A Riesz basis of wavelets and its dual with quintic deficient splines

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Abstract. In this note, the dual of the Riesz basis of quintic splines wavelets obtained in [1] is explicitly constructed.

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Introduction

It is well known that for any natural number $m$, the cardinal B-spline $N_{m+1} = \chi_{[0,1]} \ast \cdots \ast \chi_{[0,1]}$ ($m + 1$ factors) can be used as a scaling function to construct orthogonal and biorthogonal bases of wavelets in $L^2(\mathbb{R})$, with different properties (see for example [3], [8]).

But, in approximation theory for instance, other splines are also very popular: the deficient splines (see some recent results in [5], [9]). In the paper [1], one can find a direct approach of the problem of the explicit construction of scaling functions, multiresolution analysis and wavelets with symmetry properties and compact support, involving deficient splines of degree 5 and regularity 3. Other results can also be found in [6], [7].

The present paper is a continuation of [1]. It gives an explicit construction of the dual basis of the deficient splines wavelets basis obtained in [1]. The dual is also generated by two wavelets, which are deficient splines with symmetry properties and exponential decay.

1 Definitions, notations, deficient spline wavelets

For $m \in \mathbb{N}$, the set of deficient splines of degree $2m + 1$ is the set

$$V_0 = \{ f \in L_2(\mathbb{R}) : f_{|_{[k,k+1]}} = P_k^{(2m+1)}, k \in \mathbb{Z} \text{ and } f \in C_{m+1}(\mathbb{R}) \}.$$  

For $m = 1$, it is the set of classical cardinal cubic splines. For $m = 2$, we denote it as the set of deficient quintic splines.
and, in this note, we only consider this case.
In this section, we recall the explicit and direct construction of a Riesz basis of wavelets consisting of deficient splines wavelets with compact support and symmetry property of [1].

1 Proposition. The following functions \( \varphi_a \) and \( \varphi_s \)

\[
\varphi_a(x) = \begin{cases} 
  x^4 - \frac{11}{15} x^5 & \text{if } x \in [0, 1] \\
  -\frac{2}{5} (x - \frac{3}{2}) + 3(x - \frac{3}{2})^3 - \frac{28}{15} (x - \frac{3}{2})^5 & \text{if } x \in [1, 2] \\
  -(3 - x)^3 + \frac{11}{15} (3 - x)^5 & \text{if } x \in [2, 3] \\
  0 & \text{if } x < 0 \text{ or } x > 3
\end{cases}
\]

\[
\varphi_s(x) = \begin{cases} 
  x^4 - \frac{3}{2} x^5 & \text{if } x \in [0, 1] \\
  \frac{57}{80} - \frac{3}{2} (x - \frac{3}{2})^2 + (x - \frac{3}{2})^4 & \text{if } x \in [1, 2] \\
  (3 - x)^4 - \frac{3}{2} (3 - x)^5 & \text{if } x \in [2, 3] \\
  0 & \text{if } x < 0 \text{ or } x > 3
\end{cases}
\]

are respectively antisymmetric and symmetric with respect to \( \frac{3}{2} \) and the family

\[
\{ \varphi_a(\cdot - k), \ k \in \mathbb{Z} \} \cup \{ \varphi_s(\cdot - k), \ k \in \mathbb{Z} \}
\]

constitutes a Riesz basis of \( V_0 \).

For every \( j \in \mathbb{Z} \) we define

\[
V_j = \{ f \in L^2(\mathbb{R}) : f(2^{-j} \cdot) \in V_0 \}.
\]

2 Proposition. The sequence \( V_j (j \in \mathbb{Z}) \) is an increasing sequence of closed sets of \( L^2(\mathbb{R}) \) and

\[
\bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \quad \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}).
\]

Moreover, the functions \( \varphi_a, \varphi_s \) satisfy the following scaling relation

\[
\begin{pmatrix} \varphi_s(2\xi) \\ \varphi_a(2\xi) \end{pmatrix} = M_0(\xi) \begin{pmatrix} \varphi_s(\xi) \\ \varphi_a(\xi) \end{pmatrix}
\]

where \( M_0(\xi) \) is the matrix (called filter matrix)

\[
M_0(\xi) = \frac{e^{-3i\xi/2}}{64} \begin{pmatrix} 51 \cos(\frac{\xi}{2}) + 13 \cos(\frac{3\xi}{2}) & -9i(\sin(\frac{\xi}{2}) + \sin(\frac{3\xi}{2})) \\ i(11 \sin(\frac{3\xi}{2}) + 21 \sin(\frac{\xi}{2})) & -7 \cos(\frac{3\xi}{2}) + 9 \cos(\frac{\xi}{2}) \end{pmatrix}.
\]

For every \( j \in \mathbb{Z} \), we denote by \( W_j \) the orthogonal complement of \( V_j \) in \( V_{j+1} \). Using standard techniques of Fourier analysis in the context of wavelets, one obtains the following result.
3 Proposition. A function $f$ belongs to $W_0$ if and only if there exists $p, q \in L^2_{\text{loc}}, 2\pi-$ periodic such that

$$\hat{f}(2\xi) = p(\xi)\hat{\varphi}(\xi) + q(\xi)\hat{\varphi}(\xi)$$

and

$$M_0(\xi) W(\xi) \left( \begin{array}{c} p(\xi) \\ q(\xi) \end{array} \right) + M_0(\xi + \pi) W(\xi + \pi) \left( \begin{array}{c} p(\xi + \pi) \\ q(\xi + \pi) \end{array} \right) = 0 \quad \text{a.e.}$$

where $M_0$ is the filter matrix obtained in Proposition 2 and $W(\xi)$ is the matrix

$$W(\xi) = \left( \begin{array}{cc} \omega_s(\xi) & \omega_m(\xi) \\ \omega_m(\xi) & \omega_a(\xi) \end{array} \right)$$

with

$$\omega_a(\xi) = \sum_{l=-\infty}^{+\infty} |\hat{\varphi}_a(\xi + 2l\pi)|^2 = \frac{23247 - 21362 \cos \xi - 385 \cos(2\xi)}{311850}$$

$$\omega_s(\xi) = \sum_{l=-\infty}^{+\infty} |\hat{\varphi}_s(\xi + 2l\pi)|^2 = \frac{14445 + 7678 \cos \xi + 53 \cos(2\xi)}{34650}$$

$$\omega_m(\xi) = \sum_{l=-\infty}^{+\infty} \hat{\varphi}_s(\xi + 2l\pi)\hat{\varphi}_a(\xi + 2l\pi) = -\frac{i}{51975} \sin \xi \ (6910 + 193 \cos \xi).$$

4 Theorem. There exists deficient splines wavelets with support in $[0,5]$ and symmetry properties (with respect to $5/2$).

More precisely, there exists real numbers $p_j^{(s)}, q_j^{(s)}, p_j^{(a)}, q_j^{(a)}, \quad j = 0, \ldots, 7$

verifying

$$p_j^{(s)} = P_{t-j}, \quad q_j^{(s)} = -q_{t-j}^{(s)}, \quad p_j^{(a)} = -P_{t-j}^{(a)}, \quad q_j^{(a)} = q_{t-j}^{(a)}, \quad j = 0, 1, 2, 3$$

such that the family $\{\psi_s(.,-k) : k \in \mathbb{Z}\} \cup \{\psi_a(.,-k) : k \in \mathbb{Z}\}$ constitutes a Riesz basis of $W_0$, where

$$\widehat{\psi}_s(2\xi) = \sum_{j=0}^{7} p_j^{(s)} e^{-ij\xi} \hat{\varphi}_s(\xi) + \sum_{j=0}^{7} q_j^{(s)} e^{-ij\xi} \hat{\varphi}_a(\xi)$$

$$\widehat{\psi}_a(2\xi) = \sum_{j=0}^{7} p_j^{(a)} e^{-ij\xi} \hat{\varphi}_a(\xi) + \sum_{j=0}^{7} q_j^{(a)} e^{-ij\xi} \hat{\varphi}_a(\xi).$$
Explicit values of the coefficients can be found in [1]. It follows that the family
\[
\{2^{j/2}\varphi(j, -k) : j, k \in \mathbb{Z}\} \cup \{2^{j/2}\varphi(j, -k) : j, k \in \mathbb{Z}\}
\]
constitutes a Riesz basis of \(L^2(\mathbb{R})\) of deficient splines wavelets with compact support and symmetry properties. The symmetry properties can be written as follows
\[
\widehat{\varphi}(\xi) = e^{-5i\xi}\widehat{\psi}(\xi), \quad \widehat{\psi}(\xi) = -e^{-5i\xi}\widehat{\psi}(\xi).
\]

Here are pictures of \(\varphi_s, \varphi_a\)

![Picture of \(\varphi_s, \varphi_a\)]

and of \(\psi, \psi_a\) (up to a multiplicative constant)

![Picture of \(\psi, \psi_a\)]

2 The dual basis

The following result is classical in the context of frames and Riesz basis (see for example [2], [4]).

5 Proposition. If \(f_m\) \((m \in \mathbb{N})\) is a Riesz basis of an Hilbert space \(H\), there exists a unique sequence \(g_m\) \((m \in \mathbb{N})\) of elements of \(H\) such that \(<f_m, g_k> = \delta_{km}\) for every \(m, k \in \mathbb{N}\). More precisely one has
\[
g_m = S^{-1}f_m, \quad m \in \mathbb{N}
\]
where \(S\) is the frame operator

\[
S : H \rightarrow H \quad f \mapsto \sum_{m=1}^{+\infty} <f, f_m> f_m.
\]
The sequence $g_m$ ($m \in \mathbb{N}$) is also a Riesz basis and is called the dual Riesz basis of $f_m$ ($m \in \mathbb{N}$). It also satisfies

$$f = \sum_{m=1}^{+\infty} <f, f_m> g_m = \sum_{m=1}^{+\infty} <f, g_m> f_m$$

for every $f \in H$.

Now, we want to give an explicit construction of the dual basis of the Riesz basis (*).

But before doing so, let us install some notations and let us also briefly recall some additional properties concerning the wavelet basis (*). We denote by $W_\psi$ the matrix similar to $W$ (see Proposition 3) but defined using the functions $a_s, s$ instead of $a_s, s$,

$$W_\psi = \begin{pmatrix} \omega_{\psi_s, a_s}(\xi) & \omega_{\psi_s, \psi_a}(\xi) \\ \omega_{\psi_s, a_s}(\xi) & \omega_{\psi_s, \psi_a}(\xi) \end{pmatrix}$$

where

$$\omega_{\psi_s}(\xi) = \sum_{l=-\infty}^{+\infty} |\tilde{\psi}_s(\xi + 2l\pi)|^2, \quad \omega_{\psi_a}(\xi) = \sum_{l=-\infty}^{+\infty} |\tilde{\psi}_a(\xi + 2l\pi)|^2$$

$$\omega_{\psi_s, \psi_a}(\xi) = \sum_{l=-\infty}^{+\infty} \tilde{\psi}_s(\xi + 2l\pi)\tilde{\psi}_a(\xi + 2l\pi).$$

These functions have the following properties.

**5.1 Property.** The functions $\omega_{\psi_a}, \omega_{\psi_s}, \omega_{\psi_s, \psi_a}$ are $2\pi$-periodic trigonometric polynomials such that

$$\omega_{\psi_a}(\xi) \geq c > 0, \quad \omega_{\psi_s}(\xi) \geq c > 0, \quad \omega_{\psi_a}(-\xi) = \omega_{\psi_a}(\xi), \quad \omega_{\psi_s}(-\xi) = \omega_{\psi_s}(\xi)$$

and

$$\omega_{\psi_s, \psi_a}(\xi) = -\omega_{\psi_s, \psi_a}(\xi) = \omega_{\psi_s, \psi_a}(-\xi)$$

for every $\xi \in \mathbb{R}$. There are also $A, B > 0$ such that

$$A \leq \det(W_\psi(\xi)) \leq B, \quad \forall \xi \in \mathbb{R}.$$
Theorem. The functions $\tilde{\psi}_1, \tilde{\psi}_2$ defined as

$$
\tilde{\psi}_1(\xi) = \alpha_1(\xi) \tilde{\psi}_a(\xi) + \beta_1(\xi) \tilde{\psi}_s(\xi), \quad \tilde{\psi}_2(\xi) = \alpha_2(\xi) \tilde{\psi}_a(\xi) + \beta_2(\xi) \tilde{\psi}_s(\xi)
$$

where

$$
\alpha_1(\xi) = \frac{\omega_{\psi_a}(\xi)}{\det(W_\psi(\xi))}, \quad \beta_1(\xi) = \frac{\omega_{\psi_s, \psi_a}(\xi)}{\det(W_\psi(\xi))},
$$

$$
\alpha_2(\xi) = \beta_1(\xi) = -\beta_1(\xi), \quad \beta_2(\xi) = \frac{\omega_{\psi_s}(\xi)}{\det(W_\psi(\xi))}
$$

are such that the family of functions

$$\left\{ 2^{j/2} \tilde{\psi}_i(2^j \cdot - k) : i = 1, 2; j, k \in \mathbb{Z} \right\}
$$

is the dual basis of the basis of wavelets (*).

Proof. First, we look for a function $\tilde{\psi}_1$ in $W_0$ such that

$$
< \psi_a(\cdot - k), \tilde{\psi}_1 >_{L^2(\mathbb{R})} = \delta_{0k} \quad \text{and} \quad < \psi_s(\cdot - k), \tilde{\psi}_1 >_{L^2(\mathbb{R})} = 0
$$

for every $k \in \mathbb{Z}$. Since $\{\psi_a(\cdot - k) : k \in \mathbb{Z}\} \cup \{\psi_s(\cdot - k) : k \in \mathbb{Z}\}$ constitute a Riesz basis of $W_0$, we look in fact for $2\pi$-periodic and $L^2_{\text{loc}}$ functions $\alpha_1, \beta_1$ such that

$$
\tilde{\psi}_1(\xi) = \alpha_1(\xi) \tilde{\psi}_a(\xi) + \beta_1(\xi) \tilde{\psi}_s(\xi)
$$

and such that

$$
< e^{-ik} \tilde{\psi}_a, \alpha_1 \tilde{\psi}_a + \beta_1 \tilde{\psi}_s >_{L^2(\mathbb{R})} = 2\pi \delta_{0k} \quad \text{and} \quad < e^{-ik} \tilde{\psi}_s, \alpha_1 \tilde{\psi}_a + \beta_1 \tilde{\psi}_s >_{L^2(\mathbb{R})} = 0
$$

for every $k \in \mathbb{Z}$. The last equalities are equivalent to

$$
\begin{align*}
\int_0^{2\pi} e^{-ik\xi} \left( \frac{\alpha_1(\xi) \omega_{\psi_a}(\xi) + \beta_1(\xi) \omega_{\psi_s, \psi_a}(\xi)}{\alpha_1(\xi) \omega_{\psi_s, \psi_a}(\xi) + \beta_1(\xi) \omega_{\psi_s}(\xi)} \right) d\xi &= 2\pi \delta_{0k} \\
\int_0^{2\pi} e^{-ik\xi} \left( \frac{\alpha_1(\xi) \omega_{\psi_a}(\xi) + \beta_1(\xi) \omega_{\psi_s, \psi_a}(\xi)}{\alpha_1(\xi) \omega_{\psi_s, \psi_a}(\xi) + \beta_1(\xi) \omega_{\psi_s}(\xi)} \right) d\xi &= 0, \quad \forall k \in \mathbb{Z}
\end{align*}
$$

hence also to

$$
\begin{align*}
\alpha_1(\xi) \omega_{\psi_a}(\xi) + \beta_1(\xi) \omega_{\psi_s, \psi_a}(\xi) &= 1 \\
\alpha_1(\xi) \omega_{\psi_s, \psi_a}(\xi) + \beta_1(\xi) \omega_{\psi_s}(\xi) &= 0.
\end{align*}
$$

Using matrices, this can be rewritten as

$$
\begin{pmatrix}
\omega_{\psi_a}(\xi) & \omega_{\psi_s, \psi_a}(\xi) \\
\omega_{\psi_s, \psi_a}(\xi) & \omega_{\psi_s}(\xi)
\end{pmatrix}
\begin{pmatrix}
\beta_1(\xi) \\
\alpha_1(\xi)
\end{pmatrix}
= W_\psi(\xi)
\begin{pmatrix}
\beta_1(\xi) \\
\alpha_1(\xi)
\end{pmatrix}
= \begin{pmatrix}
0 \\
1
\end{pmatrix}.
$$
The solutions of this system is

\[
\begin{pmatrix}
\beta_1(\xi) \\
\alpha_1(\xi)
\end{pmatrix} = \frac{1}{\det(W_{\psi}(\xi))} \begin{pmatrix}
-\omega_{\psi_{a},\psi_{a}}(\xi) \\
\omega_{\psi_{a}}(\xi)
\end{pmatrix} = \frac{1}{\det(W_{\psi}(\xi))} \begin{pmatrix}
\omega_{\psi_{s},\psi_{s}}(\xi) \\
\omega_{\psi_{s}}(\xi)
\end{pmatrix}
\] .

We proceed exactly in the same way to find a function $\tilde{\psi}_2$ in $W_0$ such that

\[
< \psi_a(.,-k), \tilde{\psi}_2 > = 0 \quad \text{and} \quad < \psi_a(.,-k), \tilde{\psi}_2 >= \delta_{0k}
\]

for every $k \in \mathbb{Z}$. In this case, the final system is

\[
W_{\psi}(\xi) \begin{pmatrix}
\beta_2(\xi) \\
\alpha_2(\xi)
\end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

which gives the solutions.

Since the spaces $W_j$ and $W_{j'}$ are orthogonal if $j \neq j'$, we obtain, for $j, j', k, k' \in \mathbb{Z}$:

\[
< 2^{j/2} \psi_a(2^j - k), 2^{j'/2} \tilde{\psi}_1(2^{j'} - k') >= \delta_{jj'} \delta_{kk'}
\]

\[
< 2^{j/2} \psi_a(2^j - k), 2^{j'/2} \tilde{\psi}_2(2^{j'} - k') >= 0
\]

and

\[
< 2^{j/2} \psi_a(2^j - k), 2^{j'/2} \tilde{\psi}_2(2^{j'} - k') >= 0
\]

\[
< 2^{j/2} \psi_s(2^j - k), 2^{j'/2} \tilde{\psi}_1(2^{j'} - k') >= \delta_{jj'} \delta_{kk'}
\]

hence the conclusion.

\[\text{QED}\]

7 Proposition. The functions $\tilde{\psi}_2, \tilde{\psi}_2$ are deficient splines with exponential decay and symmetry properties ($\tilde{\psi}_1, \tilde{\psi}_2$ are respectively antisymmetric and symmetric relatively to $5/2$).

Proof. By construction, these functions are deficient splines. Their explicit expressions in terms of the Fourier transforms of the wavelets $\psi_a, \psi_s$ and the form of the coefficients $\alpha_i, \beta_i$ give the exponential decay and the symmetry properties.

Here are pictures of an approximation of the dual functions $\tilde{\psi}_1, \tilde{\psi}_2$ (up to a constant factor).
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References


