EXISTENCE AND UNIQUENESS OF RENORMALIZED SOLUTIONS FOR A CLASS OF DEGENERATE PARABOLIC EQUATIONS

Zhang Liqin (张丽琴)  Zhao Junning (赵俊宁)
Department of Mathematics, Xiamen University of Technology, Xiamen 361005, China
Department of Mathematics, Xiamen University, Xiamen 361005, China

Abstract This article discusses the existence and uniqueness of renormalized solutions for a class of degenerate parabolic equations $b(u)_t - \text{div}(a(u, \nabla u)) = H(u)(f + \text{div}g)$. Key words Renormalized solutions, degenerate parabolic equations

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1 Introduction

This article is devoted to the study of the following nonlinear problem:

$$b(u)_t - \text{div}(a(u, \nabla u)) = H(u)(f + \text{div}g) \quad (x, t) \in Q = \Omega \times (0, T)$$

$$u(x, t) = 0 \quad (x, t) \in \partial \Omega \times (0, T)$$

$$u(x, 0) = u_0(x) \quad x \in \Omega,$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ and suppose that

$(H_1) \quad b(s) \in C^1(\mathbb{R})$ is a strictly increasing function satisfying the normalization condition $b(0) = 0$;

$(H_2) \quad a(r, \xi) : R \times R^N \rightarrow R^N$ is a continuous vector field which satisfies, for some $1 < p < \infty$, $\alpha > 0$, and $\forall \xi, \nu \in R^N$,

$$a(r, \xi) \xi \geq \alpha |\xi|^p, \quad a(r, 0) = 0$$

$$(a(r, \xi) - a(r, \nu))(\xi - \nu) \geq 0$$

$$|a(r, \xi)| \leq C(|r|)(1 + |\xi|^{p-1}), \quad C(r) : R^+ \rightarrow R^+ \text{ nondecreasing;}$$

$(H_3) \quad H' \in C_0(R), \quad \forall s \in R$;

$(H_4) \quad f \in L^1(Q), \quad g \in (L^p(Q))^N, \quad p' = \frac{p}{p-1}.$

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Under these assumptions, problems (1)–(3) does not admit, in general, a weak solution, since the fields $a(u, \nabla u)$ do not belong to $(L^1_{loc})^N$ and the meaning of the term $H(u)(f + \text{div}g)$ is not clear. To overcome this difficulty, we used the framework of renormalized solutions in this article. This notion was introduced by Lions and Di Perna [1] for the study of Boltzmann. Lions [2] applied this notion to evolution problems in fluid mechanics. As far as the parabolic case (1)–(3) is concerned and still in the framework of renormalized solutions, the existence and uniqueness was proved in [3–5] in the case where $H(u) = 1$, $g = 0$, or $b(s) = s$, $H(s) = 1$. The nonlinear elliptic problems with the term $H(u)\mu$ were considered by Murat and Porretta (see [6], [7]), motivated by control problems arising in chemical reactions.

In this article, we first give a suitable formulation of Eq. (1), which is similar to that in [5], then the existence of weak solution is proved. Under some assumptions the uniqueness of solution of (1)-(3) is discussed.

2 Definition of Renormalized Solution and Statement of the Results

As usual, for $k > 0$, we denote by $T_k$ the truncation function defined

$$T_k(v) = \begin{cases} k, & \text{if } v \geq k, \\ v, & \text{if } |v| \leq k, \\ -k, & \text{if } v \leq -k. \end{cases}$$

**Definition 2.1** Assume that (H$_1$)–(H$_4$) hold. A measurable function $u$ is called a renormalized solution of (1)–(3) if $u$ satisfies:

$$b(u) \in L^\infty(0, T; L^1(\Omega));$$

$$T_k(u) \in L^p(0, T; W^{1,p}_0(\Omega)), \forall k > 0;$$

$$\lim_{n \to \infty} \int_{Q_n} a(u, \nabla u) \nabla ud x dt = 0;$$

for all $h \in C^1_0(R)$, $\xi \in C^1_0(Q)$,

$$\int_Q \int_0^u \xi_t \int_0^r h(r) db(r) - \int_Q a(u, \nabla u) \nabla (h(u)\xi)dx dt$$

$$= - \int_Q H(u) f h(u) \xi dx dt + \int_Q (H'(u)h(u) + H(u)h'(u)) Du \cdot g\xi dx dt;$$

and moreover,

$$b(u)|_{t=0} = b(u_0).$$

**Remark 2.1** Note that each term in (7) is well defined. Indeed, the first member of (7) is well defined as

$$\left| \int_0^u h(r) db(r) \right| \leq |h|_\infty |b(u)|$$

and $b(u) \in L^\infty(0, T; L^1(\Omega))$. The second term and the second term on the right hand of (7) are well defined as

$$\int_Q a(u, \nabla u) \nabla (h(u)\xi) dx dt = \int_{Q \cap \{|u| \leq k\}} a(u, \nabla T_k(u)) \nabla (h(T_k(u))\xi) dx dt,$$
\[
\int_Q (H'(u)h(u) + H(u)h'(u))Dug\xi dx dt = \int_Q (H'(u)h(u) + H(u)h'(u))DT_k(u) \cdot g\xi dx dt,
\]
where \( k > 0 \), such that \( \text{supp} h \subset [-k, k] \).

**Theorem 1**  Let \( u_0 : \Omega \to R \) be measurable with \( v_0 = b(u_0) \in L^1(\Omega) \). Assume that \((H_1)\)–\((H_4)\) hold. Then, there exists a function \( v = b(u) \in L^\infty(0,T;L^1(\Omega)) \) with \( u \) a renormalized solution of (1)–(3).

**Theorem 2**  Assume that the hypotheses of Theorem 1 hold and that \( b(r) = r, \ H'(s) \geq 0, \ f \geq 0, \ g = 0. \) Assume that for every \( k > 0 \) and \( |s|, |s'| \leq k \), there exist \( E_k(x,t) \in L^p(Q), \ F_k \geq 0 \) such that
\[
|a(s, \xi) - a(s', \xi)| \leq |s - s'| |(E_k(x,t) + F_k|\xi|^p).
\]

Then, the renormalized solution of (1)–(3) is unique.

3 Proof of Theorem 1

For \( \varepsilon > 0 \), we consider the following approximations of \( f, g, u_0 \)
\[
f_\varepsilon, \ g_\varepsilon \in C^1(\overline{\Omega}), \ u_{0\varepsilon}(x) \in C^1(\overline{\Omega})
\]
\[
f_\varepsilon \to f \ \text{strongly in} \ L^1(\Omega) \quad g_\varepsilon \to g \ \text{strongly in} \ L^p(\Omega)^N
\]
\[
b(u_{0\varepsilon}) \to b(u_0) \ \text{a.e. in} \ \Omega \quad \text{and strongly in} \ L^1(\Omega).
\]

Without loss of generality, we assume that \( a, b \) are appropriately smooth and such that the following problem
\[
b(u_\varepsilon)_t - \text{div}(a(u_\varepsilon, \nabla u_\varepsilon)) = H(u_\varepsilon)(f_\varepsilon + \text{div}g_\varepsilon), \quad (x,t) \in \Omega \times (0,T)
\]
\[
u_\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T)
\]
\[
u_\varepsilon(x,0) = u_{0\varepsilon}(x), \quad x \in \Omega
\]
has a classical solution \( u_\varepsilon \). Otherwise, we can use an approximate process to get the same results.

Multiply (12) by \( T_k(u_\varepsilon) \) with \( k > 0 \) and integrate it over \( Q \) to get
\[
\int_0^t \int_{\Omega} T_k(s)b'(s)ds dx + \int_Q a(u_\varepsilon, \nabla u_\varepsilon)\nabla T_k(u_\varepsilon)dx dt
= \int_Q H(u_\varepsilon)T_k(u_\varepsilon)(f_\varepsilon + \text{div}g_\varepsilon)dx dt + \int_\Omega \int_0^{u_{0\varepsilon}(x)} T_k(s)b'(s)ds dx.
\]

In view of \((H_1)\)–\((H_4)\) for some \( k > 0 \) with \( \text{supp} H'(s) \subset (-k, k),
\[
a(u_\varepsilon, \nabla u_\varepsilon)\nabla T_k(u_\varepsilon) = a(u_\varepsilon, \nabla T_k(u_\varepsilon))\nabla T_k(u_\varepsilon) \geq \alpha |\nabla T_k(u_\varepsilon)|^p,
\]
\[
\int_Q H(u_\varepsilon)T_k(u_\varepsilon)\text{div}g_\varepsilon dx dt = - \int_Q (H'(u_\varepsilon)T_k(u_\varepsilon) + H(u_\varepsilon))DT_k(u_\varepsilon) \cdot g_\varepsilon dx dt.
\]
It follows from (15) and Young’s inequality
\[
\int_{\Omega} \int_{0}^{u_{e}(x,t)} T_{k}(s)b'(s)dsdx + \frac{\alpha}{2} \int_{Q} |\nabla T_{k}(u_{e})|^p dxdt \leq C(k) + k|b(u_{0})|_{L^1(\Omega)},
\]
where \(C(k)\) is a constant independent of \(\varepsilon\).

For any \(M > 0\), let \(S_{M}\) be an increasing function of \(C^\infty\) and such that \(S_{M}(r) = r\) for \(|r| \leq \frac{M}{2}\). \(S_{M}(r) = M\text{sgn}(r)\) for \(|r| \geq M\).

We will show in the sequel that for any \(M\) the sequence \(S_{M}(b(u_{e}))\) satisfies
\[
S_{M}(b(u_{e})) \text{ is bounded in } L^p(0, T : W^{1,p}(\Omega)) \tag{17}
\]
and
\[
\frac{\partial S_{M}(b(u_{e}))}{\partial t} \text{ is bounded in } L^1(Q) + L^{p'}(0, T : W^{-1,p'}(\Omega)) \tag{18}.
\]
Once (17) and (18) are established, a Aubin’s type lemma (see [8], Corollary 4) implies that, for any \(M > 1\), \(S_{M}(b(u_{e}))\) is compact in \(L^2(Q)\).

We now establish (17) and (18). Since \(S_{M}'(b(u_{e})) = 0\) if \(|b(u_{e})| > M\), we have
\[
DS_{M}(b(u_{e})) = S_{M}'(b(u_{e}))b'(u_{e})DT_{K_{M}}(u_{e}),
\]
where \(K_{M} = \max\{-b^{-1}(-M), b^{-1}(M)\}\). This and (16) imply (17).

To show (18), we multiply (12) by \(S_{M}'(b(u_{e}))\) to obtain
\[
\frac{\partial S_{M}(b(u_{e}))}{\partial t} = \text{div}(S_{M}'(b(u_{e}))a(u_{e}, \nabla u_{e}) - S_{M}''(b(u_{e}))b'(u_{e})a(u_{e}, \nabla u_{e})\nabla T_{K_{M}}(u_{e})\)
\[
+ S_{M}'(b(u_{e}))H(u_{e})f_{\varepsilon} + \text{div}(g_{\varepsilon}S_{M}'(b(u_{e})))H(u_{e})
\]
\[
- g_{\varepsilon} \cdot DT_{K_{M}}(u_{e})(S_{M}'(b(u_{e}))H(u_{e})).
\]
This implies (18). \((H_2)\) and (16) imply
\[
a(T_{k}(u_{e}), \nabla T_{k}(u_{e})) \text{ is uniformly bounded in } (L^{p'}(Q))^N. \tag{19}
\]
To lead to (6), we prove the following estimate.

For any integer \(n \geq 1\), consider the Lipschitz continuous function \(\theta_{n}\) defined through
\[
\theta_{n}(r) = T_{n+1}(r) - T_{n}(r) = \begin{cases} 0, & \text{if } |r| \leq n, \\ (|r| - n)\text{sgn}(r), & \text{if } n \leq |r| \leq n + 1, \\ \text{sgn}(r) & \text{if } |r| \geq n + 1. \end{cases}
\]
Note that \(0 \leq |\theta_{n}| \leq 1\) for any \(n \geq 1\), and \(\theta_{n}(r) \to 0\) for any \(r\) when \(n \to \infty\).

Multiplying (12) by \(\theta_{n}(u_{e})\) and integrating it over \(Q\), lead to
\[
\int_{\Omega} \int_{0}^{u_{e}(x,t)} b'(s)\theta_{n}(s)dsdx + \int_{Q} a(u_{e}, \nabla u_{e})\nabla \theta_{n}(u_{e})dxdt
\]
\[
= \int_{Q} H(u_{e})\theta_{n}(u_{e})(f_{\varepsilon} + \text{div}g_{\varepsilon})dxdt + \int_{\Omega} \int_{0}^{u_{e}(x)} \theta_{n}(s)b'(s)dsdx. \tag{20}
\]
Note that
\[
\int_Q a(u_\varepsilon, \nabla u_\varepsilon) \nabla \theta_n(u_\varepsilon) \, dx \, dt \geq \alpha \int_Q |\nabla \theta_n(u_\varepsilon)|^p \, dx \, dt
\]
and
\[
\int_Q H(u_\varepsilon) \theta_n(u_\varepsilon) \text{div}_g \varepsilon \, dx \, dt = -\int_Q H'(u_\varepsilon) \theta_n(u_\varepsilon) D\!T_k(u_\varepsilon) \cdot g_\varepsilon \, dx \, dt - \int_Q H(u_\varepsilon) \nabla \theta(u_\varepsilon) \cdot g_\varepsilon \, dx \, dt
\]
where \(\text{supp}H' \subset [-k, k]\). Using \((H_3)\), Young’s inequality, and (16), we get from (20)
\[
\int_Q a(u_\varepsilon, \nabla u_\varepsilon) \nabla \theta_n(u_\varepsilon) \, dx \, dt \leq C, \quad \text{(21)}
\]
\[
\int_Q |\nabla \theta_n(u_\varepsilon)|^p \, dx \, dt \leq C, \quad \text{(22)}
\]
where \(C\) does not depend on \(\varepsilon\) and \(n\).

The above estimates imply that there exists a subsequence, still indexed by \(\varepsilon\), such that
\[
u_\varepsilon \to u \quad \text{a.e. in } Q, \quad \text{(23)}
\]
\[
\theta_n(u_\varepsilon) \to \theta_n(u) \quad \text{weakly in } L^p(0, T; W^{1,p}(\Omega)), \quad \text{(24)}
\]
\[
T_k(u_\varepsilon) \to T_k(u) \quad \text{weakly in } L^p(0, T; W^{1,p}(\Omega)), \quad \text{(25)}
\]
\[
a(T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \to \sigma_k \quad \text{weakly in } (L^p(\Omega))^N, \quad \text{(26)}
\]
as \(\varepsilon \to 0\), for any \(k > 0, n \geq 1, \) and \(\sigma_k \in (L^p(\Omega))^N\).

We now prove \(b(u) \in L^\infty(0, T; L^1(\Omega))\). By (16), we have
\[
\int_\Omega \int_0^{u_\varepsilon(x,T)} T_k(s)b'(s) \, ds \, dx \leq k|b(u_0)|_{L^1(\Omega)} + C(k).
\]
This implies
\[
k \int_\Omega |b(u(x, t))| \, dx \leq k|b(u_0)|_{L^1(\Omega)} + C(k) + Ck\varepsilon b(k) \text{mes}\Omega
\]
and \(b(u) \in L^\infty(0, T; L^1(\Omega))\).

From (20) and \((H_3)\), we deduce
\[
\lim_{\varepsilon \to 0} \int_Q \int_Q H(u_\varepsilon) \theta_n(u_\varepsilon) \text{div}_g \varepsilon \, dx \, dt
\]
\[
\leq \lim_{\varepsilon \to 0} \int_Q \int_Q H'(u_\varepsilon) \theta_n(u_\varepsilon) D\!T_k(u_\varepsilon) + H(u_\varepsilon) \nabla \theta_n(u_\varepsilon) \cdot g_\varepsilon \, dx \, dt.
\]
Note that
\[
\int_Q H(u_\varepsilon) \theta_n(u_\varepsilon) \text{div}_g \varepsilon \, dx \, dt = -\int_Q (H'(u_\varepsilon) \theta(u_\varepsilon) g_\varepsilon) \, dx \, dt - \int_Q H(u_\varepsilon) \nabla \theta(u_\varepsilon) \cdot g_\varepsilon \, dx \, dt,
\]
where \(\text{supp}H' \subset [-k, k]\). We have
\[
\lim_{\varepsilon \to 0} \int_Q H(u_\varepsilon) \theta_n(u_\varepsilon) \, dx \, dt
\]
\[
= \int_Q H(u) \theta_n(u) \, dx \, dt - \int_Q \theta_n(u) H'(u) \nabla T_k(u) \, dx \, dt - \int_Q H(u) \nabla \theta_n(u) \, dx \, dt.
\]
Since $\theta_n(u) \to 0$ as $n \to \infty$ and $\nabla \theta_n(u)$ converges weakly in $L^p(0, T; W^{1,p}(\Omega))$,
\[ \theta_n(u) \to 0 \quad \text{weakly in } L^p(0, T; w^{1,p}_0(\Omega)) \quad \text{as } n \to \infty. \]

Hence, combining (27), (28), and $\nabla \theta_n(u_x) = \chi_{(n \leq u_x \leq n+1)} \nabla u_x$, we get
\[ \lim_{n \to \infty} \lim_{\varepsilon \to 0} \int_{\{n \leq |u_x| \leq n+1\}} a(u_x, \nabla u_x) \nabla u_x \, dx \, dt = 0. \tag{29} \]

To prove $\sigma_k = a(T_k(u), \nabla T_k(u))$, we use the regularization method of Landes [9]. Define the regularization in time of the function $T_k(u)$ by
\[ (T_k(u))_{\nu}(x, t) = \nu \int_{-\infty}^t e^{\nu(s-t)} T_k(u(x, s)) \, ds, \tag{30} \]
where $u(x, s) = u(x, s)$ if $s > 0$; $u(x, s) = u_0(x)$ if $s \leq 0$. $(T_k(u))_{\nu}$ has the following properties [9]:
\[ \frac{\partial (T_k(u))_{\nu}}{\partial t} + \nu ((T_k(u))_{\nu} - T_k(u)) = 0 \quad \text{in } D', \tag{31} \]
\[ (T_k(u))_{\nu}|_{t=0} = T_k(u_0) \quad \text{in } \Omega; \]
\[ (T_k(u))_{\nu} \to T_k(u) \quad \text{a.e. in } Q, \text{ in } L^\infty(Q) \text{ weakly}^* \]
and strongly in $L^p(0, T; W^{1,p}_0(\Omega))$ as $\nu \to \infty$;
\[ \| (T_k(u))_{\nu} \|_{L^\infty(Q)} \leq \max(\| T_k(u) \|_{L^\infty(Q)}, \| T_k(u_0) \|_{L^\infty(Q)}) \leq k \quad \text{for any } \nu > 0. \tag{32} \]

**Lemma 3.1** Let $\sigma \in C_0^\infty(0, T)$, $\sigma \geq 0$, and let $h \in W^{1,\infty}(R)$, $h \geq 0$, supp $h$ be compact. Then,
\[ \lim_{\nu \to \infty} \lim_{\varepsilon \to 0} \int_0^T \int_0^t (b(u_x)\sigma h(u_x)(T_k(u_x) - (T_k(u))_{\nu})) \, dt \, ds \geq 0. \tag{34} \]

The proof of Lemma 3.1 is similar to that of (18) in [5]. Indeed, by integration by parts, using the properties of $(T_k(u))_{\nu}$, we have
\[ \int_0^T \int_0^t (b(u_x)\sigma h(u_x)(T_k(u_x) - (T_k(u))_{\nu})) \, ds \, dt = \int_0^T \int_0^t (B_h(u_x)\sigma (T_k(u_x) - (T_k(u))_{\nu})) \, ds \, dt \]
\[ = -\int_Q \int_0^t \sigma I_1 \, ds \, dt + \int_Q \int_0^t \sigma B_h(u_x)(T_k(u_x))_{\nu} \, ds \, dt \]
\[ = -\int_Q \int_0^t \sigma I_1 \, ds \, dt + \int_Q \int_0^t \sigma B_h(u_x)(T_k(u_x))_{\nu} \, ds \, dt \]
\[ = I_1 + I_2 + I_3, \]
where $B_h(r) = \int_0^r h(s) \, ds$. Note that
\[ I_3 = \int_Q \int_0^t \sigma B_h(u_x)(T_k(u_x) - (T_k(u))_{\nu}) \, ds \, dt \]
\[ + \int_Q \int_0^t \sigma B_h((T_k(u_x) - (T_k(u))_{\nu}) \, ds \, dt = I_3^1 + I_3^2 + I_3^3. \]
By (33) and the monotony of $B_k$ and $| (T_k(u))_\nu | \leq k$,

$$I_3^1 = \iint_{Q \cap \{ u > k \}} \sigma \nu (k - (T_k(u))_\nu)(B_k(u) - B_h(T_k(u))) \geq 0.$$  

Similarly, $I_3^2 \geq 0$. Consequently,

$$I_3 \geq \iint_Q \sigma \nu (T_k(u) - (T_k(u))_\nu)B_h((T_k(u))_\nu) = \iint_Q \sigma \frac{\partial}{\partial t}(T_k(u))_\nu B_h((T_k(u))_\nu) = \iint_Q \sigma \frac{\partial}{\partial t}(T_k(u))_\nu B_h(r)dr = - \iint_Q \sigma_\nu \int_0^T T_k(u)_\nu B_h(r)dr.$$  

Thus,

$$\lim_{\nu \to \infty} \lim_{\epsilon \to 0} \int_0^T \int_0^t \langle b(u_\epsilon)_\nu \sigma h(u_\epsilon)(T_k(u_\epsilon) - (T_k(u))_\nu)\rangle dsdt \geq - \iint_Q \sigma_\nu \int_0^u T_k(r)dB_h(r) + \iint_Q \sigma_\nu B_h(u)(T_k(u))_\nu - \iint_Q \sigma_\nu \int_0^T T_k(u)_\nu B_h(r)dr.$$  

Hence, Lemma 3.1 follows from

$$\iint_0^u T_k(r)dB_h(r) = \iint_0^T T_k(u)_\nu (B_h(u) - B_h(r))dr.$$  

**Lemma 3.2** For any $k > 0$, $u_\epsilon$ satisfies

$$\lim_{\nu \to \infty} \lim_{\epsilon \to 0} \int_0^T \sigma(t) \int_\Omega a(u_\epsilon, \nabla T_k(u_\epsilon))\nabla (T_k(u_\epsilon))dxdt \leq \int_0^T \sigma(t) \int_\Omega \sigma_k \nabla (T_k(u))dxdt,$$  

where $\sigma_k$ and $\sigma(t)$ are defined in (25) and Lemma 3.1, respectively.

**Proof** Let $S_n \in C^\infty (R)$ such that

$$S_n(r) = r \text{ for } |r| \leq n, \text{ supp} S_n \subset [-n - 1, n + 1], \| S_n'' \|_{L^\infty (R)} \leq 1, \text{ for any } n > 1. \quad (36)$$  

Let

$$\omega_\nu^\epsilon = T_k(u_\epsilon) - (T_k(u))_\nu. \quad (37)$$  

Multiplying (12) by $\sigma S_n''(u_\epsilon)\omega_\nu^\epsilon$ and integrating it over $Q$, we deduce

$$\iint_Q \omega_\nu^\epsilon \sigma(t) \frac{\partial}{\partial t} \int_0^{u_\epsilon} b'(s)S_n''dsdxdt + \iint_Q \sigma(t)S_n''(u_\epsilon)A(u_\epsilon, \nabla u_\epsilon)\nabla \omega_\nu^\epsilon dxdt$$

$$+ \iint_Q S_n''(u_\epsilon)a(u_\epsilon, \nabla u_\epsilon)\nabla u_\epsilon \omega_\nu^\epsilon \sigma(t)dxdt$$

$$= \iint_Q \sigma(t)H(u_\epsilon)(f_\epsilon + \text{div} g_\epsilon)S_n''(u_\epsilon)\omega_\nu^\epsilon dxdt.$$  

Lemma 3.1 implies

$$\lim_{\nu \to \infty} \lim_{\epsilon \to 0} \iint_Q \omega_\nu^\epsilon \sigma(t) \frac{\partial}{\partial t} \int_0^{u_\epsilon} b'(s)S_n''dsdxdt \geq 0, \quad \forall \ n \geq k. \quad (39)$$
Since $\text{supp} S_n^\nu \subset [n, n+1] \cup [-n-1, -n]$, we have
\[
\left| \int_Q S_n^\nu(u_\varepsilon) a(u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon \sigma(t) dx dt \right| 
\leq \| S_n^\nu \|_{L^\infty(R)} \| \sigma \|_{L^\infty(Q)} \int_{\{n \leq |u_\varepsilon| \leq n+1\}} a(u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon.
\]
This and (29) imply
\[
\lim_{n \to \infty} \lim_{\nu \to \infty} \lim_{\varepsilon \to 0} \left| \int_Q S_n^\nu(u_\varepsilon) a(u_\varepsilon, \nabla u_\varepsilon) \nabla u_\varepsilon \sigma(t) dx dt \right| = 0. \quad (40)
\]
Note that, if $k > n+1$,
\[
\sigma(t) H(u_\varepsilon) S_n^\nu(u_\varepsilon) \omega_\nu^\varepsilon = \sigma(t) H(T_k(u_\varepsilon)) S_n^\nu(T_k(u_\varepsilon)) \omega_\nu^\varepsilon \\
\to \sigma(t) H(u) S_n^\nu(u)(T_k(u) - T_k(u)_\nu) \text{ weakly in } L^p(0,T;W^{1,p}(\Omega)).
\]
By (10), as $\varepsilon \to 0$, we have
\[
\lim_{\varepsilon \to 0} \int_Q \sigma(t) H(u_\varepsilon) S_n^\nu(u_\varepsilon) \omega_\nu^\varepsilon (f_\varepsilon + \text{div} g_\varepsilon) dx dt \\
= \int_Q \sigma(t) H(u) S_n^\nu(u)(T_k(u) - T_k(u)_\nu) f dx dt \\
- \int_Q \sigma(t) \text{div}(H(u) S_n^\nu(u)(T_k(u) - T_k(u)_\nu)) g dx dt.
\]
Thus,
\[
\lim_{\nu \to \infty} \lim_{\varepsilon \to 0} \int_Q \sigma(t) H(u_\varepsilon) S_n^\nu(u_\varepsilon) \omega_\nu^\varepsilon (f_\varepsilon + \text{div} g_\varepsilon) dx dt = 0. \quad (41)
\]
Combining (38)–(41), we get
\[
\lim_{n \to \infty} \lim_{\nu \to \infty} \lim_{\varepsilon \to 0} \int_Q \sigma(t) S_n^\nu(u_\varepsilon) a(u_\varepsilon, \nabla u_\varepsilon)(\nabla T_k(u_\varepsilon) - \nabla T_k(u)_\nu) dx dt \leq 0. \quad (42)
\]
Since
\[
S_n^\nu(u_\varepsilon) a(u_\varepsilon, \nabla u_\varepsilon) = S_n^\nu(u_\varepsilon) a(T_{n+1}(u_\varepsilon), \nabla T_{n+1}(u_\varepsilon)) \to S_n^\nu(u) \sigma_{n+1} \text{ weakly in } L^p(Q),
\]
we have, for $k \leq n$,
\[
\lim_{\nu \to \infty} \lim_{\varepsilon \to 0} \int_Q \sigma(t) S_n^\nu(u_\varepsilon) a(u_\varepsilon, \nabla u_\varepsilon) \nabla T_k(u)_\nu dx dt \\
= \int_Q \sigma(t) S_n^\nu(u) \sigma_{n+1} \nabla T_k(u) dx dt = \int_Q \sigma(t) \sigma_{n+1} \nabla T_k(u) dx dt. \quad (43)
\]
Note that, for $k \leq n$
\[
a(T_{n+1}(u_\varepsilon), \nabla T_{n+1}(u_\varepsilon)) \chi_{\{|u_\varepsilon| \leq k\}} = a(T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \chi_{\{|u_\varepsilon| \leq k\}}.
\]
We have
\[
\sigma_{n+1} \chi_{\{|u| \leq k\}} = \sigma_k \chi_{\{|u| \leq k\}} \quad \text{a.e. in } Q \setminus \chi_{\{|u| = k\}}.
\]
This implies
\[ \sigma_{n+1} \nabla T_k(u) = \sigma_k \nabla T_k(u) \quad \text{a.e. in } Q. \] (44)

Hence, Lemma 3.2 follows from (42)–(44).

**Lemma 3.3** The subsequence \( u_\varepsilon \) satisfies, for any \( k > 0 \),
\[ \lim_{\varepsilon \to 0} \int_Q \sigma(t)[a(T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - a(T_k(u), \nabla T_k(u))] (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) \, dx \, dt = 0. \] (45)

**Proof** Let \( k \geq 0 \) be fixed. The assumption (H2) implies
\[ \int_Q \sigma(t) \left( a(T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - a(T_k(u), \nabla T_k(u)) \right) (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) \, dx \, dt \geq 0. \] (46)

By Lemma 3.3 and (47), we get
\[ a(T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \to a(T_k(u), \nabla T_k(u)) \quad \text{strongly in } (L^p(Q))^N. \] (47)

Combining (47) and Lemma 3.2, we get
\[ \lim_{\varepsilon \to 0} \int_Q \sigma(t)[a(T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - a(T_k(u), \nabla T_k(u))] (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) \, dx \, dt \leq 0. \] (48)

Hence, Lemma 3.3 follows from (46) and (48).

**Lemma 3.4** For fixed \( k \geq 0 \), we have
\[ \sigma_k = a(T_k(u), \nabla T_k(u)) \] (49)

and, as \( \varepsilon \to 0 \),
\[ a(T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla T_k(u_\varepsilon) \to a(T_k(u), \nabla T_k(u)) \nabla T_k(u) \quad \text{weakly in } L^1(Q). \] (50)

**Proof** By Lemma 3.3 and (47), we get
\[ \lim_{\varepsilon \to 0} \int_Q \sigma(t) a(T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla T_k(u_\varepsilon) \, dx \, dt = \int_Q \sigma(t) \sigma_k \nabla T_k(u) \, dx \, dt. \] (51)

From (51) and using Minty’s argument, we deduce (49).

By Lemma 3.3 and the monotone character of \( a \), we have
\[ [a(T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) - a(T_k(u), \nabla T_k(u))] (\nabla T_k(u_\varepsilon) - \nabla T_k(u)) \to 0 \quad \text{weakly in } L^1(Q). \] (52)

Moreover, (24), (25), (47), and (49) imply
\[ a(T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla T_k(u) \to a(T_k(u), \nabla T_k(u)) \nabla T_k(u) \quad \text{weakly in } L^1(Q), \]
\[ a(T_k(u_\varepsilon), \nabla T_k(u)) \nabla T_k(u_\varepsilon) \to a(T_k(u), \nabla T_k(u)) \nabla T_k(u) \quad \text{weakly in } L^1(Q), \]
and
\[ a(T_k(u_\varepsilon), \nabla T_k(u)) \nabla T_k(u) \to a(T_k(u), \nabla T_k(u)) \nabla T_k(u) \quad \text{strongly in } L^1(Q), \]
as \( \varepsilon \to 0 \).

Using the above convergence results in (52), we get, for any \( k \geq 0 \),
\[ \sigma(t)a(T_k(u_\varepsilon), \nabla T_k(u_\varepsilon)) \nabla T_k(u_\varepsilon) \to \sigma(t)a(T_k(u), \nabla T_k(u)) \nabla T_k(u) \quad \text{weakly in } L^1(Q). \]
Lemma 3.4 is proved.

Next, we prove (6). Note that
\[
\int \int_{\{(x,t)\in Q, n<|u_{\varepsilon}(x,t)|<n+1\}} a(u_{\varepsilon}, \nabla u_{\varepsilon})\nabla u_{\varepsilon} \, dx \, dt
= \int \int_{Q} a(u_{\varepsilon}, \nabla u_{\varepsilon})(\nabla T_{n+1}(u_{\varepsilon}) - \nabla T_{n}(u_{\varepsilon})) \, dx \, dt
= \int \int_{Q} a(T_{n+1}(u_{\varepsilon}), \nabla T_{n+1}(u_{\varepsilon}))\nabla T_{n+1}(u_{\varepsilon}) \, dx \, dt - \int \int_{Q} a(T_{n}(u_{\varepsilon}), \nabla T_{n}(u_{\varepsilon}))\nabla T_{n}(u_{\varepsilon}) \, dx \, dt.
\]

According to (50)
\[
\lim_{\varepsilon \to 0} \int \int_{\{(x,t)\in Q, n<|u_{\varepsilon}(x,t)|<n+1\}} a(u_{\varepsilon}, \nabla u_{\varepsilon})\nabla u_{\varepsilon} \, dx \, dt
= \int \int_{Q} a(T_{n+1}(u), \nabla T_{n+1}(u))\nabla T_{n+1}(u) \, dx \, dt - \int \int_{Q} a(T_{n}(u), \nabla T_{n}(u))\nabla T_{n}(u) \, dx \, dt
= \int \int_{\{(x,t)\in Q, n<|u(x,t)|<n+1\}} a(u, \nabla u)\nabla u \, dx \, dt. \tag{53}
\]

(53) and (29) imply (6).

Now, we prove that \( u \) satisfies (7) and (8).

Let \( h(s) \in C^{1}_{0}(R) \), \( \xi \in C^{1}_{0}(Q) \).

Multiply (12) by \( h(u_{\varepsilon})\xi \) and integrate it over \( Q \), we get
\[
\int \int_{Q} \int_{0}^{u_{\varepsilon}} b'(s)h(s)d\xi ds \, dx \, dt - \int \int_{Q} a(u_{\varepsilon}, \nabla u_{\varepsilon})\nabla \xi h(u_{\varepsilon}) \, dx \, dt
- \int \int_{Q} h'(u_{\varepsilon})a(u_{\varepsilon}, \nabla u_{\varepsilon})\nabla \xi u_{\varepsilon} \, dx \, dt
= \int \int_{Q} H(u_{\varepsilon})h(u_{\varepsilon})\xi (f_{\varepsilon} + \text{div}g_{\varepsilon}) \, dx \, dt. \tag{54}
\]

Since \( \text{supp} h \subset [-k, k] \), we have
\[
a(u_{\varepsilon}, \nabla u_{\varepsilon})h(u_{\varepsilon}) = a(T_{k}u_{\varepsilon}, \nabla T_{k}u_{\varepsilon})h(u_{\varepsilon}),
\]
\[
h'(u_{\varepsilon})a(u_{\varepsilon}, \nabla u_{\varepsilon})\nabla u_{\varepsilon} = h'(u_{\varepsilon})a(T_{k}u_{\varepsilon}, \nabla T_{k}u_{\varepsilon})\nabla T_{k}u_{\varepsilon},
\]
\[
H(u_{\varepsilon})h(u_{\varepsilon}) = H(T_{k}u_{\varepsilon})h(u_{\varepsilon}).
\]

Letting \( \varepsilon \to 0 \) in (54), we get (7).

Note that
\[
\frac{\partial}{\partial t} \int_{0}^{u_{\varepsilon}} h(s)b'(s)ds \quad \text{is uniformly bounded in} \quad L^{1}(Q) + L^{p'}(0, T; W^{-1,p'}(\Omega))
\]

with respect to \( \varepsilon \).

Similar to the argument in [5], we can conclude
\[
b(u) \big|_{t=0} = b(u_{0}).
\]

Theorem 1 is proved.
4 Proof of Theorem 2

The proof of Theorem 2 is similar to that in [5]. Similar to Lemma 5 in [5], we have the following lemmas.

**Lemma 4.1** Under the assumptions of Theorem 2, any renormalized solution of (1)–(3) satisfies the following estimate for any $s > 1$ and any $0 < \delta < 1$

$$\frac{1}{\delta} \int_{s-\delta \leq |u| \leq s+\delta} a(u, \nabla u) \nabla u \, dx \, dt \leq G(s),$$

where $G(s)$ tends to 0 when $s$ tends to $+\infty$.

**Proof** For all $s > 0$, let $S_s \in W^{2,\infty}(R)$, $S_s(0) = 0$, $S_s'(r) = 1$ if $|r| \leq s$, $S_s'(r) = s + 1 - |r|$ if $s \leq |r| \leq s + 1$, $S_s'(r) = 0$ if $|r| \geq s + 1$. For $s > 1$, $1 > \delta > 0$, let $R^\delta_s(r) = \frac{1}{\delta}(T_{s+\delta}(r) - T_{s-\delta}(r))$.

Clearly, supp($R^\delta_s$) $\subset [-s-\delta, -s+\delta] \cup [s-\delta, s+\delta]$. By Definition 2.1,

$$\int_0^T \langle \frac{\partial}{\partial t} \int_0^u S_{s+1}'(r)b'(r)dr, R^\delta_s \rangle \, dt + \int_0^T S_{s+1}'(u)a(u, \nabla u) \nabla R^\delta_s(u) \, dx \, dt$$

$$= \int_0^T S_{s+1}'(u)R^\delta_s(u)H(u) \, f \, dx \, dt. \quad (55)$$

By Lemma 2.4 of [10] and using the initial condition (3), we get

$$\int_0^T \langle \frac{\partial}{\partial t} \int_0^u S_{s+1}'(r)b'(r)dr, R^\delta_s \rangle \, dt$$

$$= \int_0^\Omega S_{s+1}'(r)b'(r)R^\delta_s(r) \, dx |t=0 \, dx - \int_0^\Omega S_{s+1}'(r)b'(r)R^\delta_s(r) \, dx |t=T \, dx$$

$$\geq - \int_0^\Omega |b(u_0)| \chi_{|u_0|>s-1} \, dx. \quad (56)$$

From the definitions of $R^\delta_s$ and $S_s$, (55), and (56), we obtain

$$\frac{1}{\delta} \int_{s-\delta \leq |u| \leq s+\delta} a(u, \nabla u) \nabla u \, dx \, dt$$

$$\leq 2 \int_{s+1 \leq |u| \leq s+2} a(u, \nabla u) \nabla u \, dx \, dt + 2 \int_{|u| \geq s} |H(u) |f| \, dx \, dt + \int_\Omega |b(u_0)| \chi_{|u_0|>s-1} \, dx. \quad (57)$$

Lemma 4.1 is proved.

**Proof of Theorem 2** Let $u$ and $v$ be two renormalized solutions of (1)–(3). Let $T^\sigma_s \in W^{2,\infty}$ be the function defined by

$$(T^\sigma_s)'(r) = \begin{cases} 
1, & \text{if } |r| \leq s, \\
\frac{1}{\sigma} (s + \sigma - |r|), & \text{if } s \leq |r| \leq s + \sigma, \\
0, & \text{if } |r| \geq s + \sigma.
\end{cases}$$
We can take $h(r) = (T^r_s)'(r), \xi = \frac{1}{2}T_k(T^r_s(u) - T^r_s(v))$ in the respective equations (7) for $u$ and $v$. Subtracting these two equations, we get

\[
\frac{1}{k} \int_0^T \int_0^t \frac{\partial}{\partial t} [T^r_s(u) - T^r_s(v)] dt ds + \frac{1}{k} \int_0^T \int_0^t \int_\Omega (T^r_s)'(u)a(u, \nabla u) - (T^r_s)'(v)a(v, \nabla v)) \nabla T_k(T^r_s(u) - T^r_s(v)) dx dz dt ds
\]

\[
= - \frac{1}{k} \int_0^T \int_0^t \int_\Omega (T^r_s)'(u)T_k(T^r_s(u) - T^r_s(v))a(u, \nabla u)\nabla u dx dz dt ds
\]

\[
+ \frac{1}{k} \int_0^T \int_0^t \int_\Omega (T^r_s)'(v)T_k(T^r_s(u) - T^r_s(v))a(v, \nabla v)\nabla v dx dz dt ds
\]

\[
+ \frac{1}{k} \int_0^T \int_0^t \int_\Omega f(H(u)(T^r_s)'(u) - H(v)(T^r_s)'(v))T_k(T^r_s(u) - T^r_s(v)) dx dz dt ds.
\]

(58)

Denote the seven integrals appeared by $I_1, I_2, I_3, I_4, I_5$, respectively, in order. In the following, we study the behaviors of these integrals.

For fixed $s > 0$, when $\sigma$ tends to 0, we have

\[
(T^\sigma_s)'(r) \to \chi_{\{|r| \leq s\}} \text{ a.e. in } Q \text{ and strongly in } L^q \text{ for any } q < +\infty,
\]

(59)

\[
T^\sigma_s(r) \to T_s(r) \text{ a.e. in } Q \text{ and strongly in } L^p(0, T; W_0^1; \Omega).
\]

(60)

Since $\text{supp}(T^\sigma_s)'(r) \subset [-s - 1, s + 1]$ for $0 < \sigma < 1$, and by (59), (60), we get

\[
\lim_{\sigma \to 0} I_2 = \frac{1}{k} \int_0^T \int_0^t \int_\Omega \chi_{\{|u| \leq s\}} a(T_s(u), \nabla T_s(u)) \nabla T_k(T_s(u) - T_s(v)) dx dz dt ds
\]

\[
- \frac{1}{k} \int_0^T \int_0^t \int_\Omega \chi_{\{|u| \leq s\}} a(T_s(v), \nabla T_s(v)) \nabla T_k(T_s(u) - T_s(v)) dx dz dt ds.
\]

(61)

Since $a(r, 0) = 0$, (61) leads to

\[
\lim_{\sigma \to 0} I_2 = \frac{1}{k} \int_0^T \int_0^t \int_\Omega [a(T_s(u), \nabla T_s(u)) - a(T_s(v), \nabla T_s(v))] \nabla T_k(T_s(u) - T_s(v)) dx dz dt ds
\]

\[
= \frac{1}{k} \int_0^T \int_0^t \int_\Omega [a(T_s(u), \nabla T_s(u)) - a(T_s(u), \nabla T_s(v))] \nabla T_k(T_s(u) - T_s(v)) dx dz dt ds
\]

\[
+ \frac{1}{k} \int_0^T \int_0^t \int_\Omega [a(T_s(u), \nabla T_s(v)) - a(T_s(v), \nabla T_s(v))] \nabla T_k(T_s(u) - T_s(v)) dx dz dt ds.
\]

(62)

Note that, by (9), we have

\[
\left| \frac{1}{k} \int_0^T \int_0^t \int_\Omega [a(T_s(u), \nabla T_s(v)) - a(T_s(v), \nabla T_s(v))] \nabla T_k(T_s(u) - T_s(v)) dx dz dt ds \right|
\]

\[
\leq \frac{T}{k} \int_{\{(T_s(u) - T_s(v)) \leq k\} \cap \{T_s(u) \neq T_s(v)\}} |T_s(u) - T_s(v)||E_s + F_s| \nabla T_s(v) |^p - 1 \]

\[
\times ||\nabla T_s(u)|| + ||\nabla T_s(v)|| dx dt.
\]
Moreover,
\[ |E_s + F_s| \leq |\nabla T_s(v)|^{p-1} \times \left( |\nabla T_s(u)| + |\nabla T_s(v)| \right) \in L^1(Q) \]
and
\[ \chi_{\{|T_s(u) - T_s(v)| \leq k\}} \chi_{\{|T_s(u) - T_s(v)| > k\}} \to 0 \quad \text{a.e. in } Q \quad \text{as } k \to 0. \]

Letting \( k \) tend to 0 makes it possible to conclude that the last integral of (62) tends to 0. This implies that
\[ \lim_{k \to 0} \lim_{\sigma \to 0} I_2 \geq 0. \quad (63) \]

In view of the definitions of \((T_s^\sigma)'(v)\), we have
\[ |I_3 + I_4| \leq \frac{T}{\sigma} \left[ \int_{s-\sigma \leq |u| \leq s+\sigma} a(u, \nabla u) \nabla u \, dx \, dt + \int_{s-\sigma \leq |v| \leq s+\sigma} a(v, \nabla v) \nabla v \, dx \, dt \right], \]
and by Lemma 4.1
\[ \lim_{\sigma \to 0} (I_3 + I_4) \leq G_1(s), \quad (64) \]
where \( G_1(s) \) tends to 0 when \( s \) tends to \(+\infty\).

In view of (59) and (60), we have for any \( \sigma \),
\[ \lim_{k \to 0} \lim_{\sigma \to 0} I_5 = \int_0^T \int_0^1 \int_0^\Omega f(H(u)) \chi_{\{|u| \leq s\}} - H(v) \chi_{\{|v| \leq s\}} \text{sgn}(T_s(u) - T_s(v)) \, dx \, ds \, dt. \quad (65) \]

Moreover,
\[ I_1 = \frac{1}{k} \int_Q \bar{T}_k(T_s^\sigma(u) - T_s^\sigma(v)) \, dx \, dt - T \int_\Omega \bar{T}_k(T_s^\sigma(u) - T_s^\sigma(v))(t = 0) \, dx, \]
where
\[ \bar{T}_k(r) = \int_0^r T_k(s) \, ds = \begin{cases} \frac{r^2}{2}, & \text{if } |r| \leq k, \\ k|r| - \frac{k^2}{2}, & \text{if } |r| \geq k. \end{cases} \]

Due to the same initial value for \( u \) and \( v \), and the properties of \( T_s^\sigma \), we have
\[ T_s^\sigma(u)(t = 0) = T_s^\sigma(v)(t = 0) = T_s^\sigma(u_0) \quad \text{a.e. in } \Omega. \]
This implies that the last term in (65) is equal to 0 for any \( \sigma > 0, s > 0, \) and \( k > 0. \)

According to (60) and the definition of \( \bar{T}_k \), we have
\[ \lim_{k \to 0} \lim_{\sigma \to 0} \frac{1}{k} \int_Q \bar{T}_k(T_s^\sigma(u) - T_s^\sigma(v)) \, dx \, dt = \int_Q \left| T_s(u) - T_s(v) \right| \, dx \, dt \quad (66) \]
for any \( s > 0. \)

In view of estimates (63)–(66), passing to limit-sup as \( \sigma \) tends to 0 and then to the limit-sup as \( k \) tends to 0 in (58) for any \( s > 0, \) we have
\[ \int_Q \left| T_s(u) - T_s(v) \right| \, dx \, dt \]
\[ \leq \int_0^T \int_0^1 \int_\Omega f(H(u)) \chi_{\{|u| \leq s\}} - H(v) \chi_{\{|v| \leq s\}} \text{sgn}(T_s(u) - T_s(v)) \, dx \, ds \, dt + G(s), \quad (67) \]
where $G(s)$ tends to 0 when $s$ tends to $+\infty$.

Passing to limit-inf as $s$ tends to $+\infty$ in (67), we get

Since $u$ and $v$ both belong to $L^\infty(0, T; L^1(\Omega))$, $T_s(u)$ and $T_s(v)$ converge, respectively, to $u$ and $v$ strongly in $L^1(Q)$ when $s$ tends to $+\infty$ so that

$$\lim_{s \to +\infty} \int_Q |T_s(u) - T_s(v)| \, dx dt = \int_Q |u - v| \, dx dt$$

$$\leq \lim_{s \to +\infty} \int_0^T \int_0^t \int_\Omega f(H(u)\chi_{\{|u| \leq s\}} - H(v)\chi_{\{|v| \leq s\}}) \text{sgn}(T_s(u) - T_s(v)) \, dx \, ds \, dt \leq 0,$$

where the assumptions in Theorem 2 are used. Theorem 2 is proved.

References


