

# RENDICONTI *del* SEMINARIO MATEMATICO *della* UNIVERSITÀ DI PADOVA

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*Rendiconti del Seminario Matematico della Università di Padova*,  
tome 79 (1988), p. 115-121

[http://www.numdam.org/item?id=RSMUP\\_1988\\_\\_79\\_\\_115\\_0](http://www.numdam.org/item?id=RSMUP_1988__79__115_0)

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## Oka-Analyticity of the Essential Spectrum.

ENRICO CASADIO TARABUSI (\*)

SUMMARY - Let  $X$  be a complex Banach space,  $G$  an open set in  $\mathbf{C}$ , and  $\lambda \mapsto T_\lambda$  a holomorphic family of closed operators on  $X$ . We show here that  $\lambda \mapsto \sigma_e(T_\lambda)$  is an analytic multifunction, where  $\sigma_e$  denotes the essential spectrum in *any* one of its several definitions.

### 1. Introduction.

Let  $X, Y$  be (nonzero) complex Banach spaces, and  $\mathcal{B}(X, Y)$  the Banach space of bounded linear operators from  $X$  to  $Y$ . It is known (see [9, Corollary 3.3 p. 371]) that, if  $G$  is an open set of  $\mathbf{C}$ , and  $\lambda \mapsto T_\lambda: G \rightarrow \mathcal{B}(X) = \mathcal{B}(X, X)$  is a holomorphic map, then the multifunction spectrum  $\lambda \mapsto \sigma(T_\lambda): G \rightarrow \text{cl}(\mathbf{C}) = \{\text{closed subsets of } \mathbf{C}\}$  is *Oka-analytic*, i.e. it is upper semicontinuous (u.s.c.; viz.  $\{\lambda \in G: \sigma(T_\lambda) \subset A\}$  is open in  $G$  if  $A$  is open in  $\mathbf{C}$  and  $\mathbf{C} \setminus A$  is compact) and the open set  $\Omega = \{(\lambda, z) \in G \times \mathbf{C}: z \notin \sigma(T_\lambda)\}$  is pseudoconvex: by abuse of terminology we say that  $\sigma$  is Oka-analytic on  $\mathcal{B}(X)$ . Denoting by  $\mathcal{C}(X, Y)$  the set of closed linear operators from  $X$  to  $Y$ , the same result holds, more generally, (see [10, Theorem 1 p. 121]) if  $G \ni \lambda \mapsto T_\lambda$  is a Kato-holomorphic family with values in  $\mathcal{C}(X) = \mathcal{C}(X, X)$ : that is (cf. [4, in Section VII.1.2 p. 366]) if, for every  $\lambda_0 \in G$  there exist  $G_0$  open neighborhood of  $\lambda_0$  in  $G$ , a Banach space  $Y$ , and holomorphic families  $\lambda \mapsto U_\lambda, V_\lambda: G_0 \rightarrow \mathcal{B}(Y, X)$  such that  $U_\lambda$  is one-to-one and  $T_\lambda = V_\lambda U_\lambda^{-1}$  for every  $\lambda \in G_0$ : we say that  $\sigma$  is Oka-analytic on  $\mathcal{C}(X)$ .

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We shall prove the same results for other multifunctions of spectral type, each one usually referred to as essential spectrum. The standard consequences of Oka-analyticity (we refer to [1], [11], [12] for precise statements and proofs) thus extend to them: for instance, several functions of each essential spectrum (such as the radius, the inverse of the distance from a fixed point, the  $k$ -th diameter, for any  $k \in \mathbf{N}$ , the capacity, etc.) are plurisubharmonic on  $\mathfrak{B}(X)$  (or  $\mathfrak{C}(X)$ ). Also, we have the analyticity of spectral sets, the finite scarcity and countable scarcity theorems; and many others.

## 2. Essential spectrum.

A linear operator  $T \in \mathfrak{C}(X, Y)$  will be said to be *Fredholm* if its range  $R(T)$  is closed, and if the dimensions of its kernel  $N(T)$  and of its co-kernel  $Y/R(T)$  are finite: such dimensions will be called *nullity* and *deficiency* of  $T$ , resp., and denoted by  $\text{nul}(T)$  and  $\text{def}(T)$ ; while the *index* of  $T$  will be  $\text{ind}(T) = \text{nul}(T) - \text{def}(T)$ . If  $\mathfrak{C}(X, Y)$  is endowed with the «gap» (metrizable) topology (see [4, in Section IV.2.4 p. 201-202]), which induces the norm topology on the open set  $\mathfrak{B}(X, Y)$ , then  $\mathcal{F}(X, Y) = \{T \in \mathfrak{C}(X, Y) \text{ that are Fredholm}\}$  is open in  $\mathfrak{C}(X, Y)$ , and on each of its connected components the function  $\text{ind}$  is constant, while  $\text{nul}$  and  $\text{def}$  are, in general, just u.s.c. (cf. [4, Theorem IV.5.17 p. 235]). By  $\mathcal{F}_0(X, Y)$  we shall denote the union of the components of  $\mathcal{F}(X, Y)$  where  $\text{ind} \equiv 0$ ; and by  $\mathcal{K}(X, Y)$  the set of compact operators from  $X$  to  $Y$ .

Each of the following sets is customarily referred to as essential spectrum of  $T \in \mathfrak{C}(X)$  (cf. [7, p. 365; 13, § 1 p. 142; 2, Definition 11 p. 107]):

a) the *Wolf spectrum*

$$\sigma_{ew}(T) = \{z \in \mathbf{C}: T - zI \notin \mathcal{F}(X) = \mathcal{F}(X, X)\};$$

b) the *Weyl spectrum*  $\sigma_{em}(T) = \bigcap_{K \in \mathcal{K}(X) = \mathcal{K}(X, X)} \sigma(T + K)$ ;

c) the *Browder spectrum*  $\sigma_{eb}(T) = \left\{ z \in \mathbf{C}: z \text{ is an accumulation point of } \sigma(T), \text{ or } R(T - zI) \text{ is not closed, or } z \text{ is an eigenvalue of } T \text{ of infinite algebraic multiplicity (i.e. } \dim \left( \bigcup_{k=1, \dots, \infty} N(T - zI)^k \right) = \infty \right\}$ .

Thus the Weyl spectrum is the largest subset of the spectrum which is invariant under compact perturbations: furthermore (see [8, Theorem VII.5.4. p. 180])

$$\sigma_{em}(T) = \{z \in \mathbf{C}: T - zI \notin \mathcal{F}_0(X) = \mathcal{F}_0(X, X)\}.$$

If  $\text{Cal}(X) = \mathcal{B}(X)/\mathcal{K}(X)$  is the Calkin algebra of  $X$ , then (see [8]) the Wolf spectrum of  $T \in \mathcal{B}(X)$  coincides with the (Banach algebra) spectrum  $\sigma_{\text{Cal}(X)}$  in  $\text{Cal}(X)$  of the coset  $[T]_{\text{Cal}(X)}$  of  $T$ .

Fixed  $T \in \mathcal{C}(X)$ , due to forementioned properties the function  $z \mapsto \text{ind}(T - zI)$  is constant on each component  $W$  of the complement of  $\sigma_{ew}(T)$ : but we also have that  $z \mapsto \text{nul}(T - zI)$ ,  $\text{def}(T - zI)$  are constant on  $W$ , except for a discrete subset of  $W$  (see [4, Theorem IV.5.31 p. 41]). This implies that  $\sigma_{ew}(T)$  is the complement of the union of those  $W$ 's which are not contained in  $\sigma(T)$ , viz. such that  $W \ni z \mapsto \text{nul}(T - zI) \equiv \text{def}(T - zI) \equiv 0$  « a.e. » (see [2]).

Thanks to the various observations made so far, we have  $\sigma_{ew}(T) \subset \sigma_{em}(T) \subset \sigma_{eb}(T) \subset \sigma(T)$  (all closed subsets), and each inclusion may be strict, even if  $T \in \mathcal{B}(X)$ . (If  $X$  is finite-dimensional, all the essential spectra are obviously empty; while if it is Hilbert and  $T$  is self-adjoint, then they coincide.)

**THEOREM 1.** *The Wolf spectrum is Oka-analytic on  $\mathcal{C}(X)$ .*

**PROOF.** The upper semicontinuity of  $\sigma_{ew}$  easily follows from the openness of  $\mathcal{F}(X)$  in  $\mathcal{C}(X)$ .

Let  $\text{Cal}(X, Y) = \mathcal{B}(X, Y)/\mathcal{K}(X, Y)$  as a Banach space: the product of composition  $\mathcal{B}(X, Y) \times \mathcal{B}(Y, X) \rightarrow \mathcal{B}(Y)$  induces a continuous bilinear product  $\text{Cal}(X, Y) \times \text{Cal}(Y, X) \rightarrow \text{Cal}(Y)$ . As in the case  $X = Y$ , one shows that  $T \in \mathcal{B}(X, Y)$  is Fredholm if and only if its coset  $[T]_{\text{Cal}(X, Y)}$  has a two-sided inverse in  $\text{Cal}(Y, X)$ : if so, such inverse is unique by the associativity of the product, and (just like when  $X = Y$ ) it depends continuously and holomorphically on  $[T]_{\text{Cal}(X, Y)}$ , so on  $T$ .

The notion of analytic multifunction being obviously local, we can assume the holomorphic families  $\lambda \mapsto U_\lambda, V_\lambda$  of operators in  $\mathbf{B}(Y, X)$  relative to  $\lambda \mapsto T_\lambda$  (see introduction) to be defined on the whole of  $G$  ( $Y$  being a suitable Banach space). If  $(\lambda, z) \in G \times \mathbf{C}$ , then  $T_\lambda - zI = (V_\lambda - zU_\lambda)U_\lambda^{-1}$ , and the range, nullity and deficiency of  $T_\lambda - zI$

are the same, resp., of  $V_\lambda - zU_\lambda$ ; so

$$\begin{aligned} \Omega_{ew} &= \{(\lambda, z) \in G \times \mathbf{C} : z \notin \sigma_{ew}(T_\lambda)\} = \\ &= \{(\lambda, z) \in G \times \mathbf{C} : V_\lambda - zU_\lambda \in \mathcal{F}(Y, X)\} = \\ &= \{(\lambda, z) \in G \times \mathbf{C} : [V_\lambda - zU_\lambda]_{\text{Cal}(Y, X)} \text{ has a two-sided inverse in } \text{Cal}(X, Y)\}. \end{aligned}$$

In view of the remarks made above, the conclusion can be drawn exactly as in [3, Proof of the Theorem, p. 1] using the nonextendability to  $(G \times \mathbf{C}) \cap \partial\Omega$  of any restriction of the holomorphic mapping

$$\lambda \mapsto ([V_\lambda - zU_\lambda]_{\text{Cal}(Y, X)})^{-1} : \Omega \rightarrow \text{Cal}(X, Y). \quad \square$$

(Examination of  $V_\lambda - zU_\lambda$  instead of  $(V_\lambda - zU_\lambda)U_\lambda^{-1}$ , as made in the preceding proof, allows a quicker proof of the Oka-analyticity of the spectrum on  $\mathbf{C}(X)$  than [10, Proof of Theorem 1 p. 123].)

**THEOREM 2.** *The Weyl spectrum is Oka-analytic on  $\mathbf{C}(X)$ .*

**PROOF.** Since Kato-holomorphic families are continuous (by [4, Theorem IV.2.29 p. 207]), if  $\varphi: G \times \mathbf{C} \rightarrow \mathcal{B}(X)$  is given by  $\varphi(\lambda, z) = T_\lambda - zI$ , then the open set  $\Omega_{em} = \{(\lambda, z) \in G \times \mathbf{C} : z \notin \sigma_{em}(T_\lambda)\} = \varphi^{-1}(\mathcal{F}_o(X))$  is a union of components of the open set  $\Omega_{ew} = \varphi^{-1}(\mathcal{F}(X))$ , which is pseudoconvex by Theorem 1.  $\square$

As to the Browder spectrum, we cannot infer its Oka-analyticity directly from that of the Wolf or the Weyl spectrum as done for Theorem 2. In fact  $\Omega_{eb} = \{(\lambda, z) \in G \times \mathbf{C} : z \notin \sigma_{eb}(T_\lambda)\}$  is not, in general, a union of components of  $\Omega_{em}$ , because of the lack of lower semicontinuity of  $\text{nul}$ ,  $\text{def}$ . Yet one could conjecture, in view of some of the properties recalled earlier, that the functions  $\text{nul}'$ ,  $\text{def}'$  are locally constant on  $\mathcal{F}_o(X)$ , where

$$\text{nul}'(T) = \text{nul}(T - zI), \quad \text{def}'(T) = \text{def}(T - zI), \quad \text{for } 0 < |z| \ll 1.$$

This conjecture is true only when  $X$  is finite-dimensional (in which case  $\text{nul}'$ ,  $\text{def}'$  vanish identically), as the following counter-example shows.

**COUNTEREXAMPLE 3.** Let:  $X$  be the Hilbert space  $l^2 = L^2(\mathbf{Z}, \nu)$  (where  $\nu$  is the counting measure);  $P, A \in \mathcal{B}(X)$  the projection on the

zeroth coordinate and the onestep shift to the right, resp.;  $G = \mathbf{C}$ ;  $\lambda \mapsto T_\lambda = A(I - \lambda P): G \rightarrow \mathcal{B}(X)$ . If  $\{e_j\}_{j \in \mathbf{Z}}$  is the canonical basis of  $l^2$ , then  $N(T_1) = [e_0]$ ,  $R(T_1) = [e_1]^\perp$ : so  $T_1 \in \mathcal{F}_0(X)$ , therefore  $T_\lambda - zI \in \mathcal{F}_0(X)$  for  $|\lambda - 1| \ll 1$ ,  $|z| \ll 1$ . But  $N(T_1 - zI) = [\sum_{j=0, \dots, +\infty} z^j e_{-j}]$  for  $|z| \ll 1$ , so  $\text{nul}'(T_1) = \text{def}'(T_1) = 1$ ; while  $T_\lambda$  is invertible for  $0 < |\lambda - 1| \ll 1$ , thus  $\text{nul}'(T_\lambda) = \text{def}'(T_\lambda) = 0$ . Hence  $(1, 0) \in \Omega_{em} \setminus \Omega_{eb}$ , whereas  $(\lambda, 0) \in \Omega_{eb}$  if  $0 < |\lambda - 1| \ll 1$ .

The upper semicontinuity of the Browder spectrum is interesting in its own, so we give it separately.

LEMMA 4. *The Browder spectrum is u.s.c. on  $\mathcal{C}(X)$ .*

PROOF. Let  $T \in \mathcal{C}(X)$ . Then  $\sigma(T) \setminus \sigma_{eb}(T)$  consists only of isolated points of  $\sigma(T)$ : if  $A$  is a compact-complemented open subset of  $\mathbf{C}$  containing  $\sigma_{eb}(T)$ , then  $\sigma(T) \setminus A = \{z_1, \dots, z_k\}$ . Since  $\sigma$  is u.s.c., for  $T' \in \mathcal{C}(X)$  near  $T$  we have  $\sigma(T') \subset A \cup (\bigcup_{j=1, \dots, k} B(z_j, \varepsilon))$  (where  $B(z_j, \varepsilon) = \{z \in \mathbf{C}: |z - z_j| < \varepsilon\}$ ), with  $\varepsilon > 0$  such that the above union is disjoint. Using the Dunford integral calculus, for every such  $T'$  [4, Theorem IV.3.16 p. 212] provides a splitting of  $X$  into a  $(k + 1)$ -ple direct sum  $X = \bigoplus_{j=0, \dots, k} X_j(T')$  of the generalized eigenspaces associated to  $\sigma(T') \cap A$ ,  $\sigma(T') \cap B(z_j, \varepsilon)$ ,  $j = 1, \dots, k$ ; furthermore  $\dim X_j(T')$  is independent of  $T'$ . Fix  $j = 1, \dots, k$ . Since the restriction of  $T - z_j I$  to  $X_j(T)$  is a quasinilpotent operator, and its approximated nullity and deficiency (defined equal to the nullity and deficiency, resp., if the range is closed, or to  $+\infty$  otherwise: cf. [4, Theorem IV.5.10 p. 233]) are finite (because  $T - z_j I$  is Fredholm), [4, Theorem IV.5.30 p. 240] yields that  $\dim X_j(T)$  is finite. Hence  $\sigma(T') \cap B(z_j, \varepsilon)$  is a finite set  $\{z_{j1}, \dots, z_{jk_j}\}$ , and for  $j' = 1, \dots, k_j$  the range of  $T' - z_{jj'} I$  is closed and the algebraic multiplicity of  $z_{jj'}$  in  $T'$  is finite; that is,  $\sigma_{eb}(T') \cap B(z_j, \varepsilon)$  is empty. Therefore  $\sigma_{eb}(T') \subset A$ .  $\square$

Let  $\lambda \mapsto \Sigma_\lambda: G \rightarrow \text{cl}(\mathbf{C})$  be an Oka-analytic multifunction, and  $\lambda \in G$ . An isolated point  $z$  of  $\Sigma_\lambda$  is a *good isolated point*, or *g.i.p.*, (for  $\Sigma$ ) at  $\lambda$  if there exists  $\delta > 0$  such that  $\sigma(T_{\lambda'}) \cap B(z, \delta)$  is finite for  $|\lambda' - \lambda| < \delta$ . The Oka-Nishino theorem (cf. [6, Corollary 5.5 p. 557]) asserts that the multifunction  $\lambda \mapsto D\Sigma'_\lambda = \Sigma_\lambda \setminus \{\text{g.i.p.'s at } \lambda\}$  is itself Oka-analytic.

THEOREM 5. *The Browder spectrum is Oka-analytic on  $\mathcal{C}(X)$ .*

PROOF. Let  $\lambda \in G$ . From the proof of Lemma 4 we get that all the points in  $\sigma(T_\lambda) \setminus \sigma_{\text{ob}}(T_\lambda)$  are g.i.p.'s at  $\lambda$ , but in general not conversely: for instance, if  $T_\lambda \equiv 0$  on  $G$ , then 0 is a g.i.p. at any  $\lambda \in G$ , but belongs to  $\sigma_{\text{ob}}(T_\lambda)$  if  $X$  is infinite-dimensional. Thus we have  $D\sigma(T_\lambda) \subset \sigma_{\text{ob}}(T_\lambda) \subset \sigma(T_\lambda)$  for each  $\lambda \in G$ , the first and third multifunction being Oka-analytic. In order to prove that  $\Omega_{\text{ob}}$  is pseudoconvex it will suffice to show that for each  $\lambda_0 \in G$  and  $z_0 \in \sigma_{\text{ob}}(T_{\lambda_0}) \setminus D\sigma(T_{\lambda_0})$  there exists a neighborhood  $U$  of  $(z_0, \lambda_0)$  in  $D\Omega = \{(\lambda, z) \in G \times \mathbf{C} : z \notin D\sigma(T_\lambda)\}$  such that  $U \cap \Omega_{\text{ob}}$  is pseudoconvex. Because  $D\Omega$  is open,  $U$  can be taken to be a bidisk; also, we will assume  $\lambda_0 = z_0 = 0$ .

Since 0 is isolated in  $\sigma(T_0)$  and  $\Omega$  is open we can choose  $\varepsilon > \delta > 0$  so that  $\sigma(T_\lambda) \cap B(0, \varepsilon) = \sigma(T_\lambda) \cap B(0, \varepsilon - \delta)$  for  $|\lambda| < \delta$ : so  $\lambda \mapsto \Sigma_\lambda = \sigma(T_\lambda) \cap B(0, \varepsilon) : B(0, \delta) \rightarrow \text{cl}(\mathbf{C})$  is still Oka-analytic. Because 0 is a g.i.p. at 0, by [1, Theorem 3.8]  $\delta$  may be taken small enough that the cardinality of  $\Sigma_\lambda$  be finite and independent of  $\lambda \in B(0, \delta) \setminus \{0\}$ , say  $k$ . Furthermore for any such  $\lambda$  there exists  $\delta_\lambda > 0$  and  $k$  holomorphic functions  $h_1, \dots, h_k : B(\lambda, \delta_\lambda) \rightarrow \mathbf{C}$  such that  $\Sigma_{\lambda'} = \{h_1(\lambda'), \dots, h_k(\lambda')\}$  for each  $\lambda' \in B(\lambda, \delta_\lambda)$ . As in the proof of Lemma 4 we have that  $h_j(\lambda') \in \sigma_{\text{ob}}(T_{\lambda'})$  if and only if the dimension of the generalized eigenspace of  $T_{\lambda'}$  associated to  $h_j(\lambda')$  is finite; such dimension being stable, either  $h_j(\lambda') \in \sigma_{\text{ob}}(T_{\lambda'})$  for all  $\lambda' \in B(\lambda, \delta_\lambda)$ , or for none of them. Therefore  $k_0$  exists, with  $1 \leq k_0 \leq k$ , such that for each  $\lambda \in B(0, \delta) \setminus \{0\}$  the function  $h_j$  can be rearranged in such a way that the former alternative holds for exactly  $h_1, \dots, h_{k_0}$ . If  $f : U = B(0, \delta) \times B(0, \varepsilon) \rightarrow \mathbf{C}$  is defined through  $f(\lambda, z) = \prod_{j=1, \dots, k_0} (z - h_j(\lambda))$  for  $\lambda \neq 0$ , and  $f(0, z) = z^{k_0}$ , such  $f$  is well-defined, holomorphic where  $\lambda \neq 0$ , and (by the upper semicontinuity of  $\lambda \mapsto \Sigma_\lambda$ , which is implied by that of  $\lambda \mapsto \sigma(T_\lambda)$ ) continuous where  $\lambda = 0$ . By Raddò's theorem the function  $f$  is holomorphic on  $U$ : since  $\{f = 0\} = U \setminus \Omega_{\text{ob}}$ , no restriction of  $1/f : U \cap \Omega_{\text{ob}} \rightarrow \mathbf{C}$  can be extended to any point of  $U \setminus \Omega_{\text{ob}}$ . Thus  $U \cap \Omega_{\text{ob}}$  is pseudoconvex, because  $U$  is.  $\square$

The following corollary of the three preceding theorems appeared in [5, Theorem 13 p. 320] for the Weyl case, while it can be easily proven directly in the Wolf case using the Oka-analyticity of the spectrum on a Banach algebra (Cal( $X$ ) here: see [1, Theorem 3.2 p. 46]).

COROLLARY 6. *The Wolf, Weyl, and Browder spectrum are all Oka-analytic on  $\mathcal{B}(X)$ .*  $\square$

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Manoscritto pervenuto in redazione il 14 Marzo 1987.