Domain Perturbation for Parabolic Equations

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Abstract

We study the effect of domain perturbation on the behaviour of parabolic equations. The first aspect considered in this thesis is the behaviour of solutions under changes of the domain. We show how solutions of linear and semilinear parabolic equations behave as a sequence of domains Ω_n converges to an open set Ω in a certain sense. In particular, we are interested in singular domain perturbations so that a change of variables is not possible on these domains. For autonomous linear equations, it is known that convergence of solutions under domain perturbation is closely related to the corresponding elliptic equations via a standard semigroup theory. We show that there is also a relation between domain perturbation for *non-autonomous* linear parabolic equations and domain perturbation for elliptic equations. The key result for this is the equivalence of Mosco convergences between various closed and convex subsets of Banach spaces. An important consequence is that the same conditions for a sequence of domains imply convergence of solutions under domain perturbation for both parabolic and elliptic equations. By applying variational methods, we obtain the convergence of solutions of initial value problems under Dirichlet or Neumann boundary conditions. A similar technique can be applied to obtain the convergence of weak solutions of parabolic variational inequalities when the underlying convex set is perturbed. Using the linear theory, we then study domain perturbation for initial boundary value problems of semilinear type. We are also interested in the behaviour of *bounded entire solutions* of parabolic equations defined on the whole real line. We establish a convergence result for bounded entire solutions of linear parabolic equations under L^2 and L^p-norms. For the L^p-theory, we also prove Hölder regularity of bounded entire solutions with respect to time. In addition, the persistence of some classes of bounded entire solutions is given for semilinear equations using the Leray-Schauder degree theory.

The second aspect is to study the dynamics of parabolic equations under domain perturbation. In this part, we consider parabolic equation as a dynamical system in an

L^2 space and study the stability of invariant manifolds near a stationary solution. In particular, we prove the continuity (upper and lower semicontinuity) of both, the local stable invariant manifolds and the local unstable invariant manifolds under domain perturbation.

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Statement of originality

This thesis contains no material which has been accepted for the award of any other degree or diploma. All work in this thesis, except where due acknowledgement is made, is believed to be original.

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Contents

Introduction

The study of domain perturbation or sometimes referred to as "perturbation of the boundary" for boundary value problems is a special topic in perturbation problems. The main characteristic is that the differential operators and the solutions of the perturbed problems live in different spaces (depending on the domain of a boundary value problem). Domain perturbation appears to be a simple problem if we are only interested in smooth perturbation of the domain. This is because we could perform a change of variables to consider the perturbed problems in a fixed domain and only perturb the coefficients. Hence, it turns back to a standard perturbation problem and we may apply standard techniques such as the implicit function theorem, the Liapunov-Schmidt method and the transversality theorem. Nevertheless, difficulties arrive when we perform a change of variables and standard tools are not enough (see [53]). When a change of variables is not possible, domain perturbation is even more challenging.

The fundamental question in domain perturbation is to look at how solutions behave upon varying domains. In particular, we would like to know when the solutions converge and what the limit problem is. This topic has been extensively studied for elliptic equations (see for example [37, 5, 39, 34, 30, 21, 22, 79] and references therein). Typically, the work of [37] provides a characterisation of domains $\Omega_n, \Omega \subset \mathbb{R}^N$ for which solutions of the perturbed problem subject to Dirichlet boundary condition

$$\begin{aligned} -\Delta u &= f_n && \text{in } \Omega_n \\ u &= 0 && \text{on } \partial\Omega_n \end{aligned} \tag{0.1}$$

converge (in some sense) to a solution of the limit problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{0.2}$$

Older techniques for the convergence of solutions of the Dirichlet problem for harmonic functions can be found in [58]. In the above work, the convergence result was referred to as "stability of solutions" under variation of the boundary of the domain. In [21, 22], a sufficient condition on domains for which the solutions converge is obtained for Neumann problems ($u = 0$ is replaced $\frac{\partial u}{\partial \nu} = 0$ on $\partial\Omega$). A more complicated behaviour occurs in the case of Robin problems ($\frac{\partial u}{\partial \nu} + \alpha u = 0$, $\alpha > 0$ on $\partial\Omega$). Dancer and Daners [34] show that both convergence of solutions and boundary homogenisation (the limit problem prescribed by a different boundary condition) are possible depending largely on how domains Ω_n converge to Ω. A variation to elliptic equations with nonlinear boundary condition is studied in [7, 8, 9]. There are also results on nonlinear equations investigated in [31, 32]. On the other hand, an explicit estimate for solutions under domain perturbation in terms of a certain "distance" between Ω_n and Ω is the subject of [72].

Another aspect of domain perturbation for elliptic equations is the dependence of the spectrum of an elliptic operator on domains. This involves the study of the eigenvalue problem

$$\begin{aligned} -\Delta u + \lambda u &= 0 \qquad \text{in } \Omega \\ Bu &= 0 \qquad \text{on } \partial\Omega \end{aligned} \tag{0.3}$$

under domain perturbation where B is a boundary condition operator. For Dirichlet problem, it is known that the spectrum behaves continuously under a large class of domain perturbation as characterised in [37]. In contrast, the spectrum for Neumann problems is not well-behaved as seen in [29, page 420] or [6, 56]. A sufficient condition on domains for which the spectrum behaves continuously is given in [10]. Besides, an explicit estimate for the variation of eigenvalues on domains is considered in [24, 25, 26, 12].

The motivation to study domain perturbation comes from various sources. The main ones include shape optimization, solutions structure of nonlinear problems and numerical analysis. A classical shape optimisation is to find minimisers of shape functionals (functions where the unknown variable runs over a class of domains). For functionals arising from differential equations, the problem involves solving and extracting convergent subsequences of solutions of differential equations defined on a sequence of domains (see [27, 20, 23, 79]). This problem leads to the concept of Γ-*convergence* for a sequence of open sets and *Mosco convergence* for a sequence of function spaces related to differential equations. In particular, the question to establish a sufficient condition for the Γ-convergence in terms of geometrical properties of the domains has attracted interest in [23, 79, 19]. The solutions structure of nonlinear elliptic equations is the main concern in [31, 32]. It is shown that the number of positive solutions and solutions structure of nonlinear elliptic equations depends on geometric properties of the domain. Examples of domains are constructed to show that there can be arbitrarily many positive solutions. Lastly, in numerical analysis, the concept of *discrete approximation* of normed spaces leads to the study of perturbation of linear operators defined on a sequence of normed spaces to another sequence of normed spaces (see [48, 78, 75, 76, 77]). The generalised perturbation results (discrete convergence, discrete compactness of linear operators) give a general framework for approximation of solutions of linear equations and eigenvalue problems. A more recent work on numerical approximation of eigenvalues of domains with multiple cracks is studied in [16].

Much less result on domain perturbation seems to be known for parabolic equations. Rauch and Taylor [70] studied domain perturbation for autonomous equations. In addition, Hale and Vegas [51] investigated the persistence of stationary (equilibrium) solutions of nonlinear autonomous parabolic equations subject to Neumann boundary condition. Dancer [32, Section 5] briefly discussed non-autonomous equations but explicit time dependent is not allowed in higher order terms (equations can be reduced to autonomous problems by a change of time variable). The methods in these works are based on domain perturbation results for linear elliptic equations. In [70] and [32], convergence of solutions under domain perturbation is proved by means of convergence of resolvents or semigroups of the corresponding elliptic operators. The persistence of equilibrium solutions in [51] is obtained by applying the *Liapunov-Schmidt method* and the implicit function theorem to a bifurcation equation derived from the corresponding elliptic equations. A crucial result to study domain perturbation for autonomous parabolic equations from the corresponding elliptic equations is the convergence of *pseudo resolvents* established by Arendt [4, Theorem 5.2] (see its application in domian perturbation in [4, Section 6] and [38, Section 6]). For quasilinear parabolic equations, there are convergence results in [45] and [73] using the concept of Mosco convergence. The main work on domain perturbation for non-autonomous parabolic equations is provided by Daners [35]. The results of [35] include convergence of solutions for both linear and semilinear parabolic initial value problems subject to Dirichlet boundary condition as well as persistence of periodic solutions under domain perturbation. There is a different perspective and theory for non-autonomous parabolic equations in the monograph by Mierczyński and Shen [64]. In particular, fundamental properties of the principal spectrum and principal Liapunov exponents for non-autonomous (as well as random) parabolic equations are developed. They also study the effect of the shape of the domain on the principal spectrum and principal Liapunov exponents and extend the Faber-Krahn inequalities for elliptic and periodic parabolic problems to general non-autonomous and random parabolic equations in [64, Section 5.4].

The aim of this thesis is to develop a general theory for domain perturbation for both autonomous and non-autonomous parabolic equations subject to either Dirichlet or Neumann boundary conditions. We are particularly interested in singular domain perturbation so that change of variables is not possible on these domains. Moreover, we mostly do not assume any smoothness of the domains.

Let Ω_n and Ω be bounded domains in $\mathbb{R}^N, N \geq 2$ such that $\Omega_n, \Omega \subset D$ for some ball $D \in \mathbb{R}^N$. We consider the following parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}(t)u = f(x,t,u) & \text{in } \Omega \times (0,T] \\ \mathcal{B}(t)u = 0 & \text{on } \partial\Omega \times (0,T] \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{0.4}$$

where $\mathcal{A}$ is an elliptic operator of the form

$$\mathcal{A}(t)u := -\partial_i[a_{ij}(x,t)\partial_j u + a_i(x,t)u] + b_i(x,t)\partial_i u + c_0(x,t)u \tag{0.5}$$

and $\mathcal{B}(t)$ is one of the following boundary operators

$$\begin{aligned} &\mathcal{B}(t)u := u && \text{Dirichlet boundary condition} \\ &\mathcal{B}(t)u := [a_{ij}(x,t)\partial_j u + a_i(x,t)u]\ \nu_i && \text{Neumann boundary condition.} \end{aligned} \tag{0.6}$$

In the above, we use summation convention with i, j running from 1 to N. The boundary condition is considered in a weak sense as the (outer) unit normal vector $\nu = (\nu_1, \dots, \nu_N)$ may not exist. Also, we assume that a_{ij}, a_i, b_i and c_0 are functions in $L^\infty(D \times (0,T))$ and that there exists a constant $\alpha > 0$ independent of $(x,t) \in D \times (0,T)$ such that

$$a_{ij}(x,t)\xi_i\xi_j \geq \alpha|\xi|^2,$$

for all $\xi \in \mathbb{R}^N$. We study the perturbed equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}_n(t)u = f_n(x,t,u) & \text{in } \Omega_n \times (0,T] \\ \mathcal{B}_n(t)u = 0 & \text{on } \partial\Omega_n \times (0,T] \\ u(\cdot,0) = u_{0,n} & \text{in } \Omega_n, \end{cases} \tag{0.7}$$

where $\mathcal{A}_n$ and $\mathcal{B}_n$ are defined similarly as in (0.5) and (0.6), respectively.

In this thesis, we investigate two aspects of the above parabolic equations under domain perturbation. The first aspect is the convergence of solutions of linear and semilinear parabolic equations under domain perturbation. Most of the time, we follow a standard bilinear form setting so that existence and uniqueness of solutions for (0.4) and (0.7) (for linear problems) are available. Our main concern is to investigate conditions on domains Ω_n and Ω for which solutions of the perturbed problems converge to a solution of the limit problem. We also study the persistence of bounded entire solutions (solutions that are defined for all time $t \in (-\infty, \infty)$ and are bounded in a suitable function space) of parabolic equations. The second aspect is to study the dynamics of parabolic equations under domain perturbation. In this part we consider parabolic equation as a dynamical system in an L^2 space and study the persistence of invariant manifolds near a stationary solution of (0.4) when the domain Ω is perturbed.

An overview of this thesis is given below.

We start our domain perturbation analysis for non-autonomous linear parabolic equations in Chapter 1. This is inspired by the work of Daners [35] for Dirichlet problems. Our main focus is hence the Neumann problem. In the case of autonomous linear parabolic equations, convergence of solutions under domain perturbation is closely related to the corresponding elliptic equations via the semigroups generated by the corresponding elliptic operators. Our results show that this also applies to non-autonomous problems using variational approach. The link between elliptic and parabolic equations is obtained by the equivalence of *Mosco convergence* established in Section 1.3. This result is an abstract generalisation of a similar equivalence of Mosco convergence of certain function spaces in [73]. The abstract approach here can be applied to both Dirichlet problem and Neumann problem. Section 1.4 presents convergence results for both boundary value problems. We note that Mosco convergence assumption for Dirichlet problem is weaker than the notion of convergence of domains used in [35]. Hence, our result allows a larger class of domain perturbation. We also include a direct application of convergence results to study final value problems in Section 1.5. In Section 1.6, we use a similar technique to obtain convergence of solutions of parabolic variational inequalities under perturbation of the underlying convex sets.

Chapter 2 provides some background and results on domain perturbation for semilinear parabolic equations. While most of the material is based on [38], our version contains an improvement. We prove in Theorem 2.3.5 that the convergence of solutions under perturbation is also possible when the nonlinear terms *converge pointwise* only. This convergence result was originally proved in [38, Theorem 4.6] when the uniform convergence of nonlinear terms is assumed. Using this improved result and the work in Chapter 1, we give domain perturbation theory for non-autonomous semilinear parabolic equations in Section 2.4. This is complementary to [38] where the application is done for autonomous semilinear parabolic equations. Nevertheless, we also include the results for autonomous problems in Section 2.4.2. In particular, we collect known results on convergence of the spectrum and resolvent operators for the corresponding elliptic equations under domain perturbation.

In Chapter 3 we study the dynamical behaviour of parabolic equations under domain perturbation. Similar studies on dynamical behaviour under domain perturbation include, for example, the works of [42] for Dirichlet problems and [10, 68] for Neumann problems. Our main focus in this thesis is the persistence of invariant manifolds near a stationary point for semilinear parabolic equations under Dirichlet boundary condition. Section 3.2 introduces the dynamical system in terms of *semiflows* induced by solutions of parabolic equations, and outlines the construction of the local stable and the local unstable invariant manifolds by the method of [14]. The main results on the persistence of unstable and stable invariant manifolds are stated in Theorem 3.3.3 and Theorem 3.3.4, respectively. The proofs are given in Section 3.5 for unstable invariant manifolds and in Section 3.6 for stable invariant manifolds. Our continuity (upper and lower semicontinuity) results for the local stable and the local unstable invariant manifolds are given in the L^2 setting using the semiflows induced by mild solutions. In [10, 68] and [42], they are only interested in continuity of local unstable invariant manifolds (and consequently continuity of attractors), but the convergence is obtained in $H^1(\mathbb{R}^N)$. Moreover, we keep track of the dependence of invariant manifolds constructed by the method of [14], whereas the existence of invariant manifolds in [10] follows from a standard construction in [52].

Finally, in Chapter 4, we turn back to study the behaviour of solutions under domain perturbation. However, we give attention to a class of *bounded entire solutions* of autonomous parabolic equations on the whole real line rather than solutions of initial value problems on a bounded interval considered in Chapter 1 and Chapter 2. This investigation is rather new in the literature. Under an *exponential dichotomy* assumption, we collect some results on the existence and uniqueness of bounded entire solutions of linear parabolic equations in Section 4.1. Roughly, the bounded entire solution can be represented in terms of semigroups (evolution systems) and projections onto stable and unstable subspaces. In Section 4.2, we give domain perturbation results for bounded entire solutions of linear parabolic equations in an L^2 setting. The results are then used in Section 4.3 to establish the convergence of solutions in L^p for a certain range of $p > 2$. The highlight of this section is the Hölder regularity (with respect to time) of bounded entire solutions in L^p proved in Proposition 4.3.7. Although a similar regularity result can be found in [40, Corollary 5.6] for evolution equations in interpolation spaces or [52, Section 3.3] for evolution equations in fractional power spaces, the same argument cannot be applied when we work in the L^p scales. In Section 4.4, we look at semilinear parabolic equations. We discuss the cases where the *Leray-Schauder degree* can or cannot be applied to establish the persistence of a known bounded entire solution of semilinear problems under domain perturbation. The technique of using the Leray-Schauder degree is not new. It appeared in [35] in the context of periodic solutions and in [31] in the context of nonlinear elliptic equations. We give a remark on the difficulty of using this technique for almost periodic solutions.

Chapter 1

Domain perturbation for linear non-autonomous parabolic equations

In this chapter, we study the behaviour of solutions of linear non-autonomous parabolic equations. We first collect preliminary results on the existence and uniqueness of solutions of non-autonomous linear parabolic equations by variational methods. In Section 1.2, we discuss a useful notion of *Mosco convergence* of closed and convex sets. It will be the main tool to deal with a sequence of functions belonging to different function spaces. Section 1.3 presents a key result that enables us to study domain perturbation for non-autonomous parabolic equations via the corresponding elliptic equations. We show the equivalence between Mosco convergences of various *closed and convex subsets* of a Banach space. This result is an abstract generalisation of the equivalence of Mosco convergences in $W_0^{1,r}(\mathbb{R}^N)$ and in $L^p((0,T), W_0^{1,r}(\mathbb{R}^N))$ proved in [73]. Section 1.4 gives some applications of the key result. In particular, we prove the convergence of solutions under domain perturbation for non-autonomous parabolic equations under Dirichlet or Neumann boundary conditions. Moreover, we include convergence results for final value problems in Section 1.5. Finally, in Section 1.6, we obtain a similar convergence of solutions for parabolic variational inequalities under perturbation of the underlying convex set.

1.1 Abstract parabolic equations

In this section, we introduce basic notations and collect standard results on linear parabolic equations.

Suppose V is a real separable and reflexive Banach space and H is a separable Hilbert space such that V is dense in H. By identifying H with its dual space H', we consider the following *evolution triple*

$$V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'.$$

For an interval $(a,b) \subset \mathbb{R}$, we denote by $L^2((a,b),V)$, the Bochner-Lebesgue space. We define the Bochner-Sobolev space

$$W((a,b),V,V') := \{u \in L^2((a,b),V) : u' \in L^2((a,b),V')\},$$

where u' is the derivative in the sense of distributions taking values in V'. The space $W((a,b),V,V')$ is a Banach space when equipped with the following norm

$$\|u\|_W := \Big(\int_a^b \|u(t)\|_V^2 \, dt + \int_a^b \|u'(t)\|_{V'}^2 \, dt \Big)^{1/2}.$$

It is well known that $W((a,b),V,V') \hookrightarrow C([a,b],H)$, where the space of H-valued continuous functions $C([a,b],H)$ is equipped with the uniform norm ([41, Theorem I1.3.1]). More precisely, if $u \in W((a,b),V,V')$, then there exists a uniquely determined H-valued continuous functions on $[a,b]$ which coincides almost everywhere on (a,b) with the function u. Hereafter, we use this uniquely determined function in $C([a,b],H)$ as a representative of a function $u \in W((a,b),V,V')$. In this sense, there exists a constant $C > 0$ such that

$$\sup_{t\in[a,b]} \|u(t)\|_H \le C\|u\|_W,$$

for all $u \in W((a,b),V,V')$. Moreover, for $u, v \in W((a,b),V,V')$ and $a_0, b_0 \in [a,b]$ with $a_0 < b_0$, we have the integration by parts formula

$$(u(b_0)|v(b_0)) - (u(a_0)|v(a_0)) = \int_{a_0}^{b_0} \langle u'(t), v(t)\rangle + \langle v'(t), u(t)\rangle \, dt. \tag{1.1}$$

Here and throughout the thesis, we denote by $(\cdot|\cdot)$, the scalar product in H and by $\langle \cdot, \cdot \rangle$, the duality paring between V' and V. Let I and J be two sets, we write $J \subset\subset I$ if $\bar{J} \subset \operatorname{Int}(I)$, where $\operatorname{Int}(I)$ denotes the interior of I. For a subset X of a Banach space V, we define the closed convex hull by

$$\overline{\operatorname{conv}}(X) := \overline{\Big\{ \sum_{i=1}^k \alpha_i x_i \mid x_i \in X, \alpha_i \in \mathbb{R}, \alpha_i \ge 0, \sum_{i=1}^k \alpha_i = 1, k = 1,2,\dots \Big\}}.$$

The framework of parabolic equations studied throughout this thesis follows a standard bilinear form setting. For each $t \in [0,T]$, suppose $a(t;\cdot,\cdot)$ is a continuous bilinear form on V satisfying the following hypothesis:

- For every $u, v \in V$, the map $t \mapsto a(t;u,v)$ is measurable.
- There exists a constant $M > 0$ independent of $t \in [0,T]$ such that
$$|a(t;u,v)| \le M\|u\|_V\|v\|_V, \tag{1.2}$$
for all $u, v \in V$ and $t \in [0,T]$.
- There exist $\alpha > 0$ and $\lambda \in \mathbb{R}$ such that
$$a(t;u,u) + \lambda\|u\|_H^2 \ge \alpha\|u\|_V^2, \tag{1.3}$$
for all $u \in V$ and $t \in [0,T]$.

For each $t \in [0,T]$ and $u \in V$, the bilinear form $a(t;\cdot,\cdot)$ induces a continuous linear operator $A(t) \in \mathscr{L}(V,V')$ with
$$\langle A(t)u, v\rangle = a(t;u,v), \tag{1.4}$$
for all $u, v \in V$. We easily see from (1.2) that $\sup_{t\in[0,T]}\|A(t)\|_{\mathscr{L}(V,V')} \le M$.

Let us consider the abstract linear parabolic equation
$$\begin{cases} u'(t) + A(t)u = f(t) & \text{for } t \in (0,T] \\ u(0) = u_0, \end{cases} \tag{1.5}$$
where $u_0 \in H$ and $f \in L^2((0,T),V')$.

Definition 1.1.1. A function $u \in W(0,T,V,V')$ satisfying (1.5) is called a *weak solution* of (1.5).

It is well known that u is a weak solution of (1.5) if and only if $u \in L^2((0,T),V)$ and
$$\begin{aligned} -\int_0^T (u(t)|v)\phi'(t)\,dt + \int_0^T a(t;u(t),v)\phi(t)\,dt \\ = (u_0|v)\phi(0) + \int_0^T \langle f(t),v\rangle\phi(t)\,dt, \end{aligned} \tag{1.6}$$
for all $v \in V$ and for all $\phi \in \mathscr{D}([0,T))$. The existence and uniqueness of solution is given in the following theorem (see, for example, [41, XVIII §3] and [81, §23.7]).

Theorem 1.1.2. *Given $f \in L^2((0,T),V')$ and $u_0 \in H$, there exists a unique weak solution of* (1.5) *satisfying*

$$\|u\|_{W(0,T,V,V')} \leq C\Big(\|u_0\|_H + \|f\|_{L^2((0,T),V')}\Big) \tag{1.7}$$

with $C > 0$ independent of f and u_0. Moreover, if $\lambda = 0$ in (1.3)*, then the weak solution satisfies*

$$\|u(t)\|_H^2 + \alpha \int_0^t \|u(s)\|_V^2 \, ds \leq \|u_0\|_H^2 + \alpha^{-1} \int_0^t \|f(s)\|_{V'}^2 \, ds, \tag{1.8}$$

for all $t \in [0,T]$.

Note that $v(t) := e^{-\lambda t}u(t)$ is a weak solution of (1.5) with $A(t)$ replaced by $A(t)+\lambda$. Hence, we can assume without loss of generality that $\lambda = 0$ in (1.3).

The remaining part of this section is devoted to an abstract formulation of linear non-autonomous parabolic equations. In other words, we consider the equation (0.4) in Introduction when the inhomogeneous term $f(t,x,u)$ is independent of u as written below

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}(t)u = f(x,t) & \text{in } \Omega \times (0,T] \\ \mathcal{B}(t)u = 0 & \text{on } \partial\Omega \times (0,T] \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{1.9}$$

For $t \in [0,T]$, we consider a bilinear form $a(t;\cdot,\cdot)$ given by

$$a(t;u,v) := \int_\Omega [a_{ij}(x,t)\partial_j u + a_i(x,t)u]\partial_i v + b_i(x,t)\partial_i uv + c_0(x,t)uv \, dx, \tag{1.10}$$

for $u, v \in V$. By assumptions on a_{ij}, a_i, b_i and c_0, it is clear that the map $t \mapsto a(t;u,v)$ is measurable for all $u,v \in V$. Moreover, it can be verified that the form $a(t;\cdot,\cdot)$ defined above satisfies (1.2) and (1.3) (see [41]).

It is well known that (1.9) can be written as an abstract equation (1.5) by taking the evolution triple $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$ with

$$\begin{aligned} &V = H_0^1(\Omega) \text{ and } H = L^2(\Omega) && \text{for Dirichlet boundary value problem or,} \\ &V = H^1(\Omega) \text{ and } H = L^2(\Omega) && \text{for Neumann boundary value problem} \end{aligned} \tag{1.11}$$

(see [81, Corollary 23.24]).

1.2 Mosco Convergence

Mosco convergence was originally introduced in [65] for a sequence of convex sets and was the main tool to establish convergence properties of solutions of elliptic variational inequalities when the underlying convex set is perturbed. We assume that V is a reflexive and separable Banach space, and K_n, K are closed and convex subsets of V. We start by giving a definition of Mosco convergence in various spaces including V, $L^2((0,T),V)$ and $W((0,T),V,V')$.

Definition 1.2.1. We say that K_n converges to K in the sense of Mosco if the following conditions hold:

$(M1)$ For every $u \in K$, there exists a sequence $u_n \in K_n$ such that $u_n \to u$ in V strongly.

$(M2)$ If (n_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in K_{n_k}$ for every k and $u_k \rightharpoonup u$ in V weakly, then $u \in K$.

There is an alternative definition of Mosco convergence defined in terms of Kuratowski limits. A general result on Mosco convergence and equivalence of these definitions can be found in [11, Chapter 3]. We give a trivial example of Mosco convergence from our domain perturbation point of view below.

Example 1.2.2. Let Ω_n and Ω be bounded open sets in $\mathbb{R}^N$ such that $\Omega_n, \Omega \subset D$ for some ball D for all $n \in \mathbb{N}$. By identifying a function $u \in H_0^1(\Omega)$ with its trivial extension (extension by zero on $D \backslash \Omega$), we regard u as a function in $H^1(D)$. By a similar identification for functions in $H_0^1(\Omega_n)$, we regard $H_0^1(\Omega)$ and $H_0^1(\Omega_n)$ as subsets of $H^1(D)$. It is easy to see that they are closed subspaces (hence, closed and convex subsets) of $H^1(D)$. If $\Omega_n \subset \Omega_{n+1} \subset \Omega$ for all $n \in \mathbb{N}$ and $\Omega = \bigcup_{n \in \mathbb{N}} \Omega_n$, then $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco.

As seen in Section 1.1, solutions of parabolic equation (1.5) is in $L^2((0,T),V)$. Thus, it is worthwhile to study Mosco convergence in $L^2((0,T),V)$. We let

$$L^2((0,T),K) := \{u \in L^2((0,T),V) \mid u(t) \in K \text{ a.e.}\},$$

and

$$C([0,T],K) := \{u \in C([0,T],V) \mid u(t) \in K \quad \forall t \in [0,T]\}.$$

It can be verified that $L^2((0,T),K)$ is a closed and convex subset of $L^2((0,T),V)$. We next state Mosco convergence of function spaces for parabolic problems.

Definition 1.2.3. We say that $L^2((0,T),K_n)$ converges to $L^2((0,T),K)$ in the sense of Mosco if the following conditions hold:

$(M1')$ For every $u \in L^2((0,T),K)$, there exists a sequence $u_n \in L^2((0,T),K_n)$ such that $u_n \to u$ in $L^2((0,T),V)$ strongly.

$(M2')$ If (n_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in L^2((0,T),K_{n_k})$ for every k and $u_k \rightharpoonup u$ in $L^2((0,T),V)$ weakly, then $u \in L^2((0,T),K)$.

Similarly, we can state Mosco convergence in $W((0,T),V,V')$.

Definition 1.2.4. We say that $W((0,T),V,V') \cap L^2((0,T),K_n)$ converges to $W((0,T),V,V') \cap L^2((0,T),K)$ in the sense of Mosco if the following conditions hold:

$(M1'')$ For every $u \in W((0,T),V,V') \cap L^2((0,T),K)$, there exists a sequence u_n in $W((0,T),V,V') \cap L^2((0,T),K_n)$ such that $u_n \to u$ in $W((0,T),V,V')$ strongly.

$(M2'')$ If (n_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in W((0,T),V,V') \cap L^2((0,T),K_{n_k})$ for every k, $u_k \rightharpoonup u$ in $L^2((0,T),V)$ weakly and $u_k' \rightharpoonup w$ in $L^2((0,T),V')$ weakly, then $u' = w$ and $u \in W((0,T),V,V') \cap L^2((0,T),K)$.

1.3 Equivalence of Mosco convergences for parabolic and elliptic equations

In this section, we prove a key result that relates domain perturbation for non-autonomous parabolic equations and domain perturbation for elliptic equations. We remark that for autonomous parabolic equations, the convergence of solutions of parabolic equations can be deduced from the convergence of solutions of the corresponding elliptic equations under domain perturbation. This is simply because we can apply semigroup methods together with the convergence result of degenerate semigroups due to Arendt [4, Theorem 5.2]. In Section 6 of the same paper, the convergence of solutions of Dirichlet heat equations is given as an application. Further examples on other boundary conditions including Neumann and Robin boundary conditions can be found in [38, Section 6].

We state our key result below.

Theorem 1.3.1. *The following assertions are equivalent:*

(i) K_n *converges to* K *in the sense of Mosco.*

(ii) $L^2((0,T),K_n)$ *converges to* $L^2((0,T),K)$ *in the sense of Mosco.*

(iii) *$W((0,T),V,V') \cap L^2((0,T),K_n)$ converges to $W((0,T),V,V') \cap L^2((0,T),K)$ in the sense of Mosco.*

Before proving the equivalence of Mosco convergences in Theorem 1.3.1, we require some technical lemmas.

Lemma 1.3.2. *For a bounded open interval $(a,b) \subset \mathbb{R}$, let $u \in L^2((a,b),K)$. If $\phi \in \mathscr{D}((a,b))$ such that $\phi \geq 0$ and $\int_a^b \phi(t)\,dt = 1$, then $\int_a^b u(t)\phi(t)\,dt \in K$.*

Proof. Since K is closed and convex, we have $\int_a^b u(t)\phi(t)\,dt \in \overline{\mathrm{conv}}\{u(t) \mid t \in (a,b)\} \subset K$ for all $u \in L^2((a,b),K)$. □

Lemma 1.3.3. *Let $I = (a,b)$ be a bounded open interval in $\mathbb{R}$. If $u \in L^2(I,V)$ and $\int_I u(t)\phi(t)\,dt \in K$ for all $\phi \in \mathscr{D}(I)$ with $\phi \geq 0$ and $\int_I \phi(t)\,dt = 1$, then $u \in L^2(J,K)$ for all $J = (c,d) \subset\subset I$.*

Proof. Let $\eta \in \mathscr{D}(\mathbb{R})$ be the standard mollifier. For $\varepsilon > 0$, we define $\eta_\varepsilon(t) = \frac{1}{\varepsilon}\eta(\frac{t}{\varepsilon})$ so that $\eta_\varepsilon \in \mathscr{D}(\mathbb{R})$ with $\int_{\mathbb{R}} \eta_\varepsilon(t)\,dt = 1$ and $\mathrm{supp}(\eta_\varepsilon) \subset (-\varepsilon,\varepsilon)$. Consider the mollified function $u_\varepsilon := \eta_\varepsilon * u$. For a.e. $t \in I$, we have

$$\begin{aligned}
\|u_\varepsilon(t) - u(t)\|_V &= \left\| \int_{t-\varepsilon}^{t+\varepsilon} \eta_\varepsilon(t-s)[u(s)-u(t)]\,ds \right\|_V \\
&\leq \frac{1}{\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} \eta\Big(\frac{t-s}{\varepsilon}\Big)\|u(s)-u(t)\|_V\,ds \\
&\leq C\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} \|u(s)-u(t)\|_V\,ds.
\end{aligned}$$

By Lebesgue's differentiation theorem for vector valued functions (Theorem III.12.8 of [44]), $u_\varepsilon(t) \to u(t)$ in V a.e. $t \in I$. By the definition of u_ε,

$$u_\varepsilon(t) = \int_I \eta_\varepsilon(t-s)u(s)\,ds =: \int_I u(s)\phi_\varepsilon(s)\,ds,$$

where we set $\phi_\varepsilon(s) := \eta_\varepsilon(t-s)$. Let $J \subset\subset I$. For $t \in J$, we can choose ε sufficiently small so that $\mathrm{supp}(\phi_\varepsilon) \subset (t-\varepsilon, t+\varepsilon) \subset I$. It follows from the assumption that $u_\varepsilon(t) \in K$ for all $t \in J$. Since K is a closed subset of V, the limit point $u(t) \in K$ a.e. $t \in J$. Hence, u belongs to $L^2(J,K)$ as required. □

Lemma 1.3.4. *The set $C([0,T],K)$ is dense in $L^2((0,T),K)$.*

Proof. Note first that the lemma is trivial if K is a subspace of V (i.e. K is a Banach space) [81, Theorem 23.2 (c)]. Let $u \in L^2((0,T),K)$ be arbitrary. We choose a function $\phi \in \mathscr{D}((0,T))$ with $\int_0^T \phi(t)\ dt = 1$. It follows from Lemma 1.3.2 that $\xi := \int_0^T u(t)\phi(t)\ dt \in K$. Define the extended function $\tilde{u} \in L^2((-1,T+1),K)$ by

$$\tilde{u}(t) := \begin{cases} \xi & \text{on } (-1,0)\cup(T,T+1) \\ u(t) & \text{on } (0,T). \end{cases}$$

By a mollification argument, the function $u_\varepsilon := \eta_\varepsilon * \tilde{u}$ belongs to $C(\mathbb{R},V)$. Moreover, u_ε converges to $\tilde{u}$ in $L^2((-1,T+1),V)$. By choosing $0<\varepsilon<1$, we have $u_\varepsilon(t)\in K$ for all $t\in[0,T]$. Therefore, the restriction of u_ε to $[0,T]$ belongs to $C([0,T],K)$ and converges to u in $L^2((0,T),V)$ as $\varepsilon\to 0$. □

Lemma 1.3.5. *The set $C^\infty([0,T],V)\cap C([0,T],K)$ is dense in $W((0,T),V,V')\cap L^2((0,T),K)$.*

Proof. Let $u\in W((0,T),V,V')\cap L^2((0,T),K)$ be arbitrary. For $\delta>0$, we define the stretching map $S_\delta:[0,T]\to[-\delta,T+\delta]$ by

$$S_\delta(t) := \Big(\frac{T+2\delta}{T}\Big)t - \delta. \tag{1.12}$$

We define $u_\delta \in W((-\delta,T+\delta),V,V')\cap L^2((-\delta,T+\delta),K)$ by $u_\delta := u\circ S_\delta^{-1}$. It can be easily seen from the dominated convergence theorem and a chain rule that the restriction of u_δ to $(0,T)$ converges to u in $W((0,T),V,V')$ as $\delta\to 0$. Let η_ε be a mollifier. For $t\in[0,T]$ and $\varepsilon<\delta$, the translation of η_ε by t (denoted by $\eta_{\varepsilon,t}$) belongs to $\mathscr{D}((-\delta,T+\delta))$. Hence, if $\varepsilon<\delta$, then $\eta_\varepsilon * u_\delta$ belongs to $C^\infty([0,T],V)\cap C([0,T],K)$. Moreover, a mollification argument shows that $\eta_\varepsilon * u_\delta$ converges to u_δ in $W((0,T),V,V')$ as $\varepsilon\to 0$. The result then follows. □

Proposition 1.3.6. *Suppose that the Mosco condition $(M1)$ is satisfied. For $\delta\ge 0$, let $A_{\delta,n} := \Big\{\sum_{i=1}^m \phi_i(t)v_i, m\in\mathbb{N}\Big\}$, where*

$$\begin{cases} v_i\in K_n, \phi_i\in C^\infty([-\delta,T+\delta]) & \textit{for all } i=1,\dots,m, \\ 0\le\phi_i(t)\le 1 & \textit{for all } t\in[-\delta,T+\delta] \textit{ and for all } i=1,\dots,m, \\ \displaystyle\sum_{i=1}^m \phi_i(t)=1 & \textit{for all } t\in[-\delta,T+\delta]. \end{cases} \tag{1.13}$$

If $u_\delta\in C([-\delta,T+\delta],K)$, then there exists a sequence of functions $u_{\delta,n}\in A_{\delta,n}$ such that $u_{\delta,n}(t)\to u_\delta(t)$ in V uniformly on $[-\delta,T+\delta]$ as $n\to\infty$.

Proof. Let u_δ be a function in $C([-\delta, T+\delta], K)$. We extend u_δ to $\tilde{u}_\delta \in C(\mathbb{R}, K)$ by

$$\tilde{u}_\delta(t) := \begin{cases} u_\delta(-\delta) & \text{on } (-\infty, -\delta) \\ u_\delta(t) & \text{on } [-\delta, T+\delta] \\ u_\delta(T+\delta) & \text{on } (T+\delta, \infty). \end{cases}$$

Let $\varepsilon > 0$ be fixed. We denote by $B(t) := B_V(\tilde{u}_\delta(t), \varepsilon/2)$ the open ball in V about $\tilde{u}_\delta(t)$ of radius $\varepsilon/2$. Let us construct an open covering $\mathscr{O}$ of $(-\delta-1, T+\delta+1)$ by

$$\mathscr{O} = \{\tilde{u}_\delta^{-1}(B(t)) \cap (-\delta-1, T+\delta+1)\}_{t\in[-\delta,T+\delta]}.$$

Since $\mathscr{O}$ is also an open covering of the compact set $[-\delta, T+\delta]$, there exists a finite subcovering

$$\tilde{\mathscr{O}} = \{\tilde{u}_\delta^{-1}(B(t_i)) \cap (-\delta-1, T+\delta+1)\}_{i=1,\ldots,m},$$

where $t_i \in [-\delta, T+\delta]$ for all $i = 1, \ldots, m$. We can assume that $t_1 < t_2 < \ldots < t_m$ and $t_1 = -\delta$, $t_m = T+\delta$ (add them if required) so that $\tilde{\mathscr{O}}$ is an open covering of $[-\delta-1/2, T+\delta+1/2]$. For each $i \in \{1, \ldots, m\}$, we have $u_\delta(t_i) \in K$. Thus, by the Mosco condition $(M1)$, there exist $v_{i,n} \in K_n$ and $N_i \in \mathbb{N}$ such that $\|v_{i,n} - u_\delta(t_i)\|_V < \varepsilon/2$ if $n > N_i$ for each $i = 1, \ldots, m$. Let $N := \max_{i=1,\ldots,m} N_i$. It follows that $\|v_{i,n} - u_\delta(t_i)\|_V < \varepsilon/2$ if $n > N$ for all $i \in \{1, \ldots, m\}$.

Choose a smooth partition of unity $\{\phi_i\}_{i=1,\ldots m}$ for $[-\delta-1/2, T+\delta+1/2]$ subordinate to $\tilde{\mathscr{O}}$. More precisely, we choose ϕ_i such that $\phi_i \in C_0^\infty(\tilde{u}_\delta^{-1}(B(t_i)) \cap (-\delta-1, T+\delta+1))$ and $\sum_{i=1}^m \phi_i(t) = 1$ for all $t \in [-\delta-1/2, T+\delta+1/2]$. Define a function $u_{\delta,n}$ on $(-\delta-1, T+\delta+1)$ by

$$u_{\delta,n}(t) := \sum_{i=1}^m \phi_i(t) v_{i,n}.$$

It is clear that the restriction of $u_{\delta,n}$ to $[-\delta, T+\delta]$ belongs to $A_{\delta,n}$ if $n > N$. Moreover, for $t \in [-\delta, T+\delta]$,

$$\begin{aligned} \|u_{\delta,n}(t) - u_\delta(t)\|_V &\leq \sum_{i=1}^m \phi_i(t)\|v_{i,n} - u_\delta(t)\|_V \\ &\leq \sum_{i=1}^m \phi_i(t)\|v_{i,n} - u_\delta(t_i)\|_V + \sum_{i=1}^m \phi_i(t)\|u_\delta(t_i) - u_\delta(t)\|_V \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

if $n > N$. Note that m and N chosen above depend on ε. As the above argument holds for each fixed ε, we conclude that for every $\varepsilon > 0$, there exist a sequence $u_{\delta,n}^\varepsilon \in A_{\delta,n}$

and $N(\varepsilon) \in \mathbb{N}$ such that

$$\|u^{\varepsilon}_{\delta,n}(t) - u_\delta(t)\|_V \leq \varepsilon,$$

for all $t \in [-\delta, T+\delta]$ if $n > N(\varepsilon)$.

In particular, for every $k \in \mathbb{N}$ we can find a sequence $u^k_{\delta,n} \in A_{\delta,n}$ and $N_k \in \mathbb{N}$ such that

$$\|u^k_{\delta,n}(t) - u_\delta(t)\|_V \leq \frac{1}{k}, \tag{1.14}$$

for all $t \in [-\delta, T+\delta]$ if $n > N_k$. By choosing inductively we can assume that $N_k < N_{k+1}$ for all $k \in \mathbb{N}$. We extract a sequence of the form

$$u^1_{\delta,1}, u^1_{\delta,2}, \ldots, u^1_{\delta,(N_1+1)}, \ldots, u^1_{\delta,N_2}, u^2_{\delta,(N_2+1)}, \ldots, u^2_{\delta,N_3}, u^3_{\delta,(N_3+1)}, \ldots, u^3_{\delta,N_4}, \ldots$$

so that the n-th element of this sequence belongs to $A_{\delta,n}$ for all $n \in \mathbb{N}$. Moreover, by (1.14), we see that this sequence converges to u_δ uniformly with respect to $t \in [-\delta, T+\delta]$ as $n \to \infty$. This proves the statement of the proposition. □

We are now in a position to prove our main result.

Proof of Theorem 1.3.1. The proof is divided into four parts including $(i) \Rightarrow (ii)$, $(ii) \Rightarrow (i)$, $(i) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$. For $(i) \Rightarrow (ii)$, we actually show that $(M1) \Rightarrow (M1')$ and $(M2) \Rightarrow (M2')$. The other three directions are proved in a similar way.

$(i) \Rightarrow (ii)$: Let $u \in L^2((0,T), K)$ be arbitrary. By the density of $C([0,T], K)$ in $L^2((0,T), K)$ (Lemma 1.3.4), we may assume that $u \in C([0,T], K)$. We apply Proposition 1.3.6 with $\delta = 0$ to obtain a sequence of functions $u_n \in L^2((0,T), K_n)$ such that $u_n(t) \to u(t)$ in V uniformly on $[0,T]$. The uniform convergence on $[0,T]$ implies that $u_n \to u$ in $L^2((0,T), V)$, showing $(M1')$. To verify condition $(M2')$, suppose (n_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in L^2((0,T), K_{n_k})$ for every k and $u_k \rightharpoonup u$ in $L^2((0,T), V)$. By the definition of weak convergence,

$$\int_0^T \langle w(t), u_k(t)\rangle \, dt \to \int_0^T \langle w(t), u(t)\rangle \, dt, \tag{1.15}$$

for all $w \in L^2((0,T), V')$. By taking w of the form $w = \xi\phi(t)$ where $\xi \in V'$ and $\phi \in \mathscr{D}((0,T))$ in (1.15) and applying a basic property of Bochner-Lebesgue space [81, Proposition 23.9(a)], it follows that

$$\int_0^T u_k(t)\phi(t) \, dt \rightharpoonup \int_0^T u(t)\phi(t) \, dt \tag{1.16}$$

in V weakly for all $\phi \in \mathscr{D}((0,T))$. Let $\phi_0 \in \mathscr{D}((0,T))$ with $\int_0^T \phi_0(t)\,dt = 1$ and define $\zeta_k := \int_0^T u_k(t)\phi_0(t)\,dt$. Lemma 1.3.2 implies that $\zeta_k \in K_{n_k}$ for all $k \in \mathbb{N}$. Since $\zeta_k \rightharpoonup \zeta := \int_0^T u(t)\phi_0(t)\,dt$ by (1.16), the Mosco condition $(M2)$ implies that $\zeta \in K$. We now extend u_k to $\tilde{u}_k \in L^2((-1,T+1),K_{n_k})$ by

$$\tilde{u}_k(t) := \begin{cases} \zeta_k & \text{on } (-1,0)\cup(T,T+1) \\ u_k(t) & \text{on } (0,T). \end{cases} \tag{1.17}$$

It can be easily seen that $\tilde{u}_k \rightharpoonup \tilde{u}$ in $L^2((-1,T+1),V)$ weakly, where $\tilde{u}$ is defined as (1.17) with k deleted. Using the definition of weak convergence in $L^2((-1,T+1),V)$ and a similar argument as above, we obtain

$$\int_{-1}^{T+1} \tilde{u}_k(t)\phi(t)\,dt \rightharpoonup \int_{-1}^{T+1} \tilde{u}(t)\phi(t)\,dt$$

in V weakly for all $\phi \in \mathscr{D}((-1,T+1))$. In particular, taking $\phi \in \mathscr{D}((-1,T+1))$ with $\int_{-1}^{T+1} \phi(t)\,dt = 1$, we have that $\int_{-1}^{T+1} \tilde{u}_k(t)\phi(t)\,dt \in K_{n_k}$ converges to $\int_{-1}^{T+1} \tilde{u}(t)\phi(t)\,dt$ in V weakly. Thus, the Mosco condition $(M2)$ implies $\int_{-1}^{T+1} \tilde{u}(t)\phi(t)\,dt \in K$ for all $\phi \in \mathscr{D}((-1,T+1))$ with $\int_{-1}^{T+1} \phi(t)\,dt = 1$. By Lemma 1.3.3, we conclude that $u \in L^2((0,T),K)$ and the Mosco condition $(M2')$ follows.

$(ii) \Rightarrow (i)$: Let $u \in K$ be arbitrary. Define $v \in L^2((0,T),K)$ by the constant function $v(t) := u$ for $t \in (0,T)$. By condition $(M1')$, there exists $(v_n)_{n\in\mathbb{N}}$ with $v_n \in L^2((0,T),K_n)$ such that $v_n \to v$ in $L^2((0,T),V)$. Let $\phi_0 \in \mathscr{D}((0,T))$ with $\int_0^T \phi_0(t)\,dt = 1$. We show that the sequence $(u_n)_{n\in\mathbb{N}}$ defined by $u_n := \int_0^T v_n(t)\phi_0(t)\,dt$ gives the Mosco condition $(M1)$. First note that $u_n \in K_n$ for all $n \in \mathbb{N}$ by Lemma 1.3.2. Moreover,

$$\begin{aligned} \|u_n - u\|_V &= \Big\| \int_0^T v_n(t)\phi_0(t)\,dt - u \Big\|_V \\ &= \Big\| \int_0^T [v_n(t)\phi_0(t) - v(t)\phi_0(t)]\,dt \Big\|_V \\ &\le \int_0^T |\phi_0(t)| \|v_n(t) - v(t)\|_V\,dt \\ &\le \sqrt{T}\Big(\int_0^T \|v_n(t) - v(t)\|_V^2\,dt \Big)^{\frac{1}{2}} \|\phi_0\|_\infty \\ &\to 0, \end{aligned}$$

as $n \to \infty$. To verify condition $(M2)$, suppose (n_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in K_{n_k}$ for every k and $u_k \rightharpoonup u$ in V. Define $v_k \in L^2((0,T),K_{n_k})$ by the constant function $v_k(t) := u_k$ for $t \in (0,T)$. It can be easily verified that $v_k \rightharpoonup v$ in $L^2((0,T),V)$, where v is the constant function $v(t) := u$ for $t \in (0,T)$. It follows from the Mosco condition $(M2')$ that $v \in L^2((0,T),K)$. Hence, u belongs to K as required.

$(i) \Rightarrow (iii)$: Let $u \in W((0,T),V,V') \cap L^2((0,T),K)$ be arbitrary. By Lemma 1.3.5, we may assume that $u \in C^\infty([0,T],V) \cap C([0,T],K)$. For $\delta > 0$, we define the stretched function $u_\delta \in C^\infty([-\delta,T+\delta],V) \cap C([-\delta,T+\delta],K)$ by $u_\delta = u \circ S_\delta^{-1}$, where S_δ is the stretching map given by (1.12). As before, it can be easily seen from the dominated convergence theorem that the restriction of u_δ to $[0,T]$ converges to u in $W((0,T),V,V')$ as $\delta \to 0$. By Proposition 1.3.6, there exists a sequence of functions $u_{\delta,n} \in A_{\delta,n}$ such that $u_{\delta,n}(t) \to u_\delta(t)$ uniformly on $[-\delta, T+\delta]$ as $n \to \infty$. Let $\eta_{1/j}$ be a mollifier. For $t \in [0,T]$ and $j > 1/\delta$, the translation of $\eta_{1/j}$ by t (denoted by $\eta_{1/j,t}$) belongs to $\mathscr{D}((-\delta,T+\delta))$. Hence, if $j > 1/\delta$, we have $\eta_{1/j} * u_{\delta,n} \in C^\infty([0,T],V) \cap C([0,T],K_n)$. By the continuity of convolution and the well known fact on the r-th order derivative that

$$\frac{d^r}{dt^r}(\eta_{1/j} * u_{\delta,n}) = \frac{d^r}{dt^r}\eta_{1/j} * u_{\delta,n} = \eta_{1/j} * \frac{d^r}{dt} u_{\delta,n},$$

we deduce that $\eta_{1/j} * u_{\delta,n} \to \eta_{1/j} * u_\delta$ in $C^\infty([0,T],V)$ as $n \to \infty$. Similarly, we have $\eta_{1/j} * u_\delta \to u_\delta$ in $C^\infty([0,T],V)$ as $j \to \infty$. The above argument shows that we can construct a function of the form $\eta_{1/j} * u_{\delta,n} \in W((0,T),V,V') \cap L^2((0,T),K_n)$ converging to u in $W((0,T),V,V')$. Hence, the Mosco condition $(M1'')$ follows. To verify condition $(M2'')$, suppose (n_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in W((0,T),V,V') \cap L^2((0,T),K_{n_k})$ for every k, $u_k \rightharpoonup u$ in $L^2((0,T),V)$ and $u_k' \rightharpoonup w$ in $L^2((0,T),V')$. Since V is continuously embedded in V', it follows immediately that $u' = w$ and hence we have $u \in W((0,T),V,V')$ (see [81, Proposition 23.19]). Using $(i) \Rightarrow (ii)$, specifically the Mosco condition $(M2')$, we conclude that $u \in W((0,T),V,V') \cap L^2((0,T),K)$.

$(iii) \Rightarrow (i)$: Let $u \in K$ be arbitrary. Define $v \in W((0,T),V,V') \cap L^2((0,T),K)$ by the constant function $v(t) := u$ for $t \in (0,T)$. By condition $(M1'')$, there exists $(v_n)_{n\in\mathbb{N}}$ with $v_n \in W((0,T),V,V') \cap L^2((0,T),K_n)$ such that $v_n \to v$ in $W((0,T),V,V')$. In particular, v_n converges to v in $L^2((0,T),V)$ strongly. By the same argument as in the proof of $(ii) \Rightarrow (i)$, we can show that $u_n := \int_0^T v_n(t)\phi_0(t)\,dt$, for some $\phi_0 \in \mathscr{D}((0,T))$ with $\int_0^T \phi_0(t)\,dt = 1$ establishes the Mosco condition $(M1)$. To verify condition $(M2)$, suppose (n_k) is a sequence of indices converging to ∞, (u_k) is a sequence such that $u_k \in K_{n_k}$ for every k and $u_k \rightharpoonup u$ in V. Define $v_k \in W((0,T),V,V') \cap L^2((0,T),K_{n_k})$ by the constant function $v_k(t) := u_k$ for $t \in (0,T)$. By the same argument as in the proof of $(ii) \Rightarrow (i)$, we have $v_k \rightharpoonup v$ in $L^2((0,T),V)$, where $v(t) := u$ for $t \in (0,T)$.

Moreover, it is clear that $v_k' = 0$ for all $k \in \mathbb{N}$ and hence $v_k' \rightharpoonup v' = 0$ in $L^2((0,T),V')$. We apply $(M2'')$ to deduce that $v \in W((0,T),V,V') \cap L^2((0,T),K)$. Hence, we have $u \in K$. □

1.4 Convergence of solutions of initial value problems

In this section, we give an application of the main theorem in Section 1.3. We consider a non-autonomous linear parabolic equation subject to Dirichlet or Neumann boundary conditions under domain perturbation.

Assumption 1.4.1. We assume that Ω_n and Ω are bounded open sets in $\mathbb{R}^N$ and that $D \subset \mathbb{R}^N$ is a ball such that $\Omega_n, \Omega \subset D$ for all $n \in \mathbb{N}$.

Suppose a_{ij}, a_i, b_i and c_0 are functions in $L^\infty(D \times (0,T))$ and a_{ij} satisfies an ellipticity condition. More precisely, there exists $\alpha > 0$ such that $a_{ij}(x,t)\xi_i\xi_j \geq \alpha|\xi|^2$ for all $\xi \in \mathbb{R}^N$. We consider the evolution triple $V_n \overset{d}{\hookrightarrow} H_n \overset{d}{\hookrightarrow} V_n'$, where we choose

- $V_n = H_0^1(\Omega_n)$ and $H_n = L^2(\Omega_n)$ for the Dirichlet problem
- $V_n = H^1(\Omega_n)$ and $H_n = L^2(\Omega_n)$ for the Neumann problem.

For $t \in (0,T)$, suppose $a_n(t;\cdot,\cdot)$ is a bilinear form on V_n defined by

$$a_n(t;u,v) := \int_{\Omega_n} [a_{ij}(x,t)\partial_j u + a_i(x,t)u]\partial_i v + b_i(x,t)\partial_i uv + c_0(x,t)uv\,dx. \tag{1.18}$$

It follows that for all $n \in \mathbb{N}$, there exist three constants $M > 0, \alpha > 0$ and $\lambda \in \mathbb{R}$ independent of $t \in [0,T]$ such that

$$|a_n(t;u,v)| \leq M\|u\|_{V_n}\|v\|_{V_n}, \tag{1.19}$$

for all $u, v \in V_n$ and

$$a_n(t;u,u) + \lambda\|u\|_{H_n}^2 \geq \alpha\|u\|_{V_n}^2, \tag{1.20}$$

for all $u \in V_n$. Given $u_{0,n} \in L^2(D)$ and $f_n \in L^2(D \times (0,T))$, let us consider the following boundary value problem in $\Omega_n \times (0,T)$.

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}_n(t)u = f_n(x,t) & \text{in } \Omega_n \times (0,T] \\ \mathcal{B}_n(t)u = 0 & \text{on } \partial\Omega_n \times (0,T] \\ u(\cdot,0) = u_{0,n} & \text{in } \Omega_n, \end{cases} \tag{1.21}$$

where $\mathcal{A}_n$ and $\mathcal{B}_n$ are operators on V_n given by

$$\mathcal{A}_n(t)u := -\partial_i[a_{ij}(x,t)\partial_j u + a_i(x,t)u] + b_i(x,t)\partial_i u + c_0(x,t)u \tag{1.22}$$

and $\mathcal{B}_n$ is one of the following

$$\begin{aligned} &\mathcal{B}_n(t)u := u && \text{Dirichlet boundary condition} \\ &\mathcal{B}_n(t)u := [a_{ij}(x,t)\partial_j u + a_i(x,t)u]\,\nu_i && \text{Neumann boundary condition.} \end{aligned} \tag{1.23}$$

We wish to show that a sequence of solutions of the above parabolic equations on $\Omega_n \times (0,T)$ converges to the solution of the following limit problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}(t)u = f(x,t) & \text{in } \Omega \times (0,T] \\ \mathcal{B}(t)u = 0 & \text{on } \partial\Omega \times (0,T] \\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{1.24}$$

However, we will consider the boundary value problems (1.21) and (1.24) in the abstract form. As discussed in Section 1.1, we can write (1.21) as

$$\begin{cases} u'(t) + A_n(t)u = f_n(t) & \text{for } t \in (0,T] \\ u(0) = u_{0,n}, \end{cases} \tag{1.25}$$

where $A_n(t) \in \mathscr{L}(V_n, V_n')$ is the operator induced by the bilinear form $a_n(t;,\cdot,\cdot)$. Similarly, we write (1.24) as

$$\begin{cases} u'(t) + A(t)u = f(t) & \text{for } t \in (0,T] \\ u(0) = u_0. \end{cases} \tag{1.26}$$

Throughout this section, we denote the weak solution of (1.25) by u_n and the weak solution of (1.26) by u.

1.4.1 Dirichlet problems

The convergence of solutions of Dirichlet problems under domain perturbation has been extensively studied in the literature. In particular, Daners [35] provides convergence results for solutions of non-autonomous parabolic equations under domain perturbation. In [35], it is assumed that a sequence of domains Ω_n converges to Ω in the following sense:

There exists a compact set $K \subset \bar{\Omega}$ of capacity zero and a compact set $K' \subset \mathbb{R}^N$ of measure zero such that

- if $\Omega' \subset\subset \Omega\backslash K$, then $\Omega' \subset\subset \Omega_n$ for large $n \in \mathbb{N}$,
- if U is an open set with $\bar{\Omega} \cup K' \subset U$, then $\Omega_n \subset U$ for large $n \in \mathbb{N}$.

A weak regularity assumption on the limit domain Ω is also imposed as follows:

$$u \in H^1(\mathbb{R}^N) \text{ and } \operatorname{supp} u \subset \bar{\Omega} \text{ imply } u \in H_0^1(\Omega).$$

In this section, we prove the convergence of solutions under domain perturbation using a technique adapted from [35]. However, we replace the notion of convergence of the domains mentioned above by Mosco convergence. In fact, it is not difficult to see that the assumptions on domains in [35] implies Mosco convergence of $H_0^1(\Omega_n)$ to $H_0^1(\Omega)$. We explain a precise concept of Mosco convergence for these Sobolev spaces below.

When the domain is perturbed, the weak solutions belong to different function spaces. We often extend functions by zero outside the domain. We embed the spaces $H_0^1(\Omega_n)$ into $H^1(D)$ by the inclusion map

$$i_n(v) := \begin{cases} v & \text{on } \Omega_n \\ 0 & \text{on } D\backslash\Omega_n, \end{cases} \tag{1.27}$$

for all $v \in H_0^1(\Omega_n)$. The map i_n also maps $L^p(\Omega_n)$ into $L^p(D)$. We may consider the embedding $L^2((0,T), H_0^1(\Omega_n)) \hookrightarrow L^2((0,T), H^1(D))$ by a similar inclusion i_n for a.e. $t \in (0,T)$. In the same way, we consider $H_0^1(\Omega)$ and $L^2((0,T), H_0^1(\Omega))$ as subspaces of $H^1(D)$ and $L^2((0,T), H^1(D))$, respectively by the trivial extension

$$i(v) := \begin{cases} v & \text{on } \Omega \\ 0 & \text{on } D\backslash\Omega, \end{cases} \tag{1.28}$$

for all $v \in H_0^1(\Omega)$. On the other hand, we have the restriction map $r_n : L^p(D) \to L^p(\Omega_n)$ defined by

$$r_n(v) := v|_{\Omega_n}. \tag{1.29}$$

We denote again by r_n the restriction map $L^2((0,T), L^p(D)) \to L^2((0,T), L^p(\Omega_n))$. Similarly, we define the restriction map

$$r(v) := v|_{\Omega} \tag{1.30}$$

when considered as a map $L^p(D) \to L^p(\Omega)$ or $L^2((0,T), L^p(D)) \to L^2((0,T), L^p(\Omega))$.

Remark 1.4.2. It is clear that $r_n i_n(v) = v$ for all $v \in L^p(\Omega_n)$. However, $i_n r_n(v) \neq v$ for most $v \in L^p(D)$ (v such that $v|_{(D\setminus\Omega_n)} \neq 0$). A similar conclusion is valid for the inclusion i and the restriction r.

Let us take $V := H^1(D)$, $K_n := H_0^1(\Omega_n)$ and $K := H_0^1(\Omega)$, and consider Mosco convergence of K_n to K. In this case, K_n and K are closed and convex subsets of V in the sense of the above embedding. In fact, K_n and K are actually closed subspaces of V.

The main application of Theorem 1.3.1 is to show that the weak solution u_n of (1.25) converges to the weak solution u of (1.26) by applying various Mosco conditions.

Theorem 1.4.3. *Suppose that $i_n f_n$ is uniformly bounded in $L^2((0,T), L^2(D))$ and $i_n u_{0,n}$ is uniformly bounded in $L^2(D)$. Assume that $r i_n f_n \rightharpoonup f$ in $L^2((0,T), L^2(\Omega))$ weakly and $r i_n u_{0,n} \rightharpoonup u_0$ in $L^2(\Omega)$ weakly. If $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco, then $i_n u_n$ converges to iu in $L^2((0,T), H^1(D))$ weakly.*

Proof. Since $i_n f_n$ is uniformly bounded in $L^2((0,T), L^2(D))$ and $i_n u_{0,n}$ is uniformly bounded in $L^2(D)$, it follows from (1.7) that $\|u_n\|_{W(0,T,V_n,V_n')}$ is uniformly bounded. Hence $i_n u_n$ is uniformly bounded in $L^2((0,T), H^1(D))$. We can extract a subsequence (denoted again by u_n) such that $i_n u_n \rightharpoonup w$ in $L^2((0,T), H^1(D))$ weakly. The Mosco condition $(M2')$ (from Theorem 1.3.1) implies that $w \in L^2((0,T), H_0^1(\Omega))$. It remains to show that $w = u$ in $L^2((0,T), H_0^1(\Omega))$.

Let $\xi \in H_0^1(\Omega)$ and $\phi \in \mathscr{D}([0,T))$ be arbitrary. The Mosco condition $(M1)$ implies that there exists $\xi_n \in H_0^1(\Omega_n)$ such that $i_n \xi_n \to i\xi$ in $H^1(D)$. As u_n is the weak solution of (1.25), we get from (1.6) that

$$\begin{aligned} -\int_0^T (u_n(t)|\xi_n)\phi'(t)\,dt + \int_0^T a_n(t; u_n(t), \xi_n)\phi(t)\,dt \\ = (u_{0,n}|\xi_n)\phi(0) + \int_0^T \langle f_n(t), \xi_n\rangle \phi(t)\,dt. \end{aligned} \tag{1.31}$$

Now

$$(i_n u_{0,n})|i_n\xi_n)_{L^2(D)} = (r i_n u_{0,n}|r i_n \xi_n)_{L^2(\Omega)} + (i_n u_{0,n}|i_n\xi_n)_{L^2(D\setminus\Omega)}.$$

Since $i_n\xi_n \to i\xi$ in $L^2(D)$, we have $r i_n \xi_n \to r i \xi = \xi$ in $L^2(\Omega)$ and $i_n\xi_n|_{(D\setminus\Omega)} \to 0$ in $L^2(D\setminus\Omega)$. Applying the dominated convergence theorem in the second term above and using the weak convergence of initial condition $r i_n u_{0,n}$ in the first term above, we see that

$$(u_{0,n}|\xi_n)_{L^2(\Omega_n)} \to (u_0|\xi)_{L^2(\Omega)}. \tag{1.32}$$

By letting $n \to \infty$ in (1.31), we get

$$\begin{aligned} -\int_0^T (w(t)|\xi)\phi'(t)\,dt + \int_0^T a(t; w(t), \xi)\phi(t)\,dt \\ = (u_0|\xi)\phi(0) + \int_0^T \langle f(t), \xi\rangle \phi(t)\,dt. \end{aligned} \tag{1.33}$$

Hence, w is a weak solution of (1.26). By the uniqueness of solution, we conclude that $w = u$ in $L^2((0,T), H_0^1(\Omega))$ and the whole sequence converges weakly. □

In fact, we can expect a better convergence result by using the following compactness from [35].

Lemma 1.4.4 ([35, Lemma 2.1]). *Let D be a ball in $\mathbb{R}^N$. Suppose that $\Omega_n \subset D$ for all $n \in \mathbb{N}$ and J is a bounded interval. If $\{v_n\}_{n\in\mathbb{N}}$ is a sequence with $v_n \in W(J, H_0^1(\Omega_n), H^{-1}(\Omega_n))$ for each $n \in \mathbb{N}$ and*

$$\|v_n\|_{W(J,H_0^1(\Omega_n),H^{-1}(\Omega_n))} \leq M$$

for some constant $M > 0$, then $\{i_n v_n\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(J, L^q(D))$ for all $q \in [1, 2N(N-2)^{-1})$.

Theorem 1.4.5. *Suppose that the hypothesis of Theorem 1.4.3 holds. Then $i_n u_n \to iu$ in $L^2((0,T), L^2(D))$ strongly.*

Proof. By Lemma 1.4.4, we can extract a subsequence u_{n_k} such that $i_{n_k} u_{n_k} \to v$ in $L^2((0,T), L^2(D))$ strongly. Since we know from Theorem 1.4.3 that the whole sequence $i_n u_n$ converges to iu in $L^2((0,T), H^1(D))$ weakly, we conclude that $v = iu$ and the whole sequence converges in $L^2((0,T), L^2(D))$ strongly. □

Lemma 1.4.6. *Suppose that the hypothesis of Theorem 1.4.3 holds. Then for each $t \in [0,T]$, we have $ri_n u_n(t) \rightharpoonup u(t)$ in $L^2(\Omega)$.*

Proof. Since $i_n f_n$ is uniformly bounded in $L^2((0,T), L^2(D))$ and $i_n u_{0,n}$ is uniformly bounded in $L^2(D)$, we have from (1.8) that

$$\sup_{n\in\mathbb{N}} \max_{t\in[0,T]} \|i_n u_n(t)\|_{L^2(D)} \leq M,$$

for some $M > 0$. Hence, for a subsequence denoted again by $u_n(t)$, there exists $w \in L^2(\Omega)$ such that $ri_n u_n(t) \rightharpoonup w$ in $L^2(\Omega)$. Let $\xi \in H_0^1(\Omega)$ and $\phi \in \mathscr{D}((0,t])$ be arbitrary.

The Mosco condition $(M1)$ implies that there exists $\xi_n \in H_0^1(\Omega_n)$ such that $i_n\xi_n \to i\xi$ in $H^1(D)$. As u_n is the weak solution of (1.25), we have

$$-\int_0^t (u_n(s)|\xi_n)\phi'(s)\,ds + \int_0^t a_n(s;u_n(s),\xi_n)\phi(s)\,ds \\ = -(u_n(t)|\xi_n)\phi(t) + \int_0^t \langle f_n(s),\xi_n\rangle\phi(s)\,ds.$$

We get in the same way as in (1.32) that

$$(i_nu_n(t)|i_n\xi_n)_{L^2(D)} = (ri_nu_n(t)|ri_n\xi_n)_{L^2(\Omega)} + (i_nu_n(t)|i_n\xi_n)_{L^2(D\setminus\Omega)} \to (w|\xi)_{L^2(\Omega)},$$

as $n \to \infty$. Hence,

$$-\int_0^t (u(s)|\xi)\phi'(s)\,ds + \int_0^t a(s;u(s),\xi)\phi(s)\,ds \\ = -(w|\xi)\phi(t) + \int_0^t \langle f(s),\xi\rangle\phi(s)\,ds, \tag{1.34}$$

as $n \to \infty$. As u is the weak solution of (1.26), a similar equation holds with $(w|\xi)_{L^2(\Omega)}$ replaced by $(u(t)|\xi)_{L^2(\Omega)}$. Therefore, $(w|\xi)_{L^2(\Omega)} = (u(t)|\xi)_{L^2(\Omega)}$ for all $\xi \in H_0^1(\Omega)$. By the density of $H_0^1(\Omega)$ in $L^2(\Omega)$, we deduce that $w = u(t)$. Hence, for subsequences, $ri_nu_n(t) \rightharpoonup u(t)$ in $L^2(\Omega)$. By the uniqueness of solutions, the whole sequence $ri_nu_n(t)$ converges to $u(t)$ in $L^2(\Omega)$ weakly. □

Next we show the strong convergence of solutions in $L^2((0,T),H^1(D))$.

Theorem 1.4.7. *Suppose that the hypothesis of Theorem 1.4.3 holds. Then for any $\delta \in (0,T]$, the solution i_nu_n converges to iu in $L^2((\delta,T),H^1(D))$ strongly. In addition, if $i_nu_{0,n} \to iu_0$ in $L^2(D)$ strongly, then the above assertion holds for $\delta = 0$.*

Proof. Recall that we have the weak convergence $i_nu_n \rightharpoonup iu$ in $L^2((0,T),H^1(D))$ from Theorem 1.4.3 and the strong convergence $i_nu_n \to iu$ in $L^2((0,T),L^2(D))$ from Theorem 1.4.5. Hence, we can extract a subsequence (denoted again by u_n) so that $i_nu_n(t,\cdot)$ converges in $L^2(D)$ for almost every $t \in (0,T)$. Choosing $\delta \in (0,T]$ arbitrarily close to zero such that $i_nu_n(\delta) \to iu(\delta)$ in $L^2(D)$ strongly. By the Mosco condition $(M1')$ (from Theorem 1.3.1), we can find $w_n \in L^2((0,T),H_0^1(\Omega_n))$ such that $i_nw_n \to iu$ in $L^2((0,T),H^1(D))$. For $t \in (\delta,T]$, we consider

$$d_n(t) = \frac{1}{2}\|i_nu_n(t) - iu(t)\|^2_{L^2(D)} + \alpha\int_\delta^t \|i_nu_n(s) - i_nw_n(s)\|^2_{H^1(D)}ds. \tag{1.35}$$

By (1.20) (with $\lambda = 0$), we have

$$\begin{aligned} d_n(t) \leq &\frac{1}{2}\|i_n u_n(t)\|^2_{L^2(D)} + \int_\delta^t a_n(s; u_n(s), u_n(s))\, ds \\ &+ \frac{1}{2}\|iu(t)\|^2_{L^2(D)} + \int_\delta^t a_n(s; w_n(s), w_n(s))\, ds \\ &- (i_n u_n(t)|iu(t))_{L^2(D)} - \int_\delta^t a_n(s; u_n(s), w_n(s))\, ds \\ &- \int_\delta^t a_n(s; w_n(s), u_n(s))\, ds, \end{aligned} \tag{1.36}$$

for all $n \in \mathbb{N}$. It can be easily seen from the weak convergence of $i_n u_n$ and the strong convergence of $i_n w_n$ to iu in $L^2((0,T), H^1(D))$ that

$$\begin{aligned} &\lim_{n\to\infty} \Big[\int_\delta^t a_n(s; u_n(s), w_n(s))\, ds + \int_\delta^t a_n(s; w_n(s), u_n(s))\, ds\Big] \\ &= 2\int_\delta^t a(s; u(s), u(s))\, ds, \end{aligned} \tag{1.37}$$

and

$$\lim_{n\to\infty} \int_\delta^t a_n(s; w_n(s), w_n(s))\, ds = \int_\delta^t a(s; u(s), u(s))\, ds. \tag{1.38}$$

Also, by Lemma 1.4.6, we have

$$\lim_{n\to\infty} (i_n u_n(t)|iu(t))_{L^2(D)} = \lim_{n\to\infty} (r i_n u_n(t)|u(t))_{L^2(\Omega)} = \|u(t)\|^2_{L^2(\Omega)}. \tag{1.39}$$

Finally, as u_n is the weak solution of (1.25) it satisfies

$$\begin{aligned} &\frac{1}{2}\|u_n(t)\|^2_{L^2(\Omega_n)} + \int_\delta^t a_n(s; u_n(s), u_n(s))\, ds \\ &= \frac{1}{2}\|u_n(\delta)\|^2_{L^2(\Omega_n)} + \int_\delta^t \langle f_n(s), u_n(s)\rangle\, ds. \end{aligned}$$

Using that $i_n u_n(\delta) \to iu(\delta)$ in $L^2(D)$ strongly, $r i_n f_n \rightharpoonup f$ in $L^2((0,T), L^2(\Omega))$ weakly and the strong convergence of solution $i_n u_n \to iu$ in $L^2((0,T), L^2(D))$, we get

$$\begin{aligned} &\lim_{n\to\infty} \Big[\frac{1}{2}\|u_n(t)\|^2_{L^2(\Omega_n)} + \int_\delta^t a_n(s; u_n(s), u_n(s))\, ds\Big] \\ &= \frac{1}{2}\|u(\delta)\|^2_{L^2(\Omega)} + \int_\delta^t \langle f(s), u(s)\rangle\, ds \\ &= \frac{1}{2}\|u(t)\|^2_{L^2(\Omega)} + \int_\delta^t a(s; u(s), u(s))\, ds. \end{aligned} \tag{1.40}$$

Hence, it follows from (1.36) – (1.40) that $d_n(t) \to 0$ for all $t \in [\delta, T]$. As $\delta > 0$ was arbitrarily chosen, this implies pointwise convergence $i_n u_n(t) \to iu(t)$ in $L^2(D)$ for all

$t \in (0, T]$. Moreover, by taking $t = T$ we get

$$\begin{aligned}\int_\delta^T \|i_n u_n(s) - iu(s)\|^2_{H^1(D)}\, ds \\ &\leq \int_\delta^T \|i_n u_n(s) - i_n w_n(s)\|^2_{H^1(D)}\, ds + \int_\delta^T \|i_n w_n(s) - iu(s)\|^2_{H^1(D)}\, ds \\ &\to 0,\end{aligned}$$

as $n \to \infty$. This proves the strong convergence $i_n u_n \to iu$ in $L^2((\delta, T), H^1(D))$ for all $\delta \in (0, T]$.

If $i_n u_{0,n} \to iu_0$ in $L^2(D)$ strongly, we repeat the above arguments with $\delta = 0$. It is trivial that all arguments remain valid. Hence, the statement of the theorem follows. □

In the next theorem, we prove convergence of solutions in a stronger norm. We show that Mosco convergence is sufficient for uniform convergence of solutions in $L^2(D)$ with respect to $t \in [0, T]$. We require the following result on Mosco convergence of $H^1_0(\Omega_n)$ to $H^1_0(\Omega)$ [37, Proposition 6.3].

Lemma 1.4.8. *The following statements are equivalent.*

1. *Mosco condition $(M1)$: for every $w \in H^1_0(\Omega)$, there exists a sequence $w_n \in H^1_0(\Omega_n)$ such that $i_n w_n \to iw$ in $H^1(D)$.*

2. $\operatorname{cap}(K \cap \Omega_n^c) \to 0$ *as $n \to \infty$ for all compact set $K \subset \Omega$.*

Theorem 1.4.9. *Suppose that the hypothesis of Theorem 1.4.3 holds. Then for any $\delta \in (0, T]$, the solution $i_n u_n$ converges to iu in $C([\delta, T], L^2(D))$ strongly. In addition, if $i_n u_{0,n} \to iu_0$ in $L^2(D)$ strongly, then the above assertion holds for $\delta = 0$.*

Proof. Let w_n be a sequence chosen as in the proof of Theorem 1.4.7. We notice from the proof of Theorem 1.4.7 that

$$\int_\delta^t \|i_n u_n(s) - i_n w_n(s)\|^2_{H^1(D)}\, ds \to 0$$

uniformly with respect to $t \in [\delta, T]$. Indeed, by (1.35), we have

$$\int_\delta^t \|i_n u_n(s) - i_n w_n(s)\|^2_{H^1(D)}\, ds \leq \alpha^{-1} d_n(T),$$

for all $n \in \mathbb{N}$ and for all $t \in [\delta, T]$. Moreover, it is clear that (1.37), (1.38) and (1.40) hold uniformly on $[\delta, T]$. It remains to show uniform convergence of (1.39).

Fix $s \in [\delta, T]$. For $\varepsilon > 0$ arbitrary, we choose a compact set $K \subset \Omega$ such that $\|u(s)\|_{L^2(\Omega\setminus K)} \leq \varepsilon/2$. Since $u \in C([0,T], L^2(\Omega))$, there exists $\eta > 0$ only depending on ε such that $\|u(t) - u(s)\|_{L^2(\Omega)} \leq \varepsilon/2$ for all $t \in (s-\eta, s+\eta) \cap [\delta, T]$. It follows that

$$\|u(t)\|_{L^2(\Omega\setminus K)} \leq \|u(t) - u(s)\|_{L^2(\Omega)} + \|u(s)\|_{L^2(\Omega\setminus K)} \leq \varepsilon/2 + \varepsilon/2 = \varepsilon, \tag{1.41}$$

for all $t \in (s-\eta, s+\eta) \cap [\delta, T]$. We next choose a cut-off function $\phi \in C_0^\infty(\Omega)$ such that $0 \leq \phi \leq 1$ and $\phi = 1$ on K. Since $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco, we have from Lemma 1.4.8 that $\operatorname{cap}(\operatorname{supp}(\phi) \cap \Omega_n^c) \to 0$. By the definition of capacity, there exists a sequence $\xi_n \in C_0^\infty(\Omega)$ such that $0 \leq \xi_n \leq 1$, $\xi_n = 1$ on a neighborhood of $\operatorname{supp}(\phi) \cap \Omega_n^c$ and $\|\xi_n\|_{H^1(\mathbb{R}^N)} \leq \operatorname{cap}(\operatorname{supp}(\phi) \cap \Omega_n^c) + 1/n$. Define $\phi_n := (1-\xi_n)\phi$. We have that $\phi_n \in C_0^\infty(\Omega_n)$ and $\phi_n \to \phi$ in $L^2(D)$. Consider

$$\begin{aligned}
&\left|(i_n u_n(t) - iu(t)|iu(t))_{L^2(D)}\right| \\
&\leq \left|(i_n u_n(t)|\phi_n iu(t))_{L^2(D)} - (iu(t)|\phi iu(t))_{L^2(D)}\right| \\
&\quad + \left|(i_n u_n(t)|(1-\phi_n) iu(t))_{L^2(D)} - (iu(t)|(1-\phi) iu(t))_{L^2(D)}\right| \\
&\leq \left|(i_n u_n(t)|\phi_n iu(t))_{L^2(D)} - (iu(t)|\phi iu(t))_{L^2(D)}\right| \\
&\quad + \left|(i_n u_n(t) - iu(t)|(1-\phi_n) iu(t))_{L^2(D)}\right| + \left|(iu(t)|(\phi - \phi_n) iu(t))_{L^2(D)}\right|.
\end{aligned} \tag{1.42}$$

We prove that each term on the right of (1.42) is uniformly small for $t \in (s-\eta, s+\eta) \cap [\delta, T]$ if n is sufficiently large. For the first term, applying the integration by parts formula (1.1) and the definition of weak solutions, we obtain

$$\begin{aligned}
&(i_n u_n(t)|\phi_n iu(t))_{L^2(D)} \\
&= (u_n(\delta)|\phi_n u(\delta))_{L^2(\Omega_n)} + \int_\delta^t \langle u_n'(s), \phi_n iu(s)\rangle\, ds + \int_\delta^t \langle u'(s), \phi_n i_n u_n(s)\rangle\, ds \\
&= (u_n(\delta)|\phi_n u(\delta))_{L^2(\Omega_n)} + \int_\delta^t \langle f_n(s), \phi_n iu(s)\rangle\, ds - \int_\delta^t a_n(s; u_n(s), \phi_n iu(s))\, ds \\
&\quad + \int_\delta^t \langle f(s), \phi_n i_n u_n(s)\rangle\, ds - \int_\delta^t a(s; u(s), \phi_n i_n u_n(s))\, ds.
\end{aligned} \tag{1.43}$$

It can be easily verified using the dominated convergence theorem that $\phi_n iu \to \phi iu$ in $L^2((0,T), L^2(D))$. Moreover,

$$\begin{aligned}
&\int_\delta^T \|\phi_n i_n u_n(t) - \phi iu(t)\|_{L^2(D)}^2\, dt \\
&\leq \int_\delta^T \|\phi_n iu(t) - \phi iu(t)\|_{L^2(D)}^2\, dt + \int_\delta^T \|\phi_n i_n u_n(t) - \phi_n iu(t)\|_{L^2(D)}^2\, dt \\
&\leq \int_\delta^T \|\phi_n iu(t) - \phi iu(t)\|_{L^2(D)}^2\, dt + \|\phi_n\|_\infty^2 \int_\delta^T \|i_n u_n(t) - iu(t)\|_{L^2(D)}^2\, dt.
\end{aligned}$$

Hence, $\phi_n i_n u_n \to \phi i u$ in $L^2((0,T), L^2(D))$. Taking into consideration that $i_n u_n(\delta) \to iu(\delta)$ in $L^2(D)$ and $r i_n f_n \rightharpoonup f$ in $L^2((0,T), L^2(\Omega))$, we conclude form (1.43) that

$$(i_n u_n(t)|\phi_n i u(t))_{L^2(D)} \to (iu(t)|\phi i u(t))_{L^2(D)} \tag{1.44}$$

uniformly with respect to $t \in [\delta, T]$. For the last term on the right of (1.42), applying a similar argument as above, we write

$$(iu(t)|\phi_n i u(t))_{L^2(D)} = (u(\delta)|\phi_n u(\delta))_{L^2(\Omega)} + 2\int_\delta^t \langle u'(s), \phi_n \tilde{u}(s)\rangle \, ds.$$

We conclude that

$$(iu(t)|\phi_n i u(t))_{L^2(D)} \to (iu(t)|\phi i u(t))_{L^2(D)} \tag{1.45}$$

uniformly with respect to $t \in [\delta, T]$. Finally, for the second term on the right of (1.42), we notice that $0 \leq 1 - \phi_n \leq 1$ on Ω and $1 - \phi_n = 1 - (1 - \xi_n)\phi = \xi_n$ on K. Moreover, by the uniform boundedness of $i_n u_{0,n}$ in $L^2(D)$ and the uniform boundedness of $i_n f_n$ in $L^2((0,T), L^2(D))$, we see from (1.8) that there exists a constant $M_0 > 0$ such that

$$\|i_n u_n(t)\|_{L^2(D)}, \|iu(t)\|_{L^2(D)} \leq M_0, \tag{1.46}$$

for all $t \in [0, T]$. Hence, by the Cauchy-Schwarz inequality and (1.41),

$$\begin{aligned} \left|(i_n u_n(t) - iu(t)|(1-\phi_n) i u(t))_{L^2(D)}\right| &\leq \|i_n u_n(t) - iu(t)\|_{L^2(D)} \|(1-\phi_n) i u(t)\|_{L^2(D)} \\ &\leq 2M_0 \left(\|u(t)\|^2_{L^2(\Omega\setminus K)} + \|\xi_n u(t)\|_{L^2(K)}\right) \\ &\leq 2M_0 \left(\varepsilon + \|\xi_n u(t)\|_{L^2(K)}\right), \end{aligned} \tag{1.47}$$

for all $t \in (s-\eta, s+\eta) \cap [\delta, T]$ and for all $n \in \mathbb{N}$. Since $\xi_n \to 0$ in $L^2(D)$, a standard argument using the dominated convergence theorem implies that $\xi_n u(s) \to 0$ in $L^2(\Omega)$. Hence, there exists $N_{s,\varepsilon} \in \mathbb{N}$ such that $\|\xi_n u(s)\|_{L^2(\Omega)} \leq \varepsilon/2$ for all $n \geq N_{s,\varepsilon}$. Therefore,

$$\begin{aligned} \|\xi_n u(t)\|_{L^2(K)} &\leq \|\xi_n u(s)\|_{L^2(K)} + \|\xi_n u(t) - \xi_n u(s)\|_{L^2(K)} \\ &\leq \|\xi_n u(s)\|_{L^2(K)} + \|u(t) - u(s)\|_{L^2(K)} \\ &\leq \varepsilon/2 + \varepsilon/2 = \varepsilon, \end{aligned}$$

for all $t \in (s-\eta, s+\eta) \cap [\delta, T]$ and for all $n \geq N_{s,\varepsilon}$. It follows from (1.47) that

$$\left|(i_n u_n(t) - iu(t)|(1-\phi_n) i u(t))_{L^2(D)}\right| \leq 2M_0(\varepsilon + \varepsilon) = 4M_0\varepsilon, \tag{1.48}$$

for all $t \in (s-\eta, s+\eta) \cap [\delta, T]$ and for all $n \geq N_{s,\varepsilon}$. Therefore, by (1.42), (1.44), (1.45), and (1.48), we conclude that there exist $\tilde{N}_{s,\varepsilon} \in \mathbb{N}$ and a positive constant C such that

$$\left|(i_n u_n(t) - iu(t)|iu(t))_{L^2(D)}\right| \leq C\varepsilon,$$

for all $t \in (s-\eta, s+\eta) \cap [\delta, T]$ and for all $n \geq \tilde{N}_{s,\varepsilon}$.

Finally, as $[\delta, T]$ is a compact interval and η only depends on ε, it follows that $(i_n u_n(t)|iu(t))_{L^2(D)} \to (iu(t)|iu(t))_{L^2(D)}$ uniformly with respect to $t \in [\delta, T]$. This proves the convergence $i_n u_n \to iu$ in $C([\delta, T], L^2(D))$ for all $\delta \in (0, T]$. If $i_n u_{0,n} \to iu_0$ in $L^2(D)$ strongly, we take $\delta = 0$ in the above arguments. The last assertion of the theorem follows. $\square$

It is known that the convergence of solution of elliptic equations subject to Dirichlet boundary condition under domain perturbation can be obtained from Mosco convergence of $H_0^1(\Omega_n)$ to $H_0^1(\Omega)$ [37]. Hence, we can use the same criterion on Ω_n and Ω to conclude the convergence of solutions of parabolic equations. In particular, the conditions on domains given in [37, Theorem 7.5] implies convergence of solutions of non-autonomous parabolic equations (1.21) subject to Dirichlet boundary condition under domain perturbation. For an open set $U \subset \mathbb{R}^N$, we define

$$\lambda_1(U) = \inf_{u \in H_0^1(U), u \neq 0} \frac{\|\nabla u\|^2}{\|u\|^2}$$

and $\lambda_1(\emptyset) = \infty$. As a consequence of [37, Theorem 7.5], we can state the following corollary.

Corollary 1.4.10. *Suppose that $i_n f_n$ is uniformly bounded in $L^2((0,T), L^2(D))$ and $i_n u_{0,n}$ is uniformly bounded in $L^2(D)$. Assume that $r i_n f_n \rightharpoonup f$ in $L^2((0,T), L^2(\Omega))$ weakly and $r i_n u_{0,n} \rightharpoonup u_0$ in $L^2(\Omega)$ weakly. If Ω_n and Ω are open and bounded sets in $\mathbb{R}^N$ satisfying*

1. *$\operatorname{cap}(K \cap \Omega_n^c) \to 0$ as $n \to \infty$ for all compact set $K \subset \Omega$;*
2. *There exists an open covering $\mathscr{O}$ of $\mathbb{R}^N \backslash \bar{\Omega}$ such that $\lambda_1(U \cap \Omega_n) \to \infty$ as $n \to \infty$ for all $U \in \mathscr{O}$;*
3. *$H_0^1(\Omega) = H_0^1(\Omega \cup \Gamma)$, where*

$$\Gamma := \bigcap_{n \in \mathbb{N}} \left(\overline{\bigcup_{k \geq n} (\Omega_k \cap \partial\Omega)} \right),$$

then for any $\delta \in (0, T]$, the solution $i_n u_n$ converges strongly to iu in $L^2((\delta, T), H^1(D))$ and in $C([\delta, T], L^2(D))$. In addition, if $i_n u_{0,n} \to iu_0$ in $L^2(D)$ strongly, then the above assertion holds for $\delta = 0$.

Examples of domains satisfying the above conditions include the following (see [37] for details and further examples):

(i) a standard dumbbell-shaped domain with shrinking handle (possibly also with a hole converging to zero in capacity) as shown in Figure 1.1;

(ii) a square with "fingers" attached in Figure 1.2. Here we increase the number of fingers and reduce their width so that the Lebesgue measure of Ω_n is preserved. This means $|\Omega_n| \not\to |\Omega|$ as $n \to \infty$.

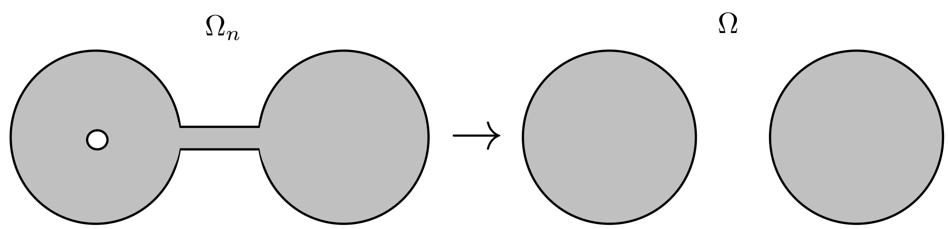

Figure 1.1: Dumbbell-shaped domains with shrinking handle.

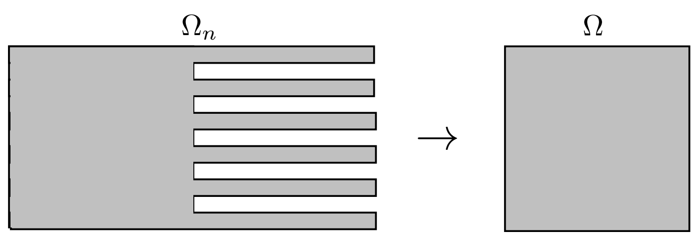

Figure 1.2: A square with fingers attached.

1.4.2 Neumann problems

It is more difficult to handle Neumann boundary condition than Dirichlet boundary condition. Convergence results for Neumann problems often require a certain regularity of the domains. For autonomous problems, it is proved in [10, Proposition 2.7] that linear semigroups converge under the following assumptions on domains (referred to as "admissible family of domains" in [10, Definition 2.5]):

- Ω_n is bounded and Lipschitz for all $n \in \mathbb{N}$;

- For every $K \subset\subset \Omega$, there exists $N_K \in \mathbb{N}$ such that $K \subset \Omega_n$ if $n > N_K$;
- Spectra of the corresponding elliptic operators behave continuously (one of the conditions (i)–(iv) of [10, Proposition 2.3]).

Our main emphasis in this work is to study *non-autonomous* problems without any smoothness of domains. We cannot simply consider the trivial extension by zero outside the domain for functions in $H^1(\Omega)$ as in the case of Dirichlet problems, because the extended function does not belong to $H^1(\mathbb{R}^N)$. Also, there is no smooth extension from $H^1(\Omega)$ to $H^1(D)$ as we do not impose any regularity of the domain. Furthermore, the compactness result in [35, Lemma 2.1] cannot be applied in the case of Neumann problems.

In order to study the limit of $u_n \in H^1(\Omega_n)$ when the domain is perturbed, we embed the space $H^1(\Omega_n)$ into the following space

$$H^1(\Omega_n) \hookrightarrow L^2(D) \times L^2(D, \mathbb{R}^N)$$

by

$$v_n \mapsto (i_n v_n, i_n \nabla v_n),$$

where i_n is the inclusion map defined in (1.27). Note that $i_n \nabla v_n$ is not the gradient of $i_n v_n$ in the sense of distribution, that is $i_n \nabla v_n \neq \nabla(i_n v_n)$. By a similar embedding for $H^1(\Omega)$, we can consider Mosco convergence of

$$K_n := \{(i_n v_n, i_n \nabla v_n) \in L^2(D) \times L^2(D, \mathbb{R}^N) \mid v_n \in H^1(\Omega_n)\}$$

to

$$K := \{(iv, i\nabla v) \in L^2(D) \times L^2(D, \mathbb{R}^N) \mid v \in H^1(\Omega)\}$$

in $V := L^2(D) \times L^2(D, \mathbb{R}^N)$. In this case, K_n and K are closed subspaces of V. For simplicity, we say $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco for K_n and K chosen above.

When dealing with solutions of parabolic equations, we identify the space

$$L^2((0,T), V) \equiv L^2((0,T), L^2(D)) \times L^2((0,T), L^2(D, \mathbb{R}^N))$$

via the isomorphism between them. Hence,

$$\begin{aligned} L^2((0,T), K_n) &\equiv \{(i_n w_n, i_n \nabla w_n) \mid w_n \in L^2((0,T), H^1(\Omega_n)\} \\ &\subset L^2((0,T), L^2(D)) \times L^2((0,T), L^2(D, \mathbb{R}^N)), \end{aligned}$$

and

$$L^2((0,T),K) \equiv \{(iw, i\nabla w) \mid w \in L^2((0,T),H^1(\Omega)\}$$
$$\subset L^2((0,T),L^2(D)) \times L^2((0,T),L^2(D,\mathbb{R}^N)).$$

We point out that the assumptions on domains in [10, Definition 2.5]) imply Mosco convergence of $H^1(\Omega_n)$ to $H^1(\Omega)$.

Lemma 1.4.11. *Suppose that Ω_n and Ω satisfy the assumptions in [10, Definiton 2.5]. Then $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco.*

Proof. We show that the Mosco conditions $(M1)$ and $(M2)$ are satisfied. Let $v \in H^1(\Omega)$ be arbitrary. Since Ω is Lipschitz, there exits an extension operator $\mathscr{E} : H^1(\Omega) \to H^1(\mathbb{R}^N)$. We extend v to a function $\mathscr{E}v$ in $H^1(\mathbb{R}^N)$. By restriction to Ω_n, we have $v_n := r_n \mathscr{E} v \in H^1(\Omega_n)$. Clearly, $\|i_n v_n - iv\|_{H^1(\Omega\cap\Omega_n)} = 0$ for all $n \in \mathbb{N}$. The condition [10, (1.4)] implies that there exists a non-increasing sequence ρ_n converging to zero such that the set

$$K_n := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \rho_n\} \subset \Omega_n,$$

for all $n \in \mathbb{N}$ (see also [10, (2.1)]). As Ω is bounded, this yields $|\Omega\backslash\Omega_n| \leq |\Omega\backslash K_n| \to 0$ as $n \to \infty$. Hence,

$$\|i_n v_n - iv\|_{L^2(\Omega\backslash\Omega_n)} + \|i_n \nabla v_n - i\nabla v\|_{L^2((\Omega\backslash\Omega_n),\mathbb{R}^N)} = \|v\|_{H^1(\Omega\backslash\Omega_n)} \to 0$$

as $n \to \infty$. Using that $|\Omega_n\backslash K_n| \to 0$ (condition (v) in [10, Proposition 2.3]), we easily see that

$$\|v_n\|_{H^1(\Omega_n\backslash\Omega)} = \|r_n \mathscr{E} v\|_{H^1(\Omega_n\backslash\Omega)} \to 0$$

as $n \to \infty$. Therefore,

$$\|i_n v_n - iv\|_{L^2(\Omega_n\backslash\Omega)} + \|i_n \nabla v_n - i\nabla v\|_{L^2((\Omega_n\backslash\Omega),\mathbb{R}^N)} = \|v_n\|_{H^1(\Omega_n\backslash\Omega)}$$
$$\to 0$$

as $n \to \infty$. The above implies

$$\|i_n v_n - iv\|_{L^2(\mathbb{R}^N)} + \|i_n \nabla v_n - i\nabla v\|_{L^2(\mathbb{R}^N,\mathbb{R}^N)} \to 0$$

as $n \to \infty$. Hence, condition $(M1)$ holds.

Next, suppose (n_k) is a sequence of indices converging to ∞, (v_k) is a sequence such that $v_k \in H^1(\Omega_{n_k})$ for every k and $(i_{n_k} v_k, i_{n_k} \nabla v_k) \rightharpoonup (v, w)$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N, \mathbb{R}^N)$

weakly. By condition (iv) in [10, Proposition 2.3], we can extract a subsequence so that there exists $\phi \in H^1(\Omega)$ such that $i_{n_{k_j}} v_{k_j} \to i\phi$ in $L^2(\mathbb{R}^N)$ as $j \to \infty$ and

$$\int_{\Omega_{n_{k_j}}} \nabla v_{k_j} \nabla \chi dx \to \int_\Omega \nabla \phi \nabla \chi dx,$$

as $j \to \infty$ for all $\chi \in H^1(\mathbb{R}^N)$. In fact, we can show by modifying the proof in [10] that

$$\int_{\Omega_{n_{k_j}}} \nabla v_{k_j} \psi dx \to \int_\Omega \nabla \phi \psi dx,$$

as $j \to \infty$ for all $\psi \in L^2(\mathbb{R}^N, \mathbb{R}^N)$. To see this, we replace $\chi \in H^1(\mathbb{R}^N)$ by $\psi \in L^2(\mathbb{R}^N, \mathbb{R}^N)$ in the second part of (iii) $\Rightarrow$ (iv) in [10, Proposition 2.3]. Since $|\Omega \backslash K_n|$ and $|\Omega_n \backslash \Omega|$ converge to zero (condition (iv) in [10, Proposition 2.3]), for any given $\eta > 0$, we can find $N \in \mathbb{N}$ sufficiently large so that $\|\psi\|_{L^2((\Omega_{n_{k_j}} \cup \Omega)\backslash K_N)} \leq \eta$ for all $j \in \mathbb{N}$ with $n_{k_j} > N$. Then

$$\begin{aligned}
&\Big| \int_{\Omega_{n_{k_j}}} \nabla v_{k_j} \psi dx - \int_\Omega \nabla \phi \psi dx \Big| \\
&\quad \leq \Big| \int_{K_N} (\nabla v_{k_j} - \nabla \phi) \psi dx \Big| + \int_{\Omega_{n_{k_j}} \backslash K_N} |\nabla v_{k_j}| |\psi| dx + \int_{\Omega \backslash K_N} |\nabla \phi| |\psi| dx \\
&\quad \leq \Big| \int_{K_N} (\nabla v_{k_j} - \nabla \phi) \psi dx \Big| + 2\eta \|\psi\|_{L^2(\mathbb{R}^N, \mathbb{R}^N)} \\
&\quad \to 2\eta \|\psi\|_{L^2(\mathbb{R}^N, \mathbb{R}^N)},
\end{aligned}$$

as $j \to \infty$. As $\eta > 0$ was arbitrary, this means $i_{n_{k_j}} \nabla v_{k_j} \rightharpoonup i \nabla \phi$ in $L^2(\mathbb{R}^N, \mathbb{R}^N)$. Therefore, we conclude that $(v, w) = (i\phi, i\nabla\phi)$ and the whole sequence $i_{n_k} v_k \to v = i\phi$ in $L^2(\mathbb{R}^N)$ and $i_{n_k} \nabla v_k \rightharpoonup w = i\nabla \phi$ in $L^2(\mathbb{R}^N, \mathbb{R}^N)$ as $k \to \infty$. This proves the Mosco condition $(M2)$. □

As in the case of Dirichlet problems, we apply various Mosco conditions from Theorem 1.3.1 to prove that the weak solution u_n of (1.25) converges to the weak solution u of (1.26). We emphasis that we do not explicitly impose any regularity of the domains and only assume the abstract Mosco convergence of $H^1(\Omega_n)$ to $H^1(\Omega)$. However, the Mosco convergence assumption requires some topological (and/or regularity) conditions on domains. We shall discuss the conditions on domains after we establish the convergence results for solutions.

Theorem 1.4.12. *Suppose that $i_n f_n$ is uniformly bounded in $L^2((0,T), L^2(D))$ and $i_n u_{0,n}$ is uniformly bounded in $L^2(D)$. Assume that $r i_n f_n \rightharpoonup f$ in $L^2((0,T), L^2(\Omega))$*

weakly and $ri_n u_{0,n} \rightharpoonup u_0$ in $L^2(\Omega)$ weakly. If $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco, then the solution $i_n u_n$ converges to iu in $L^2((0,T), L^2(D))$ weakly and $i_n \nabla u_n$ converges to $i\nabla u$ in $L^2((0,T), L^2(D, \mathbb{R}^N))$ weakly.

Proof. By a similar argument as in the proof of Theorem 1.4.3, we have the uniform boundedness of $(i_n u_n, i_n \nabla u_n)$ in $L^2((0,T), L^2(D)) \times L^2((0,T), L^2(D, \mathbb{R}^N))$. We can extract a subsequence (denoted again by u_n), such that $i_n u_n \rightharpoonup w$ in $L^2((0,T), L^2(D))$ and $i_n \nabla u_n \rightharpoonup (v_1, \ldots, v_N)$ in $L^2((0,T), L^2(D, \mathbb{R}^N))$. The Mosco condition $(M2')$ (from Theorem 1.3.1) implies that $w \in L^2((0,T), H^1(\Omega))$.

To show that $w = u$, we let $\xi \in H^1(\Omega)$ and $\phi \in \mathscr{D}([0,T))$ and then use Mosco convergence of $H^1(\Omega_n)$ to $H^1(\Omega)$. In the same way as the proof of Theorem 1.4.3, we get (1.33) holds for all $\xi \in H^1(\Omega)$ and all $\phi \in \mathscr{D}([0,T))$. Hence, by the uniqueness of solution, $w = u$ in $L^2((0,T), H^1(\Omega))$ and the whole sequence converges weakly. □

Lemma 1.4.13. *Suppose that the hypothesis of Theorem 1.4.12 holds. Then for each $t \in [0,T]$, we have $ri_n u_n(t) \rightharpoonup u(t)$ in $L^2(\Omega)$.*

Proof. We use the same argument as in the proof of Lemma 1.4.6 with Mosco convergence of $H_0^1(\Omega_n)$ to $H_0^1(\Omega)$ replaced by Mosco convegence of $H^1(\Omega_n)$ to $H^1(\Omega)$ and the fact that $H^1(\Omega)$ is also dense in $L^2(\Omega)$. □

In the remainder, we impose the strong convergences of inhomogeneous terms and initial data to obtain the strong convergence of solutions.

Theorem 1.4.14. *Assume that $i_n f_n \to if$ in $L^2((0,T), L^2(D))$ strongly and $i_n u_{0,n} \to iu_0$ in $L^2(D)$ strongly. If $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco, then the solution $i_n u_n$ converges to iu in $L^2((0,T), L^2(D))$ strongly and $i\nabla u_n$ converges to $i\nabla u$ in $L^2((0,T), L^2(D, \mathbb{R}^N))$ strongly.*

Proof. The proof is similar to the one in Theorem 1.4.7. We show some details here for the sake of completeness. By Theorem 1.4.12, $(i_n u_n, i_n \nabla u_n)$ converges to $(iu, i\nabla u)$ in $L^2((0,T), L^2(D)) \times L^2((0,T), L^2(D, \mathbb{R}^2))$ weakly. Since $u \in L^2((0,T), H^1(\Omega))$, we can find $w_n \in L^2((0,T), H^1(\Omega_n))$ from the Mosco condition $(M1')$ in Theorem 1.3.1 such that $i_n w_n \to iu$ in $L^2((0,T), L^2(D))$ and $i_n \nabla w_n \to i\nabla u$ in $L^2((0,T), L^2(D, \mathbb{R}^N))$. For

$t \in [0, T]$, we consider

$$\begin{aligned} d_n(t) &= \frac{1}{2}\|i_n u_n(t) - iu(t)\|^2_{L^2(D)} + \alpha \int_0^t \|u_n(s) - w_n(s)\|^2_{L^2(\Omega_n)}\, ds \\ &\quad + \alpha \int_0^t \|\nabla u_n(s) - \nabla w_n(s)\|^2_{L^2(\Omega_n,\mathbb{R}^N)}\, ds \\ &= \frac{1}{2}\|i_n u_n(t) - iu(t)\|^2_{L^2(D)} + \alpha \int_0^t \|u_n(s) - w_n(s)\|^2_{H^1(\Omega_n)}\, ds. \end{aligned} \tag{1.49}$$

By (1.20) (with $\lambda = 0$), we can show that d_n satisfies (1.36) with $\delta = 0$ for all $n \in \mathbb{N}$. It can be easily seen from the weak convergence of $(i_n u_n, i_n \nabla u_n)$ and the strong convergence of $(i_n w_n, i_n \nabla w_n)$ to $(iu, i\nabla u)$ in $L^2((0,T), L^2(D)) \times L^2((0,T), L^2(D, \mathbb{R}^N))$ that (1.37) and (1.38) also hold with $\delta = 0$. By using Lemma 1.4.13 instead of Lemma 1.4.6, we obtain (1.39). Finally, using that $i_n u_{0,n} \to iu_0$ in $L^2(D)$ strongly, $i_n f_n \to if$ in $L^2((0,T), L^2(D))$ strongly and the weak convergence of solution $i_n u_n \rightharpoonup iu$ in $L^2((0,T), L^2(D))$, we get (1.40) with $\delta = 0$. Hence, $d_n(t) \to 0$ for all $t \in [0,T]$. This shows pointwise convergence of $i_n u_n(t)$ to $iu(t)$ in $L^2(D)$ for all $t \in [0,T]$. Moreover, by taking $t = T$ we get

$$\begin{aligned} &\int_0^T \|i_n u_n(s) - iu(s)\|^2_{L^2(D)}\, ds \\ &\le \int_0^T \|i_n u_n(s) - i_n w_n(s)\|^2_{L^2(D)}\, ds + \int_0^T \|i_n w_n(s) - iu(s)\|^2_{L^2(D)}\, ds \\ &\to 0, \end{aligned}$$

and

$$\begin{aligned} &\int_0^T \|i_n \nabla u_n(s) - i\nabla u(s)\|^2_{L^2(D,\mathbb{R}^N)}\, ds \\ &\le \int_0^T \|i_n \nabla u_n(s) - i_n \nabla w_n(s)\|^2_{L^2(D,\mathbb{R}^N)}\, ds + \int_0^T \|i_n \nabla w_n(s) - i\nabla u(s)\|^2_{L^2(D,\mathbb{R}^N)}\, ds \\ &\to 0. \end{aligned}$$

This proves the strong convergence $i_n u_n \to iu$ in $L^2((0,T), L^2(D))$ and $i_n \nabla u_n \to i\nabla u$ in $L^2((0,T), L^2(D, \mathbb{R}^N))$. □

Recall that we embed the space $K = H^1(\Omega)$ in $V = L^2(D) \times L^2(D, \mathbb{R}^N)$. If v is a function in $W((0,T), H^1(\Omega), H^1(\Omega)')$, then $v' \in L^2((0,T), H^1(\Omega)')$. It is not always true that we can embed $v'(t)$ in $V' = L^2(D) \times L^2(D, \mathbb{R}^N)$ a.e. $t \in (0,T)$ and claim that $v \in W((0,T), V, V') \cap L^2((0,T), K)$. However, a similar argument as in the proof of Theorem 1.3.1 $(i) \Rightarrow (iii)$ for the Mosco condition $(M1'')$ gives the following result.

Lemma 1.4.15. *Suppose that $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco, If $w \in C^\infty([0,T], H^1(\Omega))$, then there exists $w_n \in C^\infty([0,T], H^1(\Omega_n))$ such that $i_n w_n$ converges to iw in $C^\infty([0,T], L^2(D))$ strongly.*

Proof. We note that Proposition 1.3.6 gives uniform convergence of the approximation sequence in $V = L^2(D) \times L^2(D, \mathbb{R}^N)$. The proof follows from the same arguments as in the proof of Theorem 1.3.1 $(i) \Rightarrow (iii)$. The only difference is that we assume here $w \in C^\infty([0,T], H^1(\Omega))$. Hence, the stretched function $w_\delta = w \circ S_\delta^{-1}$ belongs to $C^\infty([-\delta, T+\delta], H^1(\Omega))$. We point out that, by using uniform continuity of the k-th order derivative $w^{(k)}$ on $[0,T]$, the restriction of w_δ to $[0,T]$ converges to w in $C^\infty([0,T], H^1(\Omega))$. This gives the required convergence in $C^\infty([0,T], L^2(D))$. □

Using the above lemma, we show in the next theorem that the solution u_n of (1.25) indeed converges uniformly with respect to $t \in [0,T]$.

Theorem 1.4.16. *Suppose that the hypothesis of Theorem 1.4.14 holds. Then the solution $i_n u_n$ converges to iu in $C([0,T], L^2(D))$ strongly.*

Proof. As in the proof of Theorem 1.4.9, it requires to show the uniform convergence of $(i_n u_n(t) | iu(t))_{L^2(D)} \to (iu(t) | iu(t))_{L^2(D)}$.

Let $\varepsilon > 0$ arbitrary. By a similar argument as in the proof Lemma 1.3.5, we have the density of $C^\infty([0,T], H^1(\Omega))$ in $W((0,T), H^1(\Omega), H^1(\Omega)')$. Since the solution u is in the space $W((0,T), H^1(\Omega), H^1(\Omega)')$, there exists $w \in C^\infty([0,T], H^1(\Omega))$ such that

$$\|w - u\|_{W((0,T), H^1(\Omega), H^1(\Omega)')} \leq \varepsilon.$$

As $W((0,T), H^1(\Omega), H^1(\Omega)')$ is continuously embedded in $C([0,T], L^2(\Omega))$, we can indeed choose $w \in C^\infty([0,T], H^1(\Omega))$ such that

$$\|w(t) - u(t)\|_{L^2(\Omega)} \leq \varepsilon, \tag{1.50}$$

for all $t \in [0,T]$. By Lemma 1.4.15, there exists $w_n \in C^\infty([0,T], H^1(\Omega_n))$ such that $i_n w_n \to iw$ in $C^\infty([0,T], L^2(D))$. We can write

$$\begin{aligned} \left|(i_n u_n(t) - iu(t) | iu(t))_{L^2(D)}\right| &\leq \left|(i_n u_n(t) | i_n w_n(t))_{L^2(D)} - (iu(t) | iw(t))_{L^2(D)}\right| \\ &\quad + \left|(i_n u_n(t) - iu(t) | iu(t) - iw(t))_{L^2(D)}\right| \\ &\quad + \left|(i_n u_n(t) | iw(t) - i_n w_n(t))_{L^2(D)}\right|, \end{aligned} \tag{1.51}$$

for all $n \in \mathbb{N}$. Since u_n is the solution of (1.25),

$$(i_n u_n(t)|i_n w_n(t))_{L^2(D)} = (i_n u_{0,n}|i_n w_n(0))_{L^2(D)} + \int_0^t \langle f_n(s), w_n(s)\rangle \, ds + \int_0^t \langle w_n'(s), u_n(s)\rangle \, ds - \int_0^t a_n(s; u_n(s), w_n(s))\rangle \, ds,$$

for all $n \in \mathbb{N}$. Taking into consideration that $i_n u_{0,n} \to iu_0$ in $L^2(D)$, $i_n f_n \to if$ in $L^2((0,T), L^2(D))$ and $i_n w_n' \to iw'$ in $C([0,T], L^2(D))$, we conclude that

$$(i_n u_n(t)|i_n w_n(t))_{L^2(D)} \to (iu(t)|iw(t))_{L^2(D)} \tag{1.52}$$

uniformly with respect to $t \in [0,T]$. Moreover, by (1.50) and the uniform boundedness of solutions as in (1.46), we have

$$\begin{aligned} &\left|\big(i_n u_n(t) - iu(t)|iu(t) - \tilde{w}(t)\big)_{L^2(D)}\right| \\ &\leq \|i_n u_n(t) - iu(t)\|_{L^2(D)} \|iu(t) - iw(t)\|_{L^2(D)} \\ &\leq 2M_0\varepsilon, \end{aligned} \tag{1.53}$$

for all $t \in [0,T]$ and for all $n \in \mathbb{N}$. Finally, as $i_n w_n \to iw$ in $C^\infty([0,T], L^2(D))$, there exists $N_\varepsilon \in \mathbb{N}$ such that

$$\|i_n w_n(t) - iw(t)\|_{L^2(D)} \leq \varepsilon,$$

for all $t \in [0,T]$ and for all $n \geq N_\epsilon$. Hence,

$$\begin{aligned} \left|\big(i_n u_n(t)|i_n w_n(t) - iw(t)\big)_{L^2(D)}\right| &\leq \|i_n u_n(t)\|_{L^2(D)} \|\|i_n w_n(t) - iw(t)\|_{L^2(D)} \\ &\leq M_0\varepsilon, \end{aligned} \tag{1.54}$$

for all $t \in [0,T]$ and for all $n \geq N_\varepsilon$. Therefore, by (1.51) – (1.54), there exist $\tilde{N}_\varepsilon \in \mathbb{N}$ and a positive constant C such that

$$\left|\big(i_n u_n(t) - iu(t)|iu(t)\big)_{L^2(D)}\right| \leq C\varepsilon,$$

for all $t \in [0,T]$ and for all $n \geq \tilde{N}_\varepsilon$. As $\varepsilon > 0$ was arbitrary, this proves the required uniform convergence of $(i_n u_n(t)|iu(t))_{L^2(D)} \to (iu(t)|iu(t))_{L^2(D)}$ with respect to $t \in [0,T]$. □

We can use the same criterion on Ω_n and Ω for elliptic equations with Neumann boundary condition to conclude the convergence of solutions of parabolic equations subject to Neumann boundary condition under domain perturbation. In particular, in two dimensional space, conditions on domains given in [21, Theorem 3.1] imply the convergence of solutions of non-autonomous parabolic equations (1.21) subject to Neumann boundary condition.

Corollary 1.4.17. *Assume that $i_n f_n \to if$ in $L^2((0,T), L^2(D))$ strongly and $i_n u_{0,n} \to iu_0$ in $L^2(D)$ strongly. Let Ω_n and Ω be uniformly bounded open sets in $\mathbb{R}^2$ such that Ω_n converges in the Hausdorff complementary topology to Ω and the number of connected components of Ω_n^c and Ω^c are uniformly bounded. If the Lebesgue measure $|\Omega_n| \to |\Omega|$ as $n \to \infty$, then the solution $i_n u_n$ converges to iu in $C([0,T], L^2(D))$ strongly and $i_n \nabla u_n$ converges to $i\nabla u$ in $L^2((0,T), L^2(D, \mathbb{R}^2))$ strongly.*

Examples of domains in $\mathbb{R}^2$ satisfying the conditions of [21] include the following:

(i) a domain obtained by cutting a line segment and then rotating the segment about a point as shown in Figure 1.3;

(ii) a square with fingers attached in Figure 1.2. In contrast to Dirichlet problems, here we require the measure of fingers converges to zero.

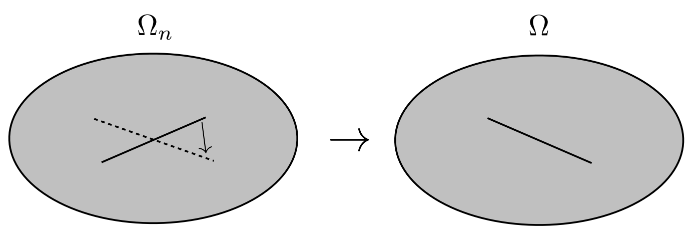

Figure 1.3: Rotating a line segment inside a domain

Remark 1.4.18. The assumptions that $i_n f_n \to if$ in $L^2((0,T), L^2(D))$ strongly and that $i_n u_{0,n} \to iu_0$ in $L^2(D)$ strongly were imposed to overcome the lack of compactness result similar to Lemma 1.4.4 for a sequence in $W((0,T), H^1(\Omega), H^1(\Omega)')$. In particular, this assumption is required to obtain the convergence $\int_0^T \langle f_n(s), u_n(s)\rangle ds \to \int_0^T \langle f(s), u(s)\rangle ds$ in the proof of Theorem 1.4.14. However, if we impose some regularity on the domains, we could only assume weak convergences of the inhomogeneous terms and the initial data to obtain the convergence of solutions in $C([\delta, T], L^2(D))$ for all $\delta \in (0,T]$ as stated in Dirichlet problems.

We give an example of domains satisfying the *cone condition* (see [1, Section 4.3]) below. Clearly, the example below is excluded from [10] as the domains are not Lipschitz.

Example 1.4.19. Let $N = 2$ and let

$$\Omega := \{x \in \mathbb{R}^2 : |x| < 1\} \backslash \{(x_1, 0) : 0 \le x_1 < 1\},$$
$$\Omega_n := \{x \in \mathbb{R}^2 : |x| < 1\} \backslash \{(x_1, 0) : \delta_n \le x_1 < 1\},$$

where $\delta_n \searrow 0$ (shown in Figure 1.4). This example is an exterior perturbation of the

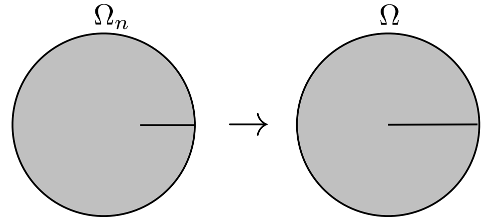

Figure 1.4: Cutting into a unit disk.

domain, that is $\Omega \subset \Omega_{n+1} \subset \Omega_n$ for all $n \in \mathbb{N}$. It is easy to see that Ω and Ω_n satisfy the cone condition uniformly with respect to $n \in \mathbb{N}$, but $H^1(\Omega)$ and $H^1(\Omega_n)$ do not have the extension property. Moreover, these domains satisfy the conditions in [21] (i.e. conditions listed in Corollary 1.4.17). Hence, $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco. Note that here we take D to be the open unit disk center at 0 in $\mathbb{R}^2$. In this example, we only need that $ri_nf_n \rightharpoonup f$ in $L^2((0,T),L^2(\Omega))$ and $ri_nu_{0,n} \rightharpoonup u_0$ in $L^2(\Omega)$ to conclude the convergence of solutions $i_nu_n \to iu$ in $C([\delta,T],L^2(D))$ for all $\delta \in (0,T]$. In addition, if $i_nu_{0,n} \to iu_0$ in $L^2(D)$, then the assertion holds for $\delta = 0$.

To see this, we note from Lemma 1.4.13 that $ri_nu_n(t) \rightharpoonup u(t)$ in $L^2(\Omega)$ weakly for all $t \in [0,T]$. Since $u \in L^2((0,T),H^1(\Omega))$, we have $u(t) \in H^1(\Omega)$ for almost everywhere $t \in (0,T)$. Fix now such $t \in (0,T)$. By the continuity of solutions $u_n \in C([0,T],L^2(\Omega_n))$, for each $n \in \mathbb{N}$ we can choose $\rho_n > 0$ such that

$$\|u_n(s) - u_n(t)\|_{L^2(\Omega_n)} \leq \frac{1}{n},$$

for all $s \in (t-\rho_n, t+\rho_n) \cap (0,T)$. As $u_n \in L^2((0,T),H^1(\Omega_n))$, we can choose $t_n \in (t-\rho_n, t+\rho_n) \cap (0,T)$ such that $u_n(t_n) \in H^1(\Omega_n)$ for all $n \in \mathbb{N}$. For these choices of t_n, we have $\|u_n(t_n) - u_n(t)\|_{L^2(\Omega_n)} \to 0$ as $n \to \infty$. It follows that $ri_nu_n(t_n) \rightharpoonup u(t)$ in $L^2(\Omega)$ weakly. Since $\Omega \subset \Omega_n$ for all $n \in \mathbb{N}$, the restriction $ru_n(t_n)$ belongs to $H^1(\Omega)$ for all $n \in \mathbb{N}$. Hence, it follows from the weak convergence of $ri_nu_n(t_n) = ru_n(t_n) \rightharpoonup u(t)$ in $L^2(\Omega)$ that

$$\int_\Omega \partial_j\big(ru_n(t_n)\big)\phi dx = -\int_\Omega ru_n(t_n)\partial_j\phi dx \to -\int_\Omega u(t)\partial_j\phi dx = \int_\Omega \partial_j u(t)\phi dx,$$

for all $\phi \in C_c^\infty(\Omega)$ for $j = 1,2$. This means $\nabla ru_n(t_n) \rightharpoonup \nabla u(t)$ in $L^2(\Omega,\mathbb{R}^2)$. Thus, $ru_n(t_n)$ is bounded in $H^1(\Omega)$. As Ω is bounded and satisfies the cone condition, we have from the Rellich-Kondrachov theorem that the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact (see [1, Theorem 6.2]). Therefore, $ru_n(t_n)$ has a subsequence which converges in $L^2(\Omega)$ strongly. Since we have a prior knowledge of weak convergence $ru_n(t_n) \rightharpoonup u(t)$ in $L^2(\Omega)$, we conclude that the whole sequence $ru_n(t_n) \to u(t)$ in $L^2(\Omega)$ strongly. By the choices of t_n, we conclude that $ru_n(t) \to u(t)$ in $L^2(\Omega)$ strongly. Since the above argument works for almost everywhere $t \in (0,T)$, we deduce from the dominated convergence theorem that $ri_n u_n = ru_n \to u$ in $L^2((0,T), L^2(\Omega))$ strongly. As the cutting line is a set of measure zero in $\mathbb{R}^2$, we have $i_n u_n \to iu$ in $L^2((0,T), L^2(D))$. By extracting a subsequence (indexed again by n), we can choose δ arbitrarily closed to zero in $(0,T]$ such that $i_n u_n(\delta) \to iu(\delta)$ in $L^2(D)$. We can now follow the proof of Theorem 1.4.14 and Theorem 1.4.16 with the integration taken over $[\delta, T]$ instead of $[0,T]$ to conclude the claim.

1.5 Convergence of solutions of final value problems

In this short section, we give some remarks on domain perturbation for final value problems. It is a direct application from initial value problems. Let $T > 0$. We consider the final value problem

$$\begin{cases} -\dfrac{\partial u}{\partial t} + \mathcal{A}(t)u = g(x,t) & \text{in } \Omega \times [0,T) \\ \mathcal{B}(t)u = 0 & \text{on } \partial\Omega \times [0,T) \\ u(\cdot, T) = u_T & \text{in } \Omega, \end{cases} \tag{1.55}$$

where $\mathcal{A}$ and $\mathcal{B}$ are defined by (0.5) and (0.6) in Introduction, respectively. As in the case of initial value problem, we choose the evolution triple $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$ as in (1.11). Let $a(t,\cdot,\cdot)$ be the corresponding bilinear form on V given by (1.10) and $A \in \mathscr{L}(V,V')$ be the operator defined by (1.4). Then we can consider the final value problem in an abstract form

$$\begin{cases} -u'(t) + A(t)u = g(t) & \text{for } t \in [0,T) \\ u(T) = u_T \end{cases} \tag{1.56}$$

in V' for a given final condition $u_T \in H$ and $g \in L^2((0,T), V')$.

Definition 1.5.1. A function $u \in W(0,T,V,V')$ is called a weak solution of (1.55) if u is a solution of (1.56), that is u satisfies (1.56) a.e. $t \in [0,T)$.

As in [2, Chapter V 2.5], we may associate (1.56) with the initial value problem

$$\begin{cases} v'(t) + A(T-t)v = g(T-t) & \text{for } t \in (0,T] \\ v(0) = u_T \end{cases} \tag{1.57}$$

in V'. Let $u(t) := v(T-t)$. Then it is trivial that u is a solution of the final value problem (1.56) if and only if v is a solution of the initial value problem (1.57). Moreover, u is a weak solution of (1.55) if and only if $u \in L^2((0,T),V)$ and

$$\begin{aligned} -\int_0^T (u(t)|v)\phi'(t)dt + \int_0^T a(t,u(t),v)\phi(t)dt \\ = (u_T|v)\phi(T) + \int_0^T \langle g(t),v\rangle\phi(t)dt, \end{aligned} \tag{1.58}$$

for all $v \in V$ and for all $\phi \in \mathscr{D}((0,T])$.

Theorem 1.5.2. *Given $g \in L^2((0,T),V')$ and $u_T \in H$, there exists a unique weak solution u of (1.55) satisfying*

$$\|u\|_{W(0,T,V,V')} \leq C\Big(\|u_T\|_H + \|g\|_{L^2((0,T),V')}\Big). \tag{1.59}$$

Moreover, if $\lambda = 0$ in (1.3), then

$$\|u(t)\|_H^2 + \alpha\int_t^T \|u(s)\|_V^2\, ds \leq \|u_T\|_H^2 + \alpha^{-1}\int_t^T \|g(s)\|_V^2\, ds, \tag{1.60}$$

for all $t \in [0,T]$.

Proof. By the above observation and Theorem 1.1.2, there exists a unique weak solution $v \in W(0,T,V,V')$ of the associated initial value problem satisfying the estimates (1.7) and (1.8) with u_0, f and $f(s)$ replaced by u_T, g and $g(T-s)$, respectively. Taking into account that $u(t) = v(T-t)$ is a weak solution of the corresponding final value problem, we obtain the estimates (1.59) and (1.60). □

We now consider domain perturbation for final value problems with Dirichlet or Neumann boundary conditions. We assume the same assumptions on the domains Ω_n and Ω, and on the bilinear forms $a_n(\cdot,u,v)$ for $u,v \in V_n$ and $a(\cdot,u,v)$ for $u,v \in V$ as in Section 1.4. Also, the operators $\mathcal{A}_n, \mathcal{A}, \mathcal{B}_n$ and $\mathcal{B}$, and the induced operators $A_n \in \mathscr{L}(V_n,V_n')$ and $A \in \mathscr{L}(V,V')$ are defined in the same way as in Section 1.4. Similar to the initial value problems, we study the following perturbation of the final value problem

$$\begin{cases} -\dfrac{\partial u}{\partial t} + \mathcal{A}_n(t)u = g_n(x,t) & \text{in } \Omega_n \times [0,T) \\ \mathcal{B}_n(t)u = 0 & \text{on } \partial\Omega_n \times [0,T) \\ u(\cdot,T) = u_{T,n} & \text{in } \Omega, \end{cases} \tag{1.61}$$

where $\mathcal{B}_n$ is either Dirichlet boundary condition or Neumann boundary condition. As usual, we write (1.61) in the abstract form as

$$\begin{cases} -u'(t) + A_n(t)u = g_n(t) & \text{for } t \in [0,T) \\ u(T) = u_{T,n}. \end{cases} \tag{1.62}$$

As a consequence of the continuity of solutions of initial value problems under domain perturbation, we can state a convergence result for final value problems.

Theorem 1.5.3. *Suppose that $i_n g_n$ is uniformly bounded in $L^2((0,T), L^2(D))$ and $i_n u_{T,n}$ is uniformly bounded in $L^2(D)$. Let u_n and u be the (unique) weak solutions of (1.62) and (1.56), respectively.*

(i) **Dirichlet problems:** *Assume that $ri_n g_n \rightharpoonup g$ in $L^2((0,T), L^2(\Omega))$ weakly and $ri_n u_{T,n} \rightharpoonup u_T$ in $L^2(\Omega)$ weakly. If $H^1_0(\Omega_n)$ converges to $H^1_0(\Omega)$ in the sense of Mosco, then $i_n u_n$ converges to iu in $L^2((0,T), H^1(D))$ and in $C([0,\delta], L^2(D))$ for every $\delta \in [0,T)$. In addition, if $i_n u_{T,n} \to iu_T$ in $L^2(D)$ strongly, the above assertion holds for $\delta = T$.*

(ii) **Neumann problems:** *Assume that $i_n g_n \to ig$ in $L^2((0,T), L^2(D))$ strongly and $i_n u_{T,n} \rightharpoonup iu_T$ in $L^2(D)$ strongly. If $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco, then $(i_n u_n, i_n \nabla u_n)$ converges to $(iu, i\nabla u)$ in $L^2((0,T), L^2(D)) \times L^2((0,T), L^2(D, \mathbb{R}^N))$ and $i_n u_n$ converges to iu in $C([0,T], L^2(D))$.*

Proof. This immediately follows from the above observation that $v_n(t) = u_n(T-t)$ and $v(t) = u(T-t)$ are weak solutions of the corresponding initial value problems and the results in Theorems 1.4.7, 1.4.9, 1.4.14 and 1.4.16. □

We remark that if $-u'(t)$ is replaced by $u'(t)$ in the final value problem (1.56), then the associated initial value problem is $v'(t) - A(T-t)v(t) = -g(T-t)$ with $v(0) = u_T$. We cannot apply the results for parabolic initial value problems in Section 1.4 as $-A(T-t)$ no longer satisfies the ellipticity assumption. We will discuss final value problems for autonomous parabolic equations of the form $u'(t) + Au(t) = g(t)$ for $t \in [0,T)$ with $u(T) = u_T$ where A is a generator of a group later. In particular, we prove convergence of the component of solutions of autonomous parabolic equations in the unstable direction in Lemma 4.2.4 (ii).

1.6 Domain perturbation for parabolic variational inequalities

In Section 1.4, the applications of the equivalence of Mosco convergences established in Theorem 1.3.1 were given when K_n and K are closed subspaces of V. The purpose of this section is to give an application when K_n and K are just closed and convex subsets of V. We show here a similar convergence of solutions of parabolic variational inequalities.

Suppose that K is a closed and convex subset of V. We denote by

$$L^2((0,T),K) := \{u \in L^2((0,T),V) \mid u(t) \in K \text{ a.e.}\}.$$

For each $t \in (0,T)$, suppose $a(t;\cdot,\cdot)$ is a continuous bilinear form on V satisfying (1.2) and (1.3). As before, we denote the induced linear operator by $A(t)$. Given $u_0 \in K$ and $f \in L^2((0,T),V')$, we wish to find u such that for a.e. $t \in (0,T)$, $u(t) \in K$ and

$$\begin{cases} \langle u'(t), v-u(t)\rangle + \langle A(t)u(t), v-u(t)\rangle - \langle f(t), v-u(t)\rangle \geq 0, \quad \forall v \in K \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u(0) = u_0. \end{cases} \tag{1.63}$$

A function $u \in W((0,T),V,V')$ satisfying (1.63) is called a *strong solution* of parabolic variational inequality (1.63). There are various (slightly different) definitions of weak solution of parabolic variational inequalities (see e.g.[18, 55, 60, 61]). We shall define a weak notion of solution similar to the one in [55] as follows.

Definition 1.6.1. A function u is a *weak solution* of parabolic variational inequality (1.63) if $u \in L^2((0,T),K)$ and

$$\begin{aligned} \int_0^T \langle v'(t), v(t)-u(t)\rangle + \langle A(t)u(t), v(t)-u(t)\rangle - \langle f(t), v(t)-u(t)\rangle \, dt \\ + \frac{1}{2}\|v(0)-u_0\|_H^2 \geq 0, \end{aligned} \tag{1.64}$$

for all $v \in W((0,T),V,V') \cap L^2((0,T),K)$.

The existence and uniqueness of weak solutions of parabolic variational inequalities have been studied by various authors according to their definitions. In our case, we can state the result in the following theorem.

Theorem 1.6.2. *Given $u_0 \in K$ and $f \in L^2((0,T),V')$. There exists a unique weak solution u of the parabolic variational inequality (1.63) satisfying $u \in L^\infty((0,T),H)$.*

Note that the existence of our weak solution follows immediately from the existence results in [60, Theorem 6.2]. The uniqueness can be proved in the same way as in [61, Theorem 2.3].

Similar to domain perturbation for parabolic equations, we examine the behaviour of weak solutions of parabolic variational inequalities when the underlying convex set K is perturbed.

Let K_n and K be closed and convex subsets of V. For each $t \in (0,T)$, suppose $a(t;\cdot,\cdot)$ is a continuous bilinear form on V satisfying (1.2) and (1.3). For simplicity, we assume that $\lambda = 0$ in (1.3). We denote by $A(t)$ the linear operator induced by $a(t;\cdot,\cdot)$. Let us consider the following parabolic variational inequalities. Given $u_{0,n} \in K_n$ and $f_n \in L^2((0,T),V')$, we want to find u_n such that for a.e. $t \in (0,T)$, $u_n(t) \in K_n$ and

$$\begin{cases} \langle u'(t), v - u(t)\rangle + \langle A(t)u(t), v - u(t)\rangle - \langle f_n(t), v - u(t)\rangle \geq 0, \quad \forall v \in K_n \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad u(0) = u_{0,n}. \end{cases} \tag{1.65}$$

When K_n, f_n and $u_{0,n}$ converge to K, f and u_0 respectively, we wish to obtain convergence results of the weak solution of (1.65) to that of the limit inequalities (1.63). In the remainder of this section, we denote the weak solution of (1.65) by u_n and the weak solution of (1.63) by u.

Theorem 1.6.3. *Let $u_{0,n} \in K_n$, $u_0 \in K$ and $f_n, f \in L^2((0,T),V')$. Suppose that $f_n \to f$ in $L^2((0,T),V')$ strongly, $u_{0,n} \rightharpoonup u_0$ in V weakly and $u_{0,n} \to u_0$ in H strongly. Then the sequence of weak solutions u_n is bounded in $L^2((0,T),V)$.*

Proof. Let $v \in W((0,T),V,V') \cap L^2((0,T),K)$ be the constant function defined by $v(t) := u_0$ for $t \in [0,T]$. Similarly, $v_n \in W((0,T),V,V') \cap L^2((0,T),K_n)$ defined by $v_n(t) := u_{0,n}$ for $t \in [0,T]$. It follows that $v_n \rightharpoonup v$ in $L^2((0,T),V)$. Since u_n is the weak solution of (1.65), we have

$$\begin{aligned}
&\int_0^T \langle A(t)u_n(t), u_n(t) - v_n(t)\rangle \, dt \\
&\quad \leq \int_0^T \langle v_n'(t), v_n(t) - u_n(t)\rangle - \langle f_n(t), v_n(t) - u_n(t)\rangle \, dt + \frac{1}{2}\|v_n(0) - u_{0,n}\|_H^2 \\
&\quad = -\int_0^T \langle f_n(t), v_n(t) - u_n(t)\rangle \, dt.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_0^T \langle A(t)u_n(t) - A(t)v_n(t), u_n(t) - v_n(t)\rangle \, dt \\
&\quad \le \int_0^T \langle A(t)v_n(t), v_n(t) - u_n(t)\rangle - \langle f_n(t), v_n(t) - u_n(t)\rangle \, dt \\
&\quad \le \|A(t)v_n - f_n\|_{L^2((0,T),V')} \|v_n - u_n\|_{L^2((0,T),V)}.
\end{aligned}$$

By the coercivity of $A(t)$, we get

$$\alpha \|u_n - v_n\|_{L^2((0,T),V)} \le \|A(t)v_n - f_n\|_{L^2((0,T),V')}.$$

By the weak convergences of v_n and f_n, we conclude that u_n is bounded in $L^2((0,T),V)$. □

Theorem 1.6.4. *Let $u_{0,n} \in K_n$, $u_0 \in K$ and $f_n, f \in L^2((0,T),V')$. Suppose that $f_n \to f$ in $L^2((0,T),V')$ strongly, $u_{0,n} \rightharpoonup u_0$ in V weakly and $u_{0,n} \to u_0$ in H strongly. If K_n converges to K in the sense of Mosco, then the sequence of weak solutions u_n converges to u in $L^2((0,T),V)$ weakly.*

Proof. By Theorem 1.6.3, we can extract a subsequence of u_n (denoted again by u_n) such that $u_n \rightharpoonup \kappa$ in $L^2((0,T),V)$ weakly. Since $u_n \in L^2((0,T),K_n)$, we apply the Mosco condition $(M2')$ (from Theorem 1.3.1) to deduce that the weak limit κ belongs to $L^2((0,T),K)$. By the uniqueness of weak solutions, it suffices to prove that κ satisfies (1.64) (with u replaced by κ) in the definition of weak solution.

By the Mosco condition $(M1')$, there exists $w_n \in L^2((0,T),K_n)$ such that $w_n \to \kappa$ in $L^2((0,T),V)$. Let $v \in W((0,T),V,V') \cap L^2((0,T),K)$ be arbitrary. We again apply Theorem 1.3.1 (Mosco condition $(M1'')$) to get a sequence of functions $v \in W((0,T),V,V') \cap L^2((0,T),K_n)$ such that $v_n \to v$ in $W((0,T),V,V')$. For each $n \in \mathbb{N}$, we can write

$$\begin{aligned}
&\langle A(t)w_n(t), v_n(t) - u_n(t)\rangle \\
&\quad = \langle A(t)u_n(t), v_n(t) - u_n(t)\rangle + \langle A(t)w_n(t) - A(t)u_n(t), v_n(t) - u_n(t)\rangle \\
&\quad = \langle A(t)u_n(t), v_n(t) - u_n(t)\rangle + \langle A(t)w_n(t) - A(t)u_n(t), w_n(t) - u_n(t)\rangle \\
&\qquad + \langle A(t)w_n(t) - A(t)u_n(t), v_n(t) - w_n(t)\rangle.
\end{aligned}$$

Hence, by the definition of weak solution on K_n and the coercivity of $A(t)$, we have

$$\begin{aligned}
&\int_0^T \langle v_n'(t), v_n(t) - u_n(t)\rangle + \langle A(t)w_n(t), v_n(t) - u_n(t)\rangle \, dt \\
&\quad - \int_0^T \langle f_n(t), v_n(t) - u_n(t)\rangle \, dt + \frac{1}{2}\|v_n(0) - u_{0,n}\|_H^2 \\
&= \int_0^T \langle v_n'(t), v_n(t) - u_n(t)\rangle + \langle A(t)u_n(t), v_n(t) - u_n(t)\rangle \, dt \\
&\quad - \int_0^T \langle f_n(t), v_n(t) - u_n(t)\rangle \, dt + \frac{1}{2}\|v_n(0) - u_{0,n}\|_H^2 \\
&\quad + \int_0^T \langle A(t)w_n(t) - A(t)u_n(t), w_n(t) - u_n(t)\rangle \, dt \\
&\quad + \int_0^T \langle A(t)w_n(t) - A(t)u_n(t), v_n(t) - w_n(t)\rangle \, dt \\
&\geq \int_0^T \langle A(t)w_n(t) - A(t)u_n(t), v_n(t) - w_n(t)\rangle \, dt,
\end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, we get

$$\begin{aligned}
&\int_0^T \langle v'(t), v(t) - \kappa(t)\rangle + \langle A(t)\kappa(t), v(t) - \kappa(t)\rangle \, dt \\
&\quad - \int_0^T \langle f(t), v(t) - \kappa(t)\rangle \, dt + \frac{1}{2}\|v(0) - u_0\|_H^2 \geq 0.
\end{aligned}$$

This implies that κ is a weak solution of (1.63) as required. □

We finally prove the strong convergence of solutions.

Theorem 1.6.5. *Let $u_{0,n} \in K_n$, $u_0 \in K$ and $f_n, f \in L^2((0,T),V')$. Suppose that $f_n \to f$ in $L^2((0,T),V')$ strongly, $u_{0,n} \rightharpoonup u_0$ in V weakly and $u_{0,n} \to u_0$ in H strongly. If K_n converges to K in the sense of Mosco, then the sequence of weak solutions u_n converges to u in $L^2((0,T),V)$ strongly.*

Proof. By the coercivity of $A(t)$, we have

$$\liminf_{n\to\infty} \int_0^T \langle A(t)u_n(t) - A(t)u(t), u_n(t) - u(t)\rangle \, dt \geq 0. \tag{1.66}$$

For each $\epsilon > 0$, we define u_ϵ by

$$\epsilon u_\epsilon' + u_\epsilon = u$$
$$u_\epsilon(0) = u_0.$$

Then $u_\epsilon \in W((0,T),V,V') \cap L^2((0,T),K)$ and $u_\epsilon \to u$ in $L^2((0,T),V)$ as $\epsilon \to 0$ (see in the proof of [61, Theorem 2.3]). For each $\epsilon > 0$, the Mosco condition $(M1'')$ (from

Theorem 1.3.1) implies that there exists $u_{\epsilon,n} \in W((0,T),V,V') \cap L^2((0,T),K_n)$ such that $u_{\epsilon,n} \to u_\epsilon$ in $W((0,T),V,V')$ as $n \to \infty$. Since u_n is the weak solution of (1.65), we have

$$\begin{aligned}
&\int_0^T \langle A(t)u_n(t), u_n(t) - u(t)\rangle \, dt \\
&\leq \int_0^T \langle v_n'(t), v_n(t) - u_n(t)\rangle \, dt - \int_0^T \langle f_n(t), v_n(t) - u_n(t)\rangle \, dt \\
&\quad + \frac{1}{2}\|v_n(0) - u_{0,n}\|_H^2 + \int_0^T \langle A(t)u_n(t), v_n(t) - u(t)\rangle \, dt,
\end{aligned}$$

for all $v \in W((0,T),V,V') \cap L^2((0,T),K)$. In particular, taking $v_n = u_{\epsilon,n}$, we get

$$\begin{aligned}
&\int_0^T \langle A(t)u_n(t), u_n(t) - u(t)\rangle \, dt \\
&\leq \int_0^T \langle u_{\epsilon,n}'(t), u_{\epsilon,n}(t) - u_n(t)\rangle \, dt - \int_0^T \langle f_n(t), u_{\epsilon,n}(t) - u_n(t)\rangle \, dt \\
&\quad + \frac{1}{2}\|u_{\epsilon,n}(0) - u_{0,n}\|_H^2 + \int_0^T \langle A(t)u_n(t), u_{\epsilon,n}(t) - u(t)\rangle \, dt.
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty} \int_0^T \langle A(t)u_n(t), u_n(t) - u(t)\rangle \, dt \\
&\leq \int_0^T \langle u_\epsilon'(t), u_\epsilon(t) - u(t)\rangle \, dt - \int_0^T \langle f(t), u_\epsilon(t) - u(t)\rangle \, dt \\
&\quad + \frac{1}{2}\|u_\epsilon(0) - u_0\|_H^2 + \int_0^T \langle A(t)u(t), u_\epsilon(t) - u(t)\rangle \, dt \\
&= -\epsilon \int_0^T \|u_\epsilon'(t)\|_H^2 \, dt - \int_0^T \langle f(t), u_\epsilon(t) - u(t)\rangle \, dt \\
&\quad + \int_0^T \langle A(t)u(t), u_\epsilon(t) - u(t)\rangle \, dt \\
&\leq -\int_0^T \langle f(t), u_\epsilon(t) - u(t)\rangle \, dt + \int_0^T \langle A(t)u(t), u_\epsilon(t) - u(t)\rangle \, dt.
\end{aligned}$$

This is true for any $\epsilon > 0$. Hence, by letting $\epsilon \to 0$, we conclude that

$$\limsup_{n\to\infty} \int_0^T \langle A(t)u_n(t), u_n(t) - u(t)\rangle \, dt \leq 0.$$

On the other hand, the weak convergence of u_n in Theorem 1.6.4 implies

$$\lim_{n\to 0} \int_0^T \langle A(t)u(t), u_n(t) - u(t)\rangle \, dt = 0.$$

Thus, we have

$$\limsup_{n\to\infty} \int_0^T \langle A(t)u_n(t) - A(t)u(t), u_n(t) - u(t)\rangle dt \leq 0. \tag{1.67}$$

It follows from the coercivity of $A(t)$, (1.66) and (1.67) that

$$\alpha\|u_n - u\|^2_{L^2((0,T),V)} \leq \int_0^T \langle A(t)u_n(t) - A(t)u(t), u_n(t) - u(t)\rangle \, dt \to 0,$$

as $n \to \infty$. Therefore, u_n converges to u in $L^2((0,T),V)$ strongly. $\square$

Chapter 2

Domain perturbation for semilinear parabolic equations

The main purpose of this chapter is to present some preliminary results on domain perturbation for semilinear parabolic equations which are required for the study of invariant manifolds in Chapter 3. Although we will only consider the dynamics of autonomous semilinear equations in $L^2(\mathbb{R}^N)$ in Chapter 3, we give an abstract domain perturbation result for non-autonomous semilinear *evolution* equations. Most of the material presented in this chapter is based on [38]. However, we point out that the convergence result in [38, Theorem 4.6] only requires a weaker assumption on convergence of the nonlinear terms. We show that the same result holds if we replace the local uniform convergence of the nonlinear terms by pointwise convergence (see Theorem 2.3.5 below). Using this improved convergence result, we consider domain perturbation for both autonomous and non-autonomous semilinear parabolic equations under pointwise convergence of the nonlinear terms in Section 2.4. We note that in [35] and [38] the nonlinear terms are fixed for the perturbed problems. As in Chapter 1, we use Mosco convergence to establish the stability of solutions under domain perturbation. The convergence results are obtained for parabolic equations under Dirichlet or Neumann boundary conditions, but stronger assumptions are required for Neumann problems. For autonomous problems, we also include some known results on convergence of the spectrum and resolvent operators under domain perturbation.

For the sake of completeness and mathematical necessity, we include the proof of some background materials from [38].

2.1 Evolution Systems

We collect some preliminary results for evolution systems arising from (abstract) non-autonomous parabolic equations. Fix $T > 0$ and $s \in [0, T)$. We denote by

$$\Delta_T := \{(s,t) \in \mathbb{R}^2 : 0 \leq s \leq t \leq T\}$$

and

$$\dot{\Delta}_T := \{(s,t) \in \mathbb{R}^2 : 0 \leq s < t \leq T\} \subset \Delta_T.$$

Consider the following homogeneous parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}(t)u = 0 & \text{in } \Omega \times (s,T] \\ \mathcal{B}(t)u = 0 & \text{on } \partial\Omega \times (s,T] \\ u(\cdot, s) = u_0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are defined by (0.5) and (0.6), respectively. By taking the evolution triple as in (1.11), we may write (2.1) as an abstract equation of the form

$$\begin{cases} u'(t) + A(t)u(t) = 0 & \text{for } t \in (s,T] \\ u(s) = u_0, \end{cases} \tag{2.2}$$

where $A(t) \in \mathscr{L}(V, V')$ is the operator induced by $a(t, \cdot, \cdot)$ as in (1.10).

By Theorem 1.1.2, for every given $u_0 \in H$, there exists a unique (weak) solution $u \in W((s,T), V, V')$ of the homogeneous non-autonomous parabolic equation.

Definition 2.1.1 (Evolution systems)**.** Denote the solution u of (2.2) by $U(\cdot, s)u_0$. Then $U(\cdot, \cdot)$ defined on Δ_T is called the *evolution system* corresponding to the family $(A(t))_{t\in[0,T]}$.

It is known that the operator $U(\cdot, \cdot)$ is in $\mathscr{L}(H)$ ([41, Section XVIII.3.4]). Since $\mathcal{A}(\cdot)$ is uniform strongly elliptic, a standard heat kernel estimates ([36, Section 7]) shows that for Dirichlet problems, $U(\cdot, \cdot) \in \mathscr{L}(L^p(\Omega), L^q(\Omega))$ satisfies the estimate

$$\|U(t,s)\|_{\mathscr{L}(L^p(\Omega), L^q(\Omega))} \leq C(t-s)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}, \tag{2.3}$$

for all $1 \leq p \leq q \leq \infty$ where the constant C is independent of Ω. If Ω satisfies the interior cone condition or has the extension property, then the estimate (2.3) is valid for Neumann problems with C depending on the cone or the norm of the extension operator, respectively. In that case, the boundary condition is considered in a weak sense as the outer unit normal may not exist.

Throughout this thesis, we write $\mathscr{L}(E,F)$ for the space of bounded linear operators from a Banach space E into a Banach space F equipped with the operator norm. We simply write $\mathscr{L}(E)$ for $\mathscr{L}(E,E)$. The notation $\mathcal{L}_s(E,F)$ represents the space of bounded linear operators under the strong operator topology. More precisely, $S_n \to S$ in $\mathcal{L}_s(E,F)$ if $S_n x \to Sx$ in F for all $x \in E$ as $n \to \infty$. Again, we simply denote by $\mathcal{L}_s(E)$ for $\mathcal{L}_s(E,E)$.

As mentioned earlier, we will give the perturbation results for evolution equations. We make the following assumptions for evolution systems.

Assumption 2.1.2. The evolution system $U(t,s), (t,s) \in \Delta_T$ on the Banach space E satisfies the following properties.

(i) $U(t,s) = U(t,\tau)U(\tau,s)$ for all $0 \le s < \tau < t \le T$;

(ii) There exists a Banach space F such that

$$U(\cdot,s) \in C((s,T],\mathcal{L}_s(E,F)) \cap C((s,T],\mathcal{L}_s(E)) \cap C((s,T],\mathcal{L}_s(F)),$$

for all $s \in [0,T)$;

(iii) There exist $C > 0$ and $\gamma \in [0,1)$ such that

$$\|U(t,s)\|_{\mathscr{L}(E)} + \|U(t,s)\|_{\mathscr{L}(F)} + (t-s)^{\gamma}\|U(t,s)\|_{\mathscr{L}(E,F)} \le C, \tag{2.4}$$

for all $(t,s) \in \dot{\Delta}_T$.

There are several problems for which the above assumptions are satisfied (see [38, Example 2.2, 2.3]). In particular, this includes the evolution systems for parabolic equations discussed above. To see this, we take $E := L^p(\Omega)$ and $F := L^q(\Omega)$ for $1 < p \le q < \infty$ satisfying

$$\gamma := \frac{N}{2}\Big(\frac{1}{p} - \frac{1}{q}\Big) \in [0,1). \tag{2.5}$$

Then by the estimate (2.3), $U(\cdot,\cdot)$ satisfies Assumption 2.1.2.

2.2 Existence of solutions of semilinear evolution equations

In this section, we prove the existence and uniqueness of solutions of the following non-autonomous semilinear evolution equation

$$\begin{cases} u'(t) + A(t)u(t) = f(t, u(t)) & \text{for } t \in (s, T] \\ u(s) = u_0, \end{cases} \tag{2.6}$$

in a Banach space E whose evolution system satisfies Assumption 2.1.2.

Definition 2.2.1. A function u is called a *mild solution* of (2.6) with an initial value $u_0 \in F$ if u satisfies the *variation of constants* formula

$$u(t) = U(t, s)u_0 + \int_s^t U(t, \tau)f(\tau, u(\tau))d\tau,$$

for all $t \in (s, T]$, where $U(t, s), 0 \leq s \leq t \leq T$ is the evolution system corresponding to the family $(A(t))_{t\in[0,T]}$.

Before studying the existence of solutions of semilinear evolution equations, we give a basic property of solutions of the inhomogeneous linear equation

$$\begin{cases} u'(t) + A(t)u(t) = g(t) & \text{for } t \in (s, T] \\ u(s) = u_0, \end{cases} \tag{2.7}$$

where $g \in L^\infty((s, T], E)$ and $u_0 \in F$. Define

$$L_s(u_0, g)(t) := U(t, s)u_0 + \int_s^t U(t, \tau)g(\tau)d\tau. \tag{2.8}$$

Then $u = L_s(u_0, g)$ is called a mild solution of (2.7). The next proposition shows that $L_s(u_0, g)$ belongs to the space

$$BC((s, T], F) := C((s, T], F) \cap L^\infty([s, T], F).$$

Proposition 2.2.2 ([38, Proposition 2.4])**.** *For every $s \in [0, T)$, we have*

$$L_s \in \mathscr{L}(F \times L^\infty((s, T], E), BC((s, T], F)).$$

Proof. See [38, Proposition 2.4] and also [40, Corollary 5.6]. □

It can easily be seen that u is a mild solution of (2.6) if u is a fixed point of the map $G := G_{u_0}$ given by

$$G(u)(t) = U(t,s)u_0 + \int_s^t U(t,\tau)f(\tau,u(\tau))d\tau$$

in the space $L^\infty([s,t],F)$.

To get the existence of solutions of semilinear equations, the following Lipschitz assumption on the nonlinearity is usually imposed.

Assumption 2.2.3. We assume that $f \in C([0,T]\times F, E)$ and that for every $R > 0$, there exists $k_R > 0$ such that

$$\|f(t,u) - f(t,v)\|_E \le k_R \|u - v\|_F,$$

for all $u, v \in F$ with $\|u\|_F, \|v\|_F \le R$ and all $t \in [0,T]$.

By the assumptions on f, we have that $f(\cdot, u(\cdot)) \in L^\infty((s,T],E)$ for all $u \in L^\infty([s,t],F)$. Hence, by Proposition 2.2.2, mild solutions of (2.6) are always in the space $BC((s,T],F)$.

We next determine an invariant subset of the space $BC((s,T],F)$ such that G is a contraction.

Proposition 2.2.4 ([38, Proposition 3.3])**.** *For all $\rho > 0$, there exist $h > 0$ and $R > 0$ such that if $\|u_0\|_F \le \rho$ and $(t_0,s) \in \dot{\Delta}_T$ with $0 < t_0 - s \le h$, then*

$$G_{u_0} : M_R(t_0,s) \to M_R(t_0,s),$$

where

$$M_R(t_0,s) := \{u \in BC((s,t_0],F) : \|u(\tau)\|_F \le R \text{ for all } \tau \in (s,t_0]\}.$$

Indeed, R can be chosen by $R := C\rho + 1$ where C is the constant from (2.4). Moreover, h depends only on γ and upper bounds for C, k_R and $M := \sup\{\|f(\tau,0)\|_E : \tau \in [0,T]\}$.

Proof. Suppose that u_0 is in F with $\|u_0\|_F \le \rho$. By the assumptions on $U(\cdot,\cdot)$, we have

$$\begin{aligned}\|G(u)(t)\|_F &\le \|U(t,s)\|_{\mathscr{L}(F)}\|u_0\|_F + \int_s^t \|U(t,\tau)\|_{\mathscr{L}(E,F)}\|f(\tau,u(\tau))\|_E d\tau \\ &\le C\rho + C \sup_{\tau\in(s,T]} \|f(\tau,u(\tau))\|_E \int_s^t (t-\tau)^{-\gamma} d\tau \qquad (2.9)\\ &= C\rho + C\frac{(t-s)^{1-\gamma}}{1-\gamma} \sup_{\tau\in(s,t_0]} \|f(\tau,u(\tau))\|_E,\end{aligned}$$

for all $u \in BC((s,t_0],F)$ and for all $t \in (s,t_0]$. In particular, for $u \in M_R(t_0,s)$, we have from Assumption 2.2.3 that

$$\begin{aligned}\|f(\tau,u(\tau))\|_E &\le \|f(\tau,u(\tau)) - f(\tau,0)\|_E + \|f(\tau,0)\|_E \\ &\le Rk_R + \sup_{\tau\in(s,t_0]} \|f(\tau,0)\|_E \\ &\le Rk_R + M,\end{aligned}$$

for all $t \in (s,t_0]$ where we set $M := \sup\{\|F(\tau,0)\|_E : \tau \in [0,T]\}$. It follows from (2.9) that

$$\|G(u)(t)\|_F \le C\rho + C\frac{(t-s)^{1-\gamma}}{1-\gamma}(Rk_R + M),$$

for all $u \in M_R(t_0,s)$ and $t \in (s,t_0)$. By setting $R := C\rho + 1$ and choosing $h > 0$ such that

$$\frac{C}{1-\gamma}(Rk_R + M)h^{1-\gamma} \le 1,$$

we conclude that if $(t_0,s) \in \dot{\Delta}_T$ with $t_0 - s \le h$, then

$$\|G(u)\|_{L^\infty((s,t_0],F)} \le C\rho + 1 = R,$$

for all $u \in M_R(t_0,s)$. □

Usually, we use the metric d_0 on $BC((s,t_0],F)$ induced by the norm, that is,

$$d_0(u,v) = \sup_{\tau\in(s,t_0]} \|u(\tau) - v(\tau)\|_F. \tag{2.10}$$

In domain perturbation problems, it is useful to consider a weaker metric. We consider a metric d_1 induced by the Fréchet space $L^\infty_{\text{loc}}((s,t_0],F)$ or a metric inducing the topology of uniform convergence on compact subsets of $(s,t_0]$. Let $\varepsilon_j := 2^{-j}$ and $h := t_0 - s$. For $j \in \mathbb{N}$, we define the seminorms

$$q_j(u) := \sup_{\tau\in[s+\varepsilon_j h,t_0]} \|u(\tau)\|_F.$$

It is well known that for $\beta_j := \varepsilon_j^{1-\gamma}$,

$$d_1(u,v) := \sum_{j=1}^{\infty} \beta_j \frac{q_j(u-v)}{1+q_j(u-v)} \tag{2.11}$$

defines a metric on $L^\infty_{\text{loc}}((s,t_0],F)$ (see [71, Section 1.44]). Note that the set $M_R(t_0,s)$ is a complete metric space with respect to the metrics d_0 and d_1 (see [38, Remark 3.4]).

Proposition 2.2.5 ([38, Proposition 3.5])**.** *For every* $R > 0, (t_0, s) \in \dot{\Delta}_T$ *and* $u_0 \in F$*, the map*

$$G_{u_0} : M_R(t_0, s) \to BC((s, t_0], F)$$

is Lipschitz with respect to d_0 *and* d_1*. Indeed,*

$$d_0(G(u), G(v)) \leq \frac{Ck_R}{1-\gamma}(t_0 - s)^{1-\gamma} d_0(u, v)$$
$$d_1(G(u), G(v)) \leq \frac{2^{1-\gamma}(1 + 2R)}{(2^{1-\gamma} - 1)} \frac{Ck_R}{1-\gamma}(t_0 - s)^{1-\gamma} d_1(u, v),$$

for all $u, v \in M_R(t_0, s)$ *where* C *is the constant from* (2.4)*.*

Proof. We give a proof here for the metric d_0 only. By the definition of $G := G_{u_0}$, we have

$$\|G(u)(t) - G(v)(t)\|_F \leq \int_s^t \|U(t, \tau)\|_{\mathscr{L}(E,F)} \|f(\tau, u(\tau)) - f(\tau, v(\tau))\|_E d\tau,$$

for all $t \in (s, t_0]$. Since $u, v \in M_R(t_0, s)$, Assumption 2.2.3 and (2.4) imply

$$\|G(u)(t) - G(v)(t)\|_F \leq Ck_R \int_s^t (t - \tau)^{-\gamma} d\tau \sup_{\tau \in (s, t_0]} \|u(\tau) - v(\tau)\|_E$$
$$\leq \frac{Ck_R}{1-\gamma}(t_0 - s)^{1-\gamma} d_0(u, v),$$

for all $t \in (s, t_0]$ and for all $u, v \in M_R(t_0, s)$. Hence, the proposition follows for the metric d_0. The proof for the metric d_1 is a bit more complicated. We refer to [38] for full details. □

We can now prove the (local) existence and uniqueness of solutions of (2.6).

Theorem 2.2.6 ([38, Theorem 3.6, Corollary 3.7])**.** *Suppose that Assumption 2.1.2 and Assumption 2.2.3 are satisfied. Given* $\rho > 0$*, there exists* $h > 0$ *such that the equation* (2.6) *has a unique mild solution* $u \in BC((s, \min\{s + h, T\}], F)$ *for every* $u_0 \in F$ *with* $\|u_0\|_F \leq \rho$*. Moreover,* h *can be chosen depending only on* γ *and upper bounds for* $C, k_{C\rho+1}$ *and* $\sup\{\|f(\tau, 0)\|_E : \tau \in [0, T]\}$*.*

Proof. We can choose R and h so that $M_R(t_0, s)$ is invariant under G_{u_0} for all $u_0 \in F$ with $\|u_0\|_F \leq \rho$. (Proposition 2.2.4). In addition, by Proposition 2.2.5, we can make sure that G_{u_0} is a contraction map on $M_R(t_0, s)$ with Lipschitz constant less than one by possibly decreasing h (this is possible for both metrics d_0 and d_1). It follows from the contraction mapping theorem that there exists a unique fixed point $u \in M_R(t_0, s)$ of G_{u_0}. Since mild solutions of (2.6) correspond to fixed points of G_{u_0}, the assertion follows. □

Theorem 2.2.7 ([38, Theorem 3.8])**.** *Suppose that Assumption 2.1.2 and Assumption 2.2.3 are satisfied. For every $u_0 \in F$ and $s \in [0, T)$, there exists $t^+(s, u_0) \in (s, T]$ such that the equation (2.6) has a unique mild solution $u \in BC((s, t^+(s, u_0)), F)$ and u cannot be extended further from $t^+(s, u_0)$.*

Proof. By extending a local solution in Theorem 2.2.6 using a standard application of Zorn's lemma, we obtain the maximal solution $u \in BC((s, t^+(s, u_0)), F)$. □

We call $t^+(s, u_0)$ the *maximal existence time* or the *positive escape time* of the solution.

2.3 Convergence of solutions of semilinear evolution equations

In this section, we study perturbations of semilinear evolution equation (2.6) in a fixed Banach space E of the form

$$\begin{cases} u'(t) + A_n(t)u(t) = f_n(t, u(t)) & \text{for } t \in (s, T] \\ u(s) = u_{0,n}. \end{cases} \tag{2.12}$$

We make the following assumptions on the corresponding evolution systems $U_n(s, t)$ on E.

Assumption 2.3.1. Suppose that the evolution system $U_n(t, s)$ for $(t, s) \in \dot{\Delta}$ satisfies the conditions below:

(i) $U_n(t, s) = U_n(t, \tau)U_n(\tau, s)$ for all $0 \le s < \tau < t \le T$ and for all $n \in \mathbb{N}$;

(ii) There exists a Banach space F such that

$$U_n(\cdot, s) \in C((s, T], \mathcal{L}_s(E, F)),$$

for all $s \in [0, T)$ and for all $n \in \mathbb{N}$;

(iii) There exist $C > 0$ and $\gamma \in [0, 1)$ such that

$$\|U_n(t, s)\|_{\mathscr{L}(E)} + \|U_n(t, s)\|_{\mathscr{L}(F)} + (t - s)^\gamma \|U_n(t, s)\|_{\mathscr{L}(E,F)} \le C, \tag{2.13}$$

for all $(t, s) \in \dot{\Delta}_T$ and for all $n \in \mathbb{N}$;

(iv) $U_n(t, s)$ converges to $U(t, s)$ in $\mathcal{L}_s(E, F)$ as $n \to \infty$ for all $(t, s) \in \dot{\Delta}_T$ with the limit $U(\cdot, s) \in C((s, T], \mathcal{L}_s(E, F))$.

Furthermore, we impose the following conditions on the nonlinearity f_n.

Assumption 2.3.2. We assume that $f_n \in C([0,T] \times F, E)$ and that for every $R > 0$, there exists $k_R > 0$ such that

$$\|f_n(t,u) - f_n(t,v)\|_E \leq k_R \|u - v\|_F,$$

for all $u, v \in F$ with $\|u\|_F, \|v\|_F \leq R$ and all $t \in [0,T]$ and $n \in \mathbb{N}$. In addition, we assume that

$$\sup\{\|f_n(t,0)\| : t \in [0,T], n \in \mathbb{N}\} < \infty.$$

We now state a convergent result under perturbation.

Theorem 2.3.3 ([38, Theorem 4.4]). *Suppose that Assumptions 2.3.1 and 2.3.2 are satisfied. Let $u_0, u_{0,n}$ be in F with $\|u_0\|_F, \|u_{0,n}\|_F \leq \rho$ for some $\rho > 0$. For $n \in \mathbb{N}$, let $G_n, G : BC((s,T], F) \to BC((s,T], F)$ be the maps defined by*

$$G_n(u)(t) := U_n(t,s)u_{0,n} + \int_s^t U_n(t,\tau) f_n(\tau, u(\tau)) d\tau$$
$$G(u)(t) := U(t,s)u_0 + \int_s^t U(t,\tau) f(\tau, u(\tau)) d\tau,$$

for all $u \in BC((s,T], F)$. Suppose that $G_n(u) \to G(u)$ in $BC((s,T], F)$ for all $u \in BC((s,T], F)$ with respect to the metric d_0 (or d_1). Then there exist $h > 0$ and $R > 0$ such that (2.23) and (2.12) have unique mild solutions $u_n, u \in BC((s, \min\{s + h, T\}], B_F(0,R))$ for all $s \in [0,T)$. Moreover, $u_n \to u$ in $BC((s, \min\{s+h, T\}], F)$ with respect to d_0 (or d_1).

Proof. By the assumptions on U_n and f_n, we have from Theorem 2.2.5 that there exist $h > 0$ and $R > 0$ such that G_n, G are contractions on $M_R(t,s)$ for all $(t,s) \in \dot{\Delta}$ with $t - s \leq h$ for all $n \in \mathbb{N}$. Moreover, G_n are uniform contractions with a Lipschitz constant independent of $n \in \mathbb{N}$. As the fixed points u_n of G_n and u of G are the unique mild solutions of (2.23) and (2.12) respectively, it remains to show that the corresponding fixed points converge. Since $G_n(u) \to G(u)$ in $M_R(t,s)$ for all $u \in M_R(t,s)$ with respect to d_0 (or d_1), a parameter dependent contraction mapping principle (see [52, Section 1.2.6]) implies that the corresponding fixed points converge with respect to d_0 (or d_1). Thus, the statement of the theorem follows. □

Theorem 2.3.4 ([38, Theorem 4.5]). *Suppose that the assumptions in Theorem 2.3.3 are satisfied. Let $t_n^+(s, u_{0,n})$ be the positive escape time for (2.12). Then*

$$t^+(s, u_0) \leq \liminf_{n \to \infty} t_n^+(s, u_{0,n}). \tag{2.14}$$

Moreover, if $G_n(u) \to G(u)$ in $BC((s,T],F)$ for all $u \in BC((s,T],F)$ with respect to the metric d_0 (or d_1), then $u_n \to u$ in $BC(s,t_0],F)$ for all $t_0 \in (s,t^+(s,u_0))$ with respect to the metric d_0 (or d_1).

Proof. See [38, Theorem 4.5]. □

Our contribution in this chapter is that we weaken a local uniform convergence assumption of the inhomogeneous terms in [38, Theorem 4.6]:

$f_n(\tau,v) \to f(\tau,v)$ in E uniformly with respect to $(\tau,v) \in [0,T] \times B_F(0,R)$ for all $R > 0$

by using only pointwise convergence. We obtain the same convergence result as follow.

Theorem 2.3.5 (c.f. [38, Theorem 4.6])**.** *Let Assumptions 2.3.1 and 2.3.2 be satisfied. Assume that*

$$U_n(\cdot,s)u_{0n} \to U(\cdot,s)u_0 \quad \text{in } BC((s,T],F) \tag{2.15}$$

with respect to d_0 (or d_1) and

$$U_n(\cdot,\cdot) \to U(\cdot,\cdot) \quad \text{in } C(\dot{\Delta}_T, \mathcal{L}_s(E,F)). \tag{2.16}$$

Finally, assume that $f_n(\tau,v) \to f(\tau,v)$ in E pointwise on $(\tau,v) \in [0,T] \times F$. Then

$$t^+(s,u_0) \leq \liminf_{n\to\infty} t_n^+(s,u_{0,n}).$$

Moreover, for every $t < t^+(s,u_0)$, we have $u_n \to u$ in $BC((s,t],F)$ with respect to the metric d_0 (or d_1).

Proof. We only need to verify the assumptions of Theorem 2.3.3 that $G_n(u) \to G(u)$ in $BC((s,T],F)$ for all $u \in BC((s,T],F)$ with respect to the metric d_0 (or d_1). The statement of the theorem then follows from Theorem 2.3.4.

Fix $s \in [0,T)$ and $u \in BC((s,T],F)$. By our assumptions, we have $U_n(\cdot,s)u_{0n} \to U(\cdot,s)u_0$ in $BC((s,T],F)$ with respect to d_0 (or d_1). Hence, we only need to show that

$$H_n(u)(t) := \int_s^t U_n(t,\tau)f_n(\tau,u(\tau))d\tau \to H(u)(t) := \int_s^t U(t,\tau)f(\tau,u(\tau))d\tau$$

in F uniformly with respect to $t \in [s,T]$, that is with respect to the metric d_0 and hence d_1. If we set $V_n(t,\tau) := U_n(t,\tau) - U(t,\tau)$, $g_n(\tau) := f_n(\tau,u(\tau))$ and $g(\tau) := f(\tau,u(\tau))$, then

$$H_n(u)(t) - H(u)(t) = \int_s^t U_n(t,\tau)(g_n(\tau) - g(\tau))d\tau + \int_s^t V_n(t,\tau)g(\tau)d\tau.$$

By the heat kernel estimates on U_n and U, we know that $\|U_n(t,\tau)\|_{\mathcal{L}(E,F)} \leq C(t-\tau)^{-\gamma}$ and $\|V_n(t,\tau)\|_{\mathcal{L}(E,F)} \leq 2C(t-\tau)^{-\gamma}$ for all $(t,\tau) \in \dot{\Delta}_T$. Since u is a bounded function, we conclude from the assumptions on f_n and f that $g_n(\tau) \to g(\tau)$ in E pointwise on $\tau \in (s,T]$. Moreover, for every $\tau \in (s,T]$, we have

$$\begin{aligned}\|g_n(\tau)\|_E &\leq \|f_n(\tau,u(\tau)) - f_n(\tau,0)\|_E + \|f_n(\tau,0)\|_E \\ &\leq k_R\|u(\tau) - 0\|_F + \|f_n(\tau,0)\|_E \\ &\leq k_R \sup_{\tau\in(s,T]} \|u(\tau)\|_F + \sup\{\|f_n(\tau,0)\|_E : \tau \in [0,T], n \in \mathbb{N}\}.\end{aligned}$$

If we set $\|g\|_\infty := \sup_{\tau\in(s,T]} \|g(\tau)\|_E$ and $\|g_n\|_\infty := \sup_{\tau\in(s,T]} \|g_n(\tau)\|_E$, then it follows from Assumption 4.3 that $\sup_{n\in\mathbb{N}} \|g_n\|_\infty < \infty$. We first show that

$$\int_s^t U_n(t,\tau)(g_n(\tau) - g(\tau))d\tau \to 0$$

in F uniformly with respect to $t \in [s,T]$. To do so, it suffices to show that for every sequence $(t_n)_{n\in\mathbb{N}}$ in $[s,T]$, we have that

$$\left\|\int_s^{t_n} U_n(t_n,\tau)(g_n(\tau) - g(\tau))d\tau\right\|_F \to 0$$

as $n \to \infty$. Indeed, we only need to show that for every sequence $(t_n)_{n\in\mathbb{N}}$ in $[s,T]$, if (t_{n_k}) is an arbitrary subsequence, then we can find a further subsequence (denoted again by t_{n_k}) such that

$$\left\|\int_s^{t_{n_k}} U_{n_k}(t_{n_k},\tau)(g_{n_k}(\tau) - g(\tau))d\tau\right\|_F \to 0 \tag{2.17}$$

as $k \to \infty$. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[s,T]$ and $(t_{n_k})_{k\in\mathbb{N}}$ be an arbitrary subsequence. Since t_{n_k} is a bounded sequence and $[s,T]$ is compact, there exists a convergent subsequence (denoted again by t_{n_k}) such that $t_{n_k} \to t_0$ as $k \to \infty$ for some $t_0 \in [s,T]$. If $t_0 = s$, then

$$\begin{aligned}&\left\|\int_s^{t_{n_k}} U_{n_k}(t_{n_k},\tau)(g_{n_k}(\tau) - g(\tau))d\tau\right\|_F \\ &\leq \int_{t_0}^{t_{n_k}} C(t_{n_k} - \tau)^{-\gamma}d\tau \left(\sup_{n\in\mathbb{N}} \|g_n\|_\infty + \|g\|_\infty\right) \\ &= C\frac{(t_{n_k} - t_0)^{1-\gamma}}{1-\gamma} \left(\sup_{n\in\mathbb{N}} \|g_n\|_\infty + \|g\|_\infty\right) \\ &\to 0,\end{aligned}$$

as $k \to \infty$. Hence, we assume $t_0 \in (s,T]$. Fix $\varepsilon > 0$ arbitrary and choose $\eta \in (0, t_0 - s)$ such that

$$C\frac{(2\eta)^{1-\gamma}}{1-\gamma} \left(\sup_{n\in\mathbb{N}} \|g_n\|_\infty + \|g\|_\infty\right) < \frac{\varepsilon}{2}.$$

By the convergence of t_{n_k}, we can find $K_1 \in \mathbb{N}$ such that $t_0 - \eta < t_{n_k} < t_0 + \eta$ for all $k > K_1$. We can write

$$\int_s^{t_{n_k}} U_{n_k}(t_{n_k}, \tau)(g_{n_k}(\tau) - g(\tau))d\tau = \int_s^{t_0-\eta} U_{n_k}(t_{n_k}, \tau)(g_{n_k}(\tau) - g(\tau))d\tau + \int_{t_0-\eta}^{t_{n_k}} U_{n_k}(t_{n_k}, \tau)(g_{n_k}(\tau) - g(\tau))d\tau,$$

for all $k > K_1$. By the choice of η, we have

$$\begin{aligned}\left\| \int_{t_0-\eta}^{t_{n_k}} U_{n_k}(t_{n_k}, \tau)(g_{n_k}(\tau) - g(\tau))d\tau \right\|_F &\leq \int_{t_0-\eta}^{t_{n_k}} C(t_{n_k} - \tau)^{-\gamma} d\tau \left(\sup_{n\in\mathbb{N}} \|g_n\|_\infty + \|g\|_\infty \right) \\ &= C\frac{(t_{n_k} - (t_0 - \eta))^{1-\gamma}}{1-\gamma} \left(\sup_{n\in\mathbb{N}} \|g_n\|_\infty + \|g\|_\infty \right) \\ &\leq C\frac{(2\eta)^{1-\gamma}}{1-\gamma} \left(\sup_{n\in\mathbb{N}} \|g_n\|_\infty + \|g\|_\infty \right) \\ &< \frac{\varepsilon}{2},\end{aligned} \tag{2.18}$$

for all $k > K_1$. In addition, for all $\tau \in (s, t_0 - \eta)$, we have

$$\begin{aligned}\|U_{n_k}(t_{n_k}, \tau)(g_{n_k}(\tau) - g(\tau))\|_F &\leq C(t_{n_k} - \tau)^{-\gamma} \|g_{n_k}(\tau) - g(\tau)\|_E \\ &\leq C(t_{n_k} - (t_0 - \eta))^{-\gamma} \|g_{n_k}(\tau) - g(\tau)\|_E\end{aligned} \tag{2.19}$$

for all $k > K_1$. By the pointwise convergence of g_n and the convergence of t_{n_k}, we get

$$C(t_{n_k} - (t_0 - \eta))^{-\gamma} \|g_{n_k}(\tau) - g(\tau)\|_E \to C\eta^{-\gamma}(0) = 0,$$

for all $\tau \in (s, t_0 - \eta)$ as $k \to \infty$. Hence, $U_{n_k}(t_{n_k}, \tau)(g_{n_k}(\tau) - g(\tau)) \to 0$ in F pointwise on $\tau \in (s, t_0 - \eta)$. We can also estimate (2.19) for $\tau \in (s, t_0 - \eta)$ by

$$\begin{aligned}\|U_{n_k}(t_{n_k}, \tau)(g_{n_k}(\tau) - g(\tau))\|_F &\leq C(t_{n_k} - \tau)^{-\gamma} \|g_{n_k}(\tau) - g(\tau)\|_E \\ &\leq C((t_0 - \eta) - \tau)^{-\gamma} \left(\sup_{n\in\mathbb{N}} \|g_n\|_\infty + \|g\|_\infty \right),\end{aligned}$$

for all $k > K_1$. Since $\int_s^{t_0-\eta} C((t_0 - \eta) - \tau)^{-\gamma} \left(\sup_{n\in\mathbb{N}} \|g_n\|_\infty + \|g\|_\infty\right) d\tau < \infty$, we conclude from the Dominated Convergence Theorem that

$$\int_s^{t_0-\eta} U_{n_k}(t_{n_k}, \tau)(g_{n_k}(\tau) - g(\tau))d\tau \to 0$$

in F as $k \to \infty$. Thus, we can find $K_2 > K_1$ in $\mathbb{N}$ such that

$$\left\| \int_s^{t_0-\eta} U_{n_k}(t_{n_k}, \tau)(g_{n_k}(\tau) - g(\tau))d\tau \right\|_F < \frac{\varepsilon}{2}, \tag{2.20}$$

for all $k > K_2$. By (2.18) and (2.20), we get

$$\left\| \int_s^{t_{n_k}} U_{n_k}(t_{n_k}, \tau)(g_{n_k}(\tau) - g(\tau))d\tau \right\|_F \leq \varepsilon,$$

for all $k > K_2$. As ε was arbitrary, we obtain (2.17) and thus the required uniform convergence holds.

It remains to show $\int_s^t V_n(t,\tau)g(\tau)d\tau \to 0$ in F unifromly with respect to $t \in [s,T]$. We follow the argument in [38]. Fix $\varepsilon > 0$ and choose $\eta \in (0, T-s)$ such that

$$\frac{2C\|g\|_\infty}{1-\gamma}\eta^{1-\gamma} < \frac{\varepsilon}{2}. \tag{2.21}$$

For $t \in [s+\eta, T]$, we have

$$\int_s^t \|V_n(t,\tau)g(\tau)\|_F d\tau \leq \int_s^{t-\eta} \|V_n(t,\tau)g(\tau)\|_F d\tau + \int_{t-\eta}^t \|V_n(t,\tau)g(\tau)\|_F d\tau.$$

By the definition of V_n, the heat kernel estimates and (2.21), we have

$$\begin{aligned}\int_{t-\eta}^t \|V_n(t,\tau)g(\tau)\|_F d\tau &\leq 2C\|g\|_\infty \int_{t-\eta}^t (t-\tau)^{-\gamma} d\tau \\ &\leq \frac{2C\|g\|_\infty}{1-\gamma}\eta^{1-\gamma} \\ &< \frac{\varepsilon}{2},\end{aligned}$$

for all $t \in [s+\eta, T]$. Hence,

$$\int_s^t \|V_n(t,\tau)g(\tau)\|_F d\tau \leq \int_s^{t-\eta} \|V_n(t,\tau)g(\tau)\|_F d\tau + \frac{\varepsilon}{2},$$

for all $t \in [s+\eta, T]$. Since $u \in BC((s,T], F)$, we have $g \in C([s+\eta, T], E)$. It follows that the set $\{g(\tau) : \tau \in [s+\eta, T]\}$ is compact in E. We also have that the set

$$K := \{(t,\tau) : s+\eta \leq t \leq T \text{ and } s \leq \tau \leq t\}$$

is a compact subset of $\dot{\Delta}_T$. By our assumptions, $V_n(t,\tau) \to 0$ in $\mathcal{L}_s(E,F)$ uniformly with respect to $(t,\tau) \in K$. Since strongly converging operators converge uniformly on compact sets, we can find $N_0 \in \mathbb{N}$ such that

$$\|V_n(t,\tau)g(\tau)\|_F \leq \frac{\varepsilon}{2T},$$

for all $(t,\tau) \in K$ and for all $n > N_0$. It follows that

$$\int_s^t \|V_n(t,\tau)g(\tau)\|_F d\tau \leq \int_s^{t-\eta} \frac{\varepsilon}{2T} d\tau + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

for all $t \in [s+\eta, T]$ and for all $n > N_0$. By the choice of η in (2.21), we obtain from the heat kernel estimates that

$$\begin{aligned}\int_s^t \|V_n(t,\tau)g(\tau)\|_F d\tau &\leq 2C\|g\|_\infty \int_s^t (t-\tau)^{-\gamma} d\tau \\ &\leq \frac{2C\|g\|_\infty}{1-\gamma}\eta^{1-\gamma} \\ &< \frac{\varepsilon}{2},\end{aligned}$$

for all $t \in [s, s+\eta]$ for all $n \in \mathbb{N}$. Therefore,

$$\int_s^t \|V_n(t,\tau)g(\tau)\|_F d\tau \leq \varepsilon,$$

for all $t \in [s,T]$ and for all $n > N_0$. Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\int_s^t \|V_n(t,\tau)g(\tau)\|_F d\tau \to 0$$

uniformly with respect to $t \in [s,T]$. This completes the proof of the theorem. □

2.4 Convergence of solutions of semilinear parabolic equations

In this section, we concentrate on semilinear parabolic equations. We apply the convergence results under the perturbation of evolution equations studied in Section 2.3 to deal with domain perturbation for semilinear parabolic equations.

2.4.1 Non-autonomous semilinear parabolic equations

Let $T > 0$ and $s \in [0,T)$. As before, we assume that Ω_n and Ω satisfy Assumption 1.4.1. Consider the following non-autonomous parabolic equation

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}(t)u = g(x,t,u) & \text{in } \Omega \times (s,T] \\ \mathcal{B}(t)u = 0 & \text{on } \partial\Omega \times (s,T] \\ u(\cdot,s) = u_0 & \text{in } \Omega, \end{cases} \tag{2.22}$$

where $\mathcal{A}(t)$ and $\mathcal{B}(t)$ are defined by (0.5) and (0.6), respectively. By taking the evolution triple as in (1.11), we may write (2.22) as an abstract equation of the form

$$\begin{cases} u'(t) + A(t)u(t) = f(t,u(t)) & \text{for } t \in (s,T] \\ u(s) = u_0, \end{cases} \tag{2.23}$$

where $f(t,u)(x) := g(t,x,u(x))$ is the substitution operator induced by g and $A(t) \in \mathscr{L}(V, V')$ is the operator induced by $a(t,\cdot,\cdot)$ defined in (1.10).

Since we have an improved convergence result in Theorem 2.3.5, we can consider a more general perturbation result. We study the following perturbation of (2.22)

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}_n(t)u = g_n(x,t,u) & \text{in } \Omega_n \times (s,T] \\ \mathcal{B}_n(t)u = 0 & \text{on } \partial\Omega_n \times (s,T] \\ u(\cdot,s) = u_{0,n} & \text{in } \Omega_n, \end{cases} \tag{2.24}$$

where $\mathcal{A}_n(t)$ and $\mathcal{B}_n(t)$ are defined by (1.22) and (1.23), respectively. As usual, we write (2.24) in the abstract form as

$$\begin{cases} u'(t) + A_n(t)u(t) = f_n(t,u(t)) & \text{for } t \in (s,T] \\ u(s) = u_{0,n}, \end{cases} \tag{2.25}$$

where $f_n(t,u)(x) := g_n(t,x,u(x))$ is the substitution operator induced by g_n and $A_n(t) \in \mathscr{L}(V_n, V_n')$ is the operator induced by $a_n(t,\cdot,\cdot)$ defined in (1.18).

Note that here we allow a perturbation g_n of the nonlinear term g, whereas in [35] the nonlinear term is fixed (see [35, Equation (6.17)]). Also, the application in domain perturbation in [38] is given for *autonomous* equations with a fixed nonlinear term (see [38, Equation (6.1)]). Possibly, the nonlinear terms are fixed under perturbation in [35] and [38] in order to have a local *uniform convergence* of the substitution operators without any further conditions on g_n and g. Here, we only need that g_n converges to g pointwise to get the pointwise convergence of the substitution operators.

We see from Theorem 1.1.2 that the evolution system $U_n(\cdot,\cdot)$ corresponding to the family $(A_n(t))_{t\in[0,T]}$ exists for all $n \in \mathbb{N}$ and $U_n(\cdot,\cdot) \in \mathscr{L}(L^2(\Omega_n))$. Similarly, the evolution system $U(\cdot,\cdot)$ corresponding to the family $(A(t))_{t\in[0,T]}$ exists and $U(\cdot,\cdot) \in \mathscr{L}(L^2(\Omega))$. Since the perturbation result in Section 2.3 is proved for evolution systems in a *fixed* Banach space E, we consider $i_nU_n(\cdot,\cdot)r_n \in \mathscr{L}(L^2(D))$ and $iU(\cdot,\cdot)r \in \mathscr{L}(L^2(D))$ where i_n and i are inclusion maps defined by (1.27) and (1.28) respectively, and r_n and r are restriction maps defined by (1.29) and (1.30), respectively. In particular, we take $E := L^2(D)$ and $F := L^2(D)$. Clearly, $iU(\cdot,\cdot)r$ satisfies Assumption 2.1.2 with $\gamma = 0$ in (2.4). Moreover, $i_nU_n(\cdot,\cdot)r_n$ satisfies Assumption 2.3.1 (i) – (iii) with $\gamma = 0$ in (2.13). We show that Assumption 2.3.1 (iv) is satisfied in the following lemma.

Lemma 2.4.1. *Suppose that $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco for the case of Dirichlet problems; or $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco* *with $|\Omega_n| \to |\Omega|$ for the case of Neumann problems. Then the corresponding evolution systems satisfy*

$$i_n U_n(t,s) r_n \to i U(t,s) r$$

in $\mathcal{L}_s(L^2(D))$ for all $(t,s) \in \dot{\Delta}_T$.

Indeed, $i_n U_n(\cdot,\cdot) r_n \to i U(\cdot,\cdot) r$ in $\mathcal{L}_s(L^2(D))$ uniformly with respect to $(t,s) \in \dot{\Delta}_T$, that is

$$i_n U_n(\cdot,\cdot) r_n \to i U(\cdot,\cdot) r$$

in $C(\dot{\Delta}_T, \mathcal{L}_s(L^2(D)))$.

Proof. Let $u_0 \in L^2(D)$ be arbitrary. By the definition of evolution systems $u_n(t) := U_n(t,s) r_n u_0$ is the unique (weak) solution of (1.25) with $f_n = 0 \in L^2((0,T), L^2(\Omega_n))$ satisfying the initial condition $r_n u_0$ at time s. Similarly, $u(t) := U(t,s) r u_0$ is the unique (weak) solution of (1.26) with $f = 0 \in L^2((0,T), L^2(\Omega))$ satisfying the initial condition $r u_0$ at time s. Clearly, we have $i_n f_n \to i f$ in $L^2((0,T), L^2(D))$ strongly as $n \to \infty$. We consider first the case of Dirichlet problems. Since we have $\operatorname{cap}(K \cap \Omega_n^c) \to 0$ for all $K \subset\subset \Omega$ from Mosco convergence (Lemma 1.4.8), it is easy to see that $r i_n r_n u_0 \to r u_0$ in $L^2(\Omega)$ strongly. As the initial time that we start with is irrelevant, we conclude from Theorem 1.4.9 that $i_n u_n(t) \to i u(t)$ in $L^2(D)$ uniformly on $(s,T]$. Since the argument holds for all $u_0 \in L^2(D)$ and $s \in [0,T)$, this implies $i_n U_n(t,s) r_n \to i U(t,s) r$ in $\mathcal{L}_s(L^2(D))$ for all $(t,s) \in \dot{\Delta}_T$ and $i_n U_n(\cdot,s) r_n \to i U(\cdot,s) r$ in $C((s,T], \mathcal{L}_s(L^2(D)))$ for all $s \in [0,T)$. Again, as the initial time that we start with is irrelevant, we also have $i_n U_n(t,\cdot) r_n \to i U(t,\cdot) r$ in $C([0,t), \mathcal{L}_s(L^2(D)))$ for all $t \in (0,T]$. Hence, the assertion of the lemma for Dirichlet problems follows.

Similarly, we use the assumption that $|\Omega_n| \to |\Omega|$ for Neumann problems to get $i_n r_n u_0 \to i r u_0$ in $L^2(D)$ strongly. We conclude from Theorem 1.4.16 that $i_n u_n(t) \to i u(t)$ in $L^2(D)$ uniformly on $(s,T]$. By a similar argument as in the case of Dirichlet problems, we can deduce the assertion of the lemma for Neumann problems. □

We require the following conditions on the nonlinearity.

Assumption 2.4.2. We assume that $g_n, g \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R})$ satisfy the following conditions:

(i) There exist $c > 0$ and $\psi \in L^2(\mathbb{R}^N)$ such that

$$\begin{aligned} |g_n(t,x,\xi)| &\le \psi(x) + c|\xi| \\ |g(t,x,\xi)| &\le \psi(x) + c|\xi|, \end{aligned} \tag{2.26}$$

for all $x \in \mathbb{R}^N$, $\xi \in \mathbb{R}$ and $n \in \mathbb{N}$.

(ii) There exists an essentially bounded function ϕ such that

$$\begin{aligned} |g_n(x,t,\xi_1) - g_n(x,t,\xi_2)| &\leq \phi(x,\zeta)|\xi_1 - \xi_2| \\ |g(x,t,\xi_1) - g(x,t,\xi_2)| &\leq \phi(x,\zeta)|\xi_1 - \xi_2|, \end{aligned} \tag{2.27}$$

for all $|\xi_1|, |\xi_2| \leq \zeta$ and all $n \in \mathbb{N}$.

(iii) g_n converges to g pointwise on $\mathbb{R}^N \times [0,T] \times \mathbb{R}$.

It is well known that the condition (2.26) is a necessary and sufficient condition for the substitution operators $f_n(t,\cdot)$ defined by $f_n(t,u)(x) := g_n(x,t,u(x))$ and $f(t,\cdot)$ defined by $f(t,u)(x) := g(x,t,u(x))$ to be in $C(L^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$ ([3, Chapter 3]). Moreover, by condition (2.27), we have $f_n(t,\cdot)$ and $f(t,\cdot)$ are locally Lipschitz in u uniformly with respect to $t \in [0,T]$ and the Lipschitz constants are independent of $n \in \mathbb{N}$ (see [3, Theorem 3.10]). Therefore, the substitution operators $f_n(t,\cdot)$ and $f(t,\cdot)$ satisfy Assumption 2.3.2 and Assumption 2.2.3 respectively with the Lipschitz constant k_R independent of $n \in \mathbb{N}$.

Let $u_{0,n} \in L^2(\Omega_n)$ and $u_0 \in L^2(\Omega)$. By Theorem 2.2.7, there exist $t_n^+(s,u_{0,n})$ and $t^+(s,u_0)$ such that the semilinear equation (2.25) has a unique maximal solution u_n in $BC((s,t_n^+(s,u_{0,n})), L^2(\Omega_n))$ and (2.23) has a unique maximal solution u in $BC((s,t^+(s,u_0)), L^2(\Omega))$. In fact, the condition (2.26) implies that the substitution operators f_n and f are linearly bounded. More precisely, for any $u_n \in L^2(\Omega_n)$ and $u \in L^2(\Omega)$, we have

$$\begin{aligned} \|f_n(t,u_n)\|_{L^2(\Omega_n)} &\leq \|\psi\|_{L^2(\Omega_n)} + c\|u_n\|_{L^2(\Omega_n)} \\ \|f(t,u)\|_{L^2(\Omega)} &\leq \|\psi\|_{L^2(\Omega)} + c\|u\|_{L^2(\Omega)}. \end{aligned}$$

Hence, the solutions exist globally, that is $t_n^+(s,u_{0,n}) = t^+(s,u_0) = \infty$ for all $n \in \mathbb{N}$.

We are now in the position to state the convergence of solutions of semilinear parabolic equations under domain perturbation.

Theorem 2.4.3. *Let Ω_n and Ω be uniformly bounded domains in $\mathbb{R}^N$ with $\Omega_n, \Omega \subset D$. Suppose that g_n and g satisfy Assumption 2.4.2. Suppose also that $i_n u_{0,n}$ is uniformly bounded in $L^2(D)$.*

(i) **Dirichlet problems:** *Assume that $r i_n u_{0,n} \rightharpoonup u_0$ in $L^2(\Omega)$ weakly. If $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco, then $i_n u_n \to iu$ in $C((s,t_0], L^2(D))$ for all $t_0 \in (s,\infty)$.*

In addition, if $i_n u_{0,n} \rightharpoonup iu_0$ in $L^2(D)$ strongly, then $i_n u_n \to iu$ in $C([s,t_0], L^2(D))$.

(i) **Neumann problems:** *Assume that $i_n u_{0,n} \to iu_0$ in $L^2(D)$ strongly. If $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco and $|\Omega_n| \to |\Omega|$ as $n \to \infty$, then $i_n u_n \to iu$ in $C([s,t_0], L^2(D))$ for all $t_0 \in (s,\infty)$.*

Proof. By Assumption 2.4.2 (iii), it is easy to conclude from the dominated convergence theorem that the substitution operator $f_n(\tau, v)$ converges pointwise to $f(\tau, v)$ in $L^2(D)$ for all $(\tau, v) \in [0,T] \times L^2(D)$. For Dirichlet problems, we can see from the convergence of solutions under domain perturbation in Theorem 1.4.9 that

$$i_n U_n(\cdot, s) u_{0,n} \to iU(\cdot, s) u_0 \tag{2.28}$$

in $C((s,T], L^2(D))$ if $ri_n u_{0,n} \rightharpoonup u_0$ in $L^2(\Omega)$ weakly. The convergence (2.28) above holds in $C([s,T], L^2(D))$ if $i_n u_{0,n} \to iu_0$ in $L^2(D)$ strongly. Taking Lemma 2.4.1 into account, we have that all assumptions in Theorem 2.3.5 are satisfied. Hence, the assertion of the theorem follows from Theorem 2.3.5 with $E = F = L^2(D)$. The statement for Neumann problems follows in the same way by using Theorem 1.4.16 instead of Theorem 1.4.9. ☐

Remark 2.4.4. We could study the convergence results in the L^p space as done in [35] and [38]. In the case of Dirichlet problems, the evolution system $U_n(\cdot,\cdot)$ corresponding to the family $(A_n(t))_{t\in[0,T]}$ exists for all $n \in \mathbb{N}$. Moreover, $U_n(\cdot,\cdot) \in \mathscr{L}(L^p(\Omega_n), L^q(\Omega_n))$ satisfies the same estimate as in (2.3) uniformly with respect to $n \in \mathbb{N}$, that is

$$\|U_n(t,s)\|_{\mathscr{L}(L^p(\Omega_n), L^q(\Omega_n))} \le C(t-s)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}, \tag{2.29}$$

for all $1 \le p \le q \le \infty$ where C is independent of $n \in \mathbb{N}$. For Neumann problems, the estimate (2.3) holds if Ω satisfies the cone condition. However, the constant C depends on the cone (see [36]). Thus, in order to have the uniform estimate (2.29) with respect to $n \in \mathbb{N}$, a sufficient condition is that Ω_n satisfies the cone condition for all $n \in \mathbb{N}$, where the same cone works for all domains Ω_n and Ω. Examples of domains satisfying this condition include the domains obtained by rotating a line segment inside an open set and the domains obtained by cutting a line into a unit disk as previously seen in Figure 1.3 and Figure 1.4, respectively. By taking $E := L^p(D)$ and $F := L^q(D)$ with $1 < p \le q < \infty$ satisfying (2.5), we see that Assumptions 2.3.1 (i) – (iii) are satisfied. To verify Assumption 2.3.1 (iv), (2.15) and (2.16), we need the L^p-theory of domain perturbation for linear parabolic equations. This can be done in the same way as in [35, Section 4] by using the interpolation arguments and the convergence results in

L^2. Note also that we need to adjust conditions (2.26) and (2.27) in order to have the substitution operators acting on $L^p(\mathbb{R}^N)$ into $L^q(\mathbb{R}^N)$. We refer to [3] for a necessary and sufficient condition. In this case, the maximal existence time $t^+(s, u_0)$ may be strictly finite. An application of Theorem 2.3.5 gives us a lower semicontinuity of the maximal existence time, that is $t^+(s, u_0) \leq \liminf_{n\to\infty} t_n^+(s, u_{0,n})$ and the convergence of solutions in $C((s, t_0], L^q(D))$ for all $t_0 \in (s, t^+(s, u_0))$.

2.4.2 Autonomous semilinear parabolic equations

Although autonomous problems can be regarded as a special case of non-autonomous problems with the equations (operators) do not depend on the time variable and the corresponding evolutions systems are just *semigroups*. We spend some time in this section to study domain perturbation for autonomous problems. In particular, we intend to discuss the convergence of semigroups and the spectrum of the corresponding elliptic operators. These results are discussed in [38, Section 6] for the heat equations. We give here domain perturbation results for parabolic equations in the abstract form using some results from [39].

We consider the following autonomous semilinear initial boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}u = g(x, u) & \text{in } \Omega \times (s, T] \\ \mathcal{B}u = 0 & \text{on } \partial\Omega \times (s, T] \\ u(\cdot, 0) = u_0 & \text{in } \Omega, \end{cases} \tag{2.30}$$

where $g \in C(\mathbb{R}^N \times \mathbb{R})$, $\mathcal{A}$ is an elliptic operator of the form

$$\mathcal{A}u := -\partial_i[a_{ij}(x)\partial_j u + a_i(x)u] + b_i(x)\partial_i u + c_0(x)u \tag{2.31}$$

and $\mathcal{B}$ is one of the following boundary conditions

$$\begin{aligned} &\mathcal{B}u := u && \text{Dirichlet boundary condition} \\ &\mathcal{B}u := [a_{ij}(x)\partial_j u + a_i(x)u]\,\nu_i && \text{Neumann boundary condition.} \end{aligned} \tag{2.32}$$

We assume that a_{ij}, a_i, b_i and c_0 are functions in $L^\infty(\mathbb{R}^N)$ and that there exists a constant $\alpha > 0$ independent of $x \in \Omega$ such that

$$a_{ij}(x)\xi_i\xi_j \geq \alpha_0|\xi|^2, \tag{2.33}$$

for all $\xi \in \mathbb{R}^N$. We consider the evolution triple $V \overset{d}{\hookrightarrow} H \overset{d}{\hookrightarrow} V'$ chosen as in 1.11. Define a bilinear form $a(\cdot, \cdot)$ associated with $\mathcal{A}$ on V by

$$a(u, v) := \int_\Omega [a_{ij}(x)\partial_j u + a_i(x)u]\partial_i v + b_i(x)\partial_i uv + c_0(x)uvdx, \tag{2.34}$$

for $u, v \in V$. Let

$$\lambda_{\mathcal{A}} := \|c_0^-\|_\infty + \frac{1}{2\alpha_0}\sum_{i=1}^N \|a_i + b_i\|_\infty,$$

where $c_0^- := \max(-c_0, 0)$ is the negative part of c_0. We set

$$\lambda_0 := \lambda_{\mathcal{A}} + \frac{\alpha_0}{2}. \tag{2.35}$$

It can be verified that $a(\cdot,\cdot)$ is continuous and

$$a(u,u) + \lambda\|u\|_H^2 \geq \frac{\alpha_0}{2}\|u\|_V^2, \tag{2.36}$$

for all $u \in V$ and for all $\lambda \geq \lambda_0$ (see [39, Proposition 2.1.6]). By the Lax-Milgram theorem, there exists $A_\Omega \in \mathscr{L}(V, V')$ such that

$$a(u,v) = \langle A_\Omega u, v\rangle, \tag{2.37}$$

for all $u, v \in V$. We may consider A_Ω as an operator on V' with the domain V. A standard argument shows that $[\lambda_0, \infty) \subset \varrho(-A_\Omega)$ (see [39, Theorem 2.2.2]). Let A be the *maximal restriction* of the operator A_Ω on $H = L^2(\Omega)$. We have that $[\lambda_0, \infty) \subset \varrho(-A_\Omega) \subset \varrho(-A)$. It is well known that $-A$ generates a strongly continuous analytic semigroup $S(t), t \geq 0$ on $L^2(\Omega)$ (see [41, Proposition 3, XVII §6]). Moreover, there exist $C > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\|_{\mathscr{L}(L^2(\Omega))} \leq Ce^{\omega t},$$

for $t \geq 0$. We consider (2.30) as an abstract equation of the form

$$\begin{cases} \dot{u}(t) + Au(t) = f(u(t)) & t \in (0,\infty) \\ u(0) = u_0 \end{cases} \tag{2.38}$$

in $L^2(\Omega)$, where $f(u)(x) := g(x, u(x))$ is the substitution operator induced by g.

We study perturbation of (2.30) given by

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}_n u = g_n(x,u) & \text{in } \Omega_n \times (s,T] \\ \mathcal{B}_n u = 0 & \text{on } \partial\Omega_n \times (s,T] \\ u(\cdot, 0) = u_{0,n} & \text{in } \Omega_n, \end{cases} \tag{2.39}$$

where $g_n \in C(\mathbb{R}^N \times \mathbb{R})$. As usual, we assume that the perturbed domains Ω_n and Ω satisfy Assumption 1.4.1. The operator $\mathcal{A}_n$ above is an elliptic operator of the form

$$\mathcal{A}_n u := -\partial_i[a_{ij,n}(x)\partial_j u + a_{i,n}(x)u] + b_{i,n}(x)\partial_i u + c_{0,n}(x)u, \tag{2.40}$$

where $a_{ij,n}, a_{i,n}, b_{i,n}$ and $c_{0,n}$ are functions in $L^\infty(\mathbb{R}^N)$ and $a_{ij,n}$ satisfies the ellipticity condition as in (2.33) with the *uniform ellipticity constant* for all $n \in \mathbb{N}$. The operator $\mathcal{B}_n$ is either Dirichlet boundary condition or Neumann boundary condition defined similarly to (2.32). We assume that the coefficients of $\mathcal{A}_n$ converges to the corresponding coefficient of $\mathcal{A}$.

Assumption 2.4.5. For all $i, j = 1, \dots, N$, we assume that $\lim_{n\to\infty} a_{ij,n} = a_{ij}$, $\lim_{n\to\infty} a_{i,n} = a_i, \lim_{n\to\infty} b_{i,n} = b_i$ and $\lim_{n\to\infty} c_{0,n} = c_0$ in $L^\infty(\mathbb{R}^N)$.

As before, the evolution triple $V_n \overset{d}{\hookrightarrow} H_n \overset{d}{\hookrightarrow} V_n'$ is chosen according to the boundary condition (either $V_n = H_0^1(\Omega_n)$ or $V_n = H^1(\Omega_n)$). In the abstract form, we can write (2.39) as

$$\begin{cases} \dot{u}(t) + A_n u(t) = f_n(u(t)) & t \in (s, T] \\ u(0) = u_{0,n} \end{cases} \tag{2.41}$$

in $L^2(\Omega_n)$, where $f_n(u)(x) := g_n(x, u(x))$ is the substitution operator induced by g_n. In the above, A_n is the maximal restriction operator on $H_n = L^2(\Omega_n)$ of the operator induced by the associated bilinear form. By Assumption 2.4.5 and the assumption that $a_{ij,n}$ satisfies (2.33) uniformly with respect to $n \in \mathbb{N}$, we can choose $\lambda_0 := \lambda_{\mathcal{A}} + \alpha_0/2$ where

$$\lambda_{\mathcal{A}} := \max\left\{\sup_{n\in\mathbb{N}}\left\{\|c_{0,n}^-\|_\infty + \frac{1}{2\alpha_0}\sum_{i=1}^N \|a_{i,n} + b_{i,n}\|_\infty\right\}, \left(\|c_0^-\|_\infty + \frac{1}{2\alpha_0}\sum_{i=1}^N \|a_i + b_i\|_\infty\right)\right\} < \infty \tag{2.42}$$

so that

$$a_n(u, u) + \lambda\|u\|_{H_n}^2 \geq \frac{\alpha_0}{2}\|u\|_{V_n}^2, \tag{2.43}$$

for all $\lambda \geq \lambda_0$, for all $u \in V_n$ and for all $n \in \mathbb{N}$. It follows from [39, Theorem 2.2.2] that $[\lambda_0, \infty) \subset \varrho(-A_n) \cap \varrho(-A)$ for all $n \in \mathbb{N}$. Moreover, the operator $-A_n$ is the generator of a strongly continuous analytic semigroup $S_n(t), t \geq 0$ on $L^2(\Omega_n)$ with the uniform exponential estimate

$$\|S_n(t)\|_{\mathscr{L}(L^2(\Omega_n))} \leq Ce^{\omega t}, \tag{2.44}$$

for all $t \geq 0$ and for all $n \in \mathbb{N}$.

We impose Assumption 2.4.2 on the nonlinearities g_n and g in $C(\mathbb{R}^N \times [0, T] \times \mathbb{R})$ (where we disregard the time variable). We note that this means the substitution operators induced by g_n and g are in $C(L^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$ and locally Lipschitz with the Lipschitz constant independent of $n \in \mathbb{N}$. Indeed, for every $R > 0$ there exists k_R independent of $n \in \mathbb{N}$ such that

$$\|f_n(u) - f_n(v)\|_{L^2(\Omega_n)} \leq k_R \|u - v\|_{L^2(\Omega_n)},$$

for all $u, v \in L^2(\Omega_n)$ with $\|u\|_{L^2(\Omega_n)}, \|v\|_{L^2(\Omega_n)} \leq R$ and for all $n \in \mathbb{N}$. In addition, we have

$$f_n(v) \to f(v)$$

in $L^2(D)$ pointwise for all $v \in L^2(D)$.

By Theorem 2.2.7 (or simply using the existence and uniqueness results for autonomous problems in [66]), we have that for every $u_0 \in L^2(\Omega)$ the equation (2.38) has a unique maximal solution $u \in C((s, t^+(s, u_0)), L^2(\Omega))$. Moreover, u can be represented by the variation of constants formula

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau) f(u(\tau)) d\tau. \tag{2.45}$$

Similarly, for every $u_{0,n} \in L^2(\Omega_n)$ the equation (2.41) has a unique maximal solution $u_n \in C((s, t_n^+(s, u_{0,n})), L^2(\Omega_n))$ which can be represented by the variation of constant formula

$$u_n(t) = S_n(t)u_{0,n} + \int_0^t S_n(t-\tau) f_n(u_n(\tau)) d\tau. \tag{2.46}$$

As noted in the case of non-autonomous equations, the conditions on f_n and f imply that u_n and u are indeed global solutions, that is $t_n^+(s, u_{0,n}) = t^+(s, u_0) = \infty$ for all $n \in \mathbb{N}$.

To deal with domain perturbation, we make use of the inclusion operators i_n and i, and the restriction operators r_n an r as in Section 2.4.1. We define a *degenerate semigroup* $\mathcal{S}_n(t)$ on $L^2(D)$ for $t \geq 0$ by

$$\mathcal{S}_n(t) := i_n S_n(t) r_n.$$

The corresponding *pseudo resolvent* is given by

$$\mathcal{R}_n(\lambda) := i_n(\lambda + A_n)^{-1} r_n$$

whenever it is defined. We define $\mathcal{S}(t)$ and $\mathcal{R}$ similarly (with n deleted).

For Dirichlet problems, the strong convergence of pseudo resolvent operators is equivalent to Mosco convergence (see [39, Theorem 5.2.4] or [37, Theorem 3.3]). For Neumann problems, we also have that Mosco convergence implies the strong convergence of pseudo resolvent operators (see [21, Proposition 3.2] for the case $A_n = -\Delta$).

It is not clear that the converse is true for the Neumann case. However, as remarked in [21, Remark 3.3], the converse follows if Ω_n converges to Ω in the Hausdorff complementary topology. We state the convergence of pseudo resolvent with respect to the strong operator topology in the following theorem.

Theorem 2.4.6. *Let Ω_n and Ω be domains satisfying Assumption 1.4.1. Suppose that $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco for the case of Dirichlet problems; or $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco for the case of Neumann problems. If $\lambda \geq \lambda_0$, then*

$$\mathcal{R}_n(\lambda)f \to \mathcal{R}(\lambda)f$$

in $L^2(D)$ as $n \to \infty$ for all $f \in L^2(D)$.

Proof. For Dirichlet problems, the statement is given in [39, Theorem 5.2.4] or [37, Theorem 3.3]. For Neumann problems, a similar argument as in the proof of [21, Proposition 3.2] for $-\Delta$ can be easily modified to a general elliptic operator in divergence form. The only nontrivial part is to prove the strong convergence. For this, we use arguments similar to the proof of [39, Proposition 5.5.2] to obtain

$$\frac{\alpha_0}{2}\|u_n - v_n\|^2_{H^1(\Omega_n)} \leq a_n(u_n - v_n, u_n - v_n) + \lambda\|u_n - v_n\|^2_{L^2(\Omega_n)} \to 0,$$

where $u_n := \mathcal{R}_n(\lambda)f$ and $v_n \in H_0^1(\Omega_n)$ is a sequence from Mosco convergence converging to $u := \mathcal{R}(\lambda)f$ (that is, $(i_n v_n, i_n \nabla v_n) \to (iu, i\nabla u)$ in $L^2(D) \times L^2(D, \mathbb{R}^N)$). Note that here we need to consider $a_n(u_n - v_n, u_n - v_n)$ in stead of $a_n(u_n - u, u_n - u)$ in the original proof because the bilinear form $a_n(\cdot,\cdot)$ is only defined on $H^1(\Omega_n)$. It is now clear that $\mathcal{R}_n(\lambda)f \to \mathcal{R}(\lambda)f$ in $L^2(D)$. □

The key result to obtain the convergence of degenerate semigroup $\mathcal{S}_n(t)$ is the following theorem.

Theorem 2.4.7 ([4, Theorem 5.2])**.** *Suppose there exists $M > 0$ such that*

$$\|\lambda\mathcal{R}_n(\lambda)\|_{\mathscr{L}(L^2(D))} \leq M,$$

for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$. If there exists $\lambda > 0$ such that $\mathcal{R}_n(\lambda) \to \mathcal{R}(\lambda)$ in the strong operator topology, then $\mathcal{S}_n(t) \to \mathcal{S}(t)$ in the strong operator topology uniformly with respect to t in compact subsets of $(0, \infty)$.

We cannot directly apply Theorem 2.4.7 to obtain the convergence of degenerate semigroups $\mathcal{S}_n(t)$ in our framework because the condition $\|\lambda\mathcal{R}_n(\lambda)\|_{\mathscr{L}(L^2(D))} \leq M$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) > 0$ does not hold for a general elliptic operator $\mathcal{A}_n$. However, we can consider the coercive form $a_{\lambda_0}(\cdot,\cdot)$ on V defined by

$$a_{\lambda_0}(u,v) := a(u,v) + \lambda_0 (u|v)_{L^2(\Omega)}, \tag{2.47}$$

for all $u, v \in V$ where $a(\cdot,\cdot)$ is given by (2.34). We define a similar coercive form $a_{\lambda_0,n}(\cdot,\cdot)$ on V_n for each $n \in \mathbb{N}$. The coercive forms $a_{\lambda_0}(\cdot,\cdot)$ and $a_{\lambda_0,n}(\cdot,\cdot)$ associate with the operators $A+\lambda_0$ and $A_n+\lambda_0$ respectively for which their negatives are generators of analytic C_0-semigroups on $L^2(\Omega)$ and $L^2(\Omega_n)$ (denoted by $T(t)$ and $T_n(t)$, respectively). Moreover, $S(t) = e^{\lambda_0 t}T(t)$ and $S_n(t) = e^{\lambda_0 t}T_n(t)$. Since the resolvent of $-(A+\lambda_0)$ satisfies the assumption of Theorem 2.4.7, we get $i_nT_n(t)r_n \to iT(t)r$ in the strong operator topology uniformly with respect to t on compact subsets of $(0,\infty)$. Hence, we obtain the strong convergence of degenerate semigroups $\mathcal{S}_n(t)$ and the convergence is uniform with respect to t on compact subsets of $(0,\infty)$. From the above discussion, we have the following result.

Theorem 2.4.8. *Suppose that the assumptions of Theorem 2.4.6 are satisfied. Then $\mathcal{S}_n \to \mathcal{S}$ in $C((0,T],\mathcal{L}_s(L^2(D)))$ for all $T > 0$.*

We state convergence of solutions under domain perturbation below.

Theorem 2.4.9. *Suppose that the assumptions of Theorem 2.4.6, Assumption 2.4.5 and Assumption 2.4.2 (with the time variable disregarded) are satisfied. Suppose also that $i_nu_{0,n}$ is uniformly bounded in $L^2(D)$. We denote the solutions of (2.38) and (2.41) by u and u_n respectively.*

(i) **Dirichlet problems:** *Assume that $ri_nu_{0,n} \rightharpoonup u_0$ in $L^2(\Omega)$ weakly. If $H^1_0(\Omega_n)$ converges to $H^1_0(\Omega)$ in the sense of Mosco, then $i_nu_n \to iu$ in $C((s,t_0],L^2(D))$ for all $t_0 \in (s,\infty)$.*

In addition, if $i_nu_{0,n} \rightharpoonup iu_0$ in $L^2(D)$ strongly, then $i_nu_n \to iu$ in $C([s,t_0],L^2(D))$.

(ii) **Neumann problems:** *Assume that $i_nu_{0,n} \to iu_0$ in $L^2(D)$ strongly. If $H^1(\Omega_n)$ converges to $H^1(\Omega)$ in the sense of Mosco, then $i_nu_n \to iu$ in $C([s,t_0],L^2(D))$ for all $t_0 \in (s,\infty)$.*

Proof. By Theorem 2.4.8, $\mathcal{S}_n \to \mathcal{S}$ in $C((0,T],\mathcal{L}_s(L^2(D)))$. For Dirichlet problems, we get from the weak convergence of $u_{0,n}$ that $\mathcal{S}_n(t)u_{0,n} \to \mathcal{S}(t)u_0$ in $L^2(D)$ uniformly with respect to t in compact subsets of $(0,\infty)$. This can be seen by considering $\mathcal{S}_n(t)u_{0,n}$

and $\mathcal{S}(t)u_0$ as solutions of homogeneous linear parabolic equations and applying the convergence results for linear parabolic equations under domain perturbation as done in the proof of Theorem 2.4.3. Therefore, the statement of theorem follows directly from Theorem 2.3.5 with $U_n(t,s) = \mathcal{S}_n(t-s)$ and $U(t,s) = \mathcal{S}(t-s)$. The assertion for Neumann problems follows in the same way. □

Remark 2.4.10. (i) We do not explicitly assume that $|\Omega_n| \to |\Omega|$ for autonomous equations under Neumann boundary condition in the above theorem. In this case, we make use of [4, Theorem 5.2] and the strong convergence of resolvent operators to get $\mathcal{S}_n \to \mathcal{S}$ in $C((0,T], \mathcal{L}_s(L^2(D)))$. However, in Theorem 2.4.3 for non-autonomous equations we require $|\Omega_n| \to |\Omega|$ to obtain (2.16) (see Lemma 2.4.1). This is because we need a strong convergence $i_n r_n u_0 \to i r u_0$ in $L^2(D)$ for a fixed function $u_0 \in L^2(D)$ in order to apply our convergence result for *non-autonomous* linear equations with Neumann boundary condition in Theorem 1.4.16.

(ii) Let $1 < p < 2$ be such that $\frac{N}{2}(\frac{1}{p} - \frac{1}{2}) \in [0,1)$. Assume that all assumptions in Theorem 2.4.9 are satisfied. If we replace (2.26) (with the time variable disregarded) by

$$|g_n(x,\xi)| \le \psi(x) + c|\xi|^{2/p}$$
$$|g(x,\xi)| \le \psi(x) + c|\xi|^{2/p}$$

where $\psi \in L^p(\mathbb{R}^N)$, then the substitution operators $f_n(t,\cdot)$ defined by $f_n(u)(x) := g_n(x,u(x))$ and $f(t,\cdot)$ defined by $f(u)(x) := g(x,u(x))$ are in $C(L^2(\mathbb{R}^N), L^p(\mathbb{R}^N))$ ([3, Chapter 3]). For Dirichlet problems, a standard heat kernel estimate (2.29) with $U_n(t,s)$ replaced by $S_n(t-s)$ holds for all $n \in \mathbb{N}$ and the constant $C > 0$ is independent of $n \in \mathbb{N}$. Taking $E := L^p(D)$ and $F := L^2(D)$, we have that $U_n(t,s) := i_n S_n(t-s) r_n$ and $U(t,s) := iS(t-s)r$ satisfy assumptions in Section 2.3 with $\gamma := \frac{N}{2}(\frac{1}{p} - \frac{1}{2})$. Unlike Theorem 2.4.9, the maximal existence time $t^+(s,u_0)$ may be strictly finite. In this case, an application of Theorem 2.3.5 implies that $t^+(s,u_0) \le \liminf_{n\to\infty} t_n^+(s,u_{0,n})$ and the weak solution $i_n u_n$ converges to iu in $C((s,t_0], L^2(D))$ for all $t_0 \in (s, t^+(s,u_0))$. In general, we could also study the convergence of solutions in an L^p space as noted in Remark 2.4.4 (see also [38, Section 6]).

We only mention above that the pseudo resolvent $\mathcal{R}_n(\lambda)$ converges to $\mathcal{R}(\lambda)$ in the strong operator topology. Since $\Omega_n, \Omega \subset D$ for all $n \in \mathbb{N}$, resolvent operators for Dirichlet problems are compact on $L^2(D)$ (from compactness of $H_0^1(\Omega_n)$ in $L^2(\Omega_n)$ by Rellich theorem). It turns out that the pseudo resolvent for Dirichlet problems converges in the operator norm as stated below.

Theorem 2.4.11 ([37, Corollary 4.7]). *Suppose that Ω_n and Ω satisfy Assumption 1.4.1 and $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco. If $\lambda \in \rho(-A)$, then $\lambda \in \rho(-A_n)$ for n suffiecntly large and $\mathcal{R}_n(\lambda) \to \mathcal{R}(\lambda)$ in $\mathscr{L}(L^2(D))$.*

An important consequence of Theorem 2.4.11 and [57, Theorem IV.3.16] is the following corollary.

Corollary 2.4.12 ([37, Corollary 4.2]). *Suppose that assumptions in Theorem 2.4.11 are satisfied. If $\Sigma \subset \sigma(-A)$ is a compact spectral set and Γ is a rectifiable closed curve enclosing Σ and separating it from the remaining of spectrum, then $\sigma(-A_n)$ is separated by Γ into a compact spectral set Σ_n and the rest of the spectrum for n sufficiently large. Moreover, for the corresponding spectral projections P and P_n, we have that the images of P and P_n have the same dimension and $i_n P_n r_n$ converges to iPr in norm.*

For Neumann problems, we do not have the uniform convergence (convergence in the operator norm) for pseudo resolvent operators if we use the corresponding Mosco convergence $H^1(\Omega_n)$ to $H^1(\Omega)$ in Theorem 2.4.11. In fact, there are several examples including a sequence of dumbbell shaped domains converging to two isolated balls (see [56] and references therein) or a more general exterior domain perturbation in Arrieta [6] show that eigenvalues do not behave continuously under domain perturbation. By [57, Theorem IV.3.16], pseudo resolvent operators cannot converge in the operator norm. Stronger assumptions on domain are required to get the uniform convergence of pseudo resolvent operators for Neumann problems. For examples, if we replace a standard dumbbell shaped domains by two touching balls opened up slightly near the touching point (see [39, Section 7] for a general assumptions) or a nonstandard dumbbell type domains constructed with a certain channel in [10], then we obtain the convergence of pseudo resolvent operators in the operator norm and consequently the convergence of spectral projections.

Chapter 3

Invariant manifolds for parabolic equations under domain perturbation

In this chapter, we consider parabolic equations as dynamical systems and study their behaviour under domain perturbation. In particular, we study stable and unstable invariant manifolds for parabolic equations near a stationary point. We only consider autonomous parabolic equations with Dirichlet boundary condition because the most complete results are known. Similar results for Neumann problems could be obtained by the same technique. However, stronger assumptions on domains are needed because convergence of spectral projections for Neumann problems requires stronger assumptions as seen in Section 2.4.2.

3.1 Introduction

The study of invariant manifolds is an important tool to understand the behaviour of a dynamical system near a stationary point. The first step is to construct invariant manifolds of various types which are characterised by their beheviour (grow, decay or neither) under the flow. For nonlinear differential equations, a fundamental way to construct invariant manifolds is done by considering the corresponding linearised equations. The main methods refer back to Hadamard [49], Liapunov [59] and Perron [67]. The Hadamard approach involves the use of splitting between various subspaces to estimate projections of the flow in the different directions. The Liapunov-Perron approach uses the fixed point argument for a certain integral equation to obtain invariant manifolds. For parabolic equations, Henry [52] obtains the existence of invariant manifolds under the assumption that the differential operator of the linearised equation generates an analytic semigroup. This allows the use of fractional power of the operator. Later, Bates and Jones [14] consider semilinear equations with the assumption that the linearised differential operator generates only a C_0-semigroup, and also allow infinite dimensional center manifolds. Once the existence of invariant manifolds is established, the fundamental problem is the *persistence* of invariant manifolds under perturbation. For finite dimensional systems, normal hyperbolicity is the main concept to deal with persistence invariant manifolds (see for example [46], [54], and [63]). For infinite dimensional systems, existence and persistence of invariant manifolds for semiflows in Banach space are obtained in [15] using normal hyperbolicity assumption.

The purpose of our work is to study the behaviour of invariant manifolds for semilinear parabolic equations under domain perturbation. There are similar results on the effect of domain variation on the dynamics of parabolic equations. In [68], upper semicontinuity of attractors is obtained for reaction-diffusion equations with Neumann boundary condition when the domain $\Omega \subset \mathbb{R}^M \times \mathbb{R}^N$ is squeezed in the $\mathbb{R}^N$-direction. Arrieta and Carvalho [10] consider a similar problem on a sequence of bounded and Lipschitz perturbed domains Ω_n. They give necessary and sufficient conditions on domains for spectral convergence of the linearised problem and obtain continuity (upper and lower semicontinuity) of local unstable manifolds and consequently continuity of attractors. For results under Dirichlet boundary condition, we refer to [42]. In [10], continuity of local unstable invariant manifolds is proved by keeping track of the construction adapted from Henry [52]. Although our framework on semilinear parabolic equations fits into [52], we use the existence of invariant manifolds from [14] to prove continuity of both stable and unstable invariant manifolds under domain perturbation.

3.2 Invariant manifolds for parabolic equations

3.2.1 Semiflows induced by abstract parabolic equations

Let Ω be a bounded open set in $\mathbb{R}^N$, $N \geq 2$. We consider the autonomous semilinear initial boundary value problem

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}u = g(x,u) & \text{in } \Omega \times (0,\infty) \\ u = 0 & \text{on } \partial\Omega \times (0,\infty) \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $g \in C(\mathbb{R}^N \times \mathbb{R})$, and $\mathcal{A}$ is an elliptic operator defined by (2.31). As seen in Section 2.4.2, we can write (3.1) in the abstract form as

$$\begin{cases} \dot{u}(t) + Au(t) = f(u(t)) & t \in (0,\infty) \\ u(0) = u_0 \end{cases} \tag{3.2}$$

in $L^2(\Omega)$, where $f(u)(x) := g(x,u(x))$ is the substitution operator induced by g. We make the following assumption on the nonlinearity f as in the framework of [14].

Assumption 3.2.1. We assume that $f : L^2(\Omega) \to L^2(\Omega)$ is locally Lipschitz and $f(0) = 0$. Moreover, for every $\varepsilon > 0$ there exists a neighbourhood $U = U(\varepsilon)$ of 0 such that f has a Lipschitz constant ε in U.

The above assumption is satisfied for g satisfying conditions in Assumption 2.4.2 (with the time variable disregarded) and $g(x,0) = 0$ for almost all $x \in \mathbb{R}^N$. Note also that Assumption 3.2.1 means $f(u)$ is a higher order term. We could think of (3.2) as a linearised problem near a fixed point.

As discussed in Section 2.4.2, for any given initial condition $u_0 \in L^2(\Omega)$ the abstract equation (3.2) has a unique maximal solution $u \in C([0,t^+(u_0)), L^2(\Omega))$. In addition, the solution u can be represented by the variation of constants formula (2.45). By our assumption that the substitution operator f belongs to $C(L^2(\Omega), L^2(\Omega))$, the nonlinearity f is linearly bounded. Thus, by a direct application of Gronwall's inequality, the estimate $\|u(t)\|_{L^2(\Omega)} \leq M$ holds for $t \in [0,T] \cap [0,t^+(u_0))$, where M is a positive constant depending on $T > 0$ and $\|u_0\|_{L^2(\Omega)}$. This implies that $[0,T] \subset [0,t^+(u_0))$ (see [40, Lemma 17.1, Corollary 17.2]). Since the above consideration holds for all $T > 0$, we have $t^+(u_0) = \infty$.

To study abstract parabolic equations as dynamical systems, we consider a *semiflow* $\Phi_t : L^2(\Omega) \to L^2(\Omega)$ defined by

$$\Phi_t(u_0) := u(t), \tag{3.3}$$

for all $t \in [0, t^+(u_0))$. Sometimes we would like to study the backwards behaviour of solutions. We call a continuous curve $u : [-t, 0] \to L^2(\Omega)$ for some $t > 0$ a *backwards solution branch* for $u_0 \in L^2(\Omega)$ if

$$\Phi_s(u(-s)) = u_0$$

for all $s \in [0, t]$. We write $\Phi_{-s}(u_0) = u(-s)$ when we look at a particular backwards solution branch.

3.2.2 Existence of invariant manifolds for parabolic equations

The purpose of this subsection is to summarise the construction of invariant manifolds in [14], which applies to our framework. Note that it only requires that $-A$ generates a C_0-semigroup in [14].

Definition 3.2.2. Let V be a subset of U.

1. V is called *positively invariant* relative to U if for each $v \in V$

   $$\bigcup_{s\in[0,t]} \Phi_s(v) \subset U \quad \text{implies} \quad \bigcup_{s\in[0,t]} \Phi_s(v) \subset V$$

   for all $t > 0$.

2. V is called *negatively invariant* relative to U if for each $v \in V$ we have that if a backwards solution branch for v exists then there exists $t_v > 0$ and a backwards branch $\Phi_{-s}(v)$ for $0 \leq s < t_v$ with t_v maximal such that

   $$\bigcup_{s\in[0,t]} \Phi_{-s}(v) \subset U \quad \text{implies} \quad \bigcup_{s\in[0,t]} \Phi_{-s}(v) \subset V$$

   for all $0 \leq t < t_v$.

3. V is *invariant* relative to U if it is positively and negatively invariant relative to U.

If the condition that orbit lies in U is removed then the set is called positively invariant, negatively invariant or invariant, respectively.

Definition 3.2.3. Let U be a neighbourhood of 0. We define

$$\begin{aligned} W^s &= \{u \in U : \Phi_t(u) \in U \text{ for all } t \geq 0 \text{ and } \Phi_t(u) \to 0 \text{ exponentially as } t \to \infty\} \\ W^u &= \{u \in U : \text{ some backwards branch } \Phi_t(u) \text{ exists for all } t < 0 \text{ and lies in } U, \\ &\qquad \text{and } \Phi_t(u) \to 0 \text{ exponentially as } t \to -\infty\} \end{aligned}$$

These sets W^s and W^u are invariant relative to U and are called stable and unstable sets, respectively. It is proved in [14] that they are indeed invariant manifolds. We sometimes write $W^s(U)$ and $W^u(U)$ to indicate their dependence on the neighbourhood U.

Recall from Section 2.4.2 that $-A$ is a generator of an analytic C_0-semigroup $S(t), t \geq 0$ on $L^2(\Omega)$. We denote the spectrum of $-A$ by $\sigma(-A)$ and decompose it as

$$\sigma(-A) = \sigma^s \cup \sigma^c \cup \sigma^u$$

where

$$\begin{aligned} \sigma^s &= \{\lambda \in \sigma(-A) : \mathrm{Re}(\lambda) < 0\} \\ \sigma^c &= \{\lambda \in \sigma(-A) : \mathrm{Re}(\lambda) = 0\} \\ \sigma^u &= \{\lambda \in \sigma(-A) : \mathrm{Re}(\lambda) > 0\}. \end{aligned} \tag{3.4}$$

Since Ω is bounded, Rellich's theorem implies that the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact. Hence the resolvent $(\lambda + A)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is also compact when it is defined. This implies that $\sigma(-A)$ consists of eigenvalues with finite multiplicities (see [57]). It is easily seen from [41, Theorem 3, XVII §6] that σ^c and σ^u are finite sets. Let Γ^c and Γ^u be rectifiable closed curves separating σ^c and σ^s respectively from the remaining spectrum. There are invariant subspaces of $L^2(\Omega)$ associated to σ^s, σ^c and σ^u via the spectral projections (see [57])

$$\begin{aligned} P^c &= \frac{1}{2\pi i} \int_{\Gamma^c} (\lambda + A)^{-1} d\lambda \\ P^u &= \frac{1}{2\pi i} \int_{\Gamma^u} (\lambda + A)^{-1} d\lambda. \end{aligned}$$

Indeed, we decompose $L^2(\Omega) = X^s \oplus X^c \oplus X^u$ where $X^s = (1 - P^c - P^u)L^2(\Omega)$, $X^c = P^c L^2(\Omega)$ and $X^u = P^u L^2(\Omega)$. Note that $\dim(X^c)$ and $\dim(X^u)$ are finite. We set $X^{cs} = X^c \oplus X^s$ and $X^{cu} = X^c \oplus X^u$. For $* = s, c, u, cs, cu$, we have that $-A^* = -A|_{X^*}$ is a generator of $S^*(t) = S(t)|_{X^*}$. Since $S(t)$ is an analytic semigroup, there exist $M > 0$ and $\sigma > 0$ such that

$$\|S^s(t)\| \leq Me^{-\sigma t},$$

for all $t > 0$.

To obtain the existence of local stable and unstable invariant manifolds, we decompose $L^2(\Omega) = X^- \oplus X^+$ with $\dim X^+ < \infty$ in two different ways; either $X^- = X^s$ and $X^+ = X^{cu}$ or $X^- = X^{cs}$ and $X^+ = X^u$. We denote a natural projection (via spectral projection) onto X^+ by P^+, a natural projection on X^- by $P^- := 1 - P^+$ and write $-A^\pm = -A|_{X^\pm}$. In both cases, we have that $-A^-$ generates a C_0-semigroup $S^-(t)$ on X^- satisfying

$$\|S^-(t)\| \leq M_1 e^{\alpha t}, \tag{3.5}$$

for all $t \geq 0$ where $M_1 > 0$ and $\alpha \in \mathbb{R}$. Similarly, $-A^+$ generates a C_0-group $S^+(t)$ on X^+ satisfying

$$\|S^+(t)\| \leq M_2 e^{\beta t}, \tag{3.6}$$

for all $t \leq 0$ where $M_2 > 0$ and $\beta > \alpha$. The parameters α and β can be chosen as follows (see proof of Theorem 1.1 case (D) and proof of Theorem 1.2 case (D) in [14]).

- If $X^- = X^s$ and $X^+ = X^{cu}$, we take $\alpha = -\sigma$ and fix β such that $-\sigma < \beta < 0$.
- If $X^- = X^{cs}$ and $X^+ = X^u$, we take $\beta > 0$ such that $\beta < \min\{\mathrm{Re}(\lambda) : \lambda \in \sigma^u\}$ and fix α such that $0 < \alpha < \beta$.

The main techniques in Bates and Jones [14] are a renorming of X^- and X^+ and a modification of nonlinearity f. Since we decompose $L^2(\Omega) = X^- \oplus X^+$, norms on X^- and X^+ are originally inherited from $L^2(\Omega)$. Indeed, if $u = v \oplus w \in L^2(\Omega)$ where $v \in X^-$ and $w \in X^+$, then

$$\frac{1}{\|P^-\| + \|P^+\|}(\|v\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}) \leq \|u\|_{L^2(\Omega)} \leq (\|v\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}). \tag{3.7}$$

However, we can renorm X^- and X^+ by

$$\|v\|_{X^-} := \sup_{t\geq 0} e^{-\alpha t}\|S^-(t)v\|_{L^2(\Omega)}, \tag{3.8}$$

for $v \in X^-$ and similarly

$$\|w\|_{X^+} := \sup_{t\leq 0} e^{-\beta t}\|S^+(t)w\|_{L^2(\Omega)}, \tag{3.9}$$

for $w \in X^+$. These norms are equivalent on X^- and X^+, respectively. It is easy to see that (see also [14, Lemma 2.1])

$$\|v\|_{L^2(\Omega)} \leq \|v\|_{X^-} \leq M_1\|v\|_{L^2(\Omega)}, \tag{3.10}$$

for all $v \in X^-$ and

$$\|w\|_{L^2(\Omega)} \leq \|w\|_{X^+} \leq M_2 \|w\|_{L^2(\Omega)}, \tag{3.11}$$

for all $w \in X^+$. The modification of nonlinearity f is done by cutting off arguments so that we obtain a globally Lipschitz function $\tilde{f}$. Let $\eta > 0$ be arbitrary. By Assumption 3.2.1, we can choose $\delta > 0$ such that f has a Lipschitz constant less than $\eta/12$ in $B_{L^2(\Omega)}(0, 2\delta)$. Let $\Psi : L^2(\Omega) \to \mathbb{R}$ be a function defined by

$$\Psi(u) = \begin{cases} 1 & \text{if } \|u\|_{L^2(\Omega)} \leq \delta \\ 2 - \dfrac{\|u\|_{L^2(\Omega)}}{\delta} & \text{if } \delta \leq \|u\|_{L^2(\Omega)} \leq 2\delta \\ 0 & \text{if } \|u\|_{L^2(\Omega)} \geq 2\delta. \end{cases}$$

By setting $\tilde{f}(u) := \Psi(u) f(u)$ for all $u \in L^2(\Omega)$, we have that $\tilde{f}$ is globally Lipschitz continuous with constant $\varepsilon < \eta/4$. This Lipschitz constant ε can be chosen as small as we require by shrinking δ.

With this modified system $\dot{u}(t) + Au(t) = \tilde{f}(u(t))$, the solution to an initial value parabolic equation $u(t)$ also exists for $t \geq 0$, that is the maximal existence time $t^+(u_0) = \infty$ for all $u_0 \in L^2(\Omega)$. Moreover, the modified system agrees with the original system (3.2) inside $B_{L^2(\Omega)}(0, \delta)$. Hence, the modification gives us a local beheviour of the original system. Although our assumptions imply that the original system has a global solution for any initial data u_0, we modify the nonlinear term f in order to gain some useful estimates of the projections of semiflows in various directions (subspaces).

In [14], invariant manifolds for the modified system are constructed as follows. We choose the Lipschitz constant ε of $\tilde{f}$ so that $\varepsilon < (\beta - \alpha)/4$ and there exists γ such that

$$-\beta + 2\varepsilon < \gamma < -\alpha - 2\varepsilon. \tag{3.12}$$

By abuse of notations, we denote again by $\Phi_t(u_0)$ the solution $u(t)$ of the modified system with the initial condition u_0. Let

$$\begin{aligned} W^- &= \{u \in L^2(\Omega) : e^{\gamma t}\Phi_t(u) \to 0 \text{ as } t \to \infty\} \\ W^+ &= \{u \in L^2(\Omega) : \text{ a backward branch } \Phi_t(u) \text{ exists for all } t \leq 0 \\ &\qquad \text{and } e^{\gamma t}\Phi_t(u) \to 0 \text{ as } t \to -\infty\}. \end{aligned}$$

The main idea to show that W^- and W^+ are invariant manifolds is that certain cones and moving cones are positively invariant, which can be determined by the difference in the growth rates on X^- and X^+. For $\lambda > 0$, we define a cone

$$K_\lambda = \{(v, w) \in X^- \times X^+ : \lambda \|v\|_{X^-} \leq \|w\|_{X^+}\}. \tag{3.13}$$

It is shown in [14, Lemma 2.4] that K_λ is positively invariant if $\lambda \in [\mu, \nu]$ where μ and ν are positive parameters with $\mu < 1 < \nu$ satisfying

$$\varepsilon < (\beta - \alpha)/(2 + \nu + \mu^{-1}). \tag{3.14}$$

Indeed, μ and ν can be further restricted so that

$$\varepsilon(1 + \mu^{-1}) - \beta < \gamma < -\varepsilon(1 + \nu) - \alpha. \tag{3.15}$$

The next two theorems give the existence of global stable and global unstable invariant manifolds for the modified system.

Theorem 3.2.4 ([14, Theorem 2.1])**.** *There exists a Lipschitz function $h^- : X^- \to X^+$ such that $W^- = \mathrm{graph}(h^-)$ and $h^-(0) = 0$.*

Sketch of the proof. Fix $v_0 \in X^-$ and let

$$B = \{w_0 \in X^+ : \|w_0\|_{X^+} \leq \mu \|v_0\|_{X^-}\}.$$

We write $\Phi_t(u_0) = u(t)$ as $u(t) = v(t) \oplus w(t)$ where $v(t) \in X^-$ and $w(t) \in X^+$. Define

$$G_t = \{w_0 \in B : \|w(t)\|_{X^+} \leq \mu \|v(t)\|_{X^-}\}.$$

It can be shown that $G_\infty := \bigcap_{t \geq 0} G_t$ contains exactly one element. A function h^- defined by $h^-(v_0) = G_\infty$ for $v_0 \in X^-$ is a Lipschitz function with $h^-(0) = 0$ and $\mathrm{graph}(h^-) = W^-$. □

Theorem 3.2.5 ([14, Theorem 2.2])**.** *There exists a Lipschitz function $h^+ : X^+ \to X^-$ such that $W^+ = \mathrm{graph}(h^+)$ and $h^+(0) = 0$.*

Sketch of the proof. The proof is based on a standard contraction mapping argument. Let

$$Y = \{h \in C(X^+, X^-) : h(0) = 0 \text{ and } h \text{ is } \nu^{-1}\text{-Lipschitz }\}.$$

Then Y is a complete metric space with the norm

$$\|h\|_{\mathrm{Lip}} = \sup_{w \neq 0} \frac{\|h(w)\|_{X^-}}{\|w\|_{X^+}}. \tag{3.16}$$

For an arbitrary $h \in Y$, it can be shown that $P^+\Phi_t(\mathrm{graph}(h)) = X^+$ and that $\Phi_t(\mathrm{graph}(h))$ is the graph of a ν^{-1}-Lipschitz function for all $t \geq 0$. Hence, the map $T_t : Y \to Y$ for $t \geq 0$ given by

$$T_t(h) = \tilde{h}$$

where $\tilde{h} \in Y$ with $\text{graph}(\tilde{h}) = \Phi_t(\text{graph}(h))$ is well-defined. Furthermore, T_t is a contraction on Y for t sufficiently large. Indeed,

$$\|T_t(h_2) - T_t(h_1)\|_{\text{Lip}} \leq \nu(\nu - \mu)^{-1} \exp\left((\alpha - \beta + \varepsilon(2 + \mu + \nu^{-1}))t\right)\|h_2 - h_1\|_{\text{Lip}}.$$

Hence, there exists a unique fixed point $h_t \in Y$ for t sufficiently large. We can show that h_t is a fixed point of T_τ for all $\tau \geq 0$ and $h^+ := h_t$ is the required Lipschitz function with $\text{graph}(h^+) = W^+$ and $h^+(0) = 0$. □

Remark 3.2.6. Let $Y_0 = \{h \in Y : h \text{ is differentiable at } 0 \text{ and } Dh(0) = 0\}$. Then Y_0 is closed in Y. As $D\tilde{f}(0) = 0$ (in fact $Df(0) = 0$ from Assumption 3.2.1), it can be shown that $T_t : Y_0 \to Y_0$ for all $t > 0$. Hence, the fixed point h^+ in Theorem 3.2.5 lies on Y_0 (see the proposition after the proof of Theorem 2.2 in [14]).

The next two theorems give the existence of the local stable and the local unstable invariant manifolds for (3.2).

Theorem 3.2.7 ([14, Theorem 1.1(i)])**.** *Under the assumptions given above, there exists an open neighbourhood U of 0 in $L^2(\Omega)$ such that W^s is a Lipschitz manifold which is tangent to X^s at 0, that is, there exists a Lipschitz function $h^s : P^s(U) \to X^{cu}$ such that $\text{graph}(h^s) = W^s$, $h^s(0) = 0$ and h^s is differentiable at 0 with $Dh^s(0) = 0$.*

Sketch of the proof. Set $X^- = X^s$ and $X^+ = X^{cu}$. We take $\alpha = -\sigma$ and fix β such that $-\sigma < \beta < 0$. Renorm X^- and X^+ by (3.8) and (3.9), respectively. By Assumption 3.2.1, there exists $\delta > 0$ such that the modification $\tilde{f}$ has a Lipschitz constant $\varepsilon < (\beta - \alpha)/4$ and the modified system agrees with the original system on $B_{L^2(\Omega)}(0, \delta)$. By applying Theorem 3.2.4, we can find a product neighbourhood $U \subset B_{L^2(\Omega)}(0, \delta)$ and prove that $W^s = W^- \cap U$ is a local stable invariant manifold. It can be shown that any local stable manifold constructed using another renorming and modification agrees on a neighbourhood on which the manifolds are both defined. The tangency condition $Dh^s(0) = 0$ follows by making $\mu \to 0$ (by letting $\varepsilon \to 0$ and possibly shrinking U). □

Theorem 3.2.8 ([14, Theorem 1.2(i)])**.** *Under the assumptions given above, there exists an open neighbourhood U of 0 in $L^2(\Omega)$ such that W^u is a Lipschitz manifold which is tangent to X^u at 0, that is, there exists a Lipschitz function $h^u : P^u(U) \to X^{cs}$ such that $\text{graph}(h^u) = W^u$, $h^u(0) = 0$ and h^u is differentiable at 0 with $Dh^u(0) = 0$.*

Sketch of the proof. Set $X^- = X^{cs}$ and $X^+ = X^u$. We take $\beta > 0$ such that $\beta < \min\{\text{Re}(\lambda) : \lambda \in \sigma^u\}$ and fix α such that $0 < \alpha < \beta$. Renorm X^- and X^+ and modify the nonlinearity f as in the proof of Theorem 3.2.7. Applying Theorem 3.2.5, we can find a product neighbourhood $U \subset B_{L^2(\Omega)}(0,\delta)$ and prove that $W^u = W^+ \cap U$ is a local unstable invariant manifold. It can be shown that any local unstable manifold constructed using another renorming and modification agrees on a neighbourhood on which the manifolds are both defined. The tangency condition $Dh^s(0) = 0$ follows from Remark 3.2.6. □

The product neigbourhood U in Theorem 3.2.7 and Theorem 3.2.8 can be chosen to be $U = V_1 \times V_2$ where $V_1 \subset X^-$ is a ball of radius δ_1 and $V_2 \subset X^+$ is a ball of radius δ_2 such that $\delta_1 < \delta_2$ for the local stable manifold and $\delta_1 > \delta_2$ for the local unstable manifold. In fact, with these choices of product neighbourhoods, W^s is positively invariant and W^u is negatively invariant (see property (P4) in [14]).

3.3 Domain perturbation for invariant manifolds

In this section, we introduce domain perturbation for invariant manifolds. We state the main result on convergence of the local stable and the local unstable invariant manifolds under changes of the domain.

Let Ω_n and Ω be bounded open sets in $\mathbb{R}^N$ satisfying Assumption 1.4.1. We consider the perturbation of (3.1) given by

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}_n u = g_n(x,u) & \text{in } \Omega_n \times (0,\infty) \\ u = 0 & \text{on } \partial\Omega_n \times (0,\infty) \\ u(\cdot,0) = u_{0,n} & \text{in } \Omega_n, \end{cases} \tag{3.17}$$

where $g_n \in C(\mathbb{R}^N \times \mathbb{R})$ and $\mathcal{A}_n$ is the operator defined by (2.40). As in Section 2.4.2, we can consider the corresponding abstract equation of (3.17) given by

$$\begin{cases} \dot{u}(t) + A_n u(t) = f_n(u(t)) & t \in (0,\infty) \\ u(0) = u_{0,n} \end{cases} \tag{3.18}$$

in $L^2(\Omega_n)$, where $f_n(u)(x) := g_n(x,u(x))$ is the substitution operator induced by g_n. As previously seen, $-A_n$ is a generator of a strongly continuous analytic semigroup $S_n(t), t \geq 0$ on $L^2(\Omega_n)$. When well-posed, the mild solution u_n of (3.18) can be represented by the variation of constants formula (2.46). We denote the local semiflow induced by parabolic equation (3.18) by $\Phi_{t,n} : L^2(\Omega_n) \to L^2(\Omega_n)$ where

$$\Phi_{t,n}(u_{0,n}) := u_n(t), \tag{3.19}$$

for $t \in (0, t^+(u_{0,n}))$.

As discussed in Section 3.1, one of the main subjects in invariant manifold theory is to study the persistence of invariant manifolds under perturbation. The focus of this work is to study the dependence of invariant manifolds on domain variation. We could think of $\Phi_{t,n}$ as a perturbation of Φ_t. However, the sequence of semiflows $\Phi_{t,n}$ is defined on different spaces for each $n \in \mathbb{N}$, namely $L^2(\Omega_n)$. As usual, we make use of the inclusion operators i_n and i (defined by (1.27) and (1.28), respectively) and the restriction operators r_n and r (defined by (1.29) and (1.30), respectively) when dealing with domain perturbation.

By the above inclusion operators, we consider the stable and unstable invariant manifolds as subsets of $L^2(D)$. The main difficulty to establish convergence of invariant manifolds under domain perturbation in this framework is that invariant manifolds are constructed using a special renorming of X^- and X^+ (see (3.8) and (3.9)). Although we have seen in Theorem 2.4.8 that under suitable assumptions on domains the degenerate semigroup $i_nS_n(t)r_n$ converges to $iS(t)r$ in the strong operator topology uniformly with respect to t in compact subsets of $(0, \infty)$, we cannot expect convergence of the projection of solutions $P_n^-u_n$ and $P_n^+u_n$ with respect to a sequence of the norms $\|\cdot\|_{X_n^-}$ and $\|\cdot\|_{X_n^+}$, respectively. This is because norms on X_n^- and X_n^+ (see (3.8) and (3.9)) involve taking the supremum of $e^{-\alpha t}\|S_n^-(t)v\|_{L^2(\Omega_n)}$ on $[0, \infty)$ and of $e^{-\beta t}\|S_n^+(t)v\|_{L^2(\Omega_n)}$ on $(-\infty, 0]$, respectively.

We next state our assumptions for the perturbed equations.

Assumption 3.3.1. We assume the following assumptions.

(i) The coefficients of the operator $\mathcal{A}_n$

converge to the corresponding coefficients of $\mathcal{A}$ as $n \to \infty$ (see Assumption 2.4.5).

(ii) The stationary point $0 \in L^2(\Omega)$ of (3.2) is hyperbolic, that is $\sigma^c = \emptyset$.

Assumption 3.3.2. We impose the following conditions on the nonlinearity.

(i) $f_n : L^2(\Omega_n) \to L^2(\Omega_n)$ is locally Lipschitz and $f_n(0) = 0$. In addition, for every $\varepsilon > 0$ there exists a neighbourhood $U_n = U_n(\varepsilon)$ of 0 such that f_n has a Lipschitz constant ε in U_n. Moreover, U_n can be chosen uniformly with respect to $n \in \mathbb{N}$ in the sense that we can take U_n to be a ball centered at 0 in $L^2(\Omega_n)$ of the same radius for all $n \in \mathbb{N}$.

(ii) $i_nf_nr_n(u) \to ifr(u)$ in $L^2(D)$ pointwise for $u \in L^2(D)$.

Note that if g_n and g satisfy conditions in Assumption 2.4.2 (with the time variable disregarded) and $g_n(x,0) = 0$ for almost all $x \in \mathbb{R}^N$, then the substitution operators f_n and f satisfy the above assumption.

Our main result can be stated as follows.

Theorem 3.3.3 (Continuity of local unstable manifolds)**.** *Suppose that Assumption 3.3.1 and Assumption 3.3.2 are satisfied. If $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco, then there exists a local unstable invariant manifold W_n^u for parabolic equation (3.18) for each $n \in \mathbb{N}$. Moreover, there exists $\delta > 0$ such that if we denote by $B_n = B_{L^2(\Omega_n)}(0,\delta)$ and $B = B_{L^2(\Omega)}(0,\delta)$, then we have*

(i) *Upper semicontinuity:*

$$\sup_{v \in W_n^u \cap B_n} \inf_{u \in W^u \cap B} \|i_n(v) - i(u)\|_{L^2(D)} \to 0$$

as $n \to \infty$.

(ii) *Lower semicontinuity:*

$$\sup_{u \in W^u \cap B} \inf_{v \in W_n^u \cap B_n} \|i_n(v) - i(u)\|_{L^2(D)} \to 0$$

as $n \to \infty$.

A similar result can be stated for local stable invariant manifolds with an additional assumption of the convergence in measure of the domains.

Theorem 3.3.4 (Continuity of local stable manifolds)**.** *Suppose that Assumption 3.3.1 and Assumption 3.3.2 are satisfied. If $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco, then there exists a local stable invariant manifold W_n^s for parabolic equation (3.18) for each $n \in \mathbb{N}$.*

In addition, assume that the Lebesgue measure $|\Omega_n| \to |\Omega|$ as $n \to \infty$. Then there exists $\delta > 0$ such that if we denote by $B_n = B_{L^2(\Omega_n)}(0,\delta)$ and $B = B_{L^2(\Omega)}(0,\delta)$, then we have

(i) *Upper semicontinuity:*

$$\sup_{v \in W_n^s \cap B_n} \inf_{u \in W^s \cap B} \|i_n(v) - i(u)\|_{L^2(D)} \to 0$$

as $n \to \infty$.

(ii) *Lower semicontinuity:*

$$\sup_{u\in W^s\cap B}\inf_{v\in W^s_n\cap B_n}\|i_n(v)-i(u)\|_{L^2(D)}\to 0$$

as $n\to\infty$.

In the remainder, we show the existence of local stable and local unstable invariant manifolds for the perturbed problem (3.18) in Theorem 3.3.3 and Theorem 3.3.4. The assertion on upper and lower semicontinuity will be proved in Section 3.5 for unstable invariant manifolds and in Section 3.6 for stable invariant manifolds.

By our assumptions in Theorem 3.3.3 or Theorem 3.3.4, we have from Theorem 2.4.9 that for every $u_{0,n}\in L^2(\Omega_n)$ and $u_0\in L^2(\Omega)$ with $ri_nu_{0,n}\rightharpoonup u_0$ in $L^2(\Omega)$ weakly, the perturbed equation (3.18) has a unique maximal solution $u_n\in C([0,t_n^+(u_{0,n})),L^2(\Omega_n))$ and $i_nu_n(t)\to iu(t)$ in $L^2(D)$ uniformly with respect to $t\in(0,t_0]$ for all $t_0\in(0,t^+(u_0))$. In terms of semiflows, the above can be stated as

$$i_n\Phi_{t,n}(u_{0,n})\to i\Phi_t(u_0) \tag{3.20}$$

in $L^2(D)$ as $n\to\infty$ uniformly with respect to $t\in(0,t_0]$ for all $t_0\in(0,t^+(u_0))$. In addition, if the initial condition $i_nu_{0,n}$ converges strongly to iu_0 in $L^2(D)$, then (3.20) holds uniformly with respect to $t\in[0,t_0]$ for all $t_0\in(0,t^+(u_0))$. Recall also that our assumptions imply convergence of pseudo resolvents in the operator norm (Theorem 2.4.11) and convergence of spectral projections (Corollary 2.4.12).

We decompose $\sigma(-A_n)=\sigma_n^s\cup\sigma_n^c\cup\sigma_n^u$ where σ_n^s,σ_n^c and σ_n^u are sets defined similarly to (3.4). By hyperbolicity assumption that $\sigma_c=\emptyset$ (Assumption 3.3.1 (ii)) and Corollary 2.4.12, we have that Γ^c and Γ^u separate σ_n^c and σ_n^u respectively from the remaining of spectrum for n sufficiently large. In fact, the convergence of spectrum in Corollary 2.4.12 implies that $\sigma_n^c=\emptyset$ for all n sufficiently large. Hence, the fixed point $0\in L^2(\Omega_n)$ of (3.18) is hyperbolic for all n sufficiently large. We define

$$\begin{aligned}
P_n^c&=\frac{1}{2\pi i}\int_{\Gamma^c}(\lambda+A_n)^{-1}d\lambda=0\\
P_n^u&=\frac{1}{2\pi i}\int_{\Gamma^u}(\lambda+A_n)^{-1}d\lambda\\
P_n^s&=1-P_n^c-P_n^u.
\end{aligned}$$

Then

$$i_nP_n^cr_n\to iP^cr \quad\text{and}\quad i_nP_n^ur_n\to iP^ur \tag{3.21}$$

in $\mathscr{L}(L^2(D))$ as $n \to \infty$. Denote by X_n^s, X_n^c and X_n^u the images of $1 - P_n^c - P_n^u, P_n^c$ and P_n^u, respectively. For n sufficiently large, we can decompose

$$L^2(\Omega_n) = X_n^s \oplus X_n^c \oplus X_n^u. \tag{3.22}$$

Indeed, from the above consideration we have that $X_n^c = \{0\}$ and X_n^u is a finite dimensional subspace with $\dim(X_n^u) = \dim(X^u)$ for all n sufficiently large.

To construct invariant manifolds for the modified system, we decompose $L^2(\Omega_n)$ as $X_n^- \oplus X_n^+$ in two different ways as in Section 3.2.2. In particular, $\dim(X_n^+) = \dim(X^+) < \infty$ for n sufficiently large and

$$i_n P_n^+ r_n \to i P^+ r \tag{3.23}$$

in $\mathscr{L}(L^2(D))$. Moreover, if $u = v \oplus w \in L^2(\Omega_n)$ where $v \in X_n^-$ and $w \in X_n^+$ then

$$\frac{1}{\|P_n^-\| + \|P_n^+\|}(\|v\|_{L^2(\Omega_n)} + \|w\|_{L^2(\Omega_n)}) \le \|u\|_{L^2(\Omega_n)} \le (\|v\|_{L^2(\Omega_n)} + \|w\|_{L^2(\Omega_n)}). \tag{3.24}$$

By Assumption 3.3.1 (i), we can choose the parameters α and β for the restriction of semigroup $S_n(t)$ to X_n^- and X_n^+ uniformly with respect to $n \in \mathbb{N}$ so that $S_n^-(t)$ and $S_n^+(t)$ satisfy similar conditions as (3.5) and (3.6), respectively. We see that

$$\|v\|_{X_n^-} := \sup_{t \ge 0} e^{-\alpha t} \|S_n^-(t) v\|_{L^2(\Omega_n)}, \tag{3.25}$$

for $v \in X_n^-$ and

$$\|w\|_{X_n^+} := \sup_{t \le 0} e^{-\beta t} \|S_n^+(t) w\|_{L^2(\Omega_n)}, \tag{3.26}$$

for $w \in X_n^+$ give equivalent norms on X_n^- and X_n^+, respectively. Indeed, we have

$$\|v\|_{L^2(\Omega_n)} \le \|v\|_{X_n^-} \le M_1 \|v\|_{L^2(\Omega_n)} \tag{3.27}$$

for all $v \in X_n^-$ and

$$\|w\|_{L^2(\Omega_n)} \le \|w\|_{X_n^+} \le M_2 \|w\|_{L^2(\Omega_n)} \tag{3.28}$$

for all $w \in X_n^+$ where M_1 and M_2 can be chosen uniformly with respect to $n \in \mathbb{N}$.

By Assumption 3.3.2 (i), there exists $\delta > 0$ independent of n such that the modification $\tilde{f}_n$ of f_n has a Lipschitz constant $\varepsilon < (\beta - \alpha)/4$ and the modified system agrees with the original system on $B_n := r_n(B)$ where $B := B_{L^2(D)}(0, \delta)$ for all $n \in \mathbb{N}$. Therefore, we can construct the stable and unstable invariant manifold for the modified system by using uniform parameters γ, μ and ν for all $n \in \mathbb{N}$. By Theorem 3.2.7, there exists a product neighbourhood $U_n \subset B_n$ such that a local stable invariant manifold is $W_n^s(U_n) = \text{graph}(h_n^-) \cap U_n$. Since the parameters α and β are chosen uniformly for the renorming of X_n^- and X_n^+ respectively, we can choose $U_n \subset B_n$ to be a product neighbourhood $V_{1,n} \times V_{2,n}$ where $V_{1,n} \subset X_n^-$ is a ball of radius δ_1 and $V_{2,n} \subset X_n^+$ is a ball of radius δ_2 with $\delta_1 < \delta_2$ for all $n \in \mathbb{N}$. Without loss of generality we may choose δ smaller so that the modified system agrees with the original system on $\overline{B}_n$ for all $n \in \mathbb{N}$. Similarly, by Theorem 3.2.8, there exists a product neighbourhood $\tilde{U}_n \subset B_n$ such that a local unstable invariant manifold is $W_n^u(\tilde{U}_n) = \text{graph}(h_n^+) \cap \tilde{U}_n$. Since the parameters α and β are chosen uniformly for the renorming of X_n^- and X_n^+ respectively, we can choose $\tilde{U}_n \subset B_n$ to be a product neighbourhood $\tilde{V}_{1,n} \times \tilde{V}_{2,n}$ where $\tilde{V}_{1,n} \subset X_n^-$ is a ball of radius $\tilde{\delta}_1$ and $\tilde{V}_{2,n} \subset X_n^+$ is a ball of radius $\tilde{\delta}_2$ with $\tilde{\delta}_1 > \tilde{\delta}_2$ for all $n \in \mathbb{N}$. Again we may choose δ smaller so that the modified system agrees with the original system on $\overline{B}_n$ for all $n \in \mathbb{N}$. Therefore, we have established the existence of local unstable manifolds and local stable manifolds for the perturbed problem (3.18).

We can assume that the choice of neighbourhoods considered above applies to the limit problem (3.2) (by possibly shrinking δ). Therefore, to prove Theorem 3.3.3 and Theorem 3.3.4, it remains to verify the continuity under domain perturbation (upper and lower semicontinuity) of local stable and local unstable invariant manifolds inside some ball $B_n = B_{L^2(\Omega_n)}(0, \hat{\delta})$ contained in $V_{1,n} \times V_{2,n}$ or $\tilde{V}_{1,n} \times \tilde{V}_{2,n}$.

3.4 Some technical results towards the proof of semicontinuity

In this section, we give some technical results required to prove upper and lower semicontinuity in Theorem 3.3.3 and Theorem 3.3.4. In particular, we prove some convergence result for a bounded sequence $(w_n)_{n\in\mathbb{N}}$ with $w_n \in X_n^+$ for each $n \in \mathbb{N}$. Moreover, we give a characterization of upper and lower semicontinuity.

3.4.1 Convergence of sequence in finite dimensional subspaces

We first give a direct consequence of convergence of spectral projections in (3.21).

Lemma 3.4.1. *Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence with $\phi_n \in L^2(\Omega_n)$ for each $n \in \mathbb{N}$ and $\phi \in L^2(\Omega)$. We decompose $\phi_n := \phi_n^s \oplus \phi_n^c \oplus \phi_n^u$ corresponding to the decomposition (3.22). Similarly, we decompose $\phi := \phi^s \oplus \phi^c \oplus \phi^u$.*

(i) *If $i_n\phi_n \to i\phi$ in $L^2(D)$ strongly, then $i_n\phi_n^* \to i\phi^*$ in $L^2(D)$ strongly for $* = s, c, u$.*

(ii) *If* $i_n\phi_n \rightharpoonup i\phi$ *in* $L^2(D)$ *weakly, then* $i_n\phi_n^* \rightharpoonup i\phi^*$ *in* $L^2(D)$ *weakly for* $* = s, c, u$.

(iii) *If* $ri_n\phi_n \to \phi$ *in* $L^2(\Omega)$ *strongly and* $\|i_n\phi_n\|_{L^2(D)}$ *is uniformly bounded, then* $ri_n\phi_n^* \to \phi^*$ *in* $L^2(\Omega)$ *strongly for* $* = s, c, u$.

(iv) *If* $ri_n\phi_n \rightharpoonup \phi$ *in* $L^2(\Omega)$ *weakly and* $\|i_n\phi_n\|_{L^2(D)}$ *is uniformly bounded, then* $ri_n\phi_n^* \rightharpoonup \phi^*$ *in* $L^2(\Omega)$ *weakly for* $* = s, c, u$.

Proof. Suppose that $i_n\phi_n \to i\phi$ in $L^2(D)$ strongly. By (3.21), we have

$$\begin{aligned}\|i_n\phi_n^* - i\phi^*\|_{L^2(D)} &\leq \|i_nP_n^*r_ni_n\phi_n - iP^*ri_n\phi_n\|_{L^2(D)} \\ &\quad + \|iP^*ri_n\phi_n - iP^*ri\phi\|_{L^2(D)} \\ &\leq \|i_nP_n^*r_n - iP^*r\|\|i_n\phi_n\|_{L^2(D)} \\ &\quad + \|iP^*r\|\|i_n\phi_n - i\phi\|_{L^2(D)} \\ &\to 0\end{aligned}$$

as $n \to \infty$ for $* = c, u$. Since $\phi_n^s = (1 - P_n^c - P_n^u)\phi_n$ and $\phi^s = (1 - P^c - P^u)\phi$, we obtain the convergence $i_n\phi_n^s \to i\phi^s$. This proves assertion (i). The statement (iii) follows from a similar argument above.

Suppose now that $i_n\phi_n \rightharpoonup i\phi$ in $L^2(D)$ weakly. By (3.21) and the weak convergence of $i_n\phi_n$, we have for $\chi \in L^2(D)$ and $* = c, u$

$$\begin{aligned}\left|(i_n\phi_n^*) - i\phi^*)|\chi)_{L^2(D)}\right| &= \left|(i_nP_n^*\phi_n - iP^*\phi|\chi)_{L^2(D)}\right| \\ &\leq \left|(i_nP_n^*\phi_n - iP^*ri_n\phi_n|\chi)_{L^2(D)}\right| \\ &\quad + \left|(iP^*ri_n\phi_n - iP^*\phi|\chi)_{L^2(D)}\right| \\ &\leq \|i_nP_n^*r_n - iP^*r\|\|i_n\phi_n\|_{L^2(D)}\|\chi\|_{L^2(D)} \\ &\quad + \left|\big(i_n\phi_n - i\phi|(iP^*r)'\chi\big)_{L^2(D)}\right| \\ &\to 0\end{aligned}$$

as $n \to \infty$ where $(iP^*r)'$ denotes the adjoint operator of iP^*r. This implies $i_n(\phi_n^*) \rightharpoonup i(\phi^*)$ in $L^2(D)$ weakly for $* = c, u$. Since $\phi_n^s = (1 - P_n^c - P_n^u)\phi_n$ and $\phi^s = (1 - P^c - P^u)\phi$, we obtain the weak convergence $i_n(\phi_n^s) \rightharpoonup i(\phi^s)$ in $L^2(D)$. This proves assertion (ii). The statement (iv) can be proved in a similar fashion. □

Note that if $i_n\phi_n$ is uniformly bounded in $L^2(D)$, then there exists a weak convergent subsequence in $L^2(D)$. In general, we do not know whether the weak limit is zero almost everywhere on $D\backslash\Omega$, that is of the form $i\phi$ for some $\phi \in L^2(\Omega)$. This is the reason we include both assertions (ii) and (iv) in Lemma 3.4.1 above.

Remark 3.4.2. The convergence of $i_n(\phi_n^s) \to i(\phi^s)$ in Lemma 3.4.1 (i) is different to convergence of the projections $i_n(1 - P_n^c - P_n^u)r_n \to i(1 - P^c - P^u)r$ in $\mathscr{L}(L^2(D))$. For example, consider a square domain Ω in $\mathbb{R}^2$ perturbed by attaching "fingers" to one of the sides as previously seen in Figure 1.2. If we increase the number of fingers so that the measure remains the same (by letting their width go to zero). Then $|\Omega_n \backslash \Omega|$ is a positive constant for all $n \in \mathbb{N}$. Recall that $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco. Let $f \in L^2(D)$ be the constant function 1. By (3.21), we have that $i_n P_n^c r_n(f) \to i P^c r(f)$ and $i_n P_n^u r_n(f) \to i P^u r(f)$ in $L^2(D)$. If $i_n(1 - P_n^c - P_n^u)r_n(f) \to i(1-P^c-P^u)r(f)$ in $L^2(D)$, then $i_n r_n(f) \to i r(f)$ in $L^2(D)$. This cannot be true because $\|i_n r_n(f) - i r(f)\|_{L^2(D)} = |\Omega_n \backslash \Omega| > 0$ for all $n \in \mathbb{N}$. Hence, $i_n(1 - P_n^c - P_n^u)r_n$ does not converge to $i(1 - P^c - P^u)r$ in $\mathscr{L}(L^2(D))$. Note that if we impose the assumption that Lebesgue measure of the domain converges, that is $|\Omega_n| \to |\Omega|$ as $n \to \infty$, then we obtain convergence of the projections $i_n(1-P_n^c-P_n^u)r_n \to i(1-P^c-P^u)r$ in $\mathscr{L}(L^2(D))$.

In the next few results, we consider an arbitrary finite dimensional subspace of $L^2(\Omega_n)$.

Lemma 3.4.3. *Let m be a positive integer. Suppose V_n is an m-dimensional subspace of $L^2(\Omega_n)$ with a basis $\{f_{1,n}, f_{2,n}, \dots f_{m,n}\}$ for each $n \in \mathbb{N}$, and V is an m-dimensional subspace of $L^2(\Omega)$ with a basis $\{f_1, f_2, \dots f_m\}$. If $i_n(f_{j,n}) \to i(f_j)$ in $L^2(D)$ as $n \to \infty$ for all $j = 1, \dots, m$, then there exists $\hat{c} > 0$ such that*

$$c_n := \inf\Big\{\Big\|\sum_{j=1}^m \xi_j f_{j,n}\Big\|_{L^2(\Omega_n)} : \xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m, |\xi| = 1\Big\} \geq \hat{c},$$

for all $n \in \mathbb{N}$.

Proof. Let $\xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m$ with $|\xi| = 1$. By convergence of the bases, we get

$$\begin{aligned}\Big\|\sum_{j=1}^m \xi_j i_n(f_{j,n}) - \sum_{j=1}^m \xi_j i(f_j)\Big\|_{L^2(D)} &\leq \sum_{j=1}^m |\xi_j| \|i_n(f_{j,n}) - i(f_j)\|_{L^2(D)} \\ &\leq \sum_{j=1}^m \|i_n(f_{j,n}) - i(f_j)\|_{L^2(D)} \\ &\to 0\end{aligned}$$

as $n \to \infty$. Notice that the above convergence does not depend on ξ. This means $\|\sum_{j=1}^m \xi_j f_{j,n}\|_{L^2(\Omega_n)} \to \|\sum_{j=1}^m \xi_j f_j\|_{L^2(\Omega)}$ uniformly with respect to $\xi \in \mathbb{R}^m$ with $|\xi| = 1$. Let

$$c := \inf\Big\{\Big\|\sum_{j=1}^m \xi_j f_j\Big\|_{L^2(\Omega)} : \xi = (\xi_1, \dots, \xi_m) \in \mathbb{R}^m, |\xi| = 1\Big\}.$$

In particular, choosing $\zeta > 0$ such that $c - \zeta > 0$, there exists $N_0 \in \mathbb{N}$ (independent of $\xi \in \mathbb{R}^m$ with $|\xi| = 1$) such that

$$\Big\| \sum_{j=1}^{m} \xi_j f_{j,n} \Big\|_{L^2(\Omega_n)} \geq \Big\| \sum_{j=1}^{m} \xi_j f_j \Big\|_{L^2(\Omega)} - \zeta,$$

for all $n > N_0$ and for all $\xi \in \mathbb{R}^m$ with $|\xi| = 1$. Since $\| \sum_{j=1}^{m} \xi_j f_j \|_{L^2(\Omega)} \geq c$, it follows that

$$\Big\| \sum_{j=1}^{m} \xi_j f_{j,n} \Big\|_{L^2(\Omega_n)} \geq c - \zeta, \tag{3.29}$$

for all $n > N_0$ and for all $\xi \in \mathbb{R}^m$ with $|\xi| = 1$. Taking the infimum over $\xi \in \mathbb{R}^m$ with $|\xi| = 1$, we obtian

$$c_n \geq c - \zeta > 0,$$

for all $n \geq N_0$. Finally, taking $\hat{c} := \min\{c_1, \dots c_{N_0}, c - \zeta\}$, the lemma follows. □

An immediate application of Lemma 3.4.3 is the following result.

Corollary 3.4.4. *Assume that V_n and V are as in Lemma 3.4.3 and that the convergence of bases $i_n(f_{j,n}) \to i(f_j)$ in $L^2(D)$ as $n \to \infty$ holds for all $j = 1, \dots, m$. Let u_n be a sequence such that $u_n \in V_n$ for each $n \in \mathbb{N}$. If $\|u_n\|_{L^2(\Omega_n)}$ is uniformly bounded then there exists a subsequence u_{n_k} such that $i_{n_k}(u_{n_k}) \to i(u)$ in $L^2(D)$ with a limit $u \in V$.*

Proof. For each $n \in \mathbb{N}$, we write $u_n = \sum_{j=1}^{m} \xi_{j,n} f_{j,n}$. By a standard argument in the proof of equivalence of norms for finite dimensional spaces,

$$\sum_{j=1}^{m} |\xi_{j,n}| \leq \frac{m}{c_n} \|u_n\|_{L^2(\Omega_n)},$$

for all $n \in \mathbb{N}$, where c_n is given in Lemma 3.4.3. It follows from the uniform boundedness of $\|u_n\|_{L^2(\Omega_n)}$ and Lemma 3.4.3 that $\sum_{j=1}^{m} |\xi_{j,n}|$ is uniform bounded. We can extract a subsequence ξ_{j,n_k} such that $\xi_{j,n_k} \to \xi_j$ for all $j = 1, \dots, m$. Hence,

$$i_{n_k}(u_{n_k}) \to i(u) := i\Big(\sum_{j=1}^{m} \xi_j f_j \Big)$$

in $L^2(D)$. □

Recall that we have $\dim(X_n^+) = \dim(X^+) < \infty$ for sufficiently large n. We set $d := \dim(X^+)$ and fix a certain basis $\{f_1, f_2, \dots, f_d\}$ of X^+. Define

$$f_{j,n} := P_n^+ r_n i(f_j), \tag{3.30}$$

for $j = 1, \dots, d$. Then we obtain a basis of X_n^+ as shown below.

Theorem 3.4.5. *There exists $N_0 \in \mathbb{N}$ such that $\{f_{1,n}, f_{2,n}, \ldots, f_{d,n}\}$ where $f_{j,n}$ defined by (3.30) is a basis of X_n^+ for each $n > N_0$. Moreover, $i_n(f_{j,n}) \to i(f_j)$ in $L^2(D)$ as $n \to \infty$ holds for all $j = 1, \ldots, d$.*

Proof. The convergence $i_n(f_{j,n}) \to i(f_j)$ is clear from the definition of $f_{j,n}$ and (3.23). Since X_n^+ is d-dimensional subspace for all n sufficiently large, it suffices to show that there exists $N_0 \in \mathbb{N}$ such that $f_{1,n}, f_{2,n}, \ldots, f_{d,n}$ are linearly independent for each $n > N_0$. We prove this by using mathematical induction on m for $m = 1, \ldots, d$ in the following statement: there exists $N_m \in \mathbb{N}$ such that $f_{1,n}, f_{2,n}, \ldots, f_{m,n}$ are linearly independent for each $n > N_m$.

The statement is trivial for $m = 1$. For the induction step, suppose that the statement is true for $1, \ldots, m$ with $m < d$, but there is no $N_{m+1} \in \mathbb{N}$ such that $f_{1,n}, f_{2,n}, \ldots, f_{m+1,n}$ are linearly independent for each $n > N_{m+1}$. This implies that for each $n \in \mathbb{N}$, there exists $l \in \mathbb{N}$ with $l > n$ such that $f_{1,l}, f_{2,l}, \ldots, f_{m+1,l}$ are linearly dependent. Thus, we can find a subsequence n_k (choosing $n_k > N_m$ for all $k \in \mathbb{N}$) such that $f_{1,n_k}, f_{2,n_k}, \ldots, f_{m+1,n_k}$ are linearly dependent. Since $f_{1,n_k}, f_{2,n_k}, \ldots, f_{m,n_k}$ are linearly independent, we can write

$$f_{m+1,n_k} = \sum_{j=1}^{m} \xi_{j,n_k} f_{j,n_k},$$

for all $k \in \mathbb{N}$. Since $i_{n_k}(f_{m+1,n_k}) \to i(f_{m+1})$ in $L^2(D)$ as $k \to \infty$, it follows that $\|f_{m+1,n_k}\|_{L^2(\Omega_{n_k})}$ is uniformly bounded. Corollary 3.4.4 implies that there exists a subsequence denoted again by f_{m+1,n_k} such that $i_{n_k}(f_{m+1,n_k}) \to i(f)$ in $L^2(D)$ as $k \to \infty$, where the limit f belongs to the m-dimensional subspace spanned by $f_1, f_2, \ldots, f_m$. By the uniqueness of a limit, we conclude that $f_{m+1} = f$. This is a contradiction to the assumption that $\{f_1, f_2, \ldots, f_d\}$ is a basis of X^+. Hence, the induction statement is true for $m + 1$ and the theorem is proved. □

As a consequence, we obtain the following convergence of a bounded sequence with each term belongs to a sequence of the spaces X_n^+.

Corollary 3.4.6. *Let $(w_n)_{n \in \mathbb{N}}$ be a sequence with $w_n \in X_n^+$ for each $n \in \mathbb{N}$. If $\|w_n\|_{L^2(\Omega_n)}$ (or $\|w_n\|_{X_n^+}$) is uniformly bounded, then there exists a subsequence w_{n_k} such that*

$$i_{n_k}(w_{n_k}) \to i(w)$$

in $L^2(D)$ with the limit $w \in X^+$.

Proof. Note that $\|w_n\|_{X_n^+} < C$ for all $n \in \mathbb{N}$ implies $\|w_n\|_{L^2(\Omega_n)} < C$ for all $n \in \mathbb{N}$ (see (3.27)). The result follows immediately from Corollary 3.4.4 and Theorem 3.4.5. □

Remark 3.4.7. The above result implies that there exists a subsequence w_{n_k} such that $\|w_{n_k}\|_{L^2(\Omega_n)} \to \|w\|_{L^2(\Omega)}$ but does not implies $\|w_{n_k}\|_{X_n^+} \to \|w\|_{X^+}$ as degenerate semigroup only converges uniformly on compact subsets of $(0, \infty)$.

We note that the convergence of components in finite dimensional subspaces in Lemma 3.4.1 (ii) is indeed a strong convergence as stated below.

Lemma 3.4.8. *Suppose that the assumptions in Lemma 3.4.1 (ii) are satisfied. If $i_n\phi_n \rightharpoonup i\phi$ in $L^2(D)$ weakly, then $i_n\phi_n^s \rightharpoonup i\phi^s$ in $L^2(D)$ weakly and $i_n\phi_n^* \to i\phi^*$ in $L^2(D)$ strongly for $* = c, u$.*

Proof. From Lemma 3.4.1, we have a weak convergence $i_n\phi_n^* \rightharpoonup i\phi^*$ in $L^2(D)$ for $* = s, c, u$. Since X_n^u is a finite dimensional subspace, Corollary 3.4.6 implies that for any subsequence n_k we can extract a further convergent subsequence $i_{n_k}\phi_{n_k}^u \to \xi^u$ in $L^2(D)$ for some $\xi^u \in L^2(D)$. The weak convergence above implies that $\xi^u = i\phi^u$ and hence $i_{n_k}\phi_{n_k}^u \to i\phi^u$ in $L^2(D)$ strongly. Since this is true for any subsequence, we conclude that the whole sequence $i_n\phi_n^u$ converges to $i\phi^u$ in $L^2(D)$ strongly. The same argument shows that $i_n(x_n^c) \to i(x^c)$ in $L^2(D)$. □

3.4.2 Characterisation of upper and lower semicontinuity

We give some equivalent statements for upper and lower semicontinuity mentioned in Theorem 3.3.3 and Theorem 3.3.4. We simplify the notations by considering bounded subsets $\mathscr{A}_n, \mathscr{A}$ of $L^2(D)$.

Lemma 3.4.9 (Characterisation of upper semicontinuity)**.** *The following statements are equivalent.*

(i) $\sup_{v\in\mathscr{A}_n} \inf_{u\in\mathscr{A}} \|v-u\|_{L^2(D)} \to 0$ *as* $n \to \infty$.

(ii) *For any sequence $\{v_n\}_{n\in\mathbb{N}}$ with $v_n \in \mathscr{A}_n$, we have $\inf_{u\in\mathscr{A}} \|v_n - u\|_{L^2(D)} \to 0$ as $n \to \infty$.*

(iii) *For any sequence $\{v_n\}_{n\in\mathbb{N}}$ with $v_n \in \mathscr{A}_n$, if $\{v_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence, then there exist a further subsequence (denoted again by v_{n_k}) and a sequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ with $u_{n_k} \in \mathscr{A}$ such that $\|v_{n_k} - u_{n_k}\|_{L^2(D)} \to 0$ as $k \to \infty$.*

Proof. The statement (i) $\Rightarrow$ (ii) is clear as

$$\inf_{u\in\mathscr{A}} \|v_n - u\|_{L^2(D)} \le \sup_{v\in\mathscr{A}_n} \inf_{u\in\mathscr{A}} \|v-u\|_{L^2(D)},$$

for any sequence $\{v_n\}_{n\in\mathbb{N}}$ with $v_n \in \mathscr{A}_n$.

For (ii) $\Rightarrow$ (i), we prove by contrapositive. Suppose that (i) fails. Then

$$\limsup_{n\to\infty} \left\{ \sup_{v\in\mathscr{A}_n} \inf_{u\in\mathscr{A}} \|v-u\|_{L^2(D)} \right\} =: a > 0.$$

Hence, there exists a subsequence $n_k \to \infty$ such that

$$\sup_{v\in\mathscr{A}_{n_k}} \inf_{u\in\mathscr{A}} \|v-u\|_{L^2(D)} \to a,$$

as $k\to\infty$. This implies that there exists $v_{n_k} \in \mathscr{A}_{n_k}$ such that

$$\inf_{u\in\mathscr{A}} \|v_{n_k} - u\|_{L^2(D)} > a/2,$$

for all $k\in\mathbb{N}$. Hence, (ii) fails.

For the statement (ii) $\Leftrightarrow$ (iii), notice first that $\inf_{u\in\mathscr{A}} \|v_n - u\|_{L^2(D)} \to 0$ as $n\to\infty$ if and only if there exists $u_n \in \mathscr{A}$ such that $\|v_n - u_n\|_{L^2(D)} \to 0$ as $n\to\infty$. To see this, we choose $u_n\in\mathscr{A}$ such that $\|v_n - u_n\|_{L^2(D)} < \inf_{u\in\mathscr{A}} \|v_n - u\|_{L^2(D)} + 1/n$ for each $n\in\mathbb{N}$. Then the forward implication follows. The backward implication is clear as $\inf_{u\in\mathscr{A}} \|v_n - u\|_{L^2(D)} < \|v_n - u_n\|_{L^2(D)}$ for all $u_n \in \mathscr{A}$. The statement (ii) $\Leftrightarrow$ (iii) then simply follows from the above and a standard subsequence characterisation of a limit. □

By a similar argument, we can state the following lemma.

Lemma 3.4.10 (Characterisation of lower semicontinuity)**.** *The following statements are equivalent.*

(i) $\sup_{u\in\mathscr{A}} \inf_{v\in\mathscr{A}_n} \|v-u\|_{L^2(D)} \to 0$ *as* $n\to\infty$.

(ii) *For any sequence* $\{u_n\}_{n\in\mathbb{N}}$ *with* $u_n\in\mathscr{A}$, *we have* $\inf_{v\in\mathscr{A}_n} \|v-u_n\|_{L^2(D)} \to 0$ *as* $n\to\infty$.

(iii) *For any sequence* $\{u_n\}_{n\in\mathbb{N}}$ *with* $u_n\in\mathscr{A}$, *if* $\{u_{n_k}\}_{k\in\mathbb{N}}$ *is a subsequence, then there exist a further subsequence (denoted again by* u_{n_k}*) and a sequence* $\{v_{n_k}\}_{k\in\mathbb{N}}$ *with* $v_{n_k}\in\mathscr{A}_{n_k}$ *such that* $\|v_{n_k} - u_{n_k}\|_{L^2(D)} \to 0$ *as* $k\to\infty$.

3.5 Convergence of unstable invariant manifolds

In this section, we prove upper and lower semicontinuity of local unstable invariant manifolds. We first show pointwise convergence of global unstable manifolds for the modified systems in Section 3.5.1. Consequently, we prove Theorem 3.3.3 in Section 3.5.2.

3.5.1 Convergence of global unstable manifolds

Let

$$Y_n = \{h \in C(X_n^+, X_n^-) : h(0) = 0 \text{ and } h \text{ is } \nu^{-1}\text{-Lipschitz } \}.$$

Then Y_n is a complete metric space with the norm

$$\|h\|_{\mathrm{Lip}} = \sup_{w \neq 0} \frac{\|h(w)\|_{X_n^-}}{\|w\|_{X_n^+}}. \tag{3.31}$$

We define $T_{t,n} : Y_n \to Y_n$ for $t \geq 0$ by

$$T_{t,n}(h) = \tilde{h},$$

where $\tilde{h} \in Y_n$ such that $\mathrm{graph}(\tilde{h}) = \Phi_{t,n}(\mathrm{graph}(h))$. Fix $t > 0$ sufficiently large such that

$$K := \nu(\nu - \mu)^{-1} \exp((\alpha - \beta + \varepsilon(2 + \mu + \nu^{-1}))t) < 1. \tag{3.32}$$

As in Theorem 3.2.5, $T_{t,n}$ is a contraction on Y_n with a uniform contraction constant K for all $n \in \mathbb{N}$. Moreover, W_n^+ is a graph of the fixed point h_n^+ of $T_{t,n}$. To prove convergence of global unstable manifolds, we show that the fixed point h_n^+ of $T_{t,n}$ converges to the fixed point h^+ of T_t.

Lemma 3.5.1. *Suppose that Ω_n and Ω satisfy Assumption 1.4.1 and $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco. Then for every $v \in X^-$, there exists $v_n \in X_n^-$ such that $i_n(v_n) \to i(v)$ in $L^2(D)$.*

Proof. Let $v \in X^- \subset L^2(\Omega)$. By the density of $H_0^1(\Omega)$ in $L^2(\Omega)$ and Mosco convergence, it follows from a standard diagonal procedure that there exists $\xi_n \in H_0^1(\Omega_n)$ such that

$$i_n(\xi_n) \to i(v)$$

in $L^2(D)$ as $n \to \infty$. By Lemma 3.4.1 (i), we get $i_n P_n^-(\xi_n) \to iP^-(v) = i(v)$ in $L^2(D)$ as $n \to \infty$. By taking $v_n := P_n^-(\xi_n)$, the lemma follows. □

Let us define $h \in Y$ by

$$h(w) := \frac{1}{C} h^+(w), \tag{3.33}$$

for all $w \in X^+$ where C is a positive constant satisfying

$$\|P^+\| \|1 - P_n^+\| M_1 M_2 \leq C, \tag{3.34}$$

for all $n \in \mathbb{N}$. Note that although $i_n(1 - P_n^+) r_n$ does not converge to $i(1 - P^+) r$ in $\mathscr{L}(L^2(D))$ with respect to the operator norm, we use $\|1 - P_n^+\| \leq 1 + \|P_n^+\|$ and (3.23) to obtain a bound C above. We will use that both $\|P_n^+\|$ and $\|P_n^-\| = \|1 - P_n^+\|$ are bounded without further notice.

In the next lemma, we obtain an approximation of h by functions in Y_n.

Lemma 3.5.2. *Let h be as in (3.33). There exists a sequence $\{h_n\}$ with $h_n \in Y_n$ for each $n \in \mathbb{N}$ such that*

(i) *$i_n h_n P_n^+ r_n(u) \to i h P^+ r(u)$ in $L^2(D)$ as $n \to \infty$ for all $u \in L^2(D)$*

(ii) *for each $m \in \mathbb{N}$, we have $i_n T_{t,n}^m(h_n) P_n^+ r_n(u) \to i T_t^m(h) P^+ r(u)$ in $L^2(D)$ as $n \to \infty$ for all $u \in L^2(D)$.*

Proof. We construct $h_n \in Y_n$ as follows. Define $h_n : X_n^+ \to X_n^-$ by

$$h_n(w) := \frac{1}{C}(1 - P_n^+) r_n i h^+ P^+ r i_n(w), \tag{3.35}$$

for $w \in X_n^+$. It is clear that $h_n(0) = 0$. Moreover, for $w_1, w_2 \in X_n^+$, it follows from the Lipschitz continuity of h^+ and the choice of C in (3.34) that

$$\begin{aligned}
&\|h_n(w_1) - h_n(w_2)\|_{X_n^-} \\
&\quad \leq M_1 \|h_n(w_1) - h_n(w_2)\|_{L^2(\Omega_n)} \\
&\quad = M_1 \Big\| \frac{1}{C}(1 - P_n^+) r_n i h^+ P^+ r i_n(w_1) - \frac{1}{C}(1 - P_n^+) r_n i h^+ P^+ r i_n(w_2) \Big\|_{L^2(\Omega_n)} \\
&\quad \leq M_1 \frac{1}{C} \|1 - P_n^+\| \, \|r_n\| \, \|i\| \, \|h^+ P^+ r i_n(w_1) - h^+ P^+ r i_n(w_2)\|_{L^2(\Omega)} \\
&\quad \leq M_1 \frac{1}{C} \|1 - P_n^+ \| \, \|h^+ P^+ r i_n(w_1) - h^+ P^+ r i_n(w_2)\|_{X^-} \\
&\quad \leq M_1 \frac{1}{C} \nu^{-1} \|1 - P_n^+\| \, \|P^+ r i_n(w_1) - P^+ r i_n(w_2)\|_{X^+} \\
&\quad \leq M_1 \frac{1}{C} \nu^{-1} M_2 \|1 - P_n^+\| \, \|P^+ r i_n(w_1) - P^+ r i_n(w_2)\|_{L^2(\Omega)} \\
&\quad \leq M_1 \frac{1}{C} \nu^{-1} M_2 \|1 - P_n^+\| \, \|P^+\| \, \|r\| \, \|i_n\| \, \|w_1 - w_2\|_{L^2(\Omega_n)} \\
&\quad \leq \nu^{-1} \|w_1 - w_2\|_{X_n^+}.
\end{aligned}$$

Hence, h_n is $\nu^{-1}-$Lipschitz and thus $h_n \in Y_n$. Note that we need to be careful about the norm used in the above calculation. In particular, we take care of the equivalence of norms on X^- and X^+ given in (3.10) and (3.11). This will be applied throughout this chapter without further comment.

We claim that h_n defined above satisfies the properties (i) and (ii). For (i), let $u \in L^2(D)$ be arbitrary. By Lemma 3.5.1, there exists $(v_n)_{n\in\mathbb{N}}$ with $v_n \in X_n^-$ such that

$$i_n(v_n) \to ih^+P^+r(u) \tag{3.36}$$

in $L^2(D)$ as $n \to \infty$. We have from the triangle inequality that

$$\begin{aligned}
&\|i_n h_n P_n^+ r_n(u) - ihP^+r(u)\|_{L^2(D)} \\
&\quad = \left\|\frac{1}{C} i_n(1-P_n^+)r_n ih^+P^+ri_nP_n^+r_n(u) - \frac{1}{C} ih^+P^+r(u)\right\|_{L^2(D)} \\
&\quad \le \frac{1}{C}\|i_n(1-P_n^+)r_n ih^+P^+ri_nP_n^+r_n(u) - i_n(1-P_n^+)r_n ih^+P^+r(u)\|_{L^2(D)} \\
&\qquad + \frac{1}{C}\|i_n(1-P_n^+)r_n ih^+P^+r(u) - ih^+P^+r(u)\|_{L^2(D)}.
\end{aligned} \tag{3.37}$$

Using the equivalence of norms on X^- and X^+, we can calculate

$$\begin{aligned}
&\frac{1}{C}\|i_n(1-P_n^+)r_n ih^+P^+ri_nP_n^+r_n(u) - i_n(1-P_n^+)r_n ih^+P^+r(u)\|_{L^2(D)} \\
&\quad \le \frac{1}{C}\|i_n\|\,\|1-P_n^+\|\,\|r_n\|\,\|ih^+P^+ri_nP_n^+r_n(u) - ih^+P^+r(u)\|_{L^2(D)} \\
&\quad \le \frac{1}{C}\|1-P_n^+\|\,\|ih^+P^+ri_nP_n^+r_n(u) - ih^+P^+riP^+r(u)\|_{L^2(D)} \\
&\quad \le \frac{1}{C}\|1-P_n^+\|\,\|h^+P^+ri_nP_n^+r_n(u) - h^+P^+riP^+r(u)\|_{X^-} \\
&\quad \le \frac{1}{C}\nu^{-1}\|1-P_n^+\|\,\|P^+ri_nP_n^+r_n(u) - P^+riP^+r(u)\|_{X^+} \\
&\quad \le \frac{1}{C}\nu^{-1}M_2\|1-P_n^+\|\,\|P^+ri_nP_n^+r_n(u) - P^+riP^+r(u)\|_{L^2(\Omega)} \\
&\quad \le \frac{1}{C}\nu^{-1}M_2\|1-P_n^+\|\,\|P^+r\|\,\|i_nP_n^+r_n(u) - iP^+r(u)\|_{L^2(D)} \\
&\quad \to 0
\end{aligned} \tag{3.38}$$

as $n \to \infty$, where we use (3.23) and the boundedness of $\|1-P_n^+\|$ in the last step. For the second term on the right of (3.37), we use (3.36) and $i_n(1-P_n^+)r_n i_n v_n = i_n v_n$ to obtain

$$\begin{aligned}
&\frac{1}{C}\|i_n(1-P_n^+)r_n ih^+P^+r(u) - ih^+P^+r(u)\|_{L^2(D)} \\
&\quad\le \frac{1}{C}\|i_n(1-P_n^+)r_n ih^+P^+r(u) - i_n(v_n)\|_{L^2(D)} \\
&\qquad + \frac{1}{C}\|i_n(v_n) - ih^+P^+r(u)\|_{L^2(D)} \\
&\quad= \frac{1}{C}\|i_n(1-P_n^+)r_n ih^+P^+r(u) - i_n(1-P_n^+)r_n i_n(v_n)\|_{L^2(D)} \\
&\qquad + \frac{1}{C}\|i_n(v_n) - ih^+P^+r(u)\|_{L^2(D)} \\
&\quad\le \frac{1}{C}\|i_n(1-P_n^+)r_n\|\;\|ih^+P^+r(u) - i_n(v_n)\|_{L^2(D)} \\
&\qquad + \frac{1}{C}\|i_n(v_n) - ih^+P^+r(u)\|_{L^2(D)} \\
&\quad\to 0
\end{aligned} \tag{3.39}$$

as $n\to\infty$. It follows from (3.37) – (3.39) that

$$\|i_n h_n P_n^+ r_n(u) - ihP^+r(u)\|_{L^2(D)} \to 0$$

as $n\to\infty$. Since the above argument is valid for any $u\in L^2(D)$, statement (i) follows.

We next prove (ii) by induction on $m\in\mathbb{N}$. By part (i) of this proof, the property (ii) is true for $m=0$. For induction step, assume that

$$i_n T^m_{t,n}(h_n)P_n^+ r_n(u) \to iT^m_t(h)P^+r(u)$$

in $L^2(D)$ as $n\to\infty$ for all $u\in L^2(D)$ holds true for $m=0,1,\ldots,k$. We need to show that

$$i_n T^{k+1}_{t,n}(h_n)P_n^+ r_n(u) \to iT^{k+1}_t(h)P^+r(u) \tag{3.40}$$

in $L^2(D)$ as $n\to\infty$ for all $u\in L^2(D)$. Let $u\in L^2(D)$ be arbitrary. We set $w := P^+r(u)\in X^+$ and $w_n := P_n^+r_n(u)\in X_n^+$. It follows from (3.23) that

$$i_n(w_n)\to i(w) \tag{3.41}$$

in $L^2(D)$ as $n\to\infty$. Since $\mathrm{graph}(T^{k+1}_t(h)) = \Phi_t(\mathrm{graph}(T^k_t(h))$, there exists $w_0\in X^+$ such that

$$\Phi_t\left(w_0\oplus T^k_t(h)(w_0)\right) = w\oplus T^{k+1}_t(h)(w).$$

For each $n\in\mathbb{N}$, we define $w_{0,n} := P_n^+r_n i(w_0)$. Again by (3.23), we have $i_n(w_{0,n})\to i(w_0)$ in $L^2(D)$ as $n\to\infty$. Moreover, by induction hypothesis,

$$i_nT^k_{t,n}(h_n)(w_{0,n}) = i_nT^k_{t,n}(h_n)P_n^+r_n i(w_0) \to iT^k_t(h)P^+ri(w_0) = iT^k_t(h)(w_0)$$

in $L^2(D)$ as $n \to \infty$. Hence, it follows from the convergence of solutions under domain perturbation in (3.20) that

$$i_n \Phi_{t,n} \left(w_{0,n} \oplus T^k_{t,n}(h_n)(w_{0,n}) \right) \to i\Phi_t \left(w_0 \oplus T^k_t(h)(w_0) \right) = i \left(w \oplus T^{k+1}_t(h)(w) \right)$$

in $L^2(D)$ as $n \to \infty$. Since $\operatorname{graph}(T^{k+1}_{t,n}(h_n)) = \Phi_{t,n}(\operatorname{graph}(T^k_{t,n}(h_n))$, there exists $\xi_n \in X^+_n$ such that

$$\Phi_{t,n} \left(w_{0,n} \oplus T^k_{t,n}(h_n)(w_{0,n}) \right) = \xi_n \oplus T^{k+1}_{t,n}(h_n)(\xi_n),$$

for each $n \in \mathbb{N}$. Hence,

$$i_n \left(\xi_n \oplus T^{k+1}_{t,n}(h_n)(\xi_n) \right) \to i \left(w \oplus T^{k+1}_t(h)(w) \right) \tag{3.42}$$

in $L^2(D)$ as $n \to \infty$. By Lemma 3.4.1 (i), it follows from (3.42) that

$$i_n(\xi_n) \to i(w) \tag{3.43}$$

and

$$i_n \left(T^{k+1}_{t,n}(h_n)(\xi_n) \right) \to i \left(T^{k+1}_t(h)(w) \right) \tag{3.44}$$

in $L^2(D)$ as $n \to \infty$. We obtain from (3.41) and (3.43) that

$$\|i_n(\xi_n) - i_n(w_n)\|_{L^2(D)} \to 0$$

as $n \to \infty$. Since $T^{k+1}_{t,n}(h_n)$ is ν^{-1}-Lipschitz, it follows that

$$\begin{aligned} \left\| T^{k+1}_{t,n}(h_n)(\xi_n) - T^{k+1}_{t,n}(h_n)(w_n) \right\|_{L^2(\Omega_n)} &\le \left\| T^{k+1}_{t,n}(h_n)(\xi_n) - T^{k+1}_{t,n}(h_n)(w_n) \right\|_{X^-_n} \\ &\le \nu^{-1} \|\xi_n - w_n\|_{X^+_n} \\ &\le \nu^{-1} M_2 \|\xi_n - w_n\|_{L^2(\Omega_n)} \\ &\to 0 \end{aligned} \tag{3.45}$$

as $n \to \infty$. By definitions of w_n and w together with (3.44) and (3.45), we conclude that

$$\begin{aligned} &\left\| i_n T^{k+1}_{t,n}(h_n) P^+_n r_n(u) - i T^{k+1}_t(h) P^+ r(u) \right\|_{L^2(D)} \\ &= \left\| i_n T^{k+1}_{t,n}(h_n)(w_n) - i T^{k+1}_t(h)(w) \right\|_{L^2(D)} \\ &\le \left\| i_n T^{k+1}_{t,n}(h_n)(w_n) - i_n T^{k+1}_{t,n}(h_n)(\xi_n) \right\|_{L^2(D)} \\ &\quad + \left\| i_n T^{k+1}_{t,n}(h_n)(\xi_n) - i T^{k+1}_t(h)(w) \right\|_{L^2(D)} \\ &\to 0 \end{aligned}$$

as $n \to \infty$. As $u \in L^2(D)$ was arbitrary, we have shown (3.40). □

We next recall a standard result on the rate of convergence to the fixed point of a contraction mapping below (see for example [74, Remark 1.2.3 (ii)]).

Lemma 3.5.3. *Let (X, d) be a complete metric space and $T : X \to X$ be a contraction mapping with a contraction constant $k \in [0, 1)$, that is,*

$$d(Tx_1, Tx_2) \leq kd(x_1, x_2),$$

for all $x_1, x_2 \in X$. Denote by x^* the unique fixed point of T. Then for any $x \in X$,

$$d(x^*, T^n x) \leq \frac{k^n}{1-k} d(x, Tx), \tag{3.46}$$

for all $n \in \mathbb{N}$.

We prove the pointwise convergence of global unstable invariant manifolds in the following theorem.

Theorem 3.5.4. *Assume that all assumptions in Theorem 3.3.3 are satisfied and $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco. Then we have*

$$i_n h_n^+ P_n^+ r_n(u) \to i h^+ P^+ r(u)$$

in $L^2(D)$ as $n \to \infty$ for all $u \in L^2(D)$.

Proof. Fix $u \in L^2(D)$ and let $\zeta > 0$ be arbitrary. We can choose $m_0 \in \mathbb{N}$ independent of n such that the contraction constant K in (3.32) satisfies

$$\frac{K^{m_0}}{1-K} 2\nu^{-1} \|P_n^+ r_n(u)\|_{X_n^+} \leq \frac{\zeta}{3}, \tag{3.47}$$

for all $n \in \mathbb{N}$ and

$$\frac{K^{m_0}}{1-K} 2\nu^{-1} \|P^+ r(u)\|_{X^+} \leq \frac{\zeta}{3}. \tag{3.48}$$

We take $h_n \in Y_n$ and $h \in Y$ as in Lemma 3.5.2. Then by the definition of Lip-norm on

Y and Y_n (see (3.16) and (3.31), respectively), we see that

$$\begin{aligned}
&\left\|i_n h_n^+ P_n^+ r_n(u) - i h^+ P^+ r(u)\right\|_{L^2(D)} \\
&\quad\le \left\|i_n h_n^+ P_n^+ r_n(u) - i_n T_{t,n}^{m_0}(h_n) P_n^+ r_n(u)\right\|_{L^2(D)} \\
&\qquad + \left\|i_n T_{t,n}^{m_0}(h_n) P_n^+ r_n(u) - i T_t^{m_0}(h) P^+ r(u)\right\|_{L^2(D)} \\
&\qquad + \left\|i T_t^{m_0}(h) P^+ r(u) - i h^+ P^+ r(u)\right\|_{L^2(D)} \\
&\quad\le \left\|h_n^+ P_n^+ r_n(u) - T_{t,n}^{m_0}(h_n) P_n^+ r_n(u)\right\|_{X_n^-} \\
&\qquad + \left\|i_n T_{t,n}^{m_0}(h_n) P_n^+ r_n(u) - i T_t^{m_0}(h) P^+ r(u)\right\|_{L^2(D)} \\
&\qquad + \left\|T_t^{m_0}(h) P^+ r(u) - h^+ P^+ r(u)\right\|_{X^-} \\
&\quad\le \|h_n^+ - T_{t,n}^{m_0}(h_n)\|_{\mathrm{Lip}} \|P_n^+ r_n(u)\|_{X_n^+} \\
&\qquad + \left\|i_n T_{t,n}^{m_0}(h_n) P_n^+ r_n(u) - i T_t^{m_0}(h) P^+ r(u)\right\|_{L^2(D)} \\
&\qquad + \|T_t^{m_0}(h) - h^+\|_{\mathrm{Lip}} \|P^+ r(u)\|_{X^+},
\end{aligned} \tag{3.49}$$

for all $n \in \mathbb{N}$. By Lemma 3.5.3, we have

$$\|h^+ - T_t^{m_0}(h)\|_{\mathrm{Lip}} \le \frac{K^{m_0}}{1-K} \|h - T_t(h)\|_{\mathrm{Lip}} \le \frac{K^{m_0}}{1-K} 2\nu^{-1}. \tag{3.50}$$

Similarly,

$$\|h_n^+ - T_{t,n}^{m_0}(h_n)\|_{\mathrm{Lip}} \le \frac{K^{m_0}}{1-K} \|h_n - T_{t,n}(h_n)\|_{\mathrm{Lip}} \le \frac{K^{m_0}}{1-K} 2\nu^{-1}, \tag{3.51}$$

for all $n \in \mathbb{N}$. Moreover, Lemma 3.5.2 (ii) implies that there exists $N_0 \in \mathbb{N}$ such that

$$\left\|i_n T_{t,n}^{m_0}(h_n) P_n^+ r_n(u) - i T_t^{m_0}(h) P^+ r(u)\right\|_{L^2(D)} \le \frac{\zeta}{3}, \tag{3.52}$$

for all $n > N_0$. It follows from (3.49) – (3.52) that

$$\begin{aligned}
\left\|i_n h_n^+ P_n^+ r_n(u) - i h^+ P^+ r(u)\right\|_{L^2(D)} \le{}& \frac{K^{m_0}}{1-K} 2\nu^{-1} \|P_n^+ r_n(u)\|_{X_n^+} + \frac{\zeta}{3} \\
&+ \frac{K^{m_0}}{1-K} 2\nu^{-1} \|P^+ r(u)\|_{X^+},
\end{aligned}$$

for all $n > N_0$. By our choice of m_0 in (3.47) and (3.48), we conclude that

$$\left\|i_n h_n^+ P_n^+ r_n(u) - i h^+ P^+ r(u)\right\|_{L^2(D)} \le \zeta,$$

for all $n > N_0$. As $\zeta > 0$ was arbitrary, we get

$$i_n h_n^+ P_n^+ r_n(u) \to i h^+ P^+ r(u)$$

in $L^2(D)$ as $n \to \infty$. Since this argument works for any $u \in L^2(D)$, the statement of the theorem follows. □

3.5.2 Upper and lower semicontinuity of local unstable manifolds

We are now in the position to prove Theorem 3.3.3.

Proof of Theorem 3.3.3 (ii). As discussed at the end of Section 3.3, there exist δ_1 and δ_2 such that $W_n^u = W_n^u(U_n)$ is a local unstable invariant manifold where $U_n = V_{1,n} \times V_{2,n}$ with $V_{1,n}$ is a ball of radius δ_1 in X_n^- and $V_{2,n}$ is a ball of radius δ_2 in X_n^+ for all $n \in \mathbb{N}$. Moreover, a similar statement holds for the unperturbed problem. By the equivalence of norms on X_n^- and X_n^+ with uniform parameters α and β, we can chose $\delta > 0$ such that $B_n := B_{L^2(\Omega_n)}(0, \delta) \subset V_{1,n} \times V_{2,n}$ for all $n \in \mathbb{N}$ and $B := B_{L^2(\Omega)}(0, \delta) \subset V_1 \times V_2$.

To prove the lower semicontinuity, we show that for every $\zeta > 0$, there exists $N_0 \in \mathbb{N}$ independent of $u \in \text{graph}(h^+) \cap B$ such that

$$\inf_{v \in \text{graph}(h_n^+) \cap B_n} \|i(u) - i_n(v)\|_{L^2(D)} < \zeta,$$

for all $n > N_0$ and for all $u \in \text{graph}(h^+) \cap B$. Let $\zeta > 0$ be arbitrary. By the Lipschitz continuity of $h^+ : X^+ \to X^-$ (taking (3.10) and (3.11) into account), we have that for every $w_0 \in X^+$, there exists $\rho > 0$ such that

$$\|(w \oplus h^+(w)) - (w_0 \oplus h^+(w_0))\|_{L^2(\Omega)} < \frac{\zeta}{2}, \tag{3.53}$$

for all $w \in B_{X^+}(w_0, \rho) := \{w \in X^+ : \|w - w_0\|_{L^2(\Omega)} < \rho\}$. Note that ρ is independent of $w_0 \in X^+$. We set

$$W := P^+ \big(\text{graph}(h^+) \cap B\big) = \{w \in X^+ : w \oplus h^+(w) \in B\}.$$

Since $\dim(X^+) < \infty$, the set $\overline{W}$ is compact. If we take the open cover $\{B_{X^+}(w, \rho) : w \in W\}$ of $\overline{W}$, then there exists a finite subcover

$$\{B_{X^+}(w_k, \rho) : w_k \in W, k = 1, \dots, m\}.$$

Hence,

$$W \subset \bigcup_{k=1}^{m} B_{X^+}(w_k, \rho). \tag{3.54}$$

Denoted by $\Delta := \min\{\delta - \|w_k \oplus h^+(w_k)\|_{L^2(\Omega)} : k = 1, \dots, m\}$. Setting $w_{k,n} := P_n^+ r_n i(w_k) \in X_n^+$ for $n \in \mathbb{N}$ and $k = 1, \dots, m$. We have from (3.23) that $i_n(w_{k,n}) \to i(w_k)$ in $L^2(D)$ as $n \to \infty$ for each $k = 1, \dots, m$. Moreover, by Theorem 3.5.4 $i_n h_n^+(w_{k,n}) \to i h^+(w_k)$ in $L^2(D)$ as $n \to \infty$ for each $k = 1, \dots, m$. Hence, for each $k = 1, \dots, m$ we can find $N_k \in \mathbb{N}$ such that

$$\|i_n(w_{k,n} \oplus h_n^+(w_{k,n})) - i(w_k \oplus h^+(w_k))\|_{L^2(D)} < \min\left\{\frac{\zeta}{2}, \Delta\right\},$$

for all $n > N_k$. Setting $N_0 = \max\{N_k : k = 1, \ldots, m\}$, we get

$$\|i_n(w_{k,n} \oplus h_n^+(w_{k,n})) - i(w_k \oplus h^+(w_k))\|_{L^2(D)} < \min\left\{\frac{\zeta}{2}, \Delta\right\}, \tag{3.55}$$

for all $n > N_0$ and for all $k = 1, \ldots, m$. Using (3.55), we obtain

$$\begin{aligned}
\|w_{k,n} \oplus h_n^+(w_{k,n})\|_{L^2(\Omega_n)} &\leq \|i_n(w_{k,n} \oplus h_n^+(w_{k,n})) - i(w_k \oplus h^+(w_k))\|_{L^2(D)} \\
&\quad + \|w_k \oplus h_n^+(w_k)\|_{L^2(\Omega)} \\
&< \|w_k \oplus h_n^+(w_k)\|_{L^2(\Omega)} + \Delta \\
&\leq \|w_k \oplus h_n^+(w_k)\|_{L^2(\Omega)} + (\delta - \|w_k \oplus h_n^+(w_k)\|_{L^2(\Omega)}) \\
&= \delta,
\end{aligned}$$

for all $n > N_0$ and for all $k = 1, \ldots, m$. Hence, $w_{k,n} \oplus h_n^+(w_{k,n}) \in \text{graph}(h_n^+) \cap B_n$ for all $n > N_0$ and for all $k = 1, \ldots, m$. Let u be in $\text{graph}(h^+) \cap B$ and write $u = w \oplus h^+(w)$ for some $w \in W$. By (3.54), there exists $k \in \{1, \ldots, m\}$ such that $w \in B_{X^+}(w_k, \rho)$. It follows from (3.53) and (3.55) that

$$\begin{aligned}
&\|i_n(w_{k,n} \oplus h_n^+(w_{k,n})) - i(w \oplus h^+(w))\|_{L^2(D)} \\
&\quad \leq \|i_n(w_{k,n} \oplus h_n^+(w_{k,n})) - i(w_k \oplus h^+(w_k))\|_{L^2(D)} \\
&\quad\quad + \|i(w_k \oplus h^+(w_k)) - i(w \oplus h^+(w))\|_{L^2(D)} \\
&\quad < \frac{\zeta}{2} + \frac{\zeta}{2} \\
&\quad = \zeta,
\end{aligned}$$

for all $n > N_0$. Since $w_{k,n} \oplus h_n^+(w_{k,n}) \in \text{graph}(h_n^+) \cap B_n$ for all $n > N_0$, we get

$$\inf_{v \in \text{graph}(h_n^+) \cap B_n} \|i(u) - i_n(v)\|_{L^2(D)} < \zeta,$$

for all $n > N_0$. The above estimate holds for every $u = w \oplus h^+(w) \in \text{graph}(h^+) \cap B$ and notice that N_0 is independent of u. Hence,

$$\inf_{v \in \text{graph}(h_n^+) \cap B_n} \|i(u) - i_n(v)\|_{L^2(D)} < \zeta,$$

for all $n > N_0$ and for all $u \in \text{graph}(h^+) \cap B$. As $\zeta > 0$ was arbitrary, we obtain the lower semicontinuity. □

Using our characterisation in Lemma 3.4.9, we can show the upper semicontinuity of unstable invariant manifolds.

Proof of Theorem 3.3.3 (i). We consider the same neighbourhood B_n and B as in the proof above. By Lemma 3.4.9, we need to show that for any sequence $\{\xi_n\}_{n\in\mathbb{N}}$ with $\xi_n \in \mathrm{graph}(h_n^+) \cap B_n$, if $\{\xi_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence then there exist a further subsequence (denoted again by ξ_{n_k}) and a sequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ with $u_{n_k} \in \mathrm{graph}(h^+) \cap B$ such that $\|i_{n_k}(\xi_{n_k}) - i(u_{n_k})\|_{L^2(D)} \to 0$ as $k \to \infty$.

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence with $\xi_n \in \mathrm{graph}(h_n^+) \cap B_n$ and $(\xi_{n_k})_{k\in\mathbb{N}}$ be an arbitrary subsequence. We write $\xi_{n_k} := w_{n_k} \oplus h_{n_k}^+(w_{n_k})$ for some $w_{n_k} \in X_{n_k}^+$. Since $\|\xi_{n_k}\|_{L^2(\Omega_{n_k})} = \|w_{n_k} \oplus h_{n_k}^+(w_{n_k})\|_{L^2(\Omega_{n_k})} < \delta$ for all $k \in \mathbb{N}$, we can apply Corollary 3.4.6 to extract a subsequence from w_{n_k} (indexed again by n_k) such that

$$i_{n_k}(w_{n_k}) \to i(w)$$

in $L^2(D)$ with the limit $w \in X^+$. Hence, by the Lipschitz continuity of h_n^+ and Theorem 3.5.4, we get

$$\begin{aligned}\|i_{n_k} h_{n_k}^+(w_{n_k}) - i h^+(w)\|_{L^2(D)} &\le \|i_{n_k} h_{n_k}^+(w_{n_k}) - i_{n_k} h_{n_k}^+ P_{n_k}^+ r_{n_k} i(w)\|_{L^2(D)} \\ &\quad + \|i_{n_k} h_{n_k}^+ P_{n_k}^+ r_{n_k} i(w) - i h^+(w)\|_{L^2(D)} \\ &\to 0\end{aligned}$$

as $k \to \infty$. If we set $u := w \oplus h^+(w) \in \mathrm{graph}(h^+)$, then $i_{n_k}(\xi_{n_k}) \to i(u)$ in $L^2(D)$ as $k \to \infty$. Since $\|i_{n_k}(\xi_{n_k})\|_{L^2(D)} < \delta$ for all $k \in \mathbb{N}$, we get $\|i(u)\|_{L^2(D)} \le \delta$. Hence, $u \in \mathrm{graph}(h^+) \cap \overline{B} = \overline{\mathrm{graph}(h^+) \cap B}$. We can find $u_{n_k} \in \mathrm{graph}(h^+) \cap B$ such that $u_{n_k} \to u$ in $L^2(\Omega)$ as $k \to \infty$. Therefore,

$$\|i_{n_k}(\xi_{n_k}) - i(u_{n_k})\|_{L^2(D)} \le \|i_{n_k}(\xi_{n_k}) - i(u)\|_{L^2(D)} + \|i(u) - i(u_{n_k})\|_{L^2(D)} \to 0$$

as $k \to \infty$ and we obtain the required subsequence. □

3.6 Convergence of stable invariant manifolds

Recall that the local stable manifold is a graph of Lipschitz function $h^- : X^- \to X^+$ inside a suitable product neighbourhood of $0 \in L^2(\Omega)$ determined by the modification in the construction (Theorem 3.2.7). In this section, we prove the upper and lower semicontinuity of local stable invariant manifolds with the following modification.

Fix the renorming of X_n^-, X_n^+, X^- and X^+ by (3.25), (3.26), (3.8) and (3.9) respectively using the same parameters α and β for all $n \in \mathbb{N}$. By shrinking the neighbourhood (choosing a smaller Lipschitz constant ε for the nonlinear terms f_n and f), we can make the following assumption.

Assumption 3.6.1. We assume that

$$0 < \mu_0 < \inf\left\{\frac{1}{2(\|P^+\| + \|P^-\|)}, \frac{1}{2(\|P_n^+\| + \|P_n^-\|)} : n \in \mathbb{N}\right\} \tag{3.56}$$

and

$$\mu := \frac{\mu_0}{M_1 M_2} \tag{3.57}$$

are parameters such that both μ_0 and μ satisfy the conditions for μ in (3.14) and (3.15).

We denote the Lipschitz functions for the modification μ_0 by $\hat{h}^-$ and for the modification μ by h^-. Let U be a smaller product neighbourhood of 0 in $L^2(\Omega)$ such that both modifications agree. Hence, the local stable manifold is $W^s(U) := \operatorname{graph}(h^-) \cap U = \operatorname{graph}(\hat{h}^-) \cap U$. Similarly, for each $n \in \mathbb{N}$, we denote the Lipschitz functions for the modification μ_0 by $\hat{h}_n^-$ and for the modification μ by h_n^-. As discussed at the end of Section 3.3, we can take a uniform product neighbourhood U_n of 0 in $L^2(\Omega_n)$ such that both modifications agree. Hence, the local stable manifold is $W_n^s(U_n) := \operatorname{graph}(h_n^-) \cap U_n = \operatorname{graph}(\hat{h}_n^-) \cap U_n$. We choose $\delta > 0$ so that $\overline{B} \subset U$ and $\overline{B}_n \subset U_n$, where $B := B_{L^2(\Omega)}(0, \delta)$ and $B_n := B_{L^2(\Omega_n)}(0, \delta)$. Hence, $h^-(v) = \hat{h}^-(v)$ on $\overline{B}$ and $h_n^-(v) = \hat{h}_n^-(v)$ on $\overline{B}_n$. We prove Theorem 3.3.4 by taking the balls of radius δ chosen above.

Lemma 3.6.2. *Let $\delta > 0$ and $\zeta_n > 0$ be a sequence with $\zeta_n \to 0$ as $n \to \infty$.*

(i) *If $z_n = y_n \oplus h^-(y_n)$ is a sequence in* $\operatorname{graph}(h^-)$ *such that $z_n \in B_{L^2(\Omega)}(0, \delta + \zeta_n)$ for each $n \in \mathbb{N}$, then there exist a subsequence z_{n_k} and a sequence u_{n_k} in* $\operatorname{graph}(h^-) \cap B$ *such that*

$$\|z_{n_k} - u_{n_k}\|_{L^2(\Omega)} \to 0$$

as $k \to \infty$.

(ii) *If $z_n = y_n \oplus h^-(y_n)$ is a sequence with $z_n \in \operatorname{graph}(h_n^-) \cap B_{L^2(\Omega_n)}(0, \delta + \zeta_n)$ for each $n \in \mathbb{N}$, then there exist a subsequence z_{n_k} and a sequence u_{n_k} with $u_{n_k} \in$* $\operatorname{graph}(h_{n_k}^-) \cap B_{n_k}$ *for each $k \in \mathbb{N}$ such that*

$$\|z_{n_k} - u_{n_k}\|_{L^2(\Omega_{n_k})} \to 0$$

as $k \to \infty$.

Proof. For assertion (i), using (3.56) we can fix $b > 0$ such that

$$b > \frac{1}{(\|P^+\| + \|P^-\|)^{-1} - 2\mu_0}. \tag{3.58}$$

Since $\zeta_n \to 0$, we can find $N_0 \in \mathbb{N}$ such that $\zeta_n < \delta/b$ for all $n > N_0$. We extract a subsequence ζ_{n_k} so that $\zeta_{n_k} < \delta/b$ for all $k \in \mathbb{N}$. Define

$$a_{n_k} := 1 - \frac{b\zeta_{n_k}}{\|y_{n_k}\|_{L^2(\Omega)} + \|h^-(y_{n_k})\|_{L^2(\Omega)}}, \tag{3.59}$$

for each $k \in \mathbb{N}$. By our assumptions, $\|z_{n_k}\|_{L^2(\Omega)} = \|y_{n_k} \oplus h^-(y_{n_k})\|_{L^2(\Omega)} < \delta + \zeta_{n_k}$ for all $k \in \mathbb{N}$. If $\|z_{n_k}\|_{L^2(\Omega)} \geq \delta$, then

$$\|y_{n_k}\|_{L^2(\Omega)} + \|h^-(y_{n_k})\|_{L^2(\Omega)} \geq \|y_{n_k} \oplus h^-(y_{n_k})\|_{L^2(\Omega)} \geq \delta.$$

Since $\zeta_{n_k} < \delta/b$, if $\|z_{n_k}\|_{L^2(\Omega)} \geq \delta$ we have that

$$\frac{b\zeta_{n_k}}{\|y_{n_k}\|_{L^2(\Omega)} + \|h^-(y_{n_k})\|_{L^2(\Omega)}} < \frac{b(\delta/b)}{\delta} = 1.$$

It follows from (3.59) that $0 < a_{n_k} \leq 1$ if $\|z_{n_k}\|_{L^2(\Omega)} \geq \delta$. For each $k \in \mathbb{N}$, we define $u_{n_k} \in \operatorname{graph}(h^-)$ by

$$u_{n_k} := \begin{cases} z_{n_k} & \text{if } \|z_{n_k}\|_{L^2(\Omega)} < \delta \\ a_{n_k} y_{n_k} \oplus h^-(a_{n_k} y_{n_k}) & \text{if } \|z_{n_k}\|_{L^2(\Omega)} \geq \delta. \end{cases} \tag{3.60}$$

Clearly, $\|z_{n_k} - u_{n_k}\|_{L^2(\Omega)} = 0$ if $\|z_{n_k}\|_{L^2(\Omega)} < \delta$. Moreover, if $\|z_{n_k}\|_{L^2(\Omega)} \geq \delta$, then

$$\begin{aligned}
\|z_{n_k} - u_{n_k}\|_{L^2(\Omega)} &= \|(y_{n_k} \oplus h^-(y_{n_k})) - (a_{n_k} y_{n_k} \oplus h^-(a_{n_k} y_{n_k}))\|_{L^2(\Omega)} \\
&\leq \|y_{n_k} - a_{n_k} y_{n_k}\|_{L^2(\Omega)} + \|h^-(y_{n_k}) - h^-(a_{n_k} y_{n_k})\|_{L^2(\Omega)} \\
&\leq \|y_{n_k} - a_{n_k} y_{n_k}\|_{L^2(\Omega)} + \|h^-(y_{n_k}) - h^-(a_{n_k} y_{n_k})\|_{X^+} \\
&\leq \|y_{n_k} - a_{n_k} y_{n_k}\|_{L^2(\Omega)} + \mu \|y_{n_k} - a_{n_k} y_{n_k}\|_{X^-} \\
&\leq (1 + \mu M_1)\|y_{n_k} - a_{n_k} y_{n_k}\|_{L^2(\Omega)} \\
&\leq (1 + \mu M_1)|1 - a_{n_k}| \; \|y_{n_k}\|_{L^2(\Omega)} \\
&\leq (1 + \mu M_1) \frac{b\zeta_{n_k}}{\|y_{n_k}\|_{L^2(\Omega)} + \|h^-(y_{n_k})\|_{L^2(\Omega)}} \|y_{n_k}\|_{L^2(\Omega)} \\
&\leq (1 + \mu M_1) b\zeta_{n_k}.
\end{aligned}$$

Hence, $\|z_{n_k} - u_{n_k}\|_{L^2(\Omega)} \leq (1 + \mu M_1) b\zeta_{n_k}$ for all $k \in \mathbb{N}$. As $\zeta_{n_k} \to 0$, we conclude that

$$\|z_{n_k} - u_{n_k}\|_{L^2(\Omega)} \to 0 \tag{3.61}$$

as $k \to \infty$. It remains to show that $u_{n_k} \in B_{L^2(\Omega)}(0, \delta)$ for all $k \in \mathbb{N}$. If $\|z_{n_k}\|_{L^2(\Omega)} < \delta$, then $u_{n_k} \in B_{L^2(\Omega)}(0,\delta)$. If $\|z_{n_k}\|_{L^2(\Omega)} \geq \delta$, we can write

$$\begin{aligned}\|u_{n_k}\|_{L^2(\Omega)} &\leq \|u_{n_k} - a_{n_k} z_{n_k}\|_{L^2(\Omega)} + \|a_{n_k} z_{n_k}\|_{L^2(\Omega)} \\ &= \|(a_{n_k} y_{n_k} \oplus h^-(a_{n_k} y_{n_k})) - a_{n_k}(y_{n_k} \oplus h^-(y_{n_k}))\|_{L^2(\Omega)} \\ &\quad + \|a_{n_k} z_{n_k}\|_{L^2(\Omega)} \\ &\leq \|h^-(a_{n_k} y_{n_k}) - a_{n_k} h^-(y_{n_k})\|_{L^2(\Omega)} + \|a_{n_k} z_{n_k}\|_{L^2(\Omega)} \\ &\leq \|h^-(a_{n_k} y_{n_k}) - h^-(y_{n_k})\|_{L^2(\Omega)} + \|h^-(y_{n_k}) - a_{n_k} h^-(y_{n_k})\|_{L^2(\Omega)} \\ &\quad + \|a_{n_k} z_{n_k}\|_{L^2(\Omega)}.\end{aligned} \tag{3.62}$$

Now, if $\|z_{n_k}\|_{L^2(\Omega)} \geq \delta$, then by the Lipschitz continuity of h^- and (3.57)

$$\begin{aligned}\|h^-(a_{n_k} y_{n_k}) - h^-(y_{n_k})\|_{L^2(\Omega)} &\leq \|h^-(a_{n_k} y_{n_k}) - h^-(y_{n_k})\|_{X^+} \\ &\leq \mu \|a_{n_k} y_{n_k} - y_{n_k}\|_{X^-} \\ &\leq \mu M_1 |a_{n_k} - 1| \, \|y_{n_k}\|_{L^2(\Omega)} \\ &= \frac{\mu_0}{M_1 M_2} M_1 \frac{b\zeta_{n_k} \|y_{n_k}\|_{L^2(\Omega)}}{\|y_{n_k}\|_{L^2(\Omega)} + \|h^-(y_{n_k})\|_{L^2(\Omega)}} \\ &\leq \mu_0 b \zeta_{n_k}.\end{aligned} \tag{3.63}$$

Similarly, if $\|z_{n_k}\|_{L^2(\Omega)} \geq \delta$, then

$$\begin{aligned}\|h^-(y_{n_k}) - a_{n_k} h^-(y_{n_k})\|_{L^2(\Omega)} &\leq |1 - a_{n_k}| \, \|h^-(y_{n_k})\|_{L^2(\Omega)} \\ &\leq |1 - a_{n_k}| \, \|h^-(y_{n_k})\|_{X^+} \\ &\leq \mu |1 - a_{n_k}| \, \|y_{n_k}\|_{X^-} \\ &\leq \mu M_1 |1 - a_{n_k}| \, \|y_{n_k}\|_{L^2(\Omega)} \\ &= \frac{\mu_0}{M_1 M_2} M_1 \frac{b\zeta_{n_k} \|y_{n_k}\|_{L^2(\Omega)}}{\|y_{n_k}\|_{L^2(\Omega)} + \|h^-(y_{n_k})\|_{L^2(\Omega)}} \\ &\leq \mu_0 b \zeta_{n_k}.\end{aligned} \tag{3.64}$$

Since $\|z_{n_k}\|_{L^2(\Omega)} \geq (\|y_{n_k}\|_{L^2(\Omega)} + \|h^-(y_{n_k})\|_{L^2(\Omega)})/(\|P^+\| + \|P^-\|)$, it follows that

$$\frac{b\zeta_{n_k} \|z_{n_k}\|_{L^2(\Omega)}}{\|y_{n_k}\|_{L^2(\Omega)} + \|h^-(y_{n_k})\|_{L^2(\Omega)}} \geq \frac{b\zeta_{n_k}}{\|P^+\| + \|P^-\|}.$$

Hence, if $\|z_{n_k}\|_{L^2(\Omega)} \geq \delta$, then

$$\begin{aligned}
\|a_{n_k} z_{n_k}\|_{L^2(\Omega)} &= |a_{n_k}|\|z_{n_k}\|_{L^2(\Omega)} \\
&= a_{n_k}\|z_{n_k}\|_{L^2(\Omega)} \\
&= \left(1 - \frac{b\zeta_{n_k}}{\|y_{n_k}\|_{L^2(\Omega)} + \|h^-(y_{n_k})\|_{L^2(\Omega)}}\right)\|z_{n_k}\|_{L^2(\Omega)} \\
&= \|z_{n_k}\|_{L^2(\Omega)} - \frac{b\zeta_{n_k}\|z_{n_k}\|_{L^2(\Omega)}}{\|y_{n_k}\|_{L^2(\Omega)} + \|h^-(y_{n_k})\|_{L^2(\Omega)}} \\
&\leq \|z_{n_k}\|_{L^2(\Omega)} - \frac{b\zeta_{n_k}}{\|P^+\| + \|P^-\|} \\
&< \delta + \zeta_{n_k} - \frac{b\zeta_{n_k}}{\|P^+\| + \|P^-\|}.
\end{aligned} \tag{3.65}$$

Therefore, by (3.62) – (3.65), if $\|z_{n_k}\|_{L^2(\Omega)} \geq \delta$, then

$$\begin{aligned}
\|u_{n_k}\|_{L^2(\Omega)} &< \mu_0 b\zeta_{n_k} + \mu_0 b\zeta_{n_k} + \delta + \zeta_{n_k} - \frac{b\zeta_{n_k}}{\|P^+\| + \|P^-\|} \\
&= \delta + \left(2\mu_0 b - \frac{b}{\|P^+\| + \|P^-\|} + 1\right)\zeta_{n_k}.
\end{aligned} \tag{3.66}$$

By the choice of b in (3.58), we get

$$\begin{aligned}
2\mu_0 b - \frac{b}{\|P^+\| + \|P^-\|} + 1 &= \left(2\mu_0 - \frac{1}{\|P^+\| + \|P^-\|}\right) b + 1 \\
&= -\Big((\|P^+\| + \|P^-\|)^{-1} - 2\mu_0\Big) b + 1 \\
&< -1 + 1 \\
&= 0.
\end{aligned}$$

It follows from (3.66) that $\|u_{n_k}\|_{L^2(\Omega)} < \delta$ if $\|z_{n_k}\|_{L^2(\Omega)} \geq \delta$. Hence, we conclude that $u_{n_k} \in \operatorname{graph}(h^-) \cap B_{L^2(\Omega)}(0, \delta)$ for all $k \in \mathbb{N}$ and statement (i) follows.

Statement (ii) can be proved similarly. The only difference is that the sequence z_n belongs to different spaces $L^2(\Omega_n)$ for each $n \in \mathbb{N}$. We only need to adjust the proof in part (i) and keep track of the dependence on n. In particular, we replace (3.58) by

$$b > \sup_{n\in\mathbb{N}} \left\{ \frac{1}{(\|P_n^+\| + \|P_n^-\|)^{-1} - 2\mu_0} \right\} > 0$$

and (3.59) by

$$a_{n_k} := 1 - \frac{b\zeta_{n_k}}{\|y_{n_k}\|_{L^2(\Omega_{n_k})} + \|h^-(y_{n_k})\|_{L^2(\Omega_{n_k})}},$$

for each $k \in \mathbb{N}$. □

We now show the upper semicontinuity of local stable invariant manifolds.

Proof of Theorem 3.3.4 (i). By Lemma 3.4.9, we need to show that for any sequence $\{\xi_n\}_{n\in\mathbb{N}}$ with $\xi_n \in \text{graph}(h_n^-)\cap B_n$, if $\{\xi_{n_k}\}_{k\in\mathbb{N}}$ is a subsequence then there exist a further subsequence (denoted again by ξ_{n_k}) and a sequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ with $u_{n_k} \in \text{graph}(h^-)\cap B$ such that $\|i_{n_k}(\xi_{n_k}) - i(u_{n_k})\|_{L^2(D)} \to 0$ as $k \to \infty$.

Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence with $\xi_n \in \text{graph}(h_n^-) \cap B_n$ and $(\xi_{n_k})_{k\in\mathbb{N}}$ be an arbitrary subsequence. We write $\xi_{n_k} := v_{n_k} \oplus h_{n_k}^-(v_{n_k})$ for some $v_{n_k} \in X_{n_k}^-$. Since $\|\xi_{n_k}\|_{L^2(\Omega_{n_k})} = \|v_{n_k} \oplus h_{n_k}^-(v_{n_k})\|_{L^2(\Omega_{n_k})} < \delta$ for all $k \in \mathbb{N}$, we can extract a subsequence of v_{n_k} (indexed again by n_k) such that

$$i_{n_k}(v_{n_k}) \rightharpoonup v$$

in $L^2(D)$ as $k \to \infty$. By the assumption that $|\Omega_n| \to |\Omega|$, we conclude that $v = 0$ almost everywhere in $D\backslash\Omega$, that is, $v \in L^2(\Omega)$. Hence,

$$i_{n_k}(v_{n_k}) \rightharpoonup i(v) \tag{3.67}$$

in $L^2(D)$ as $k \to \infty$. Moreover, by the convergence of $i_n P_n^- r_n \to iP^-r$ in $\mathscr{L}(L^2(D))$ (see Remark 3.4.2) and the weak convergence of v_{n_k}, it follows that

$$\begin{aligned}
\Big|\big(i_{n_k}(v_{n_k}) - iP^-(v)|\phi\big)_{L^2(D)}\Big| &\le \Big|\big(i_{n_k}(v_{n_k}) - iP^-ri_{n_k}(v_{n_k})|\phi\big)_{L^2(D)}\Big| \\
&\quad + \Big|\big(iP^-ri_{n_k}(v_{n_k}) - iP^-(v)|\phi\big)_{L^2(D)}\Big| \\
&\le \|i_{n_k}P_{n_k}^-r_{n_k} - iP^-r\|\|i_{n_k}(v_{n_k})\|_{L^2(D)}\|\phi\|_{L^2(D)} \\
&\quad + \Big|\big(i_{n_k}(v_{n_k}) - i(v)|(iP^-r)^*\phi\big)_{L^2(D)}\Big| \\
&\to 0
\end{aligned}$$

as $k \to \infty$ for all $\phi \in L^2(D)$, where $(iP^-r)^*$ is the adjoint operator of iP^-r. This means $i_{n_k}(v_{n_k}) \rightharpoonup iP^-(v)$ in $L^2(D)$ as $k \to \infty$. By the uniqueness of weak limit, $v = P^-(v)$ and hence $v \in X^-$. Since $\|i_{n_k}h_{n_k}^-(v_{n_k})\|_{L^2(D)}$ is uniformly bounded, we can apply Corollary 3.4.6 to extract a further subsequence (indexed again by n_k) such that

$$i_{n_k}h_{n_k}^-(v_{n_k}) \to i(w) \tag{3.68}$$

in $L^2(D)$ as $k \to \infty$ with the limit $w \in X^+$. Thus, we get

$$i_{n_k}(v_{n_k} \oplus h_{n_k}^-(v_{n_k})) \rightharpoonup i(v \oplus w) \tag{3.69}$$

in $L^2(D)$ as $k \to \infty$. By a standard property of weak convergence,

$$\begin{aligned} \|i(v \oplus w)\|_{L^2(D)} &\leq \liminf_{k\to\infty} \|i_{n_k}(v_{n_k} \oplus h^-_{n_k}(v_{n_k}))\|_{L^2(D)} \\ &\leq \limsup_{k\to\infty} \|i_{n_k}(v_{n_k} \oplus h^-_{n_k}(v_{n_k}))\|_{L^2(D)} \\ &\leq \delta. \end{aligned} \tag{3.70}$$

Hence, $u := v \oplus w$ belongs to $\overline{B}$. Applying domain perturbation result for solutions of semilinear equations in (3.20), we get from (3.69) and globally Lipschitz assumption for the modified function $\tilde{f}$ that

$$i_{n_k}\Phi_{t,n_k}(v_{n_k} \oplus h^-_{n_k}(v_{n_k})) \to i\Phi_t(v \oplus w)$$

in $L^2(D)$ as $k \to \infty$ for all $t > 0$. Lemma 3.4.1 (i) implies that

$$i_{n_k}P^-_{n_k}\Phi_{t,n_k}(v_{n_k} \oplus h^-_{n_k}(v_{n_k})) \to iP^-\Phi_t(v \oplus w)$$

and

$$i_{n_k}P^+_{n_k}\Phi_{t,n_k}(v_{n_k} \oplus h^-_{n_k}(v_{n_k})) \to iP^+\Phi_t(v \oplus w)$$

in $L^2(D)$ as $k \to \infty$ for all $t > 0$. By the construction of $h^-_{n_k}(v_{n_k})$ (see Theorem 3.2.4), we have that

$$\|P^+_{n_k}\Phi_{t,n_k}(v_{n_k} \oplus h^-_{n_k}(v_{n_k}))\|_{X^+_{n_k}} \leq \mu\|P^-_{n_k}\Phi_{t,n_k}(v_{n_k} \oplus h^-_{n_k}(v_{n_k}))\|_{X^-_{n_k}},$$

for all $t \geq 0$. Taking the norms on X^+_n and X^-_n in (3.27) and (3.28) into account, the above implies

$$\|P^+_{n_k}\Phi_{t,n_k}(v_{n_k} \oplus h^-_{n_k}(v_{n_k}))\|_{L^2(\Omega_{n_k})} \leq \mu M_1\|P^-_{n_k}\Phi_{t,n_k}(v_{n_k} \oplus h^-_{n_k}(v_{n_k}))\|_{L^2(\Omega_{n_k})},$$

for all $t \geq 0$. Passing to the limit as $k \to \infty$, we obtain

$$\|P^+\Phi_t(v \oplus w)\|_{L^2(\Omega)} \leq \mu M_1\|P^-\Phi_t(v \oplus w)\|_{L^2(\Omega)}$$

for all $t > 0$. By the assumptions on μ_0 and μ in (3.56) and (3.57), and the equivalence of norms on X^- and X^+ (see (3.10) and (3.11)), it follows that

$$\|P^+\Phi_t(v \oplus w)\|_{X^+} \leq \mu M_1 M_2\|P^-\Phi_t(v \oplus w)\|_{X^-} = \mu_0\|P^-\Phi_t(v \oplus w)\|_{X^-}, \tag{3.71}$$

for all $t > 0$. We claim that $\|w\|_{X^+} \leq \mu_0\|v\|_{X^-}$. If $\|w\|_{X^+} > \mu_0\|v\|_{X^-}$, that is $v \oplus w$ is in the interior of the cone K_{μ_0} defined by (3.13), we can find a product neighbourhood

$U(v,w)$ of $v \oplus w$ such that $U(v,w) \subset \text{Int}(K_{\mu_0})$. Since the solution of parabolic equation with the initial condition $v \oplus w$ is continuous, there exists $t_0 > 0$ such that $\Phi_t(v \oplus w) \in U(v,w)$ for $0 \le t \le t_0$. This implies that $\|P^+\Phi_t(v \oplus w)\|_{X^+} > \mu_0 \|P^-\Phi_t(v \oplus w)\|_{X^-}$ for $0 \le t \le t_0$, which is a contradiction to (3.71). Hence, by the definition of $\hat{h}^-$ (a modification with the cone K_{μ_0}), we conclude that $w = \hat{h}^-(v)$. As both modification agree on $\bar{B}$, we have $w = h^-(v)$. Therefore, (3.68) implies

$$i_{n_k} h^-_{n_k}(v_{n_k}) \to i h^-(v) \tag{3.72}$$

in $L^2(D)$ as $k \to \infty$.

The remainder of this proof deals with the existence of the required sequence $u_{n_k} \in \text{graph}(h^-) \cap B$. At this stage, we keep the index of our subsequence as in the previous part. We define

$$y_{n_k} := P^- r i_{n_k}(v_{n_k}) \in X^-,$$

for each $k \in \mathbb{N}$. By the convergence of $i_n P^-_n r_n \to i P^- r$ in $\mathscr{L}(L^2(D))$ (see Remark 3.4.2) and the boundedness of $\|i_{n_k}(v_{n_k})\|_{L^2(D)}$, we get

$$\begin{aligned} \|i(y_{n_k}) - i_{n_k}(v_{n_k})\|_{L^2(D)} &= \|iP^- r i_{n_k}(v_{n_k}) - i_{n_k}(v_{n_k})\|_{L^2(D)} \\ &\le \|iP^- r - i_{n_k} P^-_{n_k} r_{n_k}\| \|i_{n_k}(v_{n_k})\|_{L^2(D)} \\ &\to 0 \end{aligned} \tag{3.73}$$

as $k \to \infty$. In particular, $\|y_{n_k}\|_{L^2(\Omega)}$ is uniformly bounded. Moreover, by (3.73) and the weak convergence of v_{n_k},

$$\begin{aligned} \left|(i(y_{n_k}) - i(v)|\phi)_{L^2(D)}\right| &\le \left|(i(y_{n_k}) - i_{n_k}(v_{n_k})|\phi)_{L^2(D)}\right| \\ &\quad + \left|(i_{n_k}(v_{n_k}) - i(v)|\phi)_{L^2(D)}\right| \\ &\le \|i(y_{n_k}) - i_{n_k}(v_{n_k})\|_{L^2(D)} \|\phi\|_{L^2(D)} \\ &\quad + \left|(i_{n_k}(v_{n_k}) - i(v)|\phi)_{L^2(D)}\right| \\ &\to 0 \end{aligned}$$

as $k \to \infty$ for all $\phi \in L^2(D)$. Hence,

$$i(y_{n_k}) \rightharpoonup i(v) \tag{3.74}$$

in $L^2(D)$ as $k \to \infty$. By the Lipschitz continuity of h^- together with (3.10) and (3.11),

$$\|h^-(y_{n_k})\|_{L^2(\Omega)} \le \|h^-(y_{n_k})\|_{X^+} \le \mu \|y_{n_k}\|_{X^-} \le \mu M_1 \|y_{n_k}\|_{L^2(\Omega)} < \infty$$

for all $k \in \mathbb{N}$. Since X^+ is a finite dimensional space, we can extract a further subsequence (indexed again by n_k) such that

$$ih^-(y_{n_k}) \to i(\tilde{w}) \tag{3.75}$$

in $L^2(D)$ as $k \to \infty$ with the limit $\tilde{w} \in X^+$. Therefore,

$$i(y_{n_k} \oplus h^-(y_{n_k})) \rightharpoonup i(v \oplus \tilde{w}) \tag{3.76}$$

in $L^2(D)$ as $k \to \infty$. By the convergence of solutions under domain perturbation in (3.20) (with $\Omega_n = \Omega$ for all $n \in \mathbb{N}$), we get from (3.76) that

$$i\Phi_t(y_{n_k} \oplus h^-(y_{n_k})) \to i\Phi_t(v \oplus \tilde{w})$$

in $L^2(D)$ as $k \to \infty$ for all $t > 0$. Hence,

$$P^-\Phi_t(y_{n_k} \oplus h^-(y_{n_k})) \to P^-\Phi_t(v \oplus \tilde{w})$$

and

$$P^+\Phi_t(y_{n_k} \oplus h^-(y_{n_k})) \to P^+\Phi_t(v \oplus \tilde{w})$$

in $L^2(\Omega)$ as $k \to \infty$ for all $t > 0$. Since these sequences are in the fixed spaces X^- and X^+ respectively, (3.10) and (3.11) imply that they converge under $\|\cdot\|_{X^-}$ and $\|\cdot\|_{X^+}$, respectively. By the construction of $h^-(y_{n_k})$ (see Theorem 3.2.4), we have that

$$\|P^+\Phi_t(y_{n_k} \oplus h^-(y_{n_k}))\|_{X^+} \le \mu\|P^-\Phi_t(y_{n_k} \oplus h^-(y_{n_k}))\|_{X^-},$$

for all $t \ge 0$. Passing to the limit as $k \to \infty$, we obtain

$$\|P^+\Phi_t(v \oplus \tilde{w})\|_{X^+} = \mu\|P^-\Phi_t(v \oplus \tilde{w})\|_{X^-} \tag{3.77}$$

for all $t > 0$. By a similar argument appeared after (3.71), we conclude that $\|\tilde{w}\|_{X^+} \le \mu\|v\|_{X^-}$. Hence, $\tilde{w}$ agrees with $w = h^-(v)$. Therefore, (3.75) implies

$$ih^-(y_{n_k}) \to ih^-(v) \tag{3.78}$$

in $L^2(D)$ as $k \to \infty$. Recall that $\xi_{n_k} = v_{n_k} \oplus h^-_{n_k}(v_{n_k})$. If we set $z_{n_k} := y_{n_k} \oplus h^-(y_{n_k}) \in \operatorname{graph}(h^-)$, then by (3.72), (3.73) and (3.78)

$$\begin{aligned}
&\|i_{n_k}(\xi_{n_k}) - i(z_{n_k})\|_{L^2(D)} \\
&\quad = \|i_{n_k}(v_{n_k} \oplus h^-_{n_k}(v_{n_k})) - i(y_{n_k} \oplus h^-(y_{n_k}))\|_{L^2(D)} \\
&\quad \le \|i_{n_k}(v_{n_k}) - i(y_{n_k})\|_{L^2(D)} + \|i_{n_k}h^-_{n_k}(v_{n_k}) - ih^-(y_{n_k})\|_{L^2(D)} \\
&\quad \le \|i_{n_k}(v_{n_k}) - i(y_{n_k})\|_{L^2(D)} + \|i_{n_k}h^-_{n_k}(v_{n_k}) - ih^-(v)\|_{L^2(D)} \\
&\qquad + \|ih^-(v) - ih^-(y_{n_k})\|_{L^2(D)} \\
&\quad \to 0
\end{aligned} \tag{3.79}$$

as $k \to \infty$. Therefore, we can extract a further subsequence (indexed again by n_k) and $\zeta_{n_k} > 0$ with $\zeta_{n_k} \to 0$ as $k \to \infty$ such that

$$\|i_{n_k}(\xi_{n_k}) - i(z_{n_k})\|_{L^2(D)} < \zeta_{n_k},$$

for all $k \in \mathbb{N}$. It follows that

$$\|i(z_{n_k})\|_{L^2(D)} \leq \|i_{n_k}(\xi_{n_k})\|_{L^2(D)} + \zeta_{n_k} < \delta + \zeta_{n_k},$$

for all $k \in \mathbb{N}$, that is, $z_{n_k} \in \text{graph}(h^-) \cap B_{L^2(\Omega)}(0, \delta + \zeta_{n_k})$ for all $k \in \mathbb{N}$. We can apply Lemma 3.6.2 (i) to obtain a subsequence (indexed again by n_k) z_{n_k} and a sequence $u_{n_k} \in \text{graph}(h^-) \cap B$ such that $\|z_{n_k} - u_{n_k}\|_{L^2(\Omega)} \to 0$ as $k \to \infty$. It follows from (3.79) that

$$\begin{aligned}\|i_{n_k}(\xi_{n_k}) - i(u_{n_k})\|_{L^2(D)} &\leq \|i_{n_k}(\xi_{n_k}) - i(z_{n_k})\|_{L^2(D)} + \|i(z_{n_k}) - i(u_{n_k})\|_{L^2(D)} \\ &\to 0\end{aligned}$$

as $k \to \infty$. Hence, we obtain the required sequence u_{n_k}. Since we start with an arbitrary sequence $\xi_n \in \text{graph}(h_n^-) \cap B_n$, the assertion of Theorem 3.3.4 (i) follows. □

The lower semicontinuity of local stable invariant manifolds can be obtained by a similar fashion.

Proof of Theorem 3.3.4 (ii). The statement follows by a similar argument to the proof of Theorem 3.3.4 (i). We use Lemma 3.4.10 and Lemma 3.6.2 (ii) instead of Lemma 3.4.9 and 3.6.2 (i). □

Chapter 4

Persistence of bounded entire solutions of parabolic equations under domain perturbation

In this chapter, we investigate domain perturbation for a class of bounded entire solutions (solutions that are defined for all time $t \in (-\infty, \infty)$ and are bounded in a suitable function space) of parabolic equations. The existence of bounded entire solutions is commonly obtained from an *exponential dichotomy* assumption (see Definition 4.1.2 below). A study of persistence of bounded entire solutions under perturbation has also attracted interests. For example, the work of [83] deals with persistence of bounded solutions of parabolic equations under a non-autonomous perturbation of the inhomogeneous terms. Many other authors focus their studies on homoclinic and heteroclinic solutions to understand some complicated dynamical behaviour. For results on homoclinic solutions, we refer to [17] and [82]. The persistence of heteroclinic orbits of semilinear heat equation under approximation of a (non-C^1) nonlinearity by C^1 functions is obtained in [33] to calculate the Conley index. A usual technique to deal with perturbation of the nonlinearity (or inhomogeneous terms) is the use of Liapunov-Schmidt decomposition method and the implicit function theorem to derive a bifurcation equation (see [50]). When dealing with domain perturbation, a standard implicit function theorem cannot be applied because the differential operators vary upon domains.

In the literature, domain perturbation for bounded entire solutions seems only be treated for particular types of solutions including stationary (equilibrium) solutions in [51] and periodic solutions in [35, Section 5 and 7]. For periodic solutions, convergence of T-periodic solutions under domain perturbation can be obtained by applying domain perturbation results for initial value problems (see Chapter 1) and considering the *Time-T-operator*, that is the operator $U(T)$ mapping u_0 to the solution of the abstract homogeneous parabolic equation ((2.2) with $s = 0$) at time T.

The purpose of this chapter is to study domain perturbation for general bounded entire solutions of linear and semilinear parabolic equations. We restrict our attention to autonomous parabolic equations under Dirichlet boundary condition. In Section 4.1, we recall the existence and uniqueness of bounded entire solutions of linear parabolic equations under an exponential dichotomy assumption. Persistence of bounded entire solutions is studied in Section 4.2 and Section 4.3 for linear parabolic equations under L^2 and L^p-norms respectively. We consider semilinear equations in Section 4.4. The technique in this chapter is not applicable to domain perturbation for homoclinic and heteroclinic orbits. The problem is rather difficult but it is an interesting topic for future research.

4.1 Bounded entire solutions of linear parabolic equations

Let Ω be a bounded domain in $\mathbb{R}^N$. We consider the following autonomous parabolic equations subject to Dirichlet boundary condition on the whole real line

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}u = g(x,t) & \text{in } \Omega \times (-\infty, \infty) \\ u = 0 & \text{on } \partial\Omega \times (-\infty, \infty), \end{cases} \tag{4.1}$$

where $g \in C(\mathbb{R}^N \times \mathbb{R})$ and $\mathcal{A}$ is an elliptic operator defined by (2.31). As usual, we write (4.1) in the abstract form in $L^2(\Omega)$ as

$$u'(t) + Au = g(t), \quad t \in (-\infty, \infty), \tag{4.2}$$

where A is the maximal restriction operator of A_Ω given in (2.37) on $H = L^2(\Omega)$. In particular, we have from Section 2.4.2 that $-A$ is the generator of a strongly continuous analytic semigroup $S(t), t \geq 0$ on $L^2(\Omega)$.

Let $J \subset \mathbb{R}$ and X be a Banach space, we write

$$BC(J, X) := \Big\{ v : J \to X : v \text{ is continuous and } \sup_{t \in J} \|v(t)\|_X < \infty \Big\}.$$

It is well known that $BC(J, X)$ is a Banach space under the norm $\|\cdot\|_\infty$ given by

$$\|v\|_\infty := \sup_{t \in J} \|v(t)\|_X.$$

Definition 4.1.1. A function $u \in BC(\mathbb{R}, L^2(\Omega))$ is called a (mild) *bounded entire solution* of (4.2) if for all $t \geq s$

$$u(t) = S(t-s)u(s) + \int_s^t S(t-\tau)g(\tau)d\tau$$

holds for any $s \in \mathbb{R}$.

To obtain existence of bounded entire solutions, we require the notion of *exponential dichotomy*. We give a definition below for a more general case of non-autonomous problems.

Definition 4.1.2 (Exponential dichotomy (see e.g. [52, Secion 7.6]))**.** We say that the equation $u'(t) + A(t)u(t) = 0$ has an exponential dichotomy on interval $J \subset \mathbb{R}$ with exponent $\beta > 0$ and bound M if there exist projections $P(t), t \in J$ strongly continuous in t such that the evolution systems $U(t,s) \in \mathscr{L}(H), t \geq s$ satisfies

(i) $U(t,s)P(s) = P(t)U(t,s)$ for all $t \geq s$ in J;

(ii) The restriction $U(t,s)|_{\mathscr{R}(P(s))}$ is an isomorphism from $\mathscr{R}(P(s))$ onto $\mathscr{R}(P(t))$ for all $t \geq s$ in J where $\mathscr{R}(P(s))$ denotes the range of $P(s)$. We define $U(s,t)$ as the inverse map from $\mathscr{R}(P(t))$ to $\mathscr{R}(P(s))$;

(iii) $\|U(t,s)(I - P(s))\| \leq Me^{-\beta(t-s)}$ for all $t \geq s$ in J;

(iv) $\|U(t,s)P(s)\| \leq Me^{\beta(t-s)}$ for all $s \geq t$ in J.

Note that for autonomous problems, evolution systems are just semigroups $e^{-A(t-s)}$ for $t \geq s$. In this case, the projections $P(t)$ is replaced by a constant projection P.

Example 4.1.3. Consider the autonomous parabolic equation $u'(t) + Au(t) = 0$ in a Banach space $X := L^2(\Omega)$ where $-A$ is a generator of a strongly continuous analytic semigroup $S(t), t \geq 0$. The evolution systems are $U(t,s) := S(t-s)$ for $t \geq s$. The equation $u'(t) + Au(t) = 0$ has an exponential dichotomy on $\mathbb{R}$ if the spectrum $\sigma(-A)$ can be decomposed as $\sigma(-A) = \sigma^+ \cup \sigma^-$ where

$$\begin{aligned} \sigma^+ &:= \{\lambda \in \sigma(-A) : \ \mathrm{Re}(\lambda) > 0\} \\ \sigma^- &:= \{\lambda \in \sigma(-A) : \ \mathrm{Re}(\lambda) < 0\}. \end{aligned} \tag{4.3}$$

Indeed, the exponent β can be determined by the value of β such that $\sigma(A)$ is disjoint from the strip $\{\lambda : -\beta \leq \mathrm{Re}(\lambda) \leq \beta\}$. As seen in Section 3.2.2, σ^+ is a finite set. Let P^+ be the spectral projection defined by

$$P^+ = \frac{1}{2\pi i}\int_\Gamma (\lambda + A)^{-1} d\lambda, \tag{4.4}$$

where Γ is a rectifiable closed curve separating σ^+ from σ^-. We can take constant dichotomy projections $P(t) := P^+$ and $I - P(t) := I - P^+$ for all $t \in \mathbb{R}$.

In fact, the above conditions imply exponential dichotomy in a fractional power space (see examples in [52]).

We make the following assumption.

Assumption 4.1.4. We assume that the corresponding homogeneous equation of (4.2), that is $u'(t) + Au(t) = 0$ has an exponential dichotomy on $\mathbb{R}$ with exponent $\beta > 0$ and bound M.

We next prove the existence and uniqueness of bounded entire solutions of (4.2).

Theorem 4.1.5 ([69, Theorem 4])**.** *Suppose that Assumption 4.1.4 is satisfied. For every $g \in BC(\mathbb{R}, L^2(\Omega))$, there exists a unique bounded entire solution $u \in BC(\mathbb{R}, L^2(\Omega))$ of (4.2). Moreover, u can be represented by*

$$u(t) = \int_{-\infty}^{t} S(t-\tau)(I-P)g(\tau)d\tau - \int_{t}^{\infty} S(t-\tau)Pg(\tau)d\tau, \tag{4.5}$$

for all $t \in \mathbb{R}$.

Proof. Let $g \in BC(\mathbb{R}, L^2(\Omega))$ be arbitrary. Consider $u(t)$ defined by (4.5). Applying the estimates (iii) and (iv) in the definition of exponential dichotomy, we see that

$$\|u(t)\|_{L^2(\Omega)} \le \int_{-\infty}^{t} Me^{-\beta(t-\tau)}\|g\|_\infty d\tau + \int_{t}^{\infty} Me^{\beta(t-\tau)}\|g\|_\infty d\tau \le \frac{2M}{\beta}\|g\|_\infty,$$

for all $t \in \mathbb{R}$. By the change of variables, we can write (4.5) as

$$u(t) = \int_{0}^{\infty} S(\tau)(I-P)g(t-\tau)d\tau - \int_{-\infty}^{0} S(\tau)Pg(t-\tau)d\tau, \tag{4.6}$$

for all $t \in \mathbb{R}$. From (4.6), it is easy to see (using the dominated convergence theorem) that u is continuous on $\mathbb{R}$. Hence, $u \in BC(\mathbb{R}, L^2(\Omega))$. For $t \ge s$, we have

$$\begin{aligned} u(t) - S(t-s)u(s) &= \int_{-\infty}^{t} S(t-\tau)(I-P)g(\tau)d\tau - \int_{t}^{\infty} S(t-\tau)Pg(\tau)d\tau \\ &\quad - S(t-s)\int_{-\infty}^{s} S(s-\tau)(I-P)g(\tau)d\tau \\ &\quad + S(t-s)\int_{s}^{\infty} S(s-\tau)Pg(\tau)d\tau \\ &= \int_{s}^{t} S(t-\tau)(I-P)g(\tau)d\tau + \int_{s}^{t} S(t-\tau)Pg(\tau)d\tau \\ &= \int_{s}^{t} S(t-\tau)g(\tau)dt. \end{aligned}$$

Therefore, u defined by (4.5) is a bounded entire solution of (4.2).

To prove the uniqueness, let u and v be bounded entire solutions of (4.2). Setting $w := u - v$, we have w is a bounded entire solution of the homogeneous equation $w'(t) + Aw(t) = 0$. Hence, for any $t \in \mathbb{R}$, we have $w(t) = S(s)w(t-s)$ for $s \geq 0$. By the exponential dichotomy,

$$\|(I-P)w(t)\|_{L^2(\Omega)} \leq \|S(s)(I-P)\|\|w(t-s)\|_{L^2(\Omega)} \leq Me^{-\beta s}\|w\|_\infty \to 0$$

as $s \to \infty$. Hence, $(I-P)w(t) = 0$ and $Pw(t) = w(t)$ for all $t \in \mathbb{R}$. Similarly, for any $t \in \mathbb{R}$, we have $Pw(t) = S(-s)Pw(t+s)$ for $s \geq 0$. Thus,

$$\|Pw(t)\|_{L^2(\Omega)} \leq \|S(-s)P\|\|Pw(t+s)\| \leq Me^{-\beta s}\|w\|_\infty \to 0$$

as $s \to \infty$. Hence, we have $Pw(t) = 0$ for all $t \in \mathbb{R}$. This implies $w(t) = 0$ for all $t \in \mathbb{R}$ and thus $u = v$. $\square$

It follows from the above theorem that the operator

$$L^{-1} : BC(\mathbb{R}, L^2(\Omega)) \to BC(\mathbb{R}, L^2(\Omega))$$

given by

$$L^{-1}g(t) := \int_{-\infty}^{t} S(t-\tau)(I-P)g(\tau)d\tau - \int_{t}^{\infty} S(t-\tau)Pg(\tau)d\tau, \tag{4.7}$$

for $t \geq 0$ is well defined. Moreover, L^{-1} is bounded and linear with the operator norm $\|L^{-1}\| \leq 2M/\beta$. Hence, L^{-1} is continuous under the topology induced by the norm of $BC(\mathbb{R}, L^2(\Omega))$. We also remark the continuity under the topology of uniform convergence on compact subsets below.

Proposition 4.1.6. *Suppose that $g_n, g \in BC(\mathbb{R}, L^2(\Omega))$ are functions such that $g_n(t) \to g(t)$ in $L^2(\Omega)$ uniformly with respect to t on compact subsets of $\mathbb{R}$ and $\|g_n\|_\infty$ is uniformly bounded. Then $L^{-1}g_n(t) \to L^{-1}g(t)$ in $L^2(\Omega)$ uniformly with respect to t on compact subsets of $\mathbb{R}$.*

Proof. It is easy to see from the dominated convergence theorem and the representation of bounded entire solution in (4.5) that $L^{-1}g_n(t) \to L^{-1}g(t)$ in $L^2(\Omega)$ for all $t \in \mathbb{R}$. Let $J \subset \mathbb{R}$ be a compact set. We show that $L^{-1}g_n(t) \to L^{-1}g(t)$ in $L^2(\Omega)$ uniformly on J.

Let $\varepsilon > 0$ be arbitrary. By the estimates (iii) and (iv) in the definition of exponential dichotomy (Definition 4.1.2) and the uniform boundedness of $\|g_n\|_\infty$, we can choose

$T > 0$ depending only on ε such that

$$\begin{aligned} \int_T^\infty \|S(\tau)(I-P)\big(g_n(t-\tau)-g(t-\tau)\big)\|_{L^2(\Omega)}d\tau &\le \frac{\varepsilon}{4} \\ \int_{-\infty}^{-T} \|S(\tau)P\big(g_n(t-\tau)-g(t-\tau)\big)\|_{L^2(\Omega)}d\tau &\le \frac{\varepsilon}{4}, \end{aligned} \tag{4.8}$$

for all $n \in \mathbb{N}$. Using the representation (4.6) of bounded entire solutions and (4.8), we get from the exponential dichotomy that

$$\begin{aligned} \|L^{-1}g_n(t) - L^{-1}g(t)\|_{L^2(\Omega)} &\le \int_0^\infty \|S(\tau)(I-P)\big(g_n(t-\tau)-g(t-\tau)\big)\|_{L^2(\Omega)}d\tau \\ &\quad + \int_{-\infty}^0 \|S(\tau)P\big(g_n(t-\tau)-g(t-\tau)\big)\|_{L^2(\Omega)}d\tau \\ &\le \int_0^T \|S(\tau)(I-P)\big(g_n(t-\tau)-g(t-\tau)\big)\|_{L^2(\Omega)}d\tau \\ &\quad + \int_{-T}^0 \|S(\tau)P\big(g_n(t-\tau)-g(t-\tau)\big)\|_{L^2(\Omega)}d\tau + \frac{\varepsilon}{2} \\ &\le \sup_{\tau\in[0,T]} \|g_n(t-\tau)-g(t-\tau)\|_{L^2(\Omega)} \int_0^T Me^{-\beta\tau}d\tau \\ &\quad + \sup_{\tau\in[-T,0]} \|g_n(t-\tau)-g(t-\tau)\|_{L^2(\Omega)} \int_{-T}^0 Me^{\beta\tau}d\tau \\ &\quad + \frac{\varepsilon}{2}, \end{aligned} \tag{4.9}$$

for all $n \in \mathbb{N}$. Since J and $[-T,T]$ are compact, we have $\{t-\tau : t \in J, \tau \in [-T,T]\}$ is a compact subset of $\mathbb{R}$. By our assumption, $g_n(t-\tau) \to g(t-\tau)$ in $L^2(\Omega)$ uniformly with respect to $(t,\tau) \in J \times [-T,T]$. Hence, there exists $N_0 \in \mathbb{N}$ such that

$$\sup_{t\in J} \sup_{\tau\in[-T,T]} \|g_n(t-\tau)-g(t-\tau)\|_{L^2(\Omega)} \frac{M}{\beta}(1-e^{-\beta T}) \le \frac{\varepsilon}{4},$$

for all $n > N_0$. It follows from (4.9) that for every $t \in J$,

$$\|L^{-1}g_n(t) - L^{-1}g(t)\|_{L^2(\Omega)} \le \varepsilon,$$

for all $n > N_0$. As $\varepsilon > 0$ was arbitrary, this proves the required uniform convergence. □

We show below that L^{-1} leaves some subspaces of $BC(\mathbb{R}, L^2(\Omega))$ invariant. We denote the following subspaces of $BC(\mathbb{R}, L^2(\Omega))$ by

$$\begin{aligned} BUC(\mathbb{R}, L^2(\Omega)) &:= \{f \in BC(\mathbb{R}, L^2(\Omega)) : f \text{ is uniformly continuous on } \mathbb{R}\} \\ AP(\mathbb{R}, L^2(\Omega)) &:= \{f \in BC(\mathbb{R}, L^2(\Omega)) : f \text{ is almost periodic}\} \\ P(\omega, L^2(\Omega)) &:= \{f \in BC(\mathbb{R}, L^2(\Omega)) : f \text{ is } \omega\text{-periodic}\} \\ C_1(\mathbb{R}, L^2(\Omega)) &:= \{f \in BC(\mathbb{R}, L^2(\Omega)) : \lim_{t\to\pm\infty} f(t) \text{ exists}\} \\ C_0(\mathbb{R}, L^2(\Omega)) &:= \{f \in BC(\mathbb{R}, L^2(\Omega)) : \lim_{t\to\pm\infty} f(t) = 0\}. \end{aligned}$$

Recall that a function $f : \mathbb{R} \to L^2(\Omega)$ is almost periodic if $\{f_\tau\}_{\tau\in\mathbb{R}}$ is relatively compact in $BC(\mathbb{R}, L^2(\Omega))$, where

$$f_\tau(\cdot) := f(\tau + \cdot) \tag{4.10}$$

is the τ-translation of f. This condition is called Bochner's characterisation of almost periodicity. An equivalent definition can be given in terms of relative dense sets in $\mathbb{R}$ as follows: f is continuous and for every $\varepsilon > 0$ there exists $\ell(\varepsilon) > 0$ such that every interval J of length $\ell(\varepsilon)$ contains τ such that $\|f(t + \tau) - f(t)\|_{L^2(\Omega)} < \varepsilon$ for all $t \in \mathbb{R}$. We refer to [28] or [80] for the theory of almost periodic functions.

Proposition 4.1.7 ([69, Proposition 3])**.** *The operator L^{-1} has the following properties.*

(i) $L^{-1}g_\tau = (L^{-1}g)_\tau$ *for all* $\tau \in \mathbb{R}$.

(ii) *The following spaces* $BUC(\mathbb{R}, L^2(\Omega)), AP(\mathbb{R}, L^2(\Omega)), P(\omega, L^2(\Omega)), C_1(\mathbb{R}, L^2(\Omega))$ *and* $C_0(\mathbb{R}, L^2(\Omega))$ *are invariant under* L^{-1}.

Proof. The assertion (i) is almost trivial. Let $g \in BC(\mathbb{R}, L^2(\Omega))$. Then $u(t) := (L^{-1}g)(t+\tau)$ is a bounded entire solution of (4.2) with $g(t)$ replaced by $g_\tau(t) = g(t+\tau)$. By the uniqueness of bounded entire solutions, the statement follows.

We next show the invariance of the above subspaces. It is evident by (i) that $P(\omega, L^2(\Omega))$ is invariant under L^{-1}. Moreover, if $g \in AP(\mathbb{R}, L^2(\Omega))$, then by definition, $\{g_\tau\}_{\tau\in\mathbb{R}}$ is relatively compact in $BC(\mathbb{R}, L^2(\Omega))$. By (i) again, we see that $\{(L^{-1}g)_\tau\}_{\tau\in\mathbb{R}} = \{L^{-1}g_\tau\}_{\tau\in\mathbb{R}}$ is relatively compact in $BC(\mathbb{R}, L^2(\Omega))$ because L^{-1} is a bounded linear map. This proves the invariance of $AP(\mathbb{R}, L^2(\Omega))$. In addition, for any $g \in BC(\mathbb{R}, L^2(\Omega))$, we see from (i) that

$$\begin{aligned} \|(L^{-1}g)_\tau - L^{-1}g\|_\infty &= \|L^{-1}g_\tau - L^{-1}g\|_\infty \\ &\le \|L^{-1}\|\|g_\tau - g\|_\infty, \end{aligned}$$

for all $\tau \in \mathbb{R}$. Hence, $BUC(\mathbb{R}, L^2(\Omega))$ is also invariant under L^{-1}.

Now, let g be a function in $C_1(\mathbb{R}, L^2(\Omega))$. We denote by $g(\infty) := \lim_{t\to\infty} g(t)$. We define the constant function f by $f(t) := g(\infty)$ for all $t \in \mathbb{R}$. Then $g_\tau(t)$ converges to $f(t)$ in $L^2(\Omega)$ as $\tau \to \infty$ uniformly on compact subsets of $\mathbb{R}$. By Proposition 4.1.6, we have

$$(L^{-1}g)_\tau = L^{-1}g_\tau \to L^{-1}f$$

in $L^2(\Omega)$ as $\tau \to \infty$ uniformly on compact subsets of $\mathbb{R}$. In particular, this implies

$$L^{-1}g(\tau) = (L^{-1}g)_\tau(0) \to L^{-1}f(0) = A^{-1}g(\infty)$$

in $L^2(\Omega)$ as $\tau \to \infty$. By a similar argument, we see that the limit of $L^{-1}g(\tau)$ as $\tau \to -\infty$ exists. Hence, $C_1(\mathbb{R}, L^2(\Omega))$ is invariant under L^{-1}. It is now clear for the invariance of $C_0(\mathbb{R}, L^2(\Omega))$. □

For the invariance of $C_0(\mathbb{R}, L^2(\Omega))$ under L^{-1}, the proof above does not answer how fast $L^{-1}g(t)$ converges to zero as $t \to \pm\infty$. We give an alternative (direct) proof below to keep track of how large t is required for making $L^{-1}g(t)$ closed to zero.

Proposition 4.1.8. *The operator L^{-1} maps $C_0(\mathbb{R}, L^2(\Omega))$ into itself.*

Proof. Suppose $g \in C_0(\mathbb{R}, L^2(\Omega))$. Let $\varepsilon > 0$ be arbitrary. Then there exists $T > 0$ such that

$$\|g(t)\|_{L^2(\Omega)} \le \frac{\beta}{4M}\varepsilon,$$

if $|t| > T$. Fix now $t > T$. If $\tau \in (-\infty, t-T)$, then $t - \tau > T$. Hence, $\|g(t-\tau)\|_{L^2(\Omega)} \le \frac{\beta}{4M}\varepsilon$ for all $\tau \in (-\infty, t-T)$. Therefore,

$$\begin{aligned}\int_0^\infty \|S(\tau)(I-P)g(t-\tau)\|_{L^2(\Omega)}d\tau &= \int_0^{t-T} \|S(\tau)(I-P)g(t-\tau)\|_{L^2(\Omega)}d\tau \\ &\quad + \int_{t-T}^\infty \|S(\tau)(I-P)g(t-\tau)\|_{L^2(\Omega)}d\tau \\ &\le \frac{\beta}{4M}\varepsilon \int_0^{t-T} Me^{-\beta\tau}d\tau + \|g\|_\infty \int_{t-T}^\infty Me^{-\beta\tau}d\tau \\ &\le \frac{\varepsilon}{4} + \|g\|_\infty \frac{M}{\beta} e^{-\beta(t-T)}.\end{aligned} \tag{4.11}$$

Moreover,

$$\int_{-\infty}^0 \|S(\tau)Pg(t-\tau)\|_{L^2(\Omega)}d\tau \le \frac{\beta}{4M}\varepsilon \int_{-\infty}^0 Me^{\beta\tau}d\tau \le \frac{\varepsilon}{4}. \tag{4.12}$$

Combining (4.11) and (4.12), we have

$$\begin{aligned}\|L^{-1}g(t)\|_{L^2(\Omega)} &\leq \int_0^\infty \|S(\tau)(I-P)g(t-\tau)\|_{L^2(\Omega)}d\tau + \int_{-\infty}^0 \|S(\tau)Pg(t-\tau)\|_{L^2(\Omega)}d\tau \\ &\leq \frac{\varepsilon}{2} + \|g\|_\infty \frac{M}{\beta} e^{-\beta(t-T)}.\end{aligned} \tag{4.13}$$

Since the above argument works for any fixed $t > T$, we have that (4.13) holds for all $t > T$. Choosing $\tilde{T} > T$ so that

$$\|g\|_\infty \frac{M}{\beta} e^{-\beta(\tilde{T}-T)} < \frac{\varepsilon}{2},$$

it follows that $\|L^{-1}g(t)\|_{L^2(\Omega)} < \varepsilon$ if $t > \tilde{T}$. Since $\varepsilon > 0$ was arbitrary, this implies $L^{-1}g(t) \to 0$ in $L^2(\Omega)$ as $t \to \infty$.

To get $L^{-1}g(t) \to 0$ in $L^2(\Omega)$ as $t \to -\infty$, we instead consider $t < -T$ and follow a similar argument as above. Then we estimate $\|L^{-1}g(t)\|_{L^2(\Omega)}$ by splitting it into

$$\begin{aligned}&\int_{-\infty}^{t+T} \|S(\tau)Pg(t-\tau)\|_{L^2(\Omega)}d\tau + \int_{t+T}^0 \|S(\tau)Pg(t-\tau)\|_{L^2(\Omega)}d\tau \\ &\quad + \int_0^\infty \|S(\tau)(I-P)g(t-\tau)\|_{L^2(\Omega)}d\tau.\end{aligned}$$

□

4.2 Convergence of bounded entire solutions of linear equations: L^2-Theory

Let Ω_n and Ω be bounded open sets in $\mathbb{R}^N$ satisfying Assumption 1.4.1. We consider the perturbation of (4.1) given by

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathcal{A}_n u = g_n(x,t) & \text{in } \Omega_n \times (-\infty,\infty) \\ u = 0 & \text{on } \partial\Omega_n \times (-\infty,\infty), \end{cases} \tag{4.14}$$

where $g_n \in C(\mathbb{R}^N \times \mathbb{R})$ and $\mathcal{A}_n$ is an elliptic operator defined by (2.40). As previously seen in Section 2.4.2, we can write (4.14) in the abstract form in $L^2(\Omega_n)$ as

$$u'(t) + A_n u = g_n(t), \quad t \in (-\infty,\infty). \tag{4.15}$$

The operator $-A_n$ is the generator of a strongly continuous analytic semigroup $S_n(t), t \geq 0$ on $L^2(\Omega_n)$.

As before, we use the inclusion operators i_n and i defined by (1.27) and (1.28) respectively to handle domain perturbation. We also use the restriction operators r_n and r defined by (1.29) and (1.30), respectively. We make the following assumptions for the unperturbed equation.

Assumption 4.2.1. We assume that $\sigma(-A)$ can be decomposed as $\sigma(-A) = \sigma^+ \cup \sigma^-$ where σ^+ and σ^- are given as in (4.3).

We see in Example 4.1.3 that Assumption 4.2.1 implies an exponential dichotomy of $u'(t) + Au(t) = 0$ with the projection $P = P^+$ given by (4.4). We assume the conditions below for our perturbation.

Assumption 4.2.2. We assume that

(i) The coefficients of the operator $\mathcal{A}_n$ converge to the corresponding coefficients of $\mathcal{A}$ as $n \to \infty$ (see Assumption 2.4.5);

(ii) $H_0^1(\Omega_n)$ converges to $H_0^1(\Omega)$ in the sense of Mosco.

Lemma 4.2.3. *Suppose that Assumption 4.2.1 and Assumption 4.2.2 are satisfied. Then $u'(t) + A_n u(t) = 0$ has an exponential dichotomy on $\mathbb{R}$ with uniform exponent β and uniform bound M for all n sufficiently large. In particular, for n sufficiently large the dichotomy projections P_n can be taken as the spectral projection*

$$P_n := P_n^+ = \frac{1}{2\pi i}\int_\Gamma (\lambda + A_n)^{-1} d\lambda, \tag{4.16}$$

where Γ is a rectifiable closed curve separating σ^+ from σ^-. Moreover, $i_n P_n r_n \to iPr$ in $\mathscr{L}(L^2(D))$ as $n \to \infty$.

Proof. The assertion directly follows from Corollary 2.4.12 by noting that σ_n is separated by the strip $\{\lambda : -\beta \leq \mathrm{Re}(\lambda) \leq \beta\}$ with the uniform parameter β for n sufficiently large. □

We prove the convergence of solutions in the stable and unstable directions below.

Lemma 4.2.4. *Suppose that Assumption 4.2.1 and Assumption 4.2.2 are satisfied. Let $u_{0,n} \in L^2(\Omega_n)$ and $u_0 \in L^2(\Omega)$ be such that $ri_n u_{0,n} \rightharpoonup u_0$ in $L^2(\Omega)$ weakly and $\|i_n u_{0,n}\|_{L^2(D)}$ is uniformly bounded. Then*

(i) *$i_n S_n(t)(I - P_n)u_{0,n} \to iS(t)(I - P)u_0$ in $L^2(D)$ for all $t > 0$ and uniformly on compact subsets of $(0, \infty)$.*

(ii) $i_n S_n(t) P_n u_{0,n} \to i S(t) P u_0$ *in* $L^2(D)$ *for all* $t \leq 0$ *and uniformly on compact subsets of* $(-\infty, 0]$.

Moreover, if $i_n u_{0,n} \to i u_0$ *in* $L^2(D)$ *strongly, then the convergence in assertion (i) holds uniformly on compact subsets of* $[0, \infty)$.

Proof. It follows from Lemma 3.4.1 (iv) that $r i_n (I - P_n) u_{0,n} \rightharpoonup (I - P) u_0$ in $L^2(\Omega)$ weakly. Applying domain perturbation results for initial value problems in Theorem 1.4.9 (with homogeneous data $f_n, f = 0$), assertion (i) follows. Taking Lemma 3.4.1 (i) into account, the additional statement regarding to uniform convergence on compact subset of $[0, \infty)$ holds.

We next prove assertion (ii). By Lemma 3.4.1 (iv), we have $r i_n P_n u_{0n} \rightharpoonup P u_0$ in $L^2(\Omega)$ weakly. By uniform boundedness of $i_n u_{0,n}$ in $L^2(D)$, we have $\|P_n u_{0,n}\|_{L^2(\Omega_n)}$ is also uniformly bounded. Applying Corollary 3.4.6, we obtain a subsequence such that $i_{n_k} P_{n_k} u_{0,n_k} \to i w$ in $L^2(D)$ for some $w \in X^+ := P(L^2(\Omega))$. In particular, $r i_{n_k} P_{n_k} u_{0,n_k} \to w$ in $L^2(\Omega)$. Hence, we conclude that $w = P u_0$ and the whole sequence

$$i_n P_n u_{0,n} \to i P u_0 \tag{4.17}$$

in $L^2(D)$ strongly. Let now $T > 0$ be arbitrary. By Lemma 4.2.3, we have

$$i_n P_n r_n i S(-T) P u_0 \to i P r i S(-T) P u_0 = i S(-T) P u_0 \tag{4.18}$$

in $L^2(D)$. Applying domain perturbation results in Theorem 1.4.9 (for homogeneous equations), we get

$$i_n S_n(T) P_n r_n i S(-T) P u_0 \to i S(T) S(-T) P u_0 = i P u_0 \tag{4.19}$$

in $L^2(D)$. On the other hand, using (4.17) we also get

$$i_n S_n(T) P_n S_n(-T) P_n u_{0,n} = i_n P_n u_{0,n} \to i P u_0 \tag{4.20}$$

in $L^2(D)$. It follows from (4.19) and (4.20) that

$$i_n S_n(T) P_n S_n(-T) P_n u_{0,n} - i_n S_n(T) P_n r_n i S(-T) P u_0 \to 0$$

in $L^2(D)$. In particular,

$$\left\| S_n(T) P_n \big(S_n(-T) P_n u_{0,n} - P_n r_n i S(-T) P u_0 \big) \right\|_{L^2(\Omega_n)} \to 0 \tag{4.21}$$

as $n \to \infty$. By the exponential dichotomy for the perturbed equations (see Lemma 4.2.3) and (4.21), we see that

$$\begin{aligned}
&\|S_n(-T)P_n u_{0,n} - P_n r_n i S(-T) P u_0\|_{L^2(\Omega_n)} \\
&\quad \le \|S_n(-T)P_n\| \left\|S_n(T)P_n\big(S_n(-T)P_n u_{0,n} - P_n r_n i S(-T)Pu_0\big)\right\|_{L^2(\Omega_n)} \\
&\quad \le Me^{-\beta T} \left\|S_n(T)P_n\big(S_n(-T)P_n u_{0,n} - P_n r_n i S(-T)Pu_0\big)\right\|_{L^2(\Omega_n)} \\
&\quad \to 0
\end{aligned} \tag{4.22}$$

as $n \to \infty$. Combining (4.22) and (4.18), we conclude that

$$i_n S_n(-T)P_n u_{0,n} \to iS(-T)Pu_0 \tag{4.23}$$

in $L^2(D)$ as $n \to \infty$. As $T > 0$ was arbitrary and (4.17) holds, we have

$$i_n S_n(t)P_n u_{0,n} \to iS(t)Pu_0 \tag{4.24}$$

in $L^2(\Omega)$ for all $t \le 0$. The uniform convergence of (4.24) with respect to t in compact subsets of $(-\infty, 0]$ immediately follows from Theorem 1.4.9 (by applying to the forward problems with initial data $S_n(-T)P_n u_{0,n}$ for $T > 0$). □

We next prove our first result on convergence of bounded entire solutions under domain perturbation.

Theorem 4.2.5. *Suppose that Assumption 4.2.1 and Assumption 4.2.2 are satisfied. Let g_n and g be functions in $BC(\mathbb{R}, L^2(\Omega_n))$ and $BC(\mathbb{R}, L^2(\Omega))$, respectively. Assume that $\|g_n\|_\infty$ is uniformly bounded. If $ri_n g_n(t) \rightharpoonup g(t)$ weakly in $L^2(\Omega)$ uniformly with respect to $t \in \mathbb{R}$, that is*

$$\sup_{t\in\mathbb{R}} \,(ri_n g_n(t) - g(t)|v)_{L^2(\Omega)} \to 0, \tag{4.25}$$

for all $v \in L^2(\Omega)$, then $i_n L_n^{-1} g_n \to iL^{-1} g$ in $BC(\mathbb{R}, L^2(D))$.

Proof. Using the representation (4.6) of bounded entire solutions in Theorem 4.1.5, we have

$$\begin{aligned}
L_n^{-1} g_n(t) &= \int_0^\infty S_n(\tau)(I - P_n) g_n(t-\tau)\,d\tau - \int_{-\infty}^0 S_n(\tau) P_n g_n(t-\tau)\,d\tau \\
L^{-1} g(t) &= \int_0^\infty S(\tau)(I - P) g(t-\tau)\,d\tau - \int_{-\infty}^0 S(\tau) P g(t-\tau)\,d\tau,
\end{aligned}$$

for all $t \in \mathbb{R}$. To show the uniform convergence of $i_n L_n^{-1} g_n(t) \to iL^{-1}g(t)$ in $L^2(D)$ with respect to $t \in \mathbb{R}$, it suffices to show that for every sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \in \mathbb{R}$, we have

$$\int_0^\infty \|i_n S_n(\tau)(I-P_n)g_n(t_n-\tau) - iS(\tau)(I-P)g(t_n-\tau)\|_{L^2(D)} d\tau \to 0$$

$$\int_{-\infty}^0 \|i_n S_n(\tau)P_n g_n(t_n-\tau) - iS(\tau)Pg(t_n-\tau)\|_{L^2(D)} d\tau \to 0$$

as $n \to \infty$.

Let $\{t_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence in $\mathbb{R}$. We have from (4.25) that

$$(ri_n g_n(t_n-\tau) - g(t_n-\tau)|v)_{L^2(D)} \to 0, \tag{4.26}$$

for all $v \in L^2(\Omega)$ and for all $\tau \in \mathbb{R}$. This implies

$$ri_n g_n(t_n-\tau) - g(t_n-\tau) \rightharpoonup 0 \tag{4.27}$$

in $L^2(\Omega)$ weakly for all $\tau \in \mathbb{R}$. By the convergence of spectral projections $i_n P_n r_n \to iPr$ (see Lemma 4.2.3) together with (4.27) and the uniform boundedness of $\|g_n\|_\infty$, it follows that

$$\begin{aligned}
&(ri_n P_n g_n(t_n-\tau) - Pg(t_n-\tau)|v)_{L^2(\Omega)} \\
&\quad \le (ri_n P_n g_n(t_n-\tau) - Pri_n g_n(t_n-\tau)|v)_{L^2(\Omega)} \\
&\qquad + (Pri_n g_n(t_n-\tau) - Pg(t_n-\tau)|v)_{L^2(\Omega)} \\
&\quad \le \|r\|\|i_n P_n r_n - iPr\|\|i_n g_n(t_n-\tau)\|_{L^2(D)}\|v\|_{L^2(\Omega)} \\
&\qquad (ri_n g_n(t_n-\tau) - g(t_n-\tau)|P'v)_{L^2(\Omega)} \\
&\quad \to 0,
\end{aligned}$$

for all $v \in L^2(\Omega)$ and for all $\tau \in \mathbb{R}$ where P' is the adjoint operator of P. This implies

$$ri_n P_n g_n(t_n-\tau) - Pg(t_n-\tau) \rightharpoonup 0 \tag{4.28}$$

in $L^2(\Omega)$ weakly for all $\tau \in \mathbb{R}$. By (4.27), we also have that

$$ri_n(I-P_n)g_n(t_n-\tau) - (I-P)g(t_n-\tau) \rightharpoonup 0 \tag{4.29}$$

in $L^2(\Omega)$ weakly for all $\tau \in \mathbb{R}$.

We fix now an arbitrary $\tau \in (0,\infty)$. By the boundedness of $\|(I-P)g(t_n-\tau)\|_{L^2(\Omega)}$, we have that for any subsequence $\|(I-P)g(t_{n_k}-\tau)\|_{L^2(\Omega)}$, there exists a further weak convergent subsequence (indexed again by n_k) such that

$$(I-P)g(t_{n_k}-\tau) \rightharpoonup \xi_\tau$$

in $L^2(\Omega)$ for some $\xi_\tau \in L^2(\Omega)$. We get from (4.29) that

$$ri_{n_k}(I - P_{n_k})g_{n_k}(t_{n_k} - \tau) \rightharpoonup \xi_\tau$$

in $L^2(\Omega)$. By Lemma 4.2.4 (i), we have

$$\begin{aligned} i_{n_k}S_{n_k}(t)(I - P_{n_k})g_{n_k}(t_{n_k} - \tau) &\to iS(t)(I - P)\xi_\tau \\ iS(t)(I - P)g(t_{n_k} - \tau) &\to iS(t)(I - P)\xi_\tau \end{aligned}$$

in $L^2(D)$ for all $t \in (0, \infty)$. In particular, taking $t = \tau$ we have

$$\|i_{n_k}S_{n_k}(\tau)(I - P_{n_k})g_{n_k}(t_{n_k} - \tau) - iS(\tau)(I - P)g(t_{n_k} - \tau)\|_{L^2(D)} \to 0.$$

Since the above argument holds for every subsequence, we conclude that

$$\|i_nS_n(\tau)(I - P_n)g_n(t_n - \tau) - iS(\tau)(I - P)g(t_n - \tau)\|_{L^2(D)} \to 0. \tag{4.30}$$

As $\tau \in (0, \infty)$ was arbitrary, the above argument implies pointwise convergence of (4.30) for each $\tau \in (0, \infty)$. By the uniform boundedness of $\|g_n\|_\infty$ and the exponential dichotomy, we see that

$$\begin{aligned} &\int_0^\infty \|i_nS_n(\tau)(I - P_n)g_n(t_n - \tau) - iS(\tau)(I - P)g(t_n - \tau)\|_{L^2(D)}d\tau \\ &\le (\|g_n\|_\infty + \|g\|_\infty)\int_0^\infty Me^{-\beta\tau}d\tau \\ &< \infty. \end{aligned}$$

Using (4.30) and applying the dominated convergence theorem, we get

$$\int_0^\infty \|i_nS_n(\tau)(I - P_n)g_n(t_n - \tau) - iS(\tau)(I - P)g(t_n - \tau)\|_{L^2(D)}d\tau \to 0$$

as $n \to \infty$.

By a similar argument as above (using (4.28) and Lemma 4.2.4 (ii) instead of (4.29) and Lemma 4.2.4 (i), respectively), we have pointwise convergence

$$\|i_nS_n(\tau)P_ng_n(t_n - \tau) - iS(\tau)Pg(t_n - \tau)\|_{L^2(D)} \to 0, \tag{4.31}$$

for each $\tau \in (-\infty, 0)$. Applying the dominated convergence theorem, we get

$$\int_{-\infty}^0 \|i_nS_n(\tau)P_ng_n(t_n - \tau) - iS(\tau)Pg(t_n - \tau)\|_{L^2(D)}d\tau \to 0$$

as $n \to \infty$.

Since a sequence $\{t_n\}_{n\in\mathbb{N}}$ was arbitrary, we conclude that $i_nL_n^{-1}g_n(t) \to iL^{-1}g(t)$ in $L^2(D)$ uniformly on $\mathbb{R}$ as required. □

In the next convergence result, we assume only weak-$*$ convergence of the inhomogeneous terms in $L^\infty(\mathbb{R}, L^2(\Omega))$.

Theorem 4.2.6. *Suppose that Assumption 4.2.1 and Assumption 4.2.2 are satisfied. Let g_n and g be functions in $BC(\mathbb{R}, L^2(\Omega_n))$ and $BC(\mathbb{R}, L^2(\Omega))$, respectively. Assume that $\|g_n\|_\infty$ is uniformly bounded. If $ri_n g_n \overset{*}{\rightharpoonup} g$ in $L^\infty(\mathbb{R}, L^2(\Omega))$, then $i_n L_n^{-1} g_n$ converges to $iL^{-1}g$ in $BC(\mathbb{R}, L^2(D))$ under the topology of uniform convergence on compact sets.*

Proof. By the uniform boundedness of $\|g_n\|_\infty$, we have that $i_n L_n^{-1} g_n$ is also uniformly bounded in $L^\infty(\mathbb{R}, L^2(D))$. As $\mathbb{R}$ is separable, we can choose a dense sequence $\{t_1, t_2, \ldots\}$ in $\mathbb{R}$ (e.g. a sequence of rational numbers). Since $i_n L_n^{-1} g_n(t_1)$ is bounded in $L^2(D)$, we can extract a subsequence $\{i_{1,n} L_{1,n}^{-1} g_{1,n}(t_1)\}_{n\in\mathbb{N}}$ of $\{i_n L_n^{-1} g_n(t_1)\}_{n\in\mathbb{N}}$ so that it converges in $L^2(D)$ weakly. Similarly, by the boundedness of $i_{1,n} L_{1,n}^{-1} g_{1,n}(t_2)$, we can extract a subsequence $\{i_{2,n} L_{2,n}^{-1} g_{2,n}(t_2)\}_{n\in\mathbb{N}}$ of $\{i_{1,n} L_{1,n}^{-1} g_{1,n}(t_2)\}_{n\in\mathbb{N}}$ so that it converges in $L^2(D)$ weakly. Inductively, we obtain for every $k \geq 2$ a subsequence $\{i_{k,n} L_{k,n}^{-1} g_{k,n}(t_k)\}_{n\in\mathbb{N}}$ of $\{i_{(k-1),n} L_{(k-1),n}^{-1} g_{(k-1),n}(t_k)\}_{n\in\mathbb{N}}$ which converges in $L^2(D)$ weakly as $n \to \infty$. We denote the weak limit of $i_{k,n} L_{k,n}^{-1} g_{k,n}(t_k)$ in $L^2(D)$ by ψ_{t_k} for each $k \in \mathbb{N}$. In particular,

$$ri_{k,n} L_{k,n}^{-1} g_{k,n}(t_k) \rightharpoonup r\psi_{t_k} =: \phi_{t_k}$$

in $L^2(\Omega)$ as $n \to \infty$ for all $k \in \mathbb{N}$. Consider a diagonal subsequence $\{i_{n,n} L_{n,n}^{-1} g_{n,n}\}_{n\in\mathbb{N}}$. By a standard diagonal argument, $ri_{n,n} L_{n,n}^{-1} g_{n,n}(t_k) \rightharpoonup \phi_{t_k}$ in $L^2(\Omega)$ weakly for every $k \geq 1$. In other words, the diagonal subsequence converges weakly on a dense subset of $\mathbb{R}$.

By the weak $*$-convergence of $ri_n g_n \overset{*}{\rightharpoonup} g$ in $L^\infty(\mathbb{R}, L^2(\Omega))$, we have $ri_n g_n \rightharpoonup g$ in $L^2(J, L^2(\Omega))$ for every compact interval $J \subset\subset \mathbb{R}$. To see this, let $v \in L^2(J, L^2(\Omega))$ be arbitrary and denote by $\tilde{v}$ the extension of v by zero on $\mathbb{R}\backslash\bar{J}$. We have $\tilde{v} \in L^1((\mathbb{R}, L^2(\Omega))$. It follows from the definition of weak $*$-convergence that

$$\begin{aligned}\int_J (ri_n g_n(t)|v(t))_{L^2(\Omega)} dt &= \int_{-\infty}^{\infty} (ri_n g_n(t)|\tilde{v}(t))_{L^2(\Omega)} dt \\ \to \int_{-\infty}^{\infty} (g(t)|\tilde{v}(t))_{L^2(\Omega)} dt &= \int_J (g(t)|v(t))_{L^2(\Omega)} dt.\end{aligned}$$

This proves the required weak convergence in $L^2(J, L^2(\Omega))$. We note that for every $k \geq 1$, the restriction of $L_{n,n}^{-1} g_{n,n}$ on (t_k, ∞) satisfies the following initial value problem

$$\begin{cases} \dot{v}(t) + A_{n,n} v(t) = g_{n,n}(t) & \text{for } t \in (t_k, \infty) \\ v(t_k) = L_{n,n}^{-1} g_{n,n}(t_k). \end{cases} \tag{4.32}$$

Since $ri_n g_n \rightharpoonup g$ in $L^2(J, L^2(\Omega))$ for any $J \subset\subset \mathbb{R}$ and $ri_{n,n} L_{n,n}^{-1} g_{n,n}(t_k) \rightharpoonup \phi_{t_k}$ in $L^2(\Omega)$, we conclude from domain perturbation results in Theorem 1.4.9 that for every $k \geq 1$, the restriction $i_{n,n} L_{n,n}^{-1} g_{n,n}|_{(t_k,\infty)}$ converges to the unique solution of the following problem

$$\begin{cases} \dot{v}(t) + Av(t) = g(t) & \text{for } t \in (t_k, \infty) \\ v(t_k) = \phi_{t_k} \end{cases} \tag{4.33}$$

uniformly on compact subsets of (t_k, ∞). Since for every $t \in \mathbb{R}$ there exists t_k such that $t_k < t$ (from the density of $\{t_k, k \in \mathbb{N}\}$ in $\mathbb{R}$), we conclude that $i_{n,n} L_{n,n}^{-1} g_{n,n}(t)$ converges in $L^2(D)$ strongly for all $t \in \mathbb{R}$. We denote the pointwise limit of $i_{n,n} L_{n,n}^{-1} g_{n,n}(t)$ in $L^2(D)$ by $i\phi_t$ for each $t \in \mathbb{R}$ where $\phi_t \in L^2(\Omega)$. Note that for $k \geq 1$ the strong limit agrees with the weak limit ϕ_{t_k} considered previously. Since $\sup_{n\in\mathbb{N}} \sup_{t\in\mathbb{R}} \|i_n L_n^{-1} g_n(t)\|_{L^2(D)} < K$ for some $K > 0$, we see that $\|i\phi_t\|_{L^2(D)} \leq K$ for all $t \in \mathbb{R}$. Setting $\phi(t) := \phi_t$ for $t \in \mathbb{R}$, we have $\phi \in L^\infty(\mathbb{R}, L^2(\Omega))$.

Repeating the above argument for an arbitrary $s \in \mathbb{R}$ instead of $t_k, k \geq 1$, we have that the restriction of ϕ on (s, ∞) satisfies

$$\begin{cases} \dot{v}(t) + Av(t) = g(t) & \text{for } t \in (s, \infty) \\ v(s) = \phi(s). \end{cases}$$

In particular, we have $\phi|_{(s,\infty)} \in C((s,\infty), L^2(\Omega))$. Since this is true for any $s \in \mathbb{R}$, we conclude that $\phi \in BC(\mathbb{R}, L^2(\Omega))$ and ϕ is a bounded entire solution of (4.2). By the uniqueness of bounded entire solution, $\phi = L^{-1}g$ and the whole sequence converges, that is

$$i_n L_n^{-1} g_n(t) \to iL^{-1} g(t) \tag{4.34}$$

in $L^2(D)$ pointwise for all $t \in \mathbb{R}$. The uniform convergence of (4.34) with respect to t in compact subsets of $\mathbb{R}$ immediately follows from Theorem 1.4.9. □

4.3 L^p-Theory for bounded entire solutions of linear parabolic equations

In this section, we prove the convergence of bounded entire solutions under domain perturbation under the L^p-norm for a certain range of $p \geq 2$. The main ingredients for this result are the L^2 results in Section 4.2 and the interpolation arguments. In addition, we prove Hölder continuity of bounded entire solutions with respect to time under a certain L^p norm.

Recall that by our assumptions on the operator A, the semigroup $S(t)$ for Dirichlet problems satisfies the following heat kernel estimate

$$\|S(t-s)\|_{\mathscr{L}(L^p(\Omega),L^q(\Omega))} \leq C(t-s)^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}, \tag{4.35}$$

for all $1 \leq p \leq q \leq \infty$ and $t \geq s$, where C is a positive constant independent of Ω (see [36, Section 7]).

We first show that the solutions are actually bounded in L^p for some range of $p \geq 2$. For this, we need the following estimates.

Lemma 4.3.1. *Let $2 \leq p < \infty$ be such that $\alpha_p := \frac{N}{2}(\frac{1}{2} - \frac{1}{p}) \in [0, 1)$. Then there exists $M_1(p), M_2(p) > 0$ such that*

(i) $\|S(t)(I-P)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))} \leq M_1(p)t^{-\alpha_p}e^{-\frac{\beta}{2}t}$ *for all* $t > 0$.

(ii) $\|S(t)P\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))} \leq M_2(p)e^{\beta t}$ *for all* $t < 0$.

Proof. Let $t > 0$. It follows from the heat kernel estimate (4.35) and the exponential dichotomy that

$$\begin{aligned}
\|S(t)(I-P)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))} &\leq \|S(t/2)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))}\|S(t/2)(I-P)\|_{\mathscr{L}(L^2(\Omega))} \\
&\leq C\left(\frac{t}{2}\right)^{-\alpha_p} Me^{-\frac{\beta}{2}t} \\
&\leq M_1(p)t^{-\alpha_p}e^{-\frac{\beta}{2}t},
\end{aligned}$$

for all $t > 0$ where $M_1(p) := CM2^{\alpha_p}$. Similarly, for $t < 0$ we have

$$\begin{aligned}
\|S(t)P\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))} &\leq \|S(1)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))}\|S(t-1)P\|_{\mathscr{L}(L^2(\Omega))} \\
&\leq C1^{-\alpha_p}Me^{\beta(t-1)} \\
&\leq M_2(p)e^{\beta t},
\end{aligned}$$

for all $t < 0$ where $M_2(p) := CMe^{-\beta}$. □

Remark 4.3.2. The range of p satisfying the condition in the above theorem depends on N. In fact, for domains in two, three or four dimensional spaces, $2 \leq p < \infty$ is valid. For $N \geq 5$, we require $2 \leq p < \frac{2N}{N-4}$.

We next show that the bounded entire solutions are bounded with respect to a stronger L^p norm.

Theorem 4.3.3. *Let g be a function in $BC(\mathbb{R}, L^2(\Omega))$. The bounded entire solution $u = L^{-1}g$ of (4.2) belongs to $BC(\mathbb{R}, L^p(\Omega))$ for all p satisfying $2 \le p < \infty$ with $\alpha_p := \frac{N}{2}(\frac{1}{2} - \frac{1}{p}) \in [0, 1)$.*

Proof. Using the representation (4.6) of bounded entire solutions, we show that

$$\int_0^\infty \|S(\tau)(I-P)g(t-\tau)\|_{L^p(\Omega)} d\tau - \int_{-\infty}^0 \|S(\tau)Pg(t-\tau)\|_{L^p(\Omega)} d\tau$$

is uniformly bounded for all $t \in \mathbb{R}$. By Lemma 4.3.1 (i), we see that

$$\begin{aligned}
&\int_0^\infty \|S(\tau)(I-P)g(t-\tau)\|_{L^p(\Omega)} d\tau \\
&\le \int_0^\infty \|S(\tau)(I-P)\|_{\mathscr{L}(L^2(\Omega), L^p(\Omega))} \|g(t-\tau)\|_{L^2(\Omega)} d\tau \\
&\le \|g\|_\infty \int_0^\infty M_1(p) \tau^{-\alpha_p} e^{-\frac{\beta}{2}\tau} d\tau \\
&= \|g\|_\infty M_1(p) \int_0^\infty \left(\frac{2\tau}{\beta}\right)^{-\alpha_p} e^{-\tau} \frac{2}{\beta} d\tau \\
&= \|g\|_\infty M_1(p) \left(\frac{2}{\beta}\right)^{1-\alpha_p} \int_0^\infty \tau^{-\alpha_p} e^{-\tau} d\tau \\
&= \|g\|_\infty M_1(p) \left(\frac{2}{\beta}\right)^{1-\alpha_p} \Gamma(1-\alpha_p),
\end{aligned}$$

for all $t \in \mathbb{R}$. By Lemma 4.3.1 (ii), we have

$$\begin{aligned}
\int_{-\infty}^0 \|S(\tau)Pg(t-\tau)\|_{L^p(\Omega)} d\tau &\le \int_{-\infty}^0 \|S(\tau)P\|_{\mathscr{L}(L^2(\Omega), L^p(\Omega))} \|g(t-\tau)\|_{L^2(\Omega)} d\tau \\
&\le \|g\|_\infty \int_{-\infty}^0 M_2(p) e^{\beta t} d\tau \\
&= \|g\|_\infty \frac{M_2(p)}{\beta},
\end{aligned}$$

for all $t \in \mathbb{R}$. □

As a direct application of the convergence results in Section 4.2, we state the following theorems.

Theorem 4.3.4. *Suppose that all assumptions in Theorem 4.2.5 are satisfied. Then $i_n L_n^{-1} g_n \to i L^{-1} g$ in $BC(\mathbb{R}, L^p(D))$ for all p satisfying $2 \le p < \infty$ and $\frac{N}{2}(\frac{1}{2} - \frac{1}{p}) \in [0, 1)$.*

Proof. Taking Remark 4.3.2 into account, we fix p such that $2 \le p < \infty$ for $N = 2, 3, 4$ and $2 \le p < \frac{2N}{N-4}$ for $N \ge 5$. There exists q such that $p < q < \infty$ for $N = 2, 3, 4$

or $p \leq q < \frac{2N}{N-4}$ for $N \geq 5$. By the uniform estimate of semigroups $S_n(t)$ and the uniform exponential dichotomy, we see from Theorem 4.3.3 that $\|i_n L_n^{-1} g_n\|_{BC(\mathbb{R},L^q(D))}$ and $\|iL^{-1}g\|_{BC(\mathbb{R},L^q(D))}$ are uniformly bounded for all $n \in \mathbb{N}$. Hence, by Theorem 4.2.5 and interpolation inequality with $\theta := \frac{(p-2)q}{(q-2)p}$,

$$\begin{aligned}
&\|i_n L_n^{-1} g_n(t) - iL^{-1}g(t)\|_{L^p(D)} \\
&\quad \leq \|i_n L_n^{-1} g_n(t) - iL^{-1}g(t)\|^{\theta}_{L^q(D)} \|i_n L_n^{-1} g_n(t) - iL^{-1}g(t)\|^{1-\theta}_{L^2(D)} \\
&\quad \leq \|i_n L_n^{-1} g_n - iL^{-1}g\|^{\theta}_{BC(\mathbb{R},L^q(D))} \|i_n L_n^{-1} g_n - iL^{-1}g\|^{1-\theta}_{BC(\mathbb{R},L^2(D))} \\
&\quad \to 0
\end{aligned}$$

as $n \to \infty$ uniformly with respect to $t \in \mathbb{R}$. □

Theorem 4.3.5. *Suppose that all assumptions in Theorem 4.2.6 are satisfied. Then $i_n L_n^{-1} g_n$ converges to $iL^{-1}g$ in $BC(\mathbb{R}, L^p(D))$ under the topology of uniform convergence on compact subsets for all p satisfying $2 \leq p < \infty$ and $\frac{N}{2}(\frac{1}{2} - \frac{1}{p}) \in [0, 1)$.*

Proof. The result follows from Theorem 4.2.6 and a similar interpolation argument as in the proof of Theorem 4.3.4. □

In the remainder of this section, we obtain the Hölder continuity of the bounded entire solution of (4.2) with respect to time in $BC(\mathbb{R}, L^p(\Omega))$. The result is similar to the Hölder continuity proved in Daners [40] (see Lemma 5.3, Lemma 5.5, Corollary 5.6) for evolution equations in interpolation spaces, or in Henry [52, Section 3.3] for evolution equations in fractional power spaces. As we work in the L^p scales, the estimate $\|AS(t)\|_{\mathscr{L}(L^p(\Omega))} \leq \frac{c_p}{t}$ for $t > 0$ leads to a singularity when integrating near $t = 0$. This difficulty does not occur when working in interpolation spaces or fractional power spaces. This is because we can get a better estimate of $\|AS(t)\|_{\alpha,\beta}$ for certain α and β due to the use of graph norm in the definition of these spaces ($X_1 = D(A)$ for interpolation spaces, $X^\alpha = D(A^\alpha)$ for fractional power spaces). However, we exploit the boundedness and use that the solution is defined on $\mathbb{R}$ to establish Hölder continuity of bounded entire solutions under the L^p norm in Proposition 4.3.7 below. This result seems to be interesting and its proof is a little more involved than those in [40] and [52]. We first give the following lemma.

Lemma 4.3.6. *For every $2 < p < \infty$ satisfying $\alpha_p := \frac{N}{2}(\frac{1}{2} - \frac{1}{p}) \in (0, 1)$, there exists a constant $C_p > 0$ depending only on p such that*

$$\|S(t) - S(s)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))} \leq \frac{C_p}{s}|t - s|^{1-\alpha_p},$$

for all $0 < s \leq t$.

Proof. Let $0 < s < t$ be arbitrary. Using a standard heat kernel estimate (4.35) and the estimate $\|AS(t)\|_{\mathscr{L}(L^p(\Omega))} \leq \frac{c_p}{t}$ for $t > 0$ (see e.g. [40, p.21] or [66, Chapter 2, Theorem 5.2 (d)]), we compute

$$\begin{aligned}
\|S(t)x - S(s)x\|_{L^p(\Omega)} &\leq \int_s^t \left\| \frac{d}{d\tau} S(\tau)x \right\|_{L^p(\Omega)} d\tau \\
&= \int_s^t \|AS(\tau)x\|_{L^p(\Omega)} d\tau \\
&\leq \int_s^t \|AS(s)\|_{\mathscr{L}(L^p(\Omega))} \|A(\tau - s)\|_{\mathscr{L}(L^2(\Omega), L^p(\Omega))} \|x\|_{L^2(\Omega)} d\tau \\
&\leq \|x\|_{L^2(\Omega)} \frac{c_p}{s} \int_s^t C(\tau - s)^{-\alpha_p} d\tau \\
&= \|x\|_{L^2(\Omega)} \frac{c_p}{s} \frac{C}{1 - \alpha_p} (t - s)^{1-\alpha_p} \\
&\leq \frac{C_p}{s} (t - s)^{1-\alpha_p} \|x\|_{L^2(\Omega)},
\end{aligned}$$

for all $x \in L^2(\Omega)$ where we set $C_p := \frac{c_p C}{1-\alpha_p}$. This implies the assertion of the lemma. □

Proposition 4.3.7. *Let* g *be a function in* $BC(\mathbb{R}, L^2(\Omega))$*. For every* p, q *with* $2 < p < q < \infty$ *and* $\alpha_p := \frac{N}{2}(\frac{1}{2} - \frac{1}{p}), \alpha_q := \frac{N}{2}(\frac{1}{2} - \frac{1}{q}) \in (0, 1)$*, there exists a constant* $C > 0$ *depending on* $\|g\|_\infty, p, q$*, the bound* M *and the exponent* β *of the exponential dichotomy such that the bounded entire solution* $u = L^{-1}g$ *of* (4.2) *satisfies*

$$\|u(t_1) - u(t_2)\|_{L^p(\Omega)} \leq C|t_1 - t_2|^{1-\alpha_q},$$

for all $t_1, t_2 \in \mathbb{R}$.

Proof. If $|t_1 - t_2| \geq 1$, then we see from the proof of Theorem 4.3.3 that

$$\begin{aligned}
\|u(t_1) - u(t_2)\|_{L^p(\Omega)} &\leq 2 \sup_{t \in \mathbb{R}} \|u(t)\|_{L^p(\Omega)} \\
&\leq 2\|g\|_\infty \left(M_1(p) \left(\frac{2}{\beta} \right)^{1-\alpha_p} \Gamma(1 - \alpha_p) + \frac{M_2(p)}{\beta} \right) \\
&\leq 2\|g\|_\infty \left(M_1(p) \left(\frac{2}{\beta} \right)^{1-\alpha_p} \Gamma(1 - \alpha_p) + \frac{M_2(p)}{\beta} \right) |t_1 - t_2|^{1-\alpha_q} \\
&\leq C_1 |t_1 - t_2|^{1-\alpha_q},
\end{aligned} \tag{4.36}$$

for some constant $C_1 > 0$ depending on the parameters listed in the theorem (excluding q).

Let now $t_1, t_2 \in \mathbb{R}$ with $t_1 < t_2$ and $|t_1 - t_2| < 1$. We set $\delta := t_2 - t_1 > 0$ and take $s := t_1 - 1$. It is clear that

$$\delta < t_1 - s = 1. \tag{4.37}$$

By the variation of constants formula in the definition of bounded entire solution (Definition 4.1.1), we see that

$$\begin{aligned} &\|u(t_1) - u(t_2)\|_{L^p(\Omega)} \\ &\leq \|S(t_1 - s) - S(t_2 - s)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))}\|u(s)\|_{L^2(\Omega)} \\ &\quad + \int_s^{t_1} \|S(t_1 - \tau) - S(t_2 - \tau)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))}\|g(\tau)\|_{L^2(\Omega)} d\tau \\ &\quad + \int_{t_1}^{t_2} \|S(t_2 - \tau)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))}\|g(\tau)\|_{L^2(\Omega)} d\tau. \end{aligned} \tag{4.38}$$

Applying Lemma 4.3.6 and using $t_1 - s = 1$, we have

$$\begin{aligned} |S(t_1 - s) - S(t_2 - s)\|_{\mathscr{L}L^2(\Omega),L^p(\Omega))}\|u(s)\|_{L^2(\Omega)} &\leq \frac{C_p}{t_1 - s}|t_1 - t_2|^{1-\alpha_p}\|u\|_\infty \\ &\leq C_p\|u\|_\infty|t_1 - t_2|^{1-\alpha_p} \\ &\leq C_p\|L^{-1}\|\|g\|_\infty|t_1 - t_2|^{1-\alpha_p}. \end{aligned} \tag{4.39}$$

Moreover, a standard heat kernel estimate (4.35) implies

$$\begin{aligned} \int_{t_1}^{t_2} \|S(t_2 - \tau)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))}\|g(\tau)\|_{L^2(\Omega)} d\tau &\leq \int_{t_1}^{t_2} C(t_2 - \tau)^{-\alpha_p}\|g\|_\infty d\tau \\ &= \frac{C}{1 - \alpha_p}\|g\|_\infty|t_1 - t_2|^{1-\alpha_p}. \end{aligned} \tag{4.40}$$

The most difficult part is to estimate the second term on the right hand side of (4.38) which involves a singularity near $\tau = t_1$. By the change of variables, we can write

$$\begin{aligned} &\int_s^{t_1} \|S(t_1 - \tau) - S(t_2 - \tau)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))}\|g(\tau)\|_{L^2(\Omega)} d\tau \\ &= \int_0^{t_1 - s} \|S(\tau) - S(t_2 - t_1 + \tau)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))}\|g(t_1 - \tau)\|_{L^2(\Omega)} d\tau \\ &= \int_0^{\delta} \|S(\tau) - S(\delta + \tau)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))}\|g(t_1 - \tau)\|_{L^2(\Omega)} d\tau \\ &\quad + \int_\delta^1 \|S(\tau) - S(\delta + \tau)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))}\|g(t_1 - \tau)\|_{L^2(\Omega)} d\tau, \end{aligned} \tag{4.41}$$

where we use that $\delta = t_2 - t_1 < t_1 - s = 1$ from (4.37). Now, by the heat kernel estimate

(4.35), we get

$$\begin{aligned}
&\int_0^\delta \|S(\tau)-S(\delta+\tau)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))}\|g(t_1-\tau)\|_{L^2(\Omega)}d\tau \\
&\le \|g\|_\infty \int_0^\delta C[\tau^{-\alpha_p}+(\delta+\tau)^{-\alpha_p}]d\tau \\
&= \|g\|_\infty \frac{C}{1-\alpha_p}[\delta^{1-\alpha_p}+(2\delta)^{1-\alpha_p}-\delta^{1-\alpha_p}] \\
&= \|g\|_\infty \frac{2^{1-\alpha_p}C}{1-\alpha_p}|t_1-t_2|^{1-\alpha_p}.
\end{aligned} \tag{4.42}$$

By Lemma 4.3.6, we have

$$\begin{aligned}
&\int_\delta^1 \|S(\tau)-S(\delta+\tau)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))}\|g(t_1-\tau)\|_{L^2(\Omega)}d\tau \\
&\le \|g\|_\infty \int_\delta^1 \frac{C_p}{\tau}\delta^{1-\alpha_p}d\tau \\
&= \|g\|_\infty C_p|t_1-t_2|^{1-\alpha_q}\int_\delta^1 \frac{1}{\tau}\delta^{\alpha_q-\alpha_p}d\tau.
\end{aligned}$$

Note that by our assumption, $\alpha_q-\alpha_p=\frac{N}{2}(\frac{1}{p}-\frac{1}{q})\in(0,1)$ holds. Thus, $\delta^{\alpha_q-\alpha_p}\le\tau^{\alpha_q-\alpha_p}$ for all $\tau\in[\delta,1]$. Hence, it follows that

$$\begin{aligned}
&\int_\delta^1 \|S(\tau)-S(\delta+\tau)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))}\|g(t_1-\tau)\|_{L^2(\Omega)}d\tau \\
&\le \|g\|_\infty C_p|t_1-t_2|^{1-\alpha_q}\int_\delta^1 \tau^{\alpha_q-\alpha_p-1}d\tau \\
&= \|g\|_\infty \frac{C_p}{\alpha_q-\alpha_p}|t_1-t_2|^{1-\alpha_q}\left(1^{\alpha_q-\alpha_p}-\delta^{\alpha_q-\alpha_p}\right) \\
&\le \|g\|_\infty \frac{C_p}{\alpha_q-\alpha_p}|t_1-t_2|^{1-\alpha_q}.
\end{aligned} \tag{4.43}$$

Combining (4.42) and (4.43), we get from (4.41) that

$$\begin{aligned}
&\int_s^{t_1} \|S(t_1-\tau)-S(t_2-\tau)\|_{\mathscr{L}(L^2(\Omega),L^p(\Omega))}\|g(\tau)\|_{L^2(\Omega)}d\tau \\
&\le \|g\|_\infty \frac{2^{1-\alpha_p}C}{1-\alpha_p}|t_1-t_2|^{1-\alpha_p}+\|g\|_\infty\frac{C_p}{\alpha_q-\alpha_p}|t_1-t_2|^{1-\alpha_q}.
\end{aligned} \tag{4.44}$$

Since $|t_1-t_2|<1$, we see that

$$|t_1-t_2|^{1-\alpha_p}=|t_1-t_2|^{1-\alpha_q}|t_1-t_2|^{\alpha_q-\alpha_p}\le|t_1-t_2|^{1-\alpha_q}.$$

Therefore, we obtain from (4.38) – (4.40), and (4.44) that

$$\|u(t_1)-u(t_2)\|_{L^p(\Omega)}\le C_2|t_1-t_2|^{1-\alpha_q},$$

for all $t_1, t_2 \in \mathbb{R}$ with $|t_1 - t_2| < 1$ where $C_2 > 0$ is a constant depending on the parameters indicated in the theorem. Taking C to be the maximum of C_1 and C_2, the assertion of the proposition follows. □

Remark 4.3.8. Let $g_n \in BC(\mathbb{R}, L^2(\Omega))$ be such that $\|g_n\|_\infty$ is uniformly bounded. Using Hölder continuity of bounded entire solutions in Proposition 4.3.7, we have that the family $\{L^{-1}g_n : n \in \mathbb{N}\}$ is equicontinuous on any compact subset of $\mathbb{R}$. Hence, we can simplify the proof of Proposition 4.1.6. Indeed, the uniform convergence of $L^{-1}g_n(t) \to L^{-1}g(t)$ with respect to t in compact subsets of $\mathbb{R}$ follows from the above discussion and Arzelà -Ascoli theorem for vector valued functions (see [43, Proposition 7.3]).

4.4 Domain Perturbation for bounded entire solutions of semilinear parabolic equations

In this section, we study the persistence of bounded entire solutions of semilinear parabolic equations under domain perturbation. For a certain class of bounded entire solutions, we prove that the perturbed problem has a bounded entire solution closed to a *known* bounded entire solution of the unperturbed equation. The existence of bounded entire solutions of semilinear equations is generally difficult to prove. One of standard techniques to deal with nonlinear problems is applying degree theory to the equivalent fixed point problem. In particular, the *Leray-Schauder degree* ([43, Chapter 2.8] or [62, Chapter 4]) is commonly used when working in Banach spaces. In [35], persistence result is proved for periodic solutions using Leray-Schauder degree theory. Indeed, the same technique was used in the context of nonlinear elliptic equations in [31]. Since the Leray-Schauder degree is defined for completely continuous (compact) perturbations of the identity, this method relies on compactness of the corresponding fixed point map. In general, for solutions of semilinear problems in $BC(\mathbb{R}, L^2(\Omega))$, we do not have such compactness.

For example, consider the autonomous equation

$$u'(t) + Au(t) = f(u(t)) \quad \text{for } t \in \mathbb{R}$$

in $L^2(\Omega)$, where A is the operator as in the previous sections and $f : L^2(\Omega) \to L^2(\Omega)$ is globally Lipschitz continuous with $f(0) = 0$. It is easy to see that $u \in BC(\mathbb{R}, L^2(\Omega))$ is a bounded entire solution of the above equation if and only if u is a fixed point of the map $Q : BC(\mathbb{R}, L^2(\Omega)) \to BC(\mathbb{R}, L^2(\Omega))$ defined by

$$Q(u) := L^{-1} f(u(\cdot)),$$

where L^{-1} is given by (4.7). Let $u \in BC(\mathbb{R}, L^2(\Omega))$. The τ-translation u_τ of u defined by (4.10) belongs to $BC(\mathbb{R}, L^2(\Omega))$ and $\|u_\tau\|_\infty = \|u\|_\infty$ for all $\tau \in \mathbb{R}$. By Proposition 4.1.7 (i), we have $Q(u_\tau) = (Q(u))_\tau$ for all $\tau \in \mathbb{R}$. In particular, choosing a non-constant $u \in C_0(\mathbb{R}, L^2(\Omega)) \subset BC(\mathbb{R}, L^2(\Omega))$ by

$$u(t) = \begin{cases} e^{-\frac{1}{1-t^2}} u_0 & t \in (-1, 1) \\ 0 & |t| \geq 1, \end{cases}$$

where $u_0 \in L^2(\Omega)$, we have that $f(u(t)) = 0$ for $|t| \geq 1$. The family $\{Q(u_\tau)\}_{\tau \in \mathbb{R}}$ is the translation of of $Q(u)$ containing no convergent subsequence in $BC(\mathbb{R}, L^2(\Omega))$. Therefore, Q is not completely continuous.

In this work, we restrict to study the persistence of bounded entire solutions in the class of $C_0(\mathbb{R}, L^2(\Omega))$ for semilinear parabolic equations

$$u'(t) + Au(t) = f(t, u(t)) \quad \text{for } t \in \mathbb{R} \tag{4.45}$$

in $L^2(\Omega)$, where $f : \mathbb{R} \times L^2(\Omega) \to L^2(\Omega)$ is continuous. As in the case of linear problems, we assume that Assumption 4.2.1 is satisfied. Furthermore, we assume that f is of the form $f(t, \xi) = \phi(t) h(\xi)$ for some continuous functions $\phi : \mathbb{R} \to \mathbb{R}$ and $h : L^2(\Omega) \to L^2(\Omega)$.

Under suitable assumptions, we show that Q is completely continuous as a map on $C_0(\mathbb{R}, L^2(\Omega))$. Hence, the Leray-Schauder degree theory can be applied. We discuss persistence of $C_0(\mathbb{R}, L^2(\Omega))$ solutions in Section 4.4.1 below. It is also worthwhile to remark the case of almost periodic solutions where Q fails to be completely continuous. We comment on this difficulty in Section 4.4.2.

4.4.1 Persistence of $C_0(\mathbb{R}, L^2(\Omega))$ solutions

We investigate the persistence of bounded entire solutions of semilinear equations (4.45) with the nonlinear term $f(\cdot, u(\cdot))$ converges to zero as $t \to \pm\infty$.

Assumption 4.4.1. Assume that $f : \mathbb{R} \times L^2(\Omega) \to L^2(\Omega)$ is of the form $f(t, \xi) = \phi(t) h(\xi)$ for $(t, \xi) \in \mathbb{R} \times L^2(\Omega)$ where

(i) $\phi \in C_0(\mathbb{R})$;

(ii) $h : L^2(\Omega) \to L^2(\Omega)$ is locally Lipschitz, that is for any $R > 0$, there exists $k_R > 0$ such that

$$\|h(\xi_1) - h(\xi_2)\|_{L^2(\Omega)} \leq k_R \|\xi_1 - \xi_2\|_{L^2(\Omega)},$$

for all $\xi_1, \xi_2 \in L^2(\Omega)$ with $\|\xi_1\|_{L^2(\Omega)}, \|\xi_2\|_{L^2(\Omega)} \leq R$.

Note that we could assume as in Section 2.4 that h is a substitution operator of a function $\theta : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ satisfying the conditions similar to Assumption 2.4.2 (with g replaced by θ), that is $h(\xi)(x) = \theta(x, \xi(x))$ a.e. $x \in \Omega$.

By the above assumption, it follows that for $u \in BC(\mathbb{R}, L^2(\Omega))$, we have

$$\begin{aligned}\|f(t, u(t))\|_{L^2(\Omega)} &= \|\phi(t)h(u(t))\|_{L^2(\Omega)} \\ &\leq |\phi(t)|\big(\|h(0)\|_{L^2(\Omega)} + \|h(u(t)) - h(0)\|_{L^2(\Omega)}\big) \\ &\leq |\phi(t)|\big(\|h(0)\|_{L^2(\Omega)} + k_{\|u\|_\infty}\|u(t)\|_{L^2(\Omega)}\big) \\ &\leq |\phi(t)|\big(\|h(0)\|_{L^2(\Omega)} + k_{\|u\|_\infty}\|u\|_\infty\big),\end{aligned} \tag{4.46}$$

for all $t \in \mathbb{R}$. Since ϕ belongs to $C_0(\mathbb{R})$, we have that $f(\cdot, u(\cdot)) \in C_0(\mathbb{R}, L^2(\Omega))$ for all $u \in BC(\mathbb{R}, L^2(\Omega))$. Hence, Proposition 4.1.7 (ii) implies that the map Q defined by

$$Q(u) := L^{-1}f(\cdot, u(\cdot)) \tag{4.47}$$

maps $C_0(\mathbb{R}, L^2(\Omega))$ into itself, that is $Q : C_0(\mathbb{R}, L^2(\Omega)) \to C_0(\mathbb{R}, L^2(\Omega))$. Moreover, u is a solution of (4.45) in $C_0(\mathbb{R}, L^2(\Omega))$ if and only if u is a fixed point of Q.

Let Ω_n and Ω be domains satisfying Assumption 1.4.1. We consider the perturbation of (4.45) of the form

$$u'(t) + A_n u(t) = f_n(t, u(t)) \quad \text{ for } t \in \mathbb{R} \tag{4.48}$$

in $L^2(\Omega_n)$, where $f_n : \mathbb{R} \times L^2(\Omega_n) \to L^2(\Omega_n)$ is continuous. We assume that Assumption 4.2.2 is satisfied. Moreover, we impose the following assumptions on the nonlinearity f_n.

Assumption 4.4.2. Assume that $f_n : \mathbb{R} \times L^2(\Omega) \to L^2(\Omega)$ is of the form $f_n(t, \xi) = \phi_n(t)h_n(\xi)$ for $(t, \xi) \in \mathbb{R} \times L^2(\Omega_n)$ where

(i) $\phi_n \in C_0(\mathbb{R})$;

(ii) $h_n : L^2(\Omega_n) \to L^2(\Omega_n)$ is locally Lipschitz uniformly with respect to $n \in \mathbb{N}$, that is for any $R > 0$, there exists $k_R > 0$ independent of $n \in \mathbb{N}$ such that

$$\|h_n(\xi_1) - h_n(\xi_2)\|_{L^2(\Omega_n)} \leq k_R \|\xi_1 - \xi_2\|_{L^2(\Omega_n)},$$

for all $\xi_1, \xi_2 \in L^2(\Omega_n)$ with $\|\xi_1\|_{L^2(\Omega_n)}, \|\xi_2\|_{L^2(\Omega_n)} \leq R$ and for all $n \in \mathbb{N}$;

(iii) $\phi_n \to \phi$ in $BC(\mathbb{R})$ and $ri_n h_n(r_n\xi) \rightharpoonup h(r\xi)$ in $L^2(\Omega)$ weakly for all $\xi \in L^2(D)$.

One important example of functions f_n satisfying Assumption 4.4.2 is the restriction of a fixed semilinear term to the domains Ω_n. We discuss this in the example below.

Example 4.4.3. Let $\phi \in C_0(\mathbb{R})$ and $\theta : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ with $\theta(x, 0) = 0$ for almost everywhere $x \in \mathbb{R}^N$. Denoted by h the substitution operator $h(\xi)(x) = \theta(x, \xi(x))$ for $\xi \in L^2(D)$. We assume that $h \in C(L^2(D), L^p(D))$ for some $p > 2$ and h is locally Lipschitz. Note that a sufficient condition for the acting and locally Lipschitz continuity is similar to Assumption 2.4.2 with some modification to achieve the acting from $L^2(D)$ to $L^p(D)$ (see [35, Section 6] for the precise conditions and [3, Chapter 3] for general details).

Since $\theta(x, 0) = 0$ for a.e. $x \in \mathbb{R}^N$, we see that if $\xi \in L^2(\Omega)$ we have $h(i\xi) = 0$ a.e $x \in D\backslash\Omega$. This implies that h maps $L^2(\Omega)$ into itself if we identify a function in $L^2(\Omega)$ with the trivial extension $i\xi \in L^2(D)$. A similar statement holds for a function in $L^2(\Omega_n)$. We take

$$f_n(t, \xi) := \phi(t) r_n h(i_n\xi) \qquad \text{for } (t, \xi) \in \mathbb{R} \times L^2(\Omega_n)$$
$$f(t, \xi) := \phi(t) r h(i\xi) \qquad \text{for } (t, \xi) \in \mathbb{R} \times L^2(\Omega).$$

Recall from Lemma 1.4.8 that Mosco convergence implies $\operatorname{cap}(K \cap \Omega_n^c) \to 0$ as $n \to \infty$ for all compact set $K \subset \Omega$. Since $rh(ir\xi) \in L^p(\Omega)$, we have $\|rh(ir\xi)\|_{L^2(\Omega\backslash\Omega_n)} \to 0$ for all $\xi \in L^2(\Omega)$. It follows that $rh(i_n r_n\xi) \to rh(ir\xi)$ in $L^2(\Omega)$. Hence, condition (iii) of Assumption 4.4.2 is satisfied.

A similar calculation as in (4.46) shows that $f_n(\cdot, u(\cdot)) \in C_0(\mathbb{R}, L^2(\Omega_n))$ for all $u \in BC(\mathbb{R}, L^2(\Omega_n))$. Moreover, u_n is a solution of (4.48) in $C_0(\mathbb{R}, L^2(\Omega_n))$ if and only if u_n is a fixed point of $Q_n : C_0(\mathbb{R}, L^2(\Omega_n)) \to C_0(\mathbb{R}, L^2(\Omega_n))$ defined by

$$Q_n(u) := L_n^{-1} f_n(\cdot, u(\cdot)). \tag{4.49}$$

As usual, we need to consider the inclusion and the restriction operators when dealing with domain perturbation. In particular, we consider the maps $\mathcal{Q}_n$ and $\mathcal{Q}$ on $C_0(\mathbb{R}, L^2(D))$ into itself given by

$$\begin{aligned} \mathcal{Q}_n(u) &:= i_n Q_n(r_n(u)) = i_n L_n^{-1} f_n(\cdot, r_n u(\cdot)) \\ \mathcal{Q}(u) &:= iQ(r(u)) = iL^{-1} f(\cdot, ru(\cdot)), \end{aligned} \tag{4.50}$$

for all $u \in C_0(\mathbb{R}, L^2(D))$.

Our next step is to show that $\mathcal{Q}_n$ and $\mathcal{Q}$ are completely continuous.

Lemma 4.4.4. *Let* $\{v_n\}_{n\in\mathbb{N}}$ *be a bounded sequence in* $BC(\mathbb{R}, L^2(D))$. *Then for every* $t \in \mathbb{R}$, *the families* $\{\mathcal{Q}_n(v_n)(t)\}_{n\in\mathbb{N}}$ *and* $\{\mathcal{Q}(v_n)(t)\}_{n\in\mathbb{N}}$ *are relatively compact in* $L^2(D)$.

Proof. Denoted by $g_n \in C_0(\mathbb{R}, L^2(\Omega_n))$ where $g_n(t) := f_n(t, r_n v_n(t))$ for all $t \in \mathbb{R}$. By the uniform boundedness of $\|v_n\|_\infty$ and the assumptions on f_n, there exists $C > 0$ such that

$$\|g_n\|_\infty = \sup_{t\in\mathbb{R}} |\phi_n(t)| \|h_n(r_n v_n(t))\|_{L^2(\Omega_n)} < C, \tag{4.51}$$

for all $n \in \mathbb{N}$. Hence, $\|\mathcal{Q}_n(v_n)\|_\infty$ is uniformly bounded.

Fix now $t_0 \in \mathbb{R}$. We can find a subsequence $\mathcal{Q}_{n_k}(v_{n_k})(t_0 - 1)$ such that

$$r\mathcal{Q}_{n_k}(v_{n_k})(t_0 - 1) \rightharpoonup \xi$$

in $L^2(\Omega)$. By extracting a further subsequence (indexed again by n_k), we can assume that $ri_{n_k} g_{n_k} \rightharpoonup g$ in $L^2((t_0-1, t_0+1), L^2(\Omega))$ weakly. Since $Q_n(u_n) = L_n^{-1} f_n(\cdot, r_n v_n(\cdot))$, we have that the restriction of $Q_{n_k}(v_{n_k})$ to $[t_0 - 1, t_0 + 1]$ satisfies

$$\begin{cases} v'(t) + A_{n_k} v(t) = f_{n_k}(t, r_{n_k} v(t)), & t \in (t_0 - 1, t_0 + 1] \\ v(t_0 - 1) = Q_{n_k}(r_{n_k} v_{n_k})(t_0 - 1). \end{cases}$$

By Theorem 1.4.9, we have

$$i_{n_k} Q_{n_k}(r_{n_k} v_{n_k}) = \mathcal{Q}_{n_k}(v_{n_k})(t) \to iv(t)$$

in $L^2(D)$ on $t \in (t_0 - 1, t_0]$, where v is the solution of

$$\begin{cases} v'(t) + Av(t) = g(t), & t \in (t_0 - 1, t_0 + 1] \\ v(t_0 - 1) = \xi. \end{cases}$$

In particular, $\mathcal{Q}_{n_k}(v_{n_k})(t_0) \to iv(t_0)$ in $L^2(D)$. Since this is true for every $t_0 \in \mathbb{R}$, we have the relative compactness of $\{\mathcal{Q}_n(v_n)(t)\}_{n\in\mathbb{N}}$ in $L^2(D)$ for all $t \in \mathbb{R}$. The relative compactness of $\{\mathcal{Q}(v_n)(t)\}_{n\in\mathbb{N}}$ is a special case of what we have just proved (by taking a sequence of constant domains $\Omega_n = \Omega$ for all $n \in \mathbb{N}$). □

Remark 4.4.5. The assertion of Lemma 4.4.4 remains valid for a more general map $\mathcal{Q} : BC(\mathbb{R}, L^2(D)) \to BC(\mathbb{R}, L^2(D))$ (not just for the map on $C_0(\mathbb{R}, L^2(D))$ into itself) as long as

$$\|g_n\|_\infty := \sup_{t\in\mathbb{R}} \|f_n(t, r_n v_n(t))\|_{L^2(\Omega_n)}$$

is uniformly bounded, that is (4.51) holds.

Theorem 4.4.6. *For every bounded sequence $\{v_n\}_{n\in\mathbb{N}}$ in $C_0(\mathbb{R}, L^2(D))$, the sequence $\mathcal{Q}_n(v_n)$ has a convergence subsequence in $C_0(\mathbb{R}, L^2(D))$. In particular,*

$$\mathcal{Q}_n, \mathcal{Q} : C_0(\mathbb{R}, L^2(D)) \to C_0(\mathbb{R}, L^2(D))$$

are completely continuous.

Proof. Denoted by $g_n \in C_0(\mathbb{R}, L^2(\Omega_n))$ where $g_n(t) := f_n(t, r_n v_n(t))$ for all $t \in \mathbb{R}$. We have from (4.51) that $\|g_n\|_\infty$ is uniformly bounded. Moreover, by Lemma 4.4.4, $\{\mathcal{Q}_n(v_n)(t)\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(D)$ for all $t \in \mathbb{R}$. By our assumptions, we have the uniform exponential dichotomy (see Lemma 4.2.3). Hence, it follows from Theorem 4.3.7 that

$$\|\mathcal{Q}_n(v_n)(t_1) - \mathcal{Q}_n(v_n)(t_2)\|_{L^2(D)} \le C|t_1 - t_2|^{1-\alpha_q},$$

for all $t_1, t_2 \in \mathbb{R}$ and for all $n \in \mathbb{N}$, where the constant C is independent of $n \in \mathbb{N}$ and q is chosen so that $2 < p < q < \infty$ and $\alpha_p := \frac{N}{2}(\frac{1}{2} - \frac{1}{p}), \alpha_q := \frac{N}{2}(\frac{1}{2} - \frac{1}{q}) \in [0, 1)$. This means the family $\{\mathcal{Q}_n(v_n)\}_{n\in\mathbb{N}}$ is equicontinuous on $\mathbb{R}$. By Arzelà-Ascoli theorem for vector valued functions (see e.g. [43, Proposition 7.3(b)]), we can extract a subsequence $\{\mathcal{Q}_{1,n}(v_{1,n})\}_{n\in\mathbb{N}}$ of $\{\mathcal{Q}_n(v_n)\}_{n\in\mathbb{N}}$ so that $\mathcal{Q}_{1,n}(v_{1,n})$ converges uniformly on $[-1, 1]$. Applying Arzelà-Ascoli theorem again, we can extract a further subsequence $\{\mathcal{Q}_{2,n}(v_{2,n})\}_{n\in\mathbb{N}}$ of $\{\mathcal{Q}_{1,n}(v_{1,n})\}_{n\in\mathbb{N}}$ so that $\mathcal{Q}_{2,n}(v_{2,n})$ converges uniformly on $[-2, 2]$. Applying the above argument inductively, we obtain for every $k \ge 2$ a subsequence $\{\mathcal{Q}_{k,n}(v_{k,n})\}_{n\in\mathbb{N}}$ of $\{\mathcal{Q}_{(k-1),n}(v_{(k-1),n})\}_{n\in\mathbb{N}}$ such that $\{\mathcal{Q}_{k,n}(v_{k,n})\}_{n\in\mathbb{N}}$ converges uniformly on $[-k, k]$. By a standard diagonal argument, we see that $\mathcal{Q}(v_{n,n})(t)$ converges uniformly on any compact sets of $\mathbb{R}$. In particular, $\mathcal{Q}_{n,n}(v_{n,n})(t) \to v_t$ in $L^2(D)$ for all $t \in \mathbb{R}$ with the limit $v_t \in L^2(D)$. Let $v(t) := v_t$ for $t \in \mathbb{R}$. We have that v is continuous on $\mathbb{R}$ and $\sup_{t\in\mathbb{R}} \|v(t)\|_{L^2(D)}$ is finite. It remains to show that $\mathcal{Q}_{n,n}(v_{n,n})$ converges to v uniformly on $\mathbb{R}$.

Let $\varepsilon > 0$ be arbitrary. By a calculation as in (4.46) and the assumption that $\phi_n \to \phi$ in $C_0(\mathbb{R})$, we can choose $T > 0$ independent of $n \in \mathbb{N}$ so that

$$\|g_n(t)\|_{L^2(\Omega_n)} = \|f_n(t, r_n v_n(t))\|_{L^2(\Omega_n)} < \frac{\beta}{4M}\varepsilon$$

if $|t| > T$ for all $n \in \mathbb{N}$. By the uniform boundedness of $\|g_n\|_\infty$, we can find $\tilde{T} > T$ independent of $n \in \mathbb{N}$ such that

$$\|g_n\|_\infty \frac{M}{\beta} e^{-\beta(\tilde{T}-T)} = \|f_n(\cdot, r_n v_n(\cdot))\|_\infty \frac{M}{\beta} e^{-\beta(\tilde{T}-T)} < \frac{\varepsilon}{2},$$

for all $n \in \mathbb{N}$. By the argument appeared in the proof of Proposition 4.1.8, the above two inequalities imply $\|L_n^{-1} f_n(t, v_n(t))\|_{L^2(\Omega)} < \varepsilon$ if $|t| > \tilde{T}$ for all $n \in \mathbb{N}$. Therefore, the limit function v satisfies $\|v(t)\|_{L^2(\Omega)} \leq \varepsilon$ if $|t| > \tilde{T}$. Hence,

$$\|\mathcal{Q}_{n,n}(v_{n,n})(t) - v(t)\|_{L^2(\Omega)} \leq \varepsilon + \varepsilon = 2\varepsilon \tag{4.52}$$

if $|t| > \tilde{T}$ for all $n \in \mathbb{N}$. As $\mathcal{Q}_{n,n}(v_{n,n})$ converges uniformly on the compact interval $[-k, k]$ for all $k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that

$$\|\mathcal{Q}_{n,n}(v_{n,n})(t) - v(t)\|_{L^2(D)} \leq \varepsilon, \tag{4.53}$$

for all $t \in [-\tilde{T}, \tilde{T}]$ and for all $n > N$. Therefore, (4.52) and (4.53) implies

$$\|\mathcal{Q}_{n,n}(v_{n,n})(t) - v(t)\|_{L^2(D)} \leq 2\varepsilon,$$

for all $t \in \mathbb{R}$ and for all $n > N$. As $\varepsilon > 0$ was arbitrary, this proves the uniform convergence of $\mathcal{Q}_{n,n}(v_{n,n})$ on $\mathbb{R}$. Since $C_0(\mathbb{R}, L^2(D))$ is a closed subspace of $BC(\mathbb{R}, L^2(D))$, the limit v belongs to $C_0(\mathbb{R}, L^2(D))$. This gives the first assertion of the theorem.

The complete continuity of $\mathcal{Q}$ immediately follows by the same argument above as a special case with $\Omega_n := \Omega$ for all $n \in \mathbb{N}$. Similarly, by choosing a sequence of $\Omega_n := \Omega_{n_0}$ for all $n \in \mathbb{N}$ where n_0 is fixed, we obtain complete continuity of $\mathcal{Q}_{n_0}$. □

If U is a bounded open subset of $C_0(\mathbb{R}, L^2(D))$ such that $u \neq \mathcal{Q}(u)$ for all $u \in \partial U$, then by complete continuity of $\mathcal{Q}$, the Leray-Schauder degree $\deg(I - \mathcal{Q}, U, 0) \in \mathbb{Z}$ is well-defined. We show that the Leray-Schauder degree is persistent under domain perturbation. The proof is similar to the argument for periodic solutions of parabolic equations in [35, Theorem 7.1] or the argument for L^∞ solutions of elliptic equations in [5, Theorem 8.2]. However, we also consider perturbation of the nonlinear terms. We first give the following lemma.

Lemma 4.4.7. *Suppose that f_n and f satisfy Assumptions 4.4.1 and Assumption 4.4.2 respectively. If $v \in C_0(\mathbb{R}, L^2(D))$, then*

$$r i_n f_n(\cdot, r_n v(\cdot)) \overset{*}{\rightharpoonup} f(\cdot, r v(\cdot))$$

in $L^\infty(\mathbb{R}, L^2(\Omega))$ as $n \to \infty$.

Proof. By a similar calculation as in (4.46), we have

$$\|f_n(\cdot, r_n v(\cdot))\|_\infty \leq \|\phi_n\|_\infty \big(\|h_n(0)\|_{L^2(\Omega_n)} + k_{\|v\|_\infty} \|v\|_\infty\big)$$
$$\|f(\cdot, r v(\cdot))\|_\infty \leq \|\phi\|_\infty \big(\|h(0)\|_{L^2(\Omega)} + k_{\|v\|_\infty} \|v\|_\infty\big).$$

As $\phi_n \to \phi$ in $BC(\mathbb{R})$ and $ri_n h(0) \rightharpoonup h(0)$ in $L^2(\Omega)$ (condition (iii) in Assumption 4.4.2), it follows that $ri_n f_n(\cdot, r_n v(\cdot))$ and $f(\cdot, rv(\cdot))$ are uniformly bounded in $L^\infty(\mathbb{R}, L^2(\Omega))$. By Assumption 4.4.2 (iii) again, we see that

$$\begin{aligned} ri_n f_n(t, r_n v(t)) - f(t, rv(t)) &= \phi_n(t) ri_n h_n(r_n v(t)) - \phi(t) h(rv(t)) \\ &= \phi_n(t)\big(ri_n h_n(r_n v(t)) - h(rv(t))\big) \\ &\quad + \big(\phi_n(t) - \phi(t)\big) h(rv(t)) \\ &\rightharpoonup 0 \end{aligned} \tag{4.54}$$

in $L^2(\Omega)$ weakly as $n \to \infty$ for all $t \in \mathbb{R}$.

Let $g \in L^1(\mathbb{R}, L^2(\Omega))$ be arbitrary. It follows from (4.54) that

$$\big(ri_n f_n(t, r_n v(t)) - f(t, rv(t)) \mid g(t)\big)_{L^2(\Omega)} \to 0$$

as $n \to \infty$ for almost every $t \in \mathbb{R}$. Applying the dominated convergence theorem, we get

$$\int_{-\infty}^{\infty} \big(ri_n f_n(t, r_n v(t)) - f(t, rv(t)) \mid g(t)\big)_{L^2(\Omega)} dt \to 0$$

as $n \to \infty$. Since this is true for any $g \in L^1(\mathbb{R}, L^2(\Omega))$, we obtain the required weak-$*$ convergence in $L^\infty(\mathbb{R}, L^2(\Omega))$. □

Theorem 4.4.8. *Suppose that all assumptions mentioned in this subsection are satisfied. Let U be a bounded open subset of $C_0(\mathbb{R}, L^2(D))$ such that $u \neq \mathcal{Q}(u)$ for all $u \in \partial U$. Then $u \neq \mathcal{Q}_n(u)$ for all $u \in \partial U$ and*

$$\deg(I - \mathcal{Q}_n, U, 0) = \deg(I - \mathcal{Q}, U, 0),$$

for all n sufficiently large.

Proof. We apply the homotopy invariance of the Leray-Schauder degree (see [43, section 2.8.3] or [62, Theorem 4.3.4]) to prove the assertion of the theorem. For this, it suffices to show that

$$v \neq \lambda \mathcal{Q}(v) + (1 - \lambda)\mathcal{Q}_n(v)$$

if $v \in \partial U$ and $0 \leq \lambda \leq 1$ for all n sufficiently large.

By way of contradiction, we assume there exists $0 \leq \lambda_n \leq 1$ and $v_n \in \partial U$ such that

$$v_n = \lambda_n \mathcal{Q}(v_n) + (1 - \lambda_n)\mathcal{Q}_n(v_n), \tag{4.55}$$

for all $n \in \mathbb{N}$. We can assume (by possibly selecting a subsequence) that λ_n converges to some λ in $[0, 1]$. Since v_n is bounded in $C_0(\mathbb{R}, L^2(D))$ and $\mathcal{Q}$ is completely continuous

(Lemma 4.4.6), we can extract a subsequence (indexed again by n) such that $\mathcal{Q}(v_n)$ converges to some limit in $C_0(\mathbb{R}, L^2(D))$. Similarly, by Lemma 4.4.6, we extract a further subsequence (still indexed again by n) such that $\mathcal{Q}_n(v_n)$ converges to some limit in $C_0(\mathbb{R}, L^2(D))$. Hence, we get from (4.55) that (for a subsequence) $v_n \to v$ in $C_0(\mathbb{R}, L^2(D))$ for some $v \in C_0(\mathbb{R}, L^2(D))$. Moreover, $v \in \partial U$ because ∂U is closed.

Now by Assumption 4.4.1 and that $\|v_n\|_\infty < R$ for some $R > 0$, we get

$$\begin{aligned} \|f(t, rv_n(t)) - f(t, rv(t))\|_{L^2(\Omega)} &\le |\phi(t)| k_R \|rv_n(t) - rv(t)\|_{L^2(\Omega)} \\ &\le k_R \|\phi\|_\infty \sup_{t\in\mathbb{R}} \|rv_n(t) - rv(t)\|_{L^2(\Omega)}, \end{aligned} \tag{4.56}$$

for all $t \in \mathbb{R}$. As $v_n \to v$ in $C_0(\mathbb{R}, L^2(D))$, the above implies $f(\cdot, rv_n(\cdot)) \to f(\cdot, rv(\cdot))$ in $C_0(\mathbb{R}, L^2(\Omega))$. Applying Theorem 4.2.5 (with a sequence of domains $\Omega_n = \Omega$ for all $n \in \mathbb{N}$), we get

$$\mathcal{Q}(v_n) = iL^{-1} f(\cdot, rv_n(\cdot)) \to iL^{-1} f(\cdot, rv(\cdot)) = \mathcal{Q}(v) \tag{4.57}$$

in $C_0(\mathbb{R}, L^2(D))$. By a similar calculation as in (4.56) (using Assumption 4.4.2 instead), we have

$$i_n f_n(\cdot, r_n v_n(\cdot)) - i_n f_n(\cdot, r_n v(\cdot)) \to 0$$

in $C_0(\mathbb{R}, L^2(D))$ as $n \to \infty$. Together with Lemma 4.4.7, it follows that

$$r i_n f_n(\cdot, r_n v_n(\cdot)) \overset{*}{\rightharpoonup} f(\cdot, rv(\cdot))$$

in $L^\infty(\mathbb{R}, L^2(\Omega))$ as $n \to \infty$. We conclude from Theorem 4.2.6 that

$$\mathcal{Q}_n(v_n) = i_n L_n^{-1} f_n(\cdot, r_n v_n(\cdot)) \to iL^{-1} f(\cdot, rv(\cdot)) = \mathcal{Q}(v)$$

in $C_0(\mathbb{R}, L^2(D))$ under the topology of uniform convergence on compact subsets. Therefore, we get from (4.55) that $v = \lambda\mathcal{Q}(v) + (1-\lambda)\mathcal{Q}(v) = \mathcal{Q}(v)$. This contradicts our assumption that $\mathcal{Q}$ has no fixed point on ∂U. By the homotopy invariance of the Leray-Schauder degree, this implies that

$$\deg(I - \mathcal{Q}_n, U, 0) = \deg(I - \mathcal{Q}, U, 0)$$

for all n large. □

We are interested in the persistence of a *known* solution of the unperturbed problem (4.45). In Particular, if $\deg(I - \mathcal{Q}, U, 0) \neq 0$, then (4.45) has a $C_0(\mathbb{R}, L^2(\Omega))$ solution in U (see [62, Theorem 4.3.2]). We follow [5] to state a sequence of results below.

Corollary 4.4.9. *Let U be a bounded open subset of $C_0(\mathbb{R}, L^2(D))$ such that $u \neq \mathcal{Q}(u)$ for all $u \in \partial U$. If $\deg(I - \mathcal{Q}, U, 0) \neq 0$, then (4.48) has a bounded entire solution in $U \cap C_0(\mathbb{R}, L^2(\Omega_n))$ for all n sufficiently large.*

In addition, if u is an isolated solution of (4.45), then the excision property of the degree implies that for open balls $B(iu, \varepsilon)$ of radius ε and center iu in $C_0(\mathbb{R}, L^2(D))$ we have $\deg(I - \mathcal{Q}, B(u, \varepsilon), 0)$ is constant for small $\varepsilon > 0$. The index of u is defined by

$$\operatorname{index}(u) := \lim_{\varepsilon \to 0} \deg(I - \mathcal{Q}, B(iu, \varepsilon), 0).$$

Theorem 4.4.10. *Suppose that u is an isolated solution of (4.45) with $\operatorname{index}(u) \neq 0$. For n sufficiently large, there exist solutions $u_n \in C_0(\mathbb{R}, L^2(\Omega_n))$ of (4.48) such that $i_n u_n \to iu$ in $C_0(\mathbb{R}, L^2(D))$ as $n \to \infty$.*

Proof. By assumption there exists $\varepsilon_0 > 0$ such that

$$\operatorname{index}(u) = \deg(I - \mathcal{Q}, B(u, \varepsilon), 0) \neq 0,$$

for all $\varepsilon \in (0, \varepsilon_0)$. Corollary 4.4.9 implies that (4.48) has a solution in $B(iu, \varepsilon)$ for all $\varepsilon \in (0, \varepsilon_0)$ if n sufficiently large. Suppose the required sequence of solutions u_n does not exist. Then we can find $\varepsilon \in (0, \varepsilon_0)$ and a subsequence $n_k \to \infty$ such that (4.48) (with n_k in place of n) has no solution in $B(iu, \varepsilon)$ for all $k \in \mathbb{N}$. This gives a contradiction. □

4.4.2 Some remarks on almost periodic solutions

The aim of this section is to discuss some difficulties of using the Leray-Schauder degree for almost periodic solutions. We first collect some preliminary results on almost periodic functions below.

Definition 4.4.11. Let X and Y be Banach spaces. A continuous function $g : \mathbb{R} \times X \to Y$ is called *uniformly almost periodic* (with respect to x) if for every $\varepsilon > 0$ and compact set $K \subset X$ there exists $\ell(\varepsilon, K) > 0$ such that every interval J of length $\ell(\varepsilon, K)$ contains τ such that $\|g(t + \tau, x) - g(t, x)\|_Y < \varepsilon$ for all $t \in \mathbb{R}$ and for all $x \in K$.

Lemma 4.4.12. *Let X and Y be Banach spaces. Suppose that $g : \mathbb{R} \times X \to Y$ is uniformly almost periodic. If $u : \mathbb{R} \to X$ is almost periodic then the function $t \mapsto g(t, u(t))$ is also almost periodic, that is $g(\cdot, u(\cdot)) \in AP(\mathbb{R}, Y)$.*

Proof. The proof follows the same argument as either in [80, Theorem I.2.7], [47, Theorem 2.11] or [28, Theorem 2.8] for functions taking values in finite dimensional space

($\mathbb{R}^N$). The only point we need to be careful is that for almost periodic functions with values in $\mathbb{R}^N$, the compactness of $\overline{\{u(t) : t \in \mathbb{R}\}}$ in $\mathbb{R}^N$ simply follows from the boundedness of $|u(t)|$. However, we also have the compactness of $\overline{\{u(t) : t \in \mathbb{R}\}}$ in X for $u \in AP(\mathbb{R}, X)$ from [28, Theorem 6.5]. Hence, we can find a compact set $K \subset X$ containing the range of u and complete the proof in the same way as $\mathbb{R}^N$-valued functions. □

Definition 4.4.13. A family $\mathscr{F}$ of functions from $AP(\mathbb{R}, X)$ is called *equi-almost periodic* if for every $\varepsilon > 0$, there exists $\ell(\varepsilon) > 0$ such that every compact interval $J \subset \mathbb{R}$ of length $\ell(\varepsilon)$ contains τ such that

$$\|f(t+\tau) - f(t)\|_X < \varepsilon,$$

for all $t \in \mathbb{R}$ and for all $f \in \mathscr{F}$.

Lemma 4.4.14 ([28, Theorem 6.10])**.** *Let X be a Banach space. A family $\mathscr{F}$ of functions from $AP(\mathbb{R}, X)$ is relatively compact if and only if the following conditions hold:*

(i) *$\mathscr{F}$ is equicontinuous;*

(ii) *$\mathscr{F}$ is equi-almost periodic;*

(iii) *for any $t \in \mathbb{R}$, the set $\{f(t) : f \in \mathscr{F}\}$ is relatively compact in X.*

Suppose that f satisfies Assumption 4.4.1 with $\phi \in C_0(\mathbb{R})$ replaced by $\phi \in AP(\mathbb{R})$. Then

$$\begin{aligned}\|f(t+\tau,\xi) - f(t,\xi)\|_{L^2(\Omega)} &= |\phi(t+\tau) - \phi(t)|\|h(\xi)\|_{L^2(\Omega)} \\ &\le |\phi(t+\tau) - \phi(t)|\big(\|h(0)\|_{L^2(\Omega)} + k_{\|\xi\|_{L^2(\Omega)}}\|\xi\|_{L^2(\Omega)}\big),\end{aligned} \tag{4.58}$$

for all $t, \tau \in \mathbb{R}$ and $\xi \in L^2(\Omega)$. Let K be a compact subset of $L^2(\Omega)$. We have

$$\|h(0)\|_{L^2(\Omega)} + k_{\|\xi\|_{L^2(\Omega)}}\|\xi\|_{L^2(\Omega)} < C_K,$$

for all $\xi \in K$, where $C_K > 0$ is a constant depending on the set K. By the almost periodicity of ϕ, we get from (4.58) that f is uniformly almost periodic. As a consequence of Lemma 4.4.12, we have that $f(\cdot, u(\cdot)) \in AP(\mathbb{R}, L^2(\Omega))$ for all $u \in AP(\mathbb{R}, L^2(\Omega))$. Hence, Proposition 4.1.7 (ii) implies that Q defined by (4.47) maps $AP(\mathbb{R}, L^2(\Omega))$ into itself, that is $Q : AP(\mathbb{R}, L^2(\Omega)) \to AP(\mathbb{R}, L^2(\Omega))$. Moreover, u is an almost periodic solutions of (4.45) if and only if it is a fixed point of Q. We show that the technique of using the Leray-Schauder degree cannot be applied as Q is not completely continuous.

Let $\{v_n\}_{n\in\mathbb{N}}$ be a bounded sequence in $AP(\mathbb{R}, L^2(\Omega))$. We set $g_n(t) := f(t, v_n(t))$ for all $t \in \mathbb{R}$. A similar calculation as in (4.51) implies that that $\|g_n\|_\infty$ is uniformly bounded. By Remark 4.4.5, $\{Q(v_n)(t)\}_{n\in\mathbb{N}}$ is relatively compact in $L^2(\Omega)$ for all $t \in \mathbb{R}$. In addition, Theorem 4.3.7 implies that $\{Q(v_n)\}_{n\in\mathbb{N}}$ is equicontinuous on $\mathbb{R}$. The above argument shows that the family $\{Q(v_n)\}_{n\in\mathbb{N}}$ satisfies condition (i) and condition (iii) in Lemma 4.4.14. However, we do not generally have equi-almost periodicity of the family $\{Q(v_n)\}_{n\in\mathbb{N}}$ when $\{v_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $AP(\mathbb{R}, L^2(\Omega))$. To see this, let $v_1 \in AP(\mathbb{R}, L^2(\Omega))$ be a *periodic* function with (minimal) period one. For $n \geq 2$, we let $v_n(t) := v_1(t/n)$ for $t \in \mathbb{R}$. We have $v_n \in AP(\mathbb{R}, L^2(\Omega))$ is a periodic function with period n and $\|v_n\|_\infty = \|v_1\|_\infty$ for all $n \in \mathbb{N}$. If ϕ is also a periodic function, then the substitution $f(\cdot, v_n(\cdot)) = \phi(\cdot)h(v_n(\cdot))$ is also periodic with larger period as n increases. This means $\{Q(v_n)\}_{n\in\mathbb{N}}$ cannot be equi-almost periodic. By Lemma 4.4.14, the family $\{Q(v_n)\}_{n\in\mathbb{N}}$ is not relatively compact in $AP(\mathbb{R}, L^2(\Omega))$. Therefore, Q is not completely continuous.

Recall that u is an almost periodic solutions of (4.45) if and only if it is a fixed point of Q. An alternative (rather trivial) approach is using the contraction mapping theorem. This method is used to establish the existence and uniqueness of almost periodic solutions of semilinear non-autonomous problems in [13, Theorem 3.2] under the assumption that the seminlinear term is globally Lipschitz with a sufficiently small Lipschitz constant. Suppose that $\|f(t,\xi_1) - f(t,\xi_2)\|_{L^2(\Omega)} \leq k\|\xi_1 - \xi_2\|_{L^2(\Omega)}$ for all $\xi_1, \xi_2 \in L^2(\Omega)$. A straightforward calculation using (4.7) implies that

$$\|Q(u)(t) - Q(v(t)\|_{L^2(\Omega)} \leq \frac{2M}{\beta} k\|u - v\|_\infty,$$

for all $u, v \in AP(\mathbb{R}, L^2(\Omega))$. Hence, Q is a contraction map if the Lipschitz constant k is small enough. We can prove the convergence of almost periodic solutions under domain perturbation by a similar argument as in Theorem 2.3.3 provided that the family of the maps Q_n is a uniform contraction and Q_n converges to Q pointwise. In fact, this method can be applied to any class of bounded entire solutions.

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