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Abstract

A Realised Volatility GARCH model is developed within a Bayesian framework for the purpose of forecasting Value at Risk and Conditional Value at Risk. Student-t and Skewed Student-t return distributions are combined with Gaussian and Student-t distributions in the measurement equation in a GARCH framework to forecast tail risk in eight international equity index markets over a four year period. Three Realised Volatility proxies are considered within this framework. Realised Volatility GARCH models show a marked improvement compared to ordinary GARCH for both Value at Risk and Conditional Value at Risk forecasting. This improvement is consistent across a variety of data, volatility model specifications and distributions, and demonstrates that Realised Volatility is superior when producing volatility forecasts. Realised Volatility models implementing a Skewed Student-t distribution for returns in the GARCH equation are favoured.

Keywords: Realised Volatility, GARCH, Value-at-Risk, CVaR, Expected Shortfall, High-Frequency Data, Risk Management.

1 Introduction

Volatility of financial assets is a crucial aspect of risk management, and as such has attracted much attention from both academia and industry. An undisputed and key feature of returns is the change of volatility over time, and therefore it becomes paramount to model such dynamics. Seminal models such as the ARCH model by Engle (1982) and its generalisation, the GARCH model by Bollerslev (1986), were developed in order to capture such features, and are extensively used in a wide variety of applications that require the forecast of a volatility process, such as options pricing and risk management.

A large number of GARCH specifications have since been developed, with the majority utilising daily returns as the key component in their specification of conditional variance. The reasoning behind this being that the current level of volatility can be unbiasedly estimated by squared returns and used to form expectations regarding the next period, but this only offers a weak signal about the current level of volatility (Hansen, Huang, & Shek, 2012). This results in GARCH models being inefficient in reflecting the 'true' volatility during times of rapid volatility changes due to them overreacting to large shocks and being slow in going back to the long term average (Andersen, Bollerslev, Diebold, & Labys, 2003).

Far more efficient measures of realised volatility have therefore been constructed following the availability of high-frequency data, such as Realised Variance and Realised Kernel amongst others, as discussed in Andersen and Bollerslev (1998), Andersen, Bollerslev, Diebold, and Labys (2001), Barndorff-Nielsen (2002) and Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008). These measures are more accurate proxies of volatility compared to squared returns, and have therefore been used in some of the recent papers in the field.

Unfortunately these methods are not a panacea as there are issues in calculating Realised Volatility due to micro-structure noise caused by bid-ask spread and non-synchronous trading (Campbell & Lo, 1997). Methods influenced by Zhang, Mykland, and Ait-Sahalia (2005), Bandi and Russell (2008) Bandi and Russell (2011) and Barndorff-Nielsen, Hansen, Lunde, and Shephard (2011) have been developed to partially overcome such shortcomings. Non-trading hours where the availability of high-frequency returns is scarce or unavailable (such as during lunch-time or overnight) also introduces another issue in computing realised volatility. Other similar approaches have been employed by Ghysels, Santa-Clara, and Valkanov (2004) in their Mixed Data Sampling Regression Models (MIDAS), where time series at different frequencies are included into a single regression model, with a further extension being the HYBRID-GARCH model of Chen, Ghysels, and Wang (2011) where, within a GARCH framework, returns are sampled at higher frequency than the prediction horizon and assigned non identical weights. The introduction of Realised GARCH by Hansen et al. (2012) with the inclusion of a measurement equation has been shown to possess many attractive features together with substantial improvements over daily return GARCH.

This paper builds on recent work by Watanabe (2012) and Hansen et al. (2012) on Realised GARCH models for estimation of quantile forecasts, but extends it in the following ways: First, Bayesian Markov Chain Monte Carlo (MCMC) procedures are employed for the estimation of all models to account for estimation risk. As model parameters are assumed to be random, together with the addition of five extra parameters, it means that complex numerical estimation methods are required, so an MCMC approach is used. Bayes' law is used to recover the posterior distribution of the parameters, and since they have a complicated functional form, MCMC methods are used as it is impossible to simulate from them directly. Second, a Student-t distribution is proposed in the measurement equation in order to account for non-gaussianity in the distribution of the respective error term, as proposed by Watanabe (2012) as an extension to his work, for example an outlier occurring due to a sudden intra-day price movement. Third, an alternative specification of the Skewed Student-t distribution using Hansen (1994) is proposed for the returns equation. Lastly, three different Realised Volatility measures are utilised within this framework, so that they can be scrutinised and their performance compared to one another.

To the best of our knowledge this is the first implementation of a Realised Volatility GARCH model within a Bayesian framework. The proposal of a Student-t distribution for the measurement equation is also novel, together with Hansen's specification of the Skewed Student-t distribution. Student-t as well as Skewed Student-t distributions are considered for all models, and skewness and kurtosis parameters are estimated jointly with the other parameters of the volatility and measurement equations. Daily data from eight international equity indexes are used from January 2000 to February 2014. For each

data set daily returns and a variety of Realised Volatility measures are tested against a range of models and compared by the quantile forecasting performance of 1000 one-step ahead forecasts using a moving estimation window.

The main results are: Realised Volatility GARCH models perform better than their daily counterpart for Value-at-Risk and Expected Shortfall, with the choice of distribution being a significant factor. Realised Volatility models implementing a Skewed Student-t distribution for returns in the GARCH equation clearly being favoured, with the choice of Realised Volatility measure not being a large factor for quantile forecasting emerging from this analysis.

The paper is structured in the following way; Section 2 specifies the Realised Volatility GARCH model and the calculation of the various Realised Volatility Measures, Section 3 gives a brief overview of the Bayesian approach and Markov-Chain Monte Carlo methods employed, Section 4 gives an overview of the forecasting methodologies employed and the relevant backtesting, Section 5 presents the data and the empirical studies from eight international equity indexes for both Value-at-Risk and Expected Shortfall using formal and informal tests. Concluding remarks and possible extensions are given in Section 6. An Appendix and Tables with parameter estimation results follow.

2 Realised Volatility GARCH Model specifications

2.1 Calculating Realised Volatility Measures

For a log return series r_t , the Realised Volatility (RV) over the time period $[t - j, t]$, for $0 < j \leq t \leq T$ is defined as:

$$RV(t, j; m) \equiv \sum_{j=1}^m r(t - (j - 1)/m)^2, \quad (1)$$

where m is the number of intra-daily observations and $1/m$ is the sampling frequency. We can therefore treat RV as the second sample moment of a fixed interval return process of length j , in turn providing a volatility measure for the the j -period measurement interval. In this paper we consider RV calculated using the following intervals: 5 minutes (RV5), 5 minutes using 1 minute sub-samples (RV5ss) as well as the realised kernel (RK) as evaluated in Barndorff-Nielsen et al. (2008) using a Parzen weight function taking the following form:

$$K(X) = \sum_{h=-H}^H k\left(\frac{h}{H+1}\right) \gamma h, \quad \gamma h = \sum_{j=|h|+1}^n x_j x_{j-|h|}, \quad (2)$$

in which $k(x)$ is the kernel weight function, in this case using the Parzen kernel function:

$$k(x) = \begin{cases} 1 = 6x^2 + 6x^3 & 0 \geq x \geq 1/2 \\ 2(1-x)^3 & 1/2 \geq x \geq 1 \\ 0 & x > 1. \end{cases} \quad (3)$$

where x_j is the high frequency return in both (2) and (3) and H is the choice of bandwidth:

$$H^* = c^* \xi^{4/5} n^{3/5}, \quad c^* = ((12)^2 / 0.2969)^{1/5} = 3.5134,$$

$$\text{and } \xi^2 = \frac{\omega^2}{\sqrt{T \int_0^T \sigma_u^4 du}}$$

The subsampling uses a method developed by Zhang et al. (2005) in which fewer data is discarded as opposed to (1). Sampling here is carried out over subgrids of observations every 1 minute, and the results are averaged across those subgrids. Zhang et al. (2005) show that this results in a substantial decrease in bias of the estimator. By demeaning the return series so that $\mu(t) \equiv 0$, the realised volatility represents the ex-post sample variance computed from m discretely sampled (j/m) -period returns over the period $[t-j, t]$. In this case RV is (ex-ante) unbiased (except for microstructure effects) for the expected volatility $\text{Var}(t, j)$. Watanabe (2012) estimates RV-GARCH models with the following ARCH and GARCH orders: (1,1),(2,1) and (2,2) for quantile forecasts, but performance does not alter by much across these choices, and we therefore only consider RV-GARCH(1,1) in this paper.

2.2 Realised Volatility GARCH with Log-Linear Specification

The RV-GARCH(1,1) model employing a simple log-linear specification is defined by the following volatility and measurement equation respectively:

$$\begin{aligned} a_t &= r_t - \mu, \\ a_t &= \sigma_t \epsilon_t, \\ \epsilon_t &\stackrel{\text{iid}}{\sim} t_\nu, \quad \text{or} \quad \epsilon_t \stackrel{\text{iid}}{\sim} skt_{\nu, \lambda}, \\ \ln(\sigma_t^2) &= \alpha_0 + \alpha_1 \ln(RV_{t-1}) + \beta_1 \ln(\sigma_{t-1}^2) \end{aligned} \quad (4)$$

$$\begin{aligned} \ln(RV_t) &= \xi + \phi \ln(\sigma_t^2) + \tau_1 \epsilon_t + \tau_2 (\epsilon_t^2 - 1) + u_t \\ u_t &\stackrel{\text{iid}}{\sim} \mathcal{N}(0, \sigma_u^2) \quad \text{or} \quad u_t \stackrel{\text{iid}}{\sim} t_{\nu_{mes}}, \end{aligned} \quad (5)$$

This is a similar specification as employed by Hansen et al. (2012) with the addition of

a Student-t specification for the measurement equation in (5), which provides a coherent way to model the joint dependence between returns and the Realised Volatility measure. A limitation of GARCH models is their inability to distinguish between the asymmetry of returns and the respective effects on volatility, the so called "leverage effect", as only the magnitude of returns are considered and not their signs. (5) also includes a leverage function $\tau_1\epsilon_t + \tau_2(\epsilon_t^2 - 1)$ in quadratic form which captures any asymmetry in the dependence between returns and future volatility. This is similar to the EGARCH structure introduced by Nelson (1991), and therefore makes the use of an explicit leverage function in the volatility equation redundant. This quadratic form is chosen as in Hansen et al. (2012) as for the chosen distributions where $E(\epsilon_t) = 0$ and $\text{var}(\epsilon_t) = 1$ it guarantees that $E(\tau(\epsilon_t)) = 0$.

This model is therefore non-linear due to the response of $\tau_1\epsilon_t + \tau_2(\epsilon_t^2 - 1)$ to a negative or positive ϵ_t , as if $\tau_1 < 0$ and $\tau_2 > 0$ as expected, negative shocks will have larger impacts than positive ones. The size of the asymmetry between a negative and positive shock is dictated by τ_1 while the level at which this occurs is governed by τ_2 .

A logarithmic specification is chosen for both (4) and (5) as it guarantees a positive variance in (4). One drawback of using a logarithmic specification, known as LGARCH as per Geweke (1986); Pantula (1986); Milhøj (1988), is that if a zero return is observed, it will cause a runtime error in the likelihood evaluation. This is not an issue in either (4) and (5) as returns do not appear in either equation.

Microstructure noise and non-trading hours can introduce bias in RV, but this should be adjusted by ξ and ϕ in (5), while also considering RV calculated at different intervals as well as the Realised Kernel should reduce this bias to a minimum. Estimates for ϕ in (5) will suggest how much Realised Volatility is explained by conditional variance. In a scenario with zero bias and where Realised Volatility was an unbiased estimator of true volatility, we would observe $\xi = 0$ and $\phi = 1$.

We adopt a standardised Skewed Student-t distribution introduced by Hansen (1994) with takes the following PDF:

$$f(\epsilon|\nu, \lambda) = \begin{cases} bc \left(1 + \frac{1}{\nu - 2} \left(\frac{b\epsilon + a}{1 - \lambda} \right)^2 \right)^{-(\nu+1)/2} & \text{if } \epsilon < -\frac{a}{b} \\ bc \left(1 + \frac{1}{\nu - 2} \left(\frac{b\epsilon + a}{1 + \lambda} \right)^2 \right)^{-(\nu+1)/2} & \text{if } \epsilon \geq -\frac{a}{b} \end{cases} \quad (6)$$

where ν is the kurtosis parameter with range $2 < \nu < \infty$ and λ is the skewness parameter with range $-1 < \lambda < 1$ and a , b , and c are given by:

$$a = 4\lambda c \left(\frac{\nu - 2}{\nu - 1} \right), \quad b = \sqrt{1 + 3\lambda^2 - a^2}, \quad c = \frac{\Gamma\left(\frac{\nu + 1}{2}\right)}{\sqrt{\pi(\nu - 2)}\Gamma\left(\frac{\nu}{2}\right)}$$

The density above specialises to a Student-t density when $\lambda = 0$, and when $\lambda > 0$ the mode is to the left of zero and the density is skewed to the right, with the opposite occurring when $\lambda < 0$.

In this paper we propose standardised Student-t and standardised Skewed Student-t specification in (4) and standard Normal and standardised Student-t in (5), leading to four possible distribution combinations. We leave out Gaussian in (4) as a straw man.

The joint likelihood to be evaluated becomes:

$$\mathcal{L} = \prod_{t=1}^n h(r_t|r_1, \dots, r_{t-1})h(RV_t|RV_1, \dots, RV_{t-1}) \quad (7)$$

where $h(R_t|RV_1, \dots, RV_{t-1})$ is the pdf of the distribution of ϵ in (4) and $h(RV_t|RV_1, \dots, RV_{t-1})$ is the pdf of the distribution of u_t in (5).

The distributions used and parameters to estimate in the models are as follows, with detailed joint log-likelihoods provided in the Appendix:

Table 1: Distributions and Parameters to estimate

	u_t : Normal	u_t : Student-t
ϵ : Student-t	$\mu, \alpha_0, \alpha_1, \beta_1, \nu, \xi, \phi, \tau_1, \tau_2, \sigma_u^2, \sigma_0$	$\mu, \alpha_0, \alpha_1, \beta_1, \nu, \xi, \phi, \tau_1, \tau_2, \sigma_u^2, \nu_{mes}, \sigma_0$
ϵ : Skew Student-t	$\mu, \alpha_0, \alpha_1, \beta_1, \nu, \lambda, \xi, \phi, \tau_1, \tau_2, \sigma_u^2, \sigma_0$	$\mu, \alpha_0, \alpha_1, \beta_1, \nu, \lambda, \xi, \phi, \tau_1, \tau_2, \sigma_u^2, \nu_{mes}, \sigma_0$

Table 2: Distribution Notation

	u_t : Normal	u_t : Student-t
ϵ : Student-t	RGtN	RGtt
ϵ : Skew Student-t	RGSktN	RGSktt

These parameters are estimated via Bayesian MCMC methods which are further discussed in Section 3.

The long run unconditional means of (4) and (5) are respectively:

$$E(\ln(\sigma^2)) = \frac{(\alpha_0 + \alpha_1\xi)}{(1 - \beta_1 - \alpha_1\phi)}, \quad \text{and} \quad E(\ln(RV)) = \xi + \phi\ln(\sigma^2) \quad (8)$$

and hence stationarity is achieved given that

$$\phi\alpha_1 + \beta_1 < 1. \quad (9)$$

3 Markov-Chain Monte Carlo Estimation

A likelihood function is specified according to the model, with standard Student-t or Skewed Student-t distributions. A likelihood function for both distributions is provided for reference in Appendix A.

3.1 Prior

A combination of mostly uninformative and Jeffrey’s priors are chosen over the possible region for the parameters in θ , where:

$$\theta = [\alpha_0, \alpha_1, \beta_1, \nu, \lambda, \xi, \phi, \tau_1, \tau_2, \sigma_u^2, \nu_{mes}, \mu, \sigma_0^2]'$$

The prior, $\pi(\theta) \propto \frac{1}{\sigma_u^2}, \frac{1}{\nu}, \frac{1}{\nu_{mes}} I_A$, (where A is the region described by the restrictions in Table 3) is used for the degree of freedom parameters in (4) and (5) and a standard Jeffery’s prior for the volatility of u_t in (5) as we are assuming they are a priori independent. A flat prior on the Degree of Freedom parameters would lead to an improper posterior distribution as shown in Bauwens and Lubrano (1998).

During the MCMC process, the following parameter restrictions are placed to achieve stationarity.

Table 3: MCMC Parameter Restrictions

Parameter	$\mu, \alpha_0, \xi, \tau_1, \tau_2$	α_1, β_1	ν, ν_{mes}	$ \lambda $	$\phi, \sigma_u^2, \sigma_0^2$	$\phi\alpha_1 + \beta_1$
Bounds	$\pm\infty$	$\geq 0, < 1$	$> 2, < 200$	< 1	> 0	< 1

The ν and ν_{mes} parameters are restricted to be above 2 and under 200, as without an upper limit, and with a normal distribution of the data, the likelihood is not integrable in terms of degrees of freedom, so a limit of 200 was put in place, as the Student-t distribution is effectively normal at this level. The initial volatility is estimated during MCMC via the σ_0^2 parameter.

3.2 Sampling

In order to make inference about the parameters θ , the random walk Metropolis (RWM) algorithm is used to draw samples from the posterior distribution. The adaptive proposal (AP) algorithm of Haario, Saksman, and Tamminen (1999) is used during burn-in iterations to produce an efficient proposal distribution for the RWM algorithm. The high number of iterations, combined with many unknown parameters, eight data sets and 12 distribution/RV combinations plus two for daily models, all for 1000 day forecast results in a very large computational burden. Using several computers simultaneously, the

entire empirical forecasting exercise was completed in 5 days. Each single day’s forecast takes approximately 90 seconds

3.3 Burn-in

Assuming that the parameter vectors $\boldsymbol{\theta}^{(1)}, \dots, \boldsymbol{\theta}^{(i-1)}$, for $i > H$, have been generated, to generate $\boldsymbol{\theta}^{(i)}$, the AP algorithm first simulates $\boldsymbol{\theta}^p$ from the multivariate Normal proposal distribution:

$$\boldsymbol{\theta}^p \sim \mathcal{N}\left(\boldsymbol{\theta}^{(i-1)}, c_d \boldsymbol{\Sigma}_{t-1} + c_d \varepsilon \mathbf{I}_d\right),$$

where $\boldsymbol{\Sigma}_{t-1}$ is the sample covariance matrix of $\boldsymbol{\theta}_{t-H}, \dots, \boldsymbol{\theta}_{t-1}$, H is the “memory parameter”, which is the number of previously sampled parameter vectors used for evaluating the covariance matrix, c_d is a scale factor depending on the dimension, d , of the parameter vector, and ε is a small positive constant, which is used to prevent zero variances. Following Haario et al. (1999) and Gelman, Roberts, and Gilks (1996), $c_d = 2.4^2/d$. The algorithm then accepts $\boldsymbol{\theta}^p$ as $\boldsymbol{\theta}^{(i)}$, i.e., $\boldsymbol{\theta}^{(i)} = \boldsymbol{\theta}^p$ with probability:

$$\alpha = \min \left\{ \frac{p(\boldsymbol{\theta}^p | \mathbf{r})}{p(\boldsymbol{\theta}^{(i-1)} | \mathbf{r})}, 1 \right\}, \quad (10)$$

and rejects $\boldsymbol{\theta}^p$, i.e., $\boldsymbol{\theta}^{(i)} = \boldsymbol{\theta}^{(i-1)}$ with probability $1 - \alpha$.

The AP algorithm is set to run for 20,000 iterations per parameter, in order to guarantee convergence of the parameters. So a five parameter model would have a chain length of 100,000. Convergence usually occurs quickly, by 25,000 iterations or so.

3.4 Post burn-in

As the stationary distribution of the AP algorithm is not strictly ergodic, and it samples from a slightly different distribution from the target posterior distribution, it is used only during the burn-in period. After burn-in, the standard random walk Metropolis algorithm is used, whose proposal distribution is a multivariate Normal with a sample covariance matrix of the last 10,000 burn-in samples.

Assuming that the parameter vector $\boldsymbol{\theta}^{(i-1)}$, for $i > 1$, has been generated, to generate $\boldsymbol{\theta}^{(i)}$, the RWM algorithm first simulates $\boldsymbol{\theta}^p$ from the multivariate Normal proposal distribution:

$$\boldsymbol{\theta}^p \sim \mathcal{N}\left(\boldsymbol{\theta}^{(i-1)}, c_d \boldsymbol{\Sigma}\right).$$

The algorithm then accepts $\boldsymbol{\theta}^p$ as $\boldsymbol{\theta}^{(i)}$ with probability α given by equation (10), and rejects $\boldsymbol{\theta}^p$ with probability $1 - \alpha$. For post burn-in iterations, $\boldsymbol{\theta}^{(1)}$ and $\boldsymbol{\Sigma}$ are chosen to be the sample mean and the sample covariance matrix, respectively, of the parameter vectors sampled in the last 10,000 burn-in iterations. The RWM algorithm is set to run for 10,000 iterations per parameter, and a mean of the last 10,000 chains is used to generate the

parameter values, standard errors and 95% credible intervals. Tables of parameter results for two data sets are shown Tables 12 and 13.

4 Forecasting Methodology

4.1 Value-at-Risk Forecasts

Each model is used to produce 1000 one-step ahead volatility forecasts, which in turn are used to form a Value-at-Risk (VaR) forecast at both the 95% and 99% confidence level α as per (12),(13),(14) and (15). The accuracy of each model can then be measured by their relative forecast performance. The conditional 1 period VaR forecast is formally defined as:

$$\alpha = Pr(a_{t+1} < VaR_\alpha | \Omega_t) \quad (11)$$

where a_{t+1} is the one period return from time t to time $t+1$, α is the quantile level and Ω_t is the information set at time t . For a Student-t distribution, VaR is calculated via the inverse CDF (t_ν^{-1}) of the distribution as follows:

$$VaR_\alpha = \mu + \sigma t_\nu^{-1}(\alpha) \sqrt{\frac{\nu - 2}{\nu}}, \quad (12)$$

similarly, for a Skewed Student-t using its inverse CDF ($skt_{\nu,\lambda}^{-1}$) as follows

$$VaR_\alpha = \mu + \sigma skt_{\nu,\lambda}^{-1}, \quad (13)$$

where $skt_{\nu,\lambda}^{-1}$ is:

$$skt^{-1}(\alpha|\nu, \lambda) = \begin{cases} \frac{1 - \lambda}{b} \sqrt{\frac{\nu - 2}{\nu}} t_\nu^{-1} \left(\frac{\alpha}{1 - \lambda}, \nu \right) - \frac{a}{b}; & \text{if } \alpha < \frac{1 - \lambda}{2} \\ \frac{1 + \lambda}{b} \sqrt{\frac{\nu - 2}{\nu}} t_\nu^{-1} \left(0.5 + \frac{\alpha}{1 + \lambda} \left(\alpha - \frac{1 - \lambda}{2} \right), \nu \right) - \frac{a}{b}; & \text{if } \alpha \geq \frac{1 - \lambda}{2} \end{cases}$$

4.2 Conditional Value-at-Risk Forecasts

Conditional Value at Risk (CVaR) or Expected Shortfall (ES) is also used to evaluate models, as it has become preferred to VaR (McNeil, Frey, & Embrechts, 2005) due to the latter's shortcomings. For a Student-t distribution McNeil et al. (2005) define it as:

$$ES_\alpha = \mu + \sigma \frac{g_\nu(t_\nu^{-1}(\alpha))}{1 - \alpha} \left(\frac{\nu + (t_\nu^{-1}(\alpha))^2}{\nu - 1} \right) \sqrt{\frac{\nu - 2}{\nu}} \quad (14)$$

where ν is the estimated degrees of freedom restricted to $\nu > 2$ as per (4), g_ν and t_ν^{-1} are the Student-t probability density function and inverse cumulative distribution function

respectively.

We can derive the ES for a Skewed Student-t distribution for a long position given the standardised VaR value Θ :

$$ES_\alpha = \mu + \sigma E[\epsilon | \epsilon < \Theta] \quad (15)$$

where

$$\begin{aligned} E[\epsilon | \epsilon < \Theta] &= \frac{1}{skt_{\nu, \lambda}(\Theta)} \int_{-\infty}^{\Theta} bc \left(1 + \frac{1}{\nu - 2} \left(\frac{b\epsilon + a}{1 - \lambda} \right)^2 \right)^{-\frac{\nu+1}{2}} x dx \\ &= \frac{c(1 - \lambda)^2}{b \cdot skt_{\nu, \lambda}(\Theta)} \frac{\nu - 2}{1 - \nu} \left(1 + \frac{1}{\nu - 2} \left(\frac{b\Theta + a}{1 - \lambda} \right)^2 \right)^{\frac{1-\nu}{2}} - \frac{a}{b} \end{aligned}$$

where $skt_{\nu, \lambda}$ is the CDF of the Skewed Student-t distribution:

$$skt(\epsilon | \nu, \lambda) = \begin{cases} (1 - \lambda)t_\nu \left(\sqrt{\frac{\nu}{\nu - 2}} \left(\frac{b\epsilon + a}{1 - \lambda} \right), \nu \right) & \text{if } \epsilon < -\frac{a}{b} \\ \frac{1 - \lambda}{2}(1 + \lambda) \left[t_\nu \left(\sqrt{\frac{\nu}{\nu - 2}} \left(\frac{b\epsilon + a}{1 - \lambda} \right), \nu \right) - 0.5 \right] & \text{if } \epsilon \geq -\frac{a}{b} \end{cases}$$

Dokov, Stoyanov, and Rachev (2008) provide numerical approximations via Monte Carlo methods to calculate both VaR and CVaR by defining the Skewed Student-t distribution as a normal mixture with inverse gamma distribution, but the closed form solutions provided above avoid the large computational burden usually associated with using such techniques, and are able to provide a precise solution which is easily repeatable.

4.3 Back-testing VaR forecast models

An informal way to assess the performance of VaR forecasts, is to compute their Violation Rate (VRate) as per Gerlach, Chen, and Chan (2011).

$$VRate = \sum_{t=s+1}^{s+f} I(a_t < VaR_t), \quad (16)$$

where s is the estimation period, f is the forecast period, I is the indicator taking a value of 1 when $a_t < VaR_t$, with the scope of comparing individual violation ratios (VRatio) which is simply (16) divided by the respective level of α , with a target of 1. If competing models are equidistant from such target, the model that produces the lower VRatio is preferred as it indicates a more conservative model as per Wong and So (2003). Formal methods of evaluation are the Unconditional Coverage (UC) test introduced by Kupiec (1995), the Conditional Coverage (CC) of Christoffersen (1998) and the Dynamic Quantile (DQ) test using 3 lags of Engle and Manganelli (2004).

4.4 Back-testing CVaR forecast models

Expected Shortfall Rate (ESRate) is the equivalent violation rate for the CVaR forecasts, and is defined by Chen, Gerlach, and Lu (2012) as:

$$ESRate = \sum_{t=s+1}^{s+f} I(a_t < ES_t), \quad (17)$$

once again used to obtain an ESRatio, where 1 is preferred, by dividing (17) by the expected remaining quantile at the respective level δ_α , where each δ_α is obtained by taking the average degree of freedom and skewness, calculating the theoretical standardised CVaR, and running this through the CDF of that distribution. δ_α levels used are shown in Table (4). Similarly to VaR back-testing, the UC, CC and DQ test can also be applied to examine the violations from a CVaR forecast.

Table 4: δ_α level used to calculate ESRate for each distribution. Average Kurtosis and Skewness are used for each distribution per data set.

	$\delta_\alpha=5\%$						$\delta_\alpha=1\%$					
	Gt	GSkt	RGtN	RGtt	RGSktN	RGSktt	Gt	GSkt	RGtN	RGtt	RGSktN	RGSktt
CAC40	0.0182	0.0182	0.0187	0.0187	0.0188	0.0188	0.0036	0.0036	0.0037	0.0037	0.0037	0.0037
DAX	0.0182	0.0183	0.0187	0.0187	0.0189	0.0189	0.0036	0.0036	0.0037	0.0037	0.0037	0.0037
DJIA	0.0181	0.0182	0.0186	0.0185	0.0186	0.0186	0.0035	0.0036	0.0036	0.0036	0.0036	0.0036
IBEX	0.0180	0.0180	0.0186	0.0185	0.0186	0.0186	0.0035	0.0035	0.0036	0.0036	0.0036	0.0036
RUSS	0.0194	0.0194	0.0194	0.0194	0.0194	0.0194	0.0038	0.0038	0.0038	0.0038	0.0038	0.0038
SP500	0.0182	0.0183	0.0185	0.0185	0.0186	0.0185	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036
SSMI	0.0181	0.0181	0.0186	0.0186	0.0186	0.0186	0.0035	0.0035	0.0036	0.0036	0.0036	0.0036
STOXX	0.0181	0.0182	0.0188	0.0188	0.0189	0.0189	0.0035	0.0036	0.0037	0.0037	0.0037	0.0037

5 Empirical Study

5.1 Data

The models are tested by utilising both daily and RV series from eight international equity indices: Cotation Assistée en Continu 40 (CAC40) - France; Deutscher Aktien Index (DAX) - Germany; Dow Jones Industrial Average (DJIA) - USA; Índice Bursatil Español 35 (IBEX) - Spain; Russell 2000 (RUSS) - Small Cap USA; Standard & Poor's 500 (SP500) - USA; Swiss Market Index (SSMI) - Switzerland; Euro Stoxx 50 (STOXX) - Eurozone. The data was obtained from the Oxford-Man Institute of Quantitative Finance in turn sourcing data from Thompson Reuters and SIRCA, covering the period January 2000 to February 2014. Daily percentage log-returns are defined by

$$r_t = (\ln(P_t) - \ln(P_{t-1})) \times 100$$

Figure 1: Plots of Returns for all data sets. Red dotted line separates estimation and forecast period.

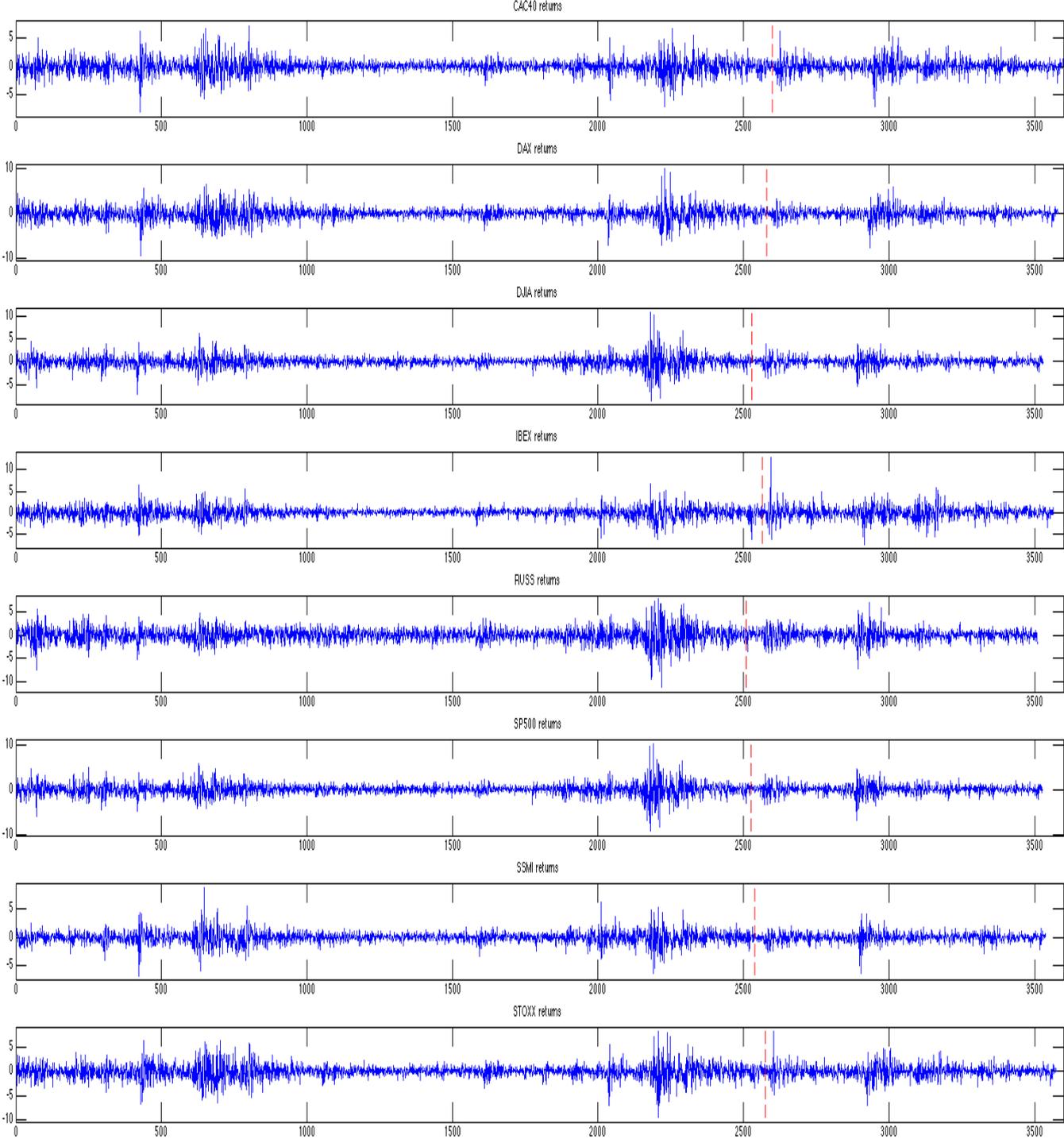


Table 5: Summary of descriptive statistics for each of the eight return series. JB is the Jarque-Bera statistic to test the null hypothesis of normality.

	Period	Mean	Std	Skewness	Kurtosis	Min	Max	JB
CAC40	Estimate (00-09)	-0.06	1.30	-0.13	7.13	-8.12	7.28	1852.3
	Forecast (10-14)	-0.01	1.18	-0.14	6.91	-7.23	6.14	640.4
DAX	Estimate (00-09)	-0.05	1.47	-0.04	7.44	-9.41	9.99	2120.2
	Forecast (10-14)	0.00	1.10	-0.25	7.68	-7.44	5.76	923.7
DJIA	Estimate (00-09)	0.01	1.29	0.10	10.77	-8.41	10.75	6370.1
	Forecast (10-14)	0.05	0.95	-0.43	6.90	-5.48	4.01	663.3
IBEX	Estimate (00-09)	-0.05	1.24	-0.23	6.11	-6.43	6.74	1056.5
	Forecast (10-14)	-0.04	1.50	0.40	10.46	-7.58	13.04	2347.5
RUSS	Estimate (00-09)	-0.01	1.59	-0.25	6.73	-11.05	7.78	1481.4
	Forecast (10-14)	0.04	1.27	-0.18	6.46	-7.15	7.04	504.2
SP500	Estimate (00-09)	-0.01	1.34	-0.07	9.90	-9.35	10.22	5010.3
	Forecast (10-14)	0.04	1.01	-0.52	7.75	-6.76	4.59	984.4
SSMI	Estimate (00-09)	-0.03	1.09	0.03	9.01	-6.73	8.68	3823.0
	Forecast (10-14)	0.00	0.81	-0.62	9.83	-6.32	4.21	2006.2
STOXX	Estimate (00-09)	-0.04	1.45	-0.17	7.33	-9.35	8.27	2026.0
	Forecast (10-14)	-0.01	1.24	0.07	7.04	-6.91	8.05	681.3

where P_t is the closing price on day t . RV is calculated as per Section 2 using the following intervals: 5 minutes, 5 minutes using 1 minute sub-samples of Zhang et al. (2005), as well as the Realised Kernel as evaluated in Barndorff-Nielsen et al. (2008) using a Parzen weight function as per (2) and (3). This generates approximately 3600 data points for each return series with small sample size variations due to trading day holidays etc. This sample is then divided into an initial learning period of 2600 days from January 2000 to early 2010 which includes the Global Financial Crisis (GFC) and a forecasting period of 1000 days from early 2010 to February 2014. A forecast during the GFC would have been beneficial and insightful, but the learning period would have been too short to provide accurate results as the data only begins in the year 2000. Figure 1 shows a plot of returns for the all data sets. Summary statistics for each data series are shown in Table 5, which clearly show that the estimation period has higher maximums and bigger losses, all periods exhibit leptokurtosis and negative skewness except for DJIA and SSMI during during the estimation period and STOXX during the forecast period. Formal tests (Jarque & Bera, 1987) for normality show that none of the series are normally distributed.

5.2 Estimation Results

In this section we present the results of the parameter estimates obtained from the MCMC chains during the learning period. Tables 12 and 13 summarise the estimation results for the CAC40 and DAX data series using the GARCH models and the log-linear RV-GARCH model with Student-t and Skewed Student-t distributions and each of the four distribution combinations for RV and three RV measures using the first 2600 returns. Other parameter result tables are omitted to save space but the findings are very similar.

What we tend to observe is the intercept parameter α_0 has a clear level shift when comparing GARCH and RV-GARCH models, as can be observed in Tables 12 and 13. In terms of statistical significance of α_0 , in GARCH it is statistically significant in all models

for all data data sets.

In terms of ARCH and GARCH parameters α_1 and β_1 , there is also a clear shift when moving from a GARCH model to the RV model, with the former going from circa 0.1 to 0.4 and the latter from 0.9 to 0.6 in the majority of data sets. While the sum of these 2 parameters always remains around unity, α_1 increases while β_1 decreases when comparing GARCH and RV-GARCH models, indicating that more emphasis is placed on RV. The sum of these parameters going beyond unity does not cause the RV-GARCH model to explode due to the restriction detailed in (9). This joint sum being close to unity is the usual characteristic of high volatility persistence in asset returns, which is present in all models and all data sets.

The degree of freedom parameter ν is consistently estimated higher when using RV-GARCH models, indicating that the data is being modelled in a superior manner when Realised Volatility is used as the tails of the residuals are not required to be as fat to fit the data as the model itself is better at capturing the excess kurtosis in the data.

Skewness parameter λ is negative implying a skew to the left and is statistically significant in all data sets, which is the expected result for financial returns data, as the tail would be longer on the left or negative side.

The parameter ξ should be close to zero if no bias is present, and in most of our data sets the value is around -0.15. Estimates of ϕ are all very close to unity, which suggest that the realised measure RV being used in (4) is roughly proportional to the conditional variance.

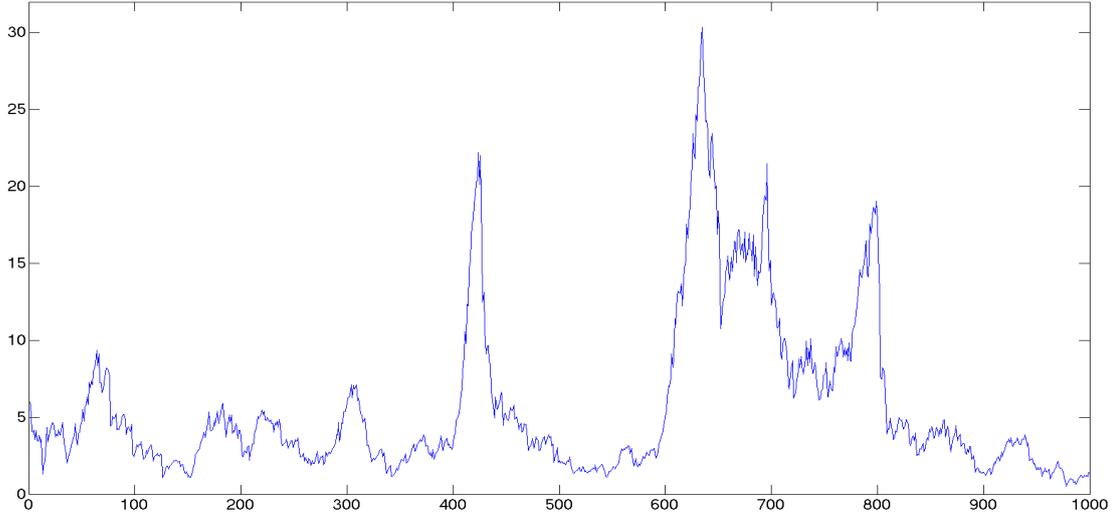
The parameters in the leverage function τ_1 and τ_2 are all of the expected sign and magnitude, with the former expected to be negative as is consistent with the well known (Nelson (1991)) phenomenon in financial returns of a negative correlation between today's returns and tomorrow's volatility, and the latter positive and the absolute value of both being circa 0.10 in the majority of our data sets. The size of the asymmetry between a negative and positive shock is dictated by τ_1 while the level at which this occurs is governed by τ_2 .

The σ_u^2 and ν_{mes} have no practical interpretation, but are required as an input for the likelihood function of the measurement equation in (5). Their value remains relatively constant across all estimates, with a level of around 0.2 and 6 to 10 respectively.

Financial return mean μ is small and not statistically significant in essentially all data sets, which is expected.

σ_0 is interesting as it shows that using a constant value for the initial volatility might not be ideal, as can be seen in Figure 2 below:

Figure 2: σ_0 GARCH Student-t CAC40



5.3 VaR forecast comparison

Table 6 shows a summary of violation ratios for each model across all data sets and with each Realised Volatility measure employed. The values which are closest to 1 are boxed while the violation ratios furthest from 1 are displayed in bold. At the $\alpha = 5\%$ and $\alpha = 1\%$ level, RV-GARCH models always provide a better VRatio compared to standard GARCH models, in many cases displaying a VRatio at exactly the 1.00 level for the former. GARCH models always have higher VRatios, therefore underestimating risk, which is an undesirable property.

Table 6: Value at Risk Violation Ratios for each distribution, RV measure and data set.

	$\alpha = 5\%$						$\alpha = 1\%$					
	Gt	GSkt	RGtN	RGtt	RGSktN	RGSktt	Gt	GSkt	RGtN	RGtt	RGSktN	RGSktt
CAC40r	1.10	1.04					2.00	1.90				
CAC40RV5			1.04	1.04	1.02	1.02			1.70	1.80	1.70	1.70
CAC40RV5ss			1.06	1.02	<u>0.98</u>	<u>0.98</u>			1.80	1.80	1.70	1.70
CAC40RK			1.06	1.06	<u>0.98</u>	0.96			1.80	1.80	1.70	<u>1.60</u>
DAXr	1.16	1.04					1.80	1.80				
DAXRV5			1.14	1.12	1.08	1.08			1.50	1.70	<u>1.10</u>	<u>1.10</u>
DAXRV5ss			1.12	1.10	0.98	<u>1.00</u>			1.50	1.40	1.20	1.20
DAXRK			1.10	1.10	<u>1.00</u>	0.98			1.50	1.50	<u>1.10</u>	<u>1.10</u>
DJIAr	1.16	1.10					2.00	1.50				
DJIARV5			1.30	1.32	<u>1.06</u>	1.12			1.70	1.90	<u>1.20</u>	<u>1.20</u>
DJIARV5ss			1.36	1.38	1.22	1.24			1.80	1.90	<u>1.20</u>	<u>1.20</u>
DJIARK			1.28	1.30	1.10	1.14			1.60	1.60	1.30	1.40
IBEXr	1.42	1.28					1.60	1.60				
IBEXRV5			1.10	1.14	<u>1.06</u>	<u>1.06</u>			1.10	1.10	<u>1.00</u>	<u>1.00</u>
IBEXRV5ss			1.18	1.18	1.10	1.10			1.10	1.10	1.10	1.10
IBEXRK			1.18	1.16	<u>1.06</u>	<u>1.06</u>			1.10	1.10	1.10	1.10
RUSRr	1.12	1.08					2.00	1.40				
RUSSRV5			1.24	1.24	1.12	1.14			1.90	2.00	1.50	1.60
RUSSRV5ss			1.14	1.12	1.02	<u>1.00</u>			2.00	2.00	<u>1.30</u>	<u>1.30</u>
RUSSRK			1.26	1.28	1.20	1.18			2.30	2.30	1.90	1.90
SP500r	1.18	1.12					1.80	1.50				
SP500RV5			1.22	1.22	<u>1.00</u>	1.04			2.30	2.30	<u>1.30</u>	1.50
SP500RV5ss			1.38	1.38	1.22	1.22			2.20	2.20	1.50	1.50
SP500RK			1.32	1.32	1.14	1.14			2.20	2.10	1.50	1.50
SSMIRr	1.28	1.10					1.40	1.30				
SSMIRV5			1.16	1.18	1.06	1.06			1.60	1.60	1.30	1.20
SSMIRV5ss			1.22	1.24	<u>0.98</u>	1.02			1.70	1.70	1.40	1.40
SSMIRK			1.18	1.16	0.94	0.94			1.60	1.60	1.20	<u>1.10</u>
STOXXr	1.22	1.12					1.60	1.30				
STOXXRV5			1.04	1.04	0.96	0.98			1.10	1.10	1.10	1.10
STOXXRV5ss			1.02	<u>1.00</u>	0.94	0.92			1.20	1.20	<u>1.00</u>	<u>1.00</u>
STOXXRK			1.02	1.02	1.02	0.98			1.30	1.30	1.10	1.10

In all data sets except for STOXX (this case is marginal) a Skewed Student-t distribution provides the best forecast, with the choice of a Student-t distribution for the measurement equation providing a similar or marginally superior forecast. It is therefore a worthwhile endeavour to use a Student-t distribution in (5), even if an additional parameter needs to be calculated. The introduction of skewness being significant is not surprising, as is a typical characteristic of data which exhibits both 'fat-tails' and asymmetric distribution.

Table 7 show the results of the UC, CC and DQ tests as described above, displaying if the model rejects the null at either the 5% and 1% significance level, where 1 indicates a rejection of the null. It can be observed that at the 5% and 1% level there is minimal model rejection for models using a Skewed Student-t distribution, and the RV5ss data set has the most model rejections when using a Student-t distribution for returns. A good VaR forecasting performance is translated into a favourable test result, with RV-GARCH models with a Skewed Student-t distribution exhibiting minimal test rejection.

Table 7: Value at Risk Unconditional Coverage, Conditional Coverage and Dynamic Quantile Test Results. 1 indicates rejection by test at relevant α level.

	$\alpha = 5\%$						$\alpha = 1\%$					
	Gt	GSkt	RGtN	RGtt	RGSktN	RGSktt	Gt	GSkt	RGtN	RGtt	RGSktN	RGSktt
Returns	1 2 1	0 1 0					3 0 4	0 0 2				
RV5			1 0 0	1 0 0	0 0 0	0 0 0			1 1 3	2 1 2	0 0 1	0 0 1
RV5ss			2 2 1	2 2 1	0 0 0	0 0 0			2 1 2	2 1 2	0 0 0	0 0 0
RK			1 0 0	2 0 1	0 0 0	0 0 0			2 2 2	2 2 2	0 0 1	0 0 1

As can be seen in Figure 3, RV-GARCH models always have preferred VRatios at the 5% and 1% quantile, regardless of distribution and frequency of RV employed, compared to their daily GARCH counterparts. There is minimal model rejection for the preferred VaR forecasts under all formal tests.

5.4 CVaR forecast comparison

Figure 4 shows 1000 one-step ahead CVaR forecast at the 1% level for the S&P500 for four models under consideration; GARCH-t, GARCH-Skt, RV5ss.t.t and RV5ss.Skt.t. The data set was chosen as it provides a clear example of the different forecasting performance of each model. The GARCH-t model consistently underestimates risk, with an ESRatio of 2.25 over the whole period, and is slow to adjust once volatility has declined. GARCH-Skt performs marginally better, with an ESRatio of 1.68, but is also slow to adjust, as is expected given the high β_1 value of circa 0.9 for both distributions. The RV5ss.t.t, with an ESRatio of 1.93, perform well compared to the GARCH-t given they are both using a Student-t distribution, indicating that the choice of RV has a favourable impact on forecasts. The RV5ss.Skt.t performs very well with an ES-ratio 1.10, and follows a close

Figure 3: Plots of Violation Ratios for Value at Risk forecasts across distributions and models for all equity indexes with $\alpha = 5\%$ on the left and $\alpha = 1\%$ on the right. Legend: x - Returns; Circle - RV5; Triangle - RV5ss; Star - RK. Ratio of 1 is best, represented by dotted line.

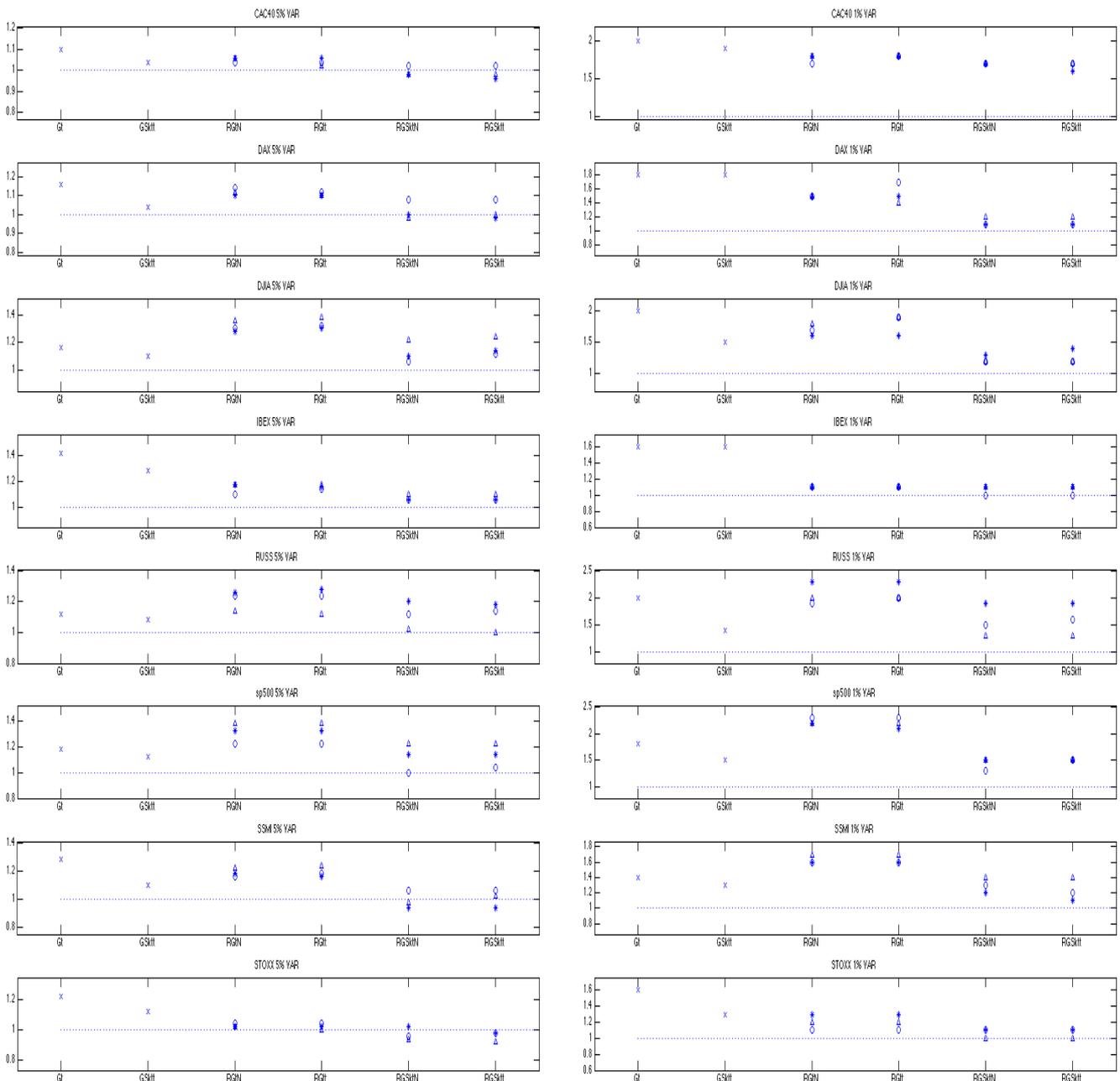


Figure 4: 1% ES Forecasts for GARCH with Student-t (black dashed dotted), GARCH with Skewed Student-t (magenta dotted), RV5ss.t.t (blue dashed) and RV5ss_Skt.t (green solid) for the S&P 500 over the whole 1000 day forecast period.

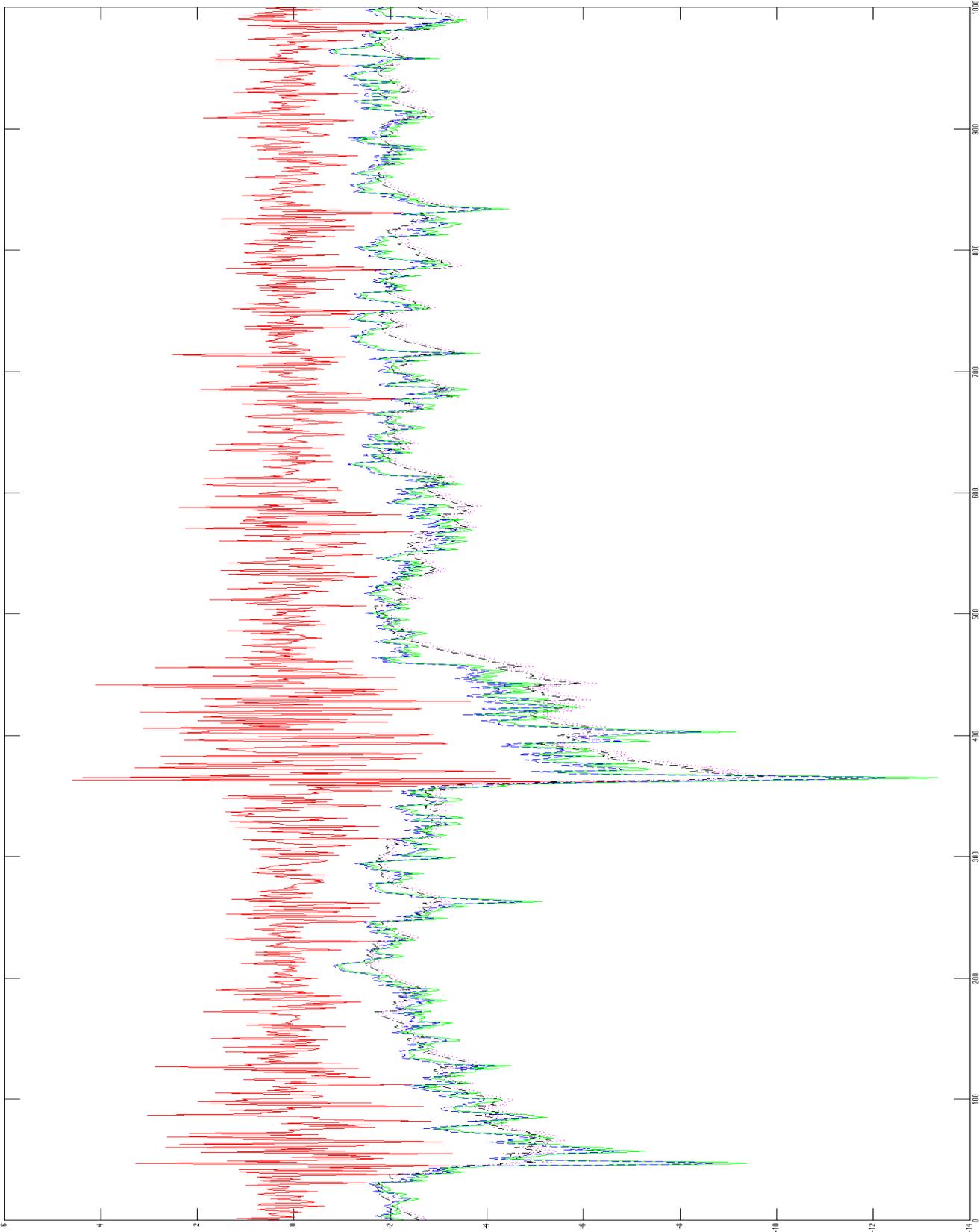
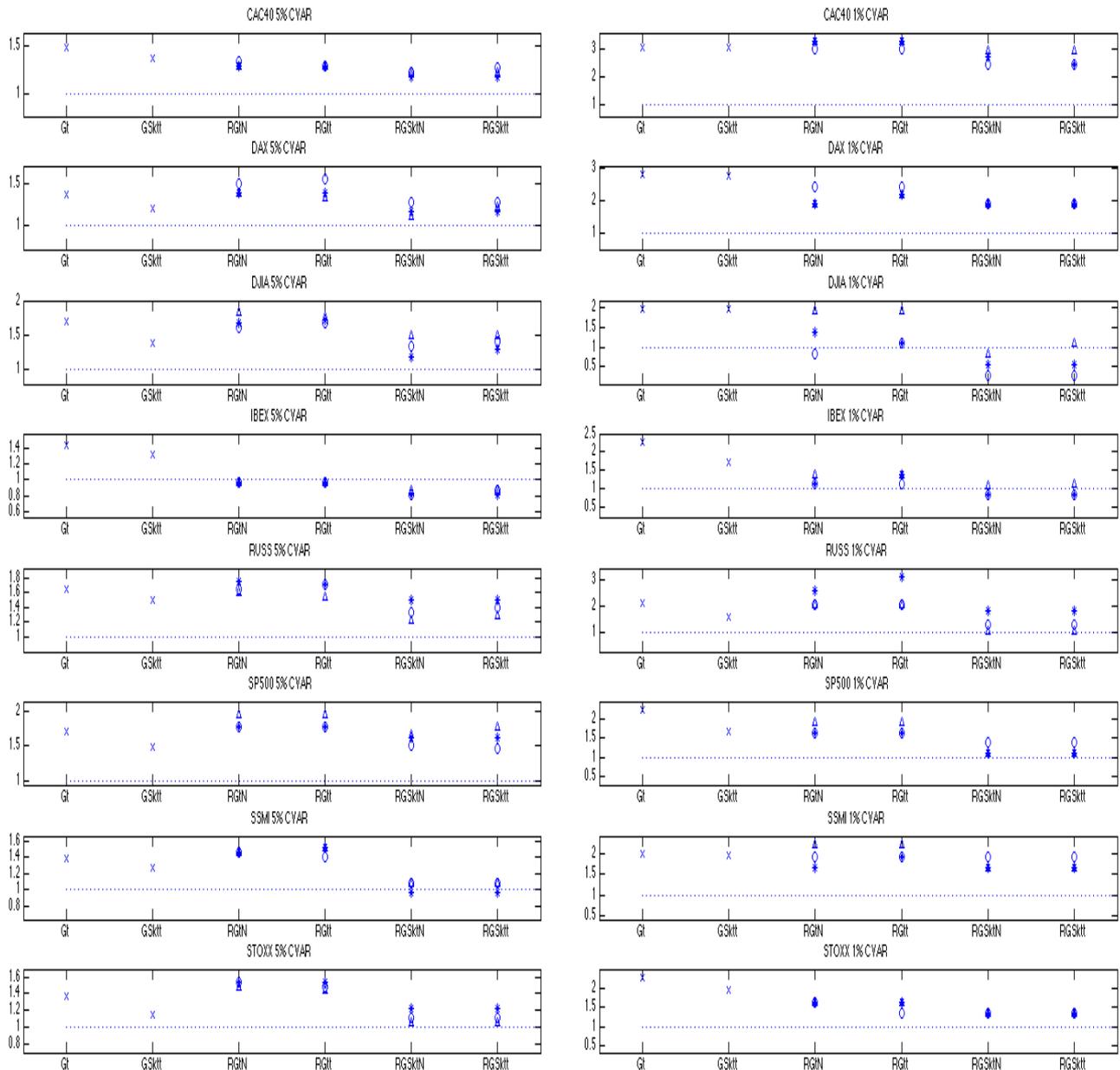


Figure 5: Plots of Violation Ratios for Expected Shortfall forecasts across distributions and models for all equity indexes with $\delta_{\alpha=5\%}$ on the left and $\delta_{\alpha=1\%}$ on the right. Legend: x - Returns; Circle - RV5; Triangle - RV5ss; Star - RK. Ratio of 1 is best, represented by dotted line.



forecast path over this period, levelling off quickly after a shock. This demonstrates RV's ability to provide a superior volatility proxy, regardless of distribution, vis-a-vis a GARCH specification using daily returns. This improvement occurred consistently across models, series and distributions. Table 8 shows a similar summary described in Section 5.3 but

Table 8: Expected Shortfall Violation Ratios for each distribution, RV measure and data set.

	$\delta_{\alpha=5\%}$						$\delta_{\alpha=1\%}$					
	Gt	GSkt	RGtN	RGtt	RGSktN	RGSktt	Gt	GSkt	RGtN	RGtt	RGSktN	RGSktt
CAC40r	1.49	1.37					3.10	3.09				
CAC40RV5			1.34	1.28	1.23	1.28			3.00	3.00	<u>2.45</u>	<u>2.45</u>
CAC40RV5ss			1.28	1.28	1.23	1.23			3.27	3.27	2.99	2.99
CAC40RK			1.28	1.28	<u>1.17</u>	<u>1.17</u>			3.27	3.27	2.72	<u>2.45</u>
DAXr	1.37	1.20					2.81	2.79				
DAXRV5			1.49	1.55	1.27	1.27			2.45	2.45	<u>1.89</u>	<u>1.89</u>
DAXRV5ss			1.39	1.34	<u>1.11</u>	1.22			1.91	2.18	<u>1.89</u>	<u>1.89</u>
DAXRK			1.39	1.39	1.16	1.16			1.91	2.18	<u>1.89</u>	<u>1.89</u>
DJIAr	1.71	1.38					1.97	1.97				
DJIARV5			1.62	1.67	1.34	1.40			0.83	1.10	0.27	0.27
DJIARV5ss			1.83	1.78	1.51	1.51			1.93	1.93	0.82	<u>1.10</u>
DJIARK			1.67	1.73	<u>1.18</u>	1.29			1.38	<u>1.10</u>	0.55	0.55
IBEXr	1.45	1.33					2.27	1.70				
IBEXRV5			<u>0.97</u>	<u>0.97</u>	0.81	0.86			1.10	1.10	0.82	0.82
IBEXRV5ss			<u>0.97</u>	<u>0.97</u>	0.86	0.86			1.38	1.38	<u>1.10</u>	<u>1.10</u>
IBEXRK			<u>0.97</u>	<u>0.97</u>	0.81	0.81			1.10	1.38	0.82	0.82
RUSSr	1.65	1.50					2.10	1.57				
RUSSRV5			1.65	1.70	1.34	1.39			2.10	2.10	1.31	1.31
RUSSRV5ss			1.60	1.55	<u>1.24</u>	1.29			2.10	2.10	<u>1.05</u>	<u>1.05</u>
RUSSRK			1.75	1.70	1.50	1.50			2.62	3.15	1.84	1.84
SP500r	1.70	1.48					2.25	1.68				
SP500RV5			1.78	1.78	1.51	<u>1.46</u>			1.65	1.65	1.37	1.37
SP500RV5ss			1.94	1.94	1.67	1.78			1.93	1.93	<u>1.10</u>	<u>1.10</u>
SP500RK			1.78	1.78	1.62	1.62			1.65	1.65	<u>1.10</u>	<u>1.10</u>
SSMlr	1.38	1.27					1.98	1.97				
SSMIRV5			1.46	1.40	1.08	1.08			1.93	1.93	1.92	1.92
SSMIRV5ss			1.46	1.51	1.08	1.08			2.20	2.20	<u>1.64</u>	<u>1.64</u>
SSMIRK			1.46	1.51	<u>0.97</u>	<u>0.97</u>			1.65	1.93	<u>1.64</u>	<u>1.64</u>
STOXXr	1.38	1.15					2.26	1.96				
STOXXRV5			1.54	1.49	1.11	1.11			1.63	1.36	<u>1.35</u>	<u>1.35</u>
STOXXRV5ss			1.49	1.44	<u>1.06</u>	<u>1.06</u>			1.63	1.63	<u>1.35</u>	<u>1.35</u>
STOXXRK			1.54	1.54	1.22	1.22			1.63	1.63	1.35	1.35

for the ESRatios. As above, the values which are closest to 1 are boxed while the values furthest from 1 are displayed in bold. The results are consistent across both 5% and 1% quantiles, with RV-GARCH models always providing the best ESRatio, and in some cases, such as with SP500, proving a very large improvement in ES forecast. At the 5% quantile, a Skewed Student-t distribution of returns always provides the best ESRatio, except for IBEX where a Student-t distribution of returns gives a preferred ESRatio. Using a Student-t distribution for the measurement equation provides either a similar or marginally improved ESRatio for all data sets except for DAX, DJIA and RUSS, where a normal distribution for the measurement equation is preferred. We observe similar findings at the 1% quantile level. A plot of these results is provided in Figure 5.

Table 9 show the results of the UC, CC and DQ tests as described above, displaying if the test rejects the null at significance level as per Table 4, where 1 indicates a rejection of the null. There is no rejection for the UC and CC tests for all models, and minimal for the DQ test at both quantile levels for RV models, but quite severe for GARCH models at the 1% quantile with 5 rejections each.

In summary, RV-GARCH models always provide superior ES forecasts compared to

Table 9: Expected Shortfall Unconditional Coverage, Conditional Coverage and Dynamic Quantile Test Results. 1 indicates rejection by test at relevant δ_α level.

	$\delta_{\alpha=5\%}$						$\delta_{\alpha=1\%}$					
	Gt	GSkt	RGtN	RGtt	RGSktN	RGSktt	Gt	GSkt	RGtN	RGtt	RGSktN	RGSktt
Returns	0 0 1	0 0 0					0 0 5	0 0 5				
RV5			0 0 0	0 0 0	0 0 0	0 0 0			0 0 1	0 0 1	0 0 1	0 0 1
RV5ss			0 0 1	0 0 0	0 0 0	0 0 0			0 0 1	0 0 1	0 0 1	0 0 1
RK			0 0 0	0 0 0	0 0 0	0 0 0			0 0 1	0 0 2	0 0 1	0 0 1

using daily GARCH models, with preferred ESRatios at both quantiles. This is an attractive quality if being used for regulatory purposes, as this would forecast an optimal amount of capital allocation for an institution.

5.5 Realised Volatility measure comparison

Following the above analysis, there does not appear to be a consistently superior RV measure, but our findings are consistent that RV measures do provide a better data source than daily returns to generate VaR and ES forecasts.

5.6 Loss function

A loss function is employed in order to further assess both VaR and ES quantile forecasts generated. As per Koenker and Bassett Jr (1978) this is done through a criterion function, which should be minimized in quantile regression estimation, defined as:

$$LF = \sum_{t=s+1}^{s+f} (a_t - Q_t)(\alpha - I_t).$$

Where I is an indicator taking the value of 1 if there is a violation for either VaR or ES ($a_t < Q_t$), Q_t is the quantile forecast, and α is the evaluation quantile for VaR, taking δ_α values described in Table 4 for ES. The model with the lowest loss function figure is preferred. Results for Value at Risk and Expected Shortfall loss function are provided in Tables 10 and 11 respectively, where lowest values are boxed and highest values are displayed in bold.

For VaR, at the 5% quantile level, the lowest loss is always associated with a RV-GARCH model, while the highest level is always associated with a daily GARCH model. At the 1% quantile level, the lowest loss is always found in a RV-GARCH model, with the highest loss being associated with a daily GARCH model in all but one data sets, the RUSS.

For ES, at the 5% and 1% quantile level, the lowest loss is always associated with an RV-GARCH model, while the highest level is always associated with a daily GARCH model.

Table 10: Value at Risk Loss Function.

	$\alpha = 0.05\%$						$\alpha = 0.01\%$					
	Gt	GSkt	RGtN	RGtt	RGSktN	RGSktt	Gt	GSkt	RGtN	RGtt	RGSktN	RGSktt
CAC40r	132.14	132.06					43.02	42.54				
CAC40RV5			127.41	127.45	127.47	127.51			38.75	38.76	38.33	38.37
CAC40RV5ss			<u>127.27</u>	127.34	127.34	127.41			38.69	38.68	38.28	38.26
CAC40RK			127.41	127.43	127.41	127.48			38.68	38.67	<u>38.23</u>	38.26
DAXr	122.15	121.87					38.45	37.69				
DAXRV5			118.38	118.47	118.10	118.11			33.34	33.53	33.23	33.33
DAXRV5ss			117.54	117.66	117.46	117.60			33.47	33.62	33.48	33.54
DAXRK			116.81	116.91	<u>116.63</u>	116.71			33.31	33.37	<u>33.17</u>	33.24
DJIAr	107.47	107.28					30.71	30.07				
DJIARV5			102.84	102.89	101.86	101.83			26.32	26.48	25.63	25.84
DJIARV5ss			101.84	101.85	100.55	100.57			25.59	25.64	<u>24.98</u>	25.04
DJIARK			101.18	101.20	<u>100.36</u>	100.37			25.63	25.68	25.14	25.23
IBEXr	162.46	161.56					49.30	48.70				
IBEXRV5			157.15	157.30	156.98	157.01			41.96	42.04	41.97	42.07
IBEXRV5ss			157.06	156.91	156.70	156.62			41.79	41.83	41.76	41.81
IBEXRK			156.72	156.76	<u>156.49</u>	156.50			41.72	41.82	<u>41.72</u>	41.82
RUSRr	139.77	139.75					38.17	37.40				
RUSSRV5			136.04	136.11	135.47	135.49			35.93	36.12	35.07	35.18
RUSSRV5ss			134.99	135.07	134.76	<u>134.54</u>			35.39	35.64	<u>34.75</u>	34.89
RUSSRK			136.88	137.15	136.08	136.34			37.76	38.21	36.45	36.87
SP500r	113.97	113.78					33.69	33.03				
SP500RV5			109.56	109.63	109.14	109.19			29.56	29.73	28.43	28.60
SP500RV5ss			110.11	110.10	108.58	108.62			28.85	28.90	<u>27.67</u>	27.76
SP500RK			109.37	109.37	<u>108.34</u>	108.35			29.30	29.43	28.18	28.32
SSMlr	94.62	94.42					29.09	28.89				
SSMIRV5			88.54	88.75	<u>88.36</u>	88.53			25.81	25.84	25.49	25.56
SSMIRV5ss			88.79	89.03	88.63	88.83			25.84	25.94	<u>25.45</u>	25.48
SSMIRK			88.90	89.09	88.87	88.99			25.86	25.93	25.61	25.69
STOXXr	134.80	134.41					40.29	40.01				
STOXXRV5			130.47	130.36	130.53	130.30			35.21	35.24	35.56	35.58
STOXXRV5ss			130.03	<u>129.81</u>	130.20	129.85			35.44	35.38	35.68	35.65
STOXXRK			130.28	130.09	130.17	130.02			35.25	<u>35.19</u>	35.37	35.41

Table 11: Expected Shortfall Loss Function.

	$\delta_{\alpha=5\%}$						$\delta_{\alpha=1\%}$					
	Gt	GSkt	RGtN	RGtt	RGSktN	RGSktt	Gt	GSkt	RGtN	RGtt	RGSktN	RGSktt
CAC40r	65.24	65.44					19.92	19.35				
CAC40RV5			60.02	60.00	61.57	61.54			17.36	17.44	17.09	17.19
CAC40RV5ss			59.93	59.86	61.56	61.43			17.24	17.26	<u>16.91</u>	16.96
CAC40RK			59.86	<u>59.75</u>	61.40	61.37			17.29	17.37	17.01	17.15
DAXr	58.98	59.49					16.82	15.96				
DAXRV5			53.35	53.36	54.86	54.87			14.87	14.95	14.76	14.84
DAXRV5ss			53.43	53.49	55.27	55.23			14.78	14.88	14.88	14.92
DAXRK			53.14	<u>53.14</u>	54.90	54.87			<u>14.43</u>	14.49	14.46	14.53
DJIAr	49.48	49.35					12.74	12.25				
DJIARV5			45.57	45.52	45.29	45.43			9.99	9.97	10.35	10.30
DJIARV5ss			44.25	44.27	<u>43.75</u>	43.77			<u>9.58</u>	9.63	9.71	9.72
DJIARK			44.05	44.05	43.77	43.82			9.91	9.91	10.16	10.15
IBEXr	76.66	76.29					20.79	19.97				
IBEXRV5			68.61	68.69	70.73	70.75			<u>16.65</u>	16.71	16.97	16.98
IBEXRV5ss			<u>68.03</u>	68.12	70.13	70.14			16.73	16.79	16.87	16.84
IBEXRK			68.09	68.21	70.34	70.36			16.74	16.75	17.01	16.96
RUSRr	63.02	65.44					16.45	16.56				
RUSSRV5			59.70	59.91	62.43	62.62			14.65	14.71	15.12	15.18
RUSSRV5ss			<u>58.90</u>	59.14	61.76	61.97			<u>14.61</u>	14.74	15.18	15.25
RUSSRK			61.09	61.52	63.68	64.07			16.02	16.29	16.32	16.53
SP500r	53.45	53.04					14.47	13.94				
SP500RV5			49.93	49.97	49.52	49.65			11.34	11.42	11.23	11.29
SP500RV5ss			49.51	49.55	<u>48.30</u>	48.41			11.09	11.16	<u>11.06</u>	11.11
SP500RK			49.77	49.82	49.07	49.18			11.38	11.44	11.25	11.32
SSMlr	44.97	44.90					13.11	12.59				
SSMIRV5			<u>40.58</u>	40.64	41.40	41.45			11.43	11.56	11.17	11.30
SSMIRV5ss			40.71	40.80	41.46	41.54			11.39	11.53	<u>11.11</u>	11.20
SSMIRK			40.63	40.74	41.61	41.69			11.51	11.63	11.36	11.48
STOXXr	62.42	62.90					17.63	16.97				
STOXXRV5			57.57	<u>57.46</u>	59.58	59.50			<u>15.04</u>	15.05	15.46	15.51
STOXXRV5ss			57.63	57.48	59.66	59.50			15.25	15.22	15.66	15.67
STOXXRK			58.00	57.68	59.39	59.33			15.13	15.13	15.51	15.56

Figure 6: Loss Function plot for each model and distribution for the Value at Risk forecast for each data set at the 5% and 1% level. Legend: Blue - Returns; Turquoise - RV5; Yellow - RV5ss; Red - RK. Lower is better.

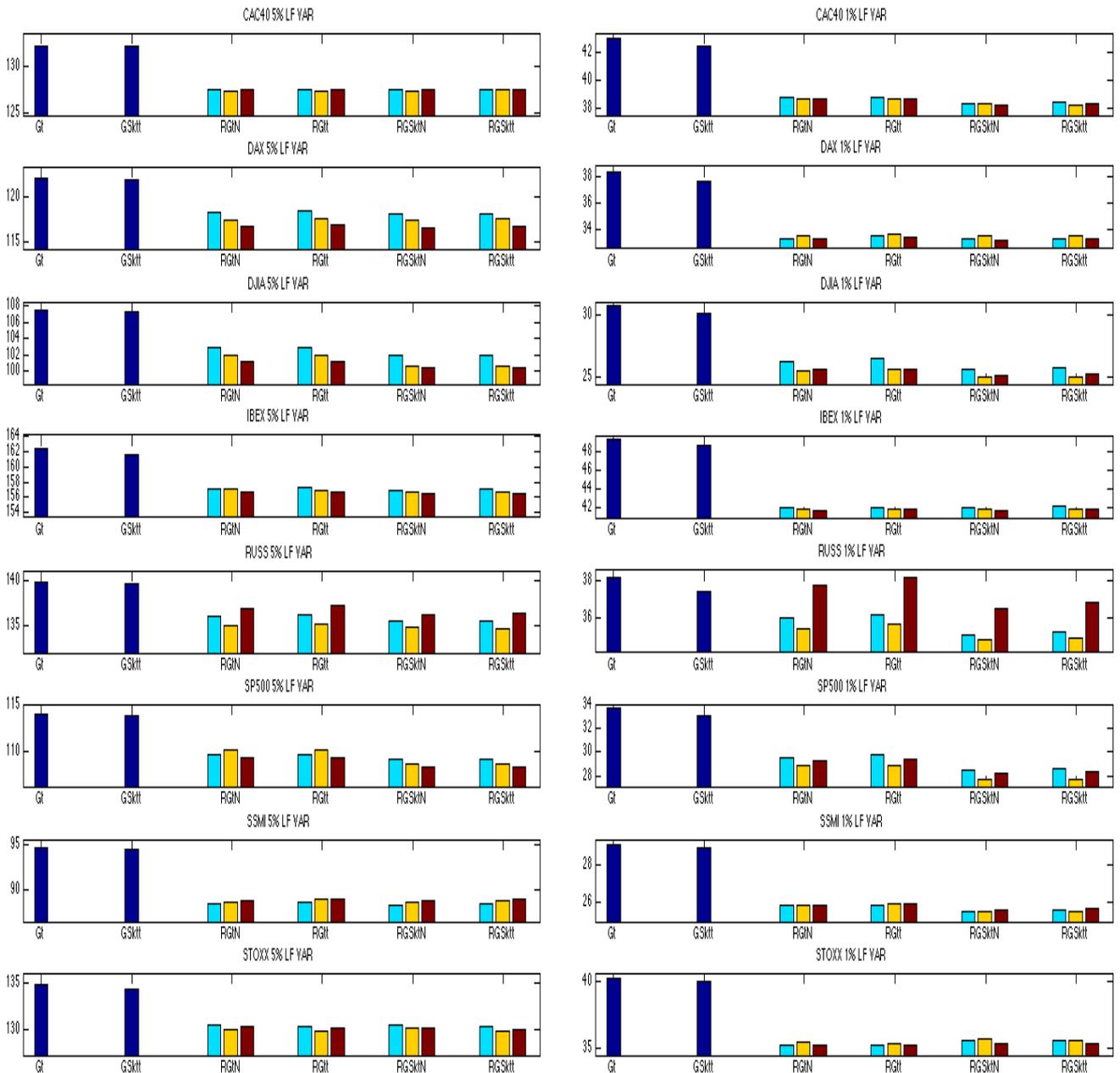
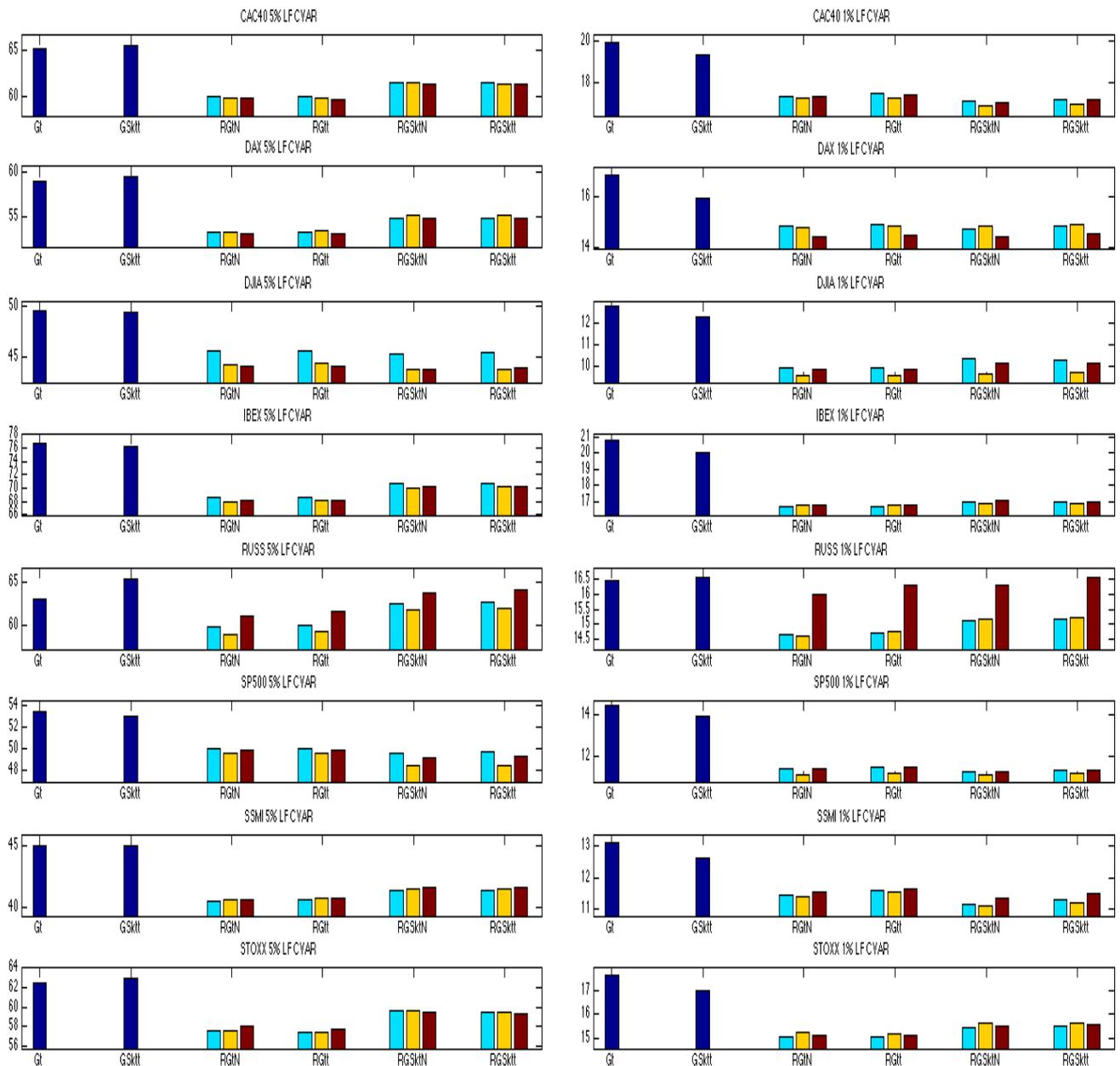


Figure 7: Loss Function plot for each model and distribution for the Expected Shortfall forecast for each data set at the $\delta_{\alpha=5\%}$ and $\delta_{\alpha=1\%}$ level. Legend: Blue - Returns; Turquoise - RV5; Yellow - RV5ss; Red - RK. Lower is better.



As per Figures 6 and 7, the loss function is plotted against the various models and distributions for every data set at each quantile level for both VaR and ES. It is apparent that models using RV tend to have far lower loss figures, with models using a Skewed Student-t distributions being the best performers in the vast majority of cases. These findings reinforce the positive forecast results, where RV models not only produce better estimates, but do so with more accuracy as reflected in the lower loss figures. The distribution of the RV model also appears to be an important factor, with a Skewed Student-t distribution of returns providing a large margin of improvement over Student-t, while the addition of a Student-t distribution for the measurement equation providing an additional marginal benefit to forecasts, formal tests and loss functions.

6 Conclusion

This paper utilises the Realised Volatility GARCH model to produce quantile forecasts, which are compared to traditional GARCH models using daily data. Our findings show that using Realised Volatility produces a more accurate forecast for both VaR and ES at the 5% and 1% quantiles when using a variety of formal and informal tests. The choice of distribution of returns is paramount to produce a valid model for forecasting, with a Skewed Student-t distribution being the preferred choice, while the choice of distribution for the measurement equation having minimal impact. Several extensions could allow for a broader range of distributions such as the Asymmetric Laplace or Weibull distribution in either or both of the returns and measurement equation, while further specifications for the volatility equation could also be considered, such as including alternative realised measures such as Realised Range. Altering the sampling frequency to daily while producing weekly or longer forecasts could also be a possible extension, as is the use of Realised Volatility GARCH models as marginals in Copulas.

Appendix A Likelihood Functions

ϵ : Student-t distribution

$$\begin{aligned} \mathcal{L}(\theta|r) = & T \left[\ln \Gamma \left(\frac{\nu + 1}{2} \right) - \ln \Gamma \left(\frac{\nu}{2} \right) - 0.5 \ln [(\nu - 2)\pi] \right] \\ & - \sum_{t=1}^T \left[\frac{\nu + 1}{2} \ln \left(1 + \frac{(r_t - \mu)^2}{(\nu - 2)\sigma_t^2} \right) + 0.5 \ln(\sigma_t^2) \right] \end{aligned}$$

ϵ : **Skewed Student-t distribution**

$$\mathcal{L}(\boldsymbol{\theta}|r) = \begin{cases} T [\mathcal{L}(b) + \mathcal{L}(c)] - \sum_{t=1}^T \left[\frac{\nu+1}{2} \ln \left(1 + \frac{1}{\nu-2} \left(\frac{b\epsilon_t + a}{1-\lambda} \right)^2 \right) + 0.5 \ln(\sigma_t^2) \right] & \text{if } \epsilon_t < -\frac{a}{b} \\ T [\mathcal{L}(b) + \mathcal{L}(c)] - \sum_{t=1}^T \left[\frac{\nu+1}{2} \ln \left(1 + \frac{1}{\nu-2} \left(\frac{b\epsilon_t + a}{1+\lambda} \right)^2 \right) + 0.5 \ln(\sigma_t^2) \right] & \text{if } \epsilon_t \geq -\frac{a}{b} \end{cases}$$

$$\mathcal{L}(b) = 0.5 \ln(1 + 3\lambda^2 - a^2)$$

$$\mathcal{L}(c) = \ln \Gamma \left(\frac{\nu+1}{2} \right) - \ln \Gamma \left(\frac{\nu}{2} \right) - 0.5 \ln[(\nu-2)\pi]$$

u_t : **Normal distribution**

$$\mathcal{L}(\boldsymbol{\theta}|RV) = -0.5 \sum_{t=1}^T \left(\ln(\sigma_{u,t}^2) + \frac{u_t^2}{\sigma_{u,t}^2} \right)$$

u_t : **Student-t distribution**

$$\begin{aligned} \mathcal{L}(\boldsymbol{\theta}|RV) = & T \left[\ln \Gamma \left(\frac{\nu_{mes} + 1}{2} \right) - \ln \Gamma \left(\frac{\nu_{mes}}{2} \right) - 0.5 \ln[(\nu_{mes} - 2)\pi] \right] \\ & - \sum_{t=1}^T \left[\frac{\nu_{mes} + 1}{2} \ln \left(1 + \frac{u_t^2}{(\nu_{mes} - 2)\sigma_{u,t}^2} \right) + 0.5 \ln(\sigma_{u,t}^2) \right] \end{aligned}$$

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7 Tables

Table 12: CAC40 MCMC Parameter Estimate with 95% Credible Interval.

CAC40	α_0	α_1	β_1	ν	λ	ξ	ϕ	τ_1	τ_2	σ_u^2	ν_{mes}	μ	σ_0
Gt	0.013	0.095	0.899	10.286								0.012	6.006
	[0.007,0.020]	[0.074,0.119]	[0.874,0.920]	[7.552,13.946]								[-0.016,0.039]	[2.026,12.578]
GSkt	0.013	0.096	0.897	10.751	-0.047							0.006	5.971
	[0.007,0.020]	[0.074,0.119]	[0.872,0.920]	[7.702,15.110]	[-0.090,-0.003]							[-0.022,0.035]	[1.950,13.181]
RGtN5	0.059	0.364	0.626	16.93		-0.161	0.982	-0.078	0.109	0.183		-0.012	1.055
	[0.039,0.079]	[0.331,0.397]	[0.594,0.656]	[10.340,28.567]		[-0.208,-0.114]	[0.943,1.024]	[-0.093,-0.063]	[0.099,0.120]	[0.175,0.192]		[-0.039,0.015]	[0.429,1.672]
RGtN5ss	0.069	0.378	0.612	17.246		-0.181	0.981	-0.079	0.108	0.177		-0.010	1.117
	[0.048,0.090]	[0.344,0.412]	[0.580,0.643]	[10.322,30.566]		[-0.229,-0.134]	[0.941,1.023]	[-0.095,-0.064]	[0.097,0.118]	[0.169,0.185]		[-0.037,0.017]	[0.486,1.735]
RGtN5RK	0.059	0.346	0.643	16.631		-0.171	0.986	-0.074	0.122	0.194		-0.008	1.212
	[0.041,0.079]	[0.315,0.379]	[0.613,0.673]	[10.262,28.065]		[-0.217,-0.126]	[0.947,1.026]	[-0.090,-0.057]	[0.111,0.133]	[0.185,0.203]		[-0.034,0.019]	[0.573,1.831]
RGtt5	0.054	0.336	0.656	16.933		-0.164	0.976	-0.077	0.102	0.185	6.269	-0.013	1.016
	[0.036,0.073]	[0.305,0.368]	[0.625,0.686]	[10.202,29.494]		[-0.211,-0.119]	[0.935,1.019]	[-0.092,-0.063]	[0.092,0.112]	[0.173,0.198]	[5.195,7.608]	[-0.040,0.014]	[0.476,1.592]
RGtt55ss	0.063	0.352	0.640	16.827		-0.183	0.977	-0.079	0.098	0.178	6.155	-0.010	1.071
	[0.045,0.083]	[0.319,0.385]	[0.607,0.671]	[10.327,28.267]		[-0.230,-0.136]	[0.936,1.018]	[-0.092,-0.065]	[0.089,0.108]	[0.166,0.190]	[5.091,7.406]	[-0.037,0.017]	[0.527,1.637]
RGtt5RK	0.055	0.318	0.673	16.500		-0.175	0.980	-0.073	0.113	0.194	6.554	-0.007	1.196
	[0.038,0.072]	[0.289,0.348]	[0.644,0.702]	[10.206,27.355]		[-0.222,-0.130]	[0.943,1.020]	[-0.087,-0.058]	[0.103,0.124]	[0.182,0.207]	[5.403,7.909]	[-0.033,0.020]	[0.683,1.717]
RGSktN5	0.058	0.364	0.625	18.508	-0.062	-0.158	0.984	-0.08	0.109	0.184		-0.018	1.047
	[0.038,0.078]	[0.331,0.397]	[0.593,0.656]	[10.685,32.013]	[-0.107,-0.017]	[-0.204,-0.110]	[0.944,1.026]	[-0.095,-0.064]	[0.098,0.119]	[0.175,0.192]		[-0.046,0.009]	[0.410,1.675]
RGSktN5ss	0.067	0.378	0.610	19.067	-0.062	-0.178	0.983	-0.081	0.107	0.177		-0.014	1.087
	[0.047,0.088]	[0.345,0.412]	[0.578,0.641]	[10.845,34.160]	[-0.107,-0.016]	[-0.225,-0.132]	[0.943,1.025]	[-0.096,-0.065]	[0.097,0.118]	[0.169,0.185]		[-0.041,0.012]	[0.458,1.722]
RGSktNRK	0.058	0.346	0.642	18.349	-0.058	-0.169	0.989	-0.075	0.122	0.194		-0.013	1.203
	[0.040,0.078]	[0.315,0.378]	[0.611,0.672]	[10.739,32.810]	[-0.101,-0.015]	[-0.215,-0.123]	[0.950,1.031]	[-0.091,-0.058]	[0.110,0.133]	[0.185,0.203]		[-0.040,0.014]	[0.574,1.814]
RGSktt5	0.054	0.337	0.654	18.239	-0.062	-0.162	0.977	-0.078	0.102	0.184	6.293	-0.017	0.996
	[0.036,0.073]	[0.307,0.369]	[0.623,0.684]	[10.618,32.156]	[-0.106,-0.018]	[-0.209,-0.115]	[0.936,1.020]	[-0.092,-0.063]	[0.092,0.112]	[0.172,0.198]	[5.169,7.646]	[-0.045,0.011]	[0.452,1.558]
RGSktt5ss	0.063	0.353	0.638	18.069	-0.059	-0.180	0.977	-0.079	0.098	0.178	6.175	-0.014	1.050
	[0.044,0.082]	[0.321,0.387]	[0.607,0.668]	[10.728,31.419]	[-0.102,-0.016]	[-0.227,-0.134]	[0.937,1.019]	[-0.094,-0.066]	[0.089,0.108]	[0.166,0.191]	[5.111,7.479]	[-0.041,0.013]	[0.504,1.630]
RGSkttRK	0.054	0.319	0.672	19.197	-0.057	-0.172	0.982	-0.074	0.113	0.193	6.651	-0.013	1.156
	[0.037,0.072]	[0.291,0.350]	[0.642,0.700]	[10.693,37.981]	[-0.101,-0.014]	[-0.218,-0.126]	[0.945,1.022]	[-0.090,-0.059]	[0.102,0.123]	[0.181,0.206]	[5.433,8.146]	[-0.040,0.014]	[0.630,1.692]

Table 13: DAX MCMC Parameter Estimate with 95% Credible Interval.

DAX	α_0	α_1	β_1	ν	λ	ξ	ϕ	τ_1	τ_2	σ_u^2	ν_{mes}	μ	σ_0
Gt	0.013	0.089	0.905	11.703								0.037	8.897
	[0.007,0.021]	[0.071,0.109]	[0.884,0.924]	[8.440,16.385]								[0.007,0.068]	[3.121,18.783]
GSkt	0.013	0.091	0.903	13.72	-0.117							0.022	9.19
	[0.007,0.020]	[0.073,0.110]	[0.883,0.921]	[9.423,20.193]	[-0.163,-0.070]							[-0.009,0.053]	[3.218,18.811]
RGtN5	0.050	0.371	0.607	21.158		-0.125	1.007	-0.100	0.104	0.201		0.004	1.822
	[0.030,0.070]	[0.339,0.405]	[0.575,0.638]	[12.342,34.929]		[-0.172,-0.077]	[0.965,1.050]	[-0.116,-0.084]	[0.093,0.115]	[0.191,0.210]		[-0.027,0.034]	[1.156,2.459]
RGtN5ss	0.077	0.387	0.598	21.551		-0.191	0.992	-0.114	0.095	0.178		0.005	1.828
	[0.056,0.099]	[0.354,0.421]	[0.566,0.630]	[12.071,39.653]		[-0.240,-0.143]	[0.951,1.036]	[-0.130,-0.099]	[0.085,0.105]	[0.170,0.187]		[-0.025,0.036]	[1.171,2.473]
RGtN5RK	0.058	0.386	0.595	23.653		-0.142	0.998	-0.107	0.103	0.187		0.004	1.871
	[0.037,0.080]	[0.353,0.421]	[0.561,0.628]	[12.794,46.819]		[-0.189,-0.094]	[0.958,1.041]	[-0.122,-0.091]	[0.093,0.114]	[0.179,0.196]		[-0.027,0.035]	[1.184,2.523]
RGtt5	0.044	0.336	0.649	21.18		-0.128	1.000	-0.101	0.100	0.201	6.907	0.008	1.759
	[0.026,0.063]	[0.305,0.369]	[0.616,0.680]	[12.084,38.414]		[-0.177,-0.080]	[0.959,1.043]	[-0.117,-0.086]	[0.089,0.111]	[0.189,0.215]	[5.606,8.482]	[-0.024,0.040]	[1.142,2.342]
RGtt55ss	0.070	0.360	0.630	20.943		-0.188	0.982	-0.114	0.092	0.179	5.924	0.007	1.833
	[0.051,0.090]	[0.328,0.395]	[0.597,0.661]	[11.969,38.725]		[-0.236,-0.141]	[0.941,1.024]	[-0.128,-0.100]	[0.081,0.102]	[0.167,0.193]	[4.951,7.118]	[-0.024,0.038]	[1.275,2.377]
RGtt5RK	0.052	0.352	0.634	22.857		-0.144	0.994	-0.107	0.100	0.189	5.788	0.006	1.888
	[0.034,0.072]	[0.318,0.385]	[0.602,0.666]	[12.426,45.507]		[-0.193,-0.098]	[0.952,1.037]	[-0.122,-0.093]	[0.089,0.110]	[0.176,0.204]	[4.832,6.961]	[-0.025,0.036]	[1.291,2.456]
RGSktN5	0.047	0.370	0.608	33.735	-0.124	-0.117	1.008	-0.102	0.103	0.201		-0.006	1.810
	[0.027,0.067]	[0.338,0.403]	[0.575,0.640]	[14.093,81.247]	[-0.169,-0.080]	[-0.165,-0.070]	[0.967,1.051]	[-0.118,-0.086]	[0.092,0.114]	[0.192,0.210]		[-0.036,0.024]	[1.141,2.447]
RGSktN5ss	0.075	0.386	0.599	30.077	-0.127	-0.186	0.994	-0.116	0.095	0.178		-0.005	1.801
	[0.054,0.096]	[0.352,0.420]	[0.566,0.631]	[14.204,66.247]	[-0.173,-0.082]	[-0.233,-0.139]	[0.954,1.035]	[-0.131,-0.100]	[0.085,0.105]	[0.170,0.186]		[-0.036,0.026]	[1.149,2.448]
RGSktNRK	0.055	0.385	0.597	35.041	-0.125	-0.135	0.999	-0.108	0.102	0.187		-0.007	1.856
	[0.036,0.076]	[0.352,0.419]	[0.564,0.629]	[14.963,85.872]	[-0.170,-0.079]	[-0.182,-0.090]	[0.958,1.041]	[-0.124,-0.092]	[0.092,0.113]	[0.179,0.196]		[-0.037,0.024]	[1.211,2.476]
RGSktt5	0.040	0.334	0.651	28.216	-0.125	-0.117	1.000	-0.103	0.098	0.201	6.839	-0.005	1.742
	[0.023,0.059]	[0.302,0.366]	[0.617,0.683]	[13.617,57.420]	[-0.170,-0.079]	[-0.168,-0.070]	[0.958,1.045]	[-0.119,-0.088]	[0.088,0.109]	[0.189,0.214]	[5.641,8.296]	[-0.034,0.024]	[1.095,2.346]
RGSktt5ss	0.067	0.358	0.631	28.46	-0.126	-0.182	0.983	-0.116	0.091	0.179	5.888	-0.003	1.826
	[0.049,0.087]	[0.325,0.393]	[0.598,0.663]	[13.929,59.379]	[-0.172,-0.081]	[-0.229,-0.135]	[0.943,1.026]	[-0.130,-0.102]	[0.081,0.101]	[0.168,0.193]	[4.934,7.058]	[-0.033,0.029]	[1.243,2.391]
RGSkttRK	0.050	0.351	0.634	32.986	-0.127	-0.136	0.995	-0.110	0.099	0.189	5.787	-0.006	1.864
	[0.031,0.069]	[0.318,0.385]	[0.601,0.666]	[14.586,75.454]	[-0.172,-0.081]	[-0.184,-0.090]	[0.953,1.040]	[-0.125,-0.095]	[0.088,0.110]	[0.176,0.203]	[4.833,6.946]	[-0.036,0.024]	[1.240,2.441]