^k} \left(\frac{\partial^2 u(x_j,t_k)}{\partial x^2} \right) \\ &= \left[\frac{1}{\Gamma(\gamma(x_j,t_k))} \frac{\partial}{\partial \xi} \int_0^{\xi} \frac{\partial^2 u(x_j,\eta)}{\partial x^2} \frac{d\eta}{(\xi-\eta)^{1-\gamma(x_j,t_k)}} \right]_{\xi=t_k} \\ &= \tau^{\gamma_j^k - 1} \sum_{l=0}^{k} \lambda_{j,k}^l \frac{\partial^2 u(x_j,t_{k-l})}{\partial x^2} + O(\tau). \end{aligned} \tag{2.6} $$

Again, because $u(x,t) \in \mathscr{U}(\Omega)$, the following formulas hold:

$$ \frac{\partial u(x_j,t_k)}{\partial t} = \frac{u(x_j,t_k) - u(x_j,t_{k-1})}{\tau} + O(\tau), \tag{2.7} $$

$$ \frac{\delta_x^2 u(x_j,t_k)}{h^2} = \frac{\partial^2 u(x_j,t_k)}{\partial x^2} + \frac{h^2}{12} \frac{\partial^4 u(x_j,t_k)}{\partial x^4} + O(h^4), \tag{2.8} $$

$$ \delta_x^2 u(x_j,t_k) = h^2 \frac{\partial^2 u(x_j,t_k)}{\partial x^2} + O(h^4), \tag{2.9} $$

where $\delta_x^2 u(x_j,t_k) = u(x_{j-1},t_k) - 2u(x_j,t_k) + u(x_{j+1},t_k)$. By replacing $u(x,t)$ with $\frac{\partial^2 u(x,t)}{\partial x^2}$ in (2.9), we arrive at

$$ \delta_x^2 \frac{\partial^2 u(x_j,t_k)}{\partial x^2} = h^2 \frac{\partial^4 u(x_j,t_k)}{\partial x^4} + O(h^4). $$

Multiplying by $\frac{1}{12}$ and adding $\frac{\partial^2 u(x_j,t_k)}{\partial x^2}$ to both sides of the above expression yields

$$ \left(1 + \frac{1}{12}\delta_x^2\right) \frac{\partial^2 u(x_j,t_k)}{\partial x^2} = \frac{\partial^2 u(x_j,t_k)}{\partial x^2} + \frac{h^2}{12} \frac{\partial^4 u(x_j,t_k)}{\partial x^4} + O(h^4). \tag{2.10} $$

Combining (2.8) and (2.10), we get

$$\frac{\delta_x^2 u(x_j,t_k)}{h^2} = \left(1+\frac{1}{12}\delta_x^2\right)\frac{\partial^2 u(x_j,t_k)}{\partial x^2} + O(h^4).$$

We then obtain the following fourth order approximation formula for the second order partial derivative:

$$\frac{\partial^2 u(x_j,t_k)}{\partial x^2} = \frac{\delta_x^2}{h^2\left(1+\frac{1}{12}\delta_x^2\right)} u(x_j,t_k) + O(h^4). \tag{2.11}$$

Applying (2.6), (2.7), and (2.11), we now have

$$u(x_j,t_k) = u(x_j,t_{k-1}) + \mu_j^k \sum_{l=0}^{k} \lambda_{j,k}^l \frac{\delta_x^2}{1+\frac{1}{12}\delta_x^2} u(x_j,t_{k-l}) + \tau f_j^k + \widetilde{R}_j^k,$$

i.e.,

$$\begin{aligned}
&\left(1+\frac{1}{12}\delta_x^2\right) u(x_j,t_k) \\
&= \left(1+\frac{1}{12}\delta_x^2\right) u(x_j,t_{k-1}) + \mu_j^k \sum_{l=0}^{k} \lambda_{j,k}^l \delta_x^2 u(x_j,t_{k-l}) \\
&\quad + \tau\left(1+\frac{1}{12}\delta_x^2\right) f_j^k + R_j^k,
\end{aligned} \tag{2.12}$$

where

$$\mu_j^k = \frac{\tau^{\gamma_j^k}}{h^2}, \qquad \widetilde{R}_j^k = O(h^4)\tau^{\gamma_j^k}\sum_{l=0}^{k}\lambda_{j,k}^l + O(\tau^2),$$

whereas

$$\begin{aligned}
R_j^k &= \left(1+\frac{1}{12}\delta_x^2\right)\widetilde{R}_j^k \\
&= \left(1+\frac{1}{12}\delta_x^2\right)\left(O(h^4)\tau^{\gamma_j^k}\sum_{l=0}^{k}\lambda_{j,k}^l + O(\tau^2)\right) \\
&= O(h^4)\tau^{\gamma_j^k}\sum_{l=0}^{k}\lambda_{j,k}^l + O(\tau^2) + O(h^4)\delta_x^2\left(\tau^{\gamma_j^k}\sum_{l=0}^{k}\lambda_{j,k}^l\right).
\end{aligned} \tag{2.13}$$

In terms of the above analysis, we present the following numerical scheme for solving (1.2) with the initial and boundary conditions (1.3)–(1.5):

$$\begin{aligned}
&\left(1+\frac{1}{12}\delta_x^2\right) u_j^k \\
&= \left(1+\frac{1}{12}\delta_x^2\right) u_j^{k-1} + \mu_j^k \sum_{l=0}^{k}\lambda_{j,k}^l \delta_x^2 u_j^{k-l} + \tau\left(1+\frac{1}{12}\delta_x^2\right) f_j^k, \\
&\qquad k=1,2,\ldots,K; \qquad j=1,2,\ldots,J-1,
\end{aligned} \tag{2.14}$$

$$u_j^0 = \phi(x_j), \qquad j = 0, 1, \ldots, J, \tag{2.15}$$

$$u_0^k = \varphi(t_k), \qquad k = 1, 2, \ldots, K, \tag{2.16}$$

$$u_J^k = \psi(t_k), \qquad k = 1, 2, \ldots, K. \tag{2.17}$$

3. Some lemmas.

In this section, we will establish two lemmas.

LEMMA 3.1. *If $0 < \gamma_{min} \le \gamma(x,t) \le \gamma_{max} < 1$, for $j = 1, 2, \ldots, J$, $k = 1, 2, \ldots, K$, $l = 0, 1, \ldots$, the coefficients $\lambda_{j,k}^l$ satisfy the following:*

(1) $\lambda_{j,k}^0 = 1$; $\lambda_{j,k}^1 = \gamma_j^k - 1 < 0$; $\lambda_{j,k}^l < 0$, $l = 2, 3, \ldots$;

(2) $\sum_{l=0}^{\infty} \lambda_{j,k}^l = 0$;

(3) *for* $n = 1, 2, \ldots, -\sum_{l=1}^{n} \lambda_{j,k}^l < 1$.

Proof. First, direct computations validate that

$$\lambda_{j,k}^0 = (-1)^0 \binom{1-\gamma_j^k}{0} = 1,$$

$$\lambda_{j,k}^1 = (-1)^1 \binom{1-\gamma_j^k}{1} = \gamma_j^k - 1 < 0.$$

We notice that $0 < \gamma_{min} \le \gamma(x,t) \le \gamma_{max} < 1$. Hence, for $j = 1, 2, \ldots, J$, $k = 1, 2, \ldots, K$,

$$0 < \gamma_j^k < 1 \tag{3.1}$$

holds; so, for $l = 2, 3, \ldots$, we have

$$\begin{aligned} \lambda_{j,k}^l &= (-1)^l \binom{1-\gamma_j^k}{l} \\ &= (-1)^l \frac{(1-\gamma_j^k)(-\gamma_j^k)(-\gamma_j^k - 1)\ldots(1-\gamma_j^k - l + 1)}{l!} \\ &= -\frac{(1-\gamma_j^k)\gamma_j^k(\gamma_j^k + 1)\ldots(\gamma_j^k + l - 2)}{l!} < 0. \end{aligned}$$

Second, taking $t = 1$ in the formula

$$\sum_{l=0}^{\infty} \lambda_{j,k}^l t^l = (1-t)^{1-\gamma_j^k},$$

we get

$$\sum_{l=0}^{\infty} \lambda_{j,k}^l = 0.$$

Finally, from the conclusions (1) and (2), for $n = 1, 2, \ldots$, it holds that

$$-\sum_{l=1}^{n} \lambda_{j,k}^l = \lambda_{j,k}^0 + \sum_{l=n+1}^{\infty} \lambda_{j,k}^l = 1 + \sum_{l=n+1}^{\infty} \lambda_{j,k}^l < 1.$$

This completes the proof of Lemma 3.1. ∎

LEMMA 3.2. *For $j = 1, 2, \ldots, J,\ k = 1, 2, \ldots, K,$ it holds that*

$$\tau^{\gamma_j^k - 1} \sum_{l=0}^{k} \lambda_{j,k}^l = \frac{1}{\Gamma(\gamma_j^k)} + O(\tau).$$

Proof. Taking $g(x,t) = 1$ and $t_k = 1$ in (2.4) gives

$$\left[{}_0D_t^{1-\gamma_j^k}(1) \right]_{t=1} = \tau^{\gamma_j^k - 1} \sum_{l=0}^{k} \lambda_{j,k}^l + O(\tau),$$

whereas

$$\left[{}_0D_t^{1-\gamma_j^k}(1) \right]_{t=1} = \left[\frac{t^{\gamma_j^k - 1}}{\Gamma(\gamma_j^k)} \right]_{t=1} = \frac{1}{\Gamma(\gamma_j^k)}.$$

Hence

$$\tau^{\gamma_j^k - 1} \sum_{l=0}^{k} \lambda_{j,k}^l = \frac{1}{\Gamma(\gamma_j^k)} + O(\tau).$$

This completes the proof of Lemma 3.2. ∎

4. Stability of the numerical scheme. In this section, we will analyze the stability of the numerical scheme (2.14)–(2.17). We consider the following difference equation:

$$\left(1 + \frac{1}{12}\delta_x^2\right) u_j^k = \left(1 + \frac{1}{12}\delta_x^2\right) u_j^{k-1} + \mu_j^k \sum_{l=0}^{k} \lambda_{j,k}^l \delta_x^2 u_j^{k-l}, \tag{4.1}$$

$$k = 1, 2, \ldots, K; \qquad j = 1, 2, \ldots, J-1.$$

For $k = 0, 1, \ldots, K$, we define the following grid function:

$$u^k(x) = \begin{cases} u_j^k & \text{when} \quad x \in \left(x_{j-\frac{1}{2}},\ x_{j+\frac{1}{2}}\right], \quad j = 1, 2, \ldots, J-1, \\ 0 & \text{when} \quad x \in [0, \frac{h}{2}] \cup (L - \frac{h}{2}, L]. \end{cases}$$

Then $u^k(x)$ has the Fourier series expansion

$$u^k(x) = \sum_{l=-\infty}^{\infty} \xi_k(l) e^{i2\pi l x/L}, \qquad k = 0, 1, \ldots, K,$$

where

$$\xi_k(l) = \frac{1}{L} \int_0^L u^k(x) e^{-i2\pi l x/L} dx.$$

Let

$$u^k = \left[u_1^k, u_2^k, \ldots, u_{J-1}^k\right]^T.$$

Then, using the Parseval identities

$$\int_0^L \left|u^k(x)\right|^2 dx = \sum_{l=-\infty}^{\infty} |\xi_k(l)|^2, \qquad k = 0, 1, \ldots, K,$$

and

$$\int_0^L \left|u^k(x)\right|^2 dx = \sum_{j=1}^{J-1} h|u_j^k|^2, \qquad k = 0, 1, \ldots, K,$$

we get

$$\|u^k\|_2 \equiv \left(\sum_{j=1}^{J-1} h|u_j^k|^2\right)^{\frac{1}{2}} = \left(\sum_{l=-\infty}^{\infty} |\xi_k(l)|^2\right)^{\frac{1}{2}}, \qquad k = 0, 1, \ldots, K. \tag{4.2}$$

Assume that the solution of the difference equation (4.1) has the form

$$u_j^k = \xi_k e^{i\sigma jh},$$

where $\sigma = 2\pi l/L$. Substituting the above expression into (4.1) gives

$$\begin{aligned} &\left(1 - \frac{1}{3}\sin^2\frac{\sigma h}{2}\right)\xi_k \\ &= \left(1 - \frac{1}{3}\sin^2\frac{\sigma h}{2}\right)\xi_{k-1} - 4\mu_j^k \sin^2\frac{\sigma h}{2}\sum_{l=0}^{k}\lambda_{j,k}^l \xi_{k-l}, \qquad k = 1, 2, \ldots, K. \end{aligned} \tag{4.3}$$

In view of Lemma 3.1, we rewrite (4.3) as

$$\begin{aligned} \xi_k = {} & \frac{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4(1-\gamma_j^k)\mu_j^k \sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k\sin^2\frac{\sigma h}{2}}\xi_{k-1} \\ & - \frac{4\mu_j^k\sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k\sin^2\frac{\sigma h}{2}}\sum_{l=2}^{k}\lambda_{j,k}^l\xi_{k-l}, \\ & k = 1, 2, \ldots, K. \end{aligned} \tag{4.4}$$

PROPOSITION 4.1. *Letting ξ_k $(k = 1, 2, \ldots, K)$ be the solution of* (4.4)*, then*

$$|\xi_k| \le |\xi_0|, \qquad k = 1, 2, \ldots, K.$$

Proof. For $k = 1$, from (4.4) we get

$$\xi_1 = \frac{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4(1-\gamma_j^1)\mu_j^1 \sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^1\sin^2\frac{\sigma h}{2}}\xi_0.$$

In view of (3.1),

$$\begin{aligned} |\xi_1| &= \left|\frac{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4(1-\gamma_j^1)\mu_j^1 \sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^1\sin^2\frac{\sigma h}{2}}\right| |\xi_0| \\ &= \frac{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4(1-\gamma_j^1)\mu_j^1 \sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^1\sin^2\frac{\sigma h}{2}}|\xi_0| \\ &\le |\xi_0|. \end{aligned}$$

Suppose that

$$|\xi_n| \le |\xi_0|, \qquad n = 1, 2, \ldots, k-1.$$

By (3.1), (4.4), and Lemma 3.1, we get

$$\begin{aligned}
|\xi_k| \le & \left| \frac{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4(1-\gamma_j^k)\mu_j^k \sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k \sin^2\frac{\sigma h}{2}} \right| |\xi_{k-1}| \\
& + \left| \frac{4\mu_j^k \sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k \sin^2\frac{\sigma h}{2}} \right| \sum_{l=2}^{k} |\lambda_{j,k}^l| |\xi_{k-l}| \\
= & \frac{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4(1-\gamma_j^k)\mu_j^k \sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k \sin^2\frac{\sigma h}{2}} |\xi_{k-1}| \\
& + \frac{4\mu_j^k \sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k \sin^2\frac{\sigma h}{2}} \sum_{l=2}^{k} |\lambda_{j,k}^l| |\xi_{k-l}| \\
\le & \left\{ \frac{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4(1-\gamma_j^k)\mu_j^k \sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k \sin^2\frac{\sigma h}{2}} \right. \\
& \left. + \frac{4\mu_j^k \sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k \sin^2\frac{\sigma h}{2}} \left(\sum_{l=1}^{k} |\lambda_{j,k}^l| - |\lambda_{j,k}^1| \right) \right\} |\xi_0| \\
= & \left\{ \frac{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4(1-\gamma_j^k)\mu_j^k \sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k \sin^2\frac{\sigma h}{2}} \right. \\
& \left. + \frac{4\mu_j^k \sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k \sin^2\frac{\sigma h}{2}} \left[-\sum_{l=1}^{k} \lambda_{j,k}^l - (1-\gamma_j^k) \right] \right\} |\xi_0| \\
\le & \left\{ \frac{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4(1-\gamma_j^k)\mu_j^k \sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k \sin^2\frac{\sigma h}{2}} \right. \\
& \left. + \frac{4\mu_j^k \sin^2\frac{\sigma h}{2}}{1 - \frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k \sin^2\frac{\sigma h}{2}} \left[1 - (1-\gamma_j^k) \right] \right\} |\xi_0| \\
= & |\xi_0|.
\end{aligned}$$

The proof of Proposition 4.1 is completed via mathematical induction. ∎

According to (4.2) and Proposition 4.1, it can be obtained that the solution of the difference equation (4.1) satisfies

$$\|u^k\|_2 \le \|u^0\|_2, \qquad k = 1, 2, \ldots, K.$$

Hence, we have the following result.

Theorem 4.2. *The numerical scheme* (2.14)–(2.17) *is unconditionally stable.*

5. Convergence of the numerical scheme.

We now carry out the convergence analysis of the numerical scheme (2.14)–(2.17). Subtracting (2.14) from (2.12), we obtain the following error equation:

$$\left(1 + \frac{1}{12}\delta_x^2\right) E_j^k = \left(1 + \frac{1}{12}\delta_x^2\right) E_j^{k-1} + \mu_j^k \sum_{l=0}^{k} \lambda_{j,k}^l \delta_x^2 E_j^{k-l} + R_j^k, \tag{5.1}$$

$$k = 1, 2, \ldots, K; \qquad j = 1, 2, \ldots, J-1,$$

where $E_j^k = u(x_j, t_k) - u_j^k$. For $k = 0, 1, \ldots, K$, we define the following grid functions, respectively:

$$E^k(x) = \begin{cases} E_j^k & \text{when} \quad x \in \left(x_{j-\frac{1}{2}},\ x_{j+\frac{1}{2}}\right], \quad j = 1, 2, \ldots, J-1, \\ 0 & \text{when} \quad x \in [0, \frac{h}{2}] \cup (L - \frac{h}{2}, L] \end{cases}$$

and

$$R^k(x) = \begin{cases} R_j^k & \text{when} \quad x \in \left(x_{j-\frac{1}{2}},\ x_{j+\frac{1}{2}}\right], \quad j = 1, 2, \ldots, J-1, \\ 0 & \text{when} \quad x \in [0, \frac{h}{2}] \cup (L - \frac{h}{2}, L]. \end{cases}$$

Then, $E^k(x)$ and $R^k(x)$ have the Fourier series expansions

$$E^k(x) = \sum_{l=-\infty}^{\infty} \alpha_k(l) e^{i2\pi l x/L}, \qquad k = 0, 1, \ldots, K,$$

and

$$R^k(x) = \sum_{l=-\infty}^{\infty} \beta_k(l) e^{i2\pi l x/L}, \qquad k = 0, 1, \ldots, K,$$

where

$$\alpha_k(l) = \frac{1}{L} \int_0^L E^k(x) e^{-i2\pi l x/L} dx, \qquad \beta_k(l) = \frac{1}{L} \int_0^L R^k(x) e^{-i2\pi l x/L} dx.$$

Letting

$$E^k = \left[E_1^k, E_2^k, \ldots, E_{J-1}^k\right]^T, \qquad R^k = \left[R_1^k, R_2^k, \ldots, R_{J-1}^k\right]^T$$

and applying the Parseval identities

$$\int_0^L \left|E^k(x)\right|^2 dx = \sum_{l=-\infty}^{\infty} |\alpha_k(l)|^2, \qquad k = 0, 1, \ldots, K,$$

$$\int_0^L \left|R^k(x)\right|^2 dx = \sum_{l=-\infty}^{\infty} |\beta_k(l)|^2, \qquad k = 0, 1, \ldots, K,$$

and

$$\int_0^L \left|E^k(x)\right|^2 dx = \sum_{j=1}^{J-1} h|E_j^k|^2, \qquad k = 0, 1, \ldots, K,$$

$$\int_0^L \left|R^k(x)\right|^2 dx = \sum_{j=1}^{J-1} h|R_j^k|^2, \qquad k = 0, 1, \ldots, K,$$

we have, respectively,

$$(5.2) \qquad \|E^k\|_2 \equiv \left(\sum_{j=1}^{J-1} h|E_j^k|^2\right)^{\frac{1}{2}} = \left(\sum_{l=-\infty}^{\infty} |\alpha_k(l)|^2\right)^{\frac{1}{2}}, \qquad k = 0, 1, \ldots, K,$$

and

$$\|R^k\|_2 \equiv \left(\sum_{j=1}^{J-1} h|R_j^k|^2\right)^{\frac{1}{2}} = \left(\sum_{l=-\infty}^{\infty} |\beta_k(l)|^2\right)^{\frac{1}{2}}, \qquad k=0,1,\ldots,K. \tag{5.3}$$

We now assume that E_j^k and R_j^k have the following form:

$$E_j^k = \alpha_k e^{i\sigma jh}, \qquad R_j^k = \beta_k e^{i\sigma jh},$$

where $\sigma = 2\pi l/L$. By substituting the above expressions into (5.1), we get

$$\begin{aligned} &\left(1-\frac{1}{3}\sin^2\frac{\sigma h}{2}\right)\alpha_k \\ &= \left(1-\frac{1}{3}\sin^2\frac{\sigma h}{2}\right)\alpha_{k-1} - 4\mu_j^k \sin^2\frac{\sigma h}{2}\sum_{l=0}^{k}\lambda_{j,k}^l \alpha_{k-l} + \beta_k, \qquad k=1,2,\ldots,K. \end{aligned} \tag{5.4}$$

Using Lemma 3.1, we rewrite (5.4) as

$$\begin{aligned} \alpha_k = {} & \frac{1-\frac{1}{3}\sin^2\frac{\sigma h}{2} + 4(1-\gamma_j^k)\mu_j^k \sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k\sin^2\frac{\sigma h}{2}}\alpha_{k-1} \\ & - \frac{4\mu_j^k \sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k\sin^2\frac{\sigma h}{2}}\sum_{l=2}^{k}\lambda_{j,k}^l\alpha_{k-l} \\ & + \frac{\beta_k}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2} + 4\mu_j^k\sin^2\frac{\sigma h}{2}}, \qquad k=1,2,\ldots,K. \end{aligned} \tag{5.5}$$

From (2.13) and Lemma 3.2, we have

$$\begin{aligned} R_j^k &= O(h^4)\tau^{\gamma_j^k}\sum_{l=0}^{k}\lambda_{j,k}^l + O(\tau^2) + O(h^4)\delta_x^2\left(\tau^{\gamma_j^k}\sum_{l=0}^{k}\lambda_{j,k}^l\right) \\ &= O(h^4)\tau\tau^{\gamma_j^k-1}\sum_{l=0}^{k}\lambda_{j,k}^l + O(\tau^2) + O(h^4)\delta_x^2\left(\tau\tau^{\gamma_j^k-1}\sum_{l=0}^{k}\lambda_{j,k}^l\right) \\ &= O(h^4)\tau\left(\frac{1}{\Gamma(\gamma_j^k)} + O(\tau)\right) \\ &\quad + O(\tau^2) + O(h^4)\delta_x^2\left[\tau\left(\frac{1}{\Gamma(\gamma_j^k)} + O(\tau)\right)\right] \\ &= O(h^4)\tau\left(\frac{1}{\Gamma(\gamma_j^k)} + O(\tau)\right) + O(\tau^2) \\ &\quad + O(h^4)\tau\delta_x^2\left(\frac{1}{\Gamma(\gamma_j^k)} + O(\tau)\right) \\ &= O\left(\tau^2 + \tau h^4\right). \end{aligned} \tag{5.6}$$

Then, there is a positive constant C_1 such that

$$|R_j^k| \le C_1(\tau^2 + \tau h^4), \quad k=1,2,\ldots,K, \quad j=1,2,\ldots,J. \tag{5.7}$$

Further, by the first equality of (5.3), we get

$$\|R^k\|_2 \le C_1\sqrt{L}(\tau^2+\tau h^4), \qquad k=1,2,\ldots,K. \tag{5.8}$$

In terms of the convergence of the series on the right-hand side of (5.3), there is a positive constant C_2 such that

$$|\beta_k| \equiv |\beta_k(l)| \le C_2|\beta_1(l)| \equiv C_2|\beta_1|, \qquad k=1,2,\ldots,K. \tag{5.9}$$

PROPOSITION 5.1. *Letting α_k $(k=1,2,\ldots,K)$ be the solution of* (5.5)*, then*

$$|\alpha_k| \le \frac{3}{2}C_2 k|\beta_1|, \qquad k=1,2,\ldots,K.$$

Proof. By $E^0=0$ and (5.2), we get

$$\alpha_0(l)=\alpha_0=0. \tag{5.10}$$

When $k=1$, from (5.5), (5.9), and (5.10), we arrive at

$$\begin{aligned}
|\alpha_1| &= \left|\frac{\beta_1}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^1\sin^2\frac{\sigma h}{2}}\right| \\
&= \frac{|\beta_1|}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^1\sin^2\frac{\sigma h}{2}} \\
&\le \frac{3}{2}|\beta_1| \le \frac{3}{2}C_2|\beta_1|.
\end{aligned}$$

Assume that

$$|\alpha_n| \le \frac{3}{2}C_2 n|\beta_1|, \qquad n=1,2,\ldots,k-1.$$

According to (3.1), (5.5), (5.9), and Lemma 3.1, we have

$$\begin{aligned}
|\alpha_k| \le{}& \left|\frac{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4(1-\gamma_j^k)\mu_j^k\sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\right||\alpha_{k-1}| \\
&+\left|\frac{4\mu_j^k\sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\right|\sum_{l=2}^{k}\left|\lambda_{j,k}^l\right||\alpha_{k-l}| \\
&+\left|\frac{\beta_k}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\right| \\
={}& \frac{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4(1-\gamma_j^k)\mu_j^k\sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}|\alpha_{k-1}| \\
&+\frac{4\mu_j^k\sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\sum_{l=2}^{k}\left|\lambda_{j,k}^l\right||\alpha_{k-l}| \\
&+\frac{|\beta_k|}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}} \\
\le{}& \left\{\frac{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4(1-\gamma_j^k)\mu_j^k\sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\frac{3}{2}(k-1)\right.
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{4\mu_j^k \sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\frac{3}{2}(k-l)\sum_{l=2}^{k}\left|\lambda_{j,k}^l\right| \\
&+ \left.\frac{1}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\right\} C_2|\beta_1| \\
\le &\left\{\frac{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4(1-\gamma_j^k)\mu_j^k\sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\frac{3}{2}(k-1)\right. \\
&+ \frac{4\mu_j^k \sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\frac{3}{2}(k-1)\sum_{l=2}^{k}\left|\lambda_{j,k}^l\right| \\
&+ \left.\frac{1}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\right\} C_2|\beta_1| \\
= &\left\{\frac{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4(1-\gamma_j^k)\mu_j^k\sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+\left(4\mu_j^k-\frac{1}{3}\right)\sin^2\frac{\sigma h}{2}}\frac{3}{2}(k-1)\right. \\
&+ \frac{4\mu_j^k \sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\frac{3}{2}(k-1)\left(\sum_{l=1}^{k}\left|\lambda_{j,k}^l\right|-\left|\lambda_{j,k}^1\right|\right) \\
&+ \left.\frac{1}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\right\} C_2|\beta_1| \\
= &\left\{\frac{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4(1-\gamma_j^k)\mu_j^k\sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\frac{3}{2}(k-1)\right. \\
&+ \frac{4\mu_j^k \sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\frac{3}{2}(k-1)\left[-\sum_{l=1}^{k}\lambda_{j,k}^l-\left(1-\gamma_j^k\right)\right] \\
&+ \left.\frac{1}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\right\} C_2|\beta_1| \\
\le &\left\{\frac{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4(1-\gamma_j^k)\mu_j^k\sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\frac{3}{2}(k-1)\right. \\
&+ \frac{4\mu_j^k \sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\frac{3}{2}(k-1)\left[1-\left(1-\gamma_j^k\right)\right] \\
&+ \left.\frac{1}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\right\} C_2|\beta_1| \\
\le &\left\{\frac{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4(1-\gamma_j^k)\mu_j^k\sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\frac{3}{2}(k-1)\right. \\
&+ \left.\frac{4\mu_j^k \sin^2\frac{\sigma h}{2}}{1-\frac{1}{3}\sin^2\frac{\sigma h}{2}+4\mu_j^k\sin^2\frac{\sigma h}{2}}\frac{3}{2}(k-1)\gamma_j^k+\frac{3}{2}\right\} C_2|\beta_1| \\
= &\frac{3}{2}C_2k|\beta_1|.
\end{aligned}
$$

The proof of Proposition 5.1 is completed via mathematical induction. ∎

Using (5.2), (5.3), (5.8), Proposition 5.1, and $k\tau \le T$, we obtain

$$\|E^k\|_2 \le \frac{3}{2}C_2 k\|R^1\|_2 \le \frac{3}{2}C_1C_2 k\tau\sqrt{L}(\tau+h^4) \le C(\tau+h^4),$$

where $C = \frac{3}{2}C_1C_2T\sqrt{L}$. So, we have the following result.

THEOREM 5.2. *Suppose that $u(x,t) \in \mathscr{U}(\Omega)$; then the numerical scheme* (2.14)–(2.17) *is convergent with order $O(\tau+h^4)$.*

6. Solvability of the numerical scheme. It can be seen that the corresponding homogeneous linear algebraic equations for the numerical scheme (2.14)–(2.17) are

$$\left(1+\frac{1}{12}\delta_x^2\right)u_j^k = \left(1+\frac{1}{12}\delta_x^2\right)u_j^{k-1} + \mu_j^k\sum_{l=0}^{k}\lambda_{j,k}^l\delta_x^2 u_j^{k-l}, \tag{6.1}$$

$$k=1,2,\dots,K; \qquad j=1,2,\dots,J-1,$$

$$u_j^0 = 0, \qquad j=0,1,\dots,J, \tag{6.2}$$

$$u_0^k = u_J^k = 0, \qquad k=0,1,\dots,K. \tag{6.3}$$

Similar to the proof of Theorem 4.2, we can also verify that the solution of (6.1)–(6.3) satisfies

$$\|u^k\|_2 \le \|u^0\|_2, \qquad k=1,2,\dots,K.$$

Since $u^0 = 0$, therefore

$$u^k = 0, \qquad k=1,2,\dots,K,$$

which indicates that (6.1)–(6.3) have only zero solutions. So, we obtain the following theorem.

THEOREM 6.1. *The numerical scheme* (2.14)–(2.17) *is uniquely solvable.*

7. An improved numerical scheme. We first present the second order approximation formula for the first order derivative. If $g(t)$ is sufficiently smooth, then

$$g(t_k) = g(t_{k-1}) + \tau g'(t_{k-1}) + \frac{\tau^2}{2}g''(t_{k-1}) + O(\tau^3),$$

i.e.,

$$\nabla_t g(t_k) = \tau g'(t_{k-1}) + \frac{\tau^2}{2}g''(t_{k-1}) + O(\tau^3),$$

or

$$\frac{\nabla_t g(t_k)}{\tau} = g'(t_{k-1}) + \frac{\tau}{2}g''(t_{k-1}) + O(\tau^2),$$

where $\nabla_t g(t_k) = g(t_k) - g(t_{k-1})$ is the first order backward difference. By replacing $g(t)$ with $g'(t)$, we get

$$\nabla_t g'(t_k) = \tau g''(t_{k-1}) + O(\tau^2).$$

Multiplying by $(-\frac{1}{2})$ and adding $g'(t_k)$ to both sides of the above equation yields

$$
\begin{aligned}
(7.1)\qquad & \left(1-\frac{1}{2}\nabla_t\right) g'(t_k) \\
&= g'(t_k) - \frac{\tau}{2} g''(t_{k-1}) + O(\tau^2) \\
&= \left(g'(t_{k-1}) + \tau g''(t_{k-1}) + O(\tau^2)\right) - \frac{\tau}{2} g''(t_{k-1}) + O(\tau^2) \\
&= g'(t_{k-1}) + \frac{\tau}{2} g''(t_{k-1}) + O(\tau^2).
\end{aligned}
$$

It then follows that

$$
\frac{\nabla_t g(t_k)}{\tau} = \left(1-\frac{1}{2}\nabla_t\right) g'(t_k) + O(\tau^2).
$$

From this, we obtain the following second order approximation formula for the first order derivative:

$$
g'(t_k) = \frac{\nabla_t}{\tau\left(1-\frac{1}{2}\nabla_t\right)} g(t_k) + O(\tau^2).
$$

In fact, we can also obtain the following second order approximation formula for the first order derivative:

$$
g'(t_{k-1}) = \frac{\Delta_t}{\tau\left(1+\frac{1}{2}\triangle_t\right)} g(t_{k-1}) + O(\tau^2),
$$

where $\triangle_t g(t_{k-1}) = g(t_k) - g(t_{k-1})$ is the first order forward difference.

In this section, we suppose that the functions $u(x,t)$ and $f(x,t)$ are sufficiently smooth. By taking $x = x_j,\ t = t_k$ in (1.2), we get

$$
\begin{aligned}
\frac{\partial u(x_j,t_k)}{\partial t} &= \frac{1}{\Gamma(\gamma(x_j,t_k))}\left[\frac{\partial}{\partial \xi}\int_0^{\xi} \frac{\partial^2 u(x_j,\eta)}{\partial x^2}\frac{d\eta}{(\xi-\eta)^{1-\gamma(x_j,t_k)}}\right]_{\xi=t_k} + f(x_j,t_k) \\
&= \left\{\frac{\partial}{\partial \xi}\left[\frac{1}{\Gamma(\gamma(x_j,t_k))}\int_0^{\xi} \frac{\partial^2 u(x_j,\eta)}{\partial x^2}\frac{d\eta}{(\xi-\eta)^{1-\gamma(x_j,t_k)}}\right]\right\}_{\xi=t_k} + f(x_j,t_k).
\end{aligned}
$$

It then follows that

$$
\begin{aligned}
(7.2)\qquad & \frac{\nabla_t}{\tau\left(1-\frac{1}{2}\nabla_t\right)} u(x_j,t_k) + O(\tau^2) \\
&= \frac{\nabla_t}{\tau\left(1-\frac{1}{2}\nabla_t\right)}\left[\frac{1}{\Gamma(\gamma(x_j,t_k))}\int_0^{t_k} \frac{\partial^2 u(x_j,\eta)}{\partial x^2}\frac{d\eta}{(t_k-\eta)^{1-\gamma(x_j,t_k)}} + O(\tau^2)\right] \\
&\quad + f(x_j,t_k),
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
(7.3)\qquad & \nabla_t u(x_j,t_k) \\
&= \nabla_t \left[\frac{1}{\Gamma(\gamma(x_j,t_k))}\int_0^{t_k} \frac{\partial^2 u(x_j,\eta)}{\partial x^2}\frac{d\eta}{(t_k-\eta)^{1-\gamma(x_j,t_k)}}\right] \\
&\quad + \tau\left(1-\frac{1}{2}\nabla_t\right) f(x_j,t_k) + O(\tau^2),
\end{aligned}
$$

or

$$
\begin{aligned}
(7.4)\quad & u(x_j,t_k)-u(x_j,t_{k-1})\\
&= \frac{1}{\Gamma(\gamma(x_j,t_k))}\int_0^{t_k}\frac{\partial^2 u(x_j,\eta)}{\partial x^2}\frac{d\eta}{(t_k-\eta)^{1-\gamma(x_j,t_k)}}\\
&\quad - \frac{1}{\Gamma(\gamma(x_j,t_{k-1}))}\int_0^{t_{k-1}}\frac{\partial^2 u(x_j,\eta)}{\partial x^2}\frac{d\eta}{(t_{k-1}-\eta)^{1-\gamma(x_j,t_{k-1})}}\\
&\quad + \tau\left(1-\frac{1}{2}\nabla_t\right)f(x_j,t_k)+O(\tau^2)\\
&= \frac{1}{\Gamma(\gamma(x_j,t_k))}\sum_{l=1}^{k}\int_{t_{l-1}}^{t_l}\frac{\partial^2 u(x_j,\eta)}{\partial x^2}\frac{d\eta}{(t_k-\eta)^{1-\gamma(x_j,t_k)}}\\
&\quad - \frac{1}{\Gamma(\gamma(x_j,t_{k-1}))}\sum_{l=1}^{k-1}\int_{t_{l-1}}^{t_l}\frac{\partial^2 u(x_j,\eta)}{\partial x^2}\frac{d\eta}{(t_{k-1}-\eta)^{1-\gamma(x_j,t_{k-1})}}\\
&\quad + \tau\left(1-\frac{1}{2}\nabla_t\right)f(x_j,t_k)+O(\tau^2)\\
&= \frac{1}{\Gamma(\gamma(x_j,t_k))}\sum_{l=1}^{k}\int_{t_{l-1}}^{t_l}\Bigg(\frac{t_l-\eta}{\tau}\frac{\partial^2 u(x_j,t_{l-1})}{\partial x^2}+\frac{\eta-t_{l-1}}{\tau}\frac{\partial^2 u(x_j,t_l)}{\partial x^2}\\
&\quad + O(\tau^2)\Bigg)\frac{d\eta}{(t_k-\eta)^{1-\gamma(x_j,t_k)}} - \frac{1}{\Gamma(\gamma(x_j,t_{k-1}))}\sum_{l=1}^{k-1}\int_{t_{l-1}}^{t_l}\Bigg(\frac{t_l-\eta}{\tau}\frac{\partial^2 u(x_j,t_{l-1})}{\partial x^2}\\
&\quad + \frac{\eta-t_{l-1}}{\tau}\frac{\partial^2 u(x_j,t_l)}{\partial x^2}+O(\tau^2)\Bigg)\frac{d\eta}{(t_{k-1}-\eta)^{1-\gamma(x_j,t_{k-1})}}\\
&\quad + \tau\left(1-\frac{1}{2}\nabla_t\right)f(x_j,t_k)+O(\tau^2)\\
&= \frac{1}{\Gamma(\gamma(x_j,t_k))}\sum_{l=1}^{k}\int_{t_{l-1}}^{t_l}\Bigg\{\frac{t_l-\eta}{\tau}\left[\frac{\delta_x^2}{h^2\left(1+\frac{1}{12}\delta_x^2\right)}u(x_j,t_{l-1})+O(h^4)\right]\\
&\quad + \frac{\eta-t_{l-1}}{\tau}\left[\frac{\delta_x^2}{h^2\left(1+\frac{1}{12}\delta_x^2\right)}u(x_j,t_l)+O(h^4)\right]+O(\tau^2)\Bigg\}\frac{d\eta}{(t_k-\eta)^{1-\gamma(x_j,t_k)}}\\
&\quad - \frac{1}{\Gamma(\gamma(x_j,t_{k-1}))}\sum_{l=1}^{k-1}\int_{t_{l-1}}^{t_l}\Bigg\{\frac{t_l-\eta}{\tau}\left[\frac{\delta_x^2}{h^2\left(1+\frac{1}{12}\delta_x^2\right)}u(x_j,t_{l-1})+O(h^4)\right]\\
&\quad + \frac{\eta-t_{l-1}}{\tau}\left[\frac{\delta_x^2}{h^2\left(1+\frac{1}{12}\delta_x^2\right)}u(x_j,t_l)+O(h^4)\right]+O(\tau^2)\Bigg\}\frac{d\eta}{(t_{k-1}-\eta)^{1-\gamma(x_j,t_{k-1})}}\\
&\quad + \tau\left(1-\frac{1}{2}\nabla_t\right)f(x_j,t_k)+O(\tau^2)\\
&= \frac{1}{h^2\left(1+\frac{1}{12}\delta_x^2\right)}\Bigg[\frac{\tau^{\gamma_j^k}}{\Gamma_j^k}\sum_{l=1}^{k}\left(a_{j,k}^l\delta_x^2 u(x_j,t_{l-1})+b_{j,k}^l\delta_x^2 u(x_j,t_l)\right)\\
&\quad - \frac{\tau^{\gamma_j^{k-1}}}{\Gamma_j^{k-1}}\sum_{l=1}^{k-1}\left(a_{j,k-1}^l\delta_x^2 u(x_j,t_{l-1})+b_{j,k-1}^l\delta_x^2 u(x_j,t_l)\right)\Bigg]\\
&\quad + \tau\left(1-\frac{1}{2}\nabla_t\right)f(x_j,t_k)+\widetilde{\Re}_j^k,
\end{aligned}
$$

where

$$\gamma_j^k \equiv \gamma(x_j, t_k), \qquad \Gamma_j^k \equiv \Gamma(\gamma(x_j, t_k) + 1),$$

$$a_{j,k}^l = (k-l+1)^{\gamma_j^k} + \frac{1}{\gamma_j^k + 1}\left[(k-l)^{\gamma_j^k+1} - (k-l+1)^{\gamma_j^k+1}\right],$$

$$b_{j,k}^l = -(k-l)^{\gamma_j^k} - \frac{1}{\gamma_j^k + 1}\left[(k-l)^{\gamma_j^k+1} - (k-l+1)^{\gamma_j^k+1}\right],$$

$$\begin{aligned}
\widetilde{\Re}_j^k &= \frac{1}{\Gamma(\gamma(x_j,t_k))}\sum_{l=1}^{k}\int_{t_{l-1}}^{t_l}\left(O(h^4)+O(\tau^2)\right)\frac{d\eta}{(t_k-\eta)^{1-\gamma(x_j,t_k)}} \\
&\quad - \frac{1}{\Gamma(\gamma(x_j,t_{k-1}))}\sum_{l=1}^{k-1}\int_{t_{l-1}}^{t_l}\left(O(h^4)+O(\tau^2)\right)\frac{d\eta}{(t_{k-1}-\eta)^{1-\gamma(x_j,t_{k-1})}} \\
&\quad + O(\tau^2) \\
&= \frac{1}{\Gamma(\gamma(x_j,t_k))}\left(O(h^4)+O(\tau^2)\right)\int_0^{t_k}\frac{d\eta}{(t_k-\eta)^{1-\gamma(x_j,t_k)}} \\
&\quad - \frac{1}{\Gamma(\gamma(x_j,t_{k-1}))}\left(O(h^4)+O(\tau^2)\right)\int_0^{t_{k-1}}\frac{d\eta}{(t_{k-1}-\eta)^{1-\gamma(x_j,t_{k-1})}} \\
&\quad + O(\tau^2) \\
&= \frac{1}{\Gamma(\gamma(x_j,t_k))}\left(O(h^4)+O(\tau^2)\right)\frac{(k\tau)^{\gamma(x_j,t_k)}}{\gamma(x_j,t_k)} \\
&\quad - \frac{1}{\Gamma(\gamma(x_j,t_{k-1}))}\left(O(h^4)+O(\tau^2)\right)\frac{((k-1)\tau)^{\gamma(x_j,t_{k-1})}}{\gamma(x_j,t_{k-1})} + O(\tau^2).
\end{aligned} \tag{7.5}$$

Since $k\tau \le T$, we get

$$\widetilde{\Re}_j^k = O\left(\tau^2 + h^4\right).$$

From (7.4), we obtain

$$\begin{aligned}
&\left(1+\frac{1}{12}\delta_x^2\right)u(x_j,t_k) - \left(1+\frac{1}{12}\delta_x^2\right)u(x_j,t_{k-1}) \\
&= \frac{\tau^{\gamma_j^k}}{h^2\Gamma_j^k}\sum_{l=1}^{k}\left(a_{j,k}^l\delta_x^2 u(x_j,t_{l-1}) + b_{j,k}^l\delta_x^2 u(x_j,t_l)\right) \\
&\quad - \frac{\tau^{\gamma_j^{k-1}}}{h^2\Gamma_j^{k-1}}\sum_{l=1}^{k-1}\left(a_{j,k-1}^l\delta_x^2 u(x_j,t_{l-1}) + b_{j,k-1}^l\delta_x^2 u(x_j,t_l)\right) \\
&\quad + \tau\left(1+\frac{1}{12}\delta_x^2\right)\left(1-\frac{1}{2}\nabla_t\right)f(x_j,t_k) + \Re_j^k,
\end{aligned} \tag{7.6}$$

where

$$\Re_j^k = \left(1+\frac{1}{12}\delta_x^2\right)\widetilde{\Re}_j^k = O\left(\tau^2+h^4\right).$$

According to the above analysis, we now present the following improved numerical scheme for solving (1.2) with the initial and boundary conditions (1.3)–(1.5):

$$\begin{aligned}(7.7)\quad &\left(1+\frac{1}{12}\delta_x^2\right)u_j^k\\ &=\left(1+\frac{1}{12}\delta_x^2\right)u_j^{k-1}+\frac{\tau^{\gamma_j^k}}{h^2\Gamma_j^k}\sum_{l=1}^{k}\left(a_{j,k}^l\delta_x^2u_j^{l-1}+b_{j,k}^l\delta_x^2u_j^l\right)\\ &\quad-\frac{\tau^{\gamma_j^{k-1}}}{h^2\Gamma_j^{k-1}}\sum_{l=1}^{k-1}\left(a_{j,k-1}^l\delta_x^2u_j^{l-1}+b_{j,k-1}^l\delta_x^2u_j^l\right)+\tau\left(1+\frac{1}{12}\delta_x^2\right)\left(1-\frac{1}{2}\nabla_t\right)f_j^k,\\ &\qquad k=1,2,\dots,K;\qquad j=1,2,\dots,J-1,\end{aligned}$$

$$u_j^0=\phi(x_j),\qquad j=0,1,\dots,J, \tag{7.8}$$

$$u_0^k=\varphi(t_k),\qquad k=1,2,\dots,K, \tag{7.9}$$

$$u_J^k=\psi(t_k),\qquad k=1,2,\dots,K, \tag{7.10}$$

where $f_j^k\equiv f(x_j,t_k)$.

8. Numerical example. In this section, we use the numerical scheme (2.14)-(2.17) and the improved numerical scheme (7.7)–(7.10) to solve the following equation:

$$\frac{\partial u(x,t)}{\partial t}={}_0D_t^{1-\gamma(x,t)}\left(\frac{\partial^2u(x,t)}{\partial x^2}\right)+2e^x\left(t-\frac{t^{1+\gamma(x,t)}}{\Gamma(2+\gamma(x,t))}\right),\qquad 0<t\le 1,\qquad 0<x<1, \tag{8.1}$$

with the initial and boundary conditions

$$u(x,0)=0,\qquad 0\le x\le 1, \tag{8.2}$$

$$u(0,t)=t^2,\quad u(1,t)=et^2,\quad 0\le t\le 1. \tag{8.3}$$

The exact solution of the problem (8.1)–(8.3) is

$$u(x,t)=e^xt^2.$$

We let

$$E_{max}=\max_{0\le k\le K}\left\{\|E^k\|_2\right\}.$$

Table 8.1 lists the maximum errors of the numerical solution for the problem (8.1)–(8.3) using the numerical scheme (2.14)–(2.17) for various $\gamma(x,t)$ and $\tau=h^4$. Table 8.2 lists the maximum errors of the numerical solution for the problem (8.1)–(8.3) using the improved numerical scheme (7.7)–(7.10) for various $\gamma(x,t)$ and $\tau^2=h^4$, where, on the finite domain $0\le x,\ t\le 1$, all $\gamma(x,t)$ satisfy $0<\gamma(x,t)<1$. A comparison of the schemes (2.14)–(2.17) and (7.7)–(7.10) is shown in Tables 8.1 and 8.2, respectively. It can be seen that the theoretical analysis results have been verified by the numerical results. The convergence order of the improved numerical scheme is improved significantly.

TABLE 8.1
The maximum error E_{max} of the numerical scheme (2.14)–(2.17).

$\gamma(x,t)$	$\tau = h^4 = \frac{1}{16}$	$\tau = h^4 = \frac{1}{81}$
$\frac{10-(xt)^4}{300}$	4.3927×10^{-4}	3.2906×10^{-4}
$\frac{15+(xt)^8}{400}$	4.9179×10^{-4}	3.7095×10^{-4}
$\frac{20-(xt)^2}{500}$	5.1910×10^{-4}	3.9264×10^{-4}
$\frac{30+(xt)^4}{750}$	5.2298×10^{-4}	3.9604×10^{-4}
$\frac{15+(\sin(xt))^8}{400}$	4.9179×10^{-4}	3.7095×10^{-4}
$\frac{\cos(xt)+\frac{xt}{2}}{28}$	5.2180×10^{-4}	3.8887×10^{-4}
$\frac{2^{\sqrt{xt}}-\sin(xt)}{50}$	3.1814×10^{-4}	2.3145×10^{-4}
$\frac{22-(xt)^2+(xt)^4}{550}$	5.1986×10^{-4}	3.9368×10^{-4}
$\frac{10-(\cos(xt))^4}{300}$	4.0926×10^{-4}	3.0519×10^{-4}
$\frac{e^{\sqrt{xt}}-1.5\sin(\sqrt{xt})}{50}$	2.7111×10^{-4}	1.9723×10^{-4}

TABLE 8.2
The maximum error E_{max} of the improved numerical scheme (7.7)–(7.10).

$\gamma(x,t)$	$\tau^2 = h^4 = \frac{1}{16}$	$\tau^2 = h^4 = \frac{1}{81}$
$\frac{10-(xt)^4}{300}$	1.9032×10^{-4}	2.9272×10^{-5}
$\frac{15+(xt)^8}{400}$	2.8569×10^{-4}	8.9489×10^{-5}
$\frac{20-(xt)^2}{500}$	2.3457×10^{-4}	2.8811×10^{-5}
$\frac{30+(xt)^4}{750}$	3.1275×10^{-4}	9.6960×10^{-5}
$\frac{15+(\sin(xt))^8}{400}$	2.7903×10^{-4}	7.6216×10^{-5}
$\frac{\cos(xt)+\frac{xt}{2}}{28}$	2.8331×10^{-4}	9.2886×10^{-5}
$\frac{2^{\sqrt{xt}}-\sin(xt)}{50}$	1.6004×10^{-4}	4.1236×10^{-5}
$\frac{22-(xt)^2+(xt)^4}{550}$	3.1368×10^{-4}	9.1263×10^{-5}
$\frac{10-(\cos(xt))^4}{300}$	3.3883×10^{-4}	1.7434×10^{-4}
$\frac{e^{\sqrt{xt}}-1.5\sin(\sqrt{xt})}{50}$	7.3110×10^{-5}	3.0528×10^{-5}

9. Conclusions. In this paper, a new numerical scheme with first order temporal accuracy and fourth order spatial accuracy for a variable-order anomalous subdiffusion equation has been proposed. Its convergence, stability, and solvability have been discussed via the technique of Fourier analysis. An improved numerical scheme has also been proposed. Some numerical examples have been given, and the results have demonstrated the effectiveness of theoretical analysis.

Acknowledgment. The authors wish to thank the referees for their constructive comments and suggestions.

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