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Riemannian geometry of noncommutative surfaces

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A Riemannian geometry of noncommutative n-dimensional surfaces is developed as a first step toward the construction of a consistent noncommutative gravitational theory. Historically, as well, Riemannian geometry was recognized to be the underlying structure of Einstein's theory of general relativity and led to further developments of the latter. The notions of metric and connections on such noncommutative surfaces are introduced, and it is shown that the connections are metric compatible, giving rise to the corresponding Riemann curvature. The latter also satisfies the noncommutative analog of the first and second Bianchi identities. As examples, noncommutative analogs of the sphere, torus, and hyperboloid are studied in detail. The problem of covariance under appropriately defined general coordinate transformations is also discussed and commented on as compared to other treatments.
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I. INTRODUCTION

In recent years there has been much progress in developing theories of noncommutative geometry and exploring their applications in physics. Many viewpoints were adopted and different mathematical approaches were followed by different researchers. Connes' theory[1] (see also Ref. 2) formulated within the framework of C^*-algebras is the most successful, which incorporates cyclic cohomology and K-theory and gives rise to noncommutative versions of index theorems. Theories generalizing aspects of algebraic geometry were also developed (see, e.g., Ref. 3, for a review and references). A notion of noncommutative schemes was formulated, which seems to provide a useful framework for developing noncommutative algebraic geometry.

A major advance in theoretical physics in recent years was the deformation quantization of Poisson manifolds by Kontsevich (see Ref. 4, for the final form of this work). This sparked intensive activities investigating applications of noncommutative geometries to quantum theory. Seiberg and Witten[5] showed that the antisymmetric tensor field arising from massless states of strings can be described by the noncommutativity of a space-time,

$$[x^\mu, x^\nu]_* = i\theta^{\mu\nu}, \theta^{\mu\nu} \text{ constant matrix}, \tag{1.1}$$

where the multiplication of the algebra of functions is governed by the Moyal product

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$$(f * g)(x) = f(x)\exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_{\mu}\overrightarrow{\partial}_{\nu}\right)g(x). \tag{1.2}$$

A considerable amount of research was done both prior and after Ref. 5, and we refer to Refs. 6 and 7 for reviews and references.

An earlier and independent work is a seminal paper by Doplicher *et al.*,[8] which laid down the fundamentals of quantum field theory on noncommutative space-time. These authors started with a theoretical examination of the long held belief by the physics community that the usual notion of space-time needed to be modified at the Planck scale and convincingly demonstrated that space-time becomes noncommutative in that the coordinates describing space-time points become operators similar to those in quantum mechanics. Therefore, noncommutative geometry is indeed a way to describe physics at Planck scale.

A consistent formulation of a noncommutative version of general relativity could give insight into a gravitational theory compatible with quantum mechanics. A unification of general relativity with quantum mechanics has long been sought after but remains as elusive as ever despite the extraordinary progress in string theory for the last two decades. The noncommutative geometrical approach to gravity could provide an alternative route. Much work has already been done in this general direction, see, e.g., Refs. 9–15 and references therein. In particular, different forms of noncommutative Riemannian geometries were proposed,[9,10,14,15] which retain some of the familiar geometric notions such as metric and curvature. Noncommutative analogs of the Hilbert–Einstein action were also suggested[9,10,12,13] by treating noncommutative gravity as gauge theories.

The noncommutative space-time with the Heisenberg-like commutation relation (1.1) violates Lorentz symmetry but was show[16,17] to have a quantum symmetry under the twisted Poincaré algebra. The Abelian twist element

$$\mathcal{F} = \exp\left(-\frac{i}{2}\theta^{\mu\nu}\partial_{\mu}\otimes\partial_{\nu}\right) \tag{1.3}$$

was used in Refs. 16 and 17 to twist the universal enveloping algebra of the Poincaré algebra, obtaining a noncommutative multiplication for the algebra of functions on the Poincaré group closely related to the Moyal product (1.2). It is then natural to try to extend the procedure to other symmetries of noncommutative field theory and investigate whether the concept of twist provides a new symmetry principle for noncommutative space-time.

The same Abelian twist element (1.3) was used in Refs. 9 and 10 for deforming the algebra of diffeomorphisms when attempting to obtain general coordinate transformations on the noncommutative space-time. It is interesting that Refs. 9 and 10 proposed gravitational theories which are different from the low energy limit of strings.[18] However, based on physical arguments one would expect the Moyal product to be frame dependent and transform under the general coordinate transformation. If the twist element is chosen as (1.3), the Moyal $*$-product is fixed once for all. This is likely to lead to problems similar to those observed in Ref. 19 when one attempted to deform the internal gauge transformations with the same twist element (1.3). Nevertheless twisting is expected to be a productive approach to the formulation of a noncommutative gravitational theory when implemented consistently. A "covariant twist" was proposed for internal gauge transformations in Ref. 20, but it turned out that the corresponding star product would not be associative.

Works on the noncommutative geometrical approach to gravity may be broadly divided into two types. One type attempts to develop noncommutative versions of Riemannian geometry axiomatically (that is, formally), while the other adapts general relativity to the noncommutative setting in an intuitive way. The problem is the lack of any safeguard against mathematical inconsistency in the latter type of works, and the same problem persists in the first type of works as well, since it is not clear whether nontrivial examples exist which satisfy all the axioms of the formally defined theories.

The aim of the present paper is to develop a theory of noncommutative Riemannian geometry by extracting an axiomatic framework from highly nontrivial and transparently consistent examples. Our approach is mathematically different from that of Refs. 9 and 10 and also quite far removed from the quantum group theoretical noncommutative Riemannian geometry[15] (see also references in Ref. 15 and subsequent publications by the same author).

Recall that the two-dimensional surfaces embedded in the Euclidean three-space provide the simplest yet nontrivial examples of Riemannian geometry. The Euclidean metric of the three-space induces a natural metric for a surface through the embedding; the Levi–Civita connection and the curvature of the tangent bundle of the surface can thus be described explicitly (for the theory of surfaces, see, e.g., the textbook Ref. 21). As a matter of fact, Riemannian geometry originated from Gauss' work on surfaces embedded in three-dimensional Euclidean space.

More generally, Whitney's theorem enables the embedding of any smooth Riemannian manifold as a high dimensional surface in a flat Euclidean space of high enough dimension (see, e.g., Theorems 9 and 11.1.1 in Ref. 22). The embedding also allows transparent construction and interpretation of all structures related to the Riemannian metric of M as in the two-dimensional case.

This paper develops noncommutative deformations of Riemannian geometry in the light of Whitney's theorem. The first step is to deform the algebra of functions on a domain of the Euclidean space. We begin Sec. II by introducing the Moyal algebra $\mathcal{A}$, which is a noncommutative deformation[23] of the algebra of smooth functions on a region of $\mathbb{R}^2$. The rest of Sec. II develops a noncommutative Riemannian geometry for noncommutative analogs of two-dimensional surfaces embedded in three-space. Working over the Moyal algebra $\mathcal{A}$, we show that much of the classical differential geometry for surfaces generalizes *naturally* to this noncommutative setting. In Sec. III, three illuminating examples are constructed, which are, respectively, noncommutative analogs of the sphere, torus, and hyperboloid. Their noncommutative geometries are studied in detail.

We emphasize that the embeddings play a crucial role in our current understanding of the geometry of the two-dimensional noncommutative surfaces. The metric of a noncommutative surface is constructed in terms of the embedding; the necessity of a left connection and also a right connection then naturally arises; even the definition of the curvature tensor is forced upon us by the context. Indeed, the extra information obtained by considering embeddings provides the guiding principles, which are lacking up to now, for building a theory of noncommutative Riemannian geometry.

Once the noncommutative Riemannian geometry of the two-dimensional surfaces is sorted out, its generalization to the noncommutative geometries corresponding to n-dimensional surfaces embedded in spaces of higher dimensions is straightforward. This is discussed in Sec. VI.

Recall that the basic principle of general relativity is general covariance. We study in Sec. V the general coordinate transformations for noncommutative surfaces, which are brought about by gauge transformations on the underlying noncommutative associative algebra $\mathcal{A}$ (over which noncommutative geometry is constructed). A new feature here is that the general coordinate transformations affect the multiplication of the underlying associative algebra $\mathcal{A}$ as well, turning it into another algebra nontrivially isomorphic to $\mathcal{A}$. We make comparison with classical Riemannian geometry, showing that the gauge transformations should be considered as noncommutative analogs of diffeomorphisms.

The theory of surfaces developed over the deformation of the algebra of smooth functions on some region in $\mathbb{R}^n$ now suggests a general theory of noncommutative Riemannian geometry of n-dimensional surfaces over arbitrary unital associative algebras with derivations. We present an outline of this general theory in Sec. IV.

We conclude this section with a remark on the presentation of the paper. As indicated above, we start from the simplest nontrivial examples of noncommutative Riemannian geometries and gradually extend the results to build up a theory of generality. This "experimental approach" is not the optimal format for presenting mathematics, as all special cases repeat the same pattern. However, it has the distinctive advantage that the general theory obtained in this way stands on firm ground.

II. NONCOMMUTATIVE SURFACES AND THEIR EMBEDDINGS

The first step in constructing noncommutative deformations of Riemannian geometry is the deformation of algebras of functions. Let us fix a region U in $\mathbb{R}^2$ and write the coordinate of a point t in U as (t_1, t_2). Let $\bar{h}$ be a real indeterminate, and denote by $\mathbb{R}[[\bar{h}]]$ the ring of formal power series in $\bar{h}$. Let $\mathcal{A}$ be the set of the formal power series in $\bar{h}$ with coefficients being real smooth functions on U. Namely, every element of $\mathcal{A}$ is of the form $\Sigma_{i\geq 0} f_i \bar{h}^i$, where f_i are smooth functions on U. Then $\mathcal{A}$ is an $\mathbb{R}[[\bar{h}]]$-module in an obvious way.

Given any two smooth functions f and g on U, we denote by fg the usual pointwise product of the two functions. We also define their star product (or more precisely, Moyal product) by

$$f * g = \lim_{t'\to t} \exp\left[\bar{h}\left(\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2'} - \frac{\partial}{\partial t_2}\frac{\partial}{\partial t_1'}\right)\right] f(t) g(t'), \tag{2.1}$$

where the exponential $\exp[\bar{h}(\partial/\partial t_1 \;\; \partial/\partial t_2' - \partial/\partial t_2 \;\; \partial/\partial t_1')]$ is to be understood as a power series in the differential operator $\partial/\partial t_1 \;\; \partial/\partial t_2' - \partial/\partial t_2 \;\; \partial/\partial t_1'$. More explicitly, let

$$\mu_p : \mathcal{A}/\bar{h}\mathcal{A} \otimes \mathcal{A}/\bar{h}\mathcal{A} \to \mathcal{A}/\bar{h}\mathcal{A}, \quad p = 0,1,2,\ldots \tag{2.2}$$

be $\mathbb{R}$-linear maps defined by

$$\mu_p(f,g) = \lim_{t'\to t} \frac{1}{p!}\left(\frac{\partial}{\partial t_1}\frac{\partial}{\partial t_2'} - \frac{\partial}{\partial t_2}\frac{\partial}{\partial t_1'}\right)^p f(t) g(t').$$

Then $f*g = \Sigma_{p=0}^{\infty} \bar{h}^p \mu_p(f,g)$. It is evident that $f*g$ lies in $\mathcal{A}$. We extend this star product $\mathbb{R}[[\bar{h}]]$ linearly to all elements in $\mathcal{A}$ by letting

$$\left(\sum f_i \bar{h}^i\right) * \left(\sum g_j \bar{h}^j\right) := \sum f_i * g_j \bar{h}^{i+j}.$$

It has been known since the early days of quantum mechanics that the Moyal product is associative (see, e.g., Ref. 4, for a reference), thus we arrive at an associative algebra over $\mathbb{R}[[\bar{h}]]$, which is a deformation[23] of the algebra of smooth functions on U. We shall usually denote this associative algebra by $\mathcal{A}$, but when it is necessary to make explicit the multiplication of the algebra, we shall write it as $(\mathcal{A}, *)$.

Remark 2.1: For the sake of being explicit, we restrict ourselves to consider the Moyal product [defined by (2.1)] only in this section. As we shall see in Secs. IV and V, the theory of noncommutative surfaces to be developed in this paper extends to more general star products over algebras of smooth functions.

Write ∂_i for $\partial/\partial t_i$ and extend $\mathbb{R}[[\bar{h}]]$-linearly the operators ∂_i to $\mathcal{A}$. One can easily verify that for smooth functions f and g,

$$\partial_i(f * g) = (\partial_i f) * g + f * (\partial_i g), \tag{2.3}$$

that is, the operators ∂_i are derivations of the algebra $\mathcal{A}$.

Let $\mathcal{A}^3 = \mathcal{A} \oplus \mathcal{A} \oplus \mathcal{A}$. There is a natural two-sided $\mathcal{A}$-module structure on $\mathcal{A}^3 \otimes_{\mathbb{R}[[\bar{h}]]} \mathcal{A}^3$, defined for all $a, b \in \mathcal{A}$, and $X \otimes Y \in \mathcal{A}^3 \otimes_{\mathbb{R}[[\bar{h}]]} \mathcal{A}^3$ by $a*(X\otimes Y)*b = a*X\otimes Y*b$. Define the map

$$\mathcal{A}^3 \otimes_{\mathbb{R}[[\bar{h}]]} \mathcal{A}^3 \to \mathcal{A}, \quad (a,b,c) \otimes (f,g,h) \mapsto a*f + b*g + c*h, \tag{2.4}$$

and denote it by $\bullet$. This is a map of two-sided $\mathcal{A}$-modules in the sense that for any $X, Y \in \mathcal{A}^3$ and $a, b \in \mathcal{A}$, $(a*X)\bullet(Y*b) = a*(X\bullet Y)*b$. We shall refer to this map as the *dot product*.

Let $X = (X^1, X^2, X^3)$ be an element of $\mathcal{A}^3$, where the superscripts of X^1, X^2, and X^3 are not powers but are indices used to label the components of a vector as in the usual convention in differential geometry. We set $\partial_i X = (\partial_i X^1, \partial_i X^2, \partial_i X^3)$ and define the following 2×2-matrix over $\mathcal{A}$,

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}, \quad g_{ij} = \partial_i X \bullet \partial_j X. \tag{2.5}$$

Let $g^0 = g \bmod \bar{h}$, which is a 2×2-matrix of smooth functions on U.

Definition 2.2: We call an element $X \in \mathcal{A}^3$ (the *noncommutative embedding* in $\mathcal{A}^3$ of) a *noncommutative surface* if g^0 is invertible for all $t \in U$. In this case, we call g the *metric* of the noncommutative surface.

Given a noncommutative surface X with a metric g, there exists a unique 2×2-matrix (g^{ij}) over $\mathcal{A}$ which is the right inverse of g, i.e.,

$$g_{ij} * g^{jk} = \delta_i^k,$$

where we have used Einstein's convention of summing over repeated indices. To see the existence of the right inverse, we write $g_{ij} = \Sigma_p \bar{h}^p g_{ij}[p]$ and $g^{ij} = \Sigma_p \bar{h}^p \tilde{g}^{ij}[p]$, where $(g^{ij}[0])$ is the inverse of $(g_{ij}[0])$. Now in terms of the maps μ_k defined by (2.2), we have

$$\delta_i^k = g_{ij} * g^{jk} = \sum_q \bar{h}^q \sum_{m+n+p=q} \mu_p(g_{ij}[m], g^{jk}[n]),$$

which is equivalent to

$$g^{ij}[q] = -\sum_{n=1}^{q} \sum_{m=0}^{q-n} g^{ik}[0] \mu_n(g_{kl}[m], g^{lj}[q-n-m]).$$

Since the right-hand side involves only $g^{lj}[r]$ with $r < q$, this equation gives a recursive formula for the right inverse of g.

In the same way, we can also show that there also exists a unique left inverse of g. It follows from the associativity of multiplication of matrices over any associative algebra that the left and right inverses of g are equal.

Definition 2.3: Given a noncommutative surface X, let

$$E_i = \partial_i X, \quad i = 1,2,$$

and call the left $\mathcal{A}$-module TX and right $\mathcal{A}$-module $\widetilde{T}X$ defined by

$$TX = \{a * E_1 + b * E_2 | a,b \in \mathcal{A}\}, \quad \widetilde{T}X = \{E_1 * a + E_2 * b | a,b \in \mathcal{A}\}$$

the *left* and *right tangent bundles* of the noncommutative surface, respectively.

Then $TX \otimes_{\mathbb{R}[[\bar{h}]]} \widetilde{T}X$ is a two-sided $\mathcal{A}$-module.

Proposition 2.4: The metric induces a homomorphism of two-sided $\mathcal{A}$-modules,

$$g: TX \otimes_{\mathbb{R}[[\bar{h}]]} \widetilde{T}X \to \mathcal{A},$$

defined for any $Z = z^i * E_i \in TX$ and $W = E_i * w^i \in \widetilde{T}X$ by

$$Z \otimes W \mapsto g(Z,W) = z^i * g_{ij} * w^j.$$

It is easy to see that the map is indeed a homomorphism of two-sided $\mathcal{A}$-modules, and it clearly coincides with the restriction of the dot product to $TX \otimes_{\mathbb{R}[[\bar{h}]]} \widetilde{T}X$.

Since the metric g is invertible, we can define

$$E^i = g^{ij} * E_j, \quad \widetilde{E}^i = E_j * g^{ji}, \tag{2.6}$$

which belong to TX and $\widetilde{T}X$, respectively. Then

$$g(E^i,E_j)=\delta^i_j=g(E_j,\widetilde{E}^i),\quad g(E^i,\widetilde{E}^j)=g^{ij}.$$

Now any $Y\in\mathcal{A}^3$ can be written as $Y=y^i*E_i+Y^\perp$, with $y^i=Y\bullet\widetilde{E}^i$ and $Y^\perp=Y-y^i*E_i$. We shall call y^i*E_i the *left tangential component* and $Y^\perp$ the *left normal component* of Y. Let

$$(TX)^\perp=\{N\in\mathcal{A}^3|N\bullet E_i=0,\ \forall\ i\},$$

which is clearly a left $\mathcal{A}$-submodule of $\mathcal{A}^3$. In a similar way, we may also decompose Y into $Y=E_i*\widetilde{y}^i+\widetilde{Y}^\perp$ with the *right tangential component* of Y given by $\widetilde{y}^i=E^i\bullet Y$ and the *right normal component* by $\widetilde{Y}^\perp=Y-E_i*\widetilde{y}^i$. Let

$$(\widetilde{T}X)^\perp=\{N\in\mathcal{A}^3|E_i\bullet N=0,\ \forall\ i\},$$

which is a right $\mathcal{A}$-submodule of $\mathcal{A}^3$. Therefore, we have the following decompositions:

$$\mathcal{A}^3=TX\oplus(TX)^\perp,\quad \text{as left } \mathcal{A}-\text{ module},$$

$$\mathcal{A}^3=\widetilde{T}X\oplus(\widetilde{T}X)^\perp,\quad \text{as right } \mathcal{A}-\text{module}. \tag{2.7}$$

It follows that the tangent bundles are finitely generated projective modules over $\mathcal{A}$. Following the general philosophy of noncommutative geometry,[1] we may regard finitely generated projective modules over $\mathcal{A}$ as vector bundles on the noncommutative surface. This justifies the terminology of left and right tangent bundles for TX and $\widetilde{T}X$.

In fact, TX and $\widetilde{T}X$ are free left and right $\mathcal{A}$-modules, respectively, as E_1 and E_2 form $\mathcal{A}$-bases for them. Consider TX, for example. If there exists a relation $a^i*E_i=0$, where $a^i\in\mathcal{A}$, we have $a^i*E_i\bullet E_j=a^i*g_{ij}=0$, $\forall j$. The invertibility of the metric then leads to $a^i=0$, $\forall i$. Since E_1 and E_2 generate TX, they indeed form an $\mathcal{A}$-basis of TX.

One can introduce connections to the tangent bundles by following the standard procedure in the theory of surfaces.[21]

Definition 2.5: Define operators

$$\nabla_i:TX\to TX,\quad i=1,2,$$

by requiring that ∇_iZ be equal to the left tangential component of ∂_iZ for all $Z\in TX$. Similarly define

$$\widetilde{\nabla}_i:\widetilde{T}X\to\widetilde{T}X,\quad i=1,2,$$

by requiring that $\widetilde{\nabla}_i\widetilde{Z}$ be equal to the right tangential component of $\partial_i\widetilde{Z}$ for all $\widetilde{Z}\in\widetilde{T}X$. Call the set consisting of the operators ∇_i ($\widetilde{\nabla}_i$) a *connection* on TX ($\widetilde{T}X$).

The following result justifies the terminology.

Lemma 2.6: For all $Z\in TX$, $W\in\widetilde{T}X$ and $f\in\mathcal{A}$,

$$\nabla_i(f*Z)=\partial_if*Z+f*\nabla_iZ,\quad \widetilde{\nabla}_i(W*f)=W*\partial_if+\widetilde{\nabla}_iW*f. \tag{2.8}$$

Proof: By the Leibniz rule (2.3) for ∂_i,

$$\partial_i(f*Z)=(\partial_if)*Z+f*(\partial_iZ),\quad \partial_i(W*f)=W*(\partial_if)+W*(\partial_if).$$

The lemma immediately follows from the tangential components of these relations under the decompositions (2.7). □

In order to describe the connections more explicitly, we note that there exist Γ^k_{ij} and $\widetilde{\Gamma}^k_{ij}$ in $\mathcal{A}$ such that

$$\nabla_i E_j = \Gamma_{ij}^k * E_k, \quad \tilde{\nabla}_i E_j = E_k * \tilde{\Gamma}_{ij}^k. \tag{2.9}$$

Because the metric is invertible, the elements Γ_{ij}^k and $\tilde{\Gamma}_{ij}^k$ are uniquely defined by Eq. (2.9). We have

$$\Gamma_{ij}^k = \partial_i E_j \bullet E^k, \quad \tilde{\Gamma}_{ij}^k = E^k \bullet \partial_i E_j. \tag{2.10}$$

It is evident that Γ_{ij}^k and $\tilde{\Gamma}_{ij}^k$ are symmetric in the indices i and j. The following closely related objects will also be useful later:

$$\Gamma_{ijk} = \partial_i E_j \bullet E_k, \quad \tilde{\Gamma}_{ijk} = E_k \bullet \partial_i E_j.$$

In contrast to the commutative case, Γ_{ij}^k and $\tilde{\Gamma}_{ij}^k$ do not coincide, in general. We have

$$\Gamma_{ij}^k = {}_c\Gamma_{ijl} * g^{lk} + \Upsilon_{ijl} * g^{lk}, \quad \tilde{\Gamma}_{ij}^k = g^{kl} * {}_c\Gamma_{ijl} - g^{kl} * \Upsilon_{ijl},$$

where

$${}_c\Gamma_{ijl} = \tfrac{1}{2}(\partial_i g_{jl} + \partial_j g_{li} - \partial_l g_{ji}),$$

$$\Upsilon_{ijl} = \tfrac{1}{2}(\partial_i E_j \bullet E_l - E_l \bullet \partial_i E_j).$$

We shall call Υ_{ijl} the *noncommutative torsion* of the noncommutative surface. Therefore the left and right connections involve two parts. The part ${}_c\Gamma_{ijl}$ depends on the metric only, while the noncommutative torsion embodies extra information. For a noncommutative surface embedded in $\mathcal{A}^3$, the noncommutative torsion depends explicitly on the embedding. In the classical limit with $\bar{h}=0$, Υ_{ij}^k vanishes and both Γ_{ij}^k and $\tilde{\Gamma}_{ij}^k$ reduce to the standard Levi–Civita connection.

We have the following result.

Proposition 2.7: The connections are metric compatible in the following sense:

$$\partial_i g(Z,\tilde{Z}) = g(\nabla_i Z,\tilde{Z}) + g(Z,\tilde{\nabla}_i \tilde{Z}), \quad \forall\, Z \in TX, \tilde{Z} \in \tilde{T}X. \tag{2.11}$$

This is equivalent to the fact that

$$\partial_i g_{jk} - \Gamma_{ijk} - \tilde{\Gamma}_{ikj} = 0. \tag{2.12}$$

Proof: Since g is a map of two-sided $\mathcal{A}$-modules, it suffices to prove (2.11) by verifying the special case with $Z=E_j$ and $\tilde{Z}=E_k$. We have

$$\partial_i g(E_j,E_k) = \partial_i(E_j \bullet E_k) = \partial_i E_j \bullet E_k + E_j \bullet \partial_i E_k = g(\nabla_i E_j,E_k) + g(E_j,\tilde{\nabla}_i E_k),$$

where the second equality is equivalent (2.12). This proves both statements of the proposition.□

Remark 2.8: In contrast to the commutative case, Eq. (2.12) by itself is not sufficient to uniquely determine the connections Γ_{ijk} and $\tilde{\Gamma}_{ijk}$; the noncommutative torsion needs to be specified independently. This is similar to the situation in supergeometry, where torsion is determined by other considerations.

At this point we should relate to the literature. The metric introduced here resembles similar notions in Refs. 14 and 24–26; also our left and right connections and their metric compatibility have much similarity with Definitions 2 and 3 in Ref. 24. However, there are crucial differences. Our left (right) tangent bundle is a left (right) $\mathcal{A}$-module only, while in Refs. 24 and 25 there is only one "tangent bundle" T which is a bimodule over some algebra (or Hopf algebra) B. The metrics defined in Refs. 14 and 24–26 are maps from $T \otimes_B T$ to B.

Remark 2.9: A noteworthy feature of the metric in Ref. 14 is that a particular moving frame can be chosen to make all the components of the metric central [see Eq. (3.22) in Ref. 14]. In the context of the Moyal algebra, this amounts to that the metric is a constant matrix.

A. Curvatures and second fundamental form

Let $[\nabla_i,\nabla_j]:=\nabla_i\nabla_j-\nabla_j\nabla_i$ and $[\tilde{\nabla}_i,\tilde{\nabla}_j]:=\tilde{\nabla}_i\tilde{\nabla}_j-\tilde{\nabla}_j\tilde{\nabla}_i$. Straightforward calculations show that for all $f\in\mathcal{A}$,

$$[\nabla_i,\nabla_j](f*Z)=f*[\nabla_i,\nabla_j]Z,\quad Z\in TX,$$

$$[\tilde{\nabla}_i,\tilde{\nabla}_j](W*f)=[\tilde{\nabla}_i,\tilde{\nabla}_j]W*f,\quad W\in\widetilde{T}X.$$

Clearly the right-hand side of the first equation belongs to TX, while that of the second equation belongs to $\widetilde{T}X$. We restate these important facts as a proposition.

Proposition 2.10: The following maps

$$[\nabla_i,\nabla_j]:TX\to TX,\quad[\tilde{\nabla}_i,\tilde{\nabla}_j]:\widetilde{T}X\to\widetilde{T}X$$

are left and right $\mathcal{A}$ *-module homomorphisms, respectively*.

Since TX ($\widetilde{T}X$) is generated by E_1 and E_2 as a left (right) $\mathcal{A}$-module, by Proposition 2.10, we can always write

$$[\nabla_i,\nabla_j]E_k=R^l_{kij}*E_l,\quad[\tilde{\nabla}_i,\tilde{\nabla}_j]E_k=E_l*\tilde{R}^l_{kij}\tag{2.13}$$

for some $R^l_{kij},\tilde{R}^l_{kij}\in\mathcal{A}$.

Definition 2.11: We refer to R^l_{kij} and $\tilde{R}^l_{kij}$, respectively, as the *Riemann curvatures* of the left and right tangent bundles of the noncommutative surface X.

The Riemann curvatures are uniquely determine by the relations (2.13). In fact, we have

$$R^l_{kij}=g([\nabla_i,\nabla_j]E_k,\tilde{E}^l),\quad\tilde{R}^l_{kij}=g(E^l,[\tilde{\nabla}_i,\tilde{\nabla}_j]E_k).\tag{2.14}$$

Simple calculations yield the following result.

Lemma 2.12:

$$R^l_{kij}=-\partial_j\Gamma^l_{ik}-\Gamma^p_{ik}*\Gamma^l_{jp}+\partial_i\Gamma^l_{jk}+\Gamma^p_{jk}*\Gamma^l_{ip},$$

$$\tilde{R}^l_{kij}=-\partial_j\tilde{\Gamma}^l_{ik}-\tilde{\Gamma}^l_{jp}*\tilde{\Gamma}^p_{ik}+\partial_i\tilde{\Gamma}^l_{jk}+\tilde{\Gamma}^l_{ip}*\tilde{\Gamma}^p_{jk}.$$

Proposition (2.13): Let $R_{lkij}=R^p_{kij}*g_{pl}$ *and* $\tilde{R}_{lkij}=-g_{kp}*\tilde{R}^p_{lij}$. *The Riemann curvatures of the left and right tangent bundles coincide in the sense that* $R_{klij}=\tilde{R}_{klij}$.

Proof: By Proposition 2.7, we have $R_{lkij}=(\nabla_i\nabla_j-\nabla_j\nabla_i)E_k\bullet E_l$, which can be rewritten as

$$\begin{aligned}R_{lkij}&=\partial_i(\nabla_jE_k\bullet E_l)-\nabla_jE_k\bullet\tilde{\nabla}_iE_l-\partial_j(\nabla_iE_k\bullet E_l)+\nabla_iE_k\bullet\tilde{\nabla}_jE_l\\&=\partial_i(\nabla_jE_k\bullet E_l+E_k\bullet\tilde{\nabla}_jE_l)-E_k\bullet\tilde{\nabla}_i\tilde{\nabla}_jE_l-\partial_j(\nabla_iE_k\bullet E_l+E_k\bullet\tilde{\nabla}_iE_l)+E_k\bullet\tilde{\nabla}_j\tilde{\nabla}_iE_l.\end{aligned}$$

Again by Proposition 2.7, the first term on the far right-hand side can be written as $\partial_i\partial_jg_{kl}$, and the third term can be written as $-\partial_i\partial_jg_{kl}$. Thus they cancel out, and we arrive at

$$R_{lkij}=-E_k\bullet(\tilde{\nabla}_i\tilde{\nabla}_j-\tilde{\nabla}_j\tilde{\nabla}_i)E_l=\tilde{R}_{lkij}.$$

□

Because of the proposition, we only need to study the Riemannian curvature on one of the tangent bundles. Note that $R_{klij}=-R_{klji}$, but there is no simple rule to relate R_{lkij} to R_{klij} in contrast to the commutative case.

Definition 2.14: Let

$$R_{ij}=R^p_{ipj},\quad R=g^{ji}*R_{ij},\tag{2.15}$$

and call them the *Ricci curvature* and *scalar curvature* of the noncommutative surface, respectively.

Then obviously

$$R_{ij} = -g([\nabla_j, \nabla_l]E_i, \tilde{E}^l), \quad R = -g([\nabla_i, \nabla_k]E^i, \tilde{E}^k). \tag{2.16}$$

In the theory of classical surfaces, the second fundamental form plays an important role. A similar notion exists for noncommutative surfaces.

Definition 2.15: We define the left and right *second fundamental forms* of the noncommutative surface X by

$$h_{ij} = \partial_i E_j - \Gamma_{ij}^k * E_k, \quad \tilde{h}_{ij} = \partial_i E_j - E_k * \tilde{\Gamma}_{ij}^k. \tag{2.17}$$

It follows from Eq. (2.9) that

$$h_{ij} \bullet E_k = 0, \quad E_k \bullet \tilde{h}_{ij} = 0. \tag{2.18}$$

Remark 2.16: Both the left and right second fundamental forms reduce to $h_{ij}^0 \mathbf{N}$ in the commutative limit, where h_{ij}^0 is the standard second fundamental form and $\mathbf{N}$ is the unit normal vector.

The Riemann curvature $R_{lkij} = (\nabla_i \nabla_j - \nabla_j \nabla_i) E_k \bullet E_l$ can be expressed in terms of the second fundamental forms. Note that

$$R_{lkij} = \partial_j E_k \bullet \partial_i E_l - \partial_j E_k \bullet \tilde{\nabla}_i E_l - \partial_i E_k \bullet \partial_j E_l + \partial_i E_k \bullet \tilde{\nabla}_j E_l.$$

By Definition 2.15,

$$R_{lkij} = \partial_j E_k \bullet \tilde{h}_{il} - \partial_i E_k \bullet \tilde{h}_{jl} = (\nabla_j E_k + h_{jk}) \bullet \tilde{h}_{il} - (\nabla_i E_k + h_{ik}) \bullet \tilde{h}_{jl}.$$

Equation (2.18) immediately leads to the following result.

Lemma 2.17: The following generalized Gauss equation holds:

$$R_{lkij} = h_{jk} \bullet \tilde{h}_{il} - h_{ik} \bullet \tilde{h}_{jl}. \tag{2.19}$$

Before closing this section, we mention that the Riemannian structure of a noncommutative surface is a deformation of the classical Riemannian structure of a surface by including quantum corrections. The embedding into $\mathcal{A}^3$ is not subject to any constraints as the general theory stands. However, one may consider particular noncommutative surfaces with embeddings satisfying extra symmetry requirements similar to the way in which various star products on $\mathbb{R}^3$ were obtained from the Moyal product on $\mathbb{R}^4$ in Secs. 4 and 5 in Ref. 27.

III. EXAMPLES

In this section, we consider in some detail three concrete examples of noncommutative surfaces: the noncommutative sphere, torus, and hyperboloid.

A. Noncommutative sphere

Let $U = (0,\pi) \times (0, 2\pi)$, and we write θ and ϕ for t_1 and t_2, respectively. Let $X(\theta,\phi) = (X^1(\theta,\phi), X^2(\theta,\phi), X^3(\theta,\phi))$ be given by

$$X(\theta,\phi) = \left(\frac{\sin\theta\cos\phi}{\cosh\bar{h}}, \frac{\sin\theta\sin\phi}{\cosh\bar{h}}, \frac{\sqrt{\cosh 2\bar{h}}\cos\theta}{\cosh\bar{h}} \right), \tag{3.1}$$

with the components being smooth functions in $(\theta,\phi) \in U$. It can be shown that X satisfies the following relation:

$$X^1 * X^1 + X^2 * X^2 + X^3 * X^3 = 1. \tag{3.2}$$

Thus we may regard the noncommutative surface defined by X as an analog of the sphere S^2. We shall denote it by $S^2_{\bar{h}}$ and refer to it as a *noncommutative sphere*. We have

$$E_1 = \left(\frac{\cos\theta\cos\phi}{\cosh\bar{h}}, \frac{\cos\theta\sin\phi}{\cosh\bar{h}}, - \frac{\sqrt{\cosh 2\bar{h}}\sin\theta}{\cosh\bar{h}} \right),$$

$$E_2 = \left(-\frac{\sin\theta\sin\phi}{\cosh\bar{h}}, \frac{\sin\theta\cos\phi}{\cosh\bar{h}}, 0 \right).$$

The components $g_{ij} = E_i \bullet E_j$ of the metric g on $S^2_{\bar{h}}$ can now be calculated, and we obtain

$$g_{11} = 1, \quad g_{22} = \sin^2\theta - \frac{\sinh^2\bar{h}}{\cosh^2\bar{h}}\cos^2\theta,$$

$$g_{12} = - g_{21} = \frac{\sinh\bar{h}}{\cosh\bar{h}}(\sin^2\theta - \cos^2\theta).$$

The components of this metric commute with one another as they depend on θ only. Thus it makes sense to consider the usual determinant G of g. We have

$$G = \sin^2\theta + \tanh^2\bar{h}(\cos^2 2\theta - \cos^2\theta) = \sin^2\theta[1 + \tanh^2\bar{h}(1 - 4\cos^2\theta)].$$

The inverse metric is given by

$$g^{11} = \frac{\sin^2\theta - \tanh^2\bar{h}\cos^2\theta}{\sin^2\theta + \tanh^2\bar{h}(\cos^2 2\theta - \cos^2\theta)},$$

$$g^{22} = \frac{1}{\sin^2\theta + \tanh^2\bar{h}(\cos^2 2\theta - \cos^2\theta)},$$

$$g^{12} = - g^{21} = \frac{\tanh\bar{h}\cos 2\theta}{\sin^2\theta + \tanh^2\bar{h}(\cos^2 2\theta - \cos^2\theta)}.$$

Now we determine the connection and curvature tensor of the noncommutative sphere. The computations are quite lengthy, thus we only record the results here. For the Christoffel symbols, we have

$$\Gamma_{111} = \tilde{\Gamma}_{111} = 0, \quad \Gamma_{112} = - \tilde{\Gamma}_{112} = \sin 2\theta\tanh\bar{h},$$

$$\Gamma_{121} = - \tilde{\Gamma}_{121} = - \sin 2\theta\tanh\bar{h}, \quad \Gamma_{122} = \tilde{\Gamma}_{122} = \tfrac{1}{2}\sin 2\theta(1 + \tanh^2\bar{h}),$$

$$\Gamma_{211} = - \tilde{\Gamma}_{211} = \sin 2\theta\tanh\bar{h}, \quad \Gamma_{212} = \tilde{\Gamma}_{212} = \tfrac{1}{2}\sin 2\theta(1 + \tanh^2\bar{h}),$$

$$\Gamma_{221}=\tilde{\Gamma}_{221}=-\tfrac{1}{2}\sin 2\theta(1+\tanh^2\bar{h}),\quad \Gamma_{222}=-\tilde{\Gamma}_{222}=\sin 2\theta\tanh\bar{h}.$$

Note that $\Gamma_{112}\neq\tilde{\Gamma}_{112}$ (cf. Remark 2.8). We now find the asymptotic expansions of the curvature tensors with respect to $\bar{h}$,

$$R_{1112}=2\bar{h}+\left(\tfrac{10}{3}+4\cos 2\theta\right)\bar{h}^3+O(\bar{h}^4),$$

$$R_{2112}=-\sin^2\theta-\tfrac{1}{2}(4+\cos 2\theta-\cos 4\theta)\bar{h}^2+O(\bar{h}^4),$$

$$R_{1212}=\sin^2\theta+\tfrac{1}{2}(4+\cos 2\theta-\cos 4\theta)\bar{h}^2+O(\bar{h}^4),$$

$$R_{2212}=-2\sin^2\theta\bar{h}-\left(\tfrac{5}{3}+\tfrac{4}{3}\cos 2\theta-4\cos 4\theta\right)\bar{h}^3+O(\bar{h}^4).$$

We can also compute asymptotic expansions of the Ricci curvature tensor,

$$R_{11}=1+(6+4\cos 2\theta)\bar{h}^2+O(\bar{h}^4),$$

$$R_{21}=(2-\cos 2\theta)\bar{h}+\tfrac{1}{3}(16+19\cos 2\theta-6\cos 4\theta)\bar{h}^3+O(\bar{h}^4),$$

$$R_{12}=(2+\cos 2\theta)\bar{h}+\tfrac{1}{3}(16+29\cos 2\theta+6\cos 4\theta)\bar{h}^3+O(\bar{h}^4),$$

$$R_{22}=\sin^2\theta+\tfrac{1}{2}(3+5\cos 2\theta-2\cos 4\theta)\bar{h}^2+O(\bar{h}^4),$$

and the scalar curvature

$$R=2+4(3+4\cos 2\theta)\bar{h}^2+O(\bar{h}^4).$$

By setting $\bar{h}=0$, we obtain from the various curvatures of $S^2_{\bar{h}}$ the corresponding objects for the usual sphere S^2. This is a useful check that our computations above are accurate.

B. Noncommutative torus

This time we shall take $U=(0,2\pi)\times(0,2\pi)$ and denote a point in U by (θ,ϕ). Let $X(\theta,\phi)=(X^1(\theta,\phi),X^2(\theta,\phi),X^3(\theta,\phi))$ be given by

$$X(\theta,\phi)=((a+\sin\theta)\cos\phi,(a+\sin\theta)\sin\phi,\cos\theta),\tag{3.3}$$

where $a>1$ is a constant. Classically X is the torus. When we extend scalars from $\mathbb{R}$ to $\mathbb{R}[[\bar{h}]]$ and impose the star product on the algebra of smooth functions, X gives rise to a noncommutative torus, which will be denoted by $T^2_{\bar{h}}$. We have

$$E_1=(\cos\theta\cos\phi,\cos\theta\sin\phi,-\sin\theta),$$

$$E_2=(-(a+\sin\theta)\sin\phi,(a+\sin\theta)\cos\phi,0).$$

The components $g_{ij}=E_i\bullet E_j$ of the metric g on $T^2_{\bar{h}}$ take the form

$$g_{11}=1+\sinh^2\bar{h}\cos 2\theta,$$

$$g_{22}=(a+\cosh\bar{h}\sin\theta)^2-\sinh^2\bar{h}\cos^2\theta,$$

$$g_{12} = - g_{21} = - \sinh \bar{h} \cosh \bar{h} \cos 2\theta + a \sinh \bar{h} \sin \theta.$$

As they depend only on θ, the components of the metric commute with one another. The inverse metric is given by

$$g^{11} = \frac{(a + \cosh \bar{h} \sin \theta)^2 - \sinh^2 \bar{h} \cos^2 \theta}{G},$$

$$g^{22} = \frac{1 + \sinh^2 \bar{h} \cos 2\theta}{G},$$

$$g^{12} = - g^{21} = \frac{\sinh \bar{h} \cosh \bar{h} \cos 2\theta + a \sinh \bar{h} \sin \theta}{G},$$

where G is the usual determinant of g given by

$$G = (\sin \theta + a \cosh \bar{h})^2 - a^2 \sin^2 \theta \sinh^2 \bar{h}.$$

Now we determine the curvature tensor of the noncommutative torus. The computations can be carried out in much the same way as in the case of the noncommutative sphere, and we merely record the results here. For the connection, we have

$$\Gamma_{111} = - \sin 2\theta \sinh^2 \bar{h}, \quad \Gamma_{112} = a \cos \theta \sinh \bar{h} + \sin 2\theta \sinh \bar{h} \cosh \bar{h},$$

$$\Gamma_{121} = - \sin 2\theta \sinh \bar{h} \cosh \bar{h}, \quad \Gamma_{122} = a \cos \theta \cosh \bar{h} + \tfrac{1}{2}\sin 2\theta \cosh 2\bar{h},$$

$$\Gamma_{211} = - \sin 2\theta \sinh \bar{h} \cosh \bar{h}, \quad \Gamma_{212} = a \cos \theta \cosh \bar{h} + \tfrac{1}{2}\sin 2\theta \cosh 2\bar{h},$$

$$\Gamma_{221} = - a \cos \theta \cosh \bar{h} - \tfrac{1}{2}\sin 2\theta \cosh \bar{h}, \quad \Gamma_{222} = 2a \cos \theta \sinh \bar{h} + \sin 2\theta \sinh \bar{h} \cosh \bar{h}.$$

We can find the asymptotic expansions of the curvature tensors with respect to $\bar{h}$,

$$R_{1112} = \frac{2 \sin \theta(1 + a \sin \theta)}{a + \sin \theta}\bar{h} + O(\bar{h}^3),$$

$$R_{2112} = - \sin \theta(a + \sin \theta) + O(\bar{h}^2),$$

$$R_{1212} = \sin \theta(a + \sin \theta) + O(\bar{h}^2),$$

$$R_{2212} = - 2 \sin^2 \theta(1 + a \sin \theta)\bar{h} + O(\bar{h}^3).$$

We can also compute asymptotic expansions of the Ricci curvature tensor,

$$R_{11} = \frac{\sin \theta}{a + \sin \theta} + O(\bar{h}^2),$$

$$R_{21} = - \frac{\sin \theta(- 3a + 5a \cos \theta - (5 + 2a^2)\sin \theta + \sin 3\theta)}{2(a + \sin \theta)^2}\bar{h} + O(\bar{h}^3),$$

$$R_{12} = \frac{\sin\,\theta(a+\cos\,2\theta + a\,\sin\theta)}{a+\sin\,\theta}\bar{h} + O(\bar{h}^3),$$

$$R_{22} = \sin\,\theta(a+\sin\,\theta) + O(\bar{h}^2),$$

and the scalar curvature,

$$R = \frac{2\,\sin\,\theta}{a+\sin\,\theta} + O(\bar{h}^2).$$

By setting $\bar{h}=0$, we obtain from the various curvatures of $T_{\bar{h}}^2$ the corresponding objects for the usual torus T^2.

C. Noncommutative hyperboloid

Another simple example is the noncommutative analog of the hyperboloid described by $X=(x,y,\sqrt{1+x^2+y^2})$. One may also change the parametrization and consider instead

$$X(r,\phi) = (\sinh\,r\,\cos\,\phi, \sinh\,r\,\sin\,\phi, \cosh\,r) \tag{3.4}$$

on $U=(0,\infty)\times(0,2\pi)$, where a point in U is denoted by (r,ϕ). When we extend scalars from $\mathbb{R}$ to $\mathbb{R}[[\bar{h}]]$ and impose the star product on the algebra of smooth functions [defined by (2.1) with $t_1=r$ and $t_2=\phi$], X gives rise to a noncommutative hyperboloid, which will be denoted by $H_{\bar{h}}^2$. We have

$$E_1 = (\cosh\,r\,\cos\,\phi, \cosh\,r\,\sin\,\phi, \sinh\,r),$$

$$E_2 = (-\sinh\,r\,\sin\,\phi, \sinh\,r\,\cos\,\phi, 0).$$

The components $g_{ij}=E_i \bullet E_j$ of the metric g on $H_{\bar{h}}^2$ take the form

$$g_{11} = \cos^2\,\bar{h}\,\cosh\,2r,$$

$$g_{22} = \tfrac{1}{2}(-1+\cos\,2\bar{h}\,\cosh\,2r),$$

$$g_{12} = -g_{21} = -\tfrac{1}{2}\sin\,2\bar{h}\,\cosh\,2r.$$

As they depend only on r, the components of the metric commute with one another. The inverse metric is given by

$$g^{11} = \frac{\sec^2\,\bar{h}}{2\,\sinh^2\,r}\left(\cos\,2\bar{h} - \frac{1}{\cosh\,2r}\right),$$

$$g^{22} = \frac{1}{\sinh^2\,r},$$

$$g^{12} = -g^{21} = \frac{\tan\,\bar{h}}{\sinh^2\,r}.$$

Now we determine the curvature tensor of the noncommutative hyperboloid. For the connection, we have

$$\Gamma_{111} = \cos^2 \bar{h} \sinh 2r, \quad \Gamma_{112} = -\tfrac{1}{2}\sin 2\bar{h} \sinh 2r,$$

$$\Gamma_{121} = \tfrac{1}{2}\sin 2\bar{h} \sinh 2r, \quad \Gamma_{122} = \tfrac{1}{2}\cos 2\bar{h} \sinh 2r,$$

$$\Gamma_{211} = \tfrac{1}{2}\sin 2\bar{h} \sinh 2r, \quad \Gamma_{212} = \tfrac{1}{2}\cos 2\bar{h} \sinh 2r,$$

$$\Gamma_{221} = -\tfrac{1}{2}\cos 2\bar{h} \sinh 2r, \quad \Gamma_{222} = \tfrac{1}{2}\sin 2\bar{h} \sinh 2r.$$

We can find the asymptotic expansions of the curvature tensors with respect to $\bar{h}$,

$$R_{1112} = \frac{2}{\cosh 2r}\bar{h} + O(\bar{h}^2),$$

$$R_{2112} = -\frac{\sinh^2 r}{\cosh 2r} + O(\bar{h}^2),$$

$$R_{1212} = \frac{\sinh^2 r}{\cosh 2r} + O(\bar{h}^2),$$

$$R_{2212} = -\frac{\cosh 2r + \sinh^2 2r}{\cosh 2r}\bar{h} + O(\bar{h}^3).$$

We can also compute asymptotic expansions of the Ricci curvature tensor,

$$R_{11} = \frac{1}{\cosh 2r} + O(\bar{h}^2),$$

$$R_{21} = \frac{\coth^2 r(2\cosh 2r - 1)}{\cosh 2r}\bar{h} + O(\bar{h}^2),$$

$$R_{12} = \frac{\cosh 2r + 2}{\cosh^2 2r}\bar{h} + O(\bar{h}^2),$$

$$R_{22} = \frac{\sinh^2 r}{\cosh^2 2r} + O(\bar{h}^2),$$

and the scalar curvature,

$$R = \frac{2}{\cosh^2 2r} + O(\bar{h}^2).$$

By setting $\bar{h}=0$, we obtain from the various curvatures of $H^2_{\bar{h}}$ the corresponding objects for the usual hyperboloid H^2.

IV. NONCOMMUTATIVE *n*-DIMENSIONAL SURFACES

One can readily generalize the theory of Sec. II to higher dimensions, and we shall do this here. Noncommutative Bianchi identities will also be obtained.

Again for the sake of explicitness we restrict attention to the Moyal product on the smooth functions. However, as we shall see in Sec. V, it will be necessary to consider more general star products in order to discuss "general coordinate transformations" of noncommutative surfaces.

A. Noncommutative *n*-dimensional surfaces

We take a region U in $\mathbb{R}^n$ for a fixed n and write the coordinate of $t \in U$ as $(t_1, t_2, \ldots, t_n)$. Let $\mathcal{A}$ denote the set of the smooth functions on U taking values in $\mathbb{R}[[\bar{h}]]$. Fix any constant skew symmetric $n \times n$ matrix θ. The Moyal product on $\mathcal{A}$ is defined by the following generalization of Eq. (2.1):

$$f * g = \lim_{t' \to t} \exp\left(\bar{h} \sum_{ij} \theta_{ij} \partial_i \partial_j'\right) f(t) g(t') \tag{4.1}$$

for any $f, g \in \mathcal{A}$. Such a multiplication is known to be associative. Since θ is a constant matrix, the Leibniz rule (2.3) remains valid in the present case,

$$\partial_i (f * g) = \partial_i f * g + f * \partial_i g .$$

For any fixed positive integer m, we can define a dot product,

$$\bullet : \mathcal{A}^m \otimes_{\mathbb{R}[[\bar{h}]]} \mathcal{A}^m \to \mathcal{A}^m \tag{4.2}$$

by generalizing (2.4) to $A \bullet B = a^i * b_i$ for all $A = (a^1, \ldots, a^m)$ and $B = (b_1, \ldots, b_m)$ in $\mathcal{A}^m$. As before, the dot product is a map of two-sided $\mathcal{A}$-modules.

Assume $m > n$. For $X \in \mathcal{A}^m$, we let $E_i = \partial_i X$ and define $g_{ij} = E_i \bullet E_j$. Denote by $g = (g_{ij})$ the $n \times n$ matrix with entries g_{ij}.

Definition 4.1: If $g \bmod \bar{h}$ is invertible over U, we shall call X a *noncommutative n-dimensional surface* embedded in $\mathcal{A}^m$ and call g the metric of X.

The discussion on the metric in Sec. II carries over to the present situation; in particular, the invertibility of $g \bmod \bar{h}$ implies that there exists a unique inverse (g^{ij}). Now as in Sec. II, we define the left tangent bundle TX (right tangent bundle $\widetilde{TX}$) of the noncommutative surface as the left (right) $\mathcal{A}$-submodule of $\mathcal{A}^m$ generated by the elements E_i. The fact that the metric g belongs to $GL_n(\mathcal{A})$ enables us to show that the left and right tangent bundles are projective $\mathcal{A}$-modules.

The connection ∇_i on the left tangent bundle will be defined in the same way as in Sec. II, namely, by the composition of the derivative ∂_i with the projection of $\mathcal{A}^m$ onto the left tangent bundle. The connection $\widetilde{\nabla}_i$ on the right tangent bundle is defined similarly. Then ∇_i and $\widetilde{\nabla}_i$ satisfy the analogous Eq. (2.8), and are compatible with the metric in the same sense as Proposition 2.7.

One can show that

$$[\nabla_i, \nabla_j] : TX \to TX, \quad [\widetilde{\nabla}_i, \widetilde{\nabla}_j] : \widetilde{TX} \to \widetilde{TX}$$

are left and right $\mathcal{A}$-module homomorphisms, respectively. This allows us to define Riemann curvatures of the tangent bundles as in Eq. (2.13). Then the formulas given in Lemma 2.12 are still valid when the indices in the formulas are assumed to take values in $\{1, 2, \ldots, n\}$. Furthermore, the left and right Riemann curvatures remain equal in the sense of Proposition 2.13.

Remark 4.2: One may define a dot product $\bullet : \mathcal{A}^m \otimes_{\mathbb{R}[[\bar{h}]]} \mathcal{A}^m \to \mathcal{A}^m$ with a Minkowski signature by

$$A \bullet B = a_0 * b_0 - \sum_{i=1}^{m-1} a_i * b_i$$

for any $A = (a_0, a_1, \ldots, a_{m-1})$ and $B = (b_0, b_1, \ldots, b_{m-1})$ in $\mathcal{A}^m$. This is still a map of two-sided $\mathcal{A}$-modules. Then the aforedeveloped theory can be adapted to this setting, leading to a theory of noncommutative surfaces embedded in $\mathcal{A}^m$ with a Minkowski signature.

For the sake of being concrete, we shall consider only noncommutative surfaces with Euclidean signature hereafter.

B. Bianchi identities

We examine properties of the Riemann curvature for arbitrary n and m. The main result in this subsection is the noncommutative analogs of Bianchi identities.

Define E^i and $\tilde{E}^l$ as in (2.6). Then

$$\nabla_p E^l = -\tilde{\Gamma}^l_{pk} * E^k, \quad \tilde{\nabla}_p \tilde{E}^l = -\tilde{E}^k * \Gamma^l_{pk}. \tag{4.3}$$

These relations will be needed presently. Let

$$R^l_{kij;p} = \partial_p R^l_{kij} - \Gamma^r_{pk} * R^l_{rij} - \Gamma^r_{pi} * R^l_{rjk} - \Gamma^r_{pj} * R^l_{rki} + R^r_{kij} * \Gamma^l_{rp}. \tag{4.4}$$

Theorem 4.3: *The Riemann curvature R^i_{jkl} satisfies the following relations:*

$$R^l_{ijk} + R^l_{jki} + R^l_{kij} = 0, \quad R^l_{kij;p} + R^l_{kjp;i} + R^l_{kpi;j} = 0, \tag{4.5}$$

which will be referred to as the first and second noncommutative Bianchi identities, respectively.

Proof: It follows from the relation $\nabla_i E_j = \nabla_j E_i$ that

$$[\nabla_i, \nabla_j]E_k + [\nabla_j, \nabla_k]E_i + [\nabla_k, \nabla_i]E_j = 0.$$

This immediately leads to

$$g([\nabla_i, \nabla_j]E_k, \tilde{E}^l) + g([\nabla_j, \nabla_k]E_i, \tilde{E}^l) + g([\nabla_k, \nabla_i]E_j, \tilde{E}^l) = 0.$$

Using the definition of the Riemann curvature in this relation, we obtain the first Bianchi identity.

To prove the second Bianchi identity, note that

$$-\partial_p R^l_{kij} + g(\nabla_p[\nabla_i, \nabla_j]E_k, \tilde{E}^l) + g([\nabla_i, \nabla_j]E_k, \tilde{\nabla}_p \tilde{E}^l) = 0.$$

Cyclic permutations of the indices p, i, j lead to two further relations. Adding all the three relations together, we arrive at

$$\begin{aligned} &-\partial_p R^l_{kij} + g([\nabla_i, \nabla_j]\nabla_p E_k, \tilde{E}^l) + g([\nabla_i, \nabla_j]E_k, \tilde{\nabla}_p \tilde{E}^l), \\ &-\partial_i R^l_{kjp} + g([\nabla_j, \nabla_p]\nabla_i E_k, \tilde{E}^l) + g([\nabla_j, \nabla_p]E_k, \tilde{\nabla}_i \tilde{E}^l), \\ &-\partial_j R^l_{kpi} + g([\nabla_p, \nabla_i]\nabla_j E_k, \tilde{E}^l) + g([\nabla_p, \nabla_i]E_k, \tilde{\nabla}_j \tilde{E}^l) = 0, \end{aligned} \tag{4.6}$$

where we have used the following variant of the Jacobian identity:

$$\nabla_p[\nabla_i, \nabla_j] + \nabla_i[\nabla_j, \nabla_p] + \nabla_j[\nabla_p, \nabla_i] = [\nabla_i, \nabla_j]\nabla_p + [\nabla_j, \nabla_p]\nabla_i + [\nabla_p, \nabla_i]\nabla_j.$$

By a tedious calculation one can show that

$$\mathcal{Q}_{ijkp} := [\nabla_j, \nabla_k]\nabla_p E_i + [\nabla_k, \nabla_i]\nabla_p E_j + [\nabla_p, \nabla_k]\nabla_i E_j + [\nabla_k, \nabla_j]\nabla_i E_p + [\nabla_i, \nabla_k]\nabla_j E_p + [\nabla_k, \nabla_p]\nabla_j E_i$$

is identically zero. Now we add $g(\mathcal{Q}_{ijkp}, \tilde{E}^l)$ to the left-hand side of (4.6), obtaining an identity with 15 terms on the left. Then the second Bianchi identity can be read off this equation by recalling (4.3). □

C. Einstein's equation

Recall that in classical Riemannian geometry, the second Bianchi identity suggests the correct form of Einstein's equation. Let us make some preliminary analysis of this point here. As we lack guiding principles for constructing an analog of Einstein's equation, the material of this subsection is of a rather speculative nature.

In Sec. II, we introduced the Ricci curvature R_{ij} and scalar curvature R. Their definitions can be generalized to higher dimensions in an obvious way. Let

$$R^i_j = g^{ik} * R_{kj}, \tag{4.7}$$

then the scalar curvature is $R = R^i_i$. Let us also introduce the following object:

$$\Theta^l_p := g([\nabla_p, \nabla_i]E^i, \tilde{E}^l) = g^{ik} * R^l_{kpi}. \tag{4.8}$$

In the commutative case, Θ^l_p coincides with R^l_p, but it is no longer true in the present setting. However, note that

$$\Theta^l_l = g^{ik} * R^l_{kli} = g^{ik} * R_{ki} = R. \tag{4.9}$$

By first contracting the indices j and l in the second Bianchi identity, then raising the index k to i by multiplying the resulting identity by g^{ik} from the left and summing over i, we obtain the identity

$$\begin{aligned} 0 = \partial_p R - \partial_i R^i_p + g([\nabla_i, \nabla_l]\nabla_p E^i, \tilde{E}^l) + g([\nabla_l, \nabla_p]\nabla_i E^i, \tilde{E}^l) - \partial_l \Theta^l_p + g([\nabla_i, \nabla_l]E^i, \tilde{\nabla}_p \tilde{E}^l) \\ + g([\nabla_p, \nabla_i]E^i, \tilde{\nabla}_l \tilde{E}^l) + g([\nabla_p, \nabla_i]\nabla_l E^i, \tilde{E}^l) + g([\nabla_l, \nabla_p]E^i, \tilde{\nabla}_i \tilde{E}^l). \end{aligned}$$

Let us denote the sum of the last two terms on the right-hand side by ϖ_p. Then

$$\varpi_p = g^{ik} * R^r_{kpl} * \Gamma^l_{ri} - \tilde{\Gamma}^i_{lr} * g^{rk} * R^l_{kpi}.$$

In the commutative case, ϖ_p vanishes identically for all p. However, in the noncommutative setting, there is no reason to expect this to happen. Let us now define

$$R^i_{p;i} = \partial_i R^i_p - \tilde{\Gamma}^i_{pr} * R^r_i + \tilde{\Gamma}^i_{ir} * R^r_p,$$

$$\Theta^l_{p;l} = \partial_l \Theta^l_p - \Theta^r_l * \Gamma^l_{rp} + \Theta^r_p * \Gamma^l_{rl} - \varpi_p. \tag{4.10}$$

Then the second Bianchi identity implies

$$R^i_{p;i} + \Theta^i_{p;i} - \partial_p R = 0. \tag{4.11}$$

The above discussions suggest that Einstein's equation no longer takes its usual form in the noncommutative setting, but we have not been able to formulate a basic principle which enables us to *derive* a noncommutative analog of Einstein's equation. However, formulas (4.11) and (4.9) seem to suggest that the following is a reasonable proposal for a noncommutative Einstein equation in the vacuum:

$$R^i_j + \Theta^i_j - \delta^i_j R = 0. \tag{4.12}$$

We were informed by Madore that in other contexts of noncommutative general relativity, it also appeared to be necessary to include an object analogous to Θ^i_j in the Einstein equation.

V. GENERAL COORDINATE TRANSFORMATIONS

We investigate the effect of "general coordinate transformations" on noncommutative n-dimensional surfaces. This requires us to consider noncommutative surfaces defined over $\mathcal{A}$

endowed with star products more general than the Moyal product. This should be compared to Refs. 9, 10, 12, and 13, where the only general coordinate transformations allowed were those keeping the Moyal product intact.

For the sake of being concrete, we assume that the noncommutative surface has Euclidean signature.

A. Gauge transformations

Denote by $\mathcal{G}(\mathcal{A})$ the set of $\mathbb{R}[[\bar{h}]]$-linear maps $\phi:\mathcal{A}\to\mathcal{A}$ satisfying the following conditions:

$$\phi(1)=1, \quad \phi=\exp\left(\sum_i \varepsilon^i\partial_i\right)\mathrm{mod}\ \bar{h}, \tag{5.1}$$

where ε^i are smooth functions on U. Then clearly we have the following result.

Lemma 5.1: The set $\mathcal{G}(\mathcal{A})$ forms a subgroup of the automorphism group of $\mathcal{A}$ as $\mathbb{R}[[\bar{h}]]$-module.

For any given $\phi\in\mathcal{G}(\mathcal{A})$, define an $\mathbb{R}[[\bar{h}]]$-linear map,

$$*_\phi:\mathcal{A}\otimes_{\mathbb{R}[[\bar{h}]]}\mathcal{A}\to\mathcal{A}, \quad f\otimes g\mapsto f*_\phi g:=\phi^{-1}(\phi(f)*\phi(g)). \tag{5.2}$$

Lemma 5.2:

(1) The *map* $*_\phi$ is associative, thus there exists the associative algebra $(\mathcal{A},*_\phi)$ over $\mathbb{R}[[\bar{h}]]$. Furthermore, $\phi:(\mathcal{A},*_\phi)\to(\mathcal{A},*)$ is an algebra isomorphism.

(2) Let $*^\phi=*_{\phi^{-1}}$, then for any $\phi,\psi\in\mathcal{G}(\mathcal{A})$

$$\psi(\psi^{-1}(f)*^\phi\psi^{-1}(g))=f*^{\psi\phi}g. \tag{5.3}$$

In this sense the definition of the new star products respects the group structure of $\mathcal{G}(\mathcal{A})$.

Proof: Because of the importance of this lemma for later discussions, we sketch a proof for it here, even though one can easily deduce a proof from Ref. 23.

For $f,g,h\in\mathcal{A}$, we have

$$\begin{aligned}(f*_\phi g)*_\phi h &= \phi^{-1}((\phi(f)*\phi(g))*\phi(h))=\phi^{-1}(\phi(f)*(\phi(g)*\phi(h)))=\phi^{-1}(\phi(f)*\phi(g*_\phi h))\\ &= f*_\phi(g*_\phi h),\end{aligned}$$

which proves the associativity of the new star product. As ϕ is an $\mathbb{R}[[\bar{h}]]$-module isomorphism by definition, we only need to show that it preserves multiplications in order to establish the isomorphism between the algebras. Now $\phi(f*_\phi g)=\phi(f)*\phi(g)$. This proves part (1).

Part (2) can be proven by unraveling the left-hand side of (5.3). □

Adopting the terminology of Drinfeld from the context of quantum groups, we call an automorphism $\phi\in\mathcal{G}(\mathcal{A})$ a *gauge transformation* and call $\mathcal{G}(\mathcal{A})$ the *gauge group*. The star product $*_\phi$ will be said to be *gauge equivalent* to the Moyal product (4.1). However, note that our notion of gauge transformations is slightly more general than that in deformation theory,[23] where the only type of gauge transformations allowed is of the special form,

$$\phi=id+\bar{h}\phi_1+\bar{h}^2\phi_2+\cdots,$$

with ϕ_i being $\mathbb{R}$-linear maps on the space of smooth functions on U such that $\phi_i(1)=0$ for all i. Such gauge transformations form a subgroup of $\mathcal{G}(\mathcal{A})$.

Remark 5.3: The prime aim of the deformation theory[23] is to classify the gauge equivalence classes of deformations in this restricted sense but for arbitrary associative algebras. The seminal paper[4] of Kontsevich provided an explicit formula for a star product from each gauge equivalence class of deformations of the algebra of functions on a Poisson manifold.

Remark 5.4: General star products gauge equivalent to the Moyal product were evaluated explicitly up to the third order in $\bar{h}$ in Ref. 28. In Ref. 29, position-dependent star products were also investigated and the ultraviolet divergences of a quantum ϕ^4 theory on four-dimensional spaces with such products were analyzed.

Given an element ϕ in the group $\mathcal{G}(\mathcal{A})$, we denote

$$u^i := \phi^{-1}(t^i), \quad i = 1,2, \ldots, n,$$

and refer to $t \mapsto u$ as a general coordinate transformation of U. Define $\mathbb{R}[[\bar{h}]]$-linear operators on $\mathcal{A}$ by

$$\partial_i^\phi = \phi^{-1} \circ \partial_i \circ \phi. \tag{5.4}$$

Lemma 5.5: The operators ∂_i^ϕ have the following properties:

$$\partial_i^\phi \circ \partial_j^\phi - \partial_j^\phi \circ \partial_i^\phi = 0, \quad \partial_i^\phi \phi^{-1}(t^j) = \delta_i^j,$$

and also satisfy the Leibniz rule

$$\partial_i^\phi(f *_\phi g) = \partial_i^\phi(f) *_\phi g + f *_\phi \partial_i^\phi(g), \quad \forall f,g \in \mathcal{A}.$$

Proof: The proof is easy but very illuminating. We have

$$\partial_i^\phi \circ \partial_j^\phi - \partial_j^\phi \circ \partial_i^\phi = \phi^{-1} \circ (\partial_i \partial_j - \partial_j \partial_i) \circ \phi = 0.$$

Also, $\partial_i^\phi \phi^{-1}(t^j) = \phi^{-1}(\partial_i t^j) = \delta_i^j$, since ϕ maps a constant function to itself.

To prove the Leibniz rule, we note that

$$\begin{aligned}\partial_i^\phi(f *_\phi g) &= \phi^{-1}(\partial_i(\phi(f) * \phi(g))) = \phi^{-1}(\partial_i \phi(f) * \phi(g)) + \phi^{-1}(\phi(f) * \partial_i \phi(g)) \\ &= \phi^{-1}(\phi(\partial_i^\phi f) * \phi(g)) + \phi^{-1}(\phi(f) * \phi(\partial_i^\phi g)) = \partial_i^\phi f *_\phi g + f *_\phi \partial_i^\phi g.\end{aligned}$$

This completes the proof of the lemma. □

The Leibniz rule plays a crucial role in constructing noncommutative surfaces over $(\mathcal{A}, *_\phi)$.

B. Reparametrizations of noncommutative surfaces

The construction of noncommutative surfaces over $(\mathcal{A}, *)$ works equally well over $(\mathcal{A}, *_\phi)$ for any $\phi \in \mathcal{G}(\mathcal{A})$. Regard $\mathcal{A}^m \otimes_{\mathbb{R}[[\bar{h}]]} \mathcal{A}^m$ as a two-sided $(\mathcal{A}, *_\phi)$-module and define the new dot product

$$\bullet_\phi : \mathcal{A}^m \otimes_{\mathbb{R}[[\bar{h}]]} \mathcal{A}^m \to \mathcal{A}$$

by $A \bullet_\phi B = a^i *_\phi b_i$ for any $A = (a^1, a^2, \ldots, a^m)$ and $B = (b_1, b_2, \ldots, b_m)$ in $\mathcal{A}^m$. It is obviously a map of two-sided $(\mathcal{A}, *_\phi)$-modules. For an element

$$X^\phi = (X^1, X^2, \ldots, X^m)$$

in the free two-sided $(\mathcal{A}, *_\phi)$-module $\mathcal{A}^m$, we define

$$E_i^\phi = \partial_i^\phi X^\phi, \quad {}^\phi g_{ij} = E_i^\phi \bullet_\phi E_j^\phi,$$

where ϕ acts on $\mathcal{A}^m$ in a componentwise way. As in Sec. II, let

$${}^\phi g = ({}^\phi g_{ij})_{i,j=1, \ldots, n}.$$

We shall say that X is an n-dimensional noncommutative surface with metric ${}^\phi g$ if ${}^\phi g$ mod $\bar{h}$ is invertible. In this case, ${}^\phi g$ has an inverse $({}^\phi g^{ij})$.

The left tangent bundle TX^{ϕ} and right tangent bundle $\widetilde{T}X^{\phi}$ of X^{ϕ} are now, respectively, the left and right $(\mathcal{A},*_{\phi})$-modules generated by E_i^{ϕ}, $i=1,2, \ldots,n$. The metric ${}^{\phi}g$ leads to a two-sided $(\mathcal{A},*_{\phi})$-module map,

$$ {}^{\phi}g:TX^{\phi}\otimes_{\mathbb{R}[[\bar{h}]]}\widetilde{T}X^{\phi} \to \mathcal{A}, \quad Z\otimes W \mapsto Z\bullet_{\phi}W, $$

which is the restriction of the dot product $\bullet_{\phi}$ to $TX^{\phi}\otimes_{\mathbb{R}[[\bar{h}]]}\widetilde{T}X^{\phi}$. By using this map, we can decompose $\mathcal{A}^m$ into

$$ \mathcal{A}^m = TX^{\phi} \oplus (TX^{\phi})^{\perp}, \quad \text{as left}(\mathcal{A},*_{\phi}) - \text{module}, $$

$$ \mathcal{A}^m = \widetilde{T}X^{\phi} \oplus (\widetilde{T}X^{\phi})^{\perp}, \quad \text{as right}(\mathcal{A},*_{\phi}) - \text{module}, $$

where $(TX^{\phi})^{\perp}$ is orthogonal to $\widetilde{T}X^{\phi}$ and $(\widetilde{T}X^{\phi})^{\perp}$ is orthogonal to TX^{ϕ} with respect to the map induced by the metric.

As in Definition 2.5, the operators

$$ \nabla_i^{\phi}:TX^{\phi} \to TX^{\phi}, \quad \widetilde{\nabla}_i^{\phi}:\widetilde{T}X^{\phi} \to \widetilde{T}X^{\phi} $$

are defined to be the compositions of ∂_i^{ϕ} with the projections of $\mathcal{A}^m$ onto the left and right tangent bundles, respectively. Thus, for any $Z\in TX^{\phi}$ and $W\in \widetilde{T}X^{\phi}$,

$$ {}^{\phi}g(\nabla_i^{\phi}Z,W) = \partial_i^{\phi}Z\bullet_{\phi}W, \quad {}^{\phi}g(Z,\widetilde{\nabla}_i^{\phi}W) = Z\bullet_{\phi}\partial_i^{\phi}W. \tag{5.5} $$

By using the Leibniz rule for ∂_i^{ϕ}, we can show that the analogous equations of (2.8) are satisfied by ∇_i^{ϕ} and $\widetilde{\nabla}_i^{\phi}$, namely, for all $Z\in TX^{\phi}$, $W\in \widetilde{T}X^{\phi}$, and $f\in\mathcal{A}$,

$$ \nabla_i^{\phi}(f*_{\phi}Z) = \partial_i^{\phi}f*_{\phi}Z + f*_{\phi}\nabla_i^{\phi}Z, $$

$$ \widetilde{\nabla}_i^{\phi}(W*_{\phi}f) = W*_{\phi}\partial_i^{\phi}f + \widetilde{\nabla}_i^{\phi}W*_{\phi}f. \tag{5.6} $$

Furthermore, the operators are metric compatible,

$$ \partial_i^{\phi}{}^{\phi}g(Z,W) = {}^{\phi}g(\nabla_i^{\phi}Z,W) + {}^{\phi}g(Z,\widetilde{\nabla}_i^{\phi}W), \quad \forall\, Z\in TX^{\phi}, W\in \widetilde{T}X^{\phi}. $$

Thus the two sets $\{\nabla_i^{\phi}\}$ and $\{\widetilde{\nabla}_i^{\phi}\}$ define connections on the left and right tangent bundles, respectively.

The Christoffel symbols ${}^{\phi}\Gamma_{ij}^k$ and ${}^{\phi}\widetilde{\Gamma}_{ij}^k$ in the present context are also defined in the same way as before,

$$ \nabla_i^{\phi}E_j^{\phi} = {}^{\phi}\Gamma_{ij}^k *_{\phi}E_k^{\phi}, \quad \widetilde{\nabla}_i^{\phi}E_j^{\phi} = E_k^{\phi}*_{\phi}{}^{\phi}\widetilde{\Gamma}_{ij}^k. $$

Then we have

$$ {}^{\phi}\Gamma_{ij}^k = \partial_i^{\phi}E_j^{\phi}\bullet_{\phi}E_l^{\phi}*_{\phi}{}^{\phi}g^{lk}, \quad {}^{\phi}\widetilde{\Gamma}_{ij}^k = {}^{\phi}g^{kl}*_{\phi}E_l^{\phi}\bullet_{\phi}\partial_i^{\phi}E_j^{\phi}. $$

These formulas are of the same form as those in Eq. (2.10).

By using (5.6), we can show that the maps $[\nabla_i^{\phi},\nabla_j^{\phi}]:TX^{\phi}\to TX^{\phi}$ and $[\widetilde{\nabla}_i^{\phi},\widetilde{\nabla}_j^{\phi}]:\widetilde{T}X^{\phi}\to\widetilde{T}X^{\phi}$ are left and right $(\mathcal{A},*_{\phi})$-module homomorphisms, respectively. Namely, for all $Z\in TX^{\phi}$, $W\in\widetilde{T}X^{\phi}$, and $f\in\mathcal{A}$,

$$ [\nabla_i^{\phi},\nabla_j^{\phi}](f*_{\phi}Z) = f*_{\phi}[\nabla_i^{\phi},\nabla_j^{\phi}]Z, $$

$$[\tilde{\nabla}_i^{\phi},\tilde{\nabla}_j^{\phi}](W*_{\phi}f)=[\tilde{\nabla}_i^{\phi},\tilde{\nabla}_j^{\phi}]W*_{\phi}f.$$

Thus we can define the curvatures ${}^{\phi}R^l_{ijk}$, ${}^{\phi}\tilde{R}^l_{ijk}$ as before by

$$[\nabla_i^{\phi},\nabla_j^{\phi}]E_k^{\phi}={}^{\phi}R^l_{kij}*_{\phi}E_l^{\phi},$$

$$[\tilde{\nabla}_i^{\phi},\tilde{\nabla}_j^{\phi}]E_k^{\phi}=E_l^{\phi}*_{\phi}{}^{\phi}\tilde{R}^l_{kij},$$

and also construct their relatives such as ${}^{\phi}R_{ijkl}$ and ${}^{\phi}\tilde{R}_{ijkl}$. Then ${}^{\phi}R^l_{ijk}$ and ${}^{\phi}\tilde{R}^l_{ijk}$ are given by the formulas of Lemma 2.12 with $*$ replaced by $*_{\phi}$, ∂_i by ∂_i^{ϕ}, Γ by ${}^{\phi}\Gamma$, and $\tilde{\Gamma}$ by ${}^{\phi}\tilde{\Gamma}$. Also, Proposition 2.13 is still valid in the present case, and ${}^{\phi}R^l_{ijk}$ and ${}^{\phi}R^l_{ijk}$ satisfy the Bianchi identities (see Theorem 4.3).

Now we examine properties of noncommutative surfaces under general coordinate transformations. Let $X=(X^1,X^2,\ \ldots,X^m)$ be an element of $\mathcal{A}^m$. We assume that X gives rise to a noncommutative surface over $(\mathcal{A},*)$. Then we have the following related noncommutative surfaces:

$$\hat{X}=(\phi(X^1),\phi(X^2),\ \ldots,\phi(X^m)),\quad \text{over}(\mathcal{A},*),$$

$$X^{\phi}=(X^1,X^2,\ \ldots,X^m),\quad \text{over}(\mathcal{A},*_{\phi}),$$

associated with X. We call X^{ϕ} over $(\mathcal{A},*_{\phi})$ the *reparametrization* of the noncommutative surface X over $(\mathcal{A},*)$ in terms of $u=\phi^{-1}(t)$.

Remark 5.6: Note that $\mathcal{A}^m$ is regarded as an $(\mathcal{A},*)$-module when we study the noncommutative surface X and regarded as an $(\mathcal{A},*_{\phi})$-module when we study X^{ϕ}. Thus even though X and X^{ϕ} are the same element in $\mathcal{A}^m$, they have quite different meanings when the module structures of $\mathcal{A}^m$ are taken into account.

Denote by $\hat{g}$ the metric, by $\hat{\Gamma}^k_{ij}$ and $\hat{\tilde{\Gamma}}^k_{ij}$ the Christoffel symbols, and by $\hat{R}^l_{ijk}$ and $\hat{\tilde{R}}^l_{ijk}$ the Riemannian curvatures of $\hat{X}$. They can be computed by using n-dimensional generalizations of the relevant formulas derived in Sec. II. The metric, curvature, and other related objects of the noncommutative surface X^{ϕ} over $(\mathcal{A},*_{\phi})$ are given in the last subsection.

Theorem 5.7: *There exist the following relations:*

$${}^{\phi}g_{ij}=\phi^{-1}(\hat{g}_{ij}),\quad {}^{\phi}g^{ij}=\phi^{-1}(\hat{g}^{ij}),$$

$${}^{\phi}\Gamma^k_{ij}=\phi^{-1}(\hat{\Gamma}^k_{ij}),\quad {}^{\phi}\tilde{\Gamma}^k_{ij}=\phi^{-1}(\hat{\tilde{\Gamma}}^k_{ij}),$$

$${}^{\phi}R^l_{ijk}=\phi^{-1}(\hat{R}^l_{ijk}),\quad {}^{\phi}\tilde{R}^l_{ijk}=\phi^{-1}(\hat{\tilde{R}}^l_{ijk}).$$

Remark 5.8: We shall see in the next section that this theorem leads to the standard transformation rules for the metric, connection, and curvature tensors in the commutative setting when we take the limit $\bar{h}\to 0$.

Proof of Theorem 5.7: Consider the first relation. Since ϕ^{-1} is an algebraic isomorphism from $(\mathcal{A},*)$ to $(\mathcal{A},*_{\phi})$, we have

$$\phi^{-1}(\hat{g}_{ij})=\phi^{-1}(\partial_i\hat{X}\bullet\partial_j\hat{X})=\phi^{-1}(\partial_i\hat{X})\bullet_{\phi}\phi^{-1}(\partial_j\hat{X}).$$

Using $\phi^{-1}(\partial_i\hat{X})=\partial_i^{\phi}X$, we obtain

$$\phi^{-1}(\hat{E}_i)=E_i^{\phi},\quad \forall\, i.$$

Thus

$$\phi^{-1}(\hat{g}_{ij}) = E_i^\phi \bullet_\phi E_j^\phi = {}^\phi g_{ij}.$$

Since ϕ maps 1 to itself, it follows that $\phi^{-1}(\hat{g}^{ij}) = {}^\phi g^{ij}$. Now

$$\phi^{-1}(\hat{\Gamma}_{ij}^k) = \phi^{-1}(\partial_i \hat{E}_j \bullet \hat{E}_l * \hat{g}^{lk}) = \phi^{-1}(\partial_i \hat{E}_j) \bullet_\phi \phi^{-1}(\hat{E}_l) *_\phi \phi^{-1}(\hat{g}^{lk}) = \partial_i^\phi E_j^\phi \bullet_\phi E_l^\phi *_\phi {}^\phi g^{lk} = {}^\phi \Gamma_{ij}^k.$$

The other relations can also be proven similarly by using the fact that ϕ^{-1} is an algebraic isomorphism. We omit the details. □

It is useful to observe how the covariant derivatives transform under general coordinate transformations. We have

$$\nabla_i^\phi E_j^\phi = \partial_i^\phi E_j^\phi + {}^\phi \Gamma_{ij}^k *_\phi E_k^\phi = \phi^{-1}(\partial_i \hat{E}_j) + \phi^{-1}(\hat{\Gamma}_{ij}^k * \hat{E}_k) = \phi^{-1}(\hat{\nabla}_i \hat{E}_j),$$

where $\hat{\nabla}_i$ is the covariant derivative in terms of the Christoffel symbols $\hat{\Gamma}_{ij}^k$.

Remark 5.9: The gauge transformation ϕ that procures the general coordinate transformation also changes the algebra $(\mathcal{A}, *)$ to $(\mathcal{A}, *_\phi)$, thus inducing a map between noncommutative surfaces defined over gauge equivalent noncommutative associative algebras. This is very different from what happens in the commutative case, but appears to be necessary in the noncommutative setting.

Remark 5.10: Although the concept of covariance under the gauge transformation ϕ is transparent and the considerations above show that our construction of noncommutative surfaces is indeed covariant under such transformations, it appears that the concept of invariance becomes more subtle. In the classical case, a scalar is a function on a manifold, which takes a value at each point of the manifold. Invariance means that when we evaluate the function at "the same point" on the manifold, we get the same value (a real or complex number) regardless of the coordinate system which we use for the calculation. In the noncommutative case, elements of $\mathcal{A}$ are not numbers. When a general coordinate transformation is performed, the algebraic structure of $\mathcal{A}$ changes. It becomes rather unclear how to compare elements in two different algebras.

C. Comparison with classical case

One obvious question is the classical analogs of the differential operators ∂_i^ϕ and the gauge transformations in $\mathcal{G}(\mathcal{A})$ which bring about the general coordinate transformations. We address this question below. Morally, one should regard $\mathcal{G}(\mathcal{A})$ as the "diffeomorphism group" of the surface and ∂_i^ϕ as "differentiation" with respect to the new coordinate $u^i := \phi^{-1}(t^i)$.

Consider an element $\phi \in \mathcal{G}(\mathcal{A})$. In the classical limit (that is, $\bar{h}=0$), ϕ obviously reduces to an element $\exp(\nu)$ in the diffeomorphism group of U, where $v = \varepsilon^i(t)\partial_i$ is a smooth tangent vector field on U classically. Then for any two smooth functions $a(t)$ and $b(t)$, the Leibniz rule $v(ab) = v(a)b + av(b)$ implies that $\exp(v)(ab) = (\exp(v)a)(\exp(v)b)$ at the classical level. By regarding a smooth function $a(t)$ as a power series in t (or t translated by some constants) we then easily see that

$$\exp(v)a(t) = a(\exp(v)t) \quad \text{at the classical level.} \tag{5.7}$$

Now for all $f(t) \in \mathcal{A}$,

$$\partial_i^\phi \phi^{-1}(f(t)) = \phi^{-1}(\partial_i f(t)) = \frac{\partial f(u)}{\partial u^i} \mod \bar{h} = \frac{\partial \phi^{-1}(f(t))}{\partial u^i} \mod \bar{h},$$

where we have used (5.7). Replacing $f(t)$ by $\phi(f(t))$ in the above computations we arrive at

$$\partial_i^\phi f(t) = \frac{\partial f(t)}{\partial u^i} \mod \bar{h}.$$

Using this result, we obtain

$$^{\phi}g_{ij} = \phi^{-1}\left(\phi\left(\frac{\partial X(t)}{\partial u^i}\right)\cdot\phi\left(\frac{\partial X(t)}{\partial u^j}\right)\right) \mod \bar{h},$$

where the $\cdot$ on the right-hand side is the usual scalar product for $\mathbb{R}^n$. Up to $\bar{h}$ terms,

$$\phi^{-1}\left(\phi\left(\frac{\partial X(t)}{\partial u^i}\right)\cdot\phi\left(\frac{\partial X(t)}{\partial u^j}\right)\right) = \frac{\partial X(t)}{\partial u^i}\cdot\frac{\partial X(t)}{\partial u^j} \mod \bar{h},$$

hence

$$^{\phi}g_{ij} = \frac{\partial t^p}{\partial u^i}g_{pq}(t)\frac{\partial t^q}{\partial u^j} \mod \bar{h}.$$

This is the usual transformation rule for the metric if we ignore terms of order $\geqslant 1$ in $\bar{h}$.

It is fairly clear now that we shall also recover the usual transformation rules for the Christoffel symbols and curvatures in the classical limit $\bar{h}\to 0$. We omit the proof.

VI. NONCOMMUTATIVE SURFACES: SKETCH OF GENERAL THEORY

In the earlier sections, we presented a theory of noncommutative n-dimensional surfaces over a deformation of the algebra of smooth functions on a region $U\subset\mathbb{R}^n$. This theory readily generalizes to arbitrary associative algebras with derivations. Below is a brief outline of the general theory.

Let $\mathcal{A}$ be an arbitrary unital associative algebra over a commutative ring k. We shall write ab as the product of any two elements $a,b\in\mathcal{A}$. Let $Z(\mathcal{A})$ be the center of $\mathcal{A}$. Then the set of derivations of $\mathcal{A}$ forms a left $Z(\mathcal{A})$-module such that for any derivation d and $z\in Z(\mathcal{A})$, zd is the derivation which maps any $a\in\mathcal{A}$ to $zd(a)$. We require $\mathcal{A}$ to have the following properties: the associative algebra $\mathcal{A}$ has a set of mutually commutative and $Z(\mathcal{A})$-linearly independent derivations ∂_i $(i=1,2,\ \dots,n)$.

Remark 6.1: The Moyal algebra satisfies these conditions. However, in general, the assumptions impose stringent constraints on the noncommutative algebras under consideration.

Let $\mathcal{A}^m$ be the free $\mathcal{A}$-module of rank m. Define a dot product

$$\bullet:\mathcal{A}^m\otimes_k\mathcal{A}^m \to \mathcal{A}, \quad A\bullet B = a_i b^i$$

for any $A=(a_1,\ \dots,a_m)$ and $B=(b^1,\ \dots,b^m)$ in $\mathcal{A}^m$. Let $X=(X^1,X^2,\ \dots,X^m)$ be an element of $\mathcal{A}^m$ for some fixed $m>n$. As before, we define

$$E_i = \partial_i X = (\partial_i X^1,\partial_i X^2,\ \dots,\partial_i X^m) \tag{6.1}$$

and construct an $n\times n$ matrix g over $\mathcal{A}$ with entries

$$g_{ij} = E_i\bullet E_j. \tag{6.2}$$

We say that X *defines a noncommutative surface over* $\mathcal{A}$ if $g\in GL_n(\mathcal{A})$ and call g the metric of the noncommutative surface.

Clearly the $Z(\mathcal{A})$-linear independence requirement on the derivations is necessary in order for any invertible g to exist. If $g\in GL_n(\mathcal{A})$, then

$$TX = \{z^i E_i | z^i \in \mathcal{A}\}, \quad \widetilde{T}X = \{E_i w^i | z^i \in \mathcal{A}\}$$

are finitely generated projective (left or right) $\mathcal{A}$-modules, which are taken to be the tangent bundles of the noncommutative surface. The metric defines a map

$$g:TX\otimes_k\widetilde{T}X \to \mathcal{A}, \quad Z\otimes W \mapsto g(Z,W) = Z\bullet W$$

of two-side $\mathcal{A}$-modules. We define connections

$$\{\nabla_i : TX \to TX | i = 1, \ldots, n\}, \quad \{\tilde{\nabla}_i : \tilde{T}X \to \tilde{T}X | i = 1, \ldots, n\}$$

on the left and right tangent bundles, respectively, by generalizing the standard procedure in the theory of surfaces,[21]

$$\nabla_i(fZ) = (\partial_i f)Z + f\nabla_i Z, \quad \forall f \in \mathcal{A}, Z \in TX,$$

$$g(\nabla_i E_j, E_l) = \partial_i E_j \bullet E_l, \tag{6.3}$$

and

$$\tilde{\nabla}_i(Wf) = W\partial_i f + \tilde{\nabla}_i W f, \quad \forall f \in \mathcal{A}, W \in \tilde{T}X,$$

$$g(E_j, \tilde{\nabla}_i E_l) = E_j \bullet \partial_i E_l. \tag{6.4}$$

Then the connections are compatible with the metric in the sense of Proposition 2.11.

It can be shown that $[\nabla_i, \nabla_j] : TX \to TX$ and $[\tilde{\nabla}_i, \tilde{\nabla}_j] : \tilde{T}X \to \tilde{T}X$ are left and right $\mathcal{A}$-module homomorphisms, respectively. Thus one can define curvatures of the connections on the left and right tangent bundles in the same way as in Secs. II and IV [see Eq. (2.13)]. We shall not present the details here, but merely point out that the various curvatures still satisfy Propositions 2.13 and 4.3.

VII. DISCUSSION AND CONCLUSIONS

Riemannian geometry is the underlying structure of Einstein's theory of general relativity, and historically the realization of this fact led to important further developments. In this paper we have developed a Riemannian geometry of noncommutative surfaces as a first step toward the construction of a consistent noncommutative gravitational theory.

Our treatment starts from the simplest nontrivial examples, on which the general theory is gradually elaborated. We begin by constructing a noncommutative Riemannian geometry for noncommutative analogs of two-dimensional surfaces embedded in three-space, working over an associative algebra $\mathcal{A}$, which is a deformation of the algebra of smooth functions on a region of $\mathbb{R}^2$. On $\mathcal{A}^3$ we define a dot product analogous to the usual scalar product for the Euclidean three-space. An embedding X of a noncommutative surface is defined to be an element of $\mathcal{A}^3$ satisfying certain conditions. Partial derivatives of X then generate a left and also a right projective $\mathcal{A}$-module, which are taken to be the tangent bundles of the noncommutative surface. Now the dot product on $\mathcal{A}^3$ induces a metric on the tangent bundles, and connections on the tangent bundles can also be introduced following the standard procedure in the theory of surfaces.[21] Much of the classical differential geometry for surfaces is shown to generalize naturally to this noncommutative setting. We point out that the embeddings greatly help the understanding of the geometry of noncommutative surfaces.

From the noncommutative Riemannian geometry of the two-dimensional surfaces we go straightforwardly to the generalization to noncommutative geometries corresponding to n-dimensional surfaces embedded in spaces of higher dimensions. In higher dimensions, the Riemannian curvature becomes much more complicated, thus it is useful to know its symmetries. A result on this is the noncommutative analogs of Bianchi identities proved in Theorem 4.3.

We also observe that there exists another object Θ^i_j [see (4.8)], which is distinct from the Ricci curvature R^i_j but also reduces to the classical Ricci curvature in the commutative case. Contracting indices in the second noncommutative Bianchi identity, we arrive at an equation involving "covariant derivatives" of both R^i_j and Θ^i_j. This appears to suggest that Einstein's equation acquires modification in the noncommutative setting, as shown in Sec. IV C. Work along this line is in progress.[30]

A special emphasis is put on the covariance under general coordinate transformations, as the fundamental principle of general relativity. It is physically natural that under general coordinate transformations, the frame-dependent Moyal star product would change. In this spirit, we introduce in Sec. V the general coordinate transformations for noncommutative surfaces, in the form of gauge transformations on the underlying noncommutative associative algebra $\mathcal{A}$, which change as well the multiplication of the underlying associative algebra $\mathcal{A}$, turning it into another algebra nontrivially isomorphic to $\mathcal{A}$. By comparison with classical Riemannian geometry, we show that the gauge transformations should be considered as noncommutative analogs of diffeomorphisms. We emphasize that in our construction we allow for all possible diffeomorphisms, and not only those preserving the θ-matrix constant, as has been done so far in most of the literature in the field.

The results are eventually generalized to a theory of noncommutative Riemannian geometry of n-dimensional surfaces over unital associative algebras with derivations. This is outlined in Sec. IV. Noncommutative surfaces should provide a useful test ground for generalizing Riemannian geometry to the noncommutative setting.

From the point of view of physics, noncommutative surfaces with Minkowski signature, which were briefly alluded to in Remark 2, are more interesting. To treat such noncommutative surfaces in depth, more care will be required. It is known in the commutative case that the realization of a pseudo-Riemannian surface in the flat Minkowski space may contain isotropic subsets with singular metrics.

The ultimate aim is to obtain a noncommutative version of gravitational theory, covariant under appropriately defined general coordinate transformations, and, possibly, compatible with the gauging of the twisted Poincaré symmetry, in analogy with the classical works of Utiyama[31] and Kibble.[32]

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