PROCEEDINGS OF SCIENCE

$\mathcal{O}(a^2)$ corrections to 1-loop matrix elements of 4-fermion operators with improved fermion/gluon actions

Martha Constantinou[a*], **Vittorio Lubicz**[b], **Haralambos Panagopoulos**[a], **Apostolos Skouroupathis**[a*], **Fotos Stylianou**[†a*]

[a] *Department of Physics, University of Cyprus*
P.O.Box 20537, Nicosia CY-1678, Cyprus
[b] *Dipartimento di Fisica, Università di Roma Tre and INFN, Sezione di Roma Tre,*
Via della Vasca Navale 84, I-00146 Roma, Italy
E-mail: marthac@ucy.ac.cy, lubicz@fis.uniroma3.it, haris@ucy.ac.cy, php4as01@ucy.ac.cy, fstyli01@ucy.ac.cy

We calculate the corrections to the amputated Green's functions of 4-fermion operators, in 1-loop Lattice Perturbation theory. The novel aspect of our calculations is that they are carried out to second order in the lattice spacing, $\mathcal{O}(a^2)$.

We employ the Wilson/clover action for massless fermions (also applicable for the twisted mass action in the chiral limit) and the Symanzik improved action for gluons. Our calculations have been carried out in a general covariant gauge. Results have been obtained for several popular choices of values for the Symanzik coefficients (Plaquette, Tree-level Symanzik, Iwasaki, TILW and DBW2 action).

We pay particular attention to $\Delta F = 2$ operators, both Parity Conserving and Parity Violating (F stands for flavour: S, C, B). We study the mixing pattern of these operators, to $\mathcal{O}(a^2)$, using the appropriate projectors. Our results for the corresponding renormalization matrices are given as a function of a large number of parameters: coupling constant, clover parameter, number of colors, lattice spacing, external momentum and gauge parameter.

The $\mathcal{O}(a^2)$ correction terms (along with our previous $\mathcal{O}(a^2)$ calculation of Z_Ψ) are essential ingredients for minimizing the lattice artifacts which are present in non-perturbative evaluations of renormalization constants with the RI′-MOM method.

A longer write-up of this work, including non-perturbative results, is in preparation together with members of the ETM Collaboration [1].

PACS: 11.15.Ha, 12.38.Gc, 11.10.Gh, 12.38.Bx

The XXVII International Symposium on Lattice Field Theory
July 26-31, 2009
Peking University, Beijing, China

*Work supported in part by the Research Promotion Foundation of Cyprus (Proposal Nr: TEXN/0308/17, ΕΝΙΣΧ/0506/17).

†Speaker.

PoS(LAT2009)260

1. Introduction

A number of flavour-changing processes are currently under study in Lattice simulations. Among the most common examples are the decay $K \to \pi\pi$ and K^0–$\bar{K}^0$ oscillations. From experimental evidence, we know that these weak processes violate the CP symmetry. In theory, the calculation of the amount of CP violation in K^0–$\bar{K}^0$ oscillations requires the knowledge of B_K.

The Kaon B_K parameter is obtained from the $\Delta S = 2$ weak matrix element:

$$B_K = \frac{\langle \bar{K}^0 | \hat{O}^{\Delta S=2} | K^0 \rangle}{\frac{8}{3} \langle \bar{K}^0 | \bar{s}\gamma_\mu d | 0 \rangle \langle 0 | \bar{s}\gamma_\mu d | K^0 \rangle}, \tag{1.1}$$

where s and d stand for strange and down quarks, and $\hat{O}^{\Delta S=2}$ is the effective 4-quark interaction renormalized operator, corresponding to the bare operator:

$$O^{\Delta S=2} = (\bar{s}\gamma_\mu^{\mathrm{L}} d)(\bar{s}\gamma_\mu^{\mathrm{L}} d), \qquad \gamma_\mu^{\mathrm{L}} = \gamma_\mu(\mathbb{1} - \gamma_5). \tag{1.2}$$

The above operator splits into parity-even and parity-odd parts; in standard notation: $O^{\Delta S=2} = O^{\Delta S=2}_{VV+AA} - O^{\Delta S=2}_{VA+AV}$. Since the above weak process is simulated in the framework of Lattice QCD, where Parity is a symmetry, the parity-odd part gives no contribution to the K^0–$\bar{K}^0$ matrix element. Thus, we conclude that B_K can be extracted from the correlator ($x_0 > 0$, $y_0 < 0$):

$$C_{KOK}(x,y) = \langle (\bar{d}\gamma_5 s)(x) \hat{O}^{\Delta S=2}_{VV+AA}(0) (\bar{d}\gamma_5 s)(y) \rangle, \qquad O^{\Delta S=2}_{VV+AA} = (\bar{s}\gamma_\mu d)(\bar{s}\gamma_\mu d) + (\bar{s}\gamma_\mu\gamma_5 d)(\bar{s}\gamma_\mu\gamma_5 d), \tag{1.3}$$

where $O^{\Delta S=2}_{VV+AA}$ is the bare operator and $\hat{O}^{\Delta S=2}_{VV+AA}$ is the respective renormalized operator.

In place of the operator in Eq. (1.3) it is advantageous to use a four-quark operator with a different flavour content (s, d, s', d'), and with $\Delta S = \Delta s + \Delta s' = 2$, namely [2]:

$$\mathcal{O}^{\Delta S=2}_{VV+AA} = (\bar{s}\gamma_\mu d)(\bar{s}'\gamma_\mu d') + (\bar{s}\gamma_\mu\gamma_5 d)(\bar{s}'\gamma_\mu\gamma_5 d') + (\bar{s}\gamma_\mu d')(\bar{s}'\gamma_\mu d) + (\bar{s}\gamma_\mu\gamma_5 d')(\bar{s}'\gamma_\mu\gamma_5 d), \tag{1.4}$$

where now the correlator is given by: $C_{K\mathcal{O}K'}(x,y) = \langle (\bar{d}\gamma_5 s)(x)\, 2\mathcal{O}^{\Delta S=2}_{VV+AA}(0) (\bar{d}'\gamma_5 s')(y) \rangle$. Making use of Wick's theorem one checks the equality: $C_{K\mathcal{O}K'}(x,y) = C_{KOK}(x,y)$, which means that both correlators contain the same physical information.

The aforementioned matrix elements are very sensitive to various systematic errors. A major issue facing Lattice Gauge Theory, since its early days, has been the reduction of effects induced by the finiteness of lattice spacing a, in order to better approach the elusive continuum limit.

In order to obtain reliable non-perturbative estimates of physical quantities (i.e. improving the accuracy of B_K) it is essential to keep under control the $\mathcal{O}(a)$ systematic errors in simulations or, additionally, reduce the lattice artifacts in numerical results. Such a reduction, regarding renormalization functions, can be achieved by subtracting appropriately the $\mathcal{O}(a^2)$ perturbative correction terms presented in this paper, from respective non-perturbative results.

In this paper we calculate the amputated Green's functions and the renormalization matrices of the complete basis of 20 four-fermion operators of dimension six which do not need power subtractions (i.e. mixing occurs only with other operators of equal dimensions). The calculations are carried out up to 1-loop in Lattice Perturbation theory and up to $\mathcal{O}(a^2)$ in lattice spacing. Our results are immediately applicable to other $\Delta F = 2$ processes of great phenomenological interest, such as $D - \bar{D}$ or $B - \bar{B}$ mixing. Let us also mention that in generic new physics models (i.e. beyond the standard model), the complete basis of 4-fermion operators contributes to neutral meson mixing amplitudes; this is the case for instance of SUSY models (see e.g. [3]).

2. Amputated Green's functions of 4-fermion $\Delta S = \Delta s + \Delta s' = 2$ operators.

In this work we evaluate, up to $\mathcal{O}(a^2)$, the 1-loop matrix element of the 4-fermion operators[1]:

$$\mathcal{O}_{XY} \equiv (\bar{s} X\, d)(\bar{s}' Y\, d') \equiv \sum_x \sum_{c,d} \sum_{k_1,k_2,k_3,k_4} \left(\bar{s}^c_{k_1}(x)\, X_{k_1k_2}\, d^c_{k_2}(x)\right)\left(\bar{s}'^d_{k_3}(x)\, Y_{k_3k_4}\, d'^d_{k_4}(x)\right) \tag{2.1}$$

$$\mathcal{O}^F_{XY} \equiv (\bar{s} X\, d')(\bar{s}' Y\, d) \equiv \sum_x \sum_{c,d} \sum_{k_1,k_2,k_3,k_4} \left(\bar{s}^c_{k_1}(x)\, X_{k_1k_2}\, d'^c_{k_2}(x)\right)\left(\bar{s}'^d_{k_3}(x)\, Y_{k_3k_4}\, d^d_{k_4}(x)\right) \tag{2.2}$$

with a generic initial state: $\bar{d}'^{a_4}_{i_4}(p_4)\, s'^{a_3}_{i_3}(p_3)|0\rangle$, and a generic final state: $\langle 0|\bar{d}^{a_2}_{i_2}(p_2)\, s^{a_1}_{i_1}(p_1)$. Spin indices are denoted by i, k, and color indices by a, c, d, while X and Y correspond to the following set of products of the Dirac matrices:

$$X, Y = \{\mathbb{1}, \gamma^5, \gamma_\mu, \gamma_\mu\gamma^5, \sigma_{\mu\nu}, \gamma^5\sigma_{\mu\nu}\} \equiv \{S, P, V, A, T, \tilde{T}\}; \qquad \sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]. \tag{2.3}$$

Our calculations are performed using massless fermions described by the Wilson/clover action. By taking $m_f = 0$, our results are identical also for the twisted mass action and the Osterwalder-Seiler action in the chiral limit (in the so called twisted mass basis). For gluons we employ a 3-parameter family of Symanzik improved actions, which comprises all common gluon actions (Plaquette, tree-level Symanzik, Iwasaki, DBW2, Lüscher-Weisz). Conventions and notations for the actions used, as well as algebraic manipulations involving the evaluation of 1-loop Feynman diagrams (up to $\mathcal{O}(a^2)$), are described in detail in Ref. [4].

To establish notation and normalization, let us first write the tree-level expression for the amputated Green's functions of the operators $\mathcal{O}_{XY}$ and $\mathcal{O}^F_{XY}$:

$$\Lambda^{XY}_{tree}(p_1,p_2,p_3,p_4,r_s,r_d,r_{s'},r_{d'})^{a_1a_2a_3a_4}_{i_1i_2i_3i_4} = X_{i_1i_2}\, Y_{i_3i_4}\, \delta_{a_1a_2}\, \delta_{a_3a_4}, \tag{2.4}$$

$$(\Lambda^F)^{XY}_{tree}(p_1,p_2,p_3,p_4,r_s,r_d,r_{s'},r_{d'})^{a_1a_2a_3a_4}_{i_1i_2i_3i_4} = -X_{i_1i_4}\, Y_{i_3i_2}\, \delta_{a_1a_4}\, \delta_{a_3a_2}, \tag{2.5}$$

where r is the Wilson parameter, one for each flavour.

We continue with the first quantum corrections. There are twelve 1-loop diagrams that enter our 4-fermion calculation, six for each operator $\mathcal{O}_{XY}$, $\mathcal{O}^F_{XY}$. The diagrams $d_1 - d_6$ corresponding to the operator $\mathcal{O}_{XY}$ are illustrated in Fig. 1. The other six diagrams, $d^F_1 - d^F_6$, involved in the Green's function of $\mathcal{O}^F_{XY}$ are similar to $d_1 - d_6$, and may be obtained from $d_1 - d_6$ by interchanging the fermionic fields d and d' along with their momenta, color and spin indices, and respective Wilson parameters.

The only diagrams that need to be calculated from first principles are d_1, d_2 and d_3, while the rest can be expressed in terms of the first three. In particular, the expressions for the amputated Green's functions $\Lambda^{XY}_{d_4} - \Lambda^{XY}_{d_6}$ can be obtained via the following relations:

$$\Lambda^{XY}_{d_4}(p_1,p_2,p_3,p_4,r_s,r_d,r_{s'},r_{d'})^{a_1a_2a_3a_4}_{i_1i_2i_3i_4} = \left(\Lambda^{XY}_{d_1}(-p_2,-p_1,-p_4,-p_3,r_d,r_s,r_{d'},r_{s'})^{a_2a_1a_4a_3}_{i_2i_1i_4i_3}\right)^\star, \tag{2.6}$$

$$\Lambda^{XY}_{d_5}(p_1,p_2,p_3,p_4,r_s,r_d,r_{s'},r_{d'})^{a_1a_2a_3a_4}_{i_1i_2i_3i_4} = \Lambda^{YX}_{d_2}(p_3,p_4,p_1,p_2,r_{s'},r_{d'},r_s,r_d)^{a_3a_4a_1a_2}_{i_3i_4i_1i_2}, \tag{2.7}$$

$$\Lambda^{XY}_{d_6}(p_1,p_2,p_3,p_4,r_s,r_d,r_{s'},r_{d'})^{a_1a_2a_3a_4}_{i_1i_2i_3i_4} = \Lambda^{YX}_{d_3}(p_3,p_4,p_1,p_2,r_{s'},r_{d'},r_s,r_d)^{a_3a_4a_1a_2}_{i_3i_4i_1i_2}. \tag{2.8}$$

Once we have constructed $\Lambda^{XY}_{d_4} - \Lambda^{XY}_{d_6}$ we can use relation:

$$(\Lambda^F)^{XY}_{d_j}(p_1,p_2,p_3,p_4,r_s,r_d,r_{s'},r_{d'})^{a_1a_2a_3a_4}_{i_1i_2i_3i_4} = -\Lambda^{XY}_{d_j}(p_1,p_4,p_3,p_2,r_s,r_{d'},r_{s'},r_d)^{a_1a_4a_3a_2}_{i_1i_4i_3i_2}, \tag{2.9}$$

[1] The superscript letter F stands for Fierz.

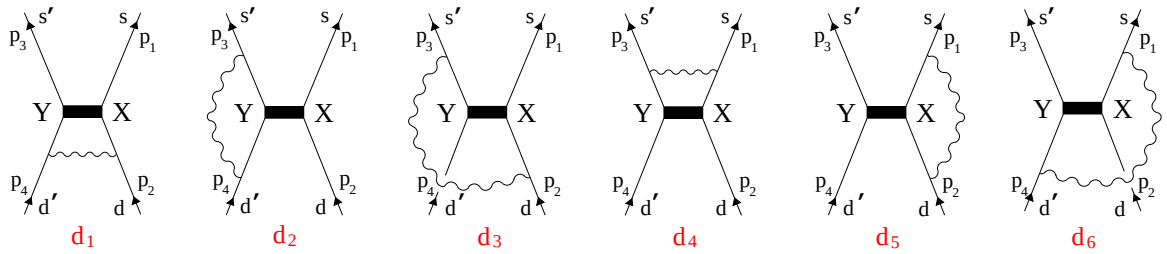

Figure 1: 1-loop diagrams contributing to the amputated Green's function of the 4-fermi operator $\mathcal{O}^{XY}$. Wavy (solid) lines represent gluons (fermions).

to derive the expressions for $(\Lambda^F)^{XY}_{d_1} - (\Lambda^F)^{XY}_{d_6}$. From the amputated Green's functions for all twelve diagrams we can write down the total 1-loop expressions for the operators $\mathcal{O}_{XY}$ and $\mathcal{O}^F_{XY}$:

$$\Lambda^{XY}_{1-loop} = \sum_{j=1}^{6} \Lambda^{XY}_{d_j}, \qquad (\Lambda^F)^{XY}_{1-loop} = \sum_{j=1}^{6} (\Lambda^F)^{XY}_{d_j}. \tag{2.10}$$

In our algebraic expressions for the 1-loop amputated Green's functions $\Lambda^{XY}_{d_1}$, $\Lambda^{XY}_{d_2}$ and $\Lambda^{XY}_{d_3}$ we kept the Wilson parameters for each quark field distinct, that is: r_s, r_d, $r_{s'}$, $r_{d'}$ for the quark fields s, d, s' and d' respectively. For the required numerical integration of the algebraic expressions of the integrands, corresponding to each Feynman diagram, we are forced to choose the square of the value for each r parameter. As in all present day simulations, we set:

$$r_s^2 = r_d^2 = r_{s'}^2 = r_{d'}^2 \equiv 1. \tag{2.11}$$

Concerning the external momenta p_i (shown explicitly in Fig. 1) we have chosen to evaluate the amputated Green's functions at the renormalization point:

$$p_1 = p_2 = p_3 = p_4 \equiv p. \tag{2.12}$$

It is easy and not time consuming to repeat the calculations for other choices of Wilson parameters and for other renormalization prescriptions. The final 1-loop expressions for $\Lambda^{XY}_{d_1}$, $\Lambda^{XY}_{d_2}$ and $\Lambda^{XY}_{d_3}$, up to $\mathcal{O}(a^2)$, are obtained as a function of: the coupling constant g, clover parameter c_{SW}, number of colors N_c, lattice spacing a, external momentum p and gauge parameter λ.

The crucial point of our calculation is the correct extraction of the full $\mathcal{O}(a^2)$ dependence from loop integrands with strong IR divergences (convergent only beyond 6 dimensions). The singularities are isolated using the procedure explained in Ref. [4]. In order to reduce the number of strong IR divergent integrals, appearing in diagram d_1, we have inserted the identity below into selected 3-point functions:

$$1 = \frac{1}{\widehat{ap}^2}\left(\widehat{k+ap}^{\,2} + \widehat{k-ap}^{\,2} - 2\hat{k}^2 + 16\sum_\sigma \sin(k_\sigma)^2 \sin(ap_\sigma)^2\right), \tag{2.13}$$

where $\hat{q}^2 = 4\sum_\mu \sin^2(\frac{q_\mu}{2})$ and $k(p)$ is the loop (external) momentum. The common factor in Eq. (2.13) can be treated by Taylor expansion. For our calculations it was necessary only to $\mathcal{O}(a^0)$:

$$\frac{1}{\widehat{ap}^2} = \frac{1}{a^2\,p^2} + \frac{\sum_\sigma p_\sigma^4}{(p^2)^2} + \mathcal{O}(a^2\,p^2). \tag{2.14}$$

Here we present one of the four integrals with strong IR divergences that enter in this calculation:

$$\int_{-\pi}^{\pi}\frac{d^4k}{(2\pi)^4}\frac{\sin(k_\mu)\sin(k_\nu)}{\hat{k}^2\widehat{k+ap}^2\widehat{k-ap}^2}=\delta_{\mu\nu}\Big[0.002457072288-\frac{\ln(a^2p^2)}{64\pi^2}+a^2p^2\Big(0.00055270353(6)-\frac{\ln(a^2p^2)}{512\pi^2}\Big)$$
$$-a^2p_\mu^2\Big(0.0001282022(1)+\frac{\ln(a^2p^2)}{768\pi^2}\Big)+0.000157122310\,a^2\frac{\sum_\sigma p_\sigma^4}{p^2}\Big]+a^2p_\mu p_\nu\Big[0.001870841540\frac{1}{a^2p^2}$$
$$-0.00029731225(4)+\frac{\ln(a^2p^2)}{768\pi^2}-0.000047949674\frac{(p_\mu^2+p_\nu^2)}{p^2}+0.000268598599\frac{\sum_\sigma p_\sigma^4}{(p^2)^2}\Big]+\mathcal{O}(a^4p^4).$$

The results for the other three integrals can be found in Ref. [4]. Integrands with simple IR divergences (convergent beyond 4 dimensions) can be handled by well-known techniques.

Due to lack of space we present only the results for $\Lambda_{d_1}^{XY}$ and for the special choices: $c_{SW}=0$, $\lambda=0$ (Landau Gauge), $r_s=r_d=r_{s'}=r_{d'}=1$, and tree-level Symanzik action:

$$\Lambda_{d_1}^{XY}(p)^{a_1a_2a_3a_4}_{i_1i_2i_3i_4}=\frac{g^2}{16\pi^2}\left(\delta_{a_1a_4}\delta_{a_3a_2}-\frac{\delta_{a_1a_2}\delta_{a_3a_4}}{N_c}\right)\times\Big\{X_{i_1i_2}Y_{i_3i_4}\Big[-\frac{1}{2}\ln(a^2p^2)-0.05294139(2)\Big]$$
$$+\sum_\mu(X\gamma^\mu)_{i_1i_2}(Y\gamma^\mu)_{i_3i_4}\left[-0.507914049(6)\right]+\sum_{\mu,\nu}(X\gamma^\mu\gamma^\nu)_{i_1i_2}(Y\gamma^\mu\gamma^\nu)_{i_3i_4}\Big[\frac{1}{8}\ln(a^2p^2)+0.0185984988(9)\Big]$$
$$+\sum_{\mu,\nu,\rho}(X\gamma^\mu\gamma^\rho)_{i_1i_2}(Y\gamma^\nu\gamma^\rho)_{i_3i_4}\Big[0.3977157268533\frac{p_\mu p_\nu}{p^2}\Big]+a(\Lambda_{\mathcal{O}(a^1)})_{d_1}^{XY}+a^2(\Lambda_{\mathcal{O}(a^2)})_{d_1}^{XY}\Big\},\qquad(2.15)$$

where:

$$(\Lambda_{\mathcal{O}(a^1)})_{d_1}^{XY}=\sum_\mu\Big((X\gamma^\mu)_{i_1i_2}Y_{i_3i_4}+X_{i_1i_2}(Y\gamma^\mu)_{i_3i_4}\Big)\times\Big[ip_\mu\Big(-\frac{1}{4}\ln(a^2p^2)+0.09460083(1)\Big)\Big]$$
$$+\sum_{\mu,\nu}\Big((X\gamma^\mu\gamma^\nu)_{i_1i_2}(Y_{i_3i_4}\gamma^\nu)+(X\gamma^\nu)_{i_1i_2}(Y\gamma^\mu\gamma^\nu)_{i_3i_4}\Big)\times\Big[ip_\mu\Big(\frac{1}{16}\ln(a^2p^2)+0.1692905881(6)\Big)\Big],\qquad(2.16)$$

and:

$$(\Lambda_{\mathcal{O}(a^2)})_{d_1}^{XY}=X_{i_1i_2}Y_{i_3i_4}\Big[p^2\Big(-\frac{17}{72}\ln(a^2p^2)+1.32362250(4)\Big)+0.06213649(4)\frac{\sum_\sigma p_\sigma^4}{p^2}\Big]$$
$$+\sum_\mu(X\gamma^\mu)_{i_1i_2}(Y\gamma^\mu)_{i_3i_4}\Big[p^2\Big(-\frac{7}{48}\ln(a^2p^2)+0.059895142(8)\Big)+1.01694823(2)p_\mu^2\Big]$$
$$+\sum_{\mu,\nu}\Big((X\gamma^\mu\gamma^\nu)_{i_1i_2}Y_{i_3i_4}+X_{i_1i_2}(Y\gamma^\mu\gamma^\nu)_{i_3i_4}\Big)\times\Big[0.00592406(2)\frac{p_\nu p_\mu^3}{p^2}\Big]$$
$$+\sum_{\mu,\nu}(X\gamma^\mu)_{i_1i_2}(Y\gamma^\nu)_{i_3i_4}\Big[p_\mu p_\nu\Big(-\frac{1}{6}\ln(a^2p^2)-0.19915360(1)\Big)\Big]$$
$$+\sum_{\mu,\nu}(X\gamma^\mu\gamma^\nu)_{i_1i_2}(Y\gamma^\mu\gamma^\nu)_{i_3i_4}\Big[p^2\Big(\frac{7}{240}\ln(a^2p^2)-0.089628048(6)\Big)-0.048180850\frac{\sum_\sigma p_\sigma^4}{p^2}$$
$$+p_\mu^2\Big(-\frac{29}{180}\ln(a^2p^2)+0.16608907(3)\Big)\Big]$$
$$+\sum_{\mu,\nu,\rho}(X\gamma^\mu\gamma^\rho)_{i_1i_2}(Y\gamma^\nu\gamma^\rho)_{i_3i_4}\Big[p_\mu p_\nu\Big(\frac{41}{360}\ln(a^2p^2)-0.21865900(2)+0.140961390\frac{\sum_\sigma p_\sigma^4}{p^2}\Big)$$
$$-0.110138790\frac{(p_\mu^3p_\nu+p_\mu p_\nu^3)}{p^2}-0.477634781(8)\frac{p_\mu p_\nu p_\rho^2}{p^2}\Big].\qquad(2.17)$$

Similar expressions exist for $\Lambda_{d_2}^{XY}$ and $\Lambda_{d_3}^{XY}$.

3. Mixing and Renormalization of $\mathcal{O}_{XY}$ and $\mathcal{O}^F_{XY}$ on the lattice.

The matrix element $\langle \bar{K}^0 | O^{\Delta S=2}_{VV+AA} | K^0 \rangle$ is very sensitive to various systematic errors. The main roots of this problem are: **a)** $\mathcal{O}(a)$ systematic errors due to numerical integration, **b)** the operator $O^{\Delta S=2}_{VV+AA}$ mixes with other 4-fermion $\Delta S = 2$ operators of dimension six. Mixing with operators of lower dimensionality is impossible because there is no candidate $\Delta S = 2$ operator.

In order to address these problems we have calculated the mixing pattern (renormalization matrices) of the Parity Conserving and Parity Violating 4-fermion $\Delta S = 2$ operators (defined below), by using the amputated Green's functions obtained in the previous section. A more extensive theoretical background and non-perturbative results, concerning renormalization matrices of 4-fermion operators, can be found in Ref. [5] (see also [2, 6, 7]). Next we summarize all important relations from Ref. [5] needed for the present calculation.

One can construct a complete basis of 20 independent operators which have the symmetries of the generic QCD Wilson lattice action (Parity P, Charge conjugation C, Flavour exchange symmetry $S{\equiv}(d \leftrightarrow d')$, Flavour Switching symmetries $S'{\equiv}(s \leftrightarrow d, s' \leftrightarrow d')$ and $S''{\equiv}(s \leftrightarrow d', d \leftrightarrow s')$), with 4 degenerate quarks. This basis can be decomposed into smaller independent bases according to the discrete symmetries P, S, CPS', CPS''. Following the notation of Ref. [5] we have 10 Parity Conserving operators, Q, ($P{=}+1, S{=}\pm1$) and 10 Parity Violating operators, $\mathcal{Q}$, ($P{=}-1, S{=}\pm1$):

$$
\begin{cases}
Q_1^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VV} \pm \mathcal{O}^F_{VV}\right] + \frac{1}{2}\left[\mathcal{O}_{AA} \pm \mathcal{O}^F_{AA}\right], \\
Q_2^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VV} \pm \mathcal{O}^F_{VV}\right] - \frac{1}{2}\left[\mathcal{O}_{AA} \pm \mathcal{O}^F_{AA}\right], \\
Q_3^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{SS} \pm \mathcal{O}^F_{SS}\right] - \frac{1}{2}\left[\mathcal{O}_{PP} \pm \mathcal{O}^F_{PP}\right], \\
Q_4^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{SS} \pm \mathcal{O}^F_{SS}\right] + \frac{1}{2}\left[\mathcal{O}_{PP} \pm \mathcal{O}^F_{PP}\right], \\
Q_5^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{TT} \pm \mathcal{O}^F_{TT}\right],
\end{cases}
\qquad
\begin{cases}
\mathcal{Q}_1^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VA} \pm \mathcal{O}^F_{VA}\right] + \frac{1}{2}\left[\mathcal{O}_{AV} \pm \mathcal{O}^F_{AV}\right], \\
\mathcal{Q}_2^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{VA} \pm \mathcal{O}^F_{VA}\right] - \frac{1}{2}\left[\mathcal{O}_{AV} \pm \mathcal{O}^F_{AV}\right], \\
\mathcal{Q}_3^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{PS} \pm \mathcal{O}^F_{PS}\right] - \frac{1}{2}\left[\mathcal{O}_{SP} \pm \mathcal{O}^F_{SP}\right], \\
\mathcal{Q}_4^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{PS} \pm \mathcal{O}^F_{PS}\right] + \frac{1}{2}\left[\mathcal{O}_{SP} \pm \mathcal{O}^F_{SP}\right], \\
\mathcal{Q}_5^{S=\pm1} \equiv \frac{1}{2}\left[\mathcal{O}_{T\tilde{T}} \pm \mathcal{O}^F_{T\tilde{T}}\right].
\end{cases}
$$

Summation over all independent Lorentz indices (if any), of the Dirac matrices, is implied. The operators shown above are grouped together according to their mixing pattern. This implies that the renormalization matrices $Z^{S=\pm1}$ ($\mathcal{Z}^{S=\pm1}$), for the Parity Conserving (Violating) operators, have the form:

$$
Z^{S=\pm1} = \begin{pmatrix}
Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} \\
Z_{21} & Z_{22} & Z_{23} & Z_{24} & Z_{25} \\
Z_{31} & Z_{32} & Z_{33} & Z_{34} & Z_{35} \\
Z_{41} & Z_{42} & Z_{43} & Z_{44} & Z_{45} \\
Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55}
\end{pmatrix}^{S=\pm1}, \quad
\mathcal{Z}^{S=\pm1} = \begin{pmatrix}
\mathcal{Z}_{11} & 0 & 0 & 0 & 0 \\
0 & \mathcal{Z}_{22} & \mathcal{Z}_{23} & 0 & 0 \\
0 & \mathcal{Z}_{32} & \mathcal{Z}_{33} & 0 & 0 \\
0 & 0 & 0 & \mathcal{Z}_{44} & \mathcal{Z}_{45} \\
0 & 0 & 0 & \mathcal{Z}_{54} & \mathcal{Z}_{55}
\end{pmatrix}^{S=\pm1}. \tag{3.1}
$$

Now the renormalized Parity Conserving (Violating) operators, $\hat{Q}^{S=\pm1}$ ($\hat{\mathcal{Q}}^{S=\pm1}$), are defined via the equations:

$$
\hat{Q}_l^{S=\pm1} = Z_{lm}^{S=\pm1} \cdot Q_m^{S=\pm1}, \quad \hat{\mathcal{Q}}_l^{S=\pm1} = \mathcal{Z}_{lm}^{S=\pm1} \cdot \mathcal{Q}_m^{S=\pm1}, \tag{3.2}
$$

where $l, m = 1, \ldots, 5$ (a sum over m is implied). The renormalized amputated Green's functions $\hat{L}^{S=\pm1}$ ($\hat{\mathcal{L}}^{S=\pm1}$) corresponding to $Q^{S=\pm1}$ ($\mathcal{Q}^{S=\pm1}$), are given in terms of their bare counterparts $L^{S=\pm1}$ ($\mathcal{L}^{S=\pm1}$) through:

$$
\hat{L}_l^{S=\pm1} = Z_\Psi^{-2} Z_{lm}^{S=\pm1} \cdot L_m^{S=\pm1}, \quad \hat{\mathcal{L}}_l^{S=\pm1} = Z_\Psi^{-2}\, \mathcal{Z}_{lm}^{S=\pm1} \cdot \mathcal{L}_m^{S=\pm1}, \tag{3.3}
$$

where Z_Ψ is the quark field renormalization constant.

In order to calculate the renormalization matrices $Z^{S=\pm1}$ ($\mathscr{Z}^{S=\pm1}$), we make use of the appropriate Parity Conserving (Violating) Projectors $P^{S=\pm1}$ ($\mathscr{P}^{S=\pm1}$):

$$
\begin{aligned}
P_1^{S=\pm1} &\equiv +\frac{\Pi_{VV}+\Pi_{AA}}{64N_c(N_c\pm1)}, & \mathscr{P}_1^{S=\pm1} &\equiv -\frac{\Pi_{VA}+\Pi_{AV}}{64N_c(N_c\pm1)},\\
P_2^{S=\pm1} &\equiv +\frac{\Pi_{VV}-\Pi_{AA}}{64(N_c^2-1)} \pm \frac{\Pi_{SS}-\Pi_{PP}}{32N_c(N_c^2-1)}, & \mathscr{P}_2^{S=\pm1} &\equiv -\frac{\Pi_{VA}-\Pi_{AV}}{64(N_c^2-1)} \mp \frac{\Pi_{SP}-\Pi_{PS}}{32N_c(N_c^2-1)},\\
P_3^{S=\pm1} &\equiv \pm\frac{\Pi_{VV}-\Pi_{AA}}{32N_c(N_c^2-1)} + \frac{\Pi_{SS}-\Pi_{PP}}{16(N_c^2-1)}, & \mathscr{P}_3^{S=\pm1} &\equiv \mp\frac{\Pi_{VA}-\Pi_{AV}}{32N_c(N_c^2-1)} - \frac{\Pi_{SP}-\Pi_{PS}}{16(N_c^2-1)},\\
P_4^{S=\pm1} &\equiv +\frac{\Pi_{SS}+\Pi_{PP}}{\frac{32N_c(N_c^2-1)}{2N_c\pm1}} \mp \frac{\Pi_{TT}}{32N_c(N_c^2-1)}, & \mathscr{P}_4^{S=\pm1} &\equiv +\frac{\Pi_{SP}+\Pi_{PS}}{\frac{32N_c(N_c^2-1)}{2N_c\pm1}} \mp \frac{\Pi_{T\tilde{T}}}{32N_c(N_c^2-1)},\\
P_5^{S=\pm1} &\equiv \mp\frac{\Pi_{SS}+\Pi_{PP}}{32N_c(N_c^2-1)} + \frac{\Pi_{TT}}{\frac{96N_c(N_c^2-1)}{2N_c\mp1}}, & \mathscr{P}_5^{S=\pm1} &\equiv \mp\frac{\Pi_{SP}+\Pi_{PS}}{32N_c(N_c^2-1)} + \frac{\Pi_{T\tilde{T}}}{\frac{96N_c(N_c^2-1)}{2N_c\mp1}},
\end{aligned}
$$

where $\Pi_{XY} \equiv (X_{i_2 i_1} \otimes Y_{i_4 i_3})\delta_{a_2 a_1}\delta_{a_4 a_3}$. Again, summation is implied over all independent Lorentz indices (if any) of the Dirac matrices) The above Projectors are chosen to obey the following orthogonality conditions:

$$
Tr(P_l^{S=\pm1}\cdot L_{m\,(tree)}^{S=\pm1}) = \delta_{lm}, \quad Tr(\mathscr{P}_l^{S=\pm1}\cdot \mathscr{L}_{m\,(tree)}^{S=\pm1}) = \delta_{lm}, \tag{3.4}
$$

where the trace is taken over spin and color indices, and $L_{(tree)}^{S=\pm1}$, $\mathscr{L}_{(tree)}^{S=\pm1}$ are the tree-level amputated Green's functions of the operators $Q^{S=\pm1}$, $\mathscr{Q}^{S=\pm1}$ respectively.

We impose the renormalization conditions:

$$
Tr(P_l^{S=\pm1}\cdot \hat{L}_m^{S=\pm1}) = \delta_{lm}, \quad Tr(\mathscr{P}_l^{S=\pm1}\cdot \hat{\mathscr{L}}_m^{S=\pm1}) = \delta_{lm}. \tag{3.5}
$$

By inserting Eqs. (3.3) in the above relations, we obtain the renormalization matrices $Z^{S=\pm1}$, $\mathscr{Z}^{S=\pm1}$ in terms of known quantities:

$$
Z^{S=\pm1} = Z_\Psi^2\left[\left(D^{S=\pm1}\right)^T\right]^{-1}, \quad \mathscr{Z}^{S=\pm1} = Z_\Psi^2\left[\left(\mathscr{D}^{S=\pm1}\right)^T\right]^{-1}, \tag{3.6}
$$

where:

$$
D_{lm}^{S=\pm1} \equiv Tr(P_l^{S=\pm1}\cdot L_m^{S=\pm1}), \quad \mathscr{D}_{lm}^{S=\pm1} \equiv Tr(\mathscr{P}_l^{S=\pm1}\cdot \mathscr{L}_m^{S=\pm1}). \tag{3.7}
$$

Note that $D^{S=\pm1}$ and $\mathscr{D}^{S=\pm1}$ have the same matrix structure as $Z^{S=\pm1}$ and $\mathscr{Z}^{S=\pm1}$ respectively.

Due to lack of space we provide only the matrix $\mathscr{D}^{S=+1}$ (Parity Violating $P=-1$, Flavour exchange symmetry $S=+1$) for the special choices: $c_{SW}=0$, $\lambda=0$ (Landau Gauge), $r_s = r_d = r_{s'} = r_{d'} = 1$, $N_c = 3$, and tree-level Symanzik action:

$$
\mathscr{D}^{S=+1} = \begin{pmatrix}
1+\mathscr{D}_{11} & 0 & 0 & 0 & 0\\
0 & 1+\mathscr{D}_{22} & \mathscr{D}_{23} & 0 & 0\\
0 & \mathscr{D}_{32} & 1+\mathscr{D}_{33} & 0 & 0\\
0 & 0 & 0 & 1+\mathscr{D}_{44} & \mathscr{D}_{45}\\
0 & 0 & 0 & \mathscr{D}_{54} & 1+\mathscr{D}_{55}
\end{pmatrix}^{S=+1} \tag{3.8}
$$

where:

$$\mathscr{D}_{11} = +\frac{g^2}{16\pi^2}\left[7.607190(2) + 2\ln(a^2p^2) + \left(2.642227(3) - \frac{19}{18}\ln(a^2p^2)\right)a^2p^2 - 2.79899088(3)\,a^2\frac{\sum_\sigma p_\sigma^4}{p^2}\right],$$

$$\mathscr{D}_{22} = +\frac{g^2}{16\pi^2}\left[2.299519(2) + \ln(a^2p^2) + \left(1.846794(4) - \frac{25}{36}\ln(a^2p^2)\right)a^2p^2 - 0.87361421(2)\,a^2\frac{\sum_\sigma p_\sigma^4}{p^2}\right],$$

$$\mathscr{D}_{23} = -\frac{g^2}{16\pi^2}\left[1.1931473(4) + \left(1.4685426(7) - \frac{1}{3}\ln(a^2p^2)\right)a^2p^2 - 0.89270364(3)\,a^2\frac{\sum_\sigma p_\sigma^4}{p^2}\right],$$

$$\mathscr{D}_{32} = -\frac{g^2}{16\pi^2}\left[10.970216(2) - 6\ln(a^2p^2) + \left(6.711307(3) - \frac{7}{6}\ln(a^2p^2)\right)a^2p^2 - 1.7027590(1)\,a^2\frac{\sum_\sigma p_\sigma^4}{p^2}\right],$$

$$\mathscr{D}_{33} = +\frac{g^2}{16\pi^2}\left[11.595959(2) - 8\ln(a^2p^2) + \left(3.102499(4) - \frac{4}{9}\ln(a^2p^2)\right)a^2p^2 + 1.92846914(2)\,a^2\frac{\sum_\sigma p_\sigma^4}{p^2}\right],$$

$$\mathscr{D}_{44} = +\frac{g^2}{16\pi^2}\left[10.269734(3) - 5\ln(a^2p^2) - \left(0.286209(5) - \frac{1}{18}\ln(a^2p^2)\right)a^2p^2 + 2.01567490(4)\,a^2\frac{\sum_\sigma p_\sigma^4}{p^2}\right],$$

$$\mathscr{D}_{45} = +\frac{g^2}{16\pi^2}\left[9.732710(2) - 5\ln(a^2p^2) + \left(4.602710(4) - \frac{7}{9}\ln(a^2p^2)\right)a^2p^2 - 1.07855465(9)\,a^2\frac{\sum_\sigma p_\sigma^4}{p^2}\right],$$

$$\mathscr{D}_{54} = +\frac{g^2}{16\pi^2}\left[1.1783609(6) + \frac{1}{3}\ln(a^2p^2) + \left(1.255191(1) - \frac{17}{54}\ln(a^2p^2)\right)a^2p^2 - 0.98220341(2)\,a^2\frac{\sum_\sigma p_\sigma^4}{p^2}\right],$$

$$\mathscr{D}_{55} = +\frac{g^2}{16\pi^2}\left[2.297078(2) + \frac{17}{3}\ln(a^2p^2) + \left(0.828945(3) - \frac{23}{27}\ln(a^2p^2)\right)a^2p^2 - 3.06215785(3)\,a^2\frac{\sum_\sigma p_\sigma^4}{p^2}\right].$$

In order to obtain Z_Ψ for a given renormalization prescription, one must make use of the inverse fermion propagator, S^{-1}, calculated (up to 1-loop and up to $\mathcal{O}(a^2)$ for massless Wilson/clover fermions and Symanzik gluons) in Ref. [4].

References

[1] M. Constantinou, V. Lubicz, H. Panagopoulos, A. Skouroupathis, F. Stylianou and members of the ETM Collaboration, in preparation.

[2] R. Frezzotti and G.C. Rossi, *Chirally improving Wilson fermions 2. Four-quark operators*, *JHEP* **10** (2004) 070. [hep-lat/0407002]

[3] M. Ciuchini, L. Conti, A. Donini, E. Franco, V. Gimenez, L. Giusti, V. Lubicz, G. Martinelli, A. Masiero, I. Scimemi, L. Silvestrini, M. Talevi and A. Vladika, ΔM_K *and* ε_K *in SUSY at the Next-to-Leading order*, *JHEP* **10** (1998) 008. [hep-ph/9808328]

[4] M. Constantinou, V. Lubicz, H. Panagopoulos, F. Stylianou, $\mathcal{O}(a^2)$ *corrections to the one-loop propagator and bilinears of clover fermions with Symanzik improved gluons*, *JHEP*, to appear [hep-lat/0907.0381].

[5] A. Donini, V. Giménez, G. Martinelli, M. Talevi and A. Vladikas, *Non-Perturbative Renormalization of Lattice Four-Fermion Operators without Power Subtractions*, *Eur. Phys. J.* **C10** (1999) 121. [hep-lat/9902030]

[6] D. Becirevic, V. Giménez, V. Lubicz, G. Martinelli, M. Papinutto and J. Reyes, *Renormalization Constants of Quark Operators for the Non-Perturbatively Improved Wilson Action*, *JHEP* **08** (2004) 022. [hep-lat/0401033]

[7] G. Martinelli, C. Pittori, C.T. Sachrajda, M. Testa and A. Vladikas, *A General Method for Non-Perturbative Renormalization of Lattice Operators*, *Nucl. Phys.* **B445** (1995) 81. [hep-lat/9411010]