On certain modified
Szasz-Mirakyan operators
in polynomial weighted spaces

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Abstract. We consider certain modified Szasz-Mirakyan operators $A_n(f;r)$ in polynomial
weighted spaces of functions of one variable and we study approximation properties of these
operators.

Keywords: Szasz-Mirakyan operator, degree of approximation, Voronovskaja type theorem.


Introduction

In the paper [1] M. Becker studied approximation problems for functions
$f \in C_p$ and Szasz-Mirakyan operators
$$S_n(f;x) := e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f \left( \frac{k}{n} \right),$$

where $x \in R_0 = [0, +\infty), \ n \in N := \{1, 2, \ldots \}$, where $C_p$ with fixed $p \in N_0 := \{0, 1, 2, \ldots \}$ is polynomial weighted space generated by the weighted function
$$w_0(x) := 1, \ w_p(x) := (1 + x^p)^{-1}, \ \text{if} \ p \geq 1,$$
i.e. $C_p$ is the set of all real-valued functions $f$, continuous on $R_0$ and such that $w_p f$ is uniformly continuous and bounded on $R_0$. The norm in $C_p$ is defined by the formula
$$\|f\|_p = \|f(\cdot)\|_p := \sup_{x \in R_0} w_p(x) |f(x)|.$$

In [1] theorems on the degree of approximation of $f \in C_p$ by the operators $S_n$ were proved. From these theorems it was deduced that
$$\lim_{n \to \infty} S_n(f;x) = f(x),$$
for every \( f \in C_p, \ p \in N_0 \) and \( x \in R_0 \). Moreover the convergence (4) is uniform on every interval \([x_1, x_2] \), \( x_2 > x_1 \geq 0 \).

In this paper we shall modify the formula (1) and we shall study certain approximation properties of introduced operators.

Let \( C_p \) be the space given above and let \( f \in C_p^1 := \{ f \in C_p : f' \in C_p \} \), where \( f' \) is the first derivative of \( f \).

For \( f \in C_p \) we define the modulus of continuity \( \omega_1(f; \cdot) \) as usual ([2]) by formula

\[
\omega_1(f; C_p; t) := \sup_{0 \leq h \leq t} \| \Delta_h f(\cdot) \|_p, \quad t \in R_0,
\]

where \( \Delta_h f(x) := f(x + h) - f(x) \), for \( x, h \in R_0 \). From the above it follows that

\[
\lim_{t \to 0^+} \omega_1(f; C_p; t) = 0,
\]

for every \( f \in C_p \). Moreover if \( f \in C_p^1 \) then there exists \( M_1 = \text{const.} > 0 \) such that

\[
\omega_1(f; C_p; t) \leq M_1 \cdot t \quad \text{for} \quad t \in R_0.
\]

We introduce the following

1 Definition. Let \( R_2 := [2, +\infty) \) and let \( r \in R_2 \) and \( p \in N_0 \) be fixed numbers. For functions \( f \in C_p \) we define the operators

\[
A_n(f; r; x) := e^{-(nx+1)r} \sum_{k=0}^{\infty} \frac{(nx+1)^r}{k!} f\left( \frac{k}{n(nx+1)^{r-1}} \right),
\]

\( x \in R_0, \ n \in N \).

Similarly as \( S_n \), the operator \( A_n \) is linear and positive. In § 2 we shall prove that \( A_n \) is an operator from the space \( C_p \) into itself for every fixed \( p \in N_0 \).

From (8) we easily derive the following formulas

\[
A_n(1; r; x) = 1,
\]

\[
A_n(t; r; x) = x + \frac{1}{n}, \quad A_n(t^2; r; x) = \left( x + \frac{1}{n} \right)^2 \left[ 1 + \frac{1}{(nx + 1)^r} \right],
\]

\[
A_n(t^3; r; x) = \left( x + \frac{1}{n} \right)^3 \left[ 1 + \frac{3}{(nx + 1)^r} + \frac{1}{(nx + 1)^{2r}} \right],
\]

for every fixed \( r \in R_2 \) and for all \( n \in N \) and \( x \in R_0 \).
1 Main results

From formulas (8), (9) and $A_n(t^k; r; x)$, $1 \leq k \leq 3$, given above we obtain

2 Lemma. Let $r \in R_2$ be a fixed number. Then for all $x \in R_0$ and $n \in N$ we have

$$A_n(t - x; r; x) = \frac{1}{n},$$

$$A_n((t - x)^2; r; x) = \frac{1}{n^2} \left[ 1 + \frac{1}{(nx + 1)^{r-2}} \right],$$

$$A_n((t - x)^3; r; x) = \frac{1}{n^3} \left[ 1 + \frac{3}{(nx + 1)^{r-2}} + \frac{1}{(nx + 1)^{2r-3}} \right].$$

Next we shall prove

3 Lemma. Let $s \in N$ and $r \in R_2$ be fixed numbers. Then there exist positive numbers $\lambda_{s,j}$, $1 \leq j \leq s$, depending only on $j$ and $s$, such that

$$A_n(t^s; r; x) = \left( x + \frac{1}{n} \right)^s \sum_{j=1}^{s} \frac{\lambda_{s,j}}{(nx + 1)^{(j-1)r}}$$

for all $n \in N$ and $x \in R_0$. Moreover $\lambda_{s,1} = \lambda_{s,s} = 1$.

Proof. We shall use the mathematical induction on $s$.

The formula (10) for $s = 1, 2, 3$ is given above.

Let (10) holds for $f(x) = x^j$, $1 \leq j \leq s$, with fixed $s \in N$. We shall prove (10) for $f(x) = x^{s+1}$. From (8) it follows that

$$A_n(t^{s+1}; r; x) = e^{-(nx+1)r} \sum_{k=1}^{\infty} \frac{(nx+1)^r}{(k-1)!} \frac{k^s}{(n(nx + 1)^{r-1})^{s+1}} =$$

$$= \frac{(nx+1)^r}{(n(nx + 1)^{r-1})^{s+1}} e^{-(nx+1)r} \sum_{k=0}^{\infty} \frac{(nx+1)^r}{k!} \frac{k^s}{(k+1)^s} =$$

$$= \frac{(nx+1)^r}{(n(nx + 1)^{r-1})^{s+1}} \sum_{k=0}^{\infty} \frac{(nx+1)^r}{k!} \sum_{\mu=0}^{s} \binom{s}{\mu} k^\mu =$$

$$= \frac{(nx+1)^r}{(n(nx + 1)^{r-1})^{s+1}} \sum_{\mu=0}^{s} \binom{s}{\mu} (n(nx + 1)^{r-1})^\mu A_n(t^\mu; r; x).$$

By our assumption we get

$$A_n(t^{s+1}; r; x) = \frac{(nx+1)^r}{(n(nx + 1)^{r-1})^{s+1}}.$$
\[ \begin{aligned}
\left\{ 1 + \sum_{\mu=1}^{s} \left( \frac{s}{\mu} \right) (nx + 1)^{\mu} \sum_{j=1}^{s} \frac{\lambda_{\mu,j}}{(nx + 1)(j-1)^{r}} \right\} &= \\
= \left( x + \frac{1}{n} \right)^{s+1} \left\{ \frac{1}{(nx + 1)^{r}} + \sum_{j=1}^{s} \frac{1}{(nx + 1)(j-1)^{r}} \right\} &= \\
= \left( x + \frac{1}{n} \right)^{s+1} \sum_{j=1}^{\lambda_{s+1}} \frac{\lambda_{s+1,j}}{(nx + 1)(j-1)^{r}} &= \\
= \left( x + \frac{1}{n} \right)^{s+1} \sum_{j=1}^{\lambda_{s+1}} \frac{\lambda_{s+1,j}}{(nx + 1)(j-1)^{r}}
\end{aligned} \]

and \( \lambda_{s+1,1} = \lambda_{s+1,s+1} = 1 \), which proves (10) for \( f(x) = x^{s+1} \). Hence the proof of (10) is completed. \( \square \)

4 Lemma. Let \( p \in N_0 \) and \( r \in R_2 \) be fixed numbers. Then there exists a positive constant \( M_2 \equiv M_2(p,r) \), depending only on the parameters \( p \) and \( r \) such that

\[ \| A_n(1/w_p(t); r; \cdot) \|_p \leq M_2, \quad n \in N. \] (11)

Moreover for every \( f \in C_p \) we have

\[ \| A_n(f; r; \cdot) \|_p \leq M_2 \| f \|_p, \quad n \in N. \] (12)

The formula (8) and inequality (12) show that \( A_n, n \in N \), is a positive linear operator from the space \( C_p \) into \( C_p \), for every \( p \in N_0 \).

Proof. The inequality (11) is obvious for \( p = 0 \) by (2), (3) and (9). Let \( p \in N \). Then by (2) and (8)-(10) we have

\[ w_p(x)A_n(1/w_p(t); r; x) = w_p(x) \{ 1 + A_n(p^p; r; x) \} = \]

\[ = \left( \frac{1}{1 + x^p} + \frac{(x + 1/n)^p}{1 + x^p} \sum_{j=1}^{p} \frac{\lambda_{p,j}}{(nx + 1)(j-1)^{r}} \right) \leq \]

\[ \leq 1 + \sum_{\mu=0}^{p} \left( \frac{p}{\mu} \right) \frac{x^p}{1 + x^p} \sum_{j=1}^{p} \frac{\lambda_{p,j}}{(nx + 1)(j-1)^{r}} \leq M_2(p,r), \]

for \( x \in R_0, n \in N \) and \( r \in R_2 \), where \( M_2(p,r) \) is a positive constant depending only \( p \) and \( r \). From this follows (11).

The formula (8) and (3) imply

\[ \| A_n(f(t); r; \cdot) \|_p \leq \| f \|_p \| A_n(1/w_p(t); r; \cdot) \|_p, \quad n \in N, \quad r \in R_2, \]

for every \( f \in C_p \). By applying (11), we obtain (12). \( \square \)
5 Lemma. Let \( p \in \mathbb{N}_0 \) and \( r \in \mathbb{R}_2 \) be fixed numbers. Then there exists a positive constant \( M_3 \equiv M_3(p, r) \) such that
\[
\left\| A_n \left( \frac{(t-x)^2}{w_p(t)}; r; x \right) \right\|_p \leq \frac{M_3}{n^2} \quad \text{for all} \quad n \in \mathbb{N}. \quad (13)
\]

Proof. The formulas given in 2 Lemma and (2), (3) imply (13) for \( p = 0 \).

By (2) and (9) we have
\[
A_n \left( (t-x)^2/w_1(t); r; x \right) = A_n \left( (t-x)^2; r; x \right) + A_n \left( t(t-x)^2; r; x \right),
\]
for \( p, n \in \mathbb{N} \) and \( r \in \mathbb{R}_2 \). If \( p = 1 \) then by the equality we get
\[
A_n \left( (t-x)^2/w_1(t); r; x \right) = A_n \left( (t-x)^2; r; x \right) + A_n \left( t(t-x)^2; r; x \right) =
\]
\[
= A_n \left( (t-x)^2; r; x \right) + (1 + x)A_n \left( (t-x)^2; r; x \right),
\]
which by (2) and (3) and 2 Lemma yields (13) for \( p = 1 \).

Let \( p \geq 2 \). By applying (10), we get
\[
w_p(x)A_n \left( t^p(t-x)^2; r; x \right) = w_p(x) \left\{ A_n \left( t^{p+2}; r; x \right) - 2xA_n \left( t^{p+1}; r; x \right) +
\]
\[
+ x^2A_n \left( t^p; r; x \right) \right\} = w_p(x) \left\{ \left( x + \frac{1}{n} \right)^{p+2} \sum_{j=1}^{p+1} \frac{\lambda_{p+2,j}}{(nx+1)(j-1)r} +
\]
\[-2x \left( x + \frac{1}{n} \right)^{p+1} \sum_{j=1}^{p+1} \frac{\lambda_{p+1,j}}{(nx+1)(j-1)r} +
\]
\[
+ x^2 \left( x + \frac{1}{n} \right)^p \sum_{j=1}^{p} \frac{\lambda_{p,j}}{(nx+1)(j-1)r} \right\} =
\]
\[
= w_p(x) \left( x + \frac{1}{n} \right)^p \left\{ \frac{1}{n^2} + \left( x + \frac{1}{n} \right)^2 \sum_{j=2}^{p+2} \frac{\lambda_{p+2,j}}{(nx+1)(j-1)r} +
\]
\[-2x \sum_{j=2}^{p+1} \frac{\lambda_{p+1,j}}{(nx+1)(j-1)r} + x^2 \sum_{j=2}^{p} \frac{\lambda_{p,j}}{(nx+1)(j-1)r} \right\}
\]
which implies
\[
w_p(x)A_n \left( t^p(t-x)^2; r; x \right) \leq \frac{1}{n^2} \left( 1 + x \right)^p \left\{ 1 + \frac{1}{(nx+1)r^2} \sum_{j=2}^{p+2} \lambda_{p+2,j} +
\]
\[+ 2x \sum_{j=2}^{p+1} \frac{\lambda_{p+1,j}}{(nx+1)(j-1)r} + x^2 \sum_{j=2}^{p} \frac{\lambda_{p,j}}{(nx+1)(j-1)r} \right\}
\]
\[
+ 2 \sum_{j=2}^{p+1} \lambda_{p+1,j} + \sum_{j=2}^{p} \lambda_{p,j} \right) \leq \frac{M_3(p,r)}{n^2}
\]

for \( x \in R_0, n \in N \) and \( r \in R_2 \). Thus the proof is completed.

Now we shall give approximation theorems for \( A_n \).

6 Theorem. Let \( p \in N_0 \) and \( r \in R_2 \) be fixed numbers. Then there exists a positive constant \( M_4 \equiv M_4(p,r) \) such that for every \( f \in C^1_p \) we have

\[
\| A_n(f;r;\cdot) - f(\cdot) \|_p \leq \frac{M_4}{n} \| f' \|_p, \quad n \in N.
\]

Proof. Let \( x \in R_0 \) be a fixed point. Then for \( f \in C^1_p \) we have

\[
f(t) - f(x) = \int_x^t f'(u)du, \quad t \in R_0.
\]

From this and by (8) and (9) we get

\[
A_n (f(t);r;x) - f(x) = A_n \left( \int_x^t f'(u)du; r; x \right), \quad n \in N.
\]

But by (2) and (3) we have

\[
\left| \int_x^t f'(u)du \right| \leq \| f' \|_p \left( \frac{1}{w_p(t)} + \frac{1}{w_p(x)} \right) |t - x|, \quad t \in R_0,
\]

which implies

\[
w_p(x)|A_n(f;r;x) - f(x)| \leq \frac{M_4}{n} \left\{ A_n \left( (|t - x|; r; x) A_n \left( \frac{|t - x|}{w_p(t)}; r; x \right) \right) \right\}
\]

for \( n \in N \). By the Hölder inequality and by (9) and 2, 4, 5 Lemmas it follows that

\[
A_n (|t - x|; r; x) \leq \left\{ A_n \left( (t - x)^2; r; x \right) A_n (1; r; x) \right\}^{1/2} \leq \frac{\sqrt{2}}{n},
\]

\[
w_p(x) A_n \left( \frac{|t - x|}{w_p(t)}; r; x \right) \leq \frac{1}{w_p(x)} \left\{ A_n \left( \frac{1}{w_p(t)}; r; x \right) \right\}^{1/2} \left\{ A_n \left( \frac{1}{w_p(t)}; r; x \right) \right\}^{1/2} \leq \frac{M_4}{n}
\]

for \( n \in N \). From this and by (15) we immediately obtain (14).
**Theorem.** Let \( p \in \mathbb{N}_0 \) and \( r \in \mathbb{R}_2 \) be fixed numbers. Then there exists \( M_6 \equiv M_6(p, r) \) such that for every \( f \in C_p \) and \( n \in \mathbb{N} \) we have

\[
\| A_n(f; r; \cdot) - f(\cdot) \|_p \leq M_6 \omega_1 \left( f; C_p; \frac{1}{n} \right).
\]

**Proof.** We use Steklov function \( f_h \) of \( f \in C_p \)

\[
f_h(x) := \frac{1}{h} \int_0^h f(x + t) dt, \quad x \in R_0, \quad h > 0.
\]

From (17) we get

\[
f_h(x) - f(x) = \frac{1}{h} \int_0^h \Delta_f(x) dt,
\]

\[
f'_h(x) = \frac{1}{h} \Delta_h f(x), \quad x \in R_0, \quad h > 0,
\]

which imply

\[
\| f_h - f \|_p \leq \omega_1 \left( f; C_p; h \right),
\]

\[
\| f'_h \|_p \leq h^{-1} \omega \left( f; C_p; h \right),
\]

for \( h > 0 \). From this we deduce that \( f_h \in C^1_p \) if \( f \in C_p \) and \( h > 0 \).

Hence we can write

\[
w_p(x) |A_n(f; x) - f(x)| \leq w_p(x) \{ |A_n(f - f_h; x)| + |A_n(f_h; x) - f_h(x)| + |f_h(x) - f(x)| \} := L_1(x) + L_2(x) + L_3(x),
\]

for \( n \in \mathbb{N}, h > 0 \) and \( x \in R_0 \). From (12) and (18) we get

\[
\| L_1 \|_p \leq M_2 \| f_h - f \|_p \leq M_2 \omega_1 \left( f; C_p; h \right),
\]

\[
\| L_3 \|_p \leq \omega_1 \left( f; C_p; h \right).
\]

By 6 Theorem and (19) it follows that

\[
\| L_2 \|_p \leq \frac{M_4}{n} \| f'_h \|_p \leq \frac{M_4}{nh} \omega_1 \left( f; C_p; h \right).
\]

Consequently

\[
\| A_n(f; r; \cdot) - f(\cdot) \|_p \leq \left( 1 + M_2 + \frac{M_4}{nh} \right) \omega_1(f; C_p; h).
\]

Now, for fixed \( n \in \mathbb{N} \), setting \( h = \frac{1}{n} \), we obtain

\[
\| A_n(f; r; \cdot) - f(\cdot) \|_p \leq M_6(p, r) \omega_1 \left( f; C_p; \frac{1}{n} \right)
\]

and we complete the proof.
From 6 Theorem and 7 Theorem we derive the following two corollaries:

8 Corollary. For every fixed \( r \in R_2 \) and \( f \in C_p \), \( p \in N_0 \), we have

\[
\lim_{n \to \infty} \|A_n(f; r; \cdot) - f(\cdot)\|_p = 0.
\]

9 Corollary. If \( f \in C^1_p \), \( p \in N_0 \) and \( r \in R_2 \), then

\[
\|A_n(f; r; \cdot) - f(\cdot)\|_p = O(1/n).
\]

Finally, we shall give the Voronovskaya type theorem for \( A_n \).

10 Theorem. Let \( f \in C^1_p \) and let \( r \in R_2 \) be fixed number. Then,

\[
\lim_{n \to \infty} n \{ A_n(f; r; x) - f(x) \} = f'(x)
\]  

(20)

for every \( x \in R_0 \).

Proof. Let \( x \in R_0 \) be a fixed point. Then by the Taylor formula we have

\[
f(t) = f(x) + f'(x)(t - x) + \varepsilon(t; x)(t - x)
\]

for \( t \in R_0 \), where \( \varepsilon(t) \equiv \varepsilon(t; x) \) is a function belonging to \( C_p \) and \( \varepsilon(x) = 0 \). Hence by (8) and (9) we get

\[
A_n(f; r; x) = f(x) + f'(x)A_n(t - x; r; x) + A_n(\varepsilon(t)(t - x); r; x), \quad n \in N, \quad (21)
\]

and by Hölder inequality

\[
|A_n(\varepsilon(t)(t - x); r; x)| \leq \left\{ A_n\left(\varepsilon^2(t); x\right)\right\}^{1/2} \left\{ A_n\left((t - x)^2; x\right)\right\}^{1/2}.
\]

By 8 Corollary we deduce that

\[
\lim_{n \to \infty} A_n(\varepsilon(t); r; x) = \varepsilon^2(x) = 0.
\]

From this and by 2 Lemma we get

\[
\lim_{n \to \infty} n A_n(\varepsilon(t)(t - x); r; x) = 0. \quad (22)
\]

Using (22) and 2 Lemma to (21), we obtain the desired assertion (20).

11 Remark. It is easily verified that the operators

\[
\mathcal{A}_n(f; r; x) := e^{-(nx+1)r} \sum_{k=0}^{\infty} \frac{(nx+1)^r}{k!n(nx+1)^r} \int_{(k+r)/(n(nx+1)^{-1})}^{(k+1+r)/(n(nx+1)^{-1})} f(t)dt,
\]

\( p \in N_0 \), \( x \in R_0 \), \( n \in N \) and \( r \in R_2 \), have analogous approximation properties in the space \( C_p \).
12 Remark. In [1] it was proved that if $f \in C_p$, $p \in N_0$, then for the Szasz-Mirakyan operators $S_n$ (defined by (1)) is satisfied the following inequality

$$w_p(x)|S_n(f; x) - f(x)| \leq M_0 \omega_2 \left( f; C_p; \frac{x}{n} \right), \quad x \in R_0, \quad n \in N_0,$$

where $M_0 = \text{const.} > 0$ and $\omega_2 (f; \cdot)$ is the modulus of smoothness defined by the formula

$$\omega_2 (f; C_p; t) := \sup_{0 \leq h < t} \| \Delta_h^2 f (\cdot) \|_p, \quad t \in R_0,$$

where $\Delta_h^2 f (x) := f(x) - 2f(x+h) + f(x+2h)$. In particular, if $f \in C^1_p$, $p \in N_0$, then

$$w_p(x)|S_n(f; x) - f(x)| \leq M_{10} \sqrt{\frac{x}{n}},$$

for $x \in R_0$ and $n \in N$ ($M_{10} = \text{const.} > 0$).

Theorem and 10 Theorem and 9 Corollary in our paper show that operators $A_n$, $n \in N$, give better degree of approximation of functions $f \in C_p$ and $f \in C^1_p$ than $S_n$.

References
