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Wilson Loops and Scattering Amplitudes in Supersymmetric Field and String Theories

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CHAPTER 1

Overview and Discussion

Within the last forty years, the most persistent concept in the realm of theoretical physics has been that of *duality*. String theory dawned itself as a dual resonance model [1] [2] after that the celebrated Veneziano amplitude [3] was obtained under the constraint of being crossing-symmetric in the s and t channels, or in today's words, self-dual. Duality is a profound statement about two apparently different theories manifesting the same behaviour in some region of their respective parameters spaces, a statement about the circumstance that, somehow, the different formulation of the two theories overshadows the fact that they are, indeed, the same theory. Often, the reason why this occurs is fathomless, but nonetheless it offers the chance to engage the same problem from different sides, and this is always an advantage. The reader may just think to the huge improvement in the understanding of $\mathcal{N} = 2$ theories at strong coupling brought by the work of Seiberg and Witten [4] [5]; or the relation between string theories and superconformal gauge theories with the AdS/CFT correspondence of Maldacena [6]; or again, more recently, the dualities between two-dimensional conformal field theories and four-dimensional $\mathcal{N} = 2$ theories introduced by Alday, Gaiotto and Tachikawa [7] [8].

A striking duality turns out to exist between two classes of objects which own a fundamental relevance in the study of gauge and string theories - Wilson loops and scattering amplitudes. Wilson loop operators were first introduced in the context of QCD [9], almost forty years ago, to investigate the dynamics of massive quarks subject to the interaction with gluon fields. Probably the most interesting observable of this kind is constituted by a couple of infinite, time-like and parallel Wilson lines, representing the trajectories of two heavy quarks, interacting between each other via the exchange of infinitely many gluon fields. This observable is known to compute the quark-antiquark potential, and thus both weak and strong coupling analysis of such loops are of primary interest. Perturbatively, the computation of QCD Wilson loops after the first few orders is a discouraging task; on the other hand non-perturbative methods based on the lattice formulation of the theory exist, but they unavoidably imply numerical evaluation of observables. The story is somewhat different if one adds supersymmetry to the gauge theory. In the case of $\mathcal{N}=4$ super Yang-Mills theory in four dimensions, the high amount of supersymmetry, which is indeed maximal, puts several constraints onto the theory, making its resolution easier. We do not want to argue about the meaning of the statement *"to solve a theory"* here. However, huge developments have been achieved in the last, say ten, years in understanding the perturbative structure of $SU(N)$ $\mathcal{N}=4$ SYM, leading to the statement that this theory is an integrable system at the planar limit $N\to\infty$, and that beyond the Lagrangean formulation there lies an hidden infinite-dimensional symmetry known as Yangian symmetry, which best manifests itself in the properties of on-shell observables like scattering amplitudes and Wilson loops [10] [11] [12] [13] [14] [15]. Yangian symmetry, or better superconformal symmetry which is its level-one realisation, was identified as being responsible of the equivalence (up to a scheme-matching) between maximally helicity violating gluon amplitudes and polygonal Wilson loops with light-like edges in $\mathcal{N}=4$ super Yang-Mills [16] [17] [13] and more recently in $\mathcal{N}=6$ super Cherns-Simons theories [18] [19] [20] [21], for which indeed the first hints of integrability came from the analysis of the Bethe ansatz [22] [23] [24] [25] and Yanghian symmetry [26] of scattering amplitudes.

Scattering amplitudes cover the remarkable role of being the medium between our formulation of fundamental interactions and nature itself. Indeed, our experimental knowledge of high energy physics necessarily goes through scattering processes, whose degree of complexity grows at mind-blowing speed as the energy involved increases. Thus the need of developing powerful methods for computing scattering amplitudes of gauge theories accurately, since the discovery of possibly new physics through beams collision implies the ability of discerning previously unobserved events shaded by a rather chaotic background and to a high level of accuracy [27]. As for the case of Wilson loops, scattering amplitudes of highly supersymmetric gauge theories embody a deep structure that makes their evaluation at least viable. And quite interestingly, they can teach us a lot about non-supersymmetric theories; gluon amplitudes in QCD at tree-level are the same as in $\mathcal{N}=4$ SYM amplitudes, and at one-loop level they can be decomposed into a sum of $\mathcal{N}=4$ and $\mathcal{N}=1$ gluon amplitudes.

Since Witten's twistor string proposal [28], a lot of developments in understanding the perturbative structure of $\mathcal{N}=4$ SYM gluon amplitudes have been made following the the same *"analytic S-matrix"* type approach which originally led to the birth of string theory [3]. On-shell recursion relations for maximally helicity violating (MHV) gluon amplitudes of $\mathcal{N}=4$ SYM were proposed in [29] by Britto, Cachazo and Feng and proved in [30], then further generalised to loop-level amplitudes [31] [32] [33]. This recursive structure can be shown to hold also for string scattering amplitudes [34] [35] where it has a nice geometrical interpretation. The last fact is reminiscent of the conformal bootstrap properties of CFT's, at least by a hand-waving argument, and opens to the concrete possibility that on-shell amplitudes in supersymmetric theories inherit such recursive structure from the conformal symmetry of the string worldsheet.

Since its first formulation [6], the conjectured correspondence of superconformal field theories in d dimensions and supersymmetric strings on anti-de Sitter spaces in $d+1$ dimensions has been an exceptionally fertile ground on which to grow our knowledge of both theories. The best known example of this duality holds between supersymmetric $\mathcal{N}=4$ Yang-Mills theory in four dimension and type IIB superstrings on the ten dimensional $AdS_5 \times S^5$ space. In more recent years other examples of such a correspondence emerged and rapidly gained more and more interest, one over all are three-dimensional Chern-Simons theories with $\mathcal{N}=6$ supersymmetry which are conjectured to capture the low-energy behaviour of a certain class of eleven-dimensional M-Theories [36] [37]. The proposal of [6] was mainly based on the observation that both $\mathcal{N}=4$ SYM in four flat dimensions and type IIB strings on $AdS_5 \times S^5$ have the same symmetry group, and in its weak formulation states that semiclassical strings solutions for $g_S \to 0$ correspond to gauge theory solutions at strong coupling $g_{YM} \gg 1$ and in the planar limit $N \to \infty$ with the 't Hooft coupling $\lambda = g_{YM}^2 N \gg 1$ held fixed. In its stronger version it states the exact equivalence between $\mathcal{N}=4$ SYM and type IIB strings on $AdS_5 \times S^5$.

The AdS/CFT correspondence provided the tool to compute gauge theoretical quantities at strong coupling by means of semiclassical computations of the corresponding strings configurations. In particular it lead to a strong coupling formulation for Wilson loops of $\mathcal{N}=4$ SYM [38] [39], stating that the expectation value $\langle W(\mathcal{C})\rangle$ of a Wilson loop defined on a contour $\mathcal{C}$ on the field theory side is given by the semiclassical area of the string worldsheet that extremises the $AdS_5 \times S^5$ string action and ends on $\mathcal{C}$ on the boundary of AdS_5. Soon after it was shown that for a particular class of Wilson loops, that are not haunted by UV divergences, a loop equation can be written and it is solved at strong coupling by the corresponding minimal area [40]. Such loops can be interpreted as "phase factors" associated with stable supersymmetric charged particles that saturate the Bogomol'nyi-Prasad-Sommerfeld bound $M = Q$, being M the mass and Q the charge of the particle in some plausible units. BPS Wilson loops then are left invariant by a certain amount of the supercharges of the theory, thing that usually makes them easier to compute. Furthermore, a stingy picture motivated the observed duality between light-like loops and scattering amplitudes [41]. The motivation relies on the self-duality properties of the space $AdS_5 \times S^5$ under a $T-$duality operation which maps the string computation of a gluon scattering amplitude into the computation of a polygonal null Wilson loop, thus enforcing the perturbative evidence. With some *caveat* being in order, what said above is strongly believed to hold also in the more recent example of gauge/gravity duality that relates $\mathcal{N}=6$ SCS theories to eleven-dimensional $M-$theory [36] [37].

Supersymmetric gauge theories in three dimensions have been the subject of a renewed interest in light of their connection to the low-energy world-volume dynamics of membranes, of which very little is known. A first attempt on this side was done in [42] but such theories shown not to have enough supersymmetries. A few years ago, Bagger and Lambert proposed a model for multiple M2-branes with $SO(8)$ $R-$symmetry, which was subsequently conjectured to be dual to a certain $\mathcal{N}=8$, $3D$ super Chern-Simons-matter gauge theory [43]. As it turned out, that theory is always strongly coupled, forbidding any consistent 't Hooft limit. This motivated the work of Aharony, Bergmann, Jafferis and Maldacena who proposed a new example of AdS/CFT correspondence between $\mathcal{N} = 6$ super Chern-Simons-matter $U(N) \times U(N)$ theory and type IIA superstrings on $AdS_4 \times \mathbb{C}P^3$ [36]. The first side of the correspondence emerges as the low energy effective theory of N $M2$ branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ orbifold singularity at the intersection of two KK monopoles in $AdS_4 \times S^7$. In the large k limit an M-theory circle in $\mathbb{C}^4/\mathbb{Z}_k$ shrinks and type IIA superstrings on $AdS_4 \times \mathbb{C}P^3$ appear (note that S^7 can be viewed as a S^1 fibration over $\mathbb{C}P^3$), giving rise to the second side of the correspondence. The $U(N) \times U(N)$ theory was soon after generalised to $U(N) \times U(M)$ theory, which still preserve $\mathcal{N} = 6$ supersymmetries of the free $M2$ theory (the $\mathbb{Z}_k$ orbifold always breaks two of them) and is thought to be dual to type IIA on $\mathbb{C}^4/\mathbb{Z}_k$ with a $B-$field flux proportional to $N - M$ [37]. This novel example of string/gauge duality is vastly unexplored and puts itself forward as a natural candidate for further studies. Quite interestingly the gauge theory side does not have maximal supersymmetry in three dimensions for k other than $1, 2$, but still possesses conformal symmetry, and this makes it a natural candidate for massive tests of AdS/CFT outside the $\mathcal{N} = 4$ SYM wonderland.

The study of Wilson loops in $\mathcal{N} = 6$ SCS theories is naturally addressed to the investigation of BPS quantities and of its stringy duals. Wilson loop operators invariant under $\frac{1}{6}$ and $\frac{1}{2}$ of the supercharges have been constructed in [44] in parallel with [45] and in [46]. An interpretation of the latter in terms of "phase factor" of a half BPS particle was given in [47]. This naturally arises a question about the interaction potential between a pair of such BPS particles, which is the straightforward generalisation of the quark-antiquark potential of QCD. Quite interestingly it turns out that in the weak coupling expansion, the one-loop expectation value of such operator receives contributions only from fermions [48]. This fact is new and unobserved in $\mathcal{N} = 4$ SYM, where fermions do not even couple directly to the loop. Also it deviates from the observation of [18] [19] [20] [21] [26] that one loop SCS polygonal loops and their dual scattering amplitudes vanish at one-loop, and that the two loop result reproduces the SYM one, up to a scheme-matching. On the other hand, the two-loop SCS computation of [48] differs from the SYM result of [49] out of the light-cone limit, but reproduces it on the light-cone, thus confining the conjecture, if any, to the realm of light-like objects.

This document is organised according to pedagogical criteria. In Chapter 2 we review general facts about scattering amplitudes in gauge theory, sketching where needed computations that will turn out to be useful, such as the one-loop IR divergence of the four-gluons maximally helicity violating amplitude. We also linger on BCFW on-shell recursion relations in field and string theory, partially reviewing the work of R.Boels, N.Obers and the author.

Chapter 3 is a review of the light-like Wilson loops/gluon amplitudes duality at both strong and weak coupling. The string computation of the minimal area governing the leading order asymptotic of the four-gluon amplitude is given in some detail for future use. The last Section is devoted to the most recent advances in the case of $\mathcal{N} = 6$ SCS theories.

Supersymmetric Wilson loop operators for SCS cited above are presented in Chapter 4, together with the derivation of the $\frac{1}{2}$ BPS operator from the low-energy theory of massive BPS particles in the broken phase of the theory.

Some important known string solutions which admit gauge duals are reviewed in Chapter 5. We also introduce the so-called cusp, or soft, anomalous dimension and its strong implications in both gauge and string theory. As we will explain, this observable is somehow related to the computation in [48]. In this Chapter we also introduce interpolating functions in SCS theories and elaborate on the fundamental role they cover, being the only known observables for any value of the coupling constant. These are partially a motivation for the work in [48].

Finally, Chapter 6 is devoted to the presentation of the results of L.Griguolo, G.Martelloni, D.Semiara and the author about the pair potential of SCS [48]. Particular emphasis is put on the geometrical and supersymmetrical deformations that make it a suitable observable for heavy tests of AdS_4/CFT_3 correspondence.

All the details about the computations and our conventions are in Appendix.

CHAPTER 2

Scattering Amplitudes at Weak Coupling

Some say scattering amplitudes are the most perfect microscopic structures in the universe [50]. Beyond subjectiveness, the past decades research has shown that a certain amount of information about the gauge theory is hidden in the deep structure of the scattering matrix and is not accessible elsewhere, at least not as easily. Moreover amplitudes provide a prime tool for testing suitability of physical models, being at present the only link between the theory of nature and nature itself, which we know through experiments. The short term interest in scattering amplitudes is mainly focused on gluon scattering. Any high-energy event, such as beam collision in a storage ring like LHC, involves a huge amount of background processes, mostly gluon-scatterings, that shade possibly new physical events behind them [27]. A great knowledge of such processes together with powerful tools for computing them to a high level of accuracy are not an option once the signatures of new physics are to be discerned and a plausible explanation is to be discriminated among hundreds of proposals. In this direction supersymmetric gauge theory amplitudes, being in some sense much easier than their QCD counterparts, offer the opportunity to study the perturbative structure of gauge theory at high loop orders and to test always newer and more effective computational methods. But quite importantly, they are not a mere toy for speculation, but indeed they offer a seriously powerful computational tool for real physics. Let us consider gluon amplitudes as a starting point. At tree-level they are the same in both QCD and in $\mathcal{N} = 4$ SYM in four dimensions, and at one-loop pure Yang-Mills gluon amplitudes can be written as

$$A_g^{YM} = (A_g + 4A_s + 3A_f)^{\mathcal{N}=4} - 4(A_f + A_s)^{\mathcal{N}=1} + A_s \tag{2.1}$$

where g, f and s mean gluons, fermions and scalars and are related to the particles circulating in loop diagrams. Evaluation of supersymmetric amplitudes is a much easier task, made so by the many advances and computational techniques; it converts hopeless tasks in back-of-an-envelope computations!

Unexpected simplicity of supersymmetric scattering amplitudes is mainly due to the large symmetry group they exhibit, which is larger than the symmetry group of the Lagrangean. For example it has been shown that tree-level scattering amplitudes in $\mathcal{N}=4$ SYM in four dimensions and in $\mathcal{N}=6$ Super-Chern-Simons (SCS) theory in three dimensions are invariant under the Yangian symmetry of the relative superconformal groups, which is strongly related to integrable properties of the two theories [11] [26], and part of this structure was indeed observed in high-loop scattering amplitudes [51]. The presence of hidden structures into scattering amplitudes and the relation they have with integrable models are important topics to investigate, as a they might be of great help in shedding light onto the theory and in developing better computational methods.

2.1 Stripping colour off of amplitudes

It turns out that the structure of gauge theory amplitudes can be highly simplified when they are written in a suitable form. In the following we will mostly deal with scattering of massless particles with light-like momenta $p_i^2=0$ and helicities $h_i=\pm1$ in four-dimensional (S)YM theories, moreover all particles are chosen to be ingoing. External momenta can be written as

$$p^{\alpha\dot{\alpha}}=\lambda^{\alpha}\lambda^{\dot{\alpha}} \tag{2.2}$$

for some spinors λ^{α}, $\lambda^{\dot{\alpha}}$ where the meaning of the double index is

$$p^{\mu}=(\gamma^{\mu})_{\alpha\dot{\alpha}}p^{\alpha\dot{\alpha}} \tag{2.3}$$

and γ^{μ} are the usual gamma matrices. The above property comes from the fact that massless solutions to the chiral Dirac equation

$$(p_{\mu}\sigma^{\mu})^{\dot{\alpha}\alpha}u_{-\alpha}(p)=0, \qquad u_{-}\overline{u}_{-}=\not{p} \tag{2.4}$$

allow for a parametrization of physical polarization vectors without imposing a non-covariant gauge (like any axial gauge). Such solutions are usually denoted by $\lambda^{\alpha}=u_{-}^{\alpha}(p)$ and $\lambda^{\dot{\alpha}}=\overline{u}_{-}^{\dot{\alpha}}(p)$ as above. In Minkowski signature they are related by complex conjugation, for which reason $p^{\alpha\dot{\alpha}}$ is invariant under a $U(1)$ phase acting on λ's. Note that the bar over $gl^{\dot{\alpha}}$ is omitted. Though, usually momenta are promoted to complex variables and the Lorentz group in four dimensions, for example, becomes $SL(2,\mathbb{C})\times SL(2,\mathbb{C})$. In this case λ^{α} and $\lambda^{\dot{\alpha}}$ become independent variables and the phase symmetry becomes a rescaling by an arbitrary complex number. Amplitudes have definite scaling properties under this transformation. Let us introduce a shorthand for multiplication of spinors

$$\langle ij\rangle=\epsilon^{\alpha\beta}\lambda_{i\alpha}\lambda_{j\beta}, \qquad [ij]=-\epsilon^{\dot{\alpha}\dot{\beta}}\lambda_{i\dot{\alpha}}\lambda_{j\dot{\beta}} \tag{2.5}$$

With these conventions it is easy to write gauge invariant quantities for null momenta

$$(p^\mu + q^\mu)^2 = 2p \cdot q = 2\langle ij\rangle [ij] \tag{2.6}$$

Gauge invariance also constrains polarization vectors of physical particles. Being external states light-like, polarization vectors must be transverse and, in a certain Lorentz frame, simply circular. This can be accomplished by choosing arbitrarily two reference spinors ξ_α and $\xi_{\dot\alpha}$ and building the two objects

$$\varepsilon^-_{\alpha\dot\alpha}(p,\xi) = -\sqrt{2}\frac{\lambda_\alpha \xi_{\dot\alpha}}{[\xi p]}, \qquad \varepsilon^+_{\alpha\dot\alpha}(p,\xi) = -\sqrt{2}\frac{\xi_\alpha \lambda_{\dot\alpha}}{\langle p\xi\rangle} \tag{2.7}$$

Polarization vectors with different reference spinors differ by a gauge transformation, so observable quantities do not depend on this choice at all.

So, once we have organized external states according to their helicities, we would like to organize colour indices as well. It is convenient to define partial amplitudes A_n for a $SU(N)$ gauge theory with gauge group generators T^a as a function of kinematical data only [52]

$$\mathcal{A}_n = \sum_{\{\sigma_i\}\in S_n/\mathbb{Z}_n} \mathrm{Tr}\left[T^{\sigma_1}T^{\sigma_2}\cdots T^{\sigma_n}\right] A_n(p_{\sigma_1}, p_{\sigma_2}, \ldots, p_{\sigma_n}) + \text{multi traces} \tag{2.8}$$

where σ_i is a shorthand for a set of collective indices that include group, momentum and helicity indices and the sum is extended to non-cyclic permutations. The coefficients $A_n(p_{\sigma_1}, p_{\sigma_2}, \ldots, p_{\sigma_n})$ are color-ordered amplitudes and are the relevant objects in the large N expansion where the single-trace dominates, for which reason they are also known as *planar partial amplitudes*. According to the polarization projectors defined above, partial amplitude can be labelled by their helicity states $A_n^{h_1,h2,\ldots,h_n}(p_1, p_2, \ldots, p_n)$. There are $(n-1)!$ of these objects but not all of them are independent, indeed there exists a complete basis of rank $(n-3)!$ as was shown in [53] and [54]. In particular note that tree level partial amplitudes for $n < 4$ vanish, and for $n = 4, 5$ there exists just one independent color-ordered amplitude.

Planar partial amplitudes also have other nice features. Under a cyclic permutation of a subset of the external legs they transform in a definite way (under a complete cyclic permutation they are simply invariant). Moreover there exist useful factorization identities relating tree level amplitudes in certain singular limits, such as whenever two adjacent external momenta, say k_1 and k_n for simplicity, become collinear [55] [56]

$$A_n^{tree}(p_1^{h_1}, \ldots, p_n^{h_n}) \to \sum_h A_{n-1}^{tree}(p_1^h, \ldots, p_{n-1}^{h_{n-1}})\, S_h^{tree}(p_1^h, p_n^{h_n}) \tag{2.9}$$

where S^{tree} is known as the tree level splitting amplitude and is a universal function in $\mathcal{N} = 4$ SYM. Another useful factorization property emerges in the multi-particle limit where $p_{1,j}^2 \to 0$

$$A_n^{tree}(p_1^{h_1}, \ldots, p_n^{h_n}) \to \sum_h A_{j+1}^{tree}(p_1^{h_1}, \ldots, p_j^{h_j}, p_{1,j}^h)\frac{\mathrm{i}}{p_{1,j}^2} A_{n+1-j}^{tree}(p_{1,j}^{-h}, p_{j+1}^{h_{j+1}}, \ldots, p_n^{h_n}) \tag{2.10}$$

These properties tightly constrain the structure of supersymmetric scattering amplitudes and have been successfully exploited in the past to actually compute a certain class of $\mathcal{N}=4$ gluon amplitudes at one-loop order [57] and even QCD amplitudes [58] [59] (and many others). Moreover they offer direct tests for the reliability of higher-points and higher-loops amplitudes computed through other techniques, and thus also a test-bed for new computational methods.

2.2 Maximally helicity violating amplitudes

The arbitrariness in choosing reference spinors for labelling helicities of external particles (2.7) can be used to show that tree-level scattering amplitudes with less than two external states in the same helicity configuration vanish identically. In particular, in $\mathcal{N}=4$ SYM all the three-point functions are zero at tree-level. The first non-vanishing amplitude would then be A_4^{++--}. This is indeed a very special object!

Following [60] one can show that the action of the supersymmetry generators Q_i relates amplitudes with different external states to all orders of perturbation theory. Moreover relations between tree-level gluon amplitudes hold in any supersymmetric gauge theory, regardless of the amount of supersymmetry. Let us denote gluons, fermions and scalars respectively with $g^\pm$, $f_i^\pm$ and s_{ij} where $\pm$ is the helicity and i,j are $R-$indices. Let us also use the shorthand $g^\pm(1)$ for the the external gluon of momentum p_1 and so on. In $\mathcal{N}=4$ SYM the action of the superconformal group gives for example

$$\begin{aligned}&Q_i(q,\theta)\left\langle 0\right| f_j^+(1)g^+(2)\ldots g^+(n)\left|0\right\rangle = \\ &= \delta_{ij}\left\langle q1\right\rangle \left\langle 0\right| g^+(1)g^+(2)\ldots g^+(n)\left|0\right\rangle + [q1]\left\langle 0\right| s_{ij}(1)g^+(2)\ldots g^+(n)\left|0\right\rangle = \\ &= 0\end{aligned} \tag{2.11}$$

The terms above vanish separately, in particular we are interested in the all-plus gluon amplitude

$$A_n^{++\ldots+}\left\langle 0\right| g^+(1)g^+(2)\ldots g^+(n)\left|0\right\rangle = 0 \tag{2.12}$$

Acting in a similar way on $\left\langle 0\right| g^-(1)f_j^+(2)g^+(3)\ldots g^+(n)\left|0\right\rangle$ one finds that also all-plus-but-one gluon amplitudes vanish identically

$$A_n^{-+\ldots+} = \left\langle 0\right| g^-(1)g^+(2)\ldots g^+(n)\left|0\right\rangle = 0 \tag{2.13}$$

It turns then out the the first non-vanishing $n-$points gluon amplitude has at least two particles with one helicity and $n-2$ with the other helicity. They are known as MHV, maximally helicity violating amplitudes

$$A^{MHV} = A^{--+..+},\ A^{-+-+..+},\ A^{-+..+-+}... \tag{2.14}$$

Note that the four-gluon amplitude encountered above A_4^{++--} is precisely the first, lower-points MHV amplitude.

As remarked above not all $n-$points MHV amplitudes are independent. For four- and five-points gluon amplitude only one independent object exists A_4^{--++} and A_5^{--+++}. Note in particular that A_5^{++---} is maximally helicity violating, more precisely is $\overline{\text{MHV}}$ in our conventions, but there is nothing physical in choosing which amplitude is MHV and which $\overline{\text{MHV}}$. Moreover all-gluon amplitudes are either MHV amplitudes or N^{k}MHV which means that k more helicities are flipped (k times next-to-MHV), and different helicity orderings are related through cyclicity properties so that only $(n-3)!$ of them are independent.

The fact that only one helicity structure is allowed at tree-level for MHV amplitudes reflects itself into the neat formula for the general $n-$gluon amplitude

$$A_n^{\text{tree}}(1^+, ..i^-, (i+1)^+, ..j^-, ..n^+) = \mathrm{i}\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle} \tag{2.15}$$

This simplicity is not lost at loop level, indeed all loop corrections are proportional to the tree-level result. So all-loop MHV gluon amplitudes can be factorized in the form

$$\mathcal{A}_n = A_n\left[1 + M_n^{(1)} + M_n^{(2)} + \dots\right] \tag{2.16}$$

where the functions $M_n^{(l)}$ depend only on gauge invariant quantities made out from external momenta and on the regularization parameter introduced when regulating divergent integrals. MHV gluon amplitudes are of particular interest in view of their duality with light-like Wilson loops. As we will discuss in the next sections, it seems that this duality does not properly hold for N^{k}MHV amplitudes, even though some attempt of extending it has been put forward [61].

2.2.1 IR divergences of the four-points MHV amplitude

It is known from long ago that on-shell gauge theory amplitudes are IR divergent at loop level. If we regulate integrals by means of dimensional regularization in $D = 4-2\epsilon$, divergences appear as poles in ϵ, and obey a quite general structure that depends only on certain universal quantities of the theory. As the easiest example, let us consider the four gluons scattering amplitude at one loop which were first computed up to two loops in [62]. The tree-level contribution can be factored and loop corrections can be recast in a unique scalar-like integral

$$A_4^{(1-loop)} = A_4^{(tree)}\left[1 - \frac{\lambda}{16\pi^4}\, stI(s,t,\epsilon) + \mathcal{O}(\epsilon^2)\right] \tag{2.17}$$

where $\lambda = g^2N$ is the 't Hooft coupling, $s = (p_1+p_2)^2$, $t = (p_1+p_4)^2$ the usual Mandelstam variables and the dimensionally regularised scalar box integral $I(s,t,\epsilon)$ reads

$$I(s,t,\epsilon)=\frac{\mu^{2\epsilon}\mathrm{e}^{\epsilon\gamma_E}}{(4\pi)^{2-\epsilon}}\int\frac{\mathrm{d}^{4-2\epsilon}k}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2} \tag{2.18}$$

For MHV amplitudes the only dependence on the helicities of external particles are factored in the tree-level term, indeed there is just one allowed helicity configuration in this case, which is $--++$. From the last expression one can easly identify the regions responsible of the infrared divergences of the amplitude. Indeed I has both soft divergences for $k^2\sim 0$, due to the emission of soft gluons, and collinear divergences $k^\mu\sim p^\mu$ arising whenever k becomes parallel to any of the on-shell momenta involved in the scattering process. Computing the integral in (2.18) one has

$$\begin{aligned}A_4^{(1-loop)}=A_4^{(tree)}\left[1-\frac{1}{\epsilon^2}\frac{\lambda}{8\pi^2}\left(\frac{\mu^2}{-s}\right)^\epsilon\right]\left[1-\frac{1}{\epsilon^2}\frac{\lambda}{8\pi}\left(\frac{\mu^2}{-t}\right)^\epsilon\right]\\ \left[1+\frac{\lambda}{8\pi}\left(\frac{1}{2}\log^2\frac{s}{t}+4\zeta_2\right)^\epsilon\right]+\mathcal{O}(\lambda^2)\end{aligned} \tag{2.19}$$

We see that divergent contributions factorise in a term which depends on s only and a one which depends on t only, and a finite contribution. For non MHV amplitudes this last term also depends on the helicity structure of the external particles, but the factorization of divergences continues to hold, even at higher loop level. The singular part of the amplitude is governed by a renormalisation group evolution equation (3.27) [63] [64] [65], which says that for conformal field theories, for which the $\beta-$function vanishes identically, at every loop level

$$\mathrm{div}(s)=\exp\left\{-\frac{1}{2}\sum_{l=1}^{\infty}(\lambda/8\pi^2)^l s^{l\epsilon}\left[\frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon)^2}+\frac{\Gamma^{(l)}}{l\epsilon}\right]\right\} \tag{2.20}$$

where

$$\Gamma_{\text{cusp}}(\lambda)=\sum_l\lambda^l\Gamma_{\text{cusp}}^{(l)},\qquad \Gamma(\lambda)=\sum_l\lambda^l\Gamma^{(l)} \tag{2.21}$$

are the so-called *cusp* and *collinear* anomalous dimensions. The former is a universal quantity that naturally appears as the coefficient of the UV divergent part of cusped Wilson loops (from which the name), it covers a fundamental role in the duality between scattering amplitudes and light-like Wilson loops. More about this topic will be said in Sections 3.2 and 5.1. The collinear anomalous dimension is, on the other hand, a scheme-dependent quantity. The fact that Γ_{cusp} enters the IR divergent part of scattering amplitude is a manifestation of the duality relation that will be reviewd in the next chapter. Note that the renormalisation group equation mentioned above was already known from the Eighties for the divergent part of Wilson loops [66] [67]. At present the cusp anomalous dimension is known up to four-loops in $\mathcal{N}=4$ SYM [68] [69] and the integrands that contribute to the five-loop correction where presented in [70]. At strong coupling it can be derived using the AdS/CFT correspondence [71] [72] [73].

2.3 Higher-points, higher-loops: the BDS ansatz

In the previous section we reviewed the structure of IR divergences of the four-point MHV gluon amplitude in $\mathcal{N}=4$ SYM. This function has been known for a while up to the second order of perturbation theory [62], until it was observed that a certain iterative structure underlaid this two-loop result [74]. Based on this observation and on an explicit three-loops computation, the authors of [75] conjectured that the finite part of the scattering amplitude exponentiates at all orders of perturbation theory. Note that the same was already known to hold for the divergent part (2.20). In particular the conjecture, known as BDS ansatz, implies that the finite part of the n-points amplitude reads

$$F_n^{BDS} = \frac{1}{2}\Gamma_{\text{cusp}}(\lambda) F_n^{(1)} + \text{constant} \tag{2.22}$$

where $F_n^{(1)}$ is the one-loop finite contribution and $\Gamma_{\text{cusp}}(\lambda)$ the cusp anomalous dimension. Defining $\mathcal{A}_n = A_n^{\text{full}}/A_n^{\text{tree}}$ the ratio of the full $n-$gluons MHV amplitude over its tree-level value, the BDS ansatz states that

$$\mathcal{A}_n^{MHV} = \mathrm{e}^{Z_n + F_n^{BDS} + C_n} \tag{2.23}$$

where the divergent part is

$$Z_n = -\frac{1}{4}\sum_{l=1}^{n}\left(\frac{\lambda}{8\pi^2}\right)^l \left[\frac{\Gamma_{\text{cusp}}^{(l)}}{(l\epsilon)^2} + \frac{\Gamma^{(l)}}{l\epsilon}\right]\sum_{i=1}^{n}\left[-\frac{s_i}{\mu^2}\right]^{-l\epsilon} \tag{2.24}$$

and generalises the divergence structure of the four-gluon aplitude (2.20) to $n-$gluons, F_n is the finite part conjectured above and C_n are constants. Also, the s_i's are the usual Mandelstam variables $s_i = x_{i,i+1}^2 = (p_i + p_{i+1})^2$. The conjecture (2.23) is mostrly a statement about the finite part of the amplitude, which is supposed to exponentiate in the same way the divergent part does.

At the origin of this apparent simplicity is the observation of [51] that the dimensionally regularised scalar box integral of (2.18) can be written in dual coordinates

$$p_1 = x_1 - x_2 = x_{12}, \quad p_2 = x_{23}, \quad p_3 = x_{34}, \quad p4 = x_{41} \tag{2.25}$$

$$I(s,t,\epsilon) = c\int \frac{\mathrm{d}^{4-2\epsilon}k}{k^2(k-p_1)^2(k-p_1-p_2)^2(k+p_4)^2} = c\int\frac{\mathrm{d}^{4-2\epsilon}x_5}{x_{15}x_{25}x_{35}x_{45}} \tag{2.26}$$

that manifest a dual conformal structure of the integrand. Indeed in the dual coordinate frame, due to dimensional regularisation, the integral I has a broken conformal symmetry. This duality was shown to hold for two-loop integrals and a dual picture was proposed [76] [77] in which the dual integral is represented as a light-like Wilson loop integral in the dual space with edges x_{ij}'s (note that $x_{ij}^2 = 0$ because momenta are light-like). Indeed the conjecture received full interest and a strong coupling motivativation for the duality was given in [41].

On the Wilson loop side, broken conformal Ward identities was also derived [16]; basing on the conformal properties of loop space integrals they constrain the structure of the expectation value of the Wilson loop so as to fully determine its value for the 4 and 5 edges cases, thus the recursive simple structure of 4 and 5-gluons MHV amplitudes observed by [75]. The uncovering of such a powerful symmetry together with explicit computations for the tree-level, all-points amplitudes [78] were a seemingly strong support to the BDS ansatz. On the other hand a deviation from this conjecture was soon observed in [79], where the six-edges light-like Wilson loop was computed at two loops, and further in a strong coupling computation of n-gluon amplitudes with n very large [80]. Soon after a direct evaluation of the six-gluons MHV scattering amplitude [17] stated the validity of the Wilson loop computation and the incompleteness of the BDS ansatz, which was then modified to account for a deviation from the original conjecture

$$\mathcal{A}_n^{MHV} = \mathrm{e}^{Z_n + F_n^{BDS} + f_n + C_n} \tag{2.27}$$

This deviation f_n in turn was attributed to so-called *reminder functions* appearing in the solution to the conformal Ward identities from six-points onwards. In fact, conformal symmetry restricts these finite contributions to be arbitrary functions of conformal ratios

$$\frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

which are evidently trivial for the 4,5 points cases but are no more when the number of external legs increases, for example at six edges one finds

$$u_1 = \frac{x_{13}^2 x_{46}^2}{x_{14}^2 x_{36}^2}, \qquad u_2 = \frac{x_{24}^2 x_{15}^2}{x_{25}^2 x_{14}^2}, \qquad u_3 = \frac{x_{35}^2 x_{26}^2}{x_{25}^2 x_{36}^2}$$

However note that the BDS ansatz (2.23) is still a particular solution to the conformal Ward identites, as was shown in [77], though definitely it is not the most general solution. On the other hand, these developments strongly pushed forward the duality between MHV amplitudes and light-like Wilson loops, which is the main topic of the next Chapter 3.

2.4 BCFW on-shell recursion relations

The collinear and multiparticle relations sketched in the previous section are hand-waving arguments to the existence of much tighter relations among scattering amplitudes with a different number of external particles. As much of the recent developments in understanding the S-matrix of field theories was inspired by Witten's twistor string [28], on-shell recursion relations for scattering amplitudes were proposed in [29] by Britto, Cachazo and Feng and then prooved in [30]. These are commonly know as BCFW recursion relations. In field theory these useful relations relate tree level amplitudes to a sum over amplitudes with a smaller number of particles, evaluated at complex values of the momenta. The elementary and elegant derivation of the relations [30] involves a complex momentum shift on two particles. Crucially, an absence of certain residues at infinite momentum shifts needs to be shown to make the relations work. Residues at infinite complex momentum are at the least by hand waving related to the UV behavior of the theory under study. This reasoning can be made more precise in field theory [81]. Absence of residues at infinity has been proven in (super)Yang-Mills and Einstein (super)gravity in any dimension from four onwards, see for example [82] and references therein.

The key observation for deriving the on-shell recursion relations is that any tree level scattering amplitude can easily be turned into a rational function of a single complex variable by deforming the momenta [30], requiring that these deformed momenta remain on-shell and obey momentum conservation. The simplest example of this is to take two particles i and j and shift their momenta by a vector q_μ

$$\begin{aligned} p_i^\mu \rightarrow \widehat{p}_i^\mu &= p_i^\mu + z q^\mu \ , \\ p_j^\mu \rightarrow \widehat{p}_j^\mu &= p_j^\mu - z q^\mu \ , \end{aligned} \tag{2.28}$$

which preserves momentum conservation. More complicated shifts are though possible. For two particle shifts linear in z as in eq. (2.28), the on-shell constraint is satisfied iff the vector q obeys

$$p_i^\mu q_\mu = p_j^\mu q_\mu = q^\mu q_\mu = 0 \ . \tag{2.29}$$

These equations do not have a solution for real q_μ, but do for complex momenta, as can easily be verified by going to the common light-cone frame. After the shift any n-point amplitude A_n becomes a function of a complex variable $A_n(z)$, where the amplitude of interest is of course $A_n(z = 0)$. This can be obtained by an elementary contour integration around a contour which only encompasses the pole at $z = 0$,

$$A_n(0) = \oint_{z=0} \frac{A_n(z)}{z} \mathrm{d}z \tag{2.30}$$

If the contour is now pulled to the other side of the Riemann sphere one encounters various poles at finite values of z and a possible residue at infinity,

$$A_n(0) = \oint_{z=0} \frac{A_n(z)}{z} \mathrm{d}z = -\left\{ \sum \mathrm{Res}_{z=\mathrm{finite}} + \mathrm{Res}_{z=\infty} \right\} \ . \tag{2.31}$$

The poles at finite values of z correspond to the exchange of physical particles. By tree level unitarity, the residues at these poles must be the product of two tree level amplitudes with each one leg containing the particle being exchanged, summed over all particles at this particular mass level. The residue at infinity does not have a similar physical interpretation. If therefore this residue vanishes then all terms on the right hand side of (2.31) are known and consist of lower point amplitudes. Therefore in this case a recursion relation is obtained between amplitudes

$$A_n(1,2,3\ldots,n) = \sum_{r,h(r)} \sum_{k=2}^{n-2} \frac{A_{k+1}(1,2,\ldots,\widehat{i},\ldots,k,\widehat{P}_r) A_{n-k+1}(\widehat{P}_r,k+1,\ldots,\widehat{j},\ldots,n)}{(p_1+p_2+\ldots+p_k)^2+m_r^2}\,, \tag{2.32}$$

where the sum is over all different mass levels r and over all polarization states at that level, denoted $h(r)$. The momentum $\widehat{P}_r$ for the 'extra' particle and its anti-particle in the amplitude is such that the particle is on-shell.

Example: pure Yang-Mills vs $\mathcal{N}=4$ amplitudes

As the large z behaviour is crucial in determining whether or not scattering amplitudes obey recursion relation, direct inspection of Feynman diagrams is often misleading, predicting a much worse scaling than the actual one. This is a strong hint of the fact that symmetries play an imporant role, indeed theories with larger Lorentz symmetry groups have a nicer behaviour under infinite complex shifts.

Consider to this purpose pure Yang-Mills theory for a gauge field fluctuating around some soft background $A^\mu = A_0^\mu + a^\mu$. The gauge-fixed YM Lagrangean for the dynamical field becomes

$$\begin{aligned} L_{YM} = &-\frac{1}{4}\mathcal{D}^{[\mu}a_{\nu]}\mathcal{D}_{[\mu}a^{\nu]} + \frac{\mathrm{i}}{2}\mathrm{tr}[a_\mu, a\nu]F^{\mu\nu} + (\mathcal{D}_\mu a^\nu)^2 \\ &= -\frac{1}{4}\mathcal{D}^\mu a_\nu \mathcal{D}_\mu a^\nu + \frac{\mathrm{i}}{2}\mathrm{tr}[a_\mu, a\nu]F^{\mu\nu} \end{aligned} \tag{2.33}$$

Now a Feynman diagram reasoning in momentum space would suggest that the first term can potentially have vertices which behave like $\mathcal{O}(z)$ under (2.28), but it also have an enhanced spin symmetry acting on a's but not on A_0's. So the general form of the amplitude would be

$$M^{ab} = (c_1 z + c_0 + c_{-1}\frac{1}{z} + \ldots)\eta^{ab} + A^{ab} + \frac{1}{z}B^{ab} + \ldots \tag{2.34}$$

where A^{ab} is anti-symmetric and a, b are some polarization indices (it is evidently convenient to change notation and use a different letter to indicate amplitudes). But this is not quite true. To get on-shell amplitudes, one actually needs to contract M^{ab} with polarization vectors $\varepsilon^a(p_i), \varepsilon^b(p_j)$ in (2.7), and using Ward identities

$$(p_i + zq)_a M^{ab}\varepsilon_{jb} = 0 \quad \Longrightarrow \quad q_a M^{ab} = \frac{-1}{z} p_{ia} M^{ab}\varepsilon_{jb} \tag{2.35}$$

one gets much better behaved amplitudes

$$
\begin{aligned}
M^{-+} &= \varepsilon^-_{ia} M^{ab} \varepsilon^+_{jb} = q_a M^{ab} q_b \\
&= \frac{-1}{z} p_{ia} \left[(c_1 z + c_0 + c_{-1}\frac{1}{z} + ...)\eta^{ab} + A^{ab} + \frac{1}{z} B^{ab} + ... \right] q_b \\
&= \frac{-1}{z} p_{ia} A^{ab} q_b \to \frac{1}{z}
\end{aligned}
\tag{2.36}
$$

Here the orthogonality condition $p_i \cdot q = 0$ is crucial for lowering the degree of divergence. Not all amplitudes of standard YM theory have good properties, hence not all of them can be recursively computed. More precisely one can show [82] that, upon choosing axial gauge to remove $\mathcal{O}(z)-$vertices from all but a finite number of diagrams, pure YM amplitudes behave like $M^{-+}, M^{--}, M^{++} \to z^{-2}$ and $M^{+-} \to z^2$ under BCFW shift. Things get better if one considers supersymmetric scattering amplitudes.

The challenge in deriving BCFW relation is the proof of absence of the residue at infinity. It was shown in [35] that a proof along these lines holds for all open string theory amplitudes in a flat background, subject to a kinematic constraint. As the derivation of the BCFW recursion relation involves a limit, the field theory limit of the resulting equations has to be treated with care [34] to avoid 'order of limits' problems. Note that any symmetry of the three-point amplitude will imply through the recursion relations a corresponding symmetry of n-point amplitudes, but this will be next section's topic.

2.4.1 Three-dimensional recursion

It is not hard to convince oneself that solutions to (2.28) subject to the constraints of momentum conservation (2.29) exist only in dimensions $D \geq 4$, therefore on-shell recursion relations in 3 dimensions must involve a non-linear shift. The spinor decomposition of external momenta

$$p^{\alpha\beta} = p^\mu (\sigma_\mu)^{\alpha\beta} \tag{2.37}$$

involves a single spinor λ, in contrast to what happens in four or grater dimensions where two spinors are needed. We preserve conventions of the previous sections

$$p^{\alpha\beta} = \lambda^\alpha \lambda^\beta \,, \qquad 2p_i \cdot p_j = -\langle i|j\rangle^2 \tag{2.38}$$

for light-like momenta. From the equation above, which is quadratic in λ, one could hint that a suitable transformation that preservers momentum conservation of on-shell particles might be achieved by a rotation of the two spinors

$$\begin{pmatrix} \lambda_i(z) \\ \lambda_j(z) \end{pmatrix} = R(z) \begin{pmatrix} \lambda_i \\ \lambda_j \end{pmatrix} \tag{2.39}$$

by a rank-two matrix depending on z. Being R a rotation, on-shellness is automatically preserved, while conservation of shifted momenta boils down to

$$(\lambda_i(z)\ \lambda_j(z))\begin{pmatrix}\lambda_i(z)\\ \lambda_j(z)\end{pmatrix}=(\lambda_i\ \lambda_j)\begin{pmatrix}\lambda_i\\ \lambda_j\end{pmatrix} \tag{2.40}$$

which gives the following constraint on R

$$^{t}R(z)\,R(z)=I \tag{2.41}$$

which is the simple statement that R is a matrix of $SO(2,\mathbb{C})$. To see that this is actually a vector rotation in two dimension it is enough to change variable to $z=\mathrm{e}^{\mathrm{i}\theta}$, then from the general parametrisation of $SO(2,\mathbb{C})$

$$R(z)=\begin{pmatrix}\frac{z+z^{-1}}{2} & -\frac{z-z^{-1}}{2\mathrm{i}}\\ \frac{z-z^{-1}}{2\mathrm{i}} & \frac{z+z^{-1}}{2}\end{pmatrix}=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix} \tag{2.42}$$

The main difference one can readily see between four and three-dimensional shifts is that in the latter case intermediate propagators in (2.32) become highly non-linear in z. For example, let us shift spinors relative to leg 1 and leg m, accordingly the momentum of the intermediate particle propagating between subamplitudes will be

$$\widehat{P}_r(z)=p_i+...+p_m(z)+...+p_j, \qquad 1<i<m<j \tag{2.43}$$

which gives the on-shell condition

$$0=\widehat{P}_r^2(z)=c_{-2}z^{-2}+c_0+c_2z^2 \tag{2.44}$$

where

$$\begin{gathered}c_{-2}=-2\widetilde{q}\cdot(P_r-p_m), \qquad c_0=(P_r+p_1)\cdot(P_r-p_m), \qquad c_{-2}=-2q\cdot(P_r-p_m)\\ q^{\alpha\beta}=\frac{1}{4}(\lambda_1+\mathrm{i}\lambda_m)^{\alpha}(\lambda_1+\mathrm{i}\lambda_m)^{\beta}, \qquad \widetilde{q}^{\alpha\beta}=\frac{1}{4}(\lambda_1-\mathrm{i}\lambda_m)^{\alpha}(\lambda_1-\mathrm{i}\lambda_m)^{\beta}\end{gathered} \tag{2.45}$$

This non-linear equation has four solutions for z, that we denote $\pm z_{\pm}^{r}$, and the corresponding contour integral must pick up the contributions of all of the four poles

$$A_n(z=1)=\oint_{z=1}\frac{A_z(z)}{z-1}\,\mathrm{d}z \tag{2.46}$$

where the contour encloses the four singularities. It follows that the analogue of the four-dimensional recursion formula (2.32) receives more than one contribution from each on-shell intermediate particle

$$A_n=B+\sum_{r,h(r)}\sum_{k=2}^{n-2}H(z_+^r,z_-^r)A_{k+1}(1,...k,r^h)\frac{1}{P_r^2}A_{n-k+1}(r^h,k+1,...n)+\{z_+^r\leftrightarrow z_-^r\} \tag{2.47}$$

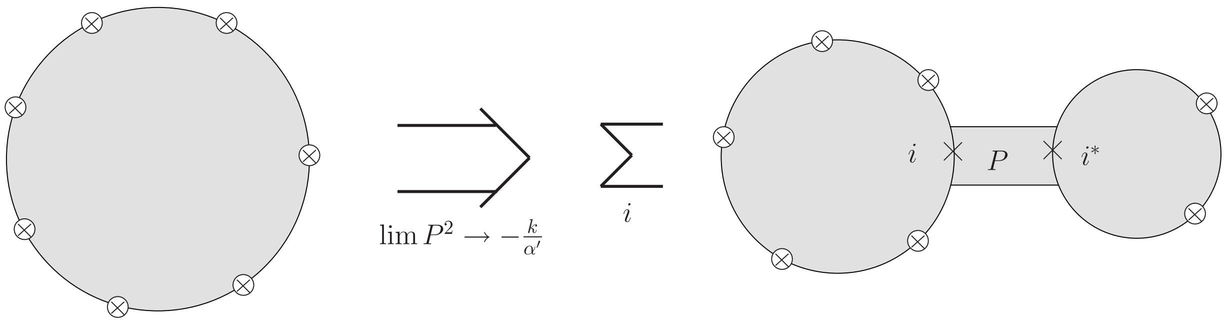


Figure 2.1: Conformal symmetry elucidates a certain kinematical limit with k a non-negative integer. The sum runs over all string states at this particular mass level.

where the sums are extended over all possible intermediate states with momentum P_r and helicity $h(r)$. Although the above expression has more than one term contributing to the recursive expansion for each on-shell propagator, the main difference with the four dimensional counterpart is in the appearance of the function

$$H(z_1, z_2) = \begin{cases} \frac{z_1^2(z_2^2-1)}{z_1^2-z_2^2} & r = \text{odd} \\ \frac{z_1\,(z_2^2-1)}{z_1^2-z_2^2} & r = \text{even} \end{cases} \tag{2.48}$$

which counts the number of solutions. At last, any boundary contributions must be taken into account. In three-dimensional theories the behaviour at large z is generally nicer since any propagator contributes a $\frac{1}{z^2}$ to the amplitude. For example note that in ABJM theory, which will shortly be the main focus of our interest, all boundary term vanish [83].

2.5 Recursion in string theory

On-shell recursion relations have a nice and intuitive interpretation in string theory. Scattering amplitudes in string theory are computed inserting vertex operators at the boundary of of the string worldsheet. For open strings this is simply a disk, as dipicted in Figure 2.5. Conformal symmetry than allows to stretch the disk along some direction until it develops a long narrow strip. When this strip is infinitely long, only on-shell string states can propagate from side to side of the worldsheet, and each half of the latter constitute by themselves a lower-point scattering amplitude. This gives a intuitive interpretation of the recursion formula (2.32) presented before.

The behavior of string theory tree-level amplitudes under a BCFW shift of two external momenta was analysed in [34] and in more details in [35]. The key idea is to apply the complex shift directly to the integral form of the string amplitude. A parallel derivation based on the study of OPE between vertex operator was also given and will be reviewed at the end of this section. For may useful reminds about old-school string theory we advice the beautiful review [84].

Let us then introduce the open string tachyon scattering amplitude for all multiplicities which is given by the well-known Koba-Nielsen formula
[85],

$$A_n = \int_{0\le y_{n-1}\le \ldots \le y_3 \le 1} \prod_{2<i<j<n} (y_i - y_j)^{2\alpha' p_i p_j} \quad . \tag{2.49}$$

In deriving this expression from the path integral the positions of 3 vertex operators have been fixed: particles 1, 2 and n at ∞, 1 and 0 respectively. Despite appearances, this expression can be shown to be cyclically symmetric in the external legs. We can therefore shift any two adjacent particles to cover all adjacent shifts and for the above expression it is convenient to choose n and 1. Through a coordinate transformation

$$u_i = \frac{y_{i+1}}{y_i} \quad , \quad 2 \le i \le n-2 \; , \tag{2.50}$$

this expression can be transformed to

$$A_n = \left(\prod_{i=2}^{n-2} \int_0^1 du_i u_i^{\alpha' s_i - 2} \right) \left(\prod_{k=2}^{n-2} \prod_{j=k+1}^{n-1} \left(1 - \prod_{l=k}^{j-1} u_l \right)^{2\alpha' p_k p_j} \right) \; , \tag{2.51}$$

with $s_i = (\sum_{k=1}^{i} p_i)^2$. From this expression it is easy to see that when particles 1 and n are shifted one can apply the integral argument given above for the Veneziano amplitude several times to obtain the limiting behavior for $z \to \infty$. Concretely, the chosen shift shifts

$$s_i \to \widehat{s}_i = s_i + 2z q_\mu \left(\sum_{k=2}^{i} p_i^\mu \right) \equiv s_i + \frac{\gamma_i}{\alpha'} z \; . \tag{2.52}$$

In line with the analysis above, change coordinates to

$$u_i = \exp\left(- \frac{\beta_i w_i}{\alpha' s_i - 2 + \gamma_i z} \right) \equiv e^{-\tilde{w}_i} \; , \tag{2.53}$$

which turns (2.51) into

$$A_n(z) = \left(\prod_{i=2}^{n-2} \int_0^\infty dw_i \left(\frac{-\beta_i e^{-\tilde{w}_i}}{\alpha' s_i + \gamma_i z - 2} \right) e^{-\beta_i w_i} \right) \left(\prod_{k=2}^{n-2} \prod_{j=k+1}^{n-1} (1 - e^{-\sum_{l=k}^{j-1} \tilde{w}_l})^{2\alpha' p_k p_j} \right) , \tag{2.54}$$

accompanied by the reality conditions

$$\text{Re} \left(\frac{\beta_i w_i}{\alpha' s_i - 2 + \gamma_i z} \right) > 0 \quad , \quad \text{Re} \; (\beta_i) > 0 \; . \tag{2.55}$$

From eq. (2.54) the large z behavior of the bosonic string tachyon amplitude follows as

$$A_n(z) \sim \left(\frac{1}{z} \right)^{\alpha'(p_1+p_n)^2 - 1} \left(G_0 + \frac{G_1}{z} + \mathcal{O}\left(\frac{1}{z} \right)^2 \right) \; , \tag{2.56}$$

which is the result from a Laurent expansion around $z = \infty$. In this expression G_i denote certain $(n-3)$-fold exponential integrals that do not depend on z. This completely isolates the large z behavior of the amplitude and we conclude that for adjacent shifts the Koba-Nielsen amplitude obeys BCFW recursion if

$$\text{Re } \left(\alpha'(p_i + p_{i+1})^2\right) > 1 \,, \tag{2.57}$$

with i and $i+1$ the labels of the shifted particles. More precisely, $n-3$ points on the contour integral must be excised. It can then be argued that their contribution vanishes because of analyticity of the integrand on the contour. In principle G_0 could integrate to zero, so the above analysis establishes a bound only, which could be better if some hidden symmetry is at work.

A similar analysis can be carried out for gluon amplitudes in both bosonic and superstrings. The main difference of the tachyon amplitudes and amplitudes involving other modes of the string are the complications caused by polarization vectors as these must be transverse to the shifted momenta. As a concrete and important example of this, adjacent shifts of the general n-point gluon amplitudes in the bosonic and superstring will be considered in this subsection. To solve the complication and obtain concrete expressions it is instructive as was done in [86] to consider the lightcone frame of the two shifted momenta,

$$p_1 = \frac{1}{\sqrt{2}}(1, 1, 0, 0; \ldots 0) \,, \qquad p_n = \frac{1}{\sqrt{2}}(1, -1, 0, 0; \ldots 0) \,, \tag{2.58}$$

where we have set the energy scale by one of the momenta to avoid cluttering formulas later. In this frame the shift vector obeying (2.29) can be chosen to be

$$q = \frac{1}{\sqrt{2}}(0, 0, 1, i; \ldots 0) \,. \tag{2.59}$$

With this shift choice it is convenient to choose the polarization vectors for unshifted momenta as

$$\zeta_1^- = \zeta_n^+ = q \,, \qquad \zeta_1^+ = \zeta_n^- = q^* \,, \qquad \zeta^T = (0, 0, 0, 0; \ldots, 1, \ldots, 0) \,. \tag{2.60}$$

These vectors are given in a lightcone gauge in which the lightcone gauge vector of one leg is the momentum of the other leg. Under a momentum shift

$$p_1 \to p_1 + qz \,, \qquad p_n \to p_n - qz \,, \tag{2.61}$$

the set of transformations that leaves the transversality constraint $\zeta_i \cdot p_i = 0$ invariant reads

$$\begin{array}{ccccccc} \zeta_1^- & = & \zeta_n^+ & = & q & \to & q \\ & & \zeta_1^+ & = & q^* & \to & q^* + zp_n \\ & & \zeta_n^- & = & q^* & \to & q^* - zp_1 \\ & & & & \zeta^T & \to & \zeta^T \end{array} \,. \tag{2.62}$$

$\zeta_1 \setminus \zeta_n$	$-$	$+$	T
$-$	$+1$	$+1$	$+1$
$+$	-3	$+1$	-1
T	-1	$+1$	-1
T2	-1	$+1$	0

Table 2.1: The leading power in $z^{-\kappa}$ for large z limit of the BCFW shift of an all gluon amplitude in field theory for all possible polarizations.

The large z behaviour of D dimensional field theory amplitude [86] generalizes the result of the previous section (2.36). It is instructive to list field theory and SYM results and compare them with data extracted from string amplitudes, for this purpose see Table 2.1.

$\zeta_1 \setminus \zeta_n$	$-$	$+$	T
$-$	-1	$+1$	$+1$
$+$	-3	-1	-1
T	-1	$+1$	-1
T2	-1	$+1$	0

$\zeta_1 \setminus \zeta_n$	$-$	$+$	T
$-$	$+1$	$+1$	$+1$
$+$	-3	$+1$	-1
T	-1	$+1$	-1
T2	-1	$+1$	0

Table 2.2: The leading power in $z^{-\alpha'(p_1+p_n)^2-\kappa}$ for large z limit of the adjacent shift of an all gluon amplitude in the bosonic string on the left and superstring on the right for all possible polarizations. For transverse polarization states TT means same polarization while $TT2$ means different polarizations, both transverse.

Without specifying the effect of the shifts of the polarization vectors ζ_1, ζ_n the large z-dependence of bosonic string gluon amplitudes can be written as

$$A_n^{\text{bosonic}} \sim \left(\frac{1}{z}\right)^{\alpha'(p_1+p_2)^2} \widehat{\zeta}_1^{\mu} \left[z \left(g_{\mu\nu} + B^3_{\mu\nu}\right) h_1\left(\frac{1}{z}\right) + \left(B^1_{\mu\nu} + B^2_{\mu\nu}\right) h_2\left(\frac{1}{z}\right) + \mathcal{O}\left(\frac{1}{z}\right)\right] \widehat{\zeta}_n^{\ \nu} \ , \quad (2.63)$$

Here the hatted quantities have been shifted and h_i are as before polynomial functions of $\frac{1}{z}$ with a non-trivial constant term. Furthermore, B^2 is the anti-symmetric matrix

$$B^2_{\mu\nu} = \sqrt{2\alpha'} \sum_{j,\,k=2}^{n-1} \left((\zeta_j)_\nu (p_k)_\mu - (\zeta_j)_\mu (p_k)_\nu\right) \ , \quad (2.64)$$

and B^3 is the symmetric matrix

$$B^3_{\mu\nu} = -2\alpha' \sum_{j,k=2}^{n-1} ((p_j)_\mu (p_k)_\nu) = -2\alpha'(p_1+p_n)_\mu (p_1+p_n)_\nu \ , \tag{2.65}$$

which is the main difference compared to the field theory behaviour in (2.36). In the same way the behaviour of superstring amplitudes can be extracted and reads

$$A_n^{\text{super}} \sim \left(\frac{1}{z}\right)^{2\alpha'(p_1\cdot p_n)} \widehat{\zeta}_1^\mu \left(z\, h_1(1+2\alpha' p_1\cdot p_n) g_{\mu\nu} + \left(h_2 B^1_{\mu\nu} + h_3 B^2_{\mu\nu}\right) + \mathcal{O}\left(\frac{1}{z}\right)\right) \widehat{\zeta}_n^\nu \ . \tag{2.66}$$

The function h_i are as before polynomial functions of $\left(\frac{1}{z}\right)$ with non-trivial constant term. Importantly, the matrix $B^2_{\mu\nu}$ is antisymmetric as is shown in [35]. By the same analogy to [86] as noted above this leads immediately to Table 2.2 for the large z-behavior of the superstring gluon amplitude. Note that one has to use Ward identities as in the YM case to soften the behaviour at infinity. It is instructive to compare field theory resuts to string theory results through Tables 2.1, 2.2. Note that somehow superstrings amplitudes have worse divergences for some combinations of external helicities. This is imputable to the fact that a non-supersymmetric shift has been performed on supersymmetric amplitudes.

CHAPTER 3

Duality between Wilson Loops and Scattering Amplitudes

3.1 Scattering amplitudes and Wilson loops at strong coupling

One of the main reasons of interest in scattering amplitudes of maximally supersymmetric theories is the hope that they could tell us something about less supersymmetric theories or theories with no supersymmetry at all, like QCD. And the reason why they have been elected to recipient of such a hope is that they are easier to compute. Indeed, MHV amplitudes in $\mathcal{N} = 4$ super Yang-Mills can be recursively computed for any number of external particles up to one-loop, which is a great deal of information about the perturbative structure of the theory. On the other hand one of the main open questions of QCD entails the behaviour of the theory in the region where it is strongly coupled, and again, supersymmetric theories could be of some help. Supersymmetric theories at strong coupling can be studied through the AdS/CFT correspondence, by means of which a strictly non-perturbative computation on the field theory side is mapped to a semiclassical computation on the string side. In particular the idea of Alday and Maldacena to relate strongly coupled gluon amplitudes to certain Wilson loops [41] [80] has been of great success in the past five years and also motivated the belief that the duality actually holds for any value of the coupling constants. Originally, this duality related a scattering amplitude in four-dimensional strongly coupled $\mathcal{N} = 4$ SYM to the minimal area of the string worldsheet ending on a light-like Wilson loop on the boundary of AdS_5. This duality is achieved by means of a symmetry of the string sigma model known as T-duality, which in turn is responsible of the observed dual superconformal symmetry of scattering amplitudes at strong coupling. Dual superconformal symmetry is also present at weak coupling and it appears to be the lower level realization of an even more hidden symmetry related to the underling integrability of the theory and known as Yangian symmetry [10] [11] [12] [13] [14] [15]. In the fist part of this chapter we will review the derivation of the aforementioned strong coupling duality for $\mathcal{N} = 4$ SYM and

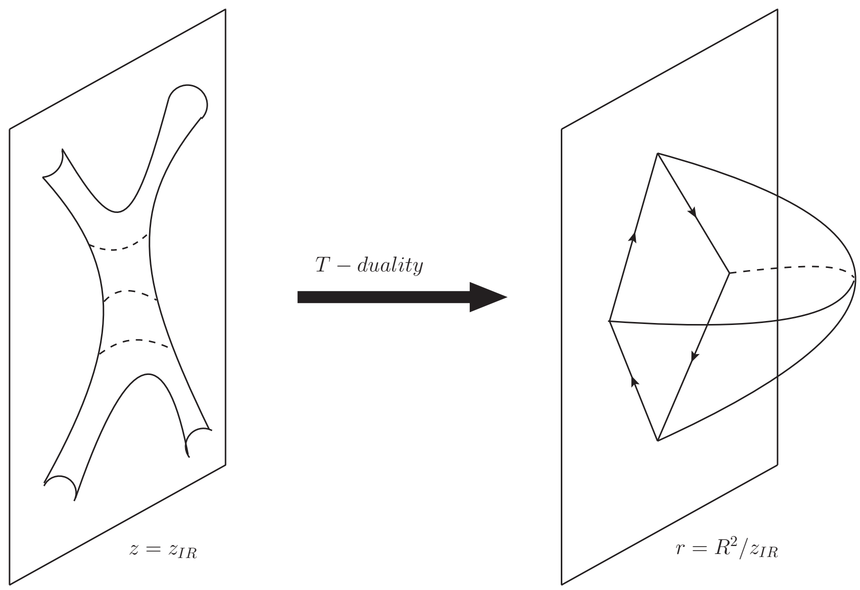


Figure 3.1: Pictorial view of the T-duality that maps the string scattering amplitude's worldsheet into the dual worldsheet ending on a null polygonal Wilson loop. The duality also maps AdS into itself and maps a D3-brane in the far infrared to a point near the dual boundary $r = 0$, where the Wilson loop is located.

type IIB superstring theory on $AdS_5 \times S^5$. Then we will focus on the weak coupling side of the amplitudes/Wilson loops duality, in particular for what concerns a more recent example of AdS/CFT which is the duality between $\mathcal{N} = 6$ supersymmetric Chern-Simons-matter theories in three dimensions and type IIA on $AdS_4 \times \mathbb{C}P^3$.

3.1.1 T-selfduality of $AdS_5 \times S^5$

Basing on the AdS/CFT correspondence [6][1], it was proposed to describe gluon scattering amplitudes in $\mathcal{N} = 4$ SYM at strong coupling by the scattering of open strings in $AdS_5 \times S^5$ [41] [80], as depicted in Figure 3.1.1. As on the gauge theory side, scattering amplitudes os string states must be properly defined, and an infrared regulator is needed. For this purpose consider the metric of AdS_5

$$\mathrm{d}s^2 = R^2 \frac{\mathrm{d}x_{3+1}^2 + \mathrm{d}z^2}{z^2} \tag{3.1}$$

and place a D3-brane extending over the coordinates x_{3+1} and localized at a large infrared point $z = z_{IR}$. The asymptotic states that we are about to scatter are open strings ending on this D3-brane. In the large N limit these strings can be considered as gluons of the $SU(N)$ SYM theory [87]. Moreover they carry proper momentum in AdS_5

[1]The author does not really understand way this paper is still being cited, though defers to the common sense.

$$k_{pr} = \frac{k\, z_{IR}}{R} \tag{3.2}$$

where k is four-dimensional momentum, which will be considered fixed in the limit where we send the regulator z_{IR} to infinity. At the end of the day we are scattering strings with very large momentum, and it turns out [88] [89] that in this limit the amplitude is dominated by the saddle point of the classical action, which means the minimal surface of the worldsheet subject to the boundary conditions in AdS_5. The worldsheet itself has the topology of a disk with vertex operators inserted at its boundary at $z = z_{IR}$. The crucial observation of [41] was that, after $T-$dualising x^μ coordinates to

$$\mathrm{d}s^2 = w^2(z)\mathrm{d}x_\mu \mathrm{d}x^\mu + ... \quad\longleftrightarrow\quad \partial_\alpha y^\mu = \mathrm{i}w^2(z)\epsilon_{\alpha\beta}\partial_\beta x^\mu \tag{3.3}$$

and redefining $r = R^2/z$, the metric of af AdS goes into itself

$$\mathrm{d}s^2 = R^2 \frac{\mathrm{d}y^2_{3+1} + \mathrm{d}r^2}{r^2} \tag{3.4}$$

The boundary of the worldsheet are $T-$dualised to a point located near the origin $r = R^2/z_{IR} \sim 0$ and boundary conditions due to the insertion of vertex operators assume the rather simple form

$$\Delta y^\mu = 2\pi k^\mu \tag{3.5}$$

Since gluons are massless, the momenta k_i are light-like. Thus, at every insertion of a vertex operator the dual coordinates get shifted by a light-like segments that equals $2\pi k_i^\mu$. This defines a polygonal loop with null edges located near the boundary of the $T-$dual metric at $r = 0$ as $z_{IR} \to \infty$, hence the computation is formally identical to the computation of the expectation value of such Wilson loop at strong coupling [38] [39]. The saddle point that dominates the string action in this regime is actually the minimal area $\mathcal{A}(k_1, .., k_n)$ of the worldsheet ending on the segments $k_1, .., k_n$ in the dual space, hence the leading exponential behaviour of both the light-like Wilson loop and the scattering amplitude reads

$$A_n \sim \mathrm{e}^{-\frac{\sqrt{\lambda}}{2\pi}\mathcal{A}(k_1,..,k_n)} \tag{3.6}$$

Note that kinematical information other then momenta of the external particles are not taken into account by this procedure. Although such information cannot alter the leading exponential, it can contribute to a prefactor depending, for instance, on the external polarisations. How to include such data in the string picture is still an open problem, which can be hopefully solved by including quantum corrections to the classical area.

3.1.2 Minimal surfaces and cusped Wilson loops

Though a semiclassical one, finding the solution to the minimal area problem is a hard task and is currently viable by analytical methods for the sole case of a four-edges loop [41]. For other cases a solution can be found in the form a set of integral equations derived by means of integrability. These equations are formally identical to the thermodynamic

Bethe equations and can be solved at least numerically through the Thermodynamic Bethe Ansatz (TBA) and the so-called $Y-$system (or $T-$system). Being integrability beyond our current purposes, we will not dip into it and we refer the reader to [90] [91] for the TBA equations, [92] [93] for the $Y-$system and other many references therein.

Let us introduce the minimal area solution for a cusped Wilson loop, which means a couple of semi-infinite null Wilson lines on a light-cone and joining in a point. The solution to this problem was first presented in [94]. It is sufficient to restrict to a AdS_3 subspace of AdS_5 where the boundary conditions imposed by the loop contour are

$$y_0 > 0, \qquad y_1 = \pm y_0, \qquad r = 0 \tag{3.7}$$

so that the metric becomes

$$\mathrm{d}s^2_{AdS_3} = \frac{-\mathrm{d}y_0^2 + \mathrm{d}y_1^2 + \mathrm{d}r^2}{r^2} \tag{3.8}$$

The classical (Nambu-Goto) action reads

$$S = \frac{R}{2\pi} \int \mathrm{d}^2\sigma \sqrt{\det(g_{\mu\nu}\eta^{ab}\partial_a X^\mu \partial_b X^\nu)} \tag{3.9}$$

writing $r = r(y_0, y_1)$ the string action can be written in an explicit form

$$S = \frac{R}{2\pi} \int \mathrm{d}y_0 \mathrm{d}y_1 \, \frac{1}{r^2} \sqrt{1 + (\partial_0 r)^2 + (\partial_1 r)^2} \tag{3.10}$$

The minimal area that extremises the classical action subject to the boundary conditions (3.7) above is readily computed

$$r(y_0, y_1) = \sqrt{2(y_0^2 + y_1^2)} \tag{3.11}$$

This is simply the are of a section of a cone ending on the two lines. Note that this area is actually unregularised, fact reflecting itself in the divergence of the classical action (3.10) computed on this solution. To overcome this issue it was argued in [41] to extend the AdS metric out of dimension in the following way

$$\mathrm{d}s^2 \sqrt{\lambda_D c_D} \left[\frac{\mathrm{d}y^2 + \mathrm{d}r^2}{r^{2+\epsilon}} \right] \tag{3.12}$$

where ϵ is a negative, small, dimensional regulator,

$$\lambda_D = \lambda \frac{\mu^{2\epsilon}}{(4\pi \mathrm{e}^{-\gamma_E})^\epsilon} \tag{3.13}$$

is the regularised coupling constant and

$$c_D = 2^4 \epsilon \pi^{3\epsilon} \Gamma(2 + \epsilon) \tag{3.14}$$

is a normalization constant. Using these conventions the infrared regularised action becomes

$$S = \frac{\sqrt{\lambda_D c_D}}{2\pi} \int \frac{\mathcal{L}_\epsilon}{r^\epsilon} \tag{3.15}$$

Such a procedure is strongly reminiscent of dimensional regularization of Feynman integrals, indeed it can be thought of as dimensional regularization at strong coupling. Though, integrals are non-isotropically taken off-dimensions, since only the internal dimensionality of AdS is modified. Moreover, solutions of the unregularised classical action (3.10) are, strictly speaking, no longer solutions of (3.15). On the other hand they are still useful in extracting the first few orders of the $\epsilon \to 0$ expansion of the regularised minimal area. To this aim it was argued in [41] that a somewhat accurate value of the minimal area can be extracted by plugging the solution of the unregularised problem into the regularised action and neglecting terms of order $\mathcal{O}(\epsilon)$ as $\epsilon \to 0$.

Getting back to our main concern, the four-cusps Wilson loop can be parametrised by a square on the boundary for $s = t$

$$r(\pm 1, y_2) = r(y_1, \pm 1) = 0, \qquad y_0(\pm 1, y_2) = \pm y_2, \qquad y_0(y_1, \pm 1) = \pm y_1 \tag{3.16}$$

being $s = (k_1 + k_2)^2$ and $t = (k_1 + k_4)^2$ ordinary Mandelstam variables for the four gluon scattering. Knowing the solution for a single cusp the generalization to four-cusps is not a big deal

$$y_0(y_1, y_2) = y_1 y_2. \qquad r(y_1, y_2) = \sqrt{(1 - y_1^2)(1 - y_2^2)} \tag{3.17}$$

At the level of the classical action (3.10), the solution for general s and t can be extracted from this particular solution by means of a conformal transformation. On the other hand, once the action has been regularised, the regulator breaks conformal invariance of the theory this argument cannot be used any more. At leading order in the 't Hooft coupling the regularised minimal area reads

$$-\frac{\sqrt{\lambda}}{2\pi}\mathcal{A}_{\text{min}} = -\frac{\sqrt{\lambda}}{2\pi}\left[\mathcal{A}_{\text{div}} + \frac{1}{2}\log^2\frac{s}{t} + \text{const}\right] \tag{3.18}$$

where the constant term is scheme dependent and the divergent term for small ϵ reads

$$\mathcal{A}_{\text{div}} = \left[-\frac{2}{\epsilon^2} - \frac{1 - \log 2}{\epsilon}\right]\left[\left(\frac{-s}{\mu^2}\right)^{-\epsilon/2} - \left(\frac{-t}{\mu^2}\right)^{-\epsilon/2}\right] \tag{3.19}$$

Note that also the subleading divergence is scheme dependent, and that, as was shown in [95], the regularisation procedure of [41] must be modified at subleading orders in $1/\sqrt{\lambda}$.

3.2 Scattering amplitudes and Wilson loops at weak coupling

3.2.1 Perturbative expansion of cusped Wilson loops

Since their first introduction in the contest of QCD quark confinement [9], Wilson loops have been the subject of massive study in field theory. There they carry information about the interaction between a massive quark moving along a certain contour $\mathcal{C}$ and interacting with the gluon field A^μ. The Wilson loop is constructed by taking the path-ordered exponential of the holonomy of the gluon field along the contour

$$W(\mathcal{C}) = \frac{1}{N}\left\langle 0|\mathrm{Tr}\mathcal{P}\mathrm{e}^{\mathrm{i}g\oint_{\mathcal{C}} A_\mu \mathrm{d}x^\mu}|0\right\rangle \tag{3.20}$$

where the trace is over the generators of the gauge group, $\mathcal{P}$ is the path-ordering operation which arranges group indices of $SU(N)$ along the integration contour $\mathcal{C}$ and $1/N$ is a normalization constant for the loop in the fundamental representation. Expanding the expectation value above for small values of the coupling constant g one recovers the full perturbative series

$$W(\mathcal{C}) \sim 1 + \mathrm{i}g\oint_{\mathcal{C}} A_\mu \mathrm{d}x^\mu + (\mathrm{i}g)^2 \oint_{\mathcal{C}} \mathrm{d}x^\mu \int_{x_j > x_i} \mathrm{d}x^\nu A_\mu(x_i) A_\nu(x_j) + \dots \tag{3.21}$$

Note that the path-ordering prescription precisely cancels the combinatoric factor coming from the expansion of the exponential in (3.20). Such Wilson loops, when computed on smooth contours, are perfectly well defined object whose only potential divergence is linear in the length L of the contour

$$W(\mathcal{C}) = \mathrm{e}^{-K\,L(\mathcal{C})} \times \text{finite} \tag{3.22}$$

and can be interpreted as the mass renormalisation of the heavy particle moving along $\mathcal{C}$ [96] [97]. This renormalisation does not depend on potential multiplicative renormalisations of the fields in the theory due to UV divergences which do not depend on the contour but are a global features of the theory itself [67], we therefore assume in the following that global renormalisation of fields have already been carried out. For example, the one loop correction to (3.21) corresponding to the exchange of a single gluon reads

$$\begin{aligned} W^{(1)}(\mathcal{C}) =\sim (\mathrm{i}g)^2 \oint_{\mathcal{C}} \mathrm{d}x_i^\mu \int_{x_j > x_i} \mathrm{d}x_j^\nu \frac{\eta_{\mu\nu}}{(x_i - x_j)^2 + a^2} \\ = (\mathrm{i}g)^2 \oint_{\mathcal{C}} \oint_{\mathcal{C}} \frac{\dot{x}(s)\cdot \dot{x}(s+t)\mathrm{d}s\mathrm{d}t}{[x(s+t) - x(s)]^2 + a^2} \end{aligned} \tag{3.23}$$

where $x(s)$ parametrises the contour and a cutoff a^2 has been introduced to regularised contact divergences emerging when the two ends of the propagator pinch over each other. There is better defined way of regulating such divergences and which is known as *framing*, we will be back on that in the following. Here the divergence comes entirely from the region where $t \sim 0$, then we can write

$$W^{(1)}(\mathcal{C}) \sim \oint \mathrm{d}s\, \dot{x}^2(s) \int_0^\Lambda \mathrm{d}t\, \frac{1}{\dot{x}^2(t) + a^2} + \text{finite}$$
$$= \frac{\pi}{a} \int \mathrm{d}s\, \sqrt{\dot{x}^2(s)} + \text{finite} = \pi \frac{L(\mathcal{C})}{a} + \text{finite} \quad (3.24)$$

which is in agreement with (3.22).

We are mostly interested in loops which are not smooth, which have cusps indeed. In such case additional UV divergences appear, localised at each cusp point and which depend in a non trivial way on the local details of the contour. Analogous arguments hold for loops having self-intersections [98]. For a Wilson loop operator in the fundamental representation of $SU(N)$ along a contour with a cusp angle θ the one loop correction reads

$$W(\mathcal{C}) = 1 - 2g^2 C_F \left[\theta \cot\theta - 1\right] \log \frac{L}{a} \quad (3.25)$$

where L is the length of the loop, a the contact divergence regulator and C_F the quadratic Casimir operator in the fundamental representation [99]. At any order of perturbation theory such divergences can be removed by multiplicative renormalisation of the Wilson loop

$$W_R(\mathcal{C}) = Z(\theta) W(\mathcal{C}) \quad (3.26)$$

and the new divergences depend only locally on the geometrical data of the loop (like θ) through $Z(\theta)$. This property can be rephrased in the statement that for such Wilson loops *there is no anomalous divergence* depending non-locally on the contour.

For Wilson loops with cusp a renormalization group equation can be derived [67]

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(g_R) \frac{\partial}{\partial g_R} + \Gamma_{\text{cusp}}(\theta, g_R)\right] W_R = 0 \quad (3.27)$$

where g_R is the renormalised coupling constant, μ is a scheme dependent mass scale and

$$\Gamma_{\text{cusp}}(\theta, g_R)) = \lim_{\epsilon \to 0} Z\, \mu \frac{\partial}{\partial \mu} Z^{-1} \quad (3.28)$$

From the two equations above one can derive a renormalization group equation for Z itself

$$\left[\beta(g_R, \epsilon) \frac{\partial}{\partial g_R} - \Gamma(\theta, g_R, \epsilon)\right] Z^{-1}(\theta, g_R, \epsilon) = 0 \quad (3.29)$$

whose solution is

$$Z^{-1}(\theta, g_R, \epsilon) = \exp\left[\int_0^{g_R} \mathrm{d}g' \, \frac{\Gamma(\theta, g', \epsilon)}{\beta(g', \epsilon)}\right] \tag{3.30}$$

This integral can be done explicitly in some cases. As an example, in $\mathcal{N} = 4$ SYM in $D = 4 - 2\epsilon$ dimensions and with the beta function $\beta(g_R, \epsilon) = g_R\epsilon$ one can show that its value takes the form

$$Z(\theta, g_R, \epsilon) = \left[\sum_{n=1}^{\infty} \frac{g^{2n}}{2n} \frac{\Gamma^{(n)}(\theta, \epsilon)}{\epsilon}\right] \tag{3.31}$$

which is a manifestation of the fact that the renormalisation group equation (3.27) implies the exponentiation of the divergent part of the Wilson loop.

3.2.2 Polygonal null Wilson loops

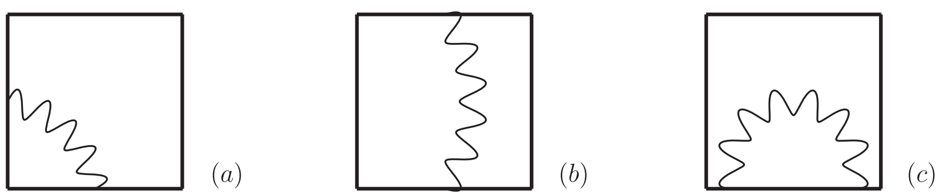


Figure 3.2: One loop graphs contributing to the expectation value of the four-edges Wilson loop in $\mathcal{N} = 4$ SYM. The divergent contribution is solely imputable to the cusp diagrams of type (a), whereas propagators ectending between more than two edges give a finite term. The last class of diagrams vanish.

Note that additional divergences can emerge in (3.25) for infinite cusp angle θ. Although this is not possible in Euclidean spacetime, it is precisely what happens in Minkowski signature to polygonal Wilson loops with light-like edges, where the hyperbolic angle between two null segments is infinite. These are indeed the kind of operators that are related to gluon scattering amplitudes by the strong coupling argument reviewed in Section 3.1, and thus we will examine them in some details in the following.

In the context of the AdS/CFT correspondence the objects that arise naturally are supersymmetric extensions of the purely gluonic Wilson loop introduced in (3.20). In analogy to the interpretation given in [9], one can construct these objects through a Higgsing of the vacuum that generates certain supersymmetric particles, the W-bosons of the theory, and extract the structure of the loop operator from the dynamics of these test particles in the limit where their mass is much bigger then the energy scale of all processes. This is outlined in Section 4.2.1 for the supersymmetric Chern-Simons-matter theory with $\mathcal{N} = 6$ supersymmetry in three dimensions.

Supersymmetric Wilson loops also involve coupling of the countour to scalars and sometimes fermions. The simplest among these objects is the Maldacena-Wilson loop of $\mathcal{N}=4$ SYM [39]

$$W(\mathcal{C}) = \frac{1}{N}\langle 0|\mathrm{Tr}\mathcal{P}\exp\left(\mathrm{i}g\oint_{\mathcal{C}}\mathrm{d}s\left[\dot{x}^\mu A_\mu + |\dot{x}|\theta^I\phi_I\right]\right)|0\rangle \tag{3.32}$$

where the θ^I coordinates parametrise the internal S^5 of $AdS_5\times S^5$ and are normalised according to $\theta^I\theta_I=1$. Depending on the shape of the contour these operators can preserve a certain amount of the supersymmetry of the vacuum, therefore are called BPS operators. For example, if the contour is an infinite straight line or a circle it was shown [100] [101] that loop corrections to the weak coupling expansion of (3.32) vanish at any order

$$W(\text{straight line}) = 1 \tag{3.33}$$

and the VEV of such operators preserved half of the supersymmetry of the vacuum, so that they are $\frac{1}{2}$ BPS Wilson loops. Also 1/4 BPS operators are known, see [102] [103] [104].

Supersymmetric Wilson loops are also known in other theories. In particular we will extensively use the 1/6 and 1/2 BPS operators of $\mathcal{N}=6$ super Chern-Simons theories introduced respectively in [44] and [46].

Polygonal contours with null edges have been first considered in [105]. Since $|\dot{x}|=0$ holds in such cases, scalars (and eventually fermions) decouple from the loop (3.32) making it completely equivalent to the purely bosonic operator. We parametrise null contours $\mathcal{C}_n$ by a set of n segments $p_i = x_{i+1}-x_i$ joining at cusp points x_i

$$\mathcal{C}_n = \bigcup_{i=1}^n p_i\,, \qquad x^\mu(s_i) = x_i^\mu + (x_{i+1}-x_i)^\mu s_i\,, \qquad s_i\in[0,1] \tag{3.34}$$

where all the p_i are obviously light-like $p_i^2=0$. Cusps cause Wilson loops to develop UV singularities, as was shown in the previous section. The fact that the contour approaches the light-cone in the neighbourhood of the cusp makes these divergences even more severe [105] [106]. Let us start from the first nontrivial term in the perturbative expansion of a light-like Wilson loop

$$W^{(1)}(\mathcal{C}_n) = (\mathrm{i}g)^2 C_F \oint_{\mathcal{C}_n}\mathrm{d}x_i^\mu \mathrm{d}x_j^\nu D_{\mu\nu}(x_i-x_j) \tag{3.35}$$

where $D_{\mu\nu}(x)$ is the gluon propagator in coordinate space, dimensionally regularised in $d=4-2\epsilon$, with $\epsilon>0$; in Feynman gauge it reads

$$D_{\mu\nu}(x) = -(\mu^2\mathrm{e}^{-\gamma_E})^\epsilon\frac{\Gamma(1-\epsilon)}{4\pi^2}\frac{\eta_{\mu\nu}}{(-x^2)^{1-\epsilon}} \tag{3.36}$$

When the gluon propagator approaches the cusp, the integrand in equation (3.35) becomes singular. Diagrams contributing to the divergent part of the integral are depicted in Figure 3.2.2 (a), while other kinds of diagrams either vanish (c) or give a finite contribution (b).

Note that the latter is regularization dependent. The contribution of a single cusp diagram is given by

$$\begin{aligned} W(p_i, p_i+1) &= -\frac{g^2 C_F}{4\pi^2}(\mu^2 \mathrm{e}^{-\gamma_E})^{\epsilon}\Gamma(1-\epsilon)\int_0^1 \mathrm{d}s_i \int_0^1 \mathrm{d}s_{i+1}\, \frac{p_i \cdot p_{i+1}}{(p_i s_i + p_{i+1}s_{i+1})^2} \\ &= -\frac{g^2 C_F}{4\pi^2}(\mu^2 \mathrm{e}^{-\gamma_E} x^2_{i,i+2})^{\epsilon}\frac{\Gamma(1-\epsilon)}{2\epsilon^2} \end{aligned} \tag{3.37}$$

where $x_{i,j} = x_j - x_i = p_i + p_{i+1} + ... + p_{j-1} = s_{i,j-1}$ is a Mandelstam kinematical invariant rephrased. Adding al cusps of a $n-$edges polygon amounts to the following divergent contribution to the one-loop correction

$$W(\mathcal{C}_n) = 1 + \frac{g^2 C_F}{4\pi^2}\left[-\frac{1}{2\epsilon^2}\sum_{i=1}^{n}(-x^2_{i-1,i+1}\mu^2)^{\epsilon} + \text{finite}\right] + \mathcal{O}(g^4) \tag{3.38}$$

The structure of divergences is somehow similar to the space-like case reviewed before (3.25), but the divergence is a double pole in ϵ rather than a single pole. This fact is due to the light-like condition which worsen the behaviour of the propagator near the cusp point. At higher loop level it is known that singularities appear as poles of order less or equal to the number of loops [107] [105]. This is a consequence of exponentiation of divergences at any loop order. Thus the expectation value of the Wilson loop can be split in a singular plus a finite contribution

$$\log W(\mathcal{C}_n) = Z_n + F_n \tag{3.39}$$

and from exponentiation it follows the structure of the divergent part

$$Z_n = -\frac{1}{4}\sum_{l\geq 1}\lambda'^l \sum_{i=1}^{n}(-x^2_{i-1,i+1}\mu^2)^{l\epsilon}\left[\frac{\Gamma^{(l)}_{\text{cusp}}}{(l\epsilon)^2} + \frac{\Gamma^{(l)}}{l\epsilon}\right] \tag{3.40}$$

where $gl' = \frac{g^2 N}{8\pi^2}$ is the (scaled) 't Hooft coupling while Γ_{cusp} and Γ respectively are the so-called cusp anomalous and collinear anomalous dimensions already encountered in the evaluation of the four-gluon MHV amplitude (2.20) in Section 2.2.1

$$\begin{aligned} \Gamma_{\text{cusp}}(\lambda') &= \sum_{l\geq 1}\lambda'^l \Gamma^{(l)}_{\text{cusp}} = 2\lambda' - \frac{\pi^2}{3}\lambda'^2 + \mathcal{O}(\lambda'^3) \\ \Gamma(\lambda') &= \sum_{l\geq 1}\lambda'^l \Gamma^{(l)} = -7\zeta_3\lambda'^2 + \mathcal{O}(\lambda'^3) \end{aligned} \tag{3.41}$$

The first few orders of their expansion in $\mathcal{N} = 4$ SYM have been listed for completeness.

3.2.3 Loops/amplitudes duality

The string construction of [41] motivated the investigation of a possible weak coupling relation between gluon MHV scattering amplitudes $M_n^{(MHV)}$ and light-like Wilson loops $W(C_n)$. From the strong coupling picture one would expect that $1/\sqrt{\lambda}$ corrections would spoil the duality relation at weak coupling. Nevertheless it was first found [76] to be true at one-loop in $\mathcal{N}=4$ SYM for the four-points Wilson loop and the four-gluon amplitudes, and soon after a confirmation arrived from the one-loop, six-points MHV amplitude and the hexagon loop [17]. More recently it was also shown to hold in three-dimensional $\mathcal{N}=6$ super Chern-Simons theory, where an intriguing similarity with the SYM result was also observed [18] [21] [20], we will get back to this point later on.

It was already known that divergences of scattering amplitudes in non-Abelian theories can be rephrased in terms of Wilson loops [108] [109] [110], more precisely they factorise in products of form factors which are in correspondence with Wilson lines at an angle [111]. This factorisation is also responsible for the exponentiation of double and single poles at each order of perturbation theory [112].

The duality relation that was observed up to two loops in $\mathcal{N}=4$ SYM goes deeper. It associates to a scattering amplitude with null external momenta p_i a polygonal contour defined by the points

$$x_{i+1}^{\mu} = x_i^{\mu} + p_i^{\mu} \tag{3.42}$$

in a dual spacetime, and where the x_1 is chosen freely, and conjectures an equivalence between the finite part of the gluon amplitude and the finite part of the corresponding Wilson loop

$$F_n^{(MHV)} = F_n^{(WL)} + \text{constant} \tag{3.43}$$

up to an irrelevant additive constant. Note that this relation is made much more precise by the absence of any regulator or cutoff since it entails really finite objects. This is in contrast with the duality between divergent parts in that the these explicitly depend on some regulator. More precisely, scattering amplitudes are IR divergent while loops are UV divergent. They are defined in different regularisation schemes. Though, leading divergent terms on the two sides coincide when defined in dimensional regularisation, since both are dominated by the cusp anomalous dimension Γ_{cusp}. Subleading singularities are scheme dependent and carry non-universal coefficients which are the collinear anomalous dimensions. Matching of these coefficients is a subtle matter [17].

Before proceeding with a more accurate analysis of the implications of this duality, let us show how $\mathcal{N}=4$ SYM rectangular Wilson loop with null edges reproduces the four-points MHV scattering amplitude of (2.19). This particular loop has also been computed at two-loops [105] [107] in the kinematical region where $x_{13}^2 = x_{24}^2$ known as the *forward limit* (in this limit the loop reduces to a square). It is quite interesting to note that it explicitly emerges from the computation a guide-principle known as *maximal trascendentality principle* [113]. Let us go back to one-loop example. Following [76], the rectangular Wilson loop reads to the lowest non-trivial order

$$W(\mathcal{C}_4) = 1 + \frac{1}{2}(\mathrm{i}g)^2 C_F \oint \mathrm{d}x_i^\mu \mathrm{d}x_j^\nu D_{\mu\nu}(x_i - x_j) \tag{3.44}$$

where $D_{\mu\nu}$ is the gluon propagator that was written in Feynman gauge in (3.35), $C_F = \frac{N^2-1}{2N}$ is again the quadratic Casimir of $SU(N)$ and the contour $\mathcal{C}_4$ is defined by light-like vectors $x_{i+1} - x_i = p_i$, $p_i^2 = 0$ (upon which an evident cyclicity condition is imposed). For this simple case the kinematical invariants x_{13}^2 and x_{24}^2 defined in the previous section correspond to the Mandelstam variable s, t respectively

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = x_{13}^2 \qquad t = (p_2 + p_3)^2 = (p_1 + p_4)^2 = x_{24}^2 \tag{3.45}$$

There are three types of diagrams contributing to this order as depicted in Figure 3.2.2. Diagrams (c) with both ends attached to the same edge of the contour vanish due to the light-likeness of the edge itself. Cusp diagrams (a), which means diagrams where the propagator extends across a cusp, were computed in (3.35), and applied to this specific case they give

$$W(\text{cusps}) = -\frac{g^2 C_F}{4\pi^2} \frac{\Gamma(1-\epsilon)}{\epsilon^2} \left[(-\mathrm{e}^{-\gamma_E} \mu^2 s)^\epsilon + (-\mathrm{e}^{-\gamma_E} \mu^2 t)^\epsilon \right] \tag{3.46}$$

which is the divergent part Z_n introduced in (3.39). The remaining diagrams (b) give a contribution amounting to

$$\begin{aligned} I(p_2, p_4) &= -\frac{g^2 C_F}{4\pi^2} \int_0^1 \frac{\mathrm{d}s_2 \mathrm{d}s_4 \, (p_2 \cdot p_4)}{(p_2 s_2 + p_4 s_4 + p_3)^2} = \frac{g^2 C_F}{8\pi^2} \int_0^1 \frac{\mathrm{d}s_2 \mathrm{d}s_4 \, (s+t)}{t s_2 + s s_4 - (s+t) s_2 s_4} \\ &= -\frac{g^2 C_F}{8\pi^2} \int_0^1 \frac{\mathrm{d}s_2}{s_2 - \frac{s}{s+t}} \left[\log \frac{s}{t} + \log \frac{1-s_2}{s_2} \right] = \frac{g^2 C_F}{16\pi^2} \left[\log^2 \frac{s}{t} + \pi^2 \right] \end{aligned} \tag{3.47}$$

The other integral $I(p_1, p_3)$ gives the same contribution once momentum conservation is taken into account. The sum of the two is the first order contribution to the finite part F_n of (3.39) including the additive constant. Note that the last integral above is well defined in $D = 4$ dimensions and therefore the dimensional regulator ϵ has been removed. Putting finite and divergent parts together and multiplicatively redefining μ one finds

$$W(\mathcal{C}_4) = 1 + \frac{g^2 C_F}{4\pi^2} \left\{ -\frac{1}{\epsilon^2} \left[(-\mu^2 x_{13}^2)^\epsilon + (-\mu^2 x_{24}^2)^\epsilon \right] + \frac{1}{2} \log^2 \frac{x_{13}^2}{x_{24}^2} + \frac{\pi^2}{3} \right\} \tag{3.48}$$

We can rewrite the finite contribution to the logarithm of the four-edges Wilson loop in the planar limit where $N \to \infty$ with $\lambda = g^2 N$ is held fixed

$$\log W(\mathcal{C}_4) = F_4^{(WL)} = \frac{\lambda}{4\pi^2} \left\{ \frac{1}{2} \log^2 \frac{x_{13}^2}{x_{24}^2} + \frac{\pi^2}{3} \right\} + \mathcal{O}(\lambda^2) \tag{3.49}$$

and compare it with the finite part of the four-point MHV gluon amplitude (2.19) and the divergent term Z_4 above with the cusp anomalous dimension (3.41). We see that there is a neat agreement with the duality conjecture.

3.3 Loops/amplitudes duality in $\mathcal{N}=6$ SCS

The duality between light-like Wilson loops and MHV scattering amplitudes seems not to be a peculiar feature of $\mathcal{N}=4$ SYM. Indeed it has also been observed in the context of the three dimensional $\mathcal{N}=6$ super Chern-Simons-matter theory introduced in [36] and which is conjectured to be dual to type IIA superstrings on $AdS_4\times\mathbb{C}P^3$. This quiver theory with gauge group $U(N)\times\overline{U(N)}$, commonly referred to as ABJM theory, includes two adjoint CS fields A and $\widehat{A}$, four fermions ψ^I_α and four complex scalars C_I respectively in the bifundamental and anti-bifundamental of $U(N)\times\overline{U(N)}$. All the details about the Lagrangean can be found in Appendix A.2.

The Yangian symmetry related to the planar integrability of $\mathcal{N}=4$ SYM was also shown to be responsible for the striking duality between null Wilson loops and MHV scattering amplitudes [13]. In the ABJM case, first hints of integrability came from the formulation of the Bethe equations for the ABJM spin chain [22] [23] [24] [25]; then the associated Yangian symmetry was identified [26], which at level one generates the three-dimensional superconformal symmetry of three-level scattering amplitudes [83].

Soon after, it was shown that the one-loop four and six-cusps light-like Wilson loops in ABJM vanish, whereas the two-loop expression for the four-cusps Wilson loop reads [18]

$$\langle W_4\rangle^{(2)}=\lambda^2\left[-\frac{(-\mu_{UV}^2x_{13}^2)^{2\epsilon}}{(2\epsilon)^2}-\frac{(-\mu_{UV}^2x_{24}^2)^{2\epsilon}}{(2\epsilon)^2}+\frac{1}{2}\log^2\left(\frac{x_{13}^2}{x_{24}^2}\right)+C+\mathcal{O}(\epsilon)\right] \tag{3.50}$$

remakably reproducing the one-loop $\mathcal{N}=4$ SYM result up to an identification of the coupling constants. The duality was then shown to hold at least at two-loops for the four-cusps loop; in [19] it was shown that the all $n-$points scattering amplitudes vanish at one-loop and subsequently it was computed the four-point amplitude at two-loops in $\mathcal{N}=2$ superspace formulation [20] and independently in [21] using generalized unitarity methods

$$\mathcal{M}^{(2)}\equiv\frac{\mathcal{A}_4^{(2\,loops)}}{\mathcal{A}_4^{tree}}=\lambda^2\left[-\frac{(s/\mu_{IR}^2)^{-2\epsilon}}{(2\,\epsilon)^2}-\frac{(t/\mu_{IR}^2)^{-2\epsilon}}{(2\,\epsilon)^2}+\frac{1}{2}\ln^2\left(\frac{s}{t}\right)+\text{const}+\mathcal{O}(\epsilon)\right] \tag{3.51}$$

Note that as pointed out in [20], the closest three-dimensional thing to a four-dimensional gluon MHV amplitude is the superamplitude with two bifundamental and two anti-bifundamental legs. Then, expressing momenta of external particles in the dual coordinate frame $p_i=x_{i+1}-x_i$, the duality relation

$$\ln\mathcal{M}_4=\ln\langle W_4\rangle+\text{const.} \tag{3.52}$$

is verified up to the identification of the UV and IR regulator and up to an irrelevant additive constant. Note the intriguing similarity with the one-loop $\mathcal{N}=4$ SYM counterpart in equations (3.48) and (2.19), which also implies that the ABJM two-loop amplitude is proportional to the tree-level times a function depending on kinematical invariants. This similarity also inspired the authors of [20] to formulate a BDS-like ansatz for the four-points amplitude in ABJM

$$\frac{\mathcal{A}_4}{\mathcal{A}_4^{tree}} = \mathrm{e}^{Div+\frac{\Gamma^{CS}_{\mathrm{cusp}}(\lambda)}{8}\left(\ln^2\left(\frac{s}{t}\right)+\mathrm{const}\right)+\Gamma^{CS}(\lambda)} \tag{3.53}$$

In analogy with the four-dimensional case (2.23) [75], $Div \equiv Z_4$ is the divergent part of the scattering amplitude, which is governed by the cusp anomalous dimension and is known to exponentiate at all orders, $\Gamma^{CS}_{\mathrm{cusp}}$ is the three-dimensional cusp anomalous dimension (scaling function) as obtained from asymptotic Bethe equations [24] up to λ^4, and Γ^{CS} is the (scheme-dependent) three-dimensional equivalent of the collinear anomalous dimension. From the previous discussion about the BDS ansatz in four dimensions, it is also clear that this expression should not receive corrections from any possible reminder function, and if so, is perturbatively exact.

CHAPTER 4

BPS Wilson Loop Operators in $\mathcal{N} = 6$ SCS Theories

Wilson loop operators preserving some supersymmetry of the vacuum are fundamental objects in any supersymmetric gauge theory. As remarked in the previous sections, the great interest they hold is partially due to the fact that they represent non-local and gauge invariant degrees of freedom, by means of which the gauge theory can be conveniently reformulated. But a more intriguing role is that they cover in the context of the AdS/CFT correspondence. On the one hand they are conjectured to be the gauge theory dual objects to supersymmetric string solutions, and on the other hand, through this strong/weak coupling duality, Wilson loops with light-like edges (formulated in a dual space) are believed to share the same structure of gluon scattering amplitudes, both perturbatively and non-perturbatively.

The search of such protected quantities in the context of supersymmetric Chern-Simons theories has been a puzzling topic for sometime. Inspired by the four-dimensional $\mathcal{N} = 4$ SYM half BPS operator of [39] [38], two parallel and equivalent proposals where made in [44] and [45] for a one-sixth BPS operator in $\mathcal{N} = 6$ SCS, and both are closely related to half BPS operator of $\mathcal{N} = 2$ SCS [114]. When calculated over an infinite straight line or a circle these operators were shown to preserve 4 out of the 24 supercharges of the theory. Quite remarkably, and in a certain sense oppositely to the SYM case, it turns out that the most supersymmetric Wilson loop operator has the structure of a superconnection which also needs to be coupled to the fermions on the contour to preserver half of the supercharges of the theory [46]. This previously unobserved feature can be understood from the low-energy dynamics of massive BPS W-bosons in the Higgsed phase of the theory, which uncovers the underlying supergroup structure. This last fact indeed admits a nice derivation in the dual M-theoretic picture [47].

Superconformal symmetry of $\mathcal{N} = 6$ SCS theories plays an important role in constraining the structure of supersymmetric loops. It was already observed in [101] that the $\frac{1}{2}$ BPD

Wilson line of four-dimensional $\mathcal{N}=4$ SYM does not receive loop corrections, as one would expect from the notion of supersymmetry-protected operators. On the other hand this is not the case for the circular Wilson loop, which is conformally equivalent to the infinite straight line. This misunderstanding arises from a mixing of the Poincare and conformal supercharges. To this aim notice that the infinite Wilson line preserves separately half of the Poicare supercharges and half of the conformal ones, whereas the circular loop preserves a mixing of them. In this sense the loop still is a half BPS operator, but *finite* contributions arise from the conformal mapping of boundary conditions. Explicit examples are given in the context of ABJ(M) theory.

4.1 Gauge theory construction of the one-sixth BPS operator

Superconformal three-dimensional Chern-Simons-matter theory with $\mathcal{N}=6$ supersymmetry and $U(N)_k \times \overline{U(N)}_{-k}$ gauge group was proposed to be dual to the theory of type IIA superstrings on the ten-dimensional background $AdS_4 \times \mathbb{C}P^3$ [36] (Appendix A.1). We will refer to this theory as ABJM theory, as is commonly done. The Chern-Simons level k takes only integer values and the quiver construction puts two different CS fields in the adjoint representation of the two gauge groups and at opposite levels k and $-k$. The theory includes scalars C_I and fermions ψ^I which transform respectively in the bifundamental, anti-bifundamental of $U(N)_k \times \overline{U(N)}_{-k}$, and carry a $R-$index $I=1,2,3,4$ of the $SU(4)_R$ symmetry group (please see Appendix A.2 for all the details). In the planar limit where N is large, the t' Hooft coupling $\lambda = \frac{N}{k}$ interpolates between weakly coupled ABJM theory for large k, $\lambda \ll 1$ and $AdS_4 \times \mathbb{C}P^3$ strings for $\lambda \gg 1$. Remind that, as mentioned in Appendix A.1, strings are a good description of strongly coupled ABJM theory for $k \ll N \ll k^5$, while beyond this region one has to consider the theory of M2 branes on $AdS_4 \times \mathbb{C}^4/\mathbb{Z}_k$ [36] [115].

Inspired by the four-dimensional Maldacena-Wilson loop [39] (3.32)

$$W(\mathcal{C}) = \frac{1}{N}\langle 0|\mathrm{Tr}\mathcal{P}\exp\left(\mathrm{i}g\oint_{\mathcal{C}} \mathrm{d}s\,\left[\dot{x}^\mu A_\mu + |\dot{x}|\theta^I\phi_I\right]\right)|0\rangle \tag{4.1}$$

where ϕ_I are the $\mathcal{N}=4$ scalars and the θ^I coordinates parametrise the internal S^5 of $AdS_5 \times S^5$, it was proposed [44] [45] to consider the following loop operator in ABJM

$$W(\mathcal{C}) = \frac{1}{N}\langle 0|\mathrm{Tr}\mathcal{P}\exp\left(\mathrm{i}\oint_{\mathcal{C}} \mathrm{d}s\,\left[\dot{x}^\mu A_\mu + \frac{2\pi}{k}|\dot{x}|M_J^I C_I \overline{C}^J\right]\right)|0\rangle \tag{4.2}$$

Differently from the four-dimensional counterpart, gauge fields are in the adjoint representation $A_\mu = A_\mu^a t^a$, where t^a are the generators of the $U(N)$ algebra in the adjoint representation. Consequently the scalar term must also transform in the adjoint, which is acheived by the quadratic coupling $M_J^I C_I \overline{C}^J$ of bifundamental and one anti-bifundamental scalar fields $C_I = (C_I)_{ij} T^i \overline{T}^j$ and $\overline{C}^J = (\overline{C}^J)_{kl} \overline{T}^k T^l$. The matrix M_J^I essentially determines the supersymmetric properties of (4.2) and in principle may depend on the contour $\mathcal{C}$ parametrised by $x(s)$. Finally lower-case Greek indices are $\mathbb{R}^{2,1}$ Lorentz indices and $I, J = 1,2,3,4$ are the $R-$indices.

The functional form of the scalar matrix coupling M_J^I is determined by the requirements that the operator (4.2), together with a suitable choice of contour $x(s)$, preserve some of the $\mathcal{N}=6$ supersymmetries of the ABJM theory. These in turn are generated by 12 Poicare supercharges $(Q_{IJ})_\alpha$ with spinor index $\alpha = 1,2$, antisymmetric in $I, J = 1,2,3,4$, and 12 Conformal supercharges $(S_{IJ})_\alpha$. The supersymmetry variation of the Wilson loop, according to the superconformal transformation of [116] (reviewed in Appendix A.3) reads [44] [45]

$$\begin{aligned}\delta W =& 2\theta_\alpha^{IJ}\left[-\dot{x}_\mu(\sigma^\mu)_{\alpha\beta}\delta_I^P + |\dot{x}|\delta_{\alpha\beta}M_I^P\right]C_P\psi_{J\beta}\\ &+ 2\epsilon_{IJKL}\theta_\alpha^{IJ}\left[\dot{x}_\mu(\sigma^\mu)_{\alpha\beta}\delta_P^K + |\dot{x}|\delta_{\alpha\beta}M_P^K\right]\overline{\psi}_\beta^L\overline{C}^P\end{aligned} \tag{4.3}$$

where the two-component spinors θ_α^{IJ} parametrise the supercharges $(Q_{IJ})_\alpha$ above. Note that this is a pointwise relation! The simplest case in which (4.3) can be seen to vanish is for an infinite straight contour and a contour-independent matrix M_I^J. In this case the spinors θ can be decomposed into eigenvectors of the chirality projector along the line

$$P_\pm = \frac{1}{2}(1+\dot{x}\cdot\sigma), \qquad \theta_\pm = P_\pm\theta \tag{4.4}$$

where $x^\mu(s) = \dot{x}^\mu s$, and accordingly the condition $\delta W = 0$ generates the two separate conditions

$$\begin{aligned}&\theta_+^{IJ}(-\delta_I^P + M_I^P) + \theta_-^{IJ}(\delta_I^P + M_I^P) = 0\\ &\epsilon_{IJKL}\theta_+^{IJ}(\delta_P^K + M_P^K) + \epsilon_{IJKL}\theta_-^{IJ}(-\delta_P^K + M_P^K) = 0\end{aligned} \tag{4.5}$$

These in turn can be solved [44] for two independent supercharges parametrized by θ_+^{12} and θ_-^{34} by choosing the following diagonal matrix, which can always be done thanks to unitarity

$$M_J^I = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix} \tag{4.6}$$

A totally analogous reasoning can be repeated for the variation of the Wilson loop under the action of the conformal supercharges S^{IJ} parametrized by spinors η^{IJ} that were constructed in [117], indeed the computation boils down to the replacement $\theta^{IJ} \to \dot{x}\cdot\sigma\eta^{IJ}$. This shows that the Wilson loop operator (4.2) coupled to an infinite straight line preserves 4 out of the 24 supercharges of ABJM theory and hence is $\frac{1}{6}$ BPS.

Several points deserve at least a brief discussion at this stage. As emerged from the analysis above, the structure of the scalar coupling M_J^I is determined by the choice of supercharges one wishes to preserve. This matrix breaks the $SU(4)_R$ symmetry of the vacuum to

$SU(2)\times SU(2)$, there is then an additional rotation acting on C_1, C_2 and C_3, C_4 which does not alter the Wilson loop operator. The supercharges are in an anti-symmetric representation of $SU(4)_R$ and are therefore neutral under this purely bosonic symmetry. Analysing the pattern of symmetry breaking from the string side [45], one can convince oneself that a fundamental string solution preserving one-sixth of the supersymmetry would be smeared along a $\mathbb{C}P^1 \in \mathbb{C}P^3$, thus braking the $R-$symmetry to $SU(2)\times SU(2)$, which is in agreement with what said above.

Conformally mapping the infinite straight line to a circle preserves the same amount of supersymmetry of (4.3), but the mapping would mix super-Poicare and superconformal charges, therefore the circular Wilson loop is invariant under some linear combination of Q^{IJ} and S^{IJ}. A possibly confusing fact arises here. The $\frac{1}{6}$ BPS Wilson line does not receive loop corrections, which fact is in agreement with the common interpretation of BPS objects as supersymmetry protected ones. Though, the same is not true for the circle Wilson loop; indeed a direct perturbative computation [44] shows that, at two-loops, it receives a purely topological term from the Chern-Simons vertex plus the contribution of scalar and gluon exchange diagrams

$$W(\text{circle}) = 1 - \frac{\pi^2 N^2}{6k^2}^{(top)} + \frac{\pi^2 N^2}{k^2}^{(exch)} \tag{4.7}$$

In this case, finite loop corrections are due to the conformal mapping of a whole class of diagrams that vanish in the infinite line case to non-vanishing ones in the circle case [101].

Note that another suitable supersymmetric Wilson loop would be

$$\overline{W}(\mathcal{C}) = \frac{1}{N}\langle 0|\mathrm{Tr}_{\overline{R}}\mathcal{P}\exp\left(\mathrm{i}\oint_{\mathcal{C}} \mathrm{d}s \left[\dot{x}^\mu \widehat{A}_\mu + \frac{2\pi}{k}|\dot{x}|\widehat{M}^I_J \overline{C}^J C^I\right]\right)|0\rangle \tag{4.8}$$

where $\widehat{A}_\mu = \widehat{A}^a_\mu \overline{T}^a$ is the second CS field in some representation $\overline{R}$ of its own gauge group $\overline{U(N)}$. Under a suitable choice of $\widehat{M}$ this operator will preserve the same supercharges of (4.2), therefore the most general $\frac{1}{6}$ BPS Wilson loop is a linear combination of the two

$$W_{R\widehat{R}} \sim \mathrm{Tr}_R \mathcal{P} e^{\mathrm{i}\oint_{\mathcal{C}} \mathrm{d}s\left[\dot{x}^\mu A_\mu + \frac{2\pi}{k}|\dot{x}|M^I_J C_I \overline{C}^J\right]} \pm \mathrm{Tr}_{\widehat{R}}\mathcal{P} e^{\mathrm{i}\oint_{\mathcal{C}} \mathrm{d}s\left[\dot{x}^\mu \widehat{A}_\mu + \frac{2\pi}{k}|\dot{x}|\widehat{M}^I_J \overline{C}^J C^I\right]} \tag{4.9}$$

labelled by two Young tableu for any two representations R and $\widehat{R}$ of $U(N)$. Generalising the last expression to ABJ theory, where the two gauge groups have different rank, is straightforward.

4.2 The one-half BPS operator

A supersymmetric Wilson loop operator that preserves half of the $\mathcal{N}=6$ supercharges of ABJ(M) theory was proposed a few time ago by Drukker ad Trancanelli in [46]. The $\frac{1}{2}$ BPS character of this loop operator relies on the extention of the ABJ gauge group

$U(N)\times U(M)$ to the supergroup $U(N|M)$, and on the subsequent introduction of the gauge superconnection

$$L=\begin{pmatrix} A_\mu\dot{x}^\mu+\frac{2\pi}{k}|\dot{x}|M_J^I C_I\overline{C}^J & \sqrt{\frac{2\pi}{k}}\,\overline{\psi}_I^\alpha\eta_\alpha^I \\ \sqrt{\frac{2\pi}{k}}\,\overline{\eta}_J^\beta\psi_\beta^J & \widehat{A}_\nu\dot{x}^\nu+\frac{2\pi}{k}|\dot{x}|\widehat{M}_L^K\overline{C}^L C_K \end{pmatrix} \tag{4.10}$$

The structure of the superconnection is that of a supermatrix - diagonal blocks are $N\times N$ and $M\times M$ bosonic operators while off-diagonal ones are $N\times M$ and $M\times N$ fermionic operators. In the ABJM case, where N and M coincide, the bosonic blocks of L become two copies of the $\frac{1}{6}$ BPS operator (4.3) introduced in [44] but with a different choice of the scalar couplings. Note that indeed the choice of M_J^I determines the amount of supercharges that are left invariant by the single block. Our conventions are as follows (all the details are in Appendix A.2). Chern-Simons gauge fields A and $\widehat{A}$ are in the adjoint represententation of $U(N)$ and $U(M)$ respectively, while the complex scalars C_I ($\overline{C}^J$) and bi-spinors $\overline{\psi}_I^\alpha$ (ψ_β^J) are in the bi-fundamental $(\boldsymbol{N},\boldsymbol{M})$ ($(\boldsymbol{M},\boldsymbol{N})$) of $U(N)\times U(M)$. Latin uppercase indices are $SU(4)$ $R-$indices and thus range from 1 to 4. With these conventions, the scalar bilinear $M_J^I C_I\overline{C}^J$ is in the adjoint of $U(N)$ and has dimension 1, so it naturally appears on-diagonal in (4.10), next to the gauge field A. A similar reasoning holds for the bilinear $\widehat{M}_L^K\overline{C}^L C_K$, which is in the adjoint of $U(M)$. The latter matrix coupling is "hatted", meaning that, in principle, it may differ from the former. However the authors of [46] chose $\widehat{M}_J^I$ to be the same matrix as M_J^I, for reasons that we will explore later. We will adopt the same choice, letting in general both M_J^I and $\widehat{M}_J^I$ be the same function of the position. Fermion fields are coupled to the spinors η_α^I, $\overline{\eta}_J^\beta$ which act like projectors of the flavour index. Also, fermions have dimension 1 by their own and carry bi-fundamental indices, so they are naturally placed off-diagonal in (4.10). The structure of the superconnection L is thus strongly constrained by the index structure of the fields, while scalars and fermion couplings are dictated by the requirements of supersymmetry.

The associated Wilson loop opearator is defined, quite expectedly, as the trace of the holonomy of L in some super-representation $\mathcal{R}$ of $U(N|M)$ [1]

$$\mathcal{W}_\mathcal{R}=\frac{1}{\dim\mathcal{R}}\mathrm{Tr}_\mathcal{R}\mathcal{P}\,\mathrm{e}^{\mathrm{i}\int_\Gamma \mathrm{d}s\,L(s)} \tag{4.11}$$

along some contour Γ.

The $\mathcal{N}=6$ Chern-Simons-matter theory of ABJ(M) has 12 Poincaré superchargers Q_{IJ}^α, parametrized by $\overline{\theta}_\alpha^{IJ}$, and 12 conformal supercharges S_{IJ}^α parametrized by $\overline{\vartheta}^{IJ}$. The supersymmetry parameters are two-component, antisymmetric spinors which obey the reality

[1] There might seem to be an abiguity here caused by the choice between the standard trace and the supertrace. In the original work [46] it is shown that supersymmetry of the Wilson loop itself requires to take the trace, instead of the supertrace, unlike intuition would have perhaps suggested. This in turn is related to the choice of boundary conditions for fermions as pointed out in [47]; the antiperiodic conditions imposed in [46] imply to take the trace.

condition $\overline{\theta}^{IJ} = \theta_{IJ}{}^*$, and similar for ϑ's. When the contour is a straight line or a circle, the Wilson loop (4.11) is 1/2 BPS and conserves the 6 supercharges parametrised by

$$\overline{\theta}^{1I}_+, \qquad \overline{\theta}^{IJ+}, \qquad I,J=2,3,4 \tag{4.12}$$

together with 6 conformal supercharges. The same set of $\overline{\theta}$'s is also conserved by known stringy solutions, like the brane constructions dual to vortex opeartors of [118], and it is likely to be the same of the half BPS string solution on in $AdS_4 \times \mathbb{C}P^3$ [119], see Section 5.2.

To make a comparison, the supersymmetric Wilson loop operator of $\mathcal{N}=4$ SYM [39] carries a $SO(6)_R$-index which, in the strong coupling regime, represents the internal S^5 position of the dual type IIB solution on $AdS_5 \times S^5$. For Wilson loop operators in the fundamental representation, the dual solution is described by a fundamental string ending on the contour of the loop on the boundary of AdS_5 and localized at a single point in S^5 [38] [39]. The analouge of this solution on $AdS_4 \times \mathbb{C}P^3$ would be localized at a point in $\mathbb{C}P^3$, and would break the $SU(4)_R$ of ABJ(M) to a $U(3)$ subgroup. This fact originally motivated the autors of [46] to consider scalar matrix coupling M^I_J that preserve a $U(3)$ of the $R-$symmetry group and will be a guide-line throughout this paper. Note further that, were the scalar couplings constant along the loop, the internal $\mathbb{C}P^3$ portion of σ-model construction could be neglected. So, quite remarkably, embedding a string solution of the AdS_5 problem into AdS_4, would give in the planar limit a solution of the corresponding $\mathcal{N}=6$ problem (when such an embedding is possible), up to the identification of the CS and SYM 't Hooft couplings. Examples exhibiting this property are given by the anti-parallel Wilson lines, describing the quark-antiquark potential [38], [39], and by the light-like cusp [94] (and also [36] for the cusp anomaluos dimension).

To prove that the loop operator (4.11) actually preserves the 12 supercharges parametrised by (4.12), the way is somewhat similar to the case of the one-sixth operator of Section 4.1. Fermions do play an important role in compensating the supersymmetry variation of scalars and hence enhancing the supersymmetric character of (4.11); a sketch of the derivation presented in [46] is given in Section 4.2.4.

Quite interestingly the half BPS operator of ABJM theory can be derived from more physical consideration. Wilson loops were first introduced in the context of QCD, where they carry the interpretation of the average interaction to which a heavy particle moving on the contour is subject. A similar fact holds in the present case too. Indeed, once the heavy particles of Higgsed theory are identified, the loop operator (4.11) emerges naturarally from their low-energy action [47]. This fact in turn has a nice interpretation in terms of a deformation of the dual M-theoretic background.

4.2.1 Heavy particles in $M2$ theory

The $\frac{1}{2}$-BPS loop operator has a neat physical interpretation in terms of the low energy dynamics of infinitely massive particles obtained by Higgsing a single M2 brane, as pointed out in [47]. We will refer to such particles as the W bosons of the theory as they preserve half of the supersymmetries. Let us then start for convenience from the theory of $N+1$ M2-branes described in Section A.1. The vacuum moduli space can be described by giving scalars the expectation value

$$\langle C_I \rangle = \mathrm{diag}(X_I^1, X_I^2, \ldots, X_I^{N+1}) \tag{4.13}$$

where X_I^i denote the positions of the $N+1$ M2's in the orbifold space $\mathbb{C}^4/\mathbb{Z}_k$ and $I = 1, 2, 3, 4$ is the $R-$index. In the low energy approximation all the M2's sit at the orbifold point and strings extending between them give rise to massless fields - this theory is dual to ABJM theory with $U(N+1) \times U(N+1)$ gauge group. As one or more M2 get separated, strings stretching between distant branes give rise to massive vector multiplets coupled to the massless theory that still leaves on the M2 near the orbifold singularity. To be more precise, let us give a vacuum expectation value to a single M2 along direction 4 in $\mathbb{C}^4$ braking the gauge group to $(N) \times U(N)$

$$\langle C_{\overline{I}} \rangle = 0, \quad \overline{I} = 2, 3, 4 \qquad \langle C_1 \rangle = \mathrm{diag}(0, 0, \ldots, v) \tag{4.14}$$

Note that this particular Higgsing preserves a $SU(3) \in SU(4)$ subgroup of the $R-$symmetry group, which in turn is the same of the half BPS string solution in $AdS_4 \times \mathbb{C}P^3$ [45]. The perturbative mass-formula arising from membrane dynamics predicts a mass term for off-diagonal modes which for large separation $v >> X_{\overline{I}}^i$ reads [120]

$$\mu = \frac{2\pi}{|k|}\left(-|X_1|^2 + |X_2|^2 + |X_3|^2 + |X_4|^2\right) + \frac{2\pi}{|k|}|v|^2 \sim \frac{2\pi}{|k|}|v|^2 \tag{4.15}$$

Considering the Lagrangean B.2 we see that off-diagonal massive modes can be racast in a couple of massive vector multiplets

$$\begin{aligned} W &= \left\{(A_\mu)_{n,N+1},\, (C_{\overline{I}})_{n,N+1},\, (\psi^I)_{n,N+1}\right\} \\ \widehat{W} &= \left\{(\widehat{A}_\mu)_{N+1,n},\, (C_{\overline{I}})_{N+1,n},\, (\psi^I)_{N+1,n}\right\} \end{aligned} \tag{4.16}$$

where $n = 1, \ldots, N$, which respectively transform in representations $(\mathbf{N}, \mathbf{1})$ and $(\mathbf{1}, \overline{\mathbf{N}})$ of the unbroken gauge group. One should expect Goldstone bosons to arise as usual, indeed off-diagonal $(C_1)_{n,N+1}$ and $(C_1)_{N+1,n}$ stay massless. There is an intuitive interpretation of the perturbative mass emerging from separating M2 branes. Remember the brane construction reviewed in Section A.1, D3 branes share three directions with NS5 and are wrapped around their third direction, which is orthogonal to fivebranes. Uplifting the construction to M-theory and upon Higgsing, one gets a M2 brane interpolating between two separated M2's and which also wraps around a projective circle of size $\frac{2\pi}{k}$ times the distance between the two M2's. The mass formula then is telling us that, near the the intresection of the two KK monopoles, the size of the interpolating brane gets bigger along directions X_2, X_3, X_4 of $\mathbb{C}^4/\mathbb{Z}_k$, while gets smaller along X_1.

From what said above is it clear that massive modes behave like external particles carrying a $U(N) \times (N)$ charge. In the low-energy approximation the initial Lagrangean can be decomposed into that of such charged particles coupled to the massless fields plus the unbroken Lagrangean

$$\mathcal{L}_{U(N+1)} \rightarrow \mathcal{L}_{U(N)}(\text{unbroken}) + \widehat{\mathcal{L}}(\text{heavy W and massless fields}) \tag{4.17}$$

The unbroken piece is simply the ABJM Lagrangean with $U(N) \times (N)$ gauge group; to get the effective dynamics of heavy particles one has to expand the original Lagrangean for large v, which thing we are doing in the next section.

4.2.2 Low-energy dynamics of massive modes

The aim of the low-energy analysis is to get a clue on the interaction between W bosons and massless fields that generate the Wilson loop operator. It is customary to think of the worldline of a heavy quark as a Wilson loop in QCD. In our case, taking the limit where the mass scale $v \to \infty$, and is much bigger than the energy, amounts to consider strictly non-relativistic (static) W bosons. In other words, these become external charged particles whose pair creation exceedes the energy bound. In these settings W bosons play the role of heavy quarks and their effective interaction determines how massless fields couple to loop contour, or otherwise stated, it determins the structure of the loop opearator. This analysis was carried out in [47] along the lines of the non-relativistic reduction of M2-branes dynamics [121] [122].

One important fact worth noting before proceeding is the choice of a gauge in which the broken Lagrangean preserves unitarity. This point is achieved turning off goldston boson $(C_1)_{n,N+1}$ and $(C_1)_{N+1,n}$ imposing the vanishing of the supersymmetry trasformation

$$\delta C_1 = \theta^{1\overline{I}\alpha}\psi^{1\overline{I}}_{\alpha} = 0 \tag{4.18}$$

for components $(N+1, n)$ and transpose. As was already pointed out in [46], the half BPS Wilson loop operator should preserve the same supercharges of the macroscopic string solution in $AdS_4 \times \mathbb{C}P^3$ [45] and shared by other supersymmetric brane constructions dual to the vortex operator of [118]. These supercharges are parametrized by two-component spinors [2]

$$\overline{\theta}^{1\overline{I}+} \quad \overline{\theta}^{\overline{I}\,\overline{J}}_{+}, \qquad \overline{I}, \overline{J} = 2, 3, 4 \tag{4.19}$$

obeying the reality condition $\overline{\theta}^{IJ} = (\theta_{IJ})^*$ and the antisymmetry condition in $R-$indices $\overline{\theta}^{IJ} = -\overline{\theta}^{JI}$. Note that this implies a choice of some helicity basis. To actually conserve this $\mathcal{N}=3$ supersymmetry one then requires that negative helicity components are projected out toghether with their complex conjugate

$$(\psi^{\overline{I}})_{N+1,n} = (\psi^{\overline{I}})_{n,N+1} = 0 \tag{4.20}$$

The free-field equations of motion arising from the kinetic part of the Lagrangean for massive particles determines the decomposition of the latter in term of non-relativistic modes, where

[2] Note that our notation for spinor indices is opposed to that of [46].

the time dependence of the wave function is purely a plane wave. These free equations can be solved under the constraint that such modes transform nicely under the $\mathcal{N} = 3$ supersymmetry we want to preserve. Indeed, the mode decomposition for upper modes

$$\begin{aligned}(A)_{n,N+1} &= \sqrt{\frac{\pi}{k}} e_- a_{+n}(x) \mathrm{e}^{-\mathrm{i}\mu t}, \qquad (C_I)_{n,N+1} = \frac{1}{\sqrt{2\mu}} \phi_{In}(x) \mathrm{e}^{-\mathrm{i}\mu t} \\ (\psi^I)_{n,N+1} &= u_- w^I_{+n}(x) \mathrm{e}^{-\mathrm{i}\mu t}, \qquad (\psi^1)_{n,N+1} = u_+ w^1_{-n}(x) \mathrm{e}^{-\mathrm{i}\mu t}\end{aligned} \tag{4.21}$$

and for lower modes

$$\begin{aligned}(\widehat{A})_{N+1,n} &= \sqrt{\frac{\pi}{k}} e_- \widehat{a}_{+n}(x) \mathrm{e}^{+\mathrm{i}\mu t}, \qquad (C_I)_{N+1,n} = \frac{1}{\sqrt{2\mu}} \widehat{\phi}_{In}(x) \mathrm{e}^{+\mathrm{i}\mu t} \\ (\psi^I)_{N+1,n} &= u_- \widehat{w}^I_{+n}(x) \mathrm{e}^{+\mathrm{i}\mu t}, \qquad (\psi^1)_{N+1,n} = u_+ \widehat{w}^1_{-n}(x) \mathrm{e}^{+\mathrm{i}\mu t}\end{aligned} \tag{4.22}$$

explicitly show the emergence of two $\mathcal{N} = 3$ vector multiplets $\{a_+, w^I_+, \phi_I, w^1_-\}$ and $\{\widehat{a}_+, \widehat{w}^I_+, \widehat{\phi}_I, \widehat{w}^1_-\}$ transforming respectively in a $(\mathrm{N}, 1)$ and $(1, \mathrm{N})$ of the unbroken gauge group $U(N) \times U(N)$. Here $e_\pm = (1 \pm \mathrm{i})$ is the gauge field polarization vector of definite helicity and $u_\pm$ are two orthonormal two-component constant spinors acting like helicity projectors. Also, upper case latin $R-$indices in massive fileds run from 2 to 4, as the index 1 denotes Goldstone bosons, while they run from 1 to 4 in massless fields. This may seem to generate confusion but it does not as they never appear in the same expression.

To actually reach the low-energy Lagrangean $\widehat{\mathcal{L}}$ for massive fields one has to insert the mode decompositions 4.21 and 4.21 into the oringinal $U(N + 1) \times U(N + 1)$ Lagrangean and expand to leading order in μ. In doing so quantum corrections could in principle afflict free-field eigenstates obtained above, and indeed they do. The result is, generally speaking, a rotation into the Hilbert space which can be reabsorbed with a ridefinition of the non-relativistic fields in the two vector multiplets. Abusing notation a bit while using the symbols A, C and ψ for the fields in the unbroken sector, the bottom line of this analysis is that the interaction between massive and massless fields can be spelled in the Lagrangean

$$\begin{aligned}\widehat{\mathcal{L}} =& \mathrm{i}\overline{a}_- \mathcal{D}_0 \overline{a}_+ + \mathrm{i}\widehat{a}_+ \widehat{\mathcal{D}}_0 \overline{\widehat{a}}_- + \mathrm{i}\overline{w}^\alpha \mathcal{D}_0 w_\alpha + \mathrm{i}\overline{\widehat{w}}^\alpha \mathcal{D}_0 \widehat{w}_\alpha + \mathrm{i}\phi_I \mathcal{D}_0 \overline{\phi}^I + \mathrm{i}\widehat{\phi}_I \mathcal{D}_0 \overline{\widehat{\phi}}^I \\ &+ \sqrt{\frac{4\pi}{k}} \left[\widehat{a}_+ \overline{\psi}_{1-} w^1_- + \widehat{w}^1_- \overline{\psi}_{1-} a_+ + \overline{w}_{1+} \psi^1_+ \overline{\widehat{a}}_- + \overline{a} \psi^1_+ \overline{\widehat{w}}_{1+} \right] \\ &+ \sqrt{\frac{4\pi}{k}} \left[\overline{\phi}^I \psi^1_+ \overline{\widehat{w}}_{I-} + \overline{w}_{I-} \psi^1_+ \overline{\widehat{\phi}}^I + \widehat{\phi}_I \overline{\psi}_{1-} w^I_+ + \widehat{w}^I_+ \overline{\psi}_{1-} \phi_I \right]\end{aligned} \tag{4.23}$$

where the covariant derivatives $\mathcal{D}_\mu = \partial_\mu - \mathrm{i}\mathcal{A}_\mu$ and $\widehat{\mathcal{D}}_\mu = \partial_\mu - \mathrm{i}\widehat{\mathcal{A}}_\mu$ are defined with respect to the two connections

$$\mathcal{A}_0 = A_0 - \frac{2\pi}{k} M^I_J C_I \overline{C}^J \qquad \widehat{\mathcal{A}}_0 = \widehat{A}_0 - \frac{2\pi}{k} M^I_J \overline{C}^J C_I \tag{4.24}$$

and the scalar bilinear term entering the equation above is just the scalar source for the W boson mass generated by the Higgsing procedure of (4.14)

$$M^I_J C_I \overline{C}^J = -|C_1|^2 + |C_2|^2 + |C_3|^2 + |C_4|^2 \tag{4.25}$$

Note that only the time derivative appears in the effective Lagrangean of W bosons as they are really treated as infinitely massive, static particles.

4.2.3 The $\frac{1}{2}$ BPS Wilson loop operator of ABJM theory

In turns out that the Lagrangean above has a deeper and nicer structure than what it might seem at a first look. Let us the collect all the massive degrees of freedom into the super-matrices W^I transforming in a adjoint representation of the gauge super-group $U(N|N)$

$$W^1 = \begin{pmatrix} a_+ & w^1_- \\ \overline{\widehat{w}}_{1+} & \overline{\widehat{a}}_- \end{pmatrix} \qquad W^{\overline{I}} = \begin{pmatrix} \phi_{\overline{I}}+ & w^{\overline{I}}_+ \\ \overline{\widehat{w}}_{\overline{I}-} & \overline{\widehat{\phi}}^{\overline{I}} \end{pmatrix} \tag{4.26}$$

still being $\overline{I} = 2, 3, 4$. The super-covariant derivative is readly constructed

$$\mathfrak{D}_0 = \partial_0 - i\mathfrak{L}_0 \tag{4.27}$$

and the super-connection $\mathfrak{L}$ can be read from (4.23)

$$\mathfrak{L}_0 = \begin{pmatrix} \mathcal{A}_0 & \sqrt{\frac{4\pi}{k}}\psi^1_+ \\ \sqrt{\frac{4\pi}{k}}\overline{\psi}_{1-} & \widehat{\mathcal{A}}_0 \end{pmatrix} \tag{4.28}$$

Then, the low-energy Lagrangean for W bosons takes the simple, rather fulfilling form

$$\widehat{\mathcal{L}} = \mathrm{Tr}\left[\mathrm{i}\overline{W}^I \mathfrak{D}_0 W_I\right] \tag{4.29}$$

Given a super-matrix $g \in U(N|N)$, the action of the gauge supergroup on the fields is a gauge symmetry of the low-energy theory

$$\mathfrak{L}_0 \to g^\dagger \mathfrak{L}_0 g + \mathrm{i} g^\dagger \partial_0 g, \qquad W \to g^\dagger W \tag{4.30}$$

From (4.29) one can read the equation of motion for massive W bosons

$$\mathfrak{D}_0 W_I = 0 \tag{4.31}$$

from which follows that the time evolution of the wavefunction is encoded in the time-like Wilson line

$$\mathcal{W}(t_1, t_2) = \mathcal{P}\exp\left\{\mathrm{i}\int_{t_1}^{t_2} \mathrm{d}\tau \mathfrak{L}_0(\tau)\right\} \tag{4.32}$$

This result can be understood as the interaction of single, massive and static W boson evolving in time while interacting with the fields of $U(N)\times U(N)$ ABJM theory. Since W is a 1/2 BPS particle, its time evolution preserves the same supercharges, hence this Wilson loop is a half BPS. This is indeed the Wilson loop operator firstly introduced in [46] on the basis of a totally different argument. The derivation of this section shows that there is a precise physical meaning behind the structure of the super-connection $\mathfrak{L}$ and that the emergence of the super-group structure is driven by the low-energy effective dynamics of the Higgsed theory.

Under the supergauge transformation $\mathfrak{L}_0 \to g^\dagger \mathfrak{L}_0 g + \mathrm{i} g^\dagger \partial_0 g$ the Wilson loop transforms covariantly

$$\mathcal{W}(t_1, t_2) \to U(t_1)^\dagger \mathcal{W}(t_1, t_2) U(t_2) \tag{4.33}$$

To have a gauge invariant operator one has to choose whether to take the trace or the supertrace of the latter. It turns out that boundary conditions impose to take the trace. To this aim note that a supersymmetry transformation under one of the conserved supercharges acts on the superconnection as

$$\delta \mathfrak{L}_0 = \partial_0 \Lambda - \mathrm{i}\,[\mathfrak{L}_0, \Lambda] \tag{4.34}$$

which remarkably is just a gauge transformation, being Λ in the Lie superalgebra $u(N|N)$

$$\Lambda = \sqrt{\frac{2\pi}{k}} \begin{pmatrix} 0 & \mathrm{i} C_I \theta_+^{1I} \\ -\mathrm{i} \overline{C}^I \theta_{1I}^+ & 0 \end{pmatrix} \tag{4.35}$$

For a closed loop one can pick either poriodic or anti-periodic boundary consitions for fermions in the loop. As suggested by [46] the former are playing the game, and are the ones naturally chosen in any quantum-mechanical problem. In this case, supersymmetry of the loop operator requires the trace, so that the gauge invariant, half BPS Wilson loop reads

$$\mathcal{W} = \mathrm{Tr} \mathcal{P} \exp \left\{ \mathrm{i} \oint \mathrm{d}s \mathfrak{L}(s) \right\} \tag{4.36}$$

Lastly, three comments are in order here. The first is about the off-diagonal terms entering in (4.28). In the infinite mass limit these can be seen as projections onto the helicitiy of massive particles. Then they can be replaced by two-component spinor projectors η_α^I and $\overline{\eta}_I^\alpha$ which in general will be functions of the integration path and whose role nothing is but projecting fermions onto states of definite helicity. For the sake of this, remember that the helicity eigenstates are determined by the elicity of the supercharges one wants to preserve, and only one state survives once the unitary gauge is imposed.

The second comment is about Lorentz covariance. Going to the non-relativistic theory we have lost Lorentz-covariance, which fact is evident for only time derivatives and time components of the connection compare in (4.32). On the other hand, covariance of the loop operator can be strikingly restored setting [46]

$$\mathfrak{L} = \begin{pmatrix} A_\mu \dot{x}^\mu + \frac{2\pi}{k} |\dot{x}| M_J^I C_I \overline{C}^J & \sqrt{\frac{2\pi}{k}}\, \overline{\psi}_I^\alpha \eta_\alpha^I \\ \sqrt{\frac{2\pi}{k}}\, \overline{\eta}_J^\beta \psi_\beta^J & \widehat{A}_\nu \dot{x}^\nu + \frac{2\pi}{k} |\dot{x}| \widehat{M}_L^K \overline{C}^L C_K \end{pmatrix} \tag{4.37}$$

and

$$\mathcal{W} = \mathrm{Tr}\mathcal{P}\exp\left\{\oint_{x^\mu(s)} \mathrm{d}s\,\mathfrak{L}(s)\right\} \tag{4.38}$$

where now the contour is an arbitrary curve parametrized by $x^\mu(s)$. As a matter of fact, such operator has been shown to be $\frac{1}{2}$BPS on a single straight line and on a circle [46] by a direct analysis of it supersymmetries. The computation is indeed particularly instructive and highlights a certain structure hidden in the perturbative expansion of the loop operator. Because of that and of the future usefulness of such computation, we will review it in the next section.

Finally the last remark. In the low energy theory a supergroup structure emerges. The heavy particles Lagrangean is symmetric under the supergauge transformations (4.34) of the gauge supergroup $U(N|N)$, of which $\mathfrak{L}$ is a connection. So far a natural question arises - what about other Chern-Simons-matter theories? Although the general question is pretty hard to answer, the genaralisation of the previous results to ABJ theory is, at least formally, straigthforward. Indeed, starting from this $\mathcal{N}=6$ theory with gauge group $U(N+1)\times U(M+1)$ and upon Higgsings, one will end up very likely with a superconnection of the supergroup $U(N|M)$. To make this reasoning more persuasive, note that the gauge field and scalar bilinear in the upper corner of (4.37) will transform in the adjoint of $U(N)$ and those in the lower corner in the adjoint of $U(M)$. Hence are $N\times N$ and $M\times M$ square matrices. Fermions, besides, are in a bifundamental or anti-bifundamental of $U(N)\times U(M)$ and thus will be rectangular matrices, indeed they lie off-diagonal, precisely in the position we would expect them to be.

4.2.4 Lines and circles

Infinite straight line

As already mentioned at the beginning of this chapter, the half BPS Wilson loop operator is dual to a fundamental string extending in $AdS_3 \in AdS_4$ and localised to a point in $\mathbb{C}P^3$. This solution thus breakes the $SU(4)$ $R-$symmetry of the vacuum to $U(1)\times SU(3)$, which would suggest to choose scalar couplings for (4.10) of the form [46]

$$M_J^I = \widehat{M}_J^I = m_1\,\delta_J^I - 2m_2\,\delta_1^I\delta_J^1 \tag{4.39}$$

which is precisely what one gets upon Higgsing of $M2-$branes (4.25) up to the choice $m_1 = m_2 = 1$. Moreover the fermionic couplings in (4.28), in the limit of infinite W-bosons mass, act like projectors on the helicity of the probe-particles and can be replaced by the fermionc couplings η of (4.10) [47]. We first consider a Wilson loop along an infinite straight line in the x_0 (time) direction, fermions must be projected according to $P\pm = \frac{1}{2}(1\pm\sigma^0)$, hence we choose fermionic projectors

$$\eta_I^\alpha = \eta\,\delta_I^1\delta_+^\alpha\,, \qquad \bar{\eta}_\alpha^I = \bar{\eta}\,\delta_1^I\delta_\alpha^+ \tag{4.40}$$

Let us remind that the fundamental string solution (5.2) [119] preserves the twelve supercharges (4.12) parametrised by

$$\bar{\theta}^{1I}_{+}\,, \qquad \bar{\theta}^{IJ+}\,, \qquad I,J=2,3,4 \tag{4.41}$$

so also the chirality of η's agrees which what is suggested by the chirality of the supercharges. Following [46], the modified gauge connections in (4.28) now take the form

$$\mathcal{A}_0 \equiv A_0 + \frac{2\pi}{k} M^I_J C_I \bar{C}^J\,, \qquad \widehat{\mathcal{A}}_0 \equiv \widehat{A}_0 + \frac{2\pi}{k} \widehat{M}^I_J \bar{C}^J C_I \tag{4.42}$$

and using the superconformal transformation in Appendix A.3, the total variation of the superconnection on an infinite straight line reads

$$\delta L = \frac{8\pi}{k}\bar{\theta}^{1I}_{+} \begin{pmatrix} C_I\,\psi^{+}_{1} & \sqrt{\frac{k}{8\pi}}\eta \mathcal{D}_0 C_I \\ 0 & \psi^{+}_{1}\,C_I \end{pmatrix} - \frac{4\pi}{k}\varepsilon_{1IJK}\,\bar{\theta}^{IJ+} \begin{pmatrix} \bar{\psi}^{1}_{+}\,\bar{C}^K & 0 \\ \sqrt{\frac{k}{8\pi}}\bar{\eta}\mathcal{D}_0 \bar{C}^K & \bar{C}^K\,\bar{\psi}^{1}_{+} \end{pmatrix} \tag{4.43}$$

where the modified covariant derivative is $\mathcal{D}_0 C_I = \partial_0 C_I + i(\mathcal{A}_0\, C_I - C_I\,\widehat{\mathcal{A}}_0)$ and analogously for $\mathcal{D}_0\bar{C}^I$. Opposedly to the $\frac{1}{6}$ BPS case, the first order variation of the super Wilson loop does not vanish on its own, but it cancels with term coming from the second order. Indeed expanding the loop operator

$$W_{\mathcal{R}} = \mathrm{Tr}_{\mathcal{R}}\left[1 + i\int_{-\infty}^{\infty} d\tau\, L(\tau) - \int_{-\infty}^{\infty} d\tau_1 \int_{\tau_1}^{\infty} d\tau_2\, L(\tau_1)L(\tau_2) + \ldots\right] \tag{4.44}$$

the off-diagonal bit of the linear (in L) term is a total derivative and can be integrated away, while the off-diagonal fermionic terms in the quadratic piece can be integrated by parts

$$\begin{aligned} \delta W_{\mathcal{R}} &= \frac{8\pi}{k}\bar{\theta}^{1I}_{+}\mathrm{Tr}_{\mathcal{R}}\Big[i\int_{-\infty}^{\infty} d\tau\, \begin{pmatrix} C_I\,\psi^{+}_{1} & \\ & \psi^{+}_{1}\,C_I \end{pmatrix} \\ &\quad - \frac{1}{2}\eta\bar{\eta}\int_{-\infty}^{\infty} d\tau_1 \int_{\tau_1}^{\infty} d\tau_2\, \begin{pmatrix} \partial_{\tau_1} C_I(\tau_1)\psi^{+}_{1}(\tau_2) & \\ & -\psi^{+}_{1}(\tau_1)\partial_{\tau_2} C_I(\tau_2) \end{pmatrix} + \ldots \Big] \\ &= \frac{8\pi}{k}\bar{\theta}^{1I}_{+}\Big[i\int_{-\infty}^{\infty} d\tau\, \begin{pmatrix} C_I\,\psi^{+}_{1} & \\ & \psi^{+}_{1}\,C_I \end{pmatrix} - \frac{1}{2}\eta\bar{\eta}\int_{-\infty}^{\infty} d\tau\, \begin{pmatrix} C_I\psi^{+}_{1} & \\ & \psi^{+}_{1}C_I \end{pmatrix}\Big] \end{aligned} \tag{4.45}$$

and cancel the diagonal pieces in the linear term if the normalisation condition $\eta\overline{\eta} = 2\mathrm{i}$ is satisfied.

Note that the appearance of the modified covariant derivative acting on off-diagonal fermions in L is necessary condition for the cancellation of the susy variation. It is remarkable that the Higgsing procedure reproduces precisely this structure. Also note that in (4.45) above we have explicitly written the variation w.r.t. a specific supercharge to show that the variation of fermions is actually a total derivative acting on scalars. This mechanism can be recursively repeated to prove that the variation of the Wilson loop operator vanishes at any order of the weak coupling expansion for the 12 supercharges above, hence the infinite Wilson line with superconnection (4.28) is a $\frac{1}{2}$ BPS operator.

Spacelike circle

ABJM theory is a superconformal theory in three dimensions - it is left untouched by conformal transformations. We mentioned in Section 4.1 that a conformal transformation that maps an infinite straight line to a circle changes the topology of certain classes of Feynman diagrams which may therefore give a finite contribution to the expectation value of the Wilson loop. In the case of the $\frac{1}{6}$ BPS Wilson loop we indeed saw that while the expectation value of the Wilson line is trivial, that of the circle receives quatum corrections (4.7) [44] , and the same is also true in the case of $\mathcal{N}=4$ SYM [101].

So consider a Wilson loop defined on a spacelike circle parametrised by the two coordinates

$$x^1 = \cos\tau\,, \qquad x^2 = \sin\tau\,. \tag{4.46}$$

and a parameter $\tau \in [0, 2\pi]$. Rotating the superconnection from timelike to spacelike configuration amounts to replace $|\dot{x}| \to -\mathrm{i}|\dot{x}|$ and is equivalent to Wick rotate from Minkowskian to Euclidean signature; it affects trivially the superconnection

$$L \to L_E = \begin{pmatrix} A_\mu \dot{x}^\mu - \mathrm{i}\frac{2\pi}{k}|\dot{x}|M_J^I C_I \overline{C}^J & -\mathrm{i}\sqrt{\frac{2\pi}{k}}\,\overline{\psi}_I^\alpha \eta_\alpha^I \\ -\mathrm{i}\sqrt{\frac{2\pi}{k}}\,\overline{\eta}_J^\beta \psi_\beta^J & \widehat{A}_\nu \dot{x}^\nu - \mathrm{i}\frac{2\pi}{k}|\dot{x}|\widehat{M}_L^K \overline{C}^L C_K \end{pmatrix} \tag{4.47}$$

The modified connection can be defined as above

$$\mathcal{A} \equiv A_\mu \dot{x}^\mu - i\frac{2\pi}{k} M_J^I C_I \bar{C}^J\,, \qquad \widehat{\mathcal{A}} \equiv \widehat{A}_\mu \dot{x}^\mu - i\frac{2\pi}{k}\widehat{M}_J^I \bar{C}^J C_I\,. \tag{4.48}$$

where the scalar couplings are determined by the same matrix M_J^I as before. The right way to define spinor coupling η_α^I and $\overline{\eta}_I^\alpha$ is again to pick up eigenvectors of the helicity projector along the loop. In the present case the prejector reads

$$1 + \dot{x}^\mu (\gamma_\mu)_\alpha^{\ \beta} = \begin{pmatrix} 1 & -ie^{-i\tau} \\ ie^{i\tau} & 1 \end{pmatrix}, \tag{4.49}$$

which has eigenstates

$$\eta_\alpha^I(\tau) = \begin{pmatrix} 1 & -ie^{-i\tau} \end{pmatrix} \eta(\tau)\, \delta_1^I\,, \qquad \bar{\eta}_I^\alpha(\tau) = i \begin{pmatrix} 1 \\ ie^{i\tau} \end{pmatrix} \bar{\eta}(\tau)\, \delta_I^1 \tag{4.50}$$

and now $\eta(\tau)$ is a function of the position along the contour. Note that at this stage it is also an arbitrary function, but will be determined by the constraints of supersymmetry. As shown in [46], the conformal transformation mapping the infinite straight line to a circle mixes the Poicare and superconformal supercharges Q^{IJ} and S^{IJ}. The variation of W under the action of the superconformal charges is the same as the variation under the action of the Poincare ones, up to the replacement of $\overline{\theta}^{IJ} \to \overline{\vartheta}^{IJ} x\cdot\sigma$, where $\overline{\theta}^{IJ}$ parametrize Q^{IJ} as before and $\overline{\vartheta}^{IJ}$ do the same job for S^{IJ}. Using some identities for sigma matrices and the explicit parametrisation of the contour, one can show that the variation of the bosonic and fermionic terms, due to te mixed charges parametrised by

$$\begin{aligned} \bar{\theta}^{1I} + \bar{\vartheta}^{1I} x^\mu \gamma_\mu &= \bar{\theta}^{1I}(1 - \dot{x}^\mu\gamma_\mu) \\ \bar{\theta}^{IJ} + \bar{\vartheta}^{IJ} x^\mu \gamma_\mu &= \bar{\theta}^{IJ}(1 + \dot{x}^\mu\gamma_\mu)\,, \qquad I, J \neq 1\,. \end{aligned} \tag{4.51}$$

reads for the circular loop

$$\begin{aligned} \delta\mathcal{A} &= \frac{8\pi i}{k}\bar{\theta}^{1I}(1-\dot{x}^\mu\gamma_\mu)C_I\psi_1 + \frac{4\pi i}{k}\varepsilon_{1IJK}\bar{\theta}^{IJ}(1+\dot{x}^\mu\gamma_\mu)\bar{\psi}^1\bar{C}^K \\ \delta\widehat{\mathcal{A}} &= \frac{8\pi i}{k}\bar{\theta}^{1I}(1-\dot{x}^\mu\gamma_\mu)\psi_1 C_I + \frac{4\pi i}{k}\varepsilon_{1IJK}\bar{\theta}^{IJ}(1+\dot{x}^\mu\gamma_\mu)\bar{C}^K\bar{\psi}^1 \\ \delta(\eta_1^\alpha(\tau)\bar{\psi}^1_\alpha) &= 4i\eta_1\bar{\theta}^{1I}\dot{x}^\mu\mathcal{D}_\mu C_I - 2\eta_1\sigma^3\bar{\theta}^{1I}C_I \\ \delta(\psi_1^\alpha\bar{\eta}^1(\tau)_\alpha) &= -\varepsilon_{1IJK}\bar{\theta}^{IJ}\big[2i\bar{\eta}^1\dot{x}^\mu\mathcal{D}_\mu\bar{C}^K + \sigma^3\bar{\eta}^1\bar{C}^K)\big] \end{aligned} \tag{4.52}$$

The requirement that the variation of fermions is again a total derivative as in the previous case implies that the following two equations are satisfied

$$\partial_\tau\eta_1 = \frac{i}{2}\eta_1\sigma^3\,, \qquad \partial_\tau\bar{\eta}^1 = -\frac{i}{2}\sigma^3\bar{\eta}^1 \tag{4.53}$$

which fixes the up-to-now arbitrary function $\eta(\tau)$. Now, expanding the loop operator and repeating the trick of partial integration, bosonic and fermioni terms cancel mutually, and only a boundary term survive arising at $\tau = 0$ from integrating over the first variation $\delta L_F(\tau_1)$ and at $\tau = 2\pi$ from the last variation $\delta L_F(\tau_p)$

$$\begin{aligned} \delta W_{\mathcal{R}} \propto\ & \mathrm{iTr}_{\mathcal{R}}\int_0^{2\pi} d\tau \begin{pmatrix} C_I\,\psi_1(1+\dot{x}^\mu\gamma_\mu)\bar{\theta}^{1I} & \\ & \psi_1\,C_I(1+\dot{x}^\mu\gamma_\mu)\bar{\theta}^{1I} \end{pmatrix} \\ & - \mathrm{Tr}_{\mathcal{R}}\int_0^{2\pi} d\tau_1 \int_{\tau_1}^{2\pi} d\tau_2 \begin{pmatrix} -(\partial_{\tau_1}\eta_1 C_I\bar{\theta}^{1I})_{(1)}(\psi_1\bar{\eta}^1)_{(2)} & \\ & -(\psi_1\bar{\eta}^1)_{(1)}(\partial_{\tau_2}\eta_1 C_I\bar{\theta}^{1I})_{(2)} \end{pmatrix} \\ & = -\mathrm{Tr}_{\mathcal{R}}\int_0^{2\pi} d\tau \begin{pmatrix} \big(\eta_1 C_I\bar{\theta}^{1I}\big)(0)\ (\psi_1\bar{\eta}^1)(\tau) & \\ & -(\psi_1\bar{\eta}^1)(\tau)\ \big(\eta_1 C_I\bar{\theta}^{1I}\big)(2\pi) \end{pmatrix} \end{aligned} \tag{4.54}$$

As mentioned before, imposing the antiperiodicity condition $\eta_1(2\pi) = -\eta_1(0)$ on fermions, implies to take the regular trace of the last expression in order to make δW vanish. The reasoning can be repeated order by order, being the mechanism totally analogous to the Wilson line case, the only difference being in boundary contributions, which indeed cancel when taking the trace. This proves that the circular Wilson loop operator is invariant under 12 mixed Poicare-superconformal supercharges and is hence $\frac{1}{2}$ BPS.

CHAPTER 5

Non-perturbative Results for Wilson Loops

5.1 The cusp anomalous dimension

The cusp anomalous dimension $\Gamma_{\text{cusp}}(\lambda)$ naturally appears in the weak coupling expantion of light-like Wilson loops with cusps and of gluon scattering amplitudes. The all-loop structure of field theory is such that divergent contributions exponentiate at all orders and $\Gamma_{\text{cusp}}(\lambda)$ itlesf is the coefficient of the leading singularity. At strong coupling, the large λ behaviour of the cusp anomalous dimention can be extracted from the energy of a rigid folded string rotating in AdS. On the gauge theory side, this string solution corresponds to a twist-two operator with large Lorentz spin M, for which the difference between the scaling dimension and spin scales logarithmically with the spin with a coefficient which is indeed $\Gamma_{\text{cusp}}(\lambda)$.

Twist operators hold main relevance in diverse fields ranging from integrability in the context of the AdS/CFT correspondence to supersymmetric gauge theories and QCD. The twist of an operator is defined as the difference between its scaling dimension and its Lorentz spin, thus any local operator having a definite value of this quantity is a twist operator in the broad sense. It is customary to focus on certain twist operators, also known as Wilson operators, belonging to a closed subsector of the theory, as they are in one-to-one correspondence with definite objects in the spin-chain picture of integrability. In the context of QCD, twist operators are involved in deep inelastic scattering processes, and their anomalous scaling dimension can be computed by considerations of Wilson loops and scattering amplitudes. A deep knowledge of such operators surely is a powerfull tool for a better understanding of field theories. Not just, because they are the natural candidate objects one would like to investigate using AdS/CFT. Indeed, the many advances achieved in understanding integrability of planar $\mathcal{N} = 4$ SYM relied on heavy tests of several conjectures, for which reason having as many quantities as possible that can be computed on different sides of the conjecture, is a necessary asset. The most evident example of this is the major role twist operators had in the developement of asymptotic Bethe equations and in revealing and understanding its inadequacies. Moreover, all-loop expressions for twist operators would not just be a notable advance, though confined to the realm of AdS/CFT and integrability, but also would give new insight in the strongly coupled region of QCD.

The quantization of rigid string solutions has been proved to be a useful tool in the investigation of quantum integrable features in the context of AdS_5/CFT_4 [123] [124] , [125]. In this case, summing up the energy of fluctuations around a classical string configuration gives the loop corrections to the classical energy of the spinning string which can be then compared to the value predicted by the Bethe ansatz. This program was successfully carried out in $AdS_5 \times S^5$ [126].

The dynamics of the $AdS_4 \times \mathbb{C}P^3$ superstring has been investigated from the point of view of integrability, indeed the coset sigma-model has been shown to be classically integrable [127] [128]. In [129] an algebrai curve formulation encoding the solution of classical $AdS_4 \times \mathbb{C}P^3$ sigma-model was given and other semiclassical features of the model where further investigated [130] [131] [132] and [133]. On the gauge thoery side, superconformal $\mathcal{N} = 6$ Chern-Simons theory was shown to be one-loop integrable [23] [116] [22], these results help an all-loop formulation of the Bethe ansatz [24] along the lines of the $\mathcal{N} = 4$ case [134] [135] [136]. Solving the all-loop Bethe equations would be an important step as it encodes the anomalous dimensions of gauge theory operators for any value of the 't Hooft coupling constant. Higher-loop tests where performed in [25] [137] and [138] and agreement with the Bethe ansatz was found. Lastly, let us mention the extention of the integrable spin chain results to super Chern-Simons theorym with $U(N) \times U(M)$ guage group of [139].

On the other hand, after those first perturbative checks, direct string theory computations [140] [141] of the one loop energy shift of the spinning folded string gave an answer which was not in complete agreement with the conjectured Bethe equations of [24]. Let us note that the string computation is direct field-teoretic computation which does not rely on the knowledgr of the algebraic curve, it is the real test-brake for testing Bethe equations' toughness, and it gives at present a trustable value for the cusp anomalous dimension.

5.1.1 Twist operators in AdS/CFT

In the context of $\mathcal{N} = 4$ SYM twist operators in the $\mathfrak{sl}(2)$ subalgebra are usually considered because they can be simply constructed applying M covariant light-cone derivatives to a set of L complex scalar fieds

$$O(x) = \mathrm{Tr}(\mathcal{D}^M Z^L) + \dots \tag{5.1}$$

where all possible permutations of derivatives and scalars are taken into account. These are single trace operators of length L and Lorentz spin M. In the large spin limit, twist operators are related to Sudakov form factors of QCD and have been shown [142] that their dimension scales at most logarithmically with the spin. This behaviour is a universal feature of gauge theories and can be traced back to one common feature, the existence in the theory of a massless spin one field (the gauge field indeed). The simplest case is that of twist-two operators in the large spin limit [63] [64], for which the anomalous dimension behaves as

$$\gamma(\lambda) = \Delta - M = 2\Gamma_{\rm cusp}(\lambda)\,\log M + \mathcal{O}(M^0) \tag{5.2}$$

where $\lambda = g^2 N$ is the 't Hooft coupling and $\Gamma_{\rm cusp}(\lambda)$ is the cusp anomalous dimension that we first presented in (3.41) of Section 3.2; in the context of integrability it also known as *universal scaling function*. This is a non-universal function and depends on the theory under consideration. As was reviewed in Section 3.2, it governs the structure of divergences of cusped Wilson loops at every order of the perturbation theory, and hence the divergences of gluon scattering amplitudes.

The all-order knowledge of $\Gamma_{\rm cusp}(\lambda)$ is one of the long-standing problems of gauge theory. Up-to-date, the cusp anomalous dimension of $cN = 4$ SYM is known at weak coupling by direct Feynman diagram computations up to the fourth order [68] [69] (see Sections 2.2.1 and 3.2.1) and there are predictions for its strong coupling asymptotics based on the AdS/CFT correspondence [94] [143] [72] [73] [144], besides the break-through of [41]. Remarkably, the Bethe ansatz predicts it in a closed integral form that can be expanded at arbitrary order [126] and which is in agreement with the direct perturbative and nonperturbative computations. For operators carrying large quantum numbers (Lorentz spin, isotopic $R-$charge,...) their scaling dimension at strong coupling can be found as the energy of dual (semi)classical string configurations propagating on the curved space [145]. In particular, the strong coupling asymptotics of twist-two operators carring large Lorents spin M can be extracted from the energy of a folded string rotating in a $AdS_3 \in AdS_5$ with angular momentum M and reads

$$\Gamma_{\rm cusp}(\lambda) = \sqrt{\frac{\lambda}{4\pi^2}} + \text{const} \tag{5.3}$$

Furthermore a brave attempt to reproduce the strong coupling behaviour of the cusp anomalous dimension by means of ressummation of Feynman diagrams contributing to a cusped Wilson loop was performed in [106]. There it was shown that ladder diagrams are not enough for reproducing the correct exponent of $\Gamma_{\rm cusp}(\lambda)$ because vertices do play an important role. A nice way to get around the issue of an all-order expression is that of [49]. There the authors deform both the loop operator and the contour and are able to obtain a strong coupling expression in terms of elliptic integrals and depending on the two deformation parameters. This expression, which was computed up to orderd $1/\sqrt{\lambda}$ corrections, reproduces in certain limit the expected cusp anomalous dimension of $\mathcal{N} = 4$ SYM. It is also related to meson-pair potential as we will discuss in Section 6.1.1.

5.1.2 Spinning strings in $AdS_4 \times \mathbb{C}P^3$

The $AdS_4 \times \mathbb{C}P^3$ correspondence is somewhat peculiar. The $AdS_5 \times S^5$ duality involves maximally supersymmetric theories with 32 supercharges and that are invariant under the superconformal group $PSU(2,2|4)$, whereas the relevant theory in the $AdS_4 \times \mathbb{C}P^3$ case is symmetric under the subgroup $OSP(2,2|6)$ bringing 24 supercharges and therefore is non-maximal. Partially fixing the $\kappa-$symmetry of the Green-Schwarz action for type IIA superstrings on $AdS_4 \times \mathbb{C}P^3$ is equivalent to add a suitable Wess-Zumino term to the coset sigma-model on the same space [127] [128]. Indeed, the GS action for type IIA involves two Majorana-Weyl fermions in ten dimension, amounting to 32 degrees of freedom; fixing the $\kappa-$symmetry gauges half of them away and one is left with 16 phisical d.o.f. On the other hand the sigma-model based on the coset space $\frac{OSP(2,2|6)}{SO(3,1)\times U(3)}$ contains 24 fermions, of which 8 are gauged away by $\kappa-$symmetry. The remaining 16 fermionic degreed of freedom together with their bosonic partners constitute the physical spectrum of the $AdS_4 \times \mathbb{C}P^3$ superstring.

It was observed in [127] that string configurations which carry only AdS spin are singular due to the fact that the rank of the corresponding $\kappa-$symmetry gets enhanced from 8 to 12. To regularize such behaviour it was suggested in [140] to consider strings which also carry a $\mathbb{C}P^3$ angular momentum J. The same operation was performed in the $AdS_5 \times S^5$ case [146] and the first energy correction to the string configuration provided the leading strong coupling correction to the generalized scaling function $\Gamma(J, \log M)$. The computation in [140] is rather technical and we will not enter into the details, and there is no reason for doing so. The spinning (M, J)-string ansatz lives in a subspace $AdS_3 \times S^1$ of $AdS_4 \times \mathbb{C}P^3$, therefore the solution must coincide with that of [123]. The energy of the long rotating string reads

$$E = M + J\sqrt{1+x^2} \tag{5.4}$$

where

$$x = \frac{\sqrt{\lambda}}{\pi J}\log\frac{M}{J} \tag{5.5}$$

and $\sqrt{\lambda} = \frac{R^2}{\alpha'}$ is related to the AdS radius. This can be further expanded according to either the *fast* limit $x \gg 1$ or the *slow* limit $x \ll 1$. This expression as a smooth $J \to 0$ limit which in turn gives back the anomalous dimension of twist-two operators.
From the fluctuation analysis around the classical configuration it emerged that the leading energy shift in the limit of slow rotation $J \ll \log M$ is

$$\delta E = -\frac{5\log 2}{2\pi}\log M \tag{5.6}$$

and the coefficient of the logarith should be interpreted as the first sub-leading correction to cusp anomalous dimension. Note that this expression is in contrast with the conjecture of [129] which would predict $-\frac{3\log 2}{2\pi}\log M$ instead. Two proposals have been made for correcting this result. Though they are mutually exclusive [147] [148], they both claim to reproduce the correct answer.

5.2 BPS string solutions

On the strongly coupled side of the AdS/CFT correspondence, supersymmetric string solutions dual to gauge theory objects are of primary importance in the understanding of the non-perturbative features of the theory and for gaining some clue about the high loop structure. Indeed, the most powerful arguments in support of the correspondence would be all-loop, interpolating functions, whose value is exact in the 't Hoof coupling constant λ both at weak and strong coupling.

It was reviews in Section 3.1 that the strong coupling behaviour of Wilson loops lying on the boundary of AdS is dominated by the area of the worldsheet of the classical string solution ending on the contour and extending to the origin of AdS. In the context of $\mathcal{N} = 6$ super Chern-Simons matter theories, the fundamental string solution ending on circular loop and preserving half of the supersymmetries of the vacuum was actually found in [119] before the formulation of the half BPS Wilson loop operator of [46] (4.10), for whom constituted hopeful inspiration.

The strong coupling flow of ABJM theory with gauge group $U(N) \times U(N)$ depends on how the planar limit $N \to \infty$ is taken with respect to the Chern-Simons level k. For $k \ll N \ll k^5$ the correct description is in terms of type IIA superstrings on $AdS_4 \times \mathbb{C}p^3$ with two-form and four-form fluxes turned on, while for higher values of N the theory is better described by the world-volume theory of N $M2-$branes probing the orbifold singularity of $AdS_4 \times \mathbb{C}^4/\mathbb{Z}_k$. The gravitational dual object to a supersymmetric Wilson loop in the fundamental representation is therefore the fundamental string solution in the $AdS_4 \times \mathbb{C}p^3$ backgroud. It is known that for Wilson loops in higher rank representation the fundamental string description fails and the loop operator should be described by probe $D2$, $D6-$branes [149]. The situation is totally analogous to the four-dimensional case where $D3$, $D5$-branes probe loops in symmetric, respectively antisymmetric, high rank representation [150] [151] [152] [103] [153] [154] [155].

In the $\alpha' \to 0$ limit the type IIA background can be written as

$$\begin{aligned} \mathrm{d}s^2_{\text{string}} &= \frac{R^3}{k}\left(\frac{1}{4}\mathrm{d}s^2_{AdS_4} + \mathrm{d}s^2_{\mathbb{C}P^3}\right) \\ \mathrm{e}^{2\phi} &= \frac{R^3}{k^3} \\ F_2 &= kJ \\ F_4 &= \frac{3}{8}R^3\epsilon_{AdS_4} \end{aligned} \tag{5.7}$$

where F_2, F_4 are the two and four-form fluxes from the Ramond-Ramond sector, ϵ_{AdS_4} is the volume-form of unit AdS_4 and J is the Käler form of $\mathbb{C}P^3$. Moreover the radius of AdS satisfies the relation

$$R^3 = 2^{5/2}\pi\sqrt{Nk} \tag{5.8}$$

It is more convenient to use Poincare coordinates and rewrite the metric as

$$\mathrm{d}s^2 = \frac{R^3}{4k}\frac{1}{y^2}\left(\mathrm{d}y^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2\right) + \frac{R^3}{k}\mathrm{d}s^2_{\mathbb{C}P^3} \tag{5.9}$$

where the conformal boundary of AdS is mapped in $y = 0$. Now let us turn to the boundary conditions. The string solution should end on a circular loop at the boundary and extend itelf into AdS_4, just as in the AdS_5 case [38] [39]. This is achieved by letting a circular contour parametrized by $x_1^2 + x_2^2 = R^2$ lie at $x_3 = 0$ on the boundary. Then, let $*g$ be the induced metric on the string worldsheet, the classical Nambu-Goto action reads

$$S = \frac{1}{2\pi}\int \mathrm{d}^2\sigma\sqrt{*g} \tag{5.10}$$

Again the AdS_5 case [156] [40] is a guide-line in the search for the solution, which in turn reads

$$x_1^2 + x_2^2 + x_3^2 = R^2 \tag{5.11}$$

and carries total action

$$-\frac{R^3}{4\pi} = -\pi\sqrt{2\lambda} \tag{5.12}$$

The leading exponential behaviour of the string solution then reads

$$\langle W_{rmstring}\rangle \sim \mathrm{e}^{\pi\sqrt{2\lambda}} \tag{5.13}$$

Again, as in the SYM counterpart this is the half BPS solution dual to the circular Wilson loop in (4.10): it preserves the same half of the $\mathcal{N} = 6$ supersymmetry ABJM theory. Quite remarkably this solution is the same as the one in the AdS_5 case, and was found just by embedding the latter into AdS_4. Note however that this picture does not hold beyond the classical limit, where it gets spoiled by quantum fluctuations around the saddle point of the string action.

A comment is in order here. On the weakly coupled side of the conjecture, each scalar fields carry an $R-$index of $SU(4)$ which represent a coordinate inside the $\mathbb{C}P^3$ part of the dual string background. As in the $\mathcal{N} = 4$ case, the most supersymmetric Wilson loop operator would be dual to a fundamental string solution localized at one point inside $\mathbb{C}P^3$. Both the $\frac{1}{2}$ and the $\frac{1}{6}$ BPS operators are coupled to two scalars C_I, $\overline{C}^J$ in the fundamental, respectively antifundamental, representation of $SU(4)$ through the matrix M_J^I. In the first case, a localized scalar source in $\mathbb{C}P^3$ would break the $R-$symmetry $SU(4) \to U(1)\times SU(3)$ and in this case the Z_k orbifold would leave half of the Killing spinors invariant, which is consistent with the loop operator being half BPS. On the other hand, in the second case the scalar coupling breaks $SU(4) \to SU(2) \times SU(2)$, which corresponds on the string side to a scalar source term *smeared* on a circle, or better a $\mathbb{C}P^1 \in \mathbb{C}P^3$. In this latter case only one sixth of the supercharges survive the orbifolding [44]. Even if the leading exponential beahviour of the two solutions is substantially the same, being dictated essentially by the AdS_4 part, a smeared scalar source can in principle generate a backreaction on the Käler potential [157] and therefore on F_2, which in turn could account on a correction of (5.13). Such a difference between the two solutions is indeed evident in the matrix model derivation presented in the next section.

5.3 Localization and matrix models

The perturbative analysis of field theoretic quantities is hardly constrained by the pestering rapidity with which Feynman diagrams become more and more cumbersome increasing the order of perturbation theory. Several attempts to reorganize the perturbation series into more tractable entities have been done in the past, among others let us cite the Bethe-Salpeter equation[1]. This method is based on the partial resummation of certain classes of diagrams, usually ladder diagrams, through a recursive integral equation. It was successfully emplyed in [101] to compute the exact contribution to a circular Wilson loop in $\mathcal{N} = 4$ SYM at any value of the coupling constant of the ladder diagrams

$$\langle W(\mathcal{C})\rangle_{\mathrm{ladd}} = \frac{2}{\sqrt{g^2N}} I_1(\sqrt{g^2N}) \tag{5.14}$$

where I_1 is the Bessel function. Quite interesting the strong coupling expansion of this partial contribution

$$\langle W(\mathcal{C})\rangle_{\mathrm{ladd}} \sim \frac{\mathrm{e}^{\sqrt{g^2N}}}{(\pi/2)^{1/2}(g^2N)^{3/4}} \tag{5.15}$$

is in remarkable agreement with the AdS/CFT prediction of [40] [156] [160]

$$\langle W(\mathcal{C})\rangle_{AdS} = \mathrm{e}^{\sqrt{g^2N}} \tag{5.16}$$

In more recent years, [161] shew how to localize infinite dimensional field theorypath-integrals of certain supersymmetric observables to ordinary matrix integrals, thus opening the way to the application of every known bit of matrix model technology to the computation of proper gauge theory objects. Promptly, the technique of localization was applied to the computation of BPS observables in $\mathcal{N} = 4$ SYM [162] [163] [164], and also in the evaluation of the partition function [165] and correlatoras [166] of Wilson loops in $\mathcal{N} = 2$ theories and $\mathcal{N} = 6$ super Chern-Simons-matter [167]. In particular [167] proposed a matrix model description for the partition function of $\mathcal{N} = 6$ super Chern-Simons-matter theory on the three-sphere which was then identified with the matrix model on a certain *lens space* derived in [168] [169]. This in turn allowed for the application of large N matrix model technique to the evaluation of all-order expressions for the partition function and the circular Wilson loop

[1] The interested reader is referred to [158] and [159] for complete and self-consistent reviews.

of ABJM theory [170] [171] [172], which constituted the first examples of exact interpolating functions between the weakly and the strongly coupled regimes of this theory.

5.3.1 Planar limit of the ABJM matrix model

The matrix model of ABJ theory with gauge group $U(N_1) \times U(N_2)$ on the three-sphere S^3 was derived in [167] using localization with respect to a specific supercharge and reads

$$Z_{\mathrm{ABJM}}(N_1, N_2, g_s) = \frac{\mathrm{i}^{-\frac{1}{2}(N_1^2 - N_2^2)}}{N_1! N_2!} \int \prod_{i=1}^{N_1} \frac{\mathrm{d}\mu_i}{2\pi} \prod_{j=1}^{N_2} \frac{\mathrm{d}\nu_j}{2\pi} \frac{\prod_{i<j} \left(2\sinh\left(\frac{\mu_i - \mu_j}{2}\right)\right)^2 \left(2\sinh\left(\frac{\nu_i - \nu_j}{2}\right)\right)^2}{\prod_{i,j} \left(2\cosh\left(\frac{\mu_i - \nu_j}{2}\right)\right)^2} \mathrm{e}^{-\frac{1}{2g_s}\left(\sum_i \mu_i^2 - \sum_j \nu_j^2\right)} \tag{5.17}$$

being the coupling $g_s = \frac{2\pi \mathrm{i}}{k}$. This is the relevant starting point for the evaluation of the partition function or expectation value of supersymmetric Wilson loops, indeed (5.17) was localized with respect to the same supercharge under which the $\frac{1}{6}$ BPS Wilson loop of [44] is left invariant. Note that the correct normalization of the integral above was introduced in [172]. In [170] it was noted that the latter matrix integral is closely related to the $L(2,1)$ lens space matrix model [168] [169]

$$\begin{aligned} Z_{L(2,1)}(N_1, N_2, g_s) = {}& \frac{\mathrm{i}^{-\frac{1}{2}(N_1^2 + N_2^2)}}{N_1! N_2!} \int \prod_{i=1}^{N_1} \frac{\mathrm{d}\mu_i}{2\pi} \prod_{j=1}^{N_2} \frac{\mathrm{d}\nu_j}{2\pi} \prod_{i<j} \left(2\sinh\left(\frac{\mu_i - \mu_j}{2}\right)\right)^2 \\ & \times \left(2\sinh\left(\frac{\nu_i - \nu_j}{2}\right)\right)^2 \times \prod_{i,j} \left(2\cosh\left(\frac{\mu_i - \nu_j}{2}\right)\right)^2 \mathrm{e}^{-\frac{1}{2g_s}\left(\sum_i \mu_i^2 + \sum_j \nu_j^2\right)}. \end{aligned} \tag{5.18}$$

by a simple analytic continuation

$$Z_{\mathrm{ABJM}}(N_1, N_2, g_s) = Z_{L(2,1)}(N_1, -N_2, g_s). \tag{5.19}$$

Since the large N expansion of the free energy gives a sequence of analytic functions of N_1, N_2, once these functions are known in one model, they can be obtained in the other by the trivial change of sign $N_2 \to -N_2$. In the large N limit the matrix potential in (5.18) developes two cuts, $\mathcal{C}_a$ and $\mathcal{C}_b$, and the two sets of eigenvalues μ_i and ν_i condense around these singularities. The solution to the large N expansion is encoded in the resolvent of the matric integral, which in the planar limit reads [173]

$$\omega_0(z) = 2\log\left(\frac{\mathrm{e}^{-\lambda/2}}{2}\left[\sqrt{(Z+b)(Z+1/b)} - \sqrt{(Z-a)(Z-1/a)}\right]\right), \tag{5.20}$$

where $Z = \mathrm{e}^z$, $\lambda = \lambda_1 + \lambda_2$ is the total 't Hooft coupling and $a, \frac{1}{a}, b, \frac{1}{b}$ are related to the endpoints of the two cuts where eigenvalues sit in the complex $z-$plane

$$\mathcal{C}_a = \left[\frac{1}{a}, a\right], \qquad \mathcal{C}_b = \left[-b, -\frac{1}{b}\right] \tag{5.21}$$

Using standard model techniques, the relevant quantities can noe be computed in terms of period integrals of $\omega_0(z)\mathrm{d}z$, for example the planar free energy F_0 reads

$$\mathcal{I} \equiv \frac{\partial F_0}{\partial t_1} - \frac{\partial F_0}{\partial t_2} - \frac{\pi \mathrm{i} t}{2} = -\frac{1}{2}\oint_{\mathcal{D}} \omega_0(z)\mathrm{d}z, \tag{5.22}$$

where $\mathcal{D}$ is cycle in the complex plane that goes through the cuts. This tool can now be used for the computation of observables in ABJ, in particular we are interested in circular Wilson loops.

5.3.2 All-orders circular Wilson loops

The vacuum expectation value of the $\frac{1}{6}$ BPS circular Wilson loop of [44] can be computed through localization using the matrix model (5.17), as was shown in [167]

$$\left\langle W_R^{1/6} \right\rangle = g_s \left\langle \mathrm{Tr}_R \left(\mathrm{e}^{\mu_i}\right)\right\rangle_{MM} \tag{5.23}$$

where R is a representation of the gauge group $U(N_1)$. It was further argued in [46] that the $\frac{1}{2}$ BPS Wilson loop constructed there also localizes to the same matrix model

$$\left\langle W_{\mathcal{R}}^{1/2} \right\rangle = g_s \left\langle \mathrm{STr}_{\mathcal{R}}\, U\right\rangle_{MM} \tag{5.24}$$

with the matrix

$$U = \begin{pmatrix} \mathrm{e}^{\mu_i} & 0 \\ 0 & -\mathrm{e}^{\nu_i} \end{pmatrix} \tag{5.25}$$

in a representation $\mathcal{R}$ of the supergroup $U(N_1|N_2)$. This remarkable feature is related to the fact that the two operators belong to the same cohomology class of the commonly conserved supercharge Q^{12+}, and fermionic contributions to the half BPS loop operator are $Q-$exact w.r.t. this supercharge [46]. Therefore, since Q^{12+} is also the localizing supercharge, they do not contribute to the matrix model.

The information needed to solve the matrix model is totally encoded in the resolvent $\omega(Z)$, indeed defining the eigenvalue distributions for the two cuts $\mathcal{C}_a$ and $\mathcal{C}_b$

$$\begin{aligned} \rho^{(1)}(Z)\mathrm{d}Z &= -\frac{1}{4\pi \mathrm{i}\lambda_1}\frac{\mathrm{d}Z}{Z}(\omega(Z+\mathrm{i}\epsilon) - \omega(Z-\mathrm{i}\epsilon)), \qquad Z \in \mathcal{C}_1 \\ \rho^{(2)}(Z)\mathrm{d}Z &= -\frac{1}{4\pi \mathrm{i}\lambda_2}\frac{\mathrm{d}Z}{Z}(\omega(Z+\mathrm{i}\epsilon) - \omega(Z-\mathrm{i}\epsilon)), \qquad Z \in \mathcal{C}_2 \end{aligned} \tag{5.26}$$

the expectation value of the loop operators in the fundamental representation of the corresponding gauge (super)groups is expressed in terms of period integrals

$$\left\langle W_{\Box}^{1/6} \right\rangle = \lambda_1 \int_{\mathcal{C}_1} \rho^{(1)}(Z) Z \mathrm{d}Z = \frac{1}{4\pi \mathrm{i}} \oint_{\mathcal{C}_1} \omega(Z) \mathrm{d}Z \tag{5.27}$$

$$\left\langle W_{\Box}^{1/2} \right\rangle = \lambda_1 \int_{\mathcal{C}_1} \rho^{(1)}(Z) Z \mathrm{d}Z - \lambda_2 \int_{\mathcal{C}_2} \rho^{(2)}(Z) Z \mathrm{d}Z = \frac{1}{4\pi \mathrm{i}} \oint_{\infty} \omega(Z) \mathrm{d}Z \tag{5.28}$$

Nothe that the second period integral is to be taken over a contour encircling both cuts, which for the analyticity properties of the resolvent is equivalent to a contour encircling the point at infinity. This last case is therefore simpler because of the fact that the leading contribution to the integral can be obtained expanding around $Z \sim \infty$. At this stage the solution yet appears quite implicit. It is convenient to define the two variables

$$\zeta = \left(a + \frac{1}{a} - b - \frac{1}{b} \right), \qquad \beta = \left(a + \frac{1}{a} + b + \frac{1}{b} \right) \tag{5.29}$$

then the dirivatives of (5.27) w.r.t. ζ, β admit an explicit expression in terms of elliptic integral functions[2]

$$\begin{aligned} \partial_\zeta \left\langle W_{\Box}^{1/6} \right\rangle &= -\frac{1}{\pi \sqrt{ab}(1+ab)} \left[aK(k) - (a+b)\Pi(n|k) \right] \\ \partial_\beta \left\langle W_{\Box}^{1/6} \right\rangle &= -\frac{2\sqrt{ab}}{\pi(a+b)} E(k) \end{aligned} \tag{5.30}$$

whereas the expression for the half BPS operator reduces to a much simpler form in the planar limit

$$\left\langle W_{\Box}^{1/2} \right\rangle_{\text{planar}} = \frac{\zeta}{2} \tag{5.31}$$

These expression hold at any order of perturbation theory, they are interpolating functions from the weakly coupled regime to the strongly coupled one and thus constitute primary importance checks for the AdS_4/CFT_3 correspondence. To this aim, following again [172], we can expand the latter expression in both regions and make contact with perturbative of Section 4.2.4 and non-perturbative results of Section 5.2. To expand (5.30) and (5.31), [172] relates the different regions of the guage theory to the expantion around different singularities of the mirror geometry dual to the matrix model. This is a standard technique in matrix models/topological strings computation and we do not want to get into details nor to introduce the needed background in special geometry. Indeed, we are just pleased by recalling the results.

In the 't Hooft limit the $\frac{1}{2}$ BPS Wilson loop obeys the perturbative expansion

[2] See [174] for a general discussion about how to derive this result.

$$\left\langle W_{\square}^{1/2}\right\rangle = \mathrm{e}^{\mathrm{i}\pi(\lambda_1-\lambda_2)}2\pi\mathrm{i}(\lambda_1+\lambda_2)\left[1-\frac{\pi^2}{6}(\lambda_1^2-4\lambda_1\lambda_2+\lambda_2^1)+\right.$$
$$\left.+\frac{\pi^4}{120}(\lambda_1^4-6\lambda_1^3\lambda_2-4\lambda_1^2\lambda_2^2-6\lambda_1\lambda_2^3+\lambda_2^4)+\mathcal{O}(\lambda^6)\right] \quad (5.32)$$

First note a phase factor appearing in front of (5.32); it is a framing dependent factor and emerges from the fact that the matrix model always reproduces the Wilson loop with framing one. Note also the discussion about this topic at the end of [46]. Secondly, there is an overall factor in (5.32) which depends upon normalization, it is an inessential factor. Reproducing this expansion from a direct field theory computation is a subtle task in that the framing should preserve the half BPS character of the Wilson loop. This can be achieved considering a couple of Hopf fibers intersecting once, but the number of graphs involved at two-loops is discouraging. the weak coupling expantion of the $\frac{1}{6}$ BPS loop in turn reads

$$\left\langle W_{\square}^{1/6}\right\rangle =$$
$$\mathrm{e}^{\mathrm{i}\pi\lambda_1}2\pi\mathrm{i}\lambda_1\left[1-\frac{\pi^2}{6}(\lambda_1^2-6\lambda_1\lambda_2)-\mathrm{i}\frac{\pi^3}{2}\lambda_1\lambda_2^2+\frac{\pi^4}{120}(\lambda_1^4-10\lambda_1^3\lambda_2-20\lambda_1\lambda_2^3)+\mathcal{O}(\lambda^5)\right] \quad (5.33)$$

Again, there is a framing dependent phase and a normalization dependent overall factor. More importantly note that this results holds for a generic gauge group $U(N_1)\times U(N_2)$ and is not confined to ABJM case. The expression above is consistent with the two-loops results of [44] [119] [45]. However the three-loops computation of [45] was done in the ABJM slice and is insensitive of odd-order terms in the perturbation theory.

At strong coupling, the exact result (5.31) reproduces the expected exponential behaviour together with a full perturbative series of $\frac{1}{\sqrt{\lambda}}$ corrections

$$\left\langle W_{\square}^{1/2}\right\rangle = \frac{1}{2}\mathrm{e}^{\mathrm{i}\pi B}\kappa(\widehat{\lambda},B) \quad (5.34)$$

where $\widehat{\lambda}$ is the 't Hooft parameter shifted by the presence of a non-zero $B-$field in the background and the function κ is defined as

$$\begin{aligned}\kappa(\widehat{\lambda},B) &= \mathrm{e}^{\pi\sqrt{2\widehat{\lambda}}}\left[1+\sum_{j\geq 1}c_i(1/\pi\sqrt{2\widehat{\lambda}},\beta)\mathrm{e}^{-2j\pi\sqrt{2\widehat{\lambda}}}\right]\\ c_j(\alpha,\beta) &= \sum_{k=0}^{2j-1}c_k^{(j)}(\beta)\alpha^k\end{aligned} \quad (5.35)$$

being $c_k^{(j)}$ Laurent polynomials of degree j in β. The shift of $\widehat{\lambda}$ has a notable geometrical meaning. Indeed ABJ theory with gauge group $U(N_1)\times U(N_2)$ is conjectured to be dual to a type IIA background where N_1 is the number of D2 branes and $N_1 - N_2$ the number of D4 branes (set $N_1 > N_2$). It was shown in [175] that the actual charge of D2's is $Q = N_1 - \frac{k}{2}(B^2 - 1/4) - \frac{1}{24}(k - 1/k)$, where $B = -\frac{N_1 - N_2}{k} - \frac{1}{2}$ is the value of the $B-$field flux in the background. At the end of the day, the 't Hooft coupling gets a shift proportional to the flux of B

$$\widehat{\lambda} = \frac{Q}{k} \tag{5.36}$$

The same analysis can be carried for the less BPS Wilson loop and it gives

$$\left\langle W_{\square}^{1/6} \right\rangle = -\frac{\sqrt{2\lambda}}{4} \mathrm{e}^{\mathrm{i}\pi\lambda_1} \mathrm{e}^{\pi\sqrt{2\lambda}} \tag{5.37}$$

Note that in both expressions (5.34) and (5.37) a framing factor under the appearance of an overall phase is present, and is precisely due to the presence of a non-vanishing value of the antisymmetric $B-$field. This feature has also been observed in [173] [176] [177].

Comparing the strong coupling expansion of $W_{\square}^{1/2}$ and $W_{\square}^{1/6}$ with the expressions derived in the Section 5.2, one can observe a perfect agreement for the leading term in the exponential, which is the action of the classical string configuration. Furthermore the $\frac{1}{6}$ BPS operator captures a $\sqrt{\lambda}$ dependent multiplicative factor which is absent in the $\frac{1}{2}$ BPS case and is possibly due to the different scalar sources that enter the string solution. As was already observed in [44], while the most supersymmetric string solution is localized to a point in $\mathbb{C}P^3$, which fact is responsible of the $R-$symmetry breaking from $SU(4) \to U(1) \times SU(3)$ in the corresponding loop operator, the less supersymmetric operator break this symmetry to $SU(2) \times SU(2)$ and therefore it is expected to be dual to a smeared scalar source on a $\mathbb{C}P^1$ inside $\mathbb{C}P^3$.

CHAPTER 6

Deformed Quark-Antiquark Potential

6.1 Interpolating functions for non-BPS observables

6.1.1 Quark-antiquark potential in $\mathcal{N} = 4$ SYM

The supersymmetric analogue of QCD's quark-antiquark potential is the interaction between a copy of BPS particles in the broken phase of the theory. Such states, protected by supersymmetry and thus perturbatively stable, are obtained via Higgsing some scalars by giving them a non-vanishing vacuum expectation value in the way shown in Section 4.2.1 (there the computation is referred to the particular case of $\mathcal{N} = 6$ SCS theories but the way of proceeding is rather general). We preserve the notation of $W-$bosons for those particles.

In the QCD case, the effective interaction of a $q\bar{q}$ pair is given by the expectation value of a rectangular Wilson loop of sides T and L extending in the time and one spatial directions

$$V(T, L, g_{QCD}) \sim \log \langle \mathcal{W}_{\square}(T, L, g_{QCD})\rangle \tag{6.1}$$

In the limit where the time-like side becomes infinitely long $T \gg L$, the Wilson loop can be approximated by a pair of anti-parallel lines representing the worldlines of the particle-antiparticle bound state. The supersymmetric case is rather similar. In $\mathcal{N} = 4$ SYM the relevant Wilson loop is given by the $\frac{1}{2}$ BPS operator of [39] (3.32) along a pair of anti-parallel lines

$$W(\mathcal{C}) = \frac{1}{N}\langle 0|\mathrm{Tr}\mathcal{P}\exp\left(\mathrm{i}g\oint_{\mathcal{C}} \mathrm{d}s\left[\dot{x}^{\mu}A_{\mu} + |\dot{x}|\theta^{I}\phi_{I}\right]\right)|0\rangle \tag{6.2}$$

and the superconformal symmetry of the theory constrains the potential to be of the Coulomb type

$$\langle \mathcal{W}_{\mathrm{SYM}}\rangle \propto \exp\left\{-TV(L, \lambda)\right\} \tag{6.3}$$

where $\lambda = g_{YM}^2 N$ is the 't Hooft coupling constant and

$$V(L,\lambda) = \frac{f(\lambda)}{L} \tag{6.4}$$

being $f(\lambda)$ some function that does not depend on the geometrical data of the Wilson loop. This operator was studied at both the perturbative and non-perturbative level in [101]. By a resummation of all planar diagrams without internal vertices, it was shown that at the second order of perturbation theory its expectation value reads

$$\left\langle \mathcal{W}_{\rm SYM}^{\rm ladders} \right\rangle = \exp\left\{ \left[\frac{\lambda}{4\pi} + \frac{\lambda^2}{4\pi^3}\log\lambda + \mathcal{O}(\lambda^3) \right] \frac{T}{L} \right\} \tag{6.5}$$

The appearance of a strictly non-perturbative term already at two-loops level can be understood as an IR divergence due to resummation of ultra-soft gluons emitted at a scale $\mathcal{O}(\lambda/L)$ which in the weak coupling approximation is much smaller than the soft, or IR, scale of order $\mathcal{O}(1/L)$ [178], see also [179]. Note that the exponent of (6.5) depends only multiplicatively on the dimensions of the contour, which dependence gives rise to a multiplicative and regularization-dependent linear divergence, in agreement with the considerations made above. Indeed, a more accurate computation shows that ultra-soft corrections to the $W-$pair potential of $\mathcal{N}=4$ SYM can be resummed by means of a RG equation giving [179]

$$V(L,\lambda) = -\frac{\lambda^{1+\lambda\frac{2}{\pi}}}{L} \tag{6.6}$$

This result is perturbatively correct up to terms $\lambda^{n+1}\log^n\lambda$ at the n^{th} order, in agreement with (6.5).

6.1.2 Deforming observables

On the other hand, on the strong coupling side AdS/CFT predicts that the behaviour of the anti-parallel Wilson lines for large values of the 't Hooft coupling is essentially regulated by the string tension [39] [38]. Stringy corrections seems hard to compute. This task was first addressed in [180] [181], where the determinants of fluctuations around the semi-classical solution were derived. At present, still, there is no analytical evaluation of these determinants; a numeric result exists [182] and a one-dimensional integral reformulation was proposed [183]. The bottom line is

$$\left\langle \mathcal{W}_{\rm SYM}^{AdS/CFT} \right\rangle = \exp\left\{ \left[\frac{4\pi^2\sqrt{\lambda}}{\Gamma^4(1/4)} + \mathcal{O}(1/\sqrt{\lambda}) \right] \frac{T}{L} \right\} \tag{6.7}$$

where $\mathcal{O}(1/\sqrt{\lambda}) = (-1.3359\ldots)/\sqrt{\lambda}$ is the numerical result of [182]. This result is not reproduced by resumming all planar non-interacting diagrams [101]

$$\left\langle \mathcal{W}_{\rm SYM}^{\rm ladders} \right\rangle = \exp\left\{ \left[\frac{\sqrt{\lambda}}{\pi} - 1 + \mathcal{O}(1/\sqrt{\lambda}) \right] \frac{T}{L} \right\} \tag{6.8}$$

but nonetheless planar ladders capture the leading exponential behaviour. Evidently (6.6) neither reproduces (6.7).

Let us recall the discussion at the beginning of Section 5.3. There we saw that that the same resummation of ladder diagrams gave in the case of the circular loop an all-order exact answer (5.14) [101]

$$\langle W(\mathcal{C})\rangle_{\text{ladd}} = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \tag{6.9}$$

which correctly reproduces the strong coupling behaviour. This particular Wilson loop preserves half of the $\mathcal{N} = 4$ supersymmetry, it is $\frac{1}{2}$ BPS, and its expectation value on the four-sphere S^4 was shown to localize to a matrix integral [161].

A rather different behaviour emerges in the perturbative expansion of the circular loop with respect to the anti-parallel Wilson lines, and possibly it is entirely due to supersymmetry. Recently, the problem of relating the two quite different expressions (6.5) and (6.7) has been faced with a different perspective in [49]. The crucial point is that one can take advantage of the BPS character of the circular loop by determining an interpolating observable that sweeps from the anti-parallel lines on one side to the circle itself on the other side. This observable was indeed determined by applying two deformations to the Wilson loop operator governed by a geometrical parameter ϕ and a coupling parameter θ. Note that the second was already introduced in [39], more on this point is said in the next section. The fact of being $\frac{1}{2}$ BPS in a certain limit of the two parameters makes this observable easier to compute, even at strong coupling. The generalized $W\overline{W}$-potential that derive therein depends on the deformations through θ and ϕ, and on the 't Hooft coupling constant λ. As already stated, superconformal symmetry bounds it to have the Coulomb form; at weak coupling it will admit the expansion

$$V(\phi, \theta, \lambda) = \sum_{n>0} \left(\frac{\sqrt{\lambda}}{4\pi}\right)^{2n} V^{(n)}(\phi, \theta), \qquad \lambda \ll 1 \tag{6.10}$$

that we have conveniently written to make contact with the strong coupling expansion, which reads

$$V(\phi, \theta, \lambda) = \frac{\sqrt{\lambda}}{4\pi} \sum_{n\geq 0} \left(\frac{4\pi}{\sqrt{\lambda}}\right)^{n} V^{(n)}_{AdS}(\phi, \theta), \qquad \lambda \gg 1 \tag{6.11}$$

The first few perturbative orders have been computed in [49] for arbitrary values of ϕ, θ and read

$$
\begin{aligned}
V^{(1)}(\phi,\theta) &= -2\frac{\cos\phi-\cos\theta}{\sin\phi}\phi \\
V^{(2)}(\phi,\theta) &= V^{(2)}_{\text{ladders}}(\phi,\theta)+V^{(2)}_{\text{vertices}}(\phi,\theta) \\
V^{(2)}_{\text{vertices}}(\phi,\theta) &= -\frac{2}{3}(\pi^2-\phi^2)V^{(1)}(\phi,\theta) \\
V^{(2)}_{\text{ladders}}(\phi,\theta) &= -\frac{1}{\phi^2}V^{(1)}(\phi,\theta)^2\left[\mathrm{Li}_3(\mathrm{e}^{2\mathrm{i}\phi})-\zeta(3)-\mathrm{i}\phi\left(\mathrm{Li}_2(\mathrm{e}^{2\mathrm{i}\phi})+\frac{\pi^2}{6}\right)+\mathrm{i}\frac{\phi^3}{3}\right]
\end{aligned}
\tag{6.12}
$$

On the strong coupling side, again one is forced to face functional determinants. An integral expression was given in [49] that can be computed numerically to arbitrary precision. But remarkably the first correction $V^{(0)}_{AdS}$ was obtained in closed form. In the limit where $\phi\to\pi,\theta\to0$, or otherwise stated, where the generalised potential approaches the true potential, it reads

$$
V^{(0)}_{AdS}(\phi,\theta)=\frac{16\pi^3}{(\pi-\phi)\Gamma(1/4)^4},\ \text{for}\ \phi\to\pi,\ \theta\to0 \tag{6.13}
$$

that precisely matches the result of [39] [38].

The aim of the next section is to study such a deformation in the case the three-dimensional $\mathcal{N}=6$ superconformal Chern-Simons theories of [36] [37] that seem to have many common features with $\mathcal{N}=4$ SYM, and that we reviewed in the previous sections.

6.2 Deforming the loop operator

6.2.1 The wedge contour

Following the authors of [49], let us consider a wedge[1] contour in Euclidean 3−dimensional spacetime

$$
\mathcal{C}(s)=P_1\,\theta(-s)\,s\cup P_2\,\theta(s)\,s=\{s\,\cos\frac{\phi}{2},|s|\,\sin\frac{\phi}{2},0\},\qquad s\in(-\infty,+\infty) \tag{6.14}
$$

parametrized by the two momenta P_1 and P_2 chosen to conveniently lie in the x_1x_2−plane

$$
\begin{aligned}
P_1^\mu &= \{\cos\frac{\phi}{2},-\sin\frac{\phi}{2},0\} \\
P_2^\mu &= \{\cos\frac{\phi}{2},\sin\frac{\phi}{2},0\}
\end{aligned}
\tag{6.15}
$$

[1] We opt to preserve the word "cusp" for the Lorentzian light-like cusp, which subtends an infinite hyperbolic angle, whereas here the angle is circular and finite.

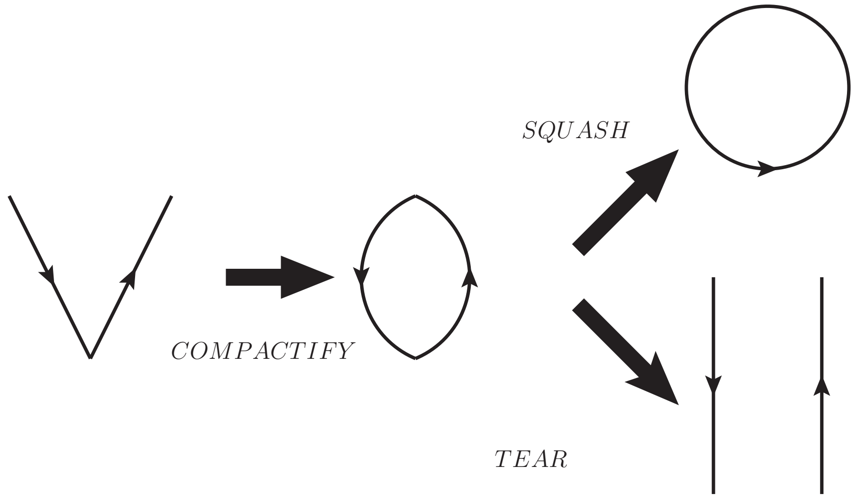


Figure 6.1: On the left side the euclidean cusp contour or wedge. Compactifing the point at infinity the wedge gets mapped to a couple of intersecting circular arcs. Mapping the centers of these arcs away leads to the cicle on the upper-right corner, whereas mapping them closer and focusing on a neighbourhood of the origin they look as the parallel straight lines on the lower-right corner.

The angle ϕ ranges in the interval $0 < \phi < \pi$ and the two semi-infinite edges of $\mathcal{C}(s)$ form an angle $\pi - \phi$ so that for $\phi = 0$ the contour is a straigth line. The wedge contour interpolates between a circle, a pair of anti-parallel lines in $\mathbb{S}^2 \times \mathbb{R}$ with Lorenzian signature and a proper cusp in Minkowski spacetime as sketched in Figure 6.2.1. This can be shown considering the following maps of Euclidean spacetime. In Euclidean signature, conformally mapping the point at infinity to a finite distance turns the two edges into symmetric arcs (with respect to the x_2-axis) of radius $r = 1/(1 - \sin\frac{\phi}{2})$ passing through the points $\{x_{(1)} = \pm 1,\ x_{(2)} = 0\}$. In the limit were the angle ϕ approaches zero, the two arcs join to form a circle centered at the origin and of radius $r = 1$; while in the opposite limit $\phi \to \infty$ the arcs stretch towards infinity in both directions appearing as a couple of parallel infinite lines. Wick rotating to Minkowsky signature, compactifying the transverse directions and applying the logaritmic map one gets $\mathbb{S}^2 \times \mathbb{R}$. The two edges get mapped to infinite parallel lines in the time ($\mathbb{R}$) direction, lying on the border of $\mathbb{S}^2$ and separated by an angle $\pi - \phi$. Again, letting $\phi \to 0$ these lines become antipodal on $\mathbb{S}^2$ and the contour such obtaind is half BPS.

6.2.2 Minimal deformation

A natural question arises so far. It concerns the possibility of coupling the loop operator (4.10) to the wedge contour in such a way that the so-formed Wilson loop preserves, at least locally, a certain ammount of the $\mathcal{N} = 6$ supersymmetries of ABJ(M) theory. The Wilson loop operator of [46] was shown to preserve half of the supersymmetries of ABJM theory when coupled to line or circular contours, see also Section 4.2.4. For $\phi \to 0$ the wedge contour appraoches a single straight line in Euclidean space, that can be conformally mapped to a circle, thus we do expect this limit to reproduce a $\frac{1}{2}$ BPS quantity. The keypoint in the recursive proof of [46] that half of the supersymmetry generators and half of the conformal generators are preserved at any order if the relation

$$M^I_J = \delta^I_J + \mathrm{i}\overline{\eta}^\alpha_J \eta^I_\alpha \tag{6.16}$$

between scalar and fermion couplings holds. We will refer to it as the *key relation*. This ensures that, at any order, the supersymmetric variation of Chern-Simons and scalars fields is cancelled by the variation of fermions at the next order. We want to deform both the scalar and fermionic couplings, under the constraint of (6.16), to some functions of an additional parameter, and study the supersymmetry properties of such deformed loop operator as the parameter varies.

The *minimal deformation* of the scalar couplings consists in rotating two of them. Consider that the matrix M^I_J breaks the $SU(4)_R$ symmetry of the ABJ Lagrangean to $SU(3)_R \times U(1)_R$. Any rotation of the scalar fields C^I that preserves $SU(4)_R$ is allowed, but rotations inside the $SU(3)_R$ are trivial and uneffective. Note in fact that, conversely to the 4-dimensional couterpart, the 3-dimensional scalar coupling of (4.10) is a quadratic form, hence any $SU(3)_R$ rotation of the C^I's is compensated by the relative, opposite rotation of the $\overline{C}_I$'s.

Starting from the scalar coupling to the infinite straight line

$$M^I_J = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.17}$$

we rotate the scalars

$$C^I \to \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} & 0 & 0 \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} C^1 \\ C^2 \\ C^3 \\ C^4 \end{pmatrix} = \begin{pmatrix} C^1\cos\frac{\theta}{2} - C^2\sin\frac{\theta}{2} \\ C^2\cos\frac{\theta}{2} + C^1\sin\frac{\theta}{2} \\ C^3 \\ C^4 \end{pmatrix} = U^I_J C^J \tag{6.18}$$

and $\overline{C}_I \to {}^tU^J_I \overline{C}_J$ of an angle $0 < \theta < \pi$. Equivalently we can rotate the matrix M by means of

$$\widetilde{M}^I_J = {}^tU^I_K M^K_L U^L_J = \begin{pmatrix} \sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2} & 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & 0 & 0 \\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6.19}$$

The idea now is to couple the loop operator to the two sides of the wedge contour using two different scalar couplings, say M^I_J for $s < 0$ and $\widetilde{M}^K_L$ for $s > 0$. The deformation above does not preserve the $SU(3)_R \times U(1)_R$ symmetry of the original M^I_J and the Wilson loop will clearly not be $\frac{1}{2}$ BPS away from $\theta = 0, \pi$, although it will stil be **locally** $\frac{1}{2}$ BPS. More precisely, for $\theta = 0$ the two sides of the loop are coupled to same scalars and in the limit where $\phi \to \pi$ this should reproduce correctly the $W-$pair potential

$$\langle \mathcal{W}(\phi \to \pi, \theta \to 0) \rangle \propto \mathrm{e}^{-LV(\lambda,\epsilon)} \tag{6.20}$$

where L is the length of the loop and is necessary to treat IR divergences, whereas ϵ is an UV regulator such as a dimensional regularization parameter. Opposedly to the SYM case, at this stage it is hard in infer on what happens to the configuration with $\theta = \pi$. In the $\mathcal{N} = 4$ analogue this would be a pair of parallel lines, which is a half BPS configuration, but the scalar couplings in (6.19) apparently do not suggest anything alike.

But one has to consider fermionic couplings also. In the case of the infinite straight line, say along x_1, the chirality of the conserved supercharges suggests that the loop operator of [46] should couple the fermionic fields ψ_α^I and $\overline{\psi}_J^\beta$ to a single bi-spinor

$$\overline{\eta}_I^\alpha = \overline{\eta}\delta_I^1\,\delta_+^\alpha \qquad \eta_\beta^J = \eta\delta_1^J\,\delta_\beta^+ \tag{6.21}$$

The $c-$number η is a global phase determined by supersymmetry, and the condition (6.16) gives the constraints $\eta\overline{\eta} = \mathrm{i}$. The chirality of (6.21) is relative to the projector along the line

$$(1 \pm \sigma^1)_\alpha^\beta \eta_\beta = \pm\eta_\alpha^{(\pm)} \tag{6.22}$$

and the $+$ chirality has been chosen according to what stated above. This is by no means the most general coupling one can write consistently with the requirements of supersymmetry, but it is by far the simplest. On the two sides of the contour the projector (6.22) gets rotated of an angle $\pm\frac{\phi}{2}$ around the x_3-axis, consistently the spinor couplings rotate as[2]

$$\begin{aligned} \eta_\alpha^I \to \eta_\alpha^I(s) &= \mathrm{e}^{\mp^s \frac{\mathrm{i}}{2}\sigma^3\frac{\phi}{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \delta_1^I \eta \\ \overline{\eta}_I^\alpha \to \overline{\eta}_I^\alpha(s) &= \mathrm{e}^{\pm^s \frac{\mathrm{i}}{2}\sigma^3\frac{\phi}{2}} (1 \;\; 1)\, \delta_I^1 \overline{\eta} \end{aligned} \tag{6.23}$$

where the sign of the exponent is dictated by the sign of s, in the sense that for $s > 0$ the line is rotated counter-clockwise while for $s < 0$ it is rotated clockwise. This procedure gives the right chirality for coupling spinors to the two edges of the wedge contour (6.14) and can be put into the rather useful pointwise relation

$$\eta_\alpha^I(s)\overline{\eta}_I^\beta(s) = \mathrm{i}\Big(|\dot{x}(s)| + \sigma^\mu x_\mu(s)\Big)_\alpha^{\;\beta} \tag{6.24}$$

where the prefactor of i is required by the normalization of η's. Finally we have to rotate $R-$indices to ensure the key relation (6.16), which in the present case reads

[2]There might seem to be an ambiguity in the sign of the exponent, reflecting itself on the handedness of the rotation. On the other hand, if we choose to rotate the projector counterclockwise as $(R^\dagger)_\alpha^\beta(1+\sigma\cdot x)_\beta^\gamma R_\gamma^\delta$, then the rotation matrix must have a minus sign $R = \mathrm{e}^{-\mathrm{i}\sigma^3\frac{\phi}{4}}$. The requirement that the rotated coupling is still an eigenvector with the same eigenvalue unambiguously states that η rotates as $R^\dagger\eta$.

$$\widetilde{M}^I_J = \delta^I_J + \mathrm{i}\overline{\eta}^\alpha_J \eta^I_\alpha = \delta^I_J + \begin{pmatrix} -2\cos^2\frac{\theta}{2} & 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & -2\sin^2\frac{\theta}{2} \end{pmatrix} \tag{6.25}$$

We conveniently restricted ourselves to the upper block $I, J = 1, 2$. This is readily solved rotating the $R-$index of the bi-spinor to

$$\begin{aligned} \delta^I_1 &\to \zeta^I = \begin{pmatrix} \cos\frac{\theta}{2} \\ -\sin\frac{\theta}{2} \end{pmatrix} \\ \delta^1_I &\to \overline{\zeta}_I = \begin{pmatrix} \cos\dfrac{\theta}{2} & -\sin\dfrac{\theta}{2} \end{pmatrix} \end{aligned} \tag{6.26}$$

while keeping $\overline{\eta}^\alpha_I \eta^I_\alpha = 2\mathrm{i}$ untouched. Arranging the various bits we find the rotated fermionic couplings

$$\begin{aligned} \eta^I_\alpha(s) &= \begin{pmatrix} \mathrm{e}^{\mp^s \frac{\phi}{4}} \\ \mathrm{e}^{\pm^s \frac{\phi}{4}} \end{pmatrix} \left(\delta^I_1 \cos\frac{\theta}{2} - \delta^I_2 \sin\frac{\theta}{2} \right) \eta \\ \overline{\eta}^\alpha_I(s) &= \begin{pmatrix} \mathrm{e}^{\mp^s \frac{\phi}{4}} & \mathrm{e}^{\mp^s \frac{\phi}{4}} \end{pmatrix} \left(\delta^1_I \cos\frac{\theta}{2} - \delta^2_I \sin\frac{\theta}{2} \right) \overline{\eta} \end{aligned} \tag{6.27}$$

Note that the choice of rotating C^1 and C^2 in (6.19) is totally equivalent to the choice of rotating any of C^3 or C^4 together with C^1. In fact this only ammounts to a permutation of the indices $I = 2, 3, 4$ and won't affect the $SU(3)_R$ invariant subgroup of $\widetilde{M}$. Indeed, any element of $SU(4)$ is allowed in line of principle, though more complicated deformation of M will give rise to a larger number of fermionic couplings to be included in order to preserve the local $\frac{1}{2}$ BPS character of the loop operator. This is the bottom line that makes this particular choice *"minimal"*.

6.3 Weak coupling expansion

In Euclidean signature the expectation value of a Wilson loop operator 4.11 with the super-connection being in a representation $\mathcal{R}$ of the supergroup $U(N|M)$ is by definition

$$\langle \mathcal{W}_\mathcal{R} \rangle = \frac{1}{\dim\mathcal{R}} \int \mathcal{D}[A, \widehat{A}, C, \overline{C}, \psi, \overline{\psi}]\, \mathrm{e}^{-S_{\mathrm{ABJ}}}\, \mathrm{Tr}_\mathcal{R} \left[\mathcal{P} \exp\left(\mathrm{i} \int_\Gamma \mathrm{d}s\, L(s) \right) \right] \tag{6.28}$$

It proves useful to rescale the gauge field $A \to \sqrt{\frac{2\pi}{k}} A$, and the same for $\widehat{A}$, so that the CS acition in (B.2) assumes a more suitable form for a weak-coupling expansion in $g = \frac{2\pi}{k}$

$$S_{CS} \to -\frac{\mathrm{i}}{2} \int \mathrm{d}^3x \epsilon^{\mu\nu\rho}\, \mathrm{Tr}(A_\mu \partial_\nu A_\rho + \frac{2}{3}\sqrt{g}\, A_\mu A_\nu A_\rho) \tag{6.29}$$

The superconnection gets rescaled too, and the first few orders of the path-ordered exponential read

$$\mathcal{P}\exp\left(\mathrm{i}\int_\Gamma \mathrm{d}s\, L(s)\right) = \mathbb{I} + \mathrm{i}g\int_\Gamma \mathrm{d}s_i\, L(s_i) + (\mathrm{i}g)^2\int_\Gamma \mathrm{d}s_i \int_{s_j<s_i}\mathrm{d}s_j\, L(s_i)L(s_j) + \dots \tag{6.30}$$

The latter are to be Wick contracted with powers of the ABJ action (B.2) in Appendix and produce effectively an expansion in integer powers of g (actually off-diagonal terms have a semi-integer power of the coupling constant, by they do not contribute to the trace). Note that for pure Chern-Simons theory the sign of the exponent of the path-exponential and the sign of the interaction term in S are related through gauge symmetry as emphasized in (B.9). In our conventions the expectation value of $\mathcal{W}$ at weak coupling becomes

$$\begin{aligned}\langle \mathcal{W}_\mathcal{R}\rangle = 1 + \frac{1}{N+M}\Bigg\{ &-(\mathrm{i}g\mu^{2\epsilon})\mathrm{Tr}\left(\int_\Gamma \mathrm{d}s_i\, L(s_i)\right)\,\mathrm{Tr}\left(\int \mathrm{d}^d x\mathcal{L}_{kin}\right) + \\ &+(\mathrm{i}g\mu^{2\epsilon})^2\mathrm{Tr}\left(\int_\Gamma \mathrm{d}s_i\mathrm{d}s_j\, L(s_i)L(s_j)\right)\frac{1}{2}\mathrm{Tr}\left(\int \mathrm{d}^d x\mathcal{L}_{kin}\right)^2 + \\ &-(\mathrm{i}g\mu^{2\epsilon})^2\mathrm{Tr}\left(\int_\Gamma \mathrm{d}s_i\mathrm{d}s_j\mathrm{d}s_k\, L(s_i)L(s_j)L(s_k)\right)\,\mathrm{Tr}\left(\int \mathrm{d}^d x\mathcal{L}_{matter}\right) + \dots\Bigg\}\end{aligned} \tag{6.31}$$

where we used the dimensionally-regularized action in $d = 3-2\epsilon$ Euclidean dimensions and μ is a mass scale that makes $g\mu^{2\epsilon}$ massless. Also note extra minus signs at odd orders of the expansion of the Euclidean action and the fact the superconnection is the one in (4.47), where $|\dot{x}| \to -\mathrm{i}|\dot{x}|$ is the effect of Wick-rotating to Euclidean signature

$$L \to L_E = \begin{pmatrix} A_\mu \dot{x}^\mu - \mathrm{i}\frac{2\pi}{k}|\dot{x}|M_J^I C_I \overline{C}^J & -\mathrm{i}\sqrt{\frac{2\pi}{k}}\,\overline{\psi}_I^\alpha \eta_\alpha^I \\ -\mathrm{i}\sqrt{\frac{2\pi}{k}}\,\overline{\eta}_J^\beta \psi_\beta^J & \widehat{A}_\nu \dot{x}^\nu - \mathrm{i}\frac{2\pi}{k}|\dot{x}|\widehat{M}_L^K \overline{C}^L C_K \end{pmatrix} \tag{6.32}$$

6.3.1 One-loop

From the discussion in Section 4.2 about the local $\frac{1}{2}$ BPS character of the Wilson loop operator, it follows that diagrams attached with both ends on the same side of the wedge contour should not contribute at any order to the weak coupling expansion. It is therefore convenient to deal with them separately, also in view of the two-loops computation, and it turns out that there is a rather definite physical reason for which they do not contribute indeed. We will extensively use dimensional regularization throughout our computations. Regularizing Chern-Simons-matter theories going off-dimensions places some matter of concern because of the presence of the anti-symmetric $\epsilon^{\mu\nu\rho}$ tensor. We will follow the DRED scheme, shifting the dimension to $d = 3-2\epsilon$ while keeping the σ algebra and $\epsilon^{\mu\nu\rho}$ tensor in strictly 3 dimensions. Note that this brakes the conformal invariance introducing a mass scale $\mu^{2l\epsilon}$, where l is the number of loops, that keeps the action dimensionless.

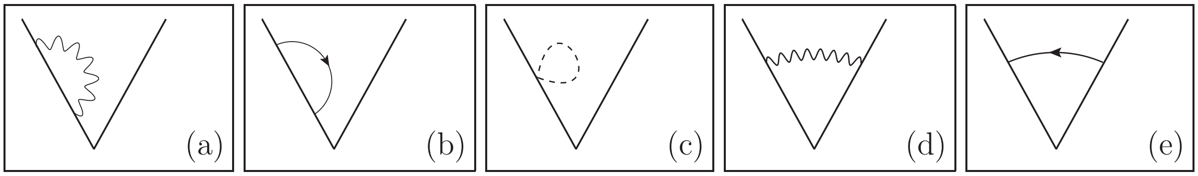


Figure 6.2: At one-loop order there are only two classes of diagrams: single-edge diagrams and and exchange diagrams. Note that scalar fild enters the loop operator through the composite bilinear $M_J^I C_I \overline{C}^J$, and its conjugate $\widehat{M}_J^I \overline{C}^J C_I$, hence the exchange of a single scalar is not permitted. Only CS and fermions propagator can contribute to the interaction at one-loop. As it turns out that CS propagators vanish on any planar contour, only fermions contribute.

At 1-loop order single-edge diagrams are either rainbow or buble diagrams. The most subtle ones are the scalar boubles of Figure 6.3.1 (c). The first order expansion of the super-connection (6.32) contracted with the kinetic term of the ABJ Lagrangean (B.2) generates a scalar bouble inserted at a single point on the cusp contour

$$\left\langle S^{(1)}_{\text{bouble}} \right\rangle = \frac{1}{(N+M)} \left\langle \left(\frac{2\pi}{k} |\dot{x}| M_J^I \underbracket{C_I \overline{C}}^J + \frac{2\pi}{k} |\dot{x}| \widehat{M}_L^K \underbracket{\overline{C}^L C_K} \right) \right\rangle = 0 \ ? \tag{6.33}$$

This nasty diagram involves the contact divergence of the scalar propagator relative to two scalar fields inserted at the same point and contracted together. However this contribution is usually considered to be zero in dimensional regularization for the simple reasoning that, whatever it could be, it should be proportional to some mass scale, but no mass scale is allowed by means of the conformal invariance of the theory. Alternatively, it was argued in [44] that assuming all fields are normal ordered removes the issue *ab initio*. One should also notice that in the $\frac{1}{6}$ BPS loop case, scalar bouble diagrams are proportional to $\text{Tr} M_J^I = 0$, which removes them anyway. This is no longer the case for the $\frac{1}{2}$ BPS loop and as the reasoning above may not appear totally fulfilling, we will be back on it soon.

No other relevant term can come from the first order expansion of the loop operator, but others arise expanding it twice. From now on, group indices will be suppressed not to uselessly weigh the notation too much; they will be restored when needed. At the second order one explicitly has

$$
\begin{aligned}
&\mathrm{i}^2\mathrm{Tr}\int_{s_2>s_1}\mathrm{d}s_1\mathrm{d}s_2 L(s_1)L(s_2)=\mathrm{i}^2\mathrm{Tr}\int_{s_2>s_1}\mathrm{d}s_1\mathrm{d}s_2\Big\{A_\mu(s_1)A_\nu(s_2)\dot{x}_1^\mu\dot{x}_2^\nu\\
&-\mathrm{i}\frac{2\pi}{k}A_\mu(s_1)[M C\overline{C}](s_2)\dot{x}_1^\mu|\dot{x}_2|-\mathrm{i}\frac{2\pi}{k}[M C\overline{C}](s_1)A_\mu(s_2)|\dot{x}_1|\dot{x}_2^\mu\\
&-\left(\frac{2\pi}{k}\right)^2|\dot{x}_1||\dot{x}_2|[M C\overline{C}](s_1)[M C\overline{C}](s_2)-\frac{2\pi}{k}[\overline{\psi}^\beta\eta_\beta](s_1)[\overline{\eta}^\alpha\psi_\alpha](s_2)\\
&+\widehat{A}_\mu(s_1)\widehat{A}_\nu(s_2)\dot{x}_1^\mu\dot{x}_2^\nu-\mathrm{i}\frac{2\pi}{k}\widehat{A}_\mu(s_1)[M\overline{C}C](s_2)\dot{x}_1^\mu|\dot{x}_2|\\
&-\mathrm{i}\frac{2\pi}{k}[M\overline{C}C](s_1)\widehat{A}_\mu(s_2)|\dot{x}_1|\dot{x}_2^\mu-\left(\frac{2\pi}{k}\right)^2|\dot{x}_1||\dot{x}_2|[M\overline{C}C](s_1)[M\overline{C}C](s_2)\\
&-\frac{2\pi}{k}[\overline{\eta}^\alpha\psi_\alpha](s_1)[\overline{\psi}^\beta\eta_\beta](s_2)\Big\}
\end{aligned}
\tag{6.34}
$$

Of these whole bunch of terms, only four admit 1-loop relevant contractions, while the other contribute at higher orders. Contracting gluons in the first and fourth lines of the above equation gives the gluon rainbow for both A and $\widehat{A}$ depicted in Figure 6.3.1 (a) and the single-gluon exchange in (d). They can be rearranged in a single integral

$$
\left\langle G^{(1)}\right\rangle=\mathrm{i}^2\mathrm{Tr}\int_{s_2>s_1}\mathrm{d}s_1\mathrm{d}s_2\left[\underbrace{A_\mu(s_1)A_\nu}(s_2)\dot{x}_1^\mu\dot{x}_2^\nu+\underbrace{\widehat{A}_\mu(s_1)\widehat{A}_\nu}(s_2)\dot{x}_1^\mu\dot{x}_2^\nu\right]
\tag{6.35}
$$

using the tree-level gluon propagator (B.13) all of the terms above can be recast together as a sum over all the possible attachments

$$
\left\langle G^{(1)}\right\rangle=\mathrm{i}N^2\frac{\mu^{2\epsilon}}{k}\frac{\Gamma(\frac{d}{2})}{\pi^{\frac{d}{2}-1}}\sum_{i,j=1}^{2}\int_{s_2>s_1}\mathrm{d}s_1\mathrm{d}s_2\,\dot{x}_i^\mu\dot{x}_j^\nu\epsilon_{\mu\nu\rho}\frac{(\dot{x}_i s_1-\dot{x}_j s_2)^\rho}{|x_i(s_1)-x_j(s_2)|^d}=0
\tag{6.36}
$$

which is zero on any planar loop for the antisymmetry of the epsilon tensor. One should also take into account the contribution of the $\widehat{A}$ field, which differs only by a trace factor of M^2 instead of N^2 in front of the integral. Note however that there will be a minus sign between the two summands of 6.35 because of an extra minus sign between the $\widehat{A}\widehat{A}$ and AA propagators, hence tree-level gluons will always cancel for any loop in the ABJM case. In the more general case of ABJ theory they give a term proportional to $\mathrm{i}(N-M)/k$ (all integrals are real) which can be interpreted quite naturally as a framing contribution. See (5.32) for comparision, where the matrix model computes the circular loop at framing one and the dependence on the framing is purely immaginary and proportional to the flux of the B field $B=(N-M)/k-1/2$ in the dual geometry.

From 6.34 also fermionic rainbow and exchange diagrams arise (Figure 6.3.1 (b) and (e))

$$
\left\langle F^{(1)}\right\rangle=-\mathrm{i}^2\frac{2\pi}{k}\mathrm{Tr}\int_{s_2>s_1}\mathrm{d}s_1\mathrm{d}s_2\left[[\overline{\eta}^\alpha\underbrace{\psi_\alpha](s_1)[\overline{\psi}^\beta}\eta_\beta](s_2)+[\eta_\alpha\underbrace{\overline{\psi}^\alpha](s_1)[\psi_\beta}\overline{\eta}^\beta](s_2)\right]
\tag{6.37}
$$

It is sufficient to consider one of the two contributions, say the first, for the second is analogous. Dimensionally regularised in $d = 3 - 2\epsilon$ dimensions it reads

$$\mathrm{Tr}\left\langle [\overline{\eta}_I^\alpha \psi_\alpha](s_2)[\overline{\psi}^\beta \eta_\beta^I](s_1)\right\rangle = \mathrm{i}MN\frac{\Gamma\left(\frac{1}{2}-\epsilon\right)}{4\pi^{3/2-\epsilon}}(\overline{\eta}_I^\alpha(s_2)\sigma^\mu \eta_\alpha^I(s_1))\frac{\partial}{\partial x_{12}^\mu}\frac{1}{[(x_1-x_2)^2]^{\frac{1}{2}-\epsilon}} \tag{6.38}$$

being obviously $x_i = x(s_i)$. Using (6.24) one can compute the trace of any number of sigma matrices sandwitched between two fermionic couplings

$$\overline{\eta}_1^\alpha(\sigma^\mu \dots \sigma^\lambda)_\alpha^\beta \eta_{2\beta} = -\frac{\mathrm{Tr}\left[(1+\dot{x}_1\cdot\sigma)(\sigma^\mu\dots\sigma^\lambda)(1+\dot{x}_2\cdot\sigma)\right]}{\overline{\eta}_2^\gamma \eta_{1\gamma}} \tag{6.39}$$

where we have simply multiplied and divided the left hand side with $\overline{\eta}_2^\gamma \eta_1^\delta \epsilon_{\gamma\delta}$, moreover according to (6.27) the dependence on the $R-$index can be factorised, so we have

$$\overline{\eta}_I^\alpha(s_2)\sigma^\mu \eta_\alpha^I(s_1) = -\overline{\zeta}_I(s_2)\zeta^I(s_1)\frac{\dot{x}^\mu(s_2)+\dot{x}^\mu(s_1)+\mathrm{i}\epsilon^{\alpha\mu\beta}\dot{x}_\beta(s_1)\dot{x}_\alpha(s_2)}{\overline{\eta}(s_1)^\alpha \eta(s_2)_\alpha} \tag{6.40}$$

We will refer to terms like this as the $R-$charges of the diagrams. The term proportional to the ϵ tensor can be dropped on any planar contour where $\dot{x}_1, \dot{x}_2$ and x_{12} lie on the same plane, so the relevant integral reads

$$-\,\mathrm{i}MN\frac{\Gamma\left(\frac{1}{2}-\epsilon\right)}{4\pi^{3/2-\epsilon}}\int_{s2>s1}\mathrm{d}s_1\,\mathrm{d}s_2\,\frac{\overline{\zeta}_I(s_2)\zeta^I(s_1)}{\overline{\eta}(s_1)^\alpha \eta(s_2)_\alpha}\frac{\partial}{\partial x_{12}^\mu}\frac{\dot{x}_1^\mu+\dot{x}_2^\mu}{[(x_1-x_2)^2]^{\frac{1}{2}-\epsilon}} \tag{6.41}$$

Note from (6.27) that the fermion couplings depend only piecewise on the contour so the factor $\frac{\overline{\zeta}_I(x_2)\zeta^I(x_1)}{\overline{\eta}(x_1)^\alpha \eta(x_2)_\alpha}$ is actually a constant depending only on which side of the wedge the propagator is attached to. It is now straightforward to see that for our parametrisation of the contour the integrand can be always recast in the difference of two total derivatives w.r.t. either s_1 or s_2, so the integral in (6.37) becomes

$$-\,\mathrm{i}MN\frac{\Gamma\left(\frac{1}{2}-\epsilon\right)}{4\pi^{3/2-\epsilon}}\int_{s2>s1}\mathrm{d}s_1\,\mathrm{d}s_2\,\frac{\overline{\zeta}_I(s_2)\zeta^I(s_1)}{\overline{\eta}(s_1)^\alpha \eta(s_2)_\alpha}\left[\frac{d}{ds_1}-\frac{d}{ds_2}\right]\frac{1}{[(x(s_1)-x(s_2))^2]^{\frac{1}{2}-\epsilon}} \tag{6.42}$$

Specialising the above to the case of single-edge diagrams we see that non-vanishing contributions arise. Introducing a regulator L for IR divergences, behold - this breakes gauge invariance, it is easy to prove that the sum of diagrams of type (b) for both $(\mathbf{N},\mathbf{M})$ and $(\mathbf{M},\mathbf{N})$ propagators in (6.37) is proportional to

$$\int_0^L \mathrm{d}s_1 \int_{s_1}^L \mathrm{d}s_2 \left[\frac{d}{ds_1}-\frac{d}{ds_2}\right]\frac{1}{[(s_1-s_2)^2]^{\frac{1}{2}-\epsilon}} \sim \frac{L^{2\epsilon}}{2\epsilon} \tag{6.43}$$

where the $R-$charge on the same leg is simply the normalisation of spinor couplings $\frac{\overline{\zeta}_I\zeta^I}{\overline{\eta}^\alpha \eta_\alpha} = 1/2\mathrm{i}$. Only one term is left, the one-fermion exchange of Figure 6.3.1 (d) relevant for the deformed potential at one-loop. The integral can be computed easly

$$\begin{aligned}\left\langle F^{(1)}_{\text{exch}}\right\rangle &\sim \int_{-L}^{0}\mathrm{d}s_1\int_0^L \mathrm{d}s_2\left[\frac{d}{ds_1}-\frac{d}{ds_2}\right]\frac{1}{[(x(s_1)-x(s_2))^2]^{\frac{1}{2}-\epsilon}}\\ &=\int_0^L\mathrm{d}s_1\int_0^L\mathrm{d}s_2\left[\frac{d}{ds_1}+\frac{d}{ds_2}\right]\frac{1}{[s_1^2+s_2^2+2s_1s_2\cos\phi]^{\frac{1}{2}-\epsilon}}\\ &=2\int_0^L\mathrm{d}s\left[\frac{1}{(s^2+L^2+2Ls\cos\phi)^{\frac{1}{2}-\epsilon}}-\frac{1}{s^{\frac{1}{2}-\epsilon}}\right]\end{aligned} \tag{6.44}$$

where the explicit parametrisation has been used and $|\dot{x}_i|=1$. It is convenient to change variables such that all integrals range from 0 to 1

$$\begin{aligned}&=2L^{2\epsilon}2\int_0^1\mathrm{d}s\left[\frac{1}{(s^2+1+2s\cos\phi)^{\frac{1}{2}-\epsilon}}-\frac{1}{s^{\frac{1}{2}-\epsilon}}\right]\\ &=-\frac{L^{2\epsilon}}{\epsilon}+2L^{2\epsilon}\int_0^1\mathrm{d}s\,\frac{1}{(s^2+1+2s\cos\phi)^{\frac{1}{2}-\epsilon}}\end{aligned} \tag{6.45}$$

the last integral can be done exactly in terms of hypergeometric functions, the finite part as $\epsilon\to 0$ reads

$$\lim_{\epsilon\to 0}\int_0^1\mathrm{d}s\,\frac{1}{(s^2+1+2s\cos\phi)^{\frac{1}{2}-\epsilon}}=\log\left(\sec\frac{\phi}{2}+1\right) \tag{6.46}$$

Putting all bits together, accounting for the $R-$charge term and for the totally equal contribution coming from the second propagator in (6.37) we have

$$\left\langle F^{(1)}_{\text{exch}}\right\rangle=\left(\frac{2\pi}{k}\right)2MN\frac{\Gamma\left(\frac{1}{2}-\epsilon\right)}{4\pi^{3/2-\epsilon}}\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}(\mu L)^{2\epsilon}\left[\frac{1}{\epsilon}+2\log\left(\sec\frac{\phi}{2}+1\right)+\mathcal{O}(\epsilon)\right] \tag{6.47}$$

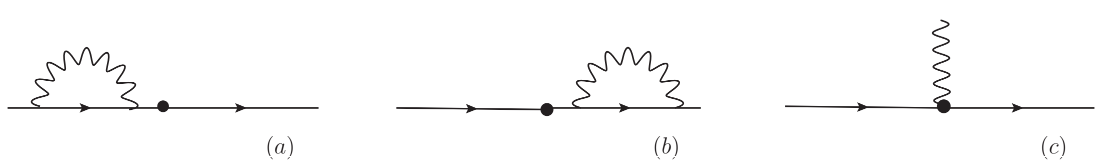


Figure 6.3: Graphs contributing to the renormalization of the Wilson loop. The wavy lines represent the propagation of a general field. In QCD propagators of external probe-particles receive self-energy contributions from (a) and (b). The renormalization of the vertex operator is given by (c). Here they represent scheme-dependent boundary terms due to the broken gauge invariance, and hence supersymmetry, of the Wilson line.

A comment is in order at this stage. Let us go back to the rainbow diagrams in (6.43). We found divergences in the IR regulator L, but since the theory is conformal, the appearance of this term may look puzzling. On the other hand there is no need to indtroduce a regulator, for the following reason that was pointed out in [67]. The term above is clearly regularisation dependent, and would also contribution to VEV of a single Wilson line, which we know from Section 4.2.4 is $\frac{1}{2}$ BPS and should not recieve quantum corrections. If we perform a simple one-loop computation in the same regularization scheme adopted above we would find

$$\langle W(\text{inf line})\rangle = \int_{-L}^{L} \mathrm{d}s_1 \int_{s_1}^{L} \mathrm{d}s_2 \left[\frac{d}{ds_1} - \frac{d}{ds_2}\right] \frac{1}{[(s_1-s_2)^2]^{\frac{1}{2}-\epsilon}} \tag{6.48}$$

that can be split into three pieces

$$\left[\int_{-L}^{0} \mathrm{d}s_1 \int_{s_1}^{0} \mathrm{d}s_2 + \int_{0}^{L} \mathrm{d}s_1 \int_{s_1}^{L} \mathrm{d}s_2 + \int_{-L}^{0} \mathrm{d}s_1 \int_{0}^{L} \mathrm{d}s_2\right] \left[\frac{d}{ds_1} - \frac{d}{ds_2}\right] \frac{1}{[(s_1-s_2)^2]^{\frac{1}{2}-\epsilon}} \tag{6.49}$$

Now, the first and second contributions above are exactly the same as (6.43) and its counterpart. A neat physical interpretation of these term exsists and actually prescribes how to deal with them as [67] pointed out. Consider in QCD an infinite Wilson line as in Figure 6.3.1 where one propagator is entering the vertex in $s = 0$ and one is going out. The first two terms in (6.49) are the equivalent of the mass renormalization of external probe particles moving along the loop, or, stated differently, the self-energies of their propagators due to the interaction with fields of the unbroken theory. The third piece in (6.49) is the renormalization of the vertex operator. These terms do not contribute to the interaction! In our supersymmetric theory we cannot interpret them in terms of corrected two and three-point functions, because probe particles moving along the loop are BPS protected, *i.e.* they do not renormalize. However the introduction of the IR regulator L breaks the gauge invariance of the infinite straight line, and hence from what we know from Section 4.2.3, also its supersymmetry. We then expect boundary terms depending on the regularisation scheme to survive for finite L. To correctly renormalise the deformed Wilson loop at one-loop we then have to subtract its undeformed value computed in the same regularisation scheme, which is the BPS Wilson line (6.49) and is far from being trivial

$$\langle \mathcal{W}(\phi,\theta)_{\text{renorm}}\rangle \equiv \langle \mathcal{W}(\phi,\theta)\rangle - \langle \mathcal{W}(0,0)\rangle \tag{6.50}$$

Note that the same argument holds for scalar boubles described at the beginning of this Section, which can be also accounted for as "renormalisation" contributions. The prescription (6.50) removes self-energy corrections to the external lines coming from (6.43) and (6.33), whatever they might be and subtracts from (6.47) the renormalisation of the vertex operator, which is the third bit of (6.49). Remarkably the final result for the wedge Wilson loop at one-loop reads

$$\begin{aligned}\left\langle \mathcal{W}^{(1)}\right\rangle_{\text{renorm}} &= \left(\frac{2\pi}{k}\right) \frac{2MN(\mu L)^{2\epsilon}}{M+N} \frac{\Gamma\left(\frac{1}{2}-\epsilon\right)}{4\pi^{3/2-\epsilon}} \\ &\times \left[\frac{1}{\epsilon}\left(\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} - 1\right) + \frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}} 2\log\left(\sec\frac{\phi}{2}+1\right) - 2\log 2 + \mathcal{O}(\epsilon)\right]\end{aligned} \tag{6.51}$$

and receives only fermionic contributions in this gauge. A final comment about the renormalization procedure. It seems that the IR regularised Wilson line receives contributions that are regularisation dependent, despite being BPS proteted and despite the computation of [46] fixes its value at 1. However, in that derivation boudary contributions to the integral of the supersymmetry variation of the Lagrangean (4.45) have been flushed, being total derivatives. Actually the value of boudary terms is fixed by gauge invariance, and only for a truely gauge invariant Wilson operator, total derivatives might be reabsorbed by means of a gauge transformation. In our case, being L large but finite, the contour is, strictly speaking, no more gauge invariant, and boundary term must be treated carefully.

6.3.2 Two-loops

The perturbative two-loop expansion of the half BPS loop operator involves a respectable amount of diagrams on non-null contours. Henceforth we present in the following relevant contributions to the expectation value of the deformed potential and refer the reader to Appendix A for all the details about computations. Looking back at (6.34) we see that order $\frac{1}{k^2}$, corrections to the Wilson loop can come already from the second order expansions of the loop operator contracted with either the kinetic part of the Lagrangean (B.2) or the interaction vertices (B.15), (B.16) and (B.17). Moreover, there are other relevant terms in the third and fourth order expansion of the loop operator. Substantially, we observe that there are three main classes of graphs contributing at order $\frac{1}{k^2}$ – double exchange graphs, corrected propagators and interaction 3-vertices, let us examine them in some details.

Double Exchange Diagrams

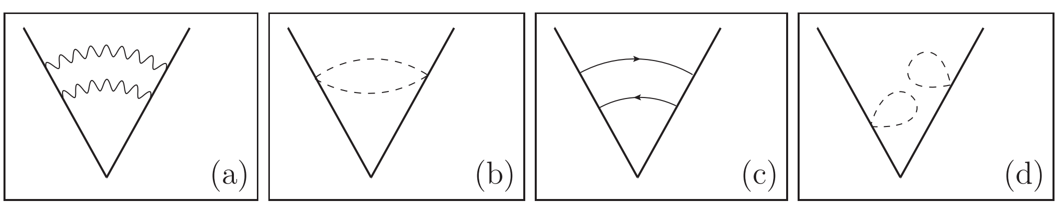


Figure 6.4: Double exchange diagrams at order $\frac{1}{k^2}$ of the VEV of the wedge Wilson loop. Only (b) and (c) contribute whereas (a) vanishes for kinematical reasons and (d) is a renormalization and scheme-dependent term.

Double gluon exchange

At any order of perturbation theory, double exchanges share the same kinematics of the one-loop single propagator exchange and the same structure of couplings to the loop contour.

Therefore gluon ladder diagrams of Figure 6.3.2 (a) vanish identically on any planar contour and independently for both gluon fields A and $\widehat{A}$ because of antisymmetry of the gluon propagator. The same holds for gluon rainbows and mixed graphs with one rainbow and one exchange propagator.

Double scalar exchange

There is a particular ladder graph generated by the BPS Wilson loops of ABJ(M) theories which does not have a SYM counterpart, it is the composite scalar exchange of Figure 6.3.2 (b). Interestingly enough, it emerges from the second order expansion of the loop operator (6.34)

$$\left\langle \mathrm{Tr}\,[MC\overline{C}](x_1)\,[\widetilde{M}C\overline{C}](x_2) \right\rangle = \left\langle \mathrm{Tr} M^I_J C_I \overline{C}^J \widehat{M}^K_L \overline{C}^L C_K \right\rangle + \left\langle \mathrm{Tr} M^I_J C_I \overline{C}^J \widehat{M}^K_L \overline{C}^L C_K \right\rangle \tag{6.52}$$

while ladder graphs typically start from the fourth order, once more highlighting the etero-geneus interplay of scalar and fermionic matter in these theories. Due to quadratic coupling of scalars in (6.32), diagrams emerging from the first summand in (6.52) represent the propagation of composite scalar bilinears. The kinematics of these diagrams is of the single-exchange type, in that they have only two insertion points despite appearing at two loops. Rainbow-like insertion of the composite scalars are regularization-dependent terms contributing to the renormalization of the Wilson loop, as explained in the previous section, which do not caputure the global structure of the loop and do not give any relevant contribution to the deformed potential. The renormalization prescription of [67] in (6.50) safely removes them. The same subtraction also eliminates double-boubles originating from the contraction of scalars within each composite insertion arising from the second summand in (6.52). Only the true exchange diagrams which capture the global structure of the Wilson loop contribute. From the expansion (6.34) we see that there are two such contributions, relative to the composite fields $[MC\overline{C}]$ and $[M\overline{C}C]$, they are computed in Section A.1.1. The first composite field is in the adjoint of $U(N)$ while the second is in the adjoint of $U(M)$, they carry trace factors N^2M and NM^2 respectively. Since the kinematical part is the same, trace factors combine and cancell the $1/(N+M)$ normalization of the Wilson loop

$$\left\langle \mathcal{W}_S^{(2)} \right\rangle = \mathrm{i}^4 NM \left(\frac{2\pi\mu^{2\epsilon}}{k}\right)^2 \frac{\Gamma\left(\frac{d}{2}-1\right)^2}{16\pi^d} \int \mathrm{d}s_1 \mathrm{d}s_2 \frac{1}{|x_1-x_2|^{2d-4}} \mathrm{Tr}[M(s_1)\widetilde{M}(s_2)] \tag{6.53}$$

In the ABJM case, where $N=M$, this reproduces the $\left(\frac{N}{k}\right)^2$ dependence which is expected in the planar limit. If we regard this as the contribution of a single adjoint field, it reproduces in $d=3$ dimensions the contribution of an adjoint scalar in $d=4$ at one-loop with the identification of the 't Hooft couplings $\lambda^2_{ABJ} = \lambda_{SYM}$. Also, we left a tilde over $\widetilde{M}(s_2)$ to underline the fact that the two matrices come from different blocks of the superconnection and are inserted at different points on opposite sides of the wedge. Using the explicit form of the deformed scalar matrix (6.19) it follows

$$\mathrm{Tr}[M(s_1)\widetilde{M}(s_2)] = 4\cos^2\frac{\theta}{2} \tag{6.54}$$

Double fermion exchange

Inserting the superconnection four times along the contour

$$\begin{aligned}(i)^4\mathcal{L}(s_1)\mathcal{L}(s_2)\mathcal{L}(s_3)\mathcal{L}(s_4) =& \\ &\left(\frac{2\pi}{k}\right)^2\begin{pmatrix}\eta_{I1}\bar{\psi}_1^I\psi_{L2}\bar{\eta}_2^L\eta_{R3}\bar{\psi}_3^R\psi_{S4}\bar{\eta}_4^S & \\ & \psi_{I1}\bar{\eta}_1^I\eta_{L2}\bar{\psi}_2^L\psi_{R3}\bar{\eta}_3^R\eta_{S4}\bar{\psi}_4^S\end{pmatrix}\end{aligned} \tag{6.55}$$

and contracting with the kinetic Lagrangean for fermions (B.2) one has diagrams with two fermionic lines. Here we avoided writing other terms in the matrix product which do not contribute to the class of diagrams under discussion. It sufficient to consider either the upper or lower terms above

$$\begin{aligned}\langle\mathrm{Tr}(\eta_{I1}\bar{\psi}_1^I\psi_{L2}\bar{\eta}_2^L\eta_{R3}\bar{\psi}_3^R\psi_{S4}\bar{\eta}_4^S)\rangle_0 =&\langle(\eta_{I1}\bar{\psi}_1^I)_{i\bar{i}}(\psi_{L2}\bar{\eta}_2^L)_{\bar{i}j}(\eta_{R3}\bar{\psi}_3^R)_{j\bar{l}}(\psi_{S4}\bar{\eta}_4^S)_{\bar{l}i}\rangle = \\ =&-\langle(\psi_{L2}\bar{\eta}_2^L)_{\bar{i}j}(\eta_{R3}\bar{\psi}_3^R)_{j\bar{l}}(\psi_{S4}\bar{\eta}_4^S)_{\bar{l}i}\ (\eta_{I1}\bar{\psi}_1^I)_{i\bar{i}}\rangle,\end{aligned} \tag{6.56}$$

explicitly written with group indices, it is clear that the only difference is a trace factor

$$-\langle(\psi_{L2}\bar{\nu}_2^L)_{\bar{i}j}(\nu_{R3}\bar{\psi}_3^R)_{j\bar{l}}(\psi_{S4}\bar{\nu}_4^S)_{\bar{l}i}\ (\nu_{I1}\bar{\psi}_1^I)_{i\bar{i}}\rangle \propto (\bar{\zeta}_2\zeta_3)(\bar{\zeta}_4\zeta_1)\delta_{\bar{i}\bar{l}}\delta_{jj}\delta_{ii}\delta_{\bar{l}\bar{i}} = MN^2(\bar{\zeta}_2\zeta_3)(\bar{\zeta}_4\zeta_1) \tag{6.57}$$

where we also factoriesed the dependence on the $R-$index. Overall, ten Feynman diagrams for each block contribute to (6.56) once all the possible insertions on the contour have been taken into account. Diagrams with no propagator going from one side of the wedge to the other are renormalisation terms and do not contribute as already explained. Here we focus on diagrams with at least one propagator going from side to side. All the details about the computations are in Appendix A.2.

Consider the configuration in Figure 6.3.2 (c). Using the same trick as in (6.39) for managing spinorial couplings and the fermion propagator in (B.13), the relevant integral is readily written

$$\begin{aligned}-\langle(\psi_2\bar{\nu}_2)(\nu_3\bar{\psi}_3)(\psi_4\bar{\nu}_4)(\nu_1\bar{\psi}_1)\rangle \sim& \frac{\Gamma^2(1/2-\epsilon)}{4\pi^{3-2\epsilon}(\eta_3\bar{\eta}_2)(\eta_1\bar{\eta}_4)}\times \\ &\times[\dot{x_2}^\mu+\dot{x_3}^\mu]\,\partial_{x_2^\mu}\left(\frac{1}{(x_{23}^2)^{1/2-\epsilon}}\right)[\dot{x_4}^\mu|\dot{x}_1|+\dot{x_1}^\mu|\dot{x}_4|]\,\partial_{x_4^\mu}\left(\frac{1}{(x_{41}^2)^{1/2-\epsilon}}\right),\end{aligned} \tag{6.58}$$

As already noted, two-loops double exchange diagrams have the same kinematical structure as one-loop diagrams, hence the integral above can be factorized

$$\frac{\Gamma^2(1/2-\epsilon)}{4\pi^{3-2\epsilon}(\eta_3\bar{\eta}_2)(\eta_1\bar{\eta}_4)}G(s_2,s_3)G(s_4,s_1) \tag{6.59}$$

where the antisymmetric function $G(s_i, s_j)$ is the same as in (6.42)

$$\begin{aligned} G(s_i,s_j) &= \left[\frac{d}{ds_i}-\frac{d}{ds_j}\right]\frac{1}{[(x(s_i)-x(s_j))^2]^{\frac{1}{2}-\epsilon}} \\ &= \left[\frac{d}{ds_i}-\frac{d}{ds_j}\right]\frac{1}{[s_i^2+s_j^2-2s_is_j\cos\phi]^{\frac{1}{2}-\epsilon}} \\ &= \left[\frac{d}{ds_i}-\frac{d}{ds_j}\right]\Delta^{\frac{1}{2}}(s_1,s_2) \end{aligned} \tag{6.60}$$

and $\Delta(s_1, s_2)$ is the scalar Feynman propagator in $d = 3 - 2\epsilon$ dimensions. At this stage it is convenient to split the integrand (6.58) in a totally symmetric part plus a pure-exchange part

$$G(s_2,s_3)G(s_4,s_1) = \underset{(A)}{[G(s_2,s_3)G(s_4,s_1)+G(s_2,s_4)G(s_3,s_1)]} - \underset{(B)}{G(s_2,s_4)G(s_3,s_1)} \tag{6.61}$$

The first piece (A) is symmetric under $s_1 \leftrightarrow s_2$ and $s_3 \leftrightarrow s_4$, then the path integral along the contour simply gives minus a half of the square of the one loop result (6.47), divided by a factor of M

$$\begin{aligned} F_A^{(2)} &= \frac{1}{2M}\left(\frac{iMN\Gamma(1/2-\epsilon)}{2\pi^{3/2-\epsilon}(\eta_1\bar{\eta}_2)}(n_{L2}\bar{n}_1^L)\int_0^L ds_1\int_{-L}^0 ds_2 G(s_1,s_2)\right)^2 \\ &= \frac{1}{2M}\left(F^{(1)}_{\text{exchange}}\right)^2 \end{aligned} \tag{6.62}$$

This is a noteworthy result. From the discussion about the massive $W-$bosons in the broken phase of the theory (4.21) (4.22) it is straightforward to show that in $U(N)\times U(M)$ SCS theories two kinds of particles arise and which transform respectively in the $(\mathbf{1},\mathbf{N}$ and $(\mathbf{1},\mathbf{M})$ bifundamental representations and their conjugate, that we will call W_N and W_M bosons. It is clear that a pair of W_N and W_M cannot form a singlet of the color indices and there is no generalization of the quark-antiquark potential in this case. On the other hand a pair of $W_N\overline{W}_n$ or $W_M\overline{W}_M$ do form color indices, hence there are two potentials in this theory! The equivalent of (6.3) in $U(N)\times U(M)$ SCS theories then reads

$$\langle \mathcal{W}(\theta,\phi)\rangle = \frac{1}{M}\mathrm{Tr}\mathcal{P}\mathrm{e}^{-V_N(\phi,\theta,\epsilon)} + \frac{1}{N}\mathrm{Tr}\mathcal{P}\mathrm{e}^{-V_M(\phi,\theta,\epsilon)} \tag{6.63}$$

The normalization are chose so that (6.63) is perturbatively identical to the definition of [46] when $M = N$. Let us concentrate on the first term on the r.h.s., applying the subtractive renormalization procedure its perturbative expansion schematically reads

$$\begin{aligned}\langle \mathcal{W}(\theta,\phi)\rangle = 1 - \frac{1}{M}g(V_N^{(1)}(\theta,\phi,\epsilon) - V_N^{(1)}(0,0,\epsilon)) - \\ \frac{1}{M}g^2\left[V_N^{(2)}(\theta,\phi,\epsilon) - V_N^{(2)}(0,0,\epsilon) + \frac{1}{2}(V_N^{(1)}(\theta,\phi,\epsilon) - V_N^{(1)}(0,0,\epsilon))^2\right] + \mathcal{O}(\lambda^3)\end{aligned} \quad (6.64)$$

where powers of the coupling constant $g = \frac{2\pi}{k}$ have been conveniently factored out. Therefore the fact that $\frac{1}{\epsilon^2}$ divergences of the Wilson loop $\mathcal{W}$ at two loops equal the square of the $\frac{1}{\epsilon}$ terms in the one-loop, means that one-loop divergences of the potential V_N exponentiate in the Wilson loop and consequently there is no real $\frac{1}{\epsilon^2}$ divergence in the two-loop expansion of V_N itself. The absence of higer order divergences in the deformed potential is dictated by superconformal symmetry of the full theory (6.3), and the only global divergence must be interpreted as a divergence of the integration region $L^\epsilon/\epsilon \propto T$ (with the T defined in (6.3)). A totally analogous reasoning holds for the corresponding term in the lower block of (6.55).

Let us go back to the contribution of the purely-exchange pieces (B) in (6.61). The computation in Appendix A.2 shows that the sum of the two purely-exchange pieces coming from (6.55) can be recast in the rather compact form

$$F^{(2)}_{(B)} = -\left(\frac{2\pi}{k}\right)^2 \frac{\Gamma^2(1/2-\epsilon)MN(N+M)}{4\pi^{3-2\epsilon}}\frac{L^{4\epsilon}}{4\epsilon}\cos^2\frac{\theta}{2}\int_0^1 dz \frac{1}{2(1-z)z\cos\varphi + z^2 + (1-z)^2} \quad (6.65)$$

Different Wick contractions of (6.56) generrate diagrams with one propagator going from side to side and one propagator with both ends on the same side, these graphs are depicted in Figure A.2 (f) trhough (i) in Appendix A.2 for the upper block and an equal amount comes from the lower block (6.55). The two blocks share the same integrals and differ only in trace factors, the sum of all these contributions can be simplified to two *master intergals* (after a lot of algebra), we will call it $F^{(2)}_{(C)}$

$$F^{(2)}_{(C)} = -\frac{1}{2\epsilon}\left(\frac{2\pi}{k}\right)^2 \frac{MN(M+N)\Gamma^2(1/2-\epsilon)}{4\pi^{3-2\epsilon}}\frac{\cos\frac{\theta}{2}}{\cos\frac{\varphi}{2}}\int_0^L ds_1 \int_{-L}^0 ds_2[(L+s_2)^{2\epsilon} + (-s_2)^{2\epsilon}]G(s_2,s_1) \quad (6.66)$$

After rescaling and flipping some integration region, the integrals above yield

$$\begin{aligned}L^{2\epsilon}\int_0^1 \mathrm{d}s_1 \int_0^1 \mathrm{d}s_2\,(1-s_2)^{2\epsilon}G(s_1,-s_2) = \\ L^{2\epsilon}\left[-\frac{1}{\epsilon} + 2\log\left(\sec\frac{\phi}{2}+1\right) + 2\zeta(2) + \mathcal{O}(\epsilon)\right]\end{aligned} \quad (6.67)$$

$$L^{2\epsilon}\int_0^1 \mathrm{d}s_1 \int_0^1 \mathrm{d}s_2\,(s_2)^{2\epsilon} G(s_1,-s_2) =$$
$$L^{2\epsilon}\left[-\frac{1}{\epsilon}+\frac{5}{2}\log\left(\sec\frac{\phi}{2}+1\right)+\log\cos^2\frac{\phi}{2}-\frac{1}{4}\log 2+\mathcal{O}(\epsilon)\right] \quad (6.68)$$

where only the divergent and finite contributions are needed for our purposes.

Self-energy diagrams

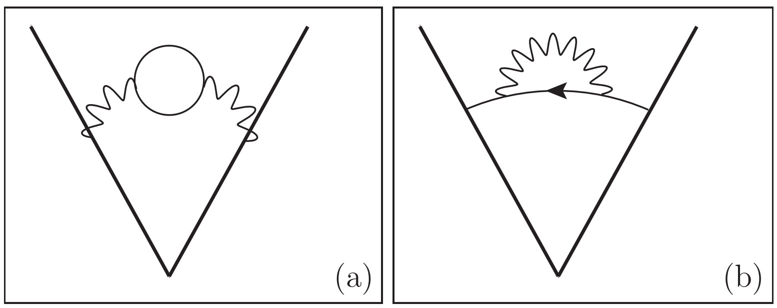


Figure 6.5:

Contracting the $< AA >$ pieces of (6.3.2) with a double copy of the interaction vertices (B.16) and (B.17)

$$\langle A_\mu(s_1)A_\nu(s_2)V_{\overline{\psi}\psi A}(w_1)V_{\overline{\psi}\psi A}(w_2)\rangle + \langle A_\mu(s_1)A_\nu(s_2)V_{C\overline{C}A}(w_1)V_{C\overline{C}A}(w_2)\rangle \quad (6.69)$$

produces the gluon self energy of Figure 6.3.2(a). The gluon-ghost and gluon-gluon 3-vertices do not play any role in the gluon self energy becuse their contributions mutually cancell, as is well known from [184]. Indeed they always cancell at any loop order so we can neglect them from now on. A short comment on this fact is present in Section A.2.2. The gluon self-energy was computed in [44] and reads

$$G^{(1)}_{\mu\nu}(x-y) = -\delta_{il}\delta_{jk}\frac{(\mu^{d-3})^2}{8\pi^d}\left(\frac{2\pi}{k}\right)^2\frac{\Gamma(1-d/2)\Gamma(d/2)^2}{\Gamma(d-1)}\times$$
$$\times\left[\frac{\Gamma(d-2)}{\Gamma(2-d/2)}\frac{\delta_{\mu\nu}}{[-(x-y)^2]^{d-2}}-\partial_\mu\partial_\nu\left(\frac{1}{4}\frac{\Gamma(d-3)}{\Gamma(3-d/2)}\frac{1}{[-(x-y)^2]^{d-3}}\right)\right] \quad (6.70)$$

There is of course an analouge of this formula for the self-enrgy of the $\widehat{A}$ gluon field. As for the case of the composite scalar the only difference between the two is in the group index structure. Indeed, from the point of view of group factors, one diagram represents the exchange of an $U(N)$ adjoint field with a nested $U(M)$ fundamental loop inside and has a $\delta_{il}\delta_{jk}$ factor in front, whereas the other is the exchange of a $U(M)$ adjoint with a $U(N)$ loop nested within, therefore its index factor is $\delta_{\widehat{il}}\delta_{\widehat{jk}}$. As already noticed in [44], when coupled to the loop contour and integrated over the insertion points, the A and $\widehat{A}$ gluons self-energies (6.70) combine with the sum of the two composite scalar diagrams (6.53) to form the effective interaction

$$-NM\left(\frac{2\pi}{k}\right)^2\int \mathrm{d}s_1\mathrm{d}s_2\frac{\Gamma\left(\frac{d}{2}-1\right)^2}{16\pi^d}\Big[\frac{2(d-1)(d-2)(\dot{x}_1\cdot\dot{x}_2)-|\dot{x}_1||\dot{x}_2|\mathrm{Tr}[M\widetilde{M}]}{|x_1-x_2|^{2(d-2)}}- \\ -(\dot{x}_1\cdot\dot{x}_2)\frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\left(\frac{2}{(d-3)(d-4)}\frac{1}{[-(x-y)^2]^{d-3}}\right)\Big] \quad (6.71)$$

Taking the $d\to 3$ limit one has the rather simple formula

$$-\left(\frac{NM}{k^2}\right)\int \mathrm{d}s_1\mathrm{d}s_2\Big[\frac{(\dot{x}_1\cdot\dot{x}_2)-|\dot{x}_1||\dot{x}_2|\frac{1}{4}\mathrm{Tr}[M\widetilde{M}]}{|x_1-x_2|^2}-(\dot{x}_1\cdot\dot{x}_2)\frac{\partial}{\partial s_1}\frac{\partial}{\partial s_2}\ln(|x_1-x_2|)\Big] \quad (6.72)$$

Since $Tr[M^2]=4$ for any $SU(4)$ deformation of the matrix couplings, the first term in the integral above cancels identically when the composite propagator is attached with both ends to the same side of the wedge contour. This is again a manifestation of the local $\frac{1}{2}$ BPS character of the Wilson loop, which prevents it from getting correction from single-edge graphs. The second term in parenthesis is a total derivative and can be removed by means of a gauge trasformation, for this reason it will be neglected in the following.

Fermion self-energy

The one loop fermion exchange in (6.36) can receive contributions from the emission and absorption of both gluon fields A and $\widehat{A}$, Figure 6.3.2 (b). On the other hand these graphs do not contribute to the expectation value of the Wilson loop. As can be seen from the contraction of the second-order expanded loop operator with the interaction Lagrangean

$$\frac{2\pi}{k}\left\langle \mathrm{Tr}\left[\eta^I_\alpha\overline{\psi}^\alpha_I(s_1)\psi^J_\beta(s_2)\overline{\eta}^\beta_J+\psi^L_\gamma(s_1)\overline{\eta}^\gamma_L\eta^K_\delta\overline{\psi}^\delta_K(s_2)\right]\left(\mathrm{Tr}\left[V_{\overline{\psi}\psi A}V_{\overline{\psi}\psi A}\right]+\mathrm{Tr}\left[V_{\overline{\psi}A\psi}V_{\overline{\psi}A\psi}\right]\right)\right\rangle \quad (6.73)$$

and paying particular attention to signs arising from the ordering of fermions, there is an overall minus sign between terms coming from the upper block and term coming from the lower block.

Nonetheless it is instructive to see what kind of contribution can arise from diagrams of this kind, as they become relevant subdiagrams at higher orders. As shown in Appendix A.1, each fermion line receives two contributions from the two gauge fields A and $\widehat{A}$ which have same kinematics but different group factors and fermionic couplings. The sum of the two contributions is

$$I^{(2)}_{\text{fermion}} = I^{(2)}_F(A) + I^{(2)}_F(\widehat{A}) = \\ = \frac{\mathrm{i}}{2\pi} NM(N-M) \left(\frac{4\pi\mu^{2\epsilon}}{k}\right)^2 |\dot{x}_1||\dot{x}_2| \int \mathrm{d}s_1\, \mathrm{d}s_2\, \frac{\overline{\eta}^\alpha_I(s_2)\eta^I_\alpha(s_1)}{(x_1-x_2)^{2-2\epsilon}} \tag{6.74}$$

and vanishes when $N = M$, so the fermion self-energy does not contribute to the 2-loops expectation value of the ABJM Wilson loop in any case. In the general case, note that coupling both ends of (6.74) to the same edge of the wedge loop produces, up to a prefactor

$$\int_{-\infty}^0 \mathrm{d}s_1 \int_{s_1}^0 \mathrm{d}s_2\, \frac{\overline{\eta}^\alpha_I \eta^I_\alpha}{(x_1-x_2)^{2(d-2)}} = \int_{-\infty}^0 \mathrm{d}s_1 \int_{s_1}^0 \mathrm{d}s_2\, \frac{2\mathrm{i}}{(x_1-x_2)^{2(d-2)}} = \\ - \int_0^{\infty} \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2\, \frac{\overline{\eta}^\alpha_I \eta^I_\alpha}{(x_1-x_2)^{2(d-2)}} \tag{6.75}$$

so their contribution to the renormalization of the infinite line vanishes as well. Here we have used the key relation for which $\overline{\eta}^\alpha_I \eta^I_\alpha = 2\mathrm{i}$ on both sides of the loop. Inserting the corrected propagator on different edges and accounting for the two possible orderings of insertion points allowed by the Wick contractions we have

$$\left\langle W^{(2)}_F \right\rangle = \frac{\mathrm{i}}{2\pi} NM \frac{N-M}{N+M} \left(\frac{4\pi\mu^{2\epsilon}}{k}\right)^2 \int_{-L}^0 \mathrm{d}s_1 \int_0^L \mathrm{d}s_2\, \frac{2\cos(\phi/2)}{(x_1-x_2)^{2-2\epsilon}} \tag{6.76}$$

where we have used the fermion couplings in (6.27). The integral above is the same $4d-$like scalar integral that appears widely throughout the computations of this Section. From a purely speculative point of view, the $\mathrm{i}(N-M)$ factor in front of the integral suggests that (6.76) is related to parity violating contributions that may appear at higher orders and in non-planar terms as observed in [185].

Interaction 3-Vertices

Now let us consired interaction vertices. In Figure 6.3.2(a) the Chern-Simons vertex is depicted, it comes from the third-order expansion of the loop operator contracted with the 3-vertex (B.15)

$$\langle V_G \rangle = \mathrm{i}^3 \int \mathrm{d}s_1 \mathrm{d}s_2 \mathrm{d}s_3 L(s_1)L(s_2)L(s_3) \int \mathrm{d}^3 w V_{AAA}(w) \tag{6.77}$$

Of course there is an analouge of it for the $\widehat{A}$ gluon 3-vertex. Let examine one of them, explicitly disregarding the ordering of the insertion points and an inessential prefactor it reads

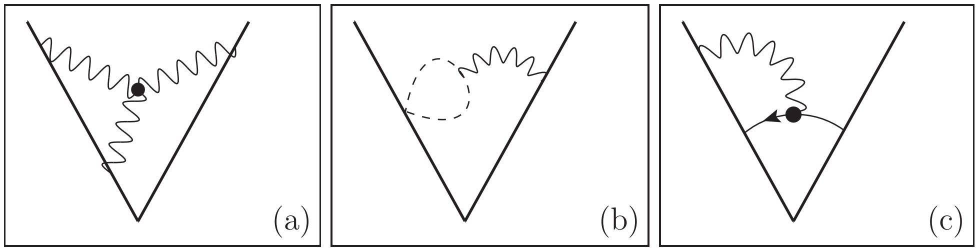


Figure 6.6:

$$\langle V_G \rangle = \int \mathrm{d}s_i \mathrm{d}s_j \mathrm{d}s_k \dot{x}_i^\mu \dot{x}_j^\nu \dot{x}_k^\rho \epsilon_{\mu\alpha\sigma}\epsilon_{\nu\beta\lambda}\epsilon_{\rho\gamma\tau}\epsilon^{\alpha\beta\gamma} \int \mathrm{d}^d w \frac{(x_i-w)^\sigma (x_j-w)^\lambda (x_k-w)^\tau}{|x_i-w|^d |x_j-w|^d |x_k-w|^d} = 0 \quad (6.78)$$

As $\dot{x}_i = P_i$ are constants and only two of them are evidently linearly independent, it is easy to see that this term vanishes for antisymmetry. As pointed out in [18] the contribution of the CS vertex to polygonal Wilson loops is non vanishing only when attached to three linearly independent edges.

The second graph in Figure 6.3.2 (b) is due to the scalar gluon interaction and arises from the contraction of the $< A(s_1)[MC\overline{C}](s_2) >$ term in (6.34) and the scalar-gluon vertex (B.17). Performing the contractions one finds

$$\langle V_S \rangle = NM \frac{(\mu^{3-d})^2}{k^2} \frac{\Gamma(d/2)\Gamma(d/2-1)}{4\pi^{d-1}} \times \dot{x}_i^\mu |\dot{x}_j| \epsilon_{\mu\nu\rho} \int \mathrm{d}s_i \mathrm{d}s_j \int \mathrm{d}^d w \frac{(x_i-w)^\rho}{|x_i-w|^d |x_j-w|} \frac{\partial}{\partial w^\nu} \frac{1}{|x_j-w|} \quad (6.79)$$

As outlined in [44], for any planar contour the derivative of the integrand w.r.t the component of w orthogonal to the loop is anti-symmetric in $w_\perp \to -w_\perp$, whereas the integration ranges from $-\infty$ to $+\infty$. Then

$$\langle V_S \rangle = 0 \quad (6.80)$$

for any Wilson loop lying on a plane.

Finally the fermion-gauge vertex diagrams of Figure 6.3.2 (c). These are the most subtle bits of the computation and a detailed derivation is presented in Appendix A.3. Such diagrams emerge from the coupling of the CS gluon fields to the fermions in the theory through the covariant derivative. Inserting three copies of the superconnection $\mathcal{L}$ along the contour and performing contractions one has

$$
\begin{aligned}
&\mathrm{Tr}[(i\mathcal{L}(\tau_1))(i\mathcal{L}(\tau_2))(i\mathcal{L}(\tau_3))] = \\
&= \frac{2\pi i}{k}\left[\psi_{I1}\bar{\nu}_1^I\mathcal{A}_2\nu_{L3}\bar{\psi}_3^L + \nu_{I1}\bar{\psi}_1^I\psi_{L2}\bar{\nu}_2^L\mathcal{A}_3 + \mathcal{A}_1\nu_{L2}\bar{\psi}_2^I\psi_{L3}\bar{\nu}_3^L\right] + \\
&+ \frac{2\pi i}{k}\left[\widehat{\mathcal{A}}_1\psi_{L2}\bar{\nu}_2^L\nu_{L3}\bar{\psi}_3^L + \psi_{I1}\bar{\nu}_1^I\nu_{L2}\bar{\psi}_2^L\widehat{\mathcal{A}}_3 + \nu_{I1}\bar{\psi}_1^I\widehat{\mathcal{A}_2}\psi_{L3}\bar{\nu}_3^L\right] + \cdots
\end{aligned}
\tag{6.81}
$$

plus other terms that do not contribute to the interaction, and we preserve the definition

$$
\mathcal{A} = A_\mu\dot{x}^\mu - i\frac{2\pi}{k}|\dot{x}|M_J{}^I C_I\bar{C}^J \quad \text{and} \quad \widehat{\mathcal{A}} = \widehat{A}_\mu\dot{x}^\mu - i\frac{2\pi}{k}|\dot{x}|\widehat{M}_J{}^I\bar{C}^J C_I. \tag{6.82}
$$

Once taking into account al the possible insertions of the fields in (6.81) one has the twelve diagrams of Figure A.3 for each gluon field. Single edge diagrams do not contribute the VEV as they are subtracted by the renormalization prescription (6.50), whereas exchange diagrams (Figure A.3 (b) (f) (m) and (n)) coming from the first line of (6.81) can be rearranged in

$$
\begin{aligned}
V_F^1\mathrm{exch} = &-\left(\frac{2\pi}{k}\right)^2 MN^2 \oint_{s_3<s_2<s_1} ds_1 ds_2 ds_3\ \Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3)\times \\
&\times\left[\bar{\nu}_1^L\gamma^\rho\gamma_\mu\gamma^\sigma\nu_{L3}\epsilon^{\nu\mu\lambda}\dot{x}_{2\nu} - \bar{\nu}_2^I\gamma^\lambda\gamma_\mu\gamma^\rho\nu_{I1}\epsilon^{\nu\mu\sigma}\dot{x}_{3\nu} - \bar{\nu}_3^I\gamma^\sigma\gamma_\mu\gamma^\lambda\nu_{I2}\epsilon^{\nu\mu\rho}\dot{x}_{1\nu}\right]
\end{aligned}
\tag{6.83}
$$

where Γ is a shorthand for the integral over the position of the interaction vertex

$$
\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) = \left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^3 \partial_{x_1^\rho}\partial_{x_2^\lambda}\partial_{x_3^\sigma}\int \frac{d^{3-2\epsilon}w}{(x_{1w}^2)^{1/2-\epsilon}(x_{2w}^2)^{1/2-\epsilon}(x_{3w}^2)^{1/2-\epsilon}} \tag{6.84}
$$

and the terms in square brackets account for the different insertion of the spinorial couplings along the contour. The computation of the integrals in (6.83) is rather cumbersome and left to the Appendix A.3. A central observation is that (6.83) can be cooked up in terms which are total derivatives and terms which apparently are not. For the latter however it is convenient to use the symmetries of the wedge contour to give them a total derivative look. As many terms vanish because of the planarity of the contour one is left with the following four contributions

$$
\left\langle V_F^{\mathrm{exch}}\right\rangle = \left(\frac{2\pi}{k}\right)^2 \frac{MN\Gamma^2(\frac{1}{2}-\epsilon)}{4\pi^{3-2\epsilon}}\left[I_1+I_2+I_3+I_4\right] \tag{6.85}
$$

where the relevant integrals are

$$
\begin{aligned}
I_1 &= -\frac{\mathrm{i}\mu^{4\epsilon}}{2\epsilon}\cos\frac{\theta}{2}\cos\frac{\phi}{2}\int_0^L ds_1\int_{-L}^0 ds_2\Delta^{\frac{1}{2}}(s_2,s_1)\frac{d}{ds_2}\left[(L+s_2)^{2\epsilon}+(-s_2)^{2\epsilon}\right]\\
I_2 &= \frac{\mathrm{i}\mu^{4\epsilon}}{4\cos\frac{\phi}{2}}\cos\frac{\theta}{2}\left[\frac{L^{4\epsilon}}{4\epsilon^2}-\left(\int_0^L ds\Delta^{\frac{1}{2}}(s,-L)\right)^2\right]\\
I_3 &= -\frac{\mathrm{i}\mu^{4\epsilon}}{2\cos\frac{\phi}{2}}\cos\frac{\theta}{2}\int_0^L ds_1\int_{-L}^0 ds_3\,\Delta^{\frac{1}{2}}(s_1,s_3)[\Delta^{\frac{1}{2}}(L,s_3)-\Delta^{\frac{1}{2}}(0,s_3)]\\
I_4 &= \frac{\mathrm{i}\mu^{4\epsilon}}{2\epsilon}\frac{1}{4\cos\frac{\phi}{2}}\cos\frac{\theta}{2}\int_0^L ds_2\int_{-L}^0 ds_3[(L-s_2)^{2\epsilon}+{s_2}^{2\epsilon}]G(s_2,s_3)
\end{aligned}
\tag{6.86}
$$

All these integrals are strongly reminiscent of those in (6.66).

Mixed Diagrams

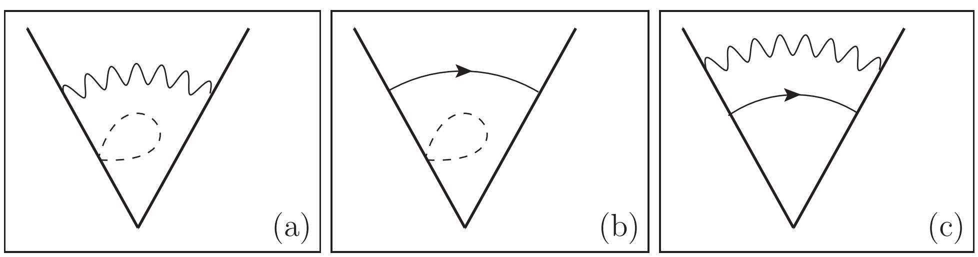


Figure 6.7: Mixed diagrams arising from the exponentiation of the one-loop single-exchange and rainbow diagrams with the scalar bouble.

Non-interacting mixed diagrams also emerge at order $\frac{1}{k^2}$. These can be divided into the three categories schematically dipicted in Figure 6.3.2. Graphs with internal scalar bouble insertions can be regarded as vanishing for the same reasons discussed in the one-loop case for the bouble alone. Besides these, mixed double exchange with one fermion and one gluon propagators annihilate on planar contours for the antisymmetry of the latter, they indeed have the same kinematical structure of the one-loop. We conclude that mixed diagrams do not contribute to the renormalised expectation value of the Wilson loop.

6.4 $W\overline{W}$ pair potential - the $\phi \to \pi$ limit

The $W\overline{W}$ pair potential is recovered from our wedge contour in the limit $\phi \to \pi$ where the the two edges aproach each other and are mapped by a conformal transformation to a pair of infinite anti-parallel lines. Indeed we have shown in (6.63) that the $U(N+1)\times U(M+1)$ quiver group structure of SCS theories in the unbroken phase implies the existence of two potentials for the two possible color singlets states made up of bifundamental $(\mathbf{1},\mathbf{N})$ and $(\mathbf{1},\mathbf{M})$ $W-$bosons with their conjugate

$$\langle \mathcal{W}(\theta,\phi)\rangle = \frac{1}{M}\mathrm{Tr}\mathcal{P}\mathrm{e}^{-V_N(\phi,\theta,\epsilon)} + \frac{1}{N}\mathrm{Tr}\mathcal{P}\mathrm{e}^{-V_M(\phi,\theta,\epsilon)} \tag{6.87}$$

From the one-loop renormalised value of the wedge loop (6.51) we can estract the one-loop renormalized potential

$$\lim_{\phi\to\pi}\left[\frac{1}{\epsilon}\left(\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}-1\right)+\frac{\cos\frac{\theta}{2}}{\cos\frac{\phi}{2}}2\log\left(\sec\frac{\phi}{2}+1\right)-2\log 2+\mathcal{O}(\epsilon)\right]=$$
$$\frac{(\mu L)^{2\epsilon}}{2\epsilon}\left[\frac{\cos\frac{\theta}{2}}{\pi-\phi}-1+\frac{4\epsilon\,\cos\frac{\theta}{2}}{\pi-\phi}\log\frac{\pi-\phi}{2}+\mathcal{O}(\pi-\phi)\right] \tag{6.88}$$

So up the identification of $\pi-\phi$ with the L (6.3) and $1/\epsilon$ with T in the same equation, in the $\epsilon\to 0$ limit we have a contribution consistent with the requirements of conformal symmetry

$$V_N^{(1)}(L,\lambda_N) = \lambda_N\frac{\cos\frac{\theta}{2}}{\pi-\phi} \tag{6.89}$$

and analogously for the other kind of particles

$$V_M^{(1)}(L,\lambda_M) = \lambda_M\frac{\cos\frac{\theta}{2}}{\pi-\phi} \tag{6.90}$$

where $\lambda_N = N/k$, $\lambda_M = M/k$ are the two 't Hooft couplings. Note that all other terms in (6.88) are subleading and scheme-dependent. Finally, note that the potential is, up to this order, of the Coulomb type, and in the case $\theta=\pi$ it vanishes and somehow the parallel line result of $\mathcal{N}=4$ SYM is recovered.

APPENDIX A

Two-loops contributions to the Wilson loop

A.1 One-loop corrected gauge and fermion propagators

Contracting the 1-loop expansion of the Wilson loop with the interaction 3-vertices of the kinetic term of the matter action yields suitable 2-loops contributions. Higher order interaction vertices in (B.2) do not contribute at this order as fermion-gluon 4-vertices enter at 3-loops, while Yukawa couplings and the sexstic scalar interaction enter at 4-loops. So the relevant contributions are just the 1-loop corrected gauge and fermion propagators depicted in Figure 6.3.2(a) and (b).

Gluon self energy

For the gluon propagator only fermion and scalar boubles contribute to the self energy at this order, once the cancellation of gluon and ghost loops are taken into account. The computation of the gluon self energy terms due to fermion and scalar running in the loop was carried out in dimensional reduction (*i.e.* keeping tensors *on-dimension*) in [44]. It involves standard field-theory techniques and we don't review it here. The final result for the 1-loop correction in $d-$dimensional euclidean space reads

$$
\begin{aligned}
G^{(1)}_{\mu\nu}(x-y) = & -\delta_{il}\delta_{jk}\delta^I_I \frac{\mu^{d-3}}{\pi^d}\left(\frac{2\pi}{k}\right)^2 \frac{\Gamma(1-d/2)\Gamma(d/2)^2}{\Gamma(d-1)}\times \\
& \times\left[\frac{\Gamma(d-2)}{\Gamma(2-d/2)}\frac{\delta_{\mu\nu}}{[-(x-y)^2]^{d-2}} - \partial_\mu\partial_\nu\left(\frac{\Gamma(d-3)}{\Gamma(3-d/2)}\frac{1}{4}\frac{1}{[-(x-y)^2]^{d-3}}\right)\right] \quad \text{(A.1)}
\end{aligned}
$$

Taking the limit $d \to 3$, accounting for four-flavours of fields running in the loop and summing to the tree-level propagator $G^{(0)}_{\mu\nu}$ one has

$$\Big\langle (A_\mu)_{ij}(x)\,(A_\nu)_{kl}(y)\Big\rangle = -\delta_{il}\delta_{jk}\frac{1}{k}\left[\frac{\epsilon_{\mu\nu\rho}(x-y)^\rho}{2|x-y|^3} + \frac{N}{k}\left(\frac{\delta_{\mu\nu}}{|x-y|^2} - \partial_\mu\partial_\nu \ln(|x-y|)\right)\right] \tag{A.2}$$

being $|x-y|$ a shothand for $[-(x-y)^2]^{\frac{1}{2}}$. There is an analogue of the latter expression for the corrected propagator of hatted gluons

$$\Big\langle (\widehat{A}_\mu)_{\widehat{i}\widehat{j}}(x)\,(\widehat{A}_\nu)_{\widehat{k}\widehat{l}}(y)\Big\rangle = -\delta_{\widehat{i}\widehat{l}}\delta_{\widehat{j}\widehat{k}}\frac{1}{k}\left[-\frac{\epsilon_{\mu\nu\rho}(x-y)^\rho}{2|x-y|^3} + \frac{M}{k}\left(\frac{\delta_{\mu\nu}}{|x-y|^2} - \partial_\mu\partial_\nu \ln(|x-y|)\right)\right] \tag{A.3}$$

In the ABJ case remember that gauge groups are different, thus the bouble contributions have different trace factors. Also note that while the tree-level pieces have opposite signs, the loop corrected ones share the same sign, meaning the two contributions to the 2-loops Wilson loop operator coming from the two gluon fields sum up even in the $M = N$ case.

Fermion self energy

The second graph in Figure 6.3.2 comes from the contraction of the 2-fermion piece in the diagonal of the loop operator (4.10) contracted with the fermion-gluon vertices

$$\frac{2\pi}{k}|\dot{x}_1||\dot{x}_2|\,\mathrm{Tr}\left[\eta^I_\alpha\overline{\psi}^\alpha_I\psi^J_\beta\overline{\eta}^\beta_J\right]\,\frac{1}{2}\left(\mathrm{Tr}\left[V_{\overline{\psi}\psi A}V_{\overline{\psi}\psi A}\right] + \mathrm{Tr}\left[V_{\overline{\psi}A\psi}V_{\overline{\psi}A\psi}\right]\right) \tag{A.4}$$

with an extra $\frac{1}{2}$ factor coming from the second order expansion of the V's. For each of the two fermionic contributions to the loop operator there are two possible contractions with the two species of gauge fields. Note from (B.10) and (B.16) that vertices and propagators of A and $\widehat{A}$ have opposite signs and different trace factors, the full self-energy will be then the sum of the two terms $I^{(2)}_{\text{fermion}} = I^{(2)}_F(A) + I^{(2)}_F(\widehat{A})$. Let us then compute the one relative to A, the full integral is

$$\begin{aligned} I^{(2)}_F(A) = \frac{1}{2}N^2M\left(\frac{2\pi}{k}\right)^2 |\dot{x}_1||\dot{x}_2|\int \mathrm{d}s_1\,\mathrm{d}s_2\,\overline{\eta}^\alpha_J(s_2)\delta^J_I\Big\{\int\frac{\mathrm{d}^dp}{(2\pi)^d}\int\frac{\mathrm{d}^dk}{(2\pi)^d} \\ \left(-\frac{\not{p}}{p^2}\right)\sigma^\mu\left(-\frac{\not{k}}{k^2}\right)\sigma^\nu\left(-\frac{\not{p}}{p^2}\right)\epsilon_{\mu\nu\rho}\frac{(p-k)^\rho}{(p-k)^2}\,\mathrm{e}^{-p\cdot(x_1-x_2)}\Big\}^\beta_\alpha\,\eta^I_\beta(s_2) \end{aligned} \tag{A.5}$$

We proceede in the standard way introducing Feynman parameters (D.30) fot the d^dk integral and shifting integration variable

$$
\begin{aligned}
\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{k_b (p-k)^\rho}{k^2 (p-k)^2} &= \int \frac{\mathrm{d}^d k}{(2\pi)^d} \int_0^1 \mathrm{d}\alpha \frac{k_b (p-k)^\rho}{\alpha k^2 + \overline{\alpha}(p-k)^2} \\
&= \int_0^1 \mathrm{d}\alpha \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{-l_b l^\rho + \alpha\overline{\alpha} p_b p^\rho}{(l^2 - \Delta)^2} + \text{terms linear in } l
\end{aligned}
\tag{A.6}
$$

where $\overline{\alpha} = (1-\alpha)$ and $\Delta = \alpha\overline{\alpha} p^2$. Then using (D.32) we find (parity of the integral drops linear terms)

$$
\begin{aligned}
&\int_0^1 \mathrm{d}\alpha \left\{ \frac{\mathrm{i}}{(4\pi)^{\frac{d}{2}}} \Gamma\left(1 - \frac{d}{2}\right) \left(\frac{1}{\alpha\overline{\alpha}}\right)^{1-\frac{d}{2}} \left[\frac{1}{2} \frac{\delta_b^\rho}{(-p^2)^{1-\frac{d}{2}}} + \left(1 - \frac{d}{2}\right) \frac{p_b p^\rho}{(-p^2)^{2-\frac{d}{2}}} \right] \right\} \\
&= -\frac{\mathrm{i}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1-\frac{d}{2})\Gamma(d/2)^2}{\Gamma(2)\Gamma(d)} \frac{p^2 \delta_b^\rho + (d+2) p_b p^\rho}{(-p^2)^{2-\frac{d}{2}}}
\end{aligned}
\tag{A.7}
$$

This is to be contracted with the the bunch of σ's in (A.5) and after some algebra with the help of (D.34) and followings, only a diagonal piece in the spinor indices survives

$$
\frac{\mathrm{i}}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1-\frac{d}{2})\Gamma(d/2)^2}{\Gamma(2)\Gamma(d)} \frac{\delta_\alpha^\beta}{(-p^2)^{2-\frac{d}{2}}}
\tag{A.8}
$$

Putting things altogether and Fourier-transforming with respect of p one has the 1-loop correction to the fermion propagator from a single A gluon

$$
\frac{1}{2} N^2 M \left(\frac{4\pi}{k}\right)^2 |\dot{x}_1||\dot{x}_2| \frac{\mathrm{i}}{8\pi^{d-1}} \frac{(d+1)\Gamma(d/2)^2}{(1-d/2)(d-1)(d-2)} \int \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\overline{\eta}_I^\alpha(s_2)\eta_\alpha^I(s_1)}{|x_1 - x_2|^{\frac{d}{2}-1}}
\tag{A.9}
$$

that assumes a quite simple form in $d = 3$ dimensions

$$
I_F^{(2)}(A) = \frac{\mathrm{i}}{2\pi} N^2 M \left(\frac{4\pi}{k}\right)^2 |\dot{x}_1||\dot{x}_2| \int \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\overline{\eta}_I^\alpha(s_2)\eta_\alpha^I(s_1)}{|x_1 - x_2|^{\frac{d}{2}-1}}
\tag{A.10}
$$

There is a second contributions coming from the emission-absorption of an $\widehat{A}$ field. The relevant vertices carry a minus sign in this case, as does the propagator. Moreover the trace factor is NM^2, being the $\widehat{A}$ field in the adjoint of $SU(M)$. What remains of the computation is left untouched, hence the full correction reads

$$
\begin{aligned}
I_{\text{fermion}}^{(2)} &= I_F^{(2)}(A) + I_F^{(2)}(\widehat{A}) = \\
&= \frac{\mathrm{i}}{2\pi} NM(N-M) \left(\frac{4\pi}{k}\right)^2 |\dot{x}_1||\dot{x}_2| \int \mathrm{d}s_1\, \mathrm{d}s_2 \frac{\overline{\eta}_I^\alpha(s_2)\eta_\alpha^I(s_1)}{|x_1 - x_2|^{\frac{d}{2}-1}}
\end{aligned}
\tag{A.11}
$$

and vanishes identically in ABJM where $N = M$.

A.1.1 Double Scalar Exchange

In the expansion of the Wilson loop operator to second order there appears a double scalar insertion which can be contracted with the kinetic term of the scalar Lagrangean

$$\left(\frac{2\pi}{k}\right)^2 \left\langle |\dot{x}_1||\dot{x}_2|[M^I_J C_I \overline{C}^J](s_1)[\widehat{M}^K_L \overline{C}^L C_K](s_2)\right\rangle \tag{A.12}$$

where the shorthand $[M^I_J C_I \overline{C}^J](s)$ means that the scalar fieds and the matrix coupling are inserted at the same point s. As for the gluon and fermion self energies there are two terms of this kind corresponding to the two different orderings of the insertion points. The only difference between the two is the trace factor, more precisely the ordering $x_1(s_1) < x_2(s_2)$ corresponds to the traces (over $R-$indices and matrix indices)

$$M^I_J (C_I)_{i\hat{i}} (\overline{C}^J)_{\hat{j}j} \widehat{M}^K_L (\overline{C}^L)_{\hat{l}l} (C_K)_{k\hat{k}} \delta^L_I \delta^J_K \delta_{il} \delta_{\hat{i}\hat{l}} \delta_{jk} \delta_{\hat{j}\hat{k}} = N^2 M \mathrm{Tr}[M\widehat{M}] \tag{A.13}$$

while the opposite ordering yelds $NM^2\mathrm{Tr}[M\widehat{M}]$. The sum of the two contributions (A.12) reads

$$NM \left(\frac{2\pi}{k}\right)^2 |\dot{x}_1||\dot{x}_2| \frac{\Gamma\left(\frac{d}{2}-1\right)^2}{4\pi^d} \int \mathrm{d}s_1 \mathrm{d}s_2 \frac{1}{|x_1-x_2|^{2d-4}} \mathrm{Tr}[M(s_1)\widehat{M}(s_2)] \tag{A.14}$$

A.2 Double Fermion Exchange

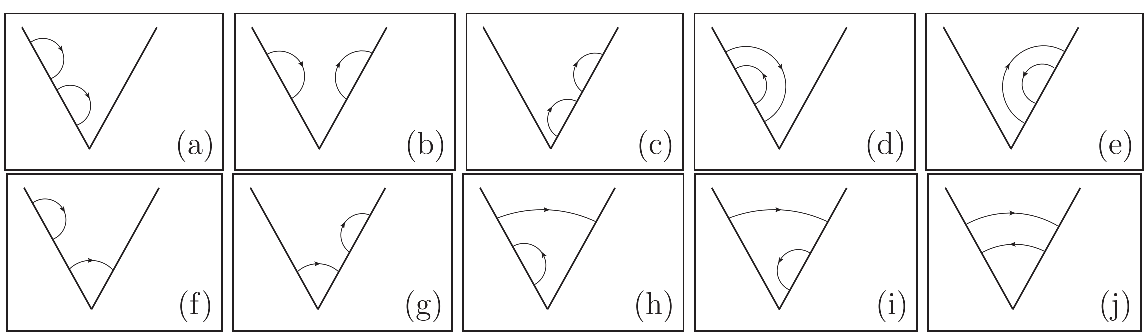


Figure A.1: Two-loops diagrams involving fermion lines only. They inhere a group factor of N^2M, but a completely analogous set of diagrams with a NM^2 factor and flipped propagators arise from the expansion and gives, on planar contours, the same contribution. In the first row, diagrams where no line is exchanged between the two sides are dipicted, these have $R-$charge $= 1$. On the second row lie single exchanges (f), (g), (h), (i) and double exchange (j). They carry respectively $R-$charge $\cos(\theta/2)$ and $\cos^2(\theta/2)$. It turns out that only the last digram gives contributes to the Wilson loop as all other graphs cancel with each other.

In a first instance one could ask whether one loop fermion diagrams exponentiate, simply yealding the two loop also to vanish. The answer is simple - No, they don't. And the reason is likely as simple - Because there are no cross-ladders. Since fermion fields are in a bifundamental representation, the trace of a correlator of any number of such fields requires a an operator with indices $\widehat{i\bar{i}}$ to be followed by an opearator with indices $\widehat{\bar{i}j}$ and so forth. Otherwise stated, the keen structure of the loop operator always places a $\bar{\psi}$ after a ψ and viceversa, meaning that at two loops cross-ladder diagrams are not permitted. They arise starting from three loops, which is however beyond our current interest.

Let us consider the so-called fermionic double exchange diagrams depicted in Figure A.2 (j) . They appear when considering the Wick contractions of the terms which are quartic in the super-connection $\mathcal{L}$. These terms are given by

$$
\begin{aligned}
&(i)^4\mathcal{L}(\tau_1)\mathcal{L}(\tau_2)\mathcal{L}(\tau_3)\mathcal{L}(\tau_4) = \\
&= \begin{pmatrix} \mathcal{A}_1\mathcal{A}_2 - \frac{2\pi}{k}\nu_{I1}\bar{\psi}_1^I\psi_{L2}\bar{\nu}_2^L & -i\sqrt{\frac{2\pi}{k}}\mathcal{A}_1\nu_{L2}\bar{\psi}_2^L - i\sqrt{\frac{2\pi}{k}}\nu_{I1}\bar{\psi}_1^I\widehat{\mathcal{A}_2} \\ -i\sqrt{\frac{2\pi}{k}}\psi_{I1}\bar{\nu}_1^I\mathcal{A}_2 - i\sqrt{\frac{2\pi}{k}}\widehat{\mathcal{A}}_1\psi_{L2}\bar{\nu}_2^L & \widehat{\mathcal{A}}_1\widehat{\mathcal{A}}_2 - \frac{2\pi}{k}\psi_{I1}\bar{\nu}_1^I\nu_{L2}\bar{\psi}_2^L \end{pmatrix} \times \\
&\quad\times \begin{pmatrix} \mathcal{A}_3\mathcal{A}_4 - \frac{2\pi}{k}\nu_{R3}\bar{\psi}_3^R\psi_{S4}\bar{\nu}_4^S & -i\sqrt{\frac{2\pi}{k}}\mathcal{A}_3\nu_{S4}\bar{\psi}_4^S - i\sqrt{\frac{2\pi}{k}}\nu_{R3}\bar{\psi}_3^R\widehat{\mathcal{A}_4} \\ -i\sqrt{\frac{2\pi}{k}}\psi_{R3}\bar{\nu}_3^R\mathcal{A}_4 - i\sqrt{\frac{2\pi}{k}}\widehat{\mathcal{A}}_3\psi_{S4}\bar{\nu}_4^S & \widehat{\mathcal{A}}_3\widehat{\mathcal{A}}_4 - \frac{2\pi}{k}\psi_{R3}\bar{\nu}_3^R\nu_{S4}\bar{\psi}_4^S \end{pmatrix} = \\
&= \left(\frac{2\pi}{k}\right)^2 \begin{pmatrix} \nu_{I1}\bar{\psi}_1^I\psi_{L2}\bar{\nu}_2^L\nu_{R3}\bar{\psi}_3^R\psi_{S4}\bar{\nu}_4^S & \\ & \psi_{I1}\bar{\nu}_1^I\nu_{L2}\bar{\psi}_2^L\psi_{R3}\bar{\nu}_3^R\nu_{S4}\bar{\psi}_4^S \end{pmatrix}.
\end{aligned}
\tag{A.15}
$$

In the last line we have neglected all the contributions which are irrelevant for the fermionic double exchange diagram. To begin with, we will evaluate the *upper* term only

$$
\begin{aligned}
\langle \mathrm{Tr}(\nu_{I1}\bar{\psi}_1^I\psi_{L2}\bar{\nu}_2^L\nu_{R3}\bar{\psi}_3^R\psi_{S4}\bar{\nu}_4^S)\rangle_0 =& \langle(\nu_{I1}\bar{\psi}_1^I)_{i\bar{i}}(\psi_{L2}\bar{\nu}_2^L)_{\bar{i}j}(\nu_{R3}\bar{\psi}_3^R)_{j\bar{l}}(\psi_{S4}\bar{\nu}_4^S)_{\bar{l}i}\rangle = \\
=& -\langle(\psi_{L2}\bar{\nu}_2^L)_{\bar{i}j}(\nu_{R3}\bar{\psi}_3^R)_{j\bar{l}}(\psi_{S4}\bar{\nu}_4^S)_{\bar{l}i}\ (\nu_{I1}\bar{\psi}_1^I)_{i\bar{i}}\rangle_0,
\end{aligned}
\tag{A.16}
$$

and moreover we will only consider the Wick-contractions in (A.16) which yield propagators connecting different edges. One has

$$
-\langle(\psi_{L2}\bar{\nu}_2^L)_{\bar{i}j}(\nu_{R3}\bar{\psi}_3^R)_{j\bar{l}}(\psi_{S4}\bar{\nu}_4^S)_{\bar{l}i}\ (\nu_{I1}\bar{\psi}_1^I)_{i\bar{i}}\rangle_0. \tag{A.17}
$$

Then, we first determine the $R-$symmentry and color factor for these contraction. Since all propagators are diagonal one immediately gets

$$
(\bar{n}_2 n_3)(\bar{n}_4 n_1)\delta_{\bar{i}\bar{l}}\delta_{jj}\delta_{ii}\delta_{\bar{l}\bar{i}} = MN^2(\bar{n}_2 n_3)(\bar{n}_4 n_1) \tag{A.18}
$$

The integrand is instead given by

$$
\begin{aligned}
&-\langle(\psi_2\bar{\nu}_2)(\nu_3\bar{\psi}_3)(\psi_4\bar{\nu}_4)(\nu_1\bar{\psi}_1)\rangle_0 = \frac{\Gamma^2(1/2-\epsilon)}{4\pi^{3-2\epsilon}(\eta_3\bar{\eta}_2)(\eta_1\bar{\eta}_4)}\times \\
&\times [\dot{x_2}^\mu + \dot{x_3}^\mu]\,\partial_{x_2^\mu}\left(\frac{1}{(x_{23}^2)^{1/2-\epsilon}}\right)[\dot{x_4}^\mu|\dot{x}_1| + \dot{x_1}^\mu|\dot{x}_4|]\,\partial_{x_4^\mu}\left(\frac{1}{(x_{41}^2)^{1/2-\epsilon}}\right),
\end{aligned}
\tag{A.19}
$$

where we used the following rule for the Wick-contraction

$$(\overbrace{\psi_1\bar{\nu}_1)(\nu_2\bar{\psi}}_2) = -\frac{i\Gamma(1/2-\epsilon)}{2\pi^{3/2-\epsilon}(\eta_1\bar{\eta}_2)}\left[\dot{x_1}^{\mu}+\dot{x_2}^{\mu}|\dot{x}_1|\right]\partial_{x_1^\mu}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right). \tag{A.20}$$

The integrand as a function of s_i with $i = 1,\ldots,4$ possesses factorized structure and it can be organized as follows

$$\frac{\Gamma^2(1/2-\epsilon)}{4\pi^{3-2\epsilon}(\eta_3\bar{\eta}_2)(\eta_1\bar{\eta}_4)}G(s_2,s_3)G(s_4,s_1) \tag{A.21}$$

where $G(s_i,s_j)$ is a function which is antisymmetric in the exchange $s_i \leftrightarrow s_j$ and whose explicit form is

$$G(s_i,s_j) = \left(\frac{d}{ds_i}-\frac{d}{ds_j}\right)\left[\frac{1}{(s_i^2+s_j^2-2s_is_j\cos\varphi)^{1/2-\epsilon}}\right] \equiv \left(\frac{d}{ds_i}-\frac{d}{ds_j}\right)H(s_i,s_j), \tag{A.22}$$

Next we shall split the integrand (A.21) in two pieces, by using the decomposition

$$G(s_2,s_3)G(s_4,s_1) = \underbrace{[G(s_2,s_3)G(s_4,s_1)+G(s_2,s_4)G(s_3,s_1)]}_{(A)} - \underbrace{G(s_2,s_4)G(s_3,s_1)}_{(B)} \tag{A.23}$$

The contribution (A) is trivially symmetric when exchanging s_3 and s_4, but it also symmetric when $s_1 \leftrightarrow s_2$. In fact

$$\begin{aligned}&[G(s_2,s_3)G(s_4,s_1)+G(s_2,s_4)G(s_3,s_1)] = [G(s_3,s_2)G(s_1,s_4)+\\&+G(s_4,s_2)G(s_1,s_3)] = [G(s_1,s_3)G(s_4,s_2)+G(s_1,s_4)G(s_3,s_2)]\,.\end{aligned} \tag{A.24}$$

Then the path-ordered integral of (A) over the contour simply yields minus one half of the square of the one-loop integral, namely

$$\int_0^L ds_1\int_0^{s_1}ds_2\int_{-L}^0 ds_3\int_{-L}^{s_3}ds_4(A) = -\frac{1}{2}\left(\int_0^L ds_1\int_{-L}^0 ds_4 G(s_1,s_4)\right)^2 \tag{A.25}$$

Restoring all the $R-$symmetry and color factors, the contribution (A) to the double exchange diagram can be rearranged as follows

$$\begin{aligned}&-\frac{1}{2}MN^2(\bar{n}_2n_3)(\bar{n}_4n_1)\frac{\Gamma^2(1/2-\epsilon)}{4\pi^{3-2\epsilon}(\eta_3\bar{\eta}_2)(\eta_1\bar{\eta}_4)}\left(\int_0^L ds_1\int_{-L}^0 ds_4G(s_1,s_4)\right)^2 =\\&=\frac{1}{2}M\left(-N\frac{i\Gamma(1/2-\epsilon)}{2\pi^{3/2-\epsilon}(\eta_1\bar{\eta}_4)}(\bar{n}_4n_1)\int_0^L ds_1\int_{-L}^0 ds_4G(s_1,s_4)\right)^2,\end{aligned} \tag{A.26}$$

where we used that $(\bar{n}_2 n_3) = (\bar{n}_4 n_1)$ and $(\eta_3 \bar{\eta}_2) = (\eta_1 \bar{\eta}_4)$. We can easily recognize that this result is $M/2$ times the square of the loop result.

$$I_u = -\frac{iMN\Gamma(1/2-\epsilon)}{2\pi^{3/2-\epsilon}(\eta_1\bar{\eta}_2)}(n_{L2}\bar{n}_1^L)\int_0^L ds_1 \int_{-L}^0 ds_2 G(s_1,s_2). \tag{A.27}$$

This is the first sign of exponentiation in our Wilson loop. We come now to discuss the contribution (B). A part from an overall constant pre-factor we have to compute

$$\begin{aligned} &-\int_0^L ds_1 \int_0^{s_1} ds_2 \int_{-L}^0 ds_3 \int_{-L}^{s_3} ds_4 G(s_2,s_4)G(s_3,s_1) = \\ &= -\int_0^L ds_1 \int_0^{s_1} ds_2 \int_0^L ds_4 \int_0^{s_4} ds_3 G(s_2,-s_4)G(-s_3,s_1). \end{aligned} \tag{A.28}$$

The singular behavior of the above integral can be extracted if we perform the following sequence of changes of variables in the integrals: (a) $s_3 \to s_4 x$, (b) $s_2 \to s_1 y$, (c) $s_1 \to L\lambda z, s_4 \to L\lambda(1-z)$. We end up with the following integral

$$\begin{aligned} &(1-2\epsilon)^2 4\cos^4\frac{\varphi}{2}L^{4\epsilon}\int_S \frac{d\lambda dz}{\lambda^{1-4\epsilon}}\int_0^1 dx \int_0^1 dy \\ &\frac{(1-z)z(x+z(1-x))((y-1)z+1)}{(x^2(z-1)^2-2xz(z-1)\cos\varphi+z^2)^{\frac{3}{2}-\epsilon}(y^2z^2-2yz(z-1)\cos\varphi+(z-1)^2)^{\frac{3}{2}-\epsilon}}. \end{aligned} \tag{A.29}$$

where the region of integration S is defined by the inequalities $0 \leq \lambda z \leq 1$ and $0 \leq \lambda(1-z) \leq 1$ and it is displayed in fig. A.2. The region is clear symmetric around $z = 1/2$ and the two curved borders are respectively given by $\lambda = 1/(1-z) \quad [0 \leq z \leq 1/2]$ and $\lambda = 1/z \quad [1/2 \leq z \leq 1]$.

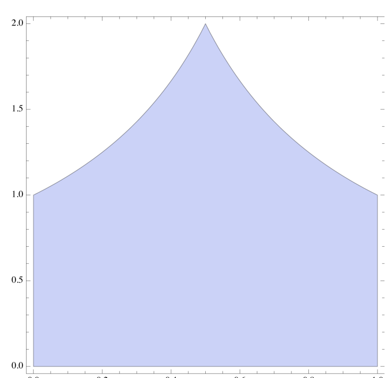


Figure A.2: Region of integration for λ and ζ

Then, using the symmetry of the integrand around $z = 1/2$, the integral (A.29) splits into two contributions

$$
\begin{aligned}
&(1-2\epsilon)^2 4\cos^4\frac{\varphi}{2}L^{4\epsilon}\int_0^1\frac{d\lambda}{\lambda^{1-4\epsilon}}\int_0^1 dz\int_0^1 dx\int_0^1 dy\\
&\frac{(1-z)z(x+z(1-x))((y-1)z+1)}{(x^2(z-1)^2-2xz(z-1)\cos\varphi+z^2)^{\frac{3}{2}-\epsilon}(y^2z^2-2yz(z-1)\cos\varphi+(z-1)^2)^{\frac{3}{2}-\epsilon}}+\\
&+2(1-2\epsilon)^2 4\cos^4\frac{\varphi}{2}L^{4\epsilon}\int_1^{1/2}dz\int_1^{\frac{1}{(1-z)}}\frac{d\lambda}{\lambda^{1-4\epsilon}}\int_0^1 dx\int_0^1 dy\\
&\frac{(1-z)z(x+z(1-x))((y-1)z+1)}{(x^2(z-1)^2-2xz(z-1)\cos\varphi+z^2)^{\frac{3}{2}-\epsilon}(y^2z^2-2yz(z-1)\cos\varphi+(z-1)^2)^{\frac{3}{2}-\epsilon}}=\\
=&(1-2\epsilon)^2\cos^4\frac{\varphi}{2}\frac{L^{4\epsilon}}{\epsilon}\int_0^1 dz\int_0^1 dx\int_0^1 dy\\
&\underbrace{\frac{(1-z)z(x+z(1-x))((y-1)z+1)}{(x^2(z-1)^2-2xz(z-1)\cos\varphi+z^2)^{\frac{3}{2}-\epsilon}(y^2z^2-2yz(z-1)\cos\varphi+(z-1)^2)^{\frac{3}{2}-\epsilon}}}_{(A)}+\\
&+2(1-2\epsilon)^2\frac{L^{4\epsilon}}{\epsilon}\cos^4\frac{\varphi}{2}\int_0^{1/2}dz\int_0^1 dx\int_0^1 dy\\
&\underbrace{\frac{(1-z)z(x+z(1-x))((y-1)z+1)[(1-z)^{-4\epsilon}-1]}{(x^2(z-1)^2-2xz(z-1)\cos\varphi+z^2)^{\frac{3}{2}-\epsilon}(y^2z^2-2yz(z-1)\cos\varphi+(z-1)^2)^{\frac{3}{2}-\epsilon}}}_{(B)}.
\end{aligned}
\tag{A.30}
$$

The above procedure has allowed us to factor out the divergent contribution. We recognize immediately that only the integral (A) possesses a divergence $1/\epsilon$, while the integral (B) is finite as $\epsilon \to 0$. Since the deformed potential is the coefficient of the single pole in ϵ, we focus our attention only on (A) and we set $\epsilon = 0$ in the integrand

$$
\begin{aligned}
(A) =& \cos^4\frac{\varphi}{2}\frac{L^{4\epsilon}}{\epsilon}\int_0^1 dz\int_0^1 dx\int_0^1 dy\times\\
&\times\frac{(1-z)z(x+z(1-x))((y-1)z+1)}{(x^2(z-1)^2-2xz(z-1)\cos\varphi+z^2)^{\frac{3}{2}}(y^2z^2-2yz(z-1)\cos\varphi+(z-1)^2)^{\frac{3}{2}}}
\end{aligned}
\tag{A.31}
$$

Since the divergence is factorized out, the integral over x, y and z are finite and so we can set $\epsilon = 0$ in the integrand and we can perform the integration over x and y. We obtain

$$
\frac{L^{4\epsilon}}{\epsilon}\cos^2\frac{\varphi}{2}\int_0^1 dz\frac{1}{2(1-z)z\cos\varphi+z^2+(1-z)^2}
\tag{A.32}
$$

This integral is reminiscent of the typical integral which appears in four dimension when computing the cusp at one loop. In fact the single exchange diagram in 4 dimensions is given by

$$\begin{aligned}&\int_0^L ds_1\int_{-L}^0 ds_2\frac{1}{((x_1-x_2)^2)^{1-\epsilon'}}=\int_0^L ds_1\int_{-L}^0 ds_2\frac{1}{(s_1^2+s_2^2-2s_1s_2\cos\varphi)^{1-\epsilon'}}=\\ =&L^{2\epsilon'}\int_0^1\frac{d\lambda}{\lambda^{1-2\epsilon'}}\int_0^1\frac{dz}{(z^2+(1-z)^2+2z(1-z)\cos\varphi)^{1-\epsilon'}}+\\ &+2L^{2\epsilon'}\int_0^{\frac{1}{2}}dz\int_1^{\frac{1}{1-z}}\frac{d\lambda}{\lambda^{1-2\epsilon'}}\frac{1}{(z^2+(1-z)^2+2z(1-z)\cos\varphi)^{1-\epsilon'}}=\\ =&\frac{L^{2\epsilon'}}{2\epsilon'}\int_0^1\frac{dz}{(z^2+(1-z)^2+2z(1-z)\cos\varphi)^{1-\epsilon'}}+\frac{L^{2\epsilon'}}{\epsilon'}\int_0^{\frac{1}{2}}\frac{[(1-z)^{-2\epsilon'}-1]dz}{(z^2+(1-z)^2+2z(1-z)\cos\varphi)^{1-\epsilon'}}.\end{aligned}\tag{A.33}$$

Thus the integral which yields the pole in ϵ takes the same form of (A.32) with $\epsilon'=2\epsilon$.

If we restore the color and $R-$symmetry factors, the fermionic double exchange diagram possesses a $1/\epsilon$ term given by

$$\begin{aligned}&\left(\frac{2\pi}{k}\right)^2\frac{\Gamma^2(1/2-\epsilon)MN^2(\bar n_2n_3)(\bar n_4n_1)}{4\pi^{3-2\epsilon}(\eta_3\bar\eta_2)(\eta_1\bar\eta_4)}\frac{L^{4\epsilon}}{\epsilon}\cos^2\frac{\varphi}{2}\int_0^1 dz\frac{1}{2(1-z)z\cos\varphi+z^2+(1-z)^2}=\\ &=-\left(\frac{2\pi}{k}\right)^2\frac{\Gamma^2(1/2-\epsilon)MN^2}{4\pi^{3-2\epsilon}}\frac{L^{4\epsilon}}{4\epsilon}\cos^2\frac{\theta}{2}\int_0^1 dz\frac{1}{2(1-z)z\cos\varphi+z^2+(1-z)^2}.\end{aligned}\tag{A.34}$$

The contribution coming from the lower diagonal block of (A.15) is now obtained by performing exchange $M\leftrightarrow N$ in (A.34). Therefore the total result for this diagram is

$$-\left(\frac{2\pi}{k}\right)^2\frac{\Gamma^2(1/2-\epsilon)MN(N+M)}{4\pi^{3-2\epsilon}}\frac{L^{4\epsilon}}{4\epsilon}\cos^2\frac{\theta}{2}\int_0^1 dz\frac{1}{2(1-z)z\cos\varphi+z^2+(1-z)^2}\tag{A.35}$$

The above diagram is the only contribution whose $R-$symmetry factor is $\cos^2\frac{\theta}{2}$. Next we consider all the fermionic double exchange diagrams which are linear in $\cos\frac{\theta}{2}$. They are given in Figure A.2 (f) (g) (h) (i) and they simply correspond to different Wick contractions of (A.15)

First we consider the contractions of (A.16) that yields the diagram (a):

$$\begin{aligned}&\langle(\nu_{I1}\bar\psi_1^I)_{i\bar i}(\psi_{L2}\bar\nu_2^L)_{\bar i j}(\nu_{R3}\bar\psi_3^R)_{j\bar l}(\psi_{S4}\bar\nu_4^S)_{\bar l i}\rangle=\langle(\psi_{L2}\bar\nu_2^L)_{\bar i j}(\nu_{I1}\bar\psi_1^I)_{i\bar i}(\psi_{S4}\bar\nu_4^S)_{\bar l i}(\nu_{R3}\bar\psi_3^R)_{j\bar l}\rangle\\ &=\langle(\psi_{L2}\bar\nu_2^L)_{\bar i j}(\nu_{I1}\bar\psi_1^I)_{i\bar i}(\psi_{S4}\bar\nu_4^S)_{\bar l i}(\nu_{R3}\bar\psi_3^R)_{j\bar l}\rangle=NM^2(\bar n_2^In_{1I})(\bar n_4^Rn_{R3})\times\\ &\times\langle(\psi_2\bar\nu_2)(\nu_1\bar\psi_1)(\psi_4\bar\nu_4)(\nu_3\bar\psi_3)\rangle=-\frac{NM^2(\bar n_2^In_{1I})(\bar n_4^Rn_{R3})\Gamma^2(1/2-\epsilon)}{4\pi^{3-2\epsilon}(\eta_2\bar\eta_1)(\eta_4\bar\eta_3)}\times\\ &\times[\dot{x}_2{}^\mu|\dot{x}_1|+\dot{x}_1{}^\mu]\,\partial_{x_2^\mu}\left(\frac{1}{(x_{21}^2)^{1/2-\epsilon}}\right)[\dot{x}_4{}^\mu+\dot{x}_3{}^\mu|\dot{x}_4|]\,\partial_{x_4^\mu}\left(\frac{1}{(x_{34}^2)^{1/2-\epsilon}}\right).\end{aligned}\tag{A.36}$$

A part from an overall constant factor, the diagram (a) is equivalent compute the path-ordered integral

$$\int_0^L ds_1\int_{-L}^0 ds_2\int_{-L}^{s_2}ds_3\int_{-L}^{s_3}ds_4\left(\frac{d}{ds_4}-\frac{d}{ds_3}\right)\left[\frac{1}{(s_3-s_4)^{1-2\epsilon}}\right]\times G(s_2,s_1),\tag{A.37}$$

where $G(s_2, s_1)$ is defined through eq (A.22). We can now easily perform the integration over s_3 and s_4 if $\epsilon > 1/2$ and we obtain

$$-\frac{1}{\epsilon}\int_0^L ds_1 \int_{-L}^0 ds_2 (L+s_2)^{2\epsilon} \times G(s_2, s_1). \tag{A.38}$$

Notice that the digram (d) originates from the same contractions, but the region of integration is different

$$\begin{aligned} &\int_0^L ds_1 \int_0^{s_1} ds_2 \int_0^{s_2} ds_3 \int_{-L}^0 ds_4 \left(\frac{d}{ds_2} - \frac{d}{ds_1}\right)\left[\frac{1}{(s_1-s_2)^{1-2\epsilon}}\right] \times G(s_4, s_3) = \\ &= \int_{-L}^0 ds_4 \int_0^L ds_3 \int_{s_3}^L ds_2 \int_{s_2}^L ds_1 \left(\frac{d}{ds_2} - \frac{d}{ds_1}\right)\left[\frac{1}{(s_1-s_2)^{1-2\epsilon}}\right] \times G(s_4, s_3) \end{aligned} \tag{A.39}$$

In this case the specific form of the integrand suggests to perform first the integration over s_1 and s_2 . We get

$$-\frac{1}{\epsilon}\int_0^L ds_3 \int_{-L}^0 ds_4 (L-s_3)^{2\epsilon} \times G(s_4, s_3). \tag{A.40}$$

If we now perform the change of variable $s_3 \mapsto -s_2$ and $s_4 \mapsto -s_1$, we can recast the integral as follows

$$-\frac{1}{\epsilon}\int_0^L ds_1 \int_{-L}^0 ds_2 (L+s_2)^{2\epsilon} \times G(-s_1, -s_2) = -\frac{1}{\epsilon}\int_0^L ds_1 \int_{-L}^0 ds_2 (L+s_2)^{2\epsilon} \times G(s_2, s_1), \tag{A.41}$$

i.e. it doubles the contribution (a). If we restore the constant factor in front of the integral we get

$$(a)+(d) = -\frac{1}{2\epsilon}\left(\frac{2\pi}{k}\right)^2 \frac{NM^2\Gamma^2(1/2-\epsilon)}{4\pi^{3-2\epsilon}} \frac{\cos\frac{\theta}{2}}{\cos\frac{\varphi}{2}} \int_0^L ds_1 \int_{-L}^0 ds_2 (L+s_2)^{2\epsilon} G(s_2, s_1). \tag{A.42}$$

Next we consider the Wick-contractions which give origin to the diagrams (b) and (c)

$$\begin{aligned} &\langle(\nu_{I1}\bar\psi_1^I)_{i\bar i}(\psi_{L2}\bar\nu_2^L)_{\bar i j}(\nu_{R3}\bar\psi_3^R)_{j\bar l}(\psi_{S4}\bar\nu_4^S)_{\bar l i}\rangle = -\langle(\psi_{L2}\bar\nu_2^L)_{\bar i j}(\nu_{R3}\bar\psi_3^R)_{j\bar l}(\psi_{S4}\bar\nu_4^S)_{\bar l i}(\nu_{I1}\bar\psi_1^I)_{i\bar i}\rangle = \\ &= -\langle(\psi_{L2}\bar\nu_2^L)_{\bar i j}(\nu_{R3}\bar\psi_3^R)_{j\bar l}(\psi_{S4}\bar\nu_4^S)_{\bar l i}(\nu_{I1}\bar\psi_1^I)_{i\bar i}\rangle = -N^2M(\bar n_3^I n_{2I})(\bar n_1^R n_{R4})\times \\ &\quad\times\langle(\psi_2\bar\nu_2)(\nu_3\bar\psi_3)(\psi_4\bar\nu_4)(\nu_1\bar\psi_1)\rangle = \frac{\Gamma^2(1/2-\epsilon)}{4\pi^{3-2\epsilon}(\eta_2\bar\eta_3)(\eta_4\bar\eta_1)} N^2M(\bar n_3^I n_{2I})(\bar n_1^R n_{R4})\times \\ &\quad\times[\dot{x_2}^\mu + \dot{x_3}^\mu]\,\partial_{x_2^\mu}\left(\frac{1}{(x_{23}^2)^{1/2-\epsilon}}\right)[\dot{x_4}^\mu|\dot x_1| + \dot{x_1}^\mu|\dot x_4|]\,\partial_{x_4^\mu}\left(\frac{1}{(x_{14}^2)^{1/2-\epsilon}}\right). \end{aligned} \tag{A.43}$$

As usual we can get rid of two integrations and we are left with

$$\begin{aligned} &\int_0^L ds_1 \int_{-L}^0 ds_2 \int_{-L}^{s_2} ds_3 \int_{-L}^{s_3} ds_4 \left(\frac{d}{ds_2} - \frac{d}{ds_3}\right)\left[\frac{1}{(s_2-s_3)^{1-2\epsilon}}\right] \times G(s_4, s_1) = \\ &= \int_0^L ds_1 \int_{-L}^0 ds_4 \int_{s_4}^0 ds_3 \int_{s_3}^0 ds_2 \left(\frac{d}{ds_2} - \frac{d}{ds_3}\right)\left[\frac{1}{(s_2-s_3)^{1-2\epsilon}}\right] \times G(s_4, s_1) = \\ &= \frac{1}{\epsilon}\int_0^L ds_1 \int_{-L}^0 ds_4 (-s_4)^{2\epsilon} G(s_4, s_1) \end{aligned} \tag{A.44}$$

The digram (c) simply doubles the previous results. Restoring all the constant coefficient we find

$$(b)+(c) = -\frac{1}{2\epsilon}\left(\frac{2\pi}{k}\right)^2 \frac{N^2 M\Gamma^2(1/2-\epsilon)}{4\pi^{3-2\epsilon}} \frac{\cos\frac{\theta}{2}}{\cos\frac{\varphi}{2}} \int_0^L ds_1 \int_{-L}^0 ds_2 (-s_2)^{2\epsilon} G(s_2,s_1). \tag{A.45}$$

The lower diagonal block is again obtained by exchanging $M \leftrightarrow N$ in the previous results. Therefore the total results is

$$-\frac{1}{2\epsilon}\left(\frac{2\pi}{k}\right)^2 \frac{MN(M+N)\Gamma^2(1/2-\epsilon)}{4\pi^{3-2\epsilon}} \frac{\cos\frac{\theta}{2}}{\cos\frac{\varphi}{2}} \int_0^L ds_1 \int_{-L}^0 ds_2 [(L+s_2)^{2\epsilon} + (-s_2)^{2\epsilon}] G(s_2,s_1) \tag{A.46}$$

A.3 Fermion-Fermion-Gauge Field vertex

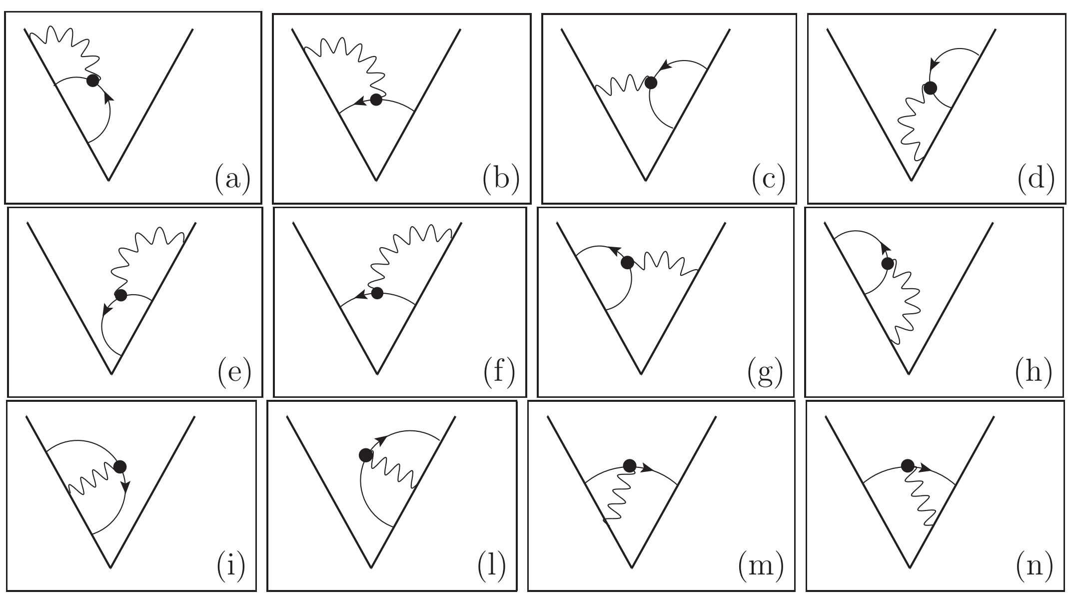


Figure A.3: There are a total of 12 graphs for each gluon field contributing to the expectation value of the fermion-gluon vertex. Above, graphs relative to the gluon A are dipicted and organized according to their integration ranges and orientation of the fermionic lines. Contributions coming from the $\widehat{A}$ field are obtained reversing all fermionic lines.

Finally we consider the diagrams which are due to the minimal coupling between the fermions and the gauge fields present in the Lagrangian. They appear when considering the cubic term in the super-connection $\mathcal{L}$:

$$\begin{aligned}
&\mathrm{Tr}[(i\mathcal{L}(\tau_1))(i\mathcal{L}(\tau_2))(i\mathcal{L}(\tau_3))] = \\
&= -i\mathrm{Tr}\left[\begin{pmatrix} \mathcal{A}_1 & -i\sqrt{\frac{2\pi}{k}}\nu_{I1}\bar{\psi}_1^I \\ -i\sqrt{\frac{2\pi}{k}}\psi_{I1}\bar{\nu}_1^I & \widehat{\mathcal{A}}_1 \end{pmatrix}\begin{pmatrix} \mathcal{A}_2 & -i\sqrt{\frac{2\pi}{k}}\nu_{L2}\bar{\psi}_2^L \\ -i\sqrt{\frac{2\pi}{k}}\psi_{L2}\bar{\nu}_2^L & \widehat{\mathcal{A}}_2 \end{pmatrix}\times\right. \\
&\quad\left.\times\begin{pmatrix} \mathcal{A}_3 & -i\sqrt{\frac{2\pi}{k}}\nu_{L3}\bar{\psi}_3^L \\ -i\sqrt{\frac{2\pi}{k}}\psi_{L3}\bar{\nu}_3^L & \widehat{\mathcal{A}}_3 \end{pmatrix}\right] \\
&= -i\mathrm{Tr}\left[\begin{pmatrix} \mathcal{A}_1\mathcal{A}_2 - \frac{2\pi}{k}\nu_{I1}\bar{\psi}_1^I\psi_{L2}\bar{\nu}_2^L & -i\sqrt{\frac{2\pi}{k}}\mathcal{A}_1\nu_{L2}\bar{\psi}_2^I - i\sqrt{\frac{2\pi}{k}}\nu_{I1}\bar{\psi}_1^I\widehat{\mathcal{A}_2} \\ -i\sqrt{\frac{2\pi}{k}}\psi_{I1}\bar{\nu}_1^I\mathcal{A}_2 - i\sqrt{\frac{2\pi}{k}}\widehat{\mathcal{A}}_1\psi_{L2}\bar{\nu}_2^L & \widehat{\mathcal{A}}_1\widehat{\mathcal{A}}_2 - \frac{2\pi}{k}\psi_{I1}\bar{\nu}_1^I\nu_{L2}\bar{\psi}_2^L \end{pmatrix}\times\right. \\
&\quad\left.\times\begin{pmatrix} \mathcal{A}_3 & -i\sqrt{\frac{2\pi}{k}}\nu_{L3}\bar{\psi}_3^L \\ -i\sqrt{\frac{2\pi}{k}}\psi_{L3}\bar{\nu}_3^L & \widehat{\mathcal{A}}_3 \end{pmatrix}\right] = \\
&= \frac{2\pi i}{k}\left[\nu_{I1}\bar{\psi}_1^I\psi_{L2}\bar{\nu}_2^L\mathcal{A}_3 + \mathcal{A}_1\nu_{L2}\bar{\psi}_2^I\psi_{L3}\bar{\nu}_3^L + \nu_{I1}\bar{\psi}_1^I\widehat{\mathcal{A}_2}\psi_{L3}\bar{\nu}_3^L\right] + \\
&\quad + \frac{2\pi i}{k}\left[\psi_{I1}\bar{\nu}_1^I\mathcal{A}_2\nu_{L3}\bar{\psi}_3^L + \widehat{\mathcal{A}}_1\psi_{L2}\bar{\nu}_2^L\nu_{L3}\bar{\psi}_3^L + \psi_{I1}\bar{\nu}_1^I\nu_{L2}\bar{\psi}_2^L\widehat{\mathcal{A}}_3\right] + \cdots \\
&= \frac{2\pi i}{k}\underbrace{\left[\psi_{I1}\bar{\nu}_1^I\mathcal{A}_2\nu_{L3}\bar{\psi}_3^L + \nu_{I1}\bar{\psi}_1^I\psi_{L2}\bar{\nu}_2^L\mathcal{A}_3 + \mathcal{A}_1\nu_{L2}\bar{\psi}_2^I\psi_{L3}\bar{\nu}_3^L\right]}_{(S_1)} + \\
&\quad + \frac{2\pi i}{k}\underbrace{\left[\widehat{\mathcal{A}}_1\psi_{L2}\bar{\nu}_2^L\nu_{L3}\bar{\psi}_3^L + \psi_{I1}\bar{\nu}_1^I\nu_{L2}\bar{\psi}_2^L\widehat{\mathcal{A}}_3 + \nu_{I1}\bar{\psi}_1^I\widehat{\mathcal{A}_2}\psi_{L3}\bar{\nu}_3^L\right]}_{(S_2)} + \cdots
\end{aligned} \tag{A.47}$$

where we have defined

$$\mathcal{A} = A_\mu\dot{x}^\mu - i\frac{2\pi}{k}|\dot{x}|M_J{}^I C_I\bar{C}^J \quad \text{and} \quad \widehat{\mathcal{A}} = \widehat{A}_\mu\dot{x}^\mu - i\frac{2\pi}{k}|\dot{x}|\widehat{M}_J{}^I\bar{C}^J C_I. \tag{A.48}$$

The dots in (A.47) stand for the terms which do not contribute to this family of diagrams. Again the relevant contributions splits into the sum of two different terms: S_1 and S_2. We focus our attention on the evaluation of S_1. Since the ABJM Lagrangian contain a minimal coupling of the form

$$-S_{int} = \int d^3w \mathrm{Tr}\left(\bar{\psi}^L\gamma^\mu\widehat{A}\psi_L - \bar{\psi}^L\gamma^\mu\psi_L A_\mu\right), \tag{A.49}$$

the value of S_1 is provided by the following correlator in the free theory

$$-\frac{2\pi i}{k}\int d^3w\langle\mathrm{Tr}\left[\underbrace{\psi_{I1}\bar{\nu}_1^I\mathcal{A}_2\nu_{L3}\bar{\psi}_3^L}_{(a)} + \underbrace{\nu_{I1}\bar{\psi}_1^I\psi_{L2}\bar{\nu}_2^L\mathcal{A}_3}_{(b)} + \underbrace{\mathcal{A}_1\nu_{L2}\bar{\psi}_2^I\psi_{L3}\bar{\nu}_3^L}_{(c)}\right]\mathrm{Tr}[\bar{\psi}_w^M\gamma^\mu\psi_{Mw}A_{\mu w}]\rangle_0 \tag{A.50}$$

For convenience we separate the evaluation of (A.50) in three steps. The monomial (a) yields the following Wick-contractions

$$\begin{aligned}
(a) =& \int d^{3-2\epsilon}w \langle \mathrm{Tr}[\psi_{I1}\bar{\nu}_1^I \mathcal{A}_2 \nu_{L3}\bar{\psi}_3^L]\mathrm{Tr}[\bar{\psi}_w^M \gamma^\mu \psi_{Mw} A_{\mu w}]\rangle_0 = \\
=& \int d^{3-2\epsilon}w \langle (\psi_{I1}\bar{\nu}_1^I)_{\bar{i}i}(\mathcal{A}_2)_{il}(\nu_{L3}\bar{\psi}_3^L)_{l\bar{i}}(\bar{\psi}_w^M)_{r\bar{s}}\gamma^\mu(\psi_{Mw})_{\bar{s}m}(A_{\mu w})_{mr}\rangle_0 = \\
=&(\gamma)_\alpha{}^\beta \int d^{3-2\epsilon}w \langle (\psi_{I1}\bar{\nu}_1^I)_{\bar{i}i}(\bar{\psi}_w^M)^\alpha_{r\bar{s}}\rangle_0 \langle (\mathcal{A}_2)_{il}(A_{\mu w})_{mr}\rangle_0 \langle (\nu_{L3}\bar{\psi}_3^L)_{l\bar{i}}(\psi_{Mw})_{\bar{s}m\beta}\rangle_0 = \\
=&(\gamma^\mu)_\alpha{}^\beta \delta_{ir}\delta_{\bar{i}\bar{s}}\delta_{ir}\delta_{lm}\delta_{lm}\delta_{\bar{i}\bar{s}} \int d^{3-2\epsilon}w \langle (\psi_1\bar{\nu}_1^L)(\bar{\psi}_w)^\alpha\rangle_0 \langle (\mathcal{A}_2)(A_{\mu w})\rangle_0 \langle (\psi_w)_\beta(\nu_{L3}\bar{\psi}_3)\rangle_0 = \\
=&MN^2(\gamma^\mu)_\alpha{}^\beta \int d^{3-2\epsilon}w \langle (\bar{\nu}_1^L\psi_1)(\bar{\psi}_w)^\alpha\rangle_0 \langle (\mathcal{A}_2)(A_{\mu w})\rangle_0 \langle (\psi_w)_\beta(\bar{\psi}_3\nu_{L3})\rangle_0 = \\
=& -\frac{2\pi i}{k} MN^2 \bar{\nu}_1^L \gamma^\rho\gamma_\mu\gamma^\sigma \nu_{L3}\epsilon^{\nu\mu\lambda}\dot{x}_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3),
\end{aligned} \tag{A.51}$$

where we have introduced the short-hand notation

$$\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) = \left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^3 \partial_{x_1^\rho}\partial_{x_2^\lambda}\partial_{x_3^\sigma} \int \frac{d^{3-2\epsilon}w}{(x_{1w}^2)^{1/2-\epsilon}(x_{2w}^2)^{1/2-\epsilon}(x_{3w}^2)^{1/2-\epsilon}} \tag{A.52}$$

and we have used the explicit form of the gauge-field propagator

$$\langle A_\mu(x_1)A_\nu(x_2)\rangle_0 = \frac{2\pi}{k}\epsilon_{\mu\nu\rho}\int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} e^{ip(x_1-x_2)}\frac{p^\rho}{p^2} = -\frac{2\pi i}{k}\epsilon_{\mu\nu\rho}\partial^{x_1^\rho}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right). \tag{A.53}$$

Next we consider the other two contributions [(b) and (c)]. Their expansion in terms of Wick-contractions is similar and one obtains the following results

$$\begin{aligned}
(b) =& \int d^{3-2\epsilon}w \langle \mathrm{Tr}\left[\nu_{I1}\bar{\psi}_1^I \psi_{L2}\bar{\nu}_2^L \mathcal{A}_3\right] \mathrm{Tr}[\bar{\psi}_w^M \gamma^\mu \psi_{Mw} A_{\mu w}]\rangle_0 = \\
=&(\gamma^\mu)_\alpha{}^\beta \int d^{3-2\epsilon}w \langle (\nu_{I1}\bar{\psi}_1^I)_{i\bar{l}}(\psi_{L2}\bar{\nu}_2^L)_{\bar{l}m}(\mathcal{A}_3)_{mi}(\bar{\psi}_w^M)^\alpha_{r\bar{s}}(\psi_{Mw})_{\bar{s}n\beta}(A_{\mu w})_{nr}\rangle_0 = \\
=& -(\gamma^\mu)_\alpha{}^\beta \int d^{3-2\epsilon}w \langle (\psi_{Mw})_{\bar{s}n\beta}(\nu_{I1}\bar{\psi}_1^I)_{i\bar{l}}\rangle_0 \langle (\psi_{L2}\bar{\nu}_2^L)_{\bar{l}m}(\bar{\psi}_w^M)^\alpha_{r\bar{s}}\rangle_0 \langle (\mathcal{A}_3)_{mi}(A_{\mu w})_{nr}\rangle_0 = \\
=& -(\gamma^\mu)_\alpha{}^\beta \delta_{\bar{s}\bar{l}}\delta_{ni}\delta_{\bar{l}\bar{s}}\delta_{mr}\delta_{in}\delta_{mr} \int d^{3-2\epsilon}w \langle (\psi_w)_\beta(\bar{\psi}_1\nu_{I1})\rangle_0 \langle (\bar{\nu}_2^I\psi_2)(\bar{\psi}_w)^\alpha\rangle_0 \langle (\mathcal{A}_3)(A_{\mu w})\rangle_0 = \\
=&\frac{2\pi i}{k} MN^2 \bar{\nu}_2^I \gamma^\rho\gamma_\mu\gamma^\sigma \nu_{I1}\Gamma_{\rho\lambda\sigma}(x_2,x_3,x_1)\epsilon^{\nu\mu\lambda}\dot{x}_{3\nu} = \\
=&\frac{2\pi i}{k} MN^2 \bar{\nu}_2^I \gamma^\lambda\gamma_\mu\gamma^\rho \nu_{I1}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3)\epsilon^{\nu\mu\sigma}\dot{x}_{3\nu}
\end{aligned} \tag{A.54}$$

$$
\begin{aligned}
(c) =& \int d^{3-2\epsilon}w \langle \mathrm{Tr}\left[\mathcal{A}_1 \nu_{L2} \bar{\psi}_2^I \psi_{L3} \bar{\nu}_3^L\right] \mathrm{Tr}[\bar{\psi}_w^M \gamma^\mu \psi_{Mw} A_{\mu w}]\rangle_0 = \\
=& (\gamma^\mu)_\alpha{}^\beta \int d^{3-2\epsilon}w \langle (\mathcal{A}_1)_{il} (\nu_{L2}\bar{\psi}_2^I)_{l\bar{m}} (\psi_{L3}\bar{\nu}_3^L)_{\bar{m}i} (\bar{\psi}_w^M)^\alpha_{n\bar{s}} (\psi_{Mw})_{\beta\bar{s}j} (A_{\mu w})_{jn}\rangle_0 = \\
=& -(\gamma^\mu)_\alpha{}^\beta \int d^{3-2\epsilon}w \langle (\psi_{Mw})_{\beta\bar{s}j} (\bar{\psi}_2^I \nu_{L2})_{l\bar{m}}\rangle_0 \langle (\bar{\nu}_3^L \psi_{L3})_{\bar{m}i} (\bar{\psi}_w^M)^\alpha_{n\bar{s}}\rangle_0 \langle (\mathcal{A}_1)_{il} (A_{\mu w})_{jn}\rangle_0 = \\
=& -(\gamma^\mu)_\alpha{}^\beta \delta_{jl}\delta_{\bar{s}\bar{m}}\delta_{\bar{m}\bar{s}}\delta_{in}\delta_{lj}\delta_{in} \int d^{3-2\epsilon}w \langle (\psi_w)_\beta (\bar{\psi}_2 \nu_{I2})\rangle_0 \langle (\bar{\nu}_3^I \psi_3)(\bar{\psi}_w)^\alpha\rangle_0 \langle (\mathcal{A}_1)(A_{\mu w})\rangle_0 = \\
=& \frac{2\pi i}{k} MN^2 \bar{\nu}_3^I \gamma^\rho \gamma_\mu \gamma^\sigma \nu_{I2} \epsilon^{\nu\mu\lambda} \dot{x}_{1\nu} \Gamma_{\rho\lambda\sigma}(x_3, x_1, x_2) = \\
=& \frac{2\pi i}{k} MN^2 \bar{\nu}_3^I \gamma^\sigma \gamma_\mu \gamma^\lambda \nu_{I2} \epsilon^{\nu\mu\rho} \dot{x}_{1\nu} \Gamma_{\rho\lambda\sigma}(x_1, x_2, x_3)
\end{aligned} \tag{A.55}
$$

Summing up the different contributions, we obtain the following compact expression for S_1

$$
\begin{aligned}
S_1 =& -\left(\frac{2\pi}{k}\right)^2 MN^2 \oint_{s_3<s_2<s_1} ds_1 ds_2 ds_3\ \Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) \times \\
& \times \left[\bar{\nu}_1^L \gamma^\rho \gamma_\mu \gamma^\sigma \nu_{L3} \epsilon^{\nu\mu\lambda} \dot{x}_{2\nu} - \bar{\nu}_2^I \gamma^\lambda \gamma_\mu \gamma^\rho \nu_{I1} \epsilon^{\nu\mu\sigma} \dot{x}_{3\nu} - \bar{\nu}_3^I \gamma^\sigma \gamma_\mu \gamma^\lambda \nu_{I2} \epsilon^{\nu\mu\rho} \dot{x}_{1\nu}\right]
\end{aligned} \tag{A.56}
$$

In the following we shall focus on the diagrams which are proportional to $\cos\frac{\theta}{2}$, namely those where the fermions are attached to different edges. They are represented in Figure A.3 (b) (f) (m) (n) and they correspond to the following integrals

$$
\begin{aligned}
(a) =& \left(\frac{2\pi}{k}\right)^2 MN^2 \int_0^L ds_1 \int_{-L}^0 ds_2 \int_{-L}^{s_2} ds_3\ \bar{\nu}_2^I \gamma^\lambda \gamma_\mu \gamma^\rho \nu_{I1} \epsilon^{\nu\mu\sigma} \dot{x}_{3\nu} \Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) \\
(b) =& -\left(\frac{2\pi}{k}\right)^2 MN^2 \int_0^L ds_1 \int_{-L}^0 ds_2 \int_{-L}^{s_2} ds_3\ \bar{\nu}_1^L \gamma^\rho \gamma_\mu \gamma^\sigma \nu_{L3} \epsilon^{\nu\mu\lambda} \dot{x}_{2\nu} \Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) \\
(c) =& -\left(\frac{2\pi}{k}\right)^2 MN^2 \int_0^L ds_1 \int_0^{s_1} ds_2 \int_{-L}^0 ds_3\ \bar{\nu}_1^L \gamma^\rho \gamma_\mu \gamma^\sigma \nu_{L3} \epsilon^{\nu\mu\lambda} \dot{x}_{2\nu} \Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) \\
(d) =& \left(\frac{2\pi}{k}\right)^2 MN^2 \int_0^L ds_1 \int_0^{s_1} ds_2 \int_{-L}^0 ds_3\ \bar{\nu}_3^I \gamma^\sigma \gamma_\mu \gamma^\lambda \nu_{I2} \epsilon^{\nu\mu\rho} \dot{x}_{1\nu} \Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3)
\end{aligned} \tag{A.57}
$$

Notice that if we perform the change of variable $s_2 \mapsto s_3$ and $s_3 \mapsto s_2$ in the integral (a) and we recall that $\bar{\eta}_2 = \bar{\eta}_3$ and $\dot{x}_2 = \dot{x}_3$ in this case, we find

$$
\begin{aligned}
& \bar{\nu}_2^I \gamma^\lambda \gamma_\mu \gamma^\rho \nu_{I1} \epsilon^{\nu\mu\sigma} \dot{x}_{3\nu} \Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) \mapsto \bar{\nu}_3^I \gamma^\lambda \gamma_\mu \gamma^\rho \nu_{I1} \epsilon^{\nu\mu\sigma} \dot{x}_{2\nu} \Gamma_{\rho\lambda\sigma}(x_1,x_3,x_2) = \\
& = \bar{\nu}_3^I \gamma^\lambda \gamma_\mu \gamma^\rho \nu_{I1} \epsilon^{\nu\mu\sigma} \dot{x}_{2\nu} \Gamma_{\rho\sigma\lambda}(x_1,x_2,x_3) = \bar{\nu}_3^I \gamma^\sigma \gamma_\mu \gamma^\rho \nu_{I1} \epsilon^{\nu\mu\lambda} \dot{x}_{2\nu} \Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3)
\end{aligned} \tag{A.58}
$$

Let us now compare

$$T_1 = \bar{\nu}_3^I \gamma^\sigma \gamma_\mu \gamma^\rho \nu_{I1} \epsilon^{\nu\mu\lambda} \dot{x}_{2\nu} \Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3)$$

with the integrand

$$T_2 = \bar{\nu}_1^L \gamma^\rho \gamma_\mu \gamma^\sigma \nu_{L3} \epsilon^{\nu\mu\lambda} \dot{x}_{2\nu} \Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3)$$

which instead appear in (b). Exploiting the identity

$$\gamma^\rho\gamma^\mu\gamma^\sigma = \delta^{\rho\mu}\gamma^\sigma + \gamma^{\rho\mu}\gamma^\sigma = \delta^{\rho\mu}\gamma^\sigma + \delta^{\mu\sigma}\gamma^\rho - \delta^{\rho\sigma}\gamma^\mu + \gamma^{\rho\mu\sigma}, \tag{A.59}$$

we can rewrite T_1 as follows

$$\begin{aligned} T_1 =& \bar\nu_3^I\gamma^\sigma\gamma_\mu\gamma^\rho\nu_{I1}\epsilon^{\nu\mu\lambda}\dot x_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) = \\ =&[\bar\nu_3^I\gamma^\rho\nu_{I1}\delta^{\mu\sigma} + \bar\nu_3^I\gamma^\sigma\nu_{I1}\delta^{\mu\rho} - \bar\nu_3^I\gamma_\mu\nu_{I1}\delta^{\rho\sigma} + \bar\nu_3^I\gamma^{\sigma\mu\rho}\nu_{I1}]\epsilon^{\nu\mu\lambda}\dot x_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) = \\ =&[\bar\nu_3^I\gamma^{\sigma\mu\rho}\nu_{I1} - \bar\nu_3^I\gamma_\mu\nu_{I1}\delta^{\rho\sigma}]\epsilon^{\nu\mu\lambda}\dot x_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) \end{aligned} \tag{A.60}$$

In the final result we have dropped all the terms which vanishes since the loop is planar. We consider now T_2

$$\begin{aligned} T_2 =& \bar\nu_1^L\gamma^\rho\gamma_\mu\gamma^\sigma\nu_{L3}\epsilon^{\nu\mu\lambda}\dot x_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) = \\ =&[\bar\nu_1^L\gamma^\sigma\nu_{L3}\delta^{\mu\rho} + \bar\nu_1^L\gamma^\rho\nu_{L3}\delta^{\mu\sigma} - \bar\nu_1^L\gamma_\mu\nu_{L3}\delta^{\rho\sigma} + \bar\nu_1^L\gamma^{\rho\mu\sigma}\nu_{L3}]\epsilon^{\nu\mu\lambda}\dot x_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) \\ =&[\bar\nu_1^L\gamma^{\rho\mu\sigma}\nu_{L3} - \bar\nu_1^L\gamma_\mu\nu_{L3}\delta^{\rho\sigma}]\epsilon^{\nu\mu\lambda}\dot x_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) \end{aligned} \tag{A.61}$$

For our contour $\bar\nu_3^I\gamma^{\sigma\mu\rho}\nu_{I1} = -\bar\nu_1^L\gamma^{\sigma\mu\rho}\nu_{L3}$ and

$$\bar\nu_3^I\gamma_\mu\nu_{I1}\delta^{\rho\sigma}\epsilon^{\nu\mu\lambda}\dot x_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) = -\bar\nu_1^L\gamma_\mu\nu_{L3}\delta^{\rho\sigma}\epsilon^{\nu\mu\lambda}\dot x_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3).$$

The origin of the last identity can be easily understood if we recall that the component of orthogonal to plane[1] of the vector $\bar\nu_i^I\gamma^\lambda\nu_{Ij}$ is anti-symmetric in the exchange (i,j). Therefore $T_1 = -T_2$, which in turn implies

$$(a)+(b) = -\left(\frac{2\pi}{k}\right)^2 MN^2\int_0^L ds_1\int_{-L}^0 ds_2\int_{-L}^0 ds_3\ \bar\nu_1^L\gamma^\rho\gamma_\mu\gamma^\sigma\nu_{L3}\epsilon^{\nu\mu\lambda}\dot x_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3). \tag{A.62}$$

The same analysis can be applied to the diagrams (c) and (d) and we find that

$$(c)+(d) = -\left(\frac{2\pi}{k}\right)^2 MN^2\int_0^L ds_1\int_0^L ds_2\int_{-L}^0 ds_3\ \bar\nu_1^L\gamma^\rho\gamma_\mu\gamma^\sigma\nu_{L3}\epsilon^{\nu\mu\lambda}\dot x_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3). \tag{A.63}$$

Now we focus our attention on the *formally* identical integrand[2]

$$\bar\nu_1^L\gamma^\rho\gamma_\mu\gamma^\sigma\nu_{L3}\epsilon^{\nu\mu\lambda}\dot x_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) = [\bar\nu_1^L\gamma^{\rho\mu\sigma}\nu_{L3} - \bar\nu_1^L\gamma_\mu\nu_{L3}\delta^{\rho\sigma}]\epsilon^{\nu\mu\lambda}\dot x_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) \tag{A.64}$$

First we separate the $R-$symmetry factor and perform some *Diracology*

$$\begin{aligned} &(\bar n_1\cdot n_3)[\bar\nu_1\gamma^{\rho\mu\sigma}\nu_3 - \bar\nu_1\gamma_\mu\nu_3\delta^{\rho\sigma}]\epsilon^{\nu\mu\lambda}\dot x_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) = \\ =&\zeta(\bar n_1\cdot n_3)\left[\bar\nu_1\nu_3\epsilon^{\rho\mu\sigma} + \frac{2}{(\eta_1\bar\eta_3)}\epsilon_{\alpha\mu\beta}\dot{x_3}^\alpha\dot{x_1}^\beta\delta^{\rho\sigma}\right]\epsilon^{\nu\mu\lambda}\dot x_{2\nu}\Gamma_{\rho\lambda\sigma}(x_1,x_2,x_3) = \\ =&\zeta(\bar n_1\cdot n_3)\left[(\bar\nu_1\nu_3)\dot x_2^\nu(\Gamma_{\nu\lambda\lambda} - \Gamma_{\lambda\lambda\nu}) + \frac{2}{(\eta_1\bar\eta_3)}[(\dot x_3\cdot\dot x_2)\dot x_1^\nu - (\dot x_1\cdot\dot x_2)\dot x_3^\nu]\Gamma_{\lambda\nu\lambda}\right] \end{aligned} \tag{A.65}$$

[1]This is the only relevant component in the above contraction.

[2]Recall that x_2 is different in the two integrals: in the former it is parallel to x_3, while in the latter it is collinear with x_1.

where we used $\gamma^{\rho\mu\sigma} = \zeta\epsilon^{\rho\mu\sigma}\mathbb{I}$ and

$$(\nu_3\gamma_\mu\bar{\nu}_1) = -\frac{2}{(\eta_1\bar{\eta}_3)}\left[x\dot{_{1\mu}}|\dot{x}_3| + x\dot{_{3\mu}}|\dot{x}_1| + \zeta\epsilon_{\lambda\mu\nu}\dot{x_3}^\lambda\dot{x_1}^\nu\right]. \tag{A.66}$$

Exploiting these result the integrals $(a)+(b)$ and $(c)+(d)$ can be written as

$$\begin{aligned}(a)+(b) =& -\left(\frac{2\pi}{k}\right)^2 MN^2\zeta(\bar{n}_1\cdot n_3)\int_0^L ds_1\int_{-L}^0 ds_2\int_{-L}^0 ds_3\\ &\left[(\bar{\nu}_1\nu_3)\dot{x}_3^\nu(\Gamma_{\nu\lambda\lambda}-\Gamma_{\lambda\lambda\nu}) + \frac{2}{(\eta_1\bar{\eta}_3)}[\dot{x}_1^\nu - (\dot{x}_1\cdot\dot{x}_3)\dot{x}_3^\nu]\Gamma_{\lambda\nu\lambda}\right];\\ (c)+(d) =& -\left(\frac{2\pi}{k}\right)^2 MN^2\zeta(\bar{n}_1\cdot n_3)\int_0^L ds_1\int_0^L ds_2\int_{-L}^0 ds_3\\ &\left[(\bar{\nu}_1\nu_3)\dot{x}_1^\nu(\Gamma_{\nu\lambda\lambda}-\Gamma_{\lambda\lambda\nu}) + \frac{2}{(\eta_1\bar{\eta}_3)}[(\dot{x}_3\cdot\dot{x}_1)\dot{x}_1^\nu - \dot{x}_3^\nu]\Gamma_{\lambda\nu\lambda}\right]\end{aligned} \tag{A.67}$$

Actually one can also relate the integral $(c)+(d)$ to $(a)+(b)$ through a finite chain of transformations. In fact if we explicitly perform the transformation $s_i \mapsto -s_i$ $[x_i \mapsto -x_i$ and $\dot{x}_i \mapsto \dot{x}_i]$ and subsequently we exchange $s_1 \leftrightarrow s_3$, we find that

$$(a)+(b) = (c)+(d). \tag{A.68}$$

We are left with the following global integral to compute

$$\begin{aligned}(a)+(b)+(c)+(d) =& -2\left(\frac{2\pi}{k}\right)^2 MN^2\zeta(\bar{n}_1\cdot n_3)\int_0^L ds_1\int_0^L ds_2\int_{-L}^0 ds_3\\ &\left[(\bar{\nu}_1\nu_3)\dot{x}_1^\nu(\Gamma_{\nu\lambda\lambda}-\Gamma_{\lambda\lambda\nu}) + \frac{2}{(\eta_1\bar{\eta}_3)}[(\dot{x}_3\cdot\dot{x}_1)\dot{x}_1^\nu - \dot{x}_3^\nu]\Gamma_{\lambda\nu\lambda}\right]\end{aligned} \tag{A.69}$$

We shall now reorganize the integrand in a more convenient way

$$\begin{aligned}&\left[(\bar{\nu}_1\nu_3)\dot{x}_1^\nu\left(\Gamma_{\nu\lambda\lambda}-\Gamma_{\lambda\lambda\nu}+\frac{2(1+(\dot{x}_1\cdot\dot{x}_3))}{(\bar{\nu}_1\nu_3)(\eta_1\bar{\eta}_3)}\Gamma_{\lambda\nu\lambda}\right) - \frac{2}{(\eta_1\bar{\eta}_3)}(\dot{x}_1^\nu+\dot{x}_3^\nu)\Gamma_{\lambda\nu\lambda}\right] =\\ =&\left[(\bar{\nu}_1\nu_3)\dot{x}_1^\nu(\Gamma_{\nu\lambda\lambda}-\underset{(R)}{\Gamma_{\lambda\lambda\nu}}+\Gamma_{\lambda\nu\lambda}) - \frac{2}{(\eta_1\bar{\eta}_3)}(\dot{x}_1^\nu+\underset{(S)}{\dot{x}_3^\nu})\Gamma_{\lambda\nu\lambda}\right]\end{aligned} \tag{A.70}$$

where we used that four our contour

$$\frac{2(1+(\dot{x}_1\cdot\dot{x}_3))}{(\bar{\nu}_1\nu_3)(\eta_1\bar{\eta}_3)} = \frac{2(1+(\dot{x}_1\cdot\dot{x}_3))}{(\bar{\nu}_1\nu_3)(\nu_1\bar{\nu}_3)} = -\frac{2(1+(\dot{x}_1\cdot\dot{x}_3))}{(\eta_3\bar{\eta}_1)(\eta_1\bar{\eta}_3)} = 1. \tag{A.71}$$

Exploiting the representation of the contractions of the 3-point function as a derivate of a scalar function $\Phi_{i,jk}$ discussed in appendix A, the combination $\dot{x}_1^\nu(\Gamma_{\nu\lambda\lambda} - \Gamma_{\lambda\lambda\nu} + \Gamma_{\lambda\nu\lambda})$ can be rewritten as a sum of total derivatives

$$
\begin{aligned}
(R) =&\dot{x}_1^\nu(\Gamma_{\nu\lambda\lambda} - \Gamma_{\lambda\lambda\nu} + \Gamma_{\lambda\nu\lambda}) = \dot{x}_1^\nu(\partial_{x_1^\nu}\Phi_{1,23} - \partial_{x_3^\nu}\Phi_{3,12} + \partial_{x_2^\nu}\Phi_{2,13}) = \\
=&\dot{x}_1^\nu[\partial_{x_1^\nu}(\Phi_{1,23} + \Phi_{3,12}) + \partial_{x_2^\nu}(\Phi_{2,13} + \Phi_{3,12})] = \\
=&\frac{d}{ds_1}(\Phi_{1,23} + \Phi_{3,12}) + \frac{d}{ds_2}(\Phi_{2,13} + \Phi_{3,12}) = \\
=&\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2 \left[\frac{d}{ds_1}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}(x_{23}^2)^{1/2-\epsilon}}\right) + 1 \leftrightarrow 2\right] = \\
=&\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2 \left[\frac{d}{ds_1}\left(\frac{1}{((s_1-s_2)^2)^{1/2-\epsilon}} H(s_2,s_3)\right) + 1 \leftrightarrow 2\right].
\end{aligned}
\tag{A.72}
$$

Since the region of integration is symmetric in the exchange $s_1 \leftrightarrow s_2$, the two terms in (A.72) gives the same result once integrated

$$
\begin{aligned}
&-2\left(\frac{2\pi}{k}\right)^2 MN^2\zeta(\bar{n}_1\cdot n_3)(\bar{\nu}_1\nu_3)\int_0^L ds_1\int_0^L ds_2\int_{-L}^0 ds_3 (R) = \\
&= -4\left(\frac{2\pi}{k}\right)^2 MN^2\zeta(\bar{n}_1\cdot n_3)(\bar{\nu}_1\nu_3)\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2\int_0^L ds_1\int_0^L ds_2\int_{-L}^0 ds_3 \\
&\qquad \frac{d}{ds_1}\left(\frac{1}{((s_1-s_2)^2)^{1/2-\epsilon}}H(s_2,s_3)\right) = \\
&= -4\left(\frac{2\pi}{k}\right)^2 MN^2\zeta(\bar{n}_1\cdot n_3)(\bar{\nu}_1\nu_3)\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2\int_0^L ds_2\int_{-L}^0 ds_3 \\
&\qquad \left(\frac{1}{(L-s_2)^{1-2\epsilon}} - \frac{1}{s_2^{1-2\epsilon}}\right)H(s_2,s_3) = \\
&= -4\left(\frac{2\pi}{k}\right)^2 MN^2\zeta(\bar{n}_1\cdot n_3)(\bar{\nu}_1\nu_3)\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2\int_0^L ds_1\int_{-L}^0 ds_2 \\
&\qquad \left(\frac{1}{(L+s_2)^{1-2\epsilon}} - \frac{1}{(-s_2)^{1-2\epsilon}}\right)H(s_2,s_1) = \\
&= -\frac{i}{2\epsilon}\left(\frac{2\pi}{k}\right)^2 MN^2\zeta\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\frac{\Gamma^2(1/2-\epsilon)}{4\pi^{3-2\epsilon}}\int_0^L ds_1\int_{-L}^0 ds_2 \\
&\qquad \frac{d}{ds_2}\left((L+s_2)^{2\epsilon} + (-s_2)^{2\epsilon}\right)H(s_2,s_1)
\end{aligned}
\tag{A.73}
$$

Now we add the similar contribution coming from the trace S_2 in (A.47) and summarize the whole result as follows

$$
-\frac{i\zeta}{2\epsilon}\left(\frac{2\pi}{k}\right)^2\cos\frac{\theta}{2}\cos\frac{\varphi}{2}\frac{MN(M+N)\Gamma^2(\frac{1}{2}-\epsilon)}{4\pi^{3-2\epsilon}}\int_0^L ds_1\int_{-L}^0 ds_2 H(s_2,s_1)\frac{d}{ds_2}\left[(L+s_2)^{2\epsilon} + (-s_2)^{2\epsilon}\right]
\tag{A.74}
$$

We analyze the contribution (S) which contains a contribution which is not, at least apparently, a sum of total derivative. We can rearrange this term as follows

$$
\begin{aligned}
\dot{x}_3^\rho \Gamma_{\lambda\rho\lambda} =& -\frac{1}{2}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2 \dot{x}_3\cdot\partial_{x_2} \\
&\left[\frac{1}{(x_{12}^2)^{1/2-\epsilon}(x_{23}^2)^{1/2-\epsilon}} - \frac{1}{(x_{13}^2)^{1/2-\epsilon}(x_{12}^2)^{1/2-\epsilon}} - \frac{1}{(x_{13}^2)^{1/2-\epsilon}(x_{23}^2)^{1/2-\epsilon}}\right] = \\
=& -\frac{1}{2}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2 \times \\
&\times\left[-\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\frac{d}{ds_3}\left(\frac{1}{(x_{23}^2)^{1/2-\epsilon}}\right) - (\dot{x}_3\cdot\dot{x}_1)\frac{d}{ds_1}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right)\frac{1}{(x_{23}^2)^{1/2-\epsilon}} - \right.\\
&\left. -(\dot{x}_3\cdot\dot{x}_1)\frac{1}{(x_{13}^2)^{1/2-\epsilon}}\frac{d}{ds_2}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right) + \frac{1}{(x_{13}^2)^{1/2-\epsilon}}\frac{d}{ds_3}\left(\frac{1}{(x_{23}^2)^{1/2-\epsilon}}\right)\right]
\end{aligned} \tag{A.75}
$$

where we have used the following identity

$$
\begin{aligned}
\dot{x}_3\cdot\partial_{x_2}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right) =&(1-2\epsilon)\frac{\dot{x}_3\cdot(x_1-x_2)}{(x_{12}^2)^{3/2-\epsilon}} = (1-2\epsilon)(\dot{x}_3\cdot\dot{x}_1)\frac{(s_1-s_2)}{(x_{12}^2)^{3/2-\epsilon}} = \\
=&(\dot{x}_3\cdot\dot{x}_1)\frac{d}{ds_2}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right) = -(\dot{x}_3\cdot\dot{x}_1)\frac{d}{ds_1}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right),
\end{aligned} \tag{A.76}
$$

which is a consequence of the fact that x_1 and x_2 lays ion the same edge in (A.69). Next we consider the combination $\dot{x}_1^\lambda\Gamma_{\rho\lambda\rho}$. Since $\dot{x}_1 = \dot{x}_2$ in (A.69), this combination can be rewritten as total derivative of $\Phi_{2,13}$ with respect to s_2. After a small manipulation, it takes the form

$$
\begin{aligned}
\dot{x}_1^\rho\Gamma_{\lambda\rho\lambda} =& -\frac{1}{2}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2 \times \\
&\times\left[\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\frac{d}{ds_2}\left(\frac{1}{(x_{23}^2)^{1/2-\epsilon}}\right) + \frac{d}{ds_2}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right)\frac{1}{(x_{23}^2)^{1/2-\epsilon}} - \right.\\
&\left. -\frac{1}{(x_{13}^2)^{1/2-\epsilon}}\frac{d}{ds_2}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right) - \frac{1}{(x_{13}^2)^{1/2-\epsilon}}\frac{d}{ds_2}\left(\frac{1}{(x_{23}^2)^{1/2-\epsilon}}\right)\right] = \\
=& -\frac{1}{2}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2 \times \\
&\times\left[\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\frac{d}{ds_2}\left(\frac{1}{(x_{23}^2)^{1/2-\epsilon}}\right) - \frac{d}{ds_1}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right)\frac{1}{(x_{23}^2)^{1/2-\epsilon}} - \right.\\
&\left. -\frac{1}{(x_{13}^2)^{1/2-\epsilon}}\frac{d}{ds_2}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right) - \frac{1}{(x_{13}^2)^{1/2-\epsilon}}\frac{d}{ds_2}\left(\frac{1}{(x_{23}^2)^{1/2-\epsilon}}\right)\right].
\end{aligned} \tag{A.77}
$$

Combining the two contributions we get

$$\begin{aligned}(\dot{x}_1^\rho + \dot{x}_3^\rho)\Gamma_{\lambda\rho\lambda} = &-\frac{1}{2}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2 \times \\ &\times\left[\frac{G(s_2,s_3)}{(x_{12}^2)^{1/2-\epsilon}} - [1+(\dot{x}_3\cdot\dot{x}_1)]\left[\frac{d}{ds_1}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right)\frac{1}{(x_{23}^2)^{1/2-\epsilon}} + \right.\right.\\ &\left.\left. + \frac{1}{(x_{13}^2)^{1/2-\epsilon}}\frac{d}{ds_2}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right)\right] + \frac{G(s_3,s_2)}{(x_{13}^2)^{1/2-\epsilon}}\right]. \end{aligned} \tag{A.78}$$

We first examine the integral of the term proportional to $[1+(\dot{x}_3\cdot\dot{x}_1)]$

$$-\frac{2[1+(\dot{x}_3\cdot\dot{x}_1)]}{(\eta_1\bar{\eta}_3)}\frac{1}{2}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2\left[\frac{d}{ds_1}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right)\frac{1}{(x_{23}^2)^{1/2-\epsilon}} + 1 \leftrightarrow 2\right]$$

The overall coefficient dependent on the circuit is

$$-2\frac{[1+(\dot{x}_3\cdot\dot{x}_1)]}{\eta_1\bar{\eta}_3} = \frac{\eta_1\bar{\eta}_3\eta_3\bar{\eta}_1}{\eta_1\bar{\eta}_3} = \eta_3\bar{\eta}_1 = -\bar{\nu}_1\nu_3 \tag{A.79}$$

Then the integrand contains two terms which yields the same result, once integrated, because of the symmetry $1 \leftrightarrow 2$. We are left with the following quantity to compute

$$\begin{aligned} &-(\bar{\nu}_1\nu_3)\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2\int_0^L ds_1\int_0^L ds_2\int_{-L}^0 ds_3\frac{d}{ds_1}\left(\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right)\frac{1}{(x_{23}^2)^{1/2-\epsilon}} = \\ =&-(\bar{\nu}_1\nu_3)\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2\int_0^L ds_1\int_0^L ds_2\int_{-L}^0 ds_3\frac{d}{ds_1}\left[\frac{1}{((s_1-s_2)^2)^{1/2-\epsilon}}H(s_2,s_3)\right], \end{aligned} \tag{A.80}$$

which is the same integral which appear in the (R) contribution (A.72).

The remaining quantity to compute is

$$-\frac{1}{\eta_1\bar{\eta}_3}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2\int_0^L ds_1\int_0^L ds_2\int_{-L}^0 ds_3\left[\frac{1}{(x_{12}^2)^{1/2-\epsilon}} - \frac{1}{(x_{13}^2)^{1/2-\epsilon}}\right]G(s_2,s_3) \tag{A.81}$$

The three integral above are given by:

(I)

$$\begin{aligned} &\int_0^L ds_1\int_0^L ds_2\int_{-L}^0 ds_3 H(s_1,s_3)\frac{d}{ds_3}H(s_2,s_3) = \frac{1}{2}\int_{-L}^0 ds_3\frac{d}{ds_3}\left(\int_0^L dsH(s,s_3)\right)^2 = \\ &= \frac{1}{2}\left[\left(\int_0^L dsH(s,0)\right)^2 - \left(\int_0^L dsH(s,-L)\right)^2\right] = \frac{1}{2}\left[\frac{L^{4\epsilon}}{4\epsilon^2} - \left(\int_0^L dsH(s,-L)\right)^2\right]. \end{aligned} \tag{A.82}$$

(II)

$$\begin{aligned} &-\int_0^L ds_1\int_0^L ds_2\int_{-L}^0 ds_3 H(s_1,s_3)\frac{d}{ds_2}H(s_2,s_3) = \\ &= -\int_0^L ds_1\int_{-L}^0 ds_3 H(s_1,s_3)[H(L,s_3) - H(0,s_3)] \end{aligned} \tag{A.83}$$

(III)

$$
\begin{aligned}
&\int_0^L ds_1 \int_0^L ds_2 \int_{-L}^0 ds_3 \frac{1}{((s_1-s_2)^2)^{1/2-\epsilon}} G(s_2,s_3) = \\
&= \frac{1}{2\epsilon} \int_0^L ds_2 \int_{-L}^0 ds_3 [(L-s_2)^{2\epsilon} + {s_2}^{2\epsilon}] G(s_2,s_3)
\end{aligned}
\tag{A.84}
$$

.1

In the following we shall prove some useful identities for manipulating an integrand which contain the three point function. To begin with, we shall notice that the scalar Green function can be cast as follows

$$G(x_1,x_2)=\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right)=\int\frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}\frac{e^{ip(x_1-x_2)}}{p^2}. \tag{A.1}$$

Then we can easily evaluate the d'Alembertian of $G(x_1,x_2)$,

$$\Box_{x_1}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\frac{1}{(x_{12}^2)^{1/2-\epsilon}}\right)=-\int\frac{d^{3-2\epsilon}p}{(2\pi)^{3-2\epsilon}}e^{ip(x_1-x_2)}=-\delta^{3-2\epsilon}(x_1-x_2). \tag{A.2}$$

Next consider the 3−point function $\Gamma_{\lambda\mu\nu}(x_1,x_2,x_2)$. It can be written as a multiple gradient of a scalar function, namely $\Gamma_{\lambda\mu\nu}(x_1,x_2,x_2)=\partial_{x_1^\lambda}\partial_{x_2^\mu}\partial_{x_3^\nu}\Phi$ where

$$\Phi=\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^3\int\frac{d^{3-2\epsilon}w}{(x_{1w}^2)^{1/2-\epsilon}(x_{2w}^2)^{1/2-\epsilon}(x_{3w}^2)^{1/2-\epsilon}}. \tag{A.3}$$

Moreover Φ is a translational invariant function, *i.e.* $(\partial_{x_1^\lambda}+\partial_{x_2^\lambda}+\partial_{x_3^\lambda})\Phi=0$. With the help of this representation we can determine the different contractions of the vertex function Γ, which appear in (A.65), in terms of the derivatives of scalar propagators

$$\begin{aligned}\Gamma_{\lambda\lambda\rho}=&\partial_{x_3^\rho}(\partial_{x^1}\cdot\partial_{x_2})\Phi=\frac{1}{2}\partial_{x_3^\rho}[(\partial_{x_1}+\partial_{x_2})\cdot(\partial_{x_1}+\partial_{x_2})-\partial_{x_1}\cdot\partial_{x_1}-\partial_{x_2}\cdot\partial_{x_2}]\Phi=\\=&\frac{1}{2}\partial_{x_3^\rho}[\Box_{x_3}-\Box_{x_1}-\Box_{x_2}]\Phi=-\frac{1}{2}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2\partial_{x_3^\rho}\times\\&\times\left[\frac{1}{(x_{13}^2)^{1/2-\epsilon}(x_{23}^2)^{1/2-\epsilon}}-\frac{1}{(x_{13}^2)^{1/2-\epsilon}(x_{12}^2)^{1/2-\epsilon}}-\frac{1}{(x_{12}^2)^{1/2-\epsilon}(x_{23}^2)^{1/2-\epsilon}}\right]=\\=&\partial_{x_3^\rho}\Phi_{3,12},\end{aligned} \tag{A.4}$$

$$\begin{aligned}\Gamma_{\lambda\rho\lambda}=&\frac{1}{2}\partial_{x_2^\rho}[\Box_{x_2}-\Box_{x_1}-\Box_{x_3}]\Phi=-\frac{1}{2}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2\partial_{x_2^\rho}\times\\&\times\left[\frac{1}{(x_{12}^2)^{1/2-\epsilon}(x_{23}^2)^{1/2-\epsilon}}-\frac{1}{(x_{13}^2)^{1/2-\epsilon}(x_{12}^2)^{1/2-\epsilon}}-\frac{1}{(x_{13}^2)^{1/2-\epsilon}(x_{23}^2)^{1/2-\epsilon}}\right]=\\=&\partial_{x_2^\rho}\Phi_{2,13},\end{aligned} \tag{A.5}$$

$$\begin{aligned}\Gamma_{\rho\lambda\lambda}=&\frac{1}{2}\partial_{x_1^\rho}[\Box_{x_1}-\Box_{x_2}-\Box_{x_3}]\Phi=-\frac{1}{2}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2\partial_{x_1^\rho}\times\\&\times\left[\frac{1}{(x_{12}^2)^{1/2-\epsilon}(x_{13}^2)^{1/2-\epsilon}}-\frac{1}{(x_{23}^2)^{1/2-\epsilon}(x_{12}^2)^{1/2-\epsilon}}-\frac{1}{(x_{13}^2)^{1/2-\epsilon}(x_{23}^2)^{1/2-\epsilon}}\right]=\\=&\partial_{x_1^\rho}\Phi_{1,23},\end{aligned} \tag{A.6}$$

where we found convenient to introduce a short-hand notation

$$\begin{aligned}\Phi_{i,jk} =& -\frac{1}{2}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2\left[\frac{1}{(x_{ij}^2)^{1/2-\epsilon}(x_{ik}^2)^{1/2-\epsilon}}-\frac{1}{(x_{ij}^2)^{1/2-\epsilon}(x_{kj}^2)^{1/2-\epsilon}}-\right.\\ &\left.-\frac{1}{(x_{ik}^2)^{1/2-\epsilon}(x_{jk}^2)^{1/2-\epsilon}}\right].\end{aligned} \tag{A.7}$$

It will be also useful to consider the coincidence limit of $\Phi_{i,jk}$. For $\epsilon > 1/2$ they are finite and they are given by

$$\begin{aligned}\Phi_{i,ik} =& \frac{1}{2}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2\frac{1}{(x_{ik}^2)^{1-2\epsilon}},\\ \Phi_{i,jj} =& -\frac{1}{2}\left(\frac{\Gamma(1/2-\epsilon)}{4\pi^{3/2-\epsilon}}\right)^2\frac{1}{(x_{ij}^2)^{1-2\epsilon}}.\end{aligned} \tag{A.8}$$

APPENDIX A

$M2-$branes/Type IIA/$\mathcal{N} = 6$ SCS for Dummies

A.1 Effective action for M2-branes in $AdS_4 \times \mathbb{C}^4/\mathbb{Z}_k$

Recently, a super Chern-Simons theory with $\mathcal{N} = 8$ supersymmetry [186] [43] and a three-algebra structure [187] was related to the world-volume theory of $M2-$branes when the CS level k equated $1, 2$. [188] [189]. On the gauge theory side there is no viable 't Hooft limit and this theory, known as BL(G)[1], turns out to be always strongly coupled. This motivated the work of [36], that provided a class of brane constructions which in the IR are thought to flow to superconformal $\mathcal{N} = 6$ SCS theory at level k. Moreover for $k = 1, 2$ the supersymmetry of these theories get enhanced and the $\mathcal{N} = 8$ BLG is recovered.

Type IIB $D3/NS5-$brane constructions that produce three-dimensional theories with $U(N)$ CS terms were already known since [190] [191], but they have at most $\mathcal{N} = 3$ supersymmetry. The construction of [36] generalises the latter, introducing a mass deformation, to Yang-Mills-Chern-Simons quiver theories with gauge group $U(N)_k \times U(N)_{-k}$, where k is the CS level. Uplifting to $M-$theory through a $T-$duality, one ends up with N $M2-$branes extending in AdS_4 and probing the $\mathbb{C}^4/\mathbb{Z}_k$ orbifold singularity at the intersection of two Kaluza-Klein monopoles, where the supersymmetry gets enhanced to $\mathcal{N} = 6$. A careful analysis of the moduli space of this this theory hints to a low-energy RG flow to the superconformal $\mathcal{N} = 6$ CS-matter theory (A.2) with gauge group $U(N)_k \times U(N)_{-k}$. This is actually a class of theories labelled by two integers N and k. They admit a large N (planar) limit in which they are strongly coupled for $\lambda = \frac{N}{k} \gg 1$ and weakly coupled for $\lambda \ll 1$. Moreover, for $k \to \infty$, the $\mathbb{Z}_k$ orbifold shrinks a cicle in $\mathbb{C}^4 \sim S^1 \times \mathbb{C}P^3$, and one is left with type IIA superstrings on $AdS_4 \times \mathbb{C}P^3$. The string coupling constant and tension are given

[1]Some authors prefer the caption BL, stating that the three-algebra formulation of the $\mathcal{N} = 8$ SCS Lagrangean does not play a key role in the theory. However, the acronym BLG is more frequent in the litterature.

by

$$g_s \sim \left(\frac{N}{k^5}\right)^{1/4} = \frac{\lambda^{5/4}}{N}\,, \qquad \frac{R^2}{\alpha'} = 4\pi\sqrt{2\lambda} \tag{A.1}$$

where R is the radius of $\mathbb{C}P^3$, which equals twice as much the radius of AdS_4, so that these backgrounds do not really look four-dimensional. The $M-$theory/ABJM duality holds for any value of the coupling; the string theory picture becomes effective only in the $k^5 \gg N$ regime where $M2$ branes wrapping S^1 become well approximated by weakly coupled strings. In the opposite case $N \gg k^5$ one has strongly coupled string theory, which is indeed $M-$theory. So, the $\mathcal{N}=6$ SCS theory in the weakly coupled regime and in the 't Hooft limit is also conjectured to be dual to strongly coupled type IIA strings on $AdS_4 \times \mathbb{C}P^3$. This constitutes a further example of AdS/CFT duality.

The relation above can be generalised to SCS theory with gauge group $U(N)_k \times U(M)_{-k}$ and $\mathcal{N}=6$ supersymmetry [37] and the theory of N $M2-$branes plus $N-M$ fractional $M2$'s and a nonvanishing $B-$field flux. Also, orbifold projection of the $(U(N)\times U(N))^n$ kind have been studied and conjectured to be dual to certain $AdS_4 \times \mathbb{C}^4/(\mathbb{Z}_n \times \mathbb{Z}_k)$ $M-$theory backgrounds [115].

A.2 Chern-Simons-matter theories in three dimensions

A.2.1 Lagrangean of $\mathcal{N}=6$ ABJ(M)

The Lagrangean of $\mathcal{N}=6$ superconformal Chern-Simons-matter theory in 3 dimensions [37] describes two CS gauge fields $(A_\mu)_{ij}$ and $(\widehat{A}_\mu)_{\hat{i}\hat{j}}$ in the adjoint of $U(N)$ and $U(M)$ gauge groups respectively, coupled to bifundamental scalars $(C_I)_{i\hat{i}}$, $(\overline{C}^I)_{\hat{i}i}$ and fermions $(\psi_\alpha^I)_{\hat{i}i}$, $(\overline{\psi}_I^\alpha)_{i\hat{i}}$ in a supersymmetric fashion. In euclidean spacetime it reads:

$$S_{\rm ABJ} = S_{\rm CS}(k) + S_{\rm CS}(-k) + S_{\rm gfix} + S_{\rm matter} \tag{B.2}$$

where

$$S_{\rm CS}(k) = -\mathrm{i}\frac{k}{4\pi}\int \mathrm{d}^3x\ \epsilon^{\mu\nu\rho}\mathrm{Tr}(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho)$$

$$S_{\rm gfix} = \frac{k}{4\pi}\int \mathrm{d}^3x \left[\frac{1}{\alpha}\mathrm{Tr}(\partial_\mu A^\mu)^2 + \mathrm{Tr}(\partial_\mu \bar{c} D^\mu c) - \frac{1}{\alpha}\mathrm{Tr}(\partial_\mu \widehat{A}^\mu)^2 - \mathrm{Tr}(\partial_\mu \bar{\hat{c}} D^\mu \hat{c})\right]$$

$$S_{\rm matter} = \int \mathrm{d}^3x \left[\mathrm{Tr}(D_\mu C_I D^\mu \overline{C}^I) + \mathrm{i}\mathrm{Tr}(\overline{\psi}_I \not{D}\psi^I)\right] + S_{\rm int}$$

The covariant gauge fixing action involves two sets of ghosts $(c,\bar{c})$ and $(\bar{\hat{c}},\hat{c})$, they have a crucial role in the cancellation of gauge field loops [184]. The interacting Lagrangean in $S_{\rm int}$ is a sextic-order scalar potential plus scalar-fermion Yukawa couplings. It is better written in terms of $\mathcal{N}=2$ superspace variables while it assumes a quite cumbersome appearence in components (see [36] or [115] for the more general case). On the other hand these couplings do not enter before four loops in the weak-coupling expansion of the Wilson loop, so we skip them on the easy street.

Some remarks are worth once and for all. Lower case latin indices belong to a vector $\boldsymbol{N}$ representation of the gauge group $U(N)$ or $SU(N)$, analogously lower case hatted latin indices belong to a $\boldsymbol{M}$ of $U(M)$ or $SU(M)$. Throughout the paper we avoid using indices in the adjoint and keep the double index notation for clarity. Thus $(A_\mu)_{ij}$ and $(\widehat{A}_\mu)_{\hat{i}\hat{j}}$ transform in the adjoint of their respective gauge groups, $(C_I)_{i\hat{i}}$, $(\overline{\psi}_I^\alpha)_{i\hat{i}}$ transform in the bifundamental $(\boldsymbol{N},\boldsymbol{M})$ and lastly $(\overline{C}^I)_{\hat{i}i}$, $(\psi_\alpha^I)_{\hat{i}i}$ are in $(\boldsymbol{M},\boldsymbol{N})$. Note that the CS levels may differ in the most general case [115], though the gauge invariance of the CS action constraints them to be two integers. The symmetry group of this general class of theories is however smaller. Setting $k_N=-k_M$ the global symmetry has an enhancement to the $SU(4)_R$ symmetry of (B.2) and one recovers the $\mathcal{N}=6$ ABJ action. Further setting $M=N$ the ABJM action is obtained [36]. In these settings latin (hatted) indices $i, j, \ldots$ $(\hat{i}, \hat{j}, \ldots)$ belong to the fundamental $\boldsymbol{N}$ (anti-fundamental $\overline{\boldsymbol{N}}$) of $U(N)$ or $SU(N)$, upper case latin letters are $SU(4)$ $R-$symmetry indices spanning $I=1,2,3,4$ and greek lower case letters are Weyl spinor indices $\alpha=+,-$ and are chosen such that ψ_α^I are columns and $\overline{\psi}_I^\alpha$ are rows. The 3-dimensional Clifford algebra

$$\begin{aligned}[\sigma^\mu,\,\sigma^\nu]&=2\mathrm{i}\epsilon^{\mu\nu\rho}\sigma^\rho\\ \{\sigma^\mu,\,\sigma^\nu\}&=2\delta^{\mu\nu}\end{aligned} \tag{B.3}$$

is represented by ordinary euclidean Pauli matrices

$$(\sigma^\mu)_\alpha^{\ \beta}=\{\sigma^1,\,\sigma^2,\,\sigma^3\}_\alpha^{\ \beta} \tag{B.4}$$

Spinor indices are rised and lowerd with the antisymmetric tensor

$$\psi^\alpha=\epsilon^{\alpha\beta}\psi_\beta,\quad \epsilon^{+-}=-\epsilon_{-+}=1 \tag{B.5}$$

so lowering the upper index, σ matrices become symmetric

$$(\sigma^\mu)_{\alpha\beta}=\{-\sigma^3,\,1,\,\sigma^1\}_{\alpha\beta} \tag{B.6}$$

Spinor indices are always contracted from the upper right to lower left corner

$$\overline{\chi}\sigma^\mu\psi=\overline{\chi}^\alpha(\sigma^\mu)_\alpha^{\ \beta}\psi_\beta=-\psi\sigma^\mu\overline{\chi} \tag{B.7}$$

for any two spinors χ and ψ. This rule is assumed to hold when spinor indices are not written explicitly.

The two gauge fields A and $\widehat{A}$ act from opposite sides on scalars and fermions, as an example the covariant derivatives read

$$
\begin{aligned}
D_\mu\psi_{\hat{i}i} &= \partial_\mu\psi_{\hat{i}i} + \mathrm{i}\left((\widehat{A}_\mu)_{\hat{i}\hat{j}}\psi_{\hat{j}i} - \psi_{\hat{i}j}(A_\mu)_{ji}\right) \\
D_\mu\overline{\psi}_{i\hat{i}} &= \partial_\mu\overline{\psi}_{i\hat{i}} - \mathrm{i}\left(\overline{\psi}_{i\hat{j}}(\widehat{A}_\mu)_{\hat{j}\hat{i}} - (A_\mu)_{ij}\overline{\psi}_{j\hat{i}}\right)
\end{aligned}
\tag{B.8}
$$

and the same for scalars.

Finally note that the sign of the interaction 3-form in the CS Lagrangean is related to the sign of the exponent in the Wilson loop operator through gauge symmetry. It is well known that the CS action

$$
S_{\mathrm{CS}}(k) = -\mathrm{i}\frac{k}{4\pi}\int \mathrm{d}^3x\ \epsilon^{\mu\nu\rho}\mathrm{Tr}(A_\mu\partial_\nu A_\rho + \frac{2}{3}\sigma A_\mu A_\nu A_\rho)
$$

with $\sigma = \pm 1$ receives, under a gauge transformation $g(x) \in SU(N)$ acting as

$$
A_\mu \to g(x)(A_\mu + \mathrm{i}\sigma\,\partial_\mu)g^{-1}(x)
$$

a shift $S \to S + 2k\pi\,\delta S$, being

$$
\delta S = -\frac{1}{24\pi^2}\int \mathrm{d}^3x\epsilon^{\mu\nu\rho}\mathrm{Tr}\left[(\partial_\mu g^{-1})g(\partial_\nu g^{-1})g(\partial_\rho g^{-1})g\right] \in \mathbb{N}
$$

Thus the path integral is left invariant if k itself is an integer. Correspondingly the Wilson loop operator is invariant under the same $g \in SU(N)$ if the sign of the exponent is σ

$$
\mathcal{W} = \frac{1}{N}\mathrm{Tr}\mathcal{P}\exp\left(\mathrm{i}\sigma\oint A_\mu\,\mathrm{d}x^\mu\right)
\tag{B.9}
$$

A.2.2 Feynman rules

From the Lagrangean in (B.2) we can read the momentum space propagators in Landau gauge ($\alpha \to \infty$)

$$
\begin{aligned}
\left\langle (A_\mu)_{ij}(p)\,(A_\nu)_{kl}(-p)\right\rangle &= \frac{2\pi}{k}\delta_{il}\delta_{jk}\,\epsilon_{\mu\nu\rho}\frac{p^\rho}{p^2} \\
\left\langle (\widehat{A}_\mu)_{\hat{i}\hat{j}}(p)\,(\widehat{A}_\nu)_{\hat{k}\hat{l}}(-p)\right\rangle &= -\frac{2\pi}{k}\delta_{\hat{i}\hat{l}}\delta_{\hat{j}\hat{k}}\,\epsilon_{\mu\nu\rho}\frac{p^\rho}{p^2} \\
\left\langle (\psi^I_\alpha)_{\hat{i}i}(p)\,(\overline{\psi}^\beta_J)_{j\hat{j}}(-p)\right\rangle &= -\delta^J_I\delta^\beta_\alpha\delta_{ij}\delta_{\hat{i}\hat{j}}\,\frac{\not{p}}{p^2} \\
\left\langle (C_I)_{i\hat{i}}(p)\,(\overline{C}^J)_{\hat{j}j}(-p)\right\rangle &= \delta^J_I\delta_{ij}\delta_{\hat{i}\hat{j}}\,\frac{1}{p^2}
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
\left\langle c_{ij}(p)\,\overline{c}_{kl}(-p)\right\rangle &= \frac{2\pi}{k}\delta_{il}\delta_{jk}\,\frac{1}{p^2} \\
\left\langle \widehat{c}_{\hat{i}\hat{j}}(p)\,\overline{\widehat{c}}_{\hat{k}\hat{l}}(-p)\right\rangle &= -\frac{2\pi}{k}\delta_{\hat{i}\hat{l}}\delta_{\hat{j}\hat{k}}\,\frac{1}{p^2}
\end{aligned}
\tag{B.11}
$$

Using the $d-$dimensional Fourier transform

$$\int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{e}^{-\mathrm{i}p\cdot x}}{(p^2)^k} = \mathrm{i}\frac{\Gamma(\frac{d}{2}-k)}{\Gamma(k)} \frac{1}{2^{2k}\pi^{\frac{d}{2}}} \frac{1}{(x^2)^{\frac{d}{2}-k}} \tag{B.12}$$

we obtain the coordinate space propagators, relevant for the evaluation of the coordinate (euclidean) space Wilson loop operator

$$\begin{aligned}
\Big\langle (A_\mu)_{ij}(x)\,(A_\nu)_{kl}(y)\Big\rangle &= \mathrm{i}\delta_{il}\delta_{jk}\frac{\mu^{3-d}}{2k}\frac{\Gamma(\frac{d}{2}-1)}{\pi^{\frac{d}{2}-1}}\,\epsilon_{\mu\nu\rho}\partial^\rho \frac{1}{[(x-y)^2]^{\frac{d}{2}-1}} \\
&= -\mathrm{i}\delta_{il}\delta_{jk}\frac{\mu^{3-d}}{k}\frac{\Gamma(\frac{d}{2})}{\pi^{\frac{d}{2}-1}}\,\epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{[(x-y)^2]^{\frac{d}{2}}} \\
\Big\langle (\widehat{A}_\mu)_{ij}(x)\,(\widehat{A}_\nu)_{kl}(y)\Big\rangle &= -\mathrm{i}\delta_{il}\delta_{jk}\frac{\mu^{3-d}}{2k}\frac{\Gamma(\frac{d}{2}-1)}{\pi^{\frac{d}{2}-1}}\,\epsilon_{\mu\nu\rho}\partial^\rho \frac{1}{[(x-y)^2]^{\frac{d}{2}-1}} \\
&= \mathrm{i}\delta_{il}\delta_{jk}\frac{\mu^{3-d}}{k}\frac{\Gamma(\frac{d}{2})}{\pi^{\frac{d}{2}-1}}\,\epsilon_{\mu\nu\rho}\frac{(x-y)^\rho}{[(x-y)^2]^{\frac{d}{2}}} \\
\Big\langle (\psi^I_\alpha)_{\hat{i}i}(x)\,(\overline{\psi}^\beta_J)_{j\hat{j}}(y)\Big\rangle &= \mathrm{i}\delta^J_I\delta^\beta_\alpha\delta_{ij}\delta_{\hat{i}\hat{j}}\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}\partial\!\!\!/ \frac{1}{[(x-y)^2]^{\frac{d}{2}-1}} \\
&= -\mathrm{i}\delta^J_I\delta^\beta_\alpha\delta_{ij}\delta_{\hat{i}\hat{j}}\frac{\Gamma(\frac{d}{2})}{2\pi^{\frac{d}{2}}}\frac{\sigma^\mu(x-y)_\mu}{[(x-y)^2]^{\frac{d}{2}}} \\
\Big\langle (C_I)_{i\hat{i}}(x)\,(\overline{C}^J)_{\hat{j}j}(y)\Big\rangle &= \delta^J_I\delta_{ij}\delta_{\hat{i}\hat{j}}\frac{\Gamma(\frac{d}{2}-1)}{4\pi^{\frac{d}{2}}}\frac{1}{[(x-y)^2]^{\frac{d}{2}-1}}
\end{aligned} \tag{B.13}$$

For the ghost field propagator the following relation holds

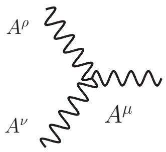
$$\Big\langle (A_\mu)_{ij}(x)\,(A_\nu)_{kl}(y)\Big\rangle = \frac{1}{2}\epsilon_{\mu\nu\rho}\partial^\rho\Big\langle c_{ij}(x)\,\overline{c}_{kl}(y)\Big\rangle \tag{B.14}$$

which is the key point in demonstrating the cancellation, mentioned above, of the gauge field and ghost loops [184]. An analogous relation holds between the $<\widehat{A}\widehat{A}>$ and $<\widehat{c}\widehat{\overline{c}}>$ propagators. Finally, let us consciously omit the mass scale μ throughout the computations and restore it at the end, when needed.

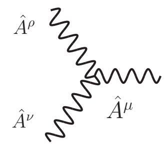
Expecially in the ABJM case there's a subtle interplay of cancellations between 2-loops Feynman diagrams, for this reason we highlight the different signs in front of different interaction 3-vertices:

A.2.3 Trace rules

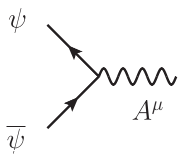
The computations below involve possibly four kinds of indices and thus four kinds of traces that arise from the definition of the expectation value of the Wilson loop operator. Consider as an example the 2-loops double scalar excange of Section A.1.1; it arises from the second order expansion of the loop operator



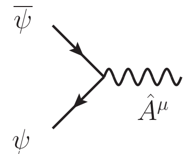
$= V_{AAA} = \frac{2\pi}{3\sqrt{k}}\epsilon^{\mu\nu\rho}$



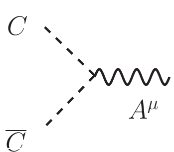
$= V_{\hat{A}\hat{A}\hat{A}} = -\frac{2\pi}{3\sqrt{k}}\epsilon^{\mu\nu\rho}$ (B.15)



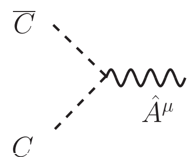
$= V_{\overline{\psi}\psi A} = +\sigma^\mu$



$= V_{\overline{\psi}\hat{A}\psi} = -\sigma^\mu$ (B.16)



$= V_{C\overline{C}A} = -\mathrm{i}\partial^\mu$



$= V_{C\hat{A}\overline{C}} = \mathrm{i}\partial^\mu$ (B.17)

$$\left(\frac{2\pi}{k}\right)^2 \left\langle |\dot{x}_1||\dot{x}_2|[M^I_J C_I \overline{C}^J][\widehat{M}^K_L \overline{C}^L C_K]\right\rangle \tag{B.18}$$

Contracting fields inserted at the same point

$$\mathrm{Tr} M^I_J \underbrace{C_I \overline{C}^J} \widehat{M}^K_L \underbrace{\overline{C}^L C_K} \tag{B.19}$$

gives the double-bouble graph, which vanishes for the same reasons pointed out in Section 6.3.1 and that make the single bouble graph vanishing. On the other hand contracting fields inserted at different points

$$\mathrm{Tr} M^I_J C_I \overline{C}^J \widehat{M}^K_L \overline{C}^L C_K \tag{B.20}$$

returns the composite exchange of Figure 6.3.2 (b). Inserting the propagators between scalar fields as indicated by the Wick contraction above, and flushing all but the tensorial structure, the traces over $U(N) \times U(M)$ and $SU(4)_R$ idices become

$$M^I_J (C_I)_{\hat{i}i} (\overline{C}^J)_{\hat{j}j} \widehat{M}^K_L (\overline{C}^L)_{\hat{l}l} (C_K)_{k\hat{k}} \delta^L_I \delta^J_K \delta_{il} \delta_{\hat{i}\hat{l}} \delta_{jk} \delta_{\hat{j}\hat{k}} = N^2 M \mathrm{Tr}[M\widehat{M}] \tag{B.21}$$

This is in our opinion the best way for keeping track of trace factors.

A.3 Superconformal symmetry in three dimensions

Chern-Simons-matter theories described in Section A.1 are invariant under the superconformal group in three-dimensions $OSp(6|4)$ containing 24 supercharges instead of the 32 of $PSU(2,2|4)$ of $\mathcal{N}=4$ SYM. Therefore they are not maximally supersymmetric. The $\mathfrak{osp}(6|4)$ algebra is generated by the $\mathfrak{sp}(4)$ translations $P^{\alpha\beta}$, $\alpha,\beta=1,2$, rotations J_α, Lorentz transformation L^α_β, special conformal transformation $K_{\alpha\beta}$ and dilatation D, $\mathfrak{so}(6)$ $R-$symmetries R^{AB}, $A,B=1,2,3$, and 24 supercharges $Q^{\alpha A}, Q^\alpha_A, S_{\alpha A}, S^A_\alpha$. We use $\mathfrak{so}(6)$ indices for convenience in writing the commutators. The Lorentz and internal rotations algebra reads

$$\begin{aligned}
&[L^\alpha{}_\beta, J^\gamma] = +\delta^\gamma_\beta J^\alpha - \frac{1}{2}\delta^\alpha_\beta J^\gamma\,, && [L^\alpha{}_\beta, J_\gamma] = -\delta^\alpha_\gamma J_\beta + \frac{1}{2}\delta^\alpha_\beta J_\gamma\,,\\
&[R^A{}_B, J^C] = +\delta^C_B J^A\,, && [R^A{}_B, J_C] = -\delta^A_C J_B\,,\\
&[R_{AB}, J^C] = \delta^C_B J_A - \delta^C_A J_B\,, && [R^{AB}, J_C] = \delta^B_C J^A - \delta^A_C J^B
\end{aligned}\tag{C.22}$$

Commutators of special conformal tranformations and translations give a Lorentz transformation and a dilation

$$[K_{\alpha\beta}, P^{\gamma\delta}] = \delta^\delta_\beta L^\gamma{}_\alpha + \delta^\gamma_\beta L^\delta{}_\alpha + \delta^\delta_\alpha L^\gamma{}_\beta + \delta^\gamma_\alpha L^\delta{}_\beta + 2\delta^\delta_\beta\delta^\gamma_\alpha D + 2\delta^\gamma_\beta\delta^\delta_\alpha D\,,\tag{C.23}$$

Commutators including a supercharge are again a supercharge

$$\begin{aligned}
&[P^{\alpha\beta}, S^A_\gamma] = -\delta^\alpha_\gamma Q^{\beta A} - \delta^\beta_\gamma Q^{\alpha A}\,, && [K_{\alpha\beta}, Q^{\gamma A}] = \delta^\gamma_\beta S^A_\alpha + \delta^\gamma_\alpha S^A_\beta\,,\\
&[P^{\alpha\beta}, S_{\gamma A}] = -\delta^\alpha_\gamma Q^\beta_A - \delta^\beta_\gamma Q^\alpha_A\,, && [K_{\alpha\beta}, Q^\gamma_A] = \delta^\gamma_\alpha S_{\beta A} + \delta^\gamma_\beta S_{\alpha A}
\end{aligned}\tag{C.24}$$

and two supercharges commute into translations and rotations

$$\{Q^{\alpha A}, Q^\beta_B\} = \delta^A_B P^{\alpha\beta}\,, \qquad \{S_{\alpha A}, S^B_\beta\} = \delta^B_A K_{\alpha\beta}\,,\tag{C.25}$$

$$\begin{aligned}
&\{Q^{\alpha A}, S_{\beta B}\} = \delta^A_B L^\alpha{}_\beta - \delta^\alpha_\beta R^A{}_B + \delta^A_B\delta^\alpha_\beta D\,, && \{Q^{\alpha A}, S^B_\beta\} = -\delta^\alpha_\beta R^{AB}\,,\\
&\{Q^\alpha_A, S^B_\beta\} = \delta^B_A L^\alpha{}_\beta + \delta^\alpha_\beta R^B{}_A + \delta^A_B\delta^\alpha_\beta D\,, && \{Q^\alpha_A, S_{\beta B}\} = -\delta^\alpha_\beta R_{AB}
\end{aligned}\tag{C.26}$$

The superconformal transformations of SCS theories of the ABJM kind are generated by the twelve Poicare supercharges Q^{IJ}_α, parametrised by two-component spinors θ^{IJ}_α, and the twelve superconformal ones S^{IJ}_α, parametrised by ϑ^{IJ}_α. These parameters are antisymmetric, $\bar\theta^{IJ} = -\bar\theta^{JI}$, and obey the reality condition

$$\bar\theta^{IJ} = (\theta_{IJ})^*\tag{C.27}$$

moreover

$$\theta_{IJ} = \frac{1}{2}\varepsilon_{IJKL}\bar\theta^{KL}\,.\tag{C.28}$$

The supersymmetry trasformations act on the fields of the theory in the following way

$$
\begin{aligned}
\delta A_\mu &= \frac{4\pi i}{k}\bar\theta^{IJ\alpha}(\gamma_\mu)_\alpha^{\ \beta}\left(C_I\psi_{J\beta}+\frac{1}{2}\varepsilon_{IJKL}\bar\psi^K_\beta\bar C^L\right),\\
\delta\widehat{A}_\mu &= \frac{4\pi i}{k}\bar\theta^{IJ\alpha}(\gamma_\mu)_\alpha^{\ \beta}\left(\psi_{J\beta}C_I+\frac{1}{2}\varepsilon_{IJKL}\bar C^L\bar\psi^K_\beta\right),\\
\delta C_K &= \bar\theta^{IJ\alpha}\varepsilon_{IJKL}\bar\psi^L_\alpha\,,\\
\delta\bar C^K &= 2\bar\theta^{KL\alpha}\psi_{L\alpha}\,,\\
\delta\psi^\beta_K &= -i\bar\theta^{IJ\alpha}\varepsilon_{IJKL}(\gamma^\mu)_\alpha^{\ \beta}D_\mu\bar C^L\\
&\qquad+\frac{2\pi i}{k}\bar\theta^{IJ\beta}\varepsilon_{IJKL}(\bar C^LC_P\bar C^P-\bar C^PC_P\bar C^L)+\frac{4\pi i}{k}\bar\theta^{IJ\beta}\varepsilon_{IJML}\bar C^MC_K\bar C^L,\\
\delta\bar\psi^K_\beta &= -2i\bar\theta^{KL\alpha}(\gamma^\mu)_{\alpha\beta}D_\mu C_L-\frac{4\pi i}{k}\bar\theta^{KL}_\beta(C_L\bar C^MC_M-C_M\bar C^MC_L)-\frac{8\pi i}{k}\bar\theta^{IJ}_\beta C_I\bar C^KC_J\,,
\end{aligned}
\tag{C.29}
$$

and analogous expressions for the superconformal transformation up to $\theta\to\vartheta$. Nothe that here $\mathfrak{su}(4)$ $R-$indices have been used instead.

A.4 Toolbox

Momentum integrals in $d-$dimensions

Momentum space integrals are conveniently managed using Feynman paramerization

$$
\frac{1}{A_1^{m_1}A_2^{m_2}\dots A_n^{m_n}}=\int_0^1 \mathrm{d}\alpha_i\delta\left(\sum\alpha_i-1\right)\frac{\prod\alpha_i^{m_i-1}}{[\sum\alpha_iA_i]^{\sum m_i}}\frac{\Gamma(\sum m_i)}{\prod\Gamma(m_i)}
\tag{D.30}
$$

then momentum integrals are performed in dimensional regularization when possible, or eventually are dimensional reduced, meaning that tensors are kept on-dimension while kinematics is taken off-dimension. The general d-dimensional integral rotated reduced to the form

$$
\begin{aligned}
\int\frac{\mathrm{d}^dl}{(2\pi)^d}\frac{l^{2n}}{(l^2-\Delta)^m} &= \mathrm{i}(-1)^m\int\mathrm{d}\Omega_d\int\frac{\mathrm{d}|l_E|}{(2\pi)^d}\frac{|l_E|^{d-1+2n}}{(l_E^2-\Delta)^m}\\
&= \frac{\mathrm{i}(-1)^m}{2^d\pi^{d/2}}\frac{\Gamma(m-n-d/2)\Gamma(n+d/2)}{2\Gamma(m)}\left(\frac{1}{-\Delta}\right)^{n-m+\frac{d}{2}}
\end{aligned}
\tag{D.31}
$$

of which two common examples are

$$
\begin{aligned}
\int\frac{\mathrm{d}^dl}{(2\pi)^d}\frac{1}{(l^2-\Delta)^2} &= \frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{(\Delta)^{2-\frac{d}{2}}}\\
\int\frac{\mathrm{d}^dl}{(2\pi)^d}\frac{l_\mu l_\nu}{(l^2-\Delta)^2} &= \int\frac{\mathrm{d}^dl}{(2\pi)^d}\frac{\frac{1}{d}l^2g_{\mu\nu}}{(l^2-\Delta)^2}=-\frac{\mathrm{i}}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(1-d/2)}{(\Delta)^{1-\frac{d}{2}}}
\end{aligned}
\tag{D.32}
$$

Fourier transform in d-dimensions is carried out by means of the standard formula

$$\int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{\mathrm{e}^{-\mathrm{i}k\cdot x}}{(-k^2)^n} = \frac{\mathrm{i}}{4^n \pi^{\frac{d}{2}}} \frac{\Gamma\left(\frac{d}{2}-n\right)}{\Gamma(n)} \frac{1}{(-x^2)^{\frac{d}{2}-n}} \tag{D.33}$$

Traces of σ matrices and tensor identities

In the following we list some useful identities between σ matrices and Levi-Civita tensors.

$$\begin{aligned} \sigma^\mu\sigma^\nu &= \frac{1}{2}[\sigma^\mu,\, \sigma^\nu] + \frac{1}{2}\{\sigma^\mu,\, \sigma^\nu\} = \delta^{\mu\nu} + \mathrm{i}\epsilon^{\mu\nu}{}_\rho\sigma^\rho \\ \sigma^\mu\sigma^\alpha\sigma^\nu &= -\delta^{\mu\nu}\sigma^\alpha + \delta^{\mu\alpha}\sigma^\nu + \delta^{\nu\alpha}\sigma^\mu + \mathrm{i}\epsilon^{\mu\alpha\nu} \\ \sigma^\mu\sigma^\alpha\sigma^\nu\sigma^\beta &= \delta^{\mu\alpha}\delta^{\nu\beta} + \delta^{\mu\beta}\delta^{\nu\alpha} - \delta^{\mu\nu}\delta^{\alpha\beta} + \mathrm{i}\epsilon^{\alpha\nu\beta}\sigma^\mu + \\ &\quad + \mathrm{i}(\delta^{\nu\beta}\epsilon^{\mu\alpha}{}_\gamma + \delta^{\alpha\nu}\epsilon^{\mu\beta}{}_\gamma - \delta^{\alpha\beta}\epsilon^{\mu\nu}{}_\gamma)\sigma^\gamma \end{aligned} \tag{D.34}$$

and so on acting repeatidly with the first identity. From these one can easily compute the traces

$$\begin{aligned} \mathrm{Tr}[\sigma^\mu\sigma^\nu] &= 2\delta^{\mu\nu} \\ \mathrm{Tr}[\sigma^\mu\sigma^\alpha\sigma^\nu] &= 2\mathrm{i}\epsilon^{\mu\alpha\nu} \\ \mathrm{Tr}[\sigma^\mu\sigma^\alpha\sigma^\nu\sigma^\beta] &= 2(\delta^{\mu\alpha}\delta^{\nu\beta} - \delta^{\alpha\beta}\delta^{\mu\nu} + \delta^{\nu\alpha}\delta^{\mu\beta}) \end{aligned} \tag{D.35}$$

Contractions of antisymmetric Levi-Civita ϵ tensors are also useful ...sometimes ...

$$\begin{aligned} \epsilon_{ijk}\epsilon^{jkl} &= 2\delta_k^l \\ \epsilon_{ijk}\epsilon^{ihl} &= \delta_j^h\delta_k^l - \delta_j^l\delta_k^h \\ \epsilon_{ijk}\epsilon_{lmn} &= \delta_{il}(\delta_{jm}\sigma_{kn} - \delta_{jn}\delta_{km}) - \delta_{im}(\delta_{jl}\sigma_{kn} - \delta_{jn}\delta_{km}) \\ &\quad + \delta_{in}(\delta_{jl}\sigma_{km} - \delta_{jm}\delta_{kl}) \end{aligned} \tag{D.36}$$

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