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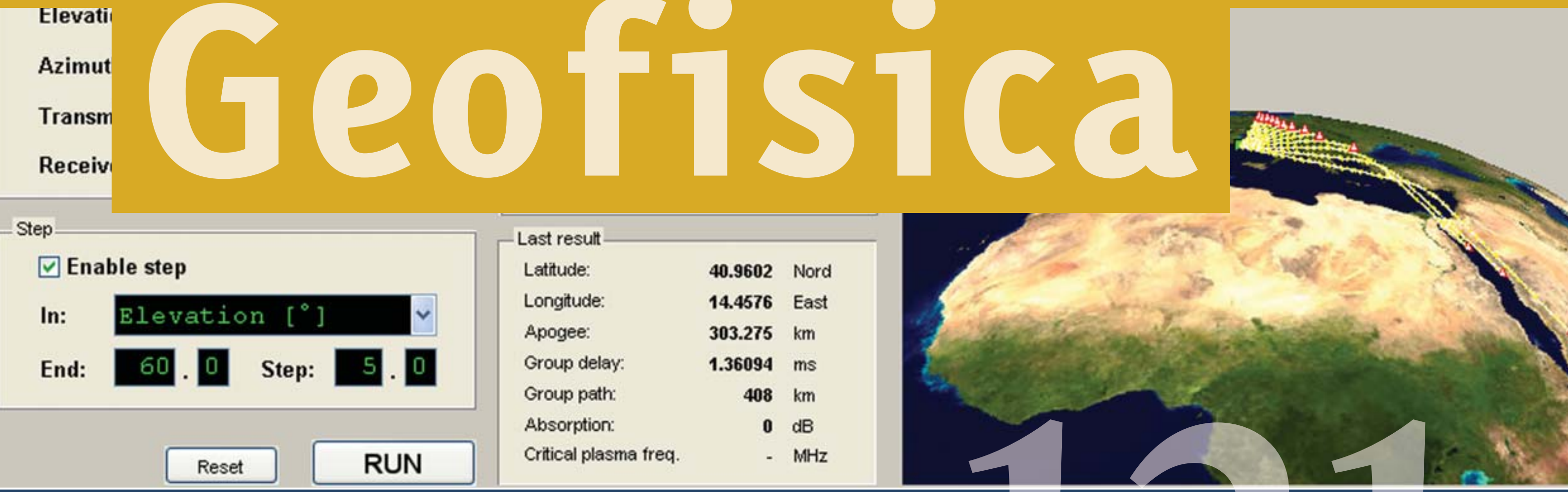

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Ray theory formulation and ray tracing method. Application in ionospheric propagation

Formulazione della *ray theory* e metodo del *ray tracing*. Applicazione nella propagazione ionosferica

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In copertina The graphical user interface of a ray-tracing program (particolare)

Ray theory formulation and ray tracing method. Application in ionospheric propagation

This work will lead to ray theory and ray tracing formulation. To deal with this problem the theory of classical geometrical optics is presented, and applications to ionospheric propagation will be described. This provides useful theoretical basis for scientists involved in research on radio propagation in inhomogeneous anisotropic media, especially in a magneto-plasma. Application in high frequencies (HF) radio propagation, radio communication, over-the-horizon-radar (OTHR) coordinate registration and related homing techniques for direction finding of HF wave, all rely on ray tracing computational algorithm. In this theory the formulation of the canonical, or Hamiltonian, equations related to the ray, which allow calculating the wave direction of propagation in a continuous, inhomogeneous and anisotropic medium with minor gradient, will be dealt. At least six Hamilton's equations will be written both in Cartesian and spherical coordinates in the simplest way. These will be achieved by introducing the refractive surface index equations and the ray surface equations in an appropriate free-dimensional space. By the combination of these equations even the Fermat's principle will be derived to give more generality to the formulation of ray theory. It will be shown that the canonical equations are dependent on a constant quantity H and the Cartesian coordinates and components of wave vector along the ray path. These quantities respectively indicated as $r_i(\tau)$, $p_i(\tau)$ are dependent on the parameter τ, that must increase monotonically along the path. Effectively, the procedure described above is the ray tracing formulation. In ray tracing computational techniques, the most convenient Hamiltonian describing the medium can be adopted, and the simplest way to choose properly H will be discussed. Finally, a system of equations, which can be numerically solved, is generated.

Questo lavoro descrive la teoria del raggio d'onda indicata, nella letteratura internazionale, come "ray theory" *e la tecnica del ray tracing. Per affrontare questo problema, viene presentata la teoria dell'ottica classica relativa al raggio d'onda e viene descritta l'applicazione nella propagazione ionosferica. Così vengono fornite utili basi teoriche per i ricercatori coinvolti nelle ricerche della radio propagazione nei mezzi disomogenei e anisotropi, quale è il magneto-plasma ionosferico. Le applicazioni nella radio propagazione ad alta frequenza (HF), nelle radio comunicazioni, nelle tecniche radar* over-the-horizon-radar *(OTHR) per la tecnica di* coordinate registration, *nella relativa tecnica di* homing *per la* direction finding *dell'onda HF si basano tutte su algoritmi di* ray tracing *numerici. Questa teoria tratta la formulazione delle equazioni canoniche o Hamiltoniane relative al raggio d'onda che permettono di calcolare la direzione di propagazione nei mezzi anisotropi, continui o con piccoli gradienti di disomogeneità. Sono necessarie almeno sei equazioni che possono essere scritte in coordinate cartesiane o sferiche nella maniera più semplice. Queste possono essere ottenute introducendo l'equazione della superficie degli indici di fase e della superficie dei raggi, in un opportuno spazio adimensionale. Dalla combinazione di quest'ultime equazioni può essere derivato anche il principio di Fermat per conferire più generalità alla* ray theory. *Viene dimostrato che le equazioni canoniche dipendono da una quantità costante* H, *dalle coordinate cartesiane e dalle componenti del vettore d'onda lungo l'intero* ray path. *Queste quantità, rispettivamente indicate come* $r_i(\tau)$, $p_i(\tau)$, *sono dipendenti dal parametro* τ *che è una quantità che cresce monotonamente lungo il* ray path. *Di fatto, la procedura sopra descritta contiene il concetto stesso di* ray tracing. *Nella tecnica computazionale del* ray tracing, *può essere adottata la Hamiltoniana più conveniente per descrivere il mezzo e viene discusso il modo più semplice per scegliere* H *opportunamente. Infine, viene generato un sistema di equazioni che può essere numericamente risolto.*

Introduction

This paper describes the ray propagation of electromagnetic waves, which is generally referred to in international literature as "ray theory". This theory often includes the method of ray-tracing formulation, which typically applies numerical analysis techniques. The theory is based on some fundamental concepts and on the approximate solution of partial differential equations. These are the well-known WKB approximations, an acronym from Wentzel, Kramers, and Brillouin the three authors who independently proposed the technique in 1926. This approximation leads to the eikonal equation, a phase integral, which was used to define the canonical equations of the ray [Budden, 1988].

The theory described here leads to the formulation of the canonical, or Hamiltonian, equations related to the ray of the wave. It deals with the propagation of the ray in a continuous, inhomogeneous and anisotropic medium with minor gradient. When the discontinuities of the medium are high, the calculation of the ray path using this technique is no longer valid and Snell's law should be applied at the interface surface discontinuities.
The ray theory formulation, therefore, allows definition of a number of differential equations equal to the variables, which affect the three canonical equations in generalized coordinates (q_1, q_2, q_3) and the 3 components of the wave vector that can be assimilated to the moments (p_1, p_2, p_3). If a non time-variant medium is considered, six equations are sufficient to describe the phenomena. Conversely, if it is assumed that the medium is time-variant, then two more equations need to be added to the group of six equations, since the Hamiltonian also depends on time and frequency. In the initial paragraphs a correct but less formal method will be followed to arrive at the formulation of the canonical equations. This is achieved by introducing the equation of the refractive surface index $G(x, y, z, p_x, p_y, p_z)=1$ which depends on both the spatial coordinate, and components of the refractive index expressed in a special "space index". The latter equation combined with the "ray surface" derived from the "ray space" is utilized to derive the canonical ray equations $F(x, y, z, \dot{x}, \dot{y}, \dot{z}) = 1$. This is the simplest mathematical approach found in literature [Budden, 1961].
Mention will also be made of a very formal approach (paragraph 9), which takes into account the asymptotic expansion of the ray of the wave, which in different degrees of approximation leads to the Hamiltonian formulation, then to the determination of ray tracing. Moreover, returning to the first point, an introduction is provided for the method of combining the surfaces *G* and *F* to derive Fermat's principle. This gives a certain generality to the formulation of ray theory. On the basis of this principle only a particular path where the ray takes the minimum time is actually ray-path permitted, while the ray does not propagate along any of the other possible paths. Once the canonical equations are derived, a constant quantity *H* dependent on x, y, z, p_x, p_y, p_z must be chosen in order to derive from six differential equations, the coordinate and wave vector direction along the whole path. These quantities generically indicated as $r_i(\tau)$, $p_i(\tau)$ are dependent on the parameter τ, which must increase monotonically along the path. Effectively, the procedure described above is the ray tracing concept. In the ionosphere the refractive index is a complex quantity that can assume two different expressions since the medium is bi-refractive and two different paths are permitted (ordinary and extraordinary ray paths) for the two different propagating modes. In ray tracing computational techniques, the most convenient Hamiltonian can be adopted describing the medium and the mode simply by choosing *H*. Finally, a system of equations, which can be numerically solved, is generated.

1. Phase and ray velocity

In the ray tracing technique, ray velocity v_r is considered rather than phase velocity. Ray velocity is the velocity at which the energy of a monochromatic wave propagates in a medium. This concept does not apply to a wave packet which propagates at group velocity. The latter velocity is always lower than *c* (velocity of light in vacuum) as each of the packet components propagates at its own speed. In a monochrome wave, the speed at which the energy propagates is the speed of the ray. In general when dealing with a problem of ray tracing, the parameter of interest is the path followed by the ray of the wave in the medium in which it propagates. If the medium is isotropic, the problem is simple since the phase and ray velocities propagate in the same direction. Propagation in an anisotropic medium is different because the phase and ray velocities have different directions. If the medium is inhomogeneous, the wave is subject to the phenomenon of refraction but, at the wave front, both phase and group velocities have the same direction.
The phase velocity, in anisotropic media, forms an angle α relative to the speed of the ray as the first is always perpendicular to the wave front (phase constant surface) while the second coincides with the propagation direction of energy (see figure 1).
Figure 2 shows the same phenomenon from another point of view. A radio wave propagates in an anisotropic medium from point T to point R, which is the path along which the energy propagates, with the velocity vector of the ray always tangent to the trajectory along the path TR. The phase velocity vector, perpendicular to the wave front, forms an angle α with ray vector dependent on the index of refraction of phase and group at the generic points x, y, z.
In an anisotropic medium at the points *x, y, z* of the ray-path, the relations between the ray and phase velocities and the corresponding refractive indices, are linked to each other through the angle α [Fowles, 1989] as in the following:

$$v = v_r \cos\alpha, \tag{1.1}$$

$$n_r = \mu \cos\alpha. \tag{1.2}$$

One of the most intuitive derivations of equations (1.1) and

(1.2) can be found in texts discussing radio ionospheric propagation [Davies, 1990]. A simple derivation is given on [Kelso, 1964; Kelso, 1968]. The Poynting $\vec{I}$ vector (energy that flows into the surface unit in the unit of time) can be written:

$$\vec{I} = w\vec{v}_r \,, \tag{1.3}$$

where w indicates the energy and therefore it appears that $v_r = I / w$. Now, knowing that $v = c / \mu$ it can be written as follows:

$$v = \frac{\vec{I} \cdot \vec{p}}{w} \,, \tag{1.4}$$

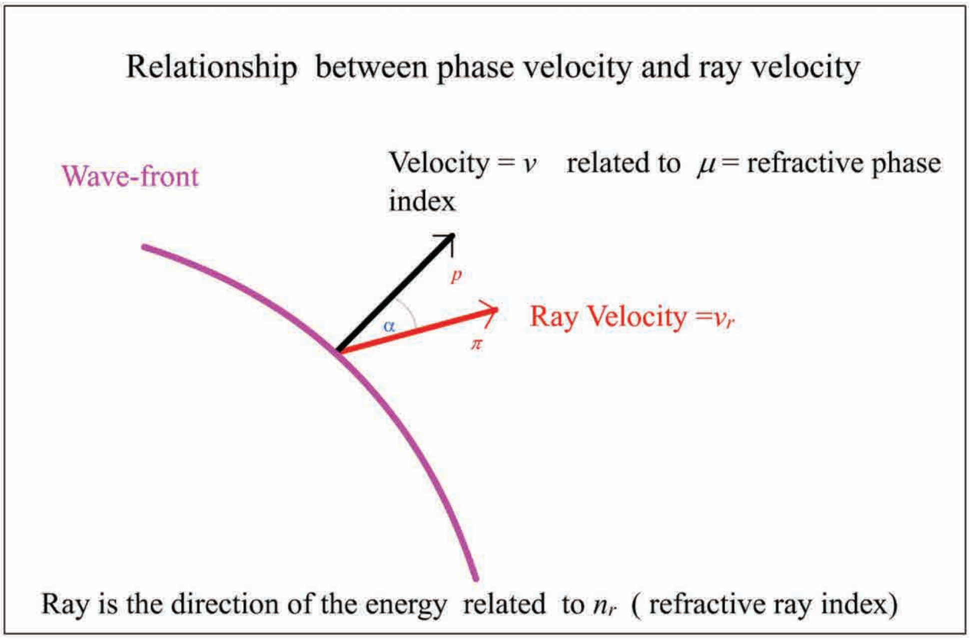

Figure 1. The perpendicular to the wave front is the propagation direction of the phase velocity $\vec{p}$. In an anisotropic medium it forms an angle α with the ray's velocity $\vec{\pi}$.
Figura 1. La perpendicolare al fronte d'onda è la direzione di propagazione della velocità di fase $\vec{p}$. In un mezzo anisotropo, forma un angolo α con la velocità del raggio d'onda $\vec{\pi}$.

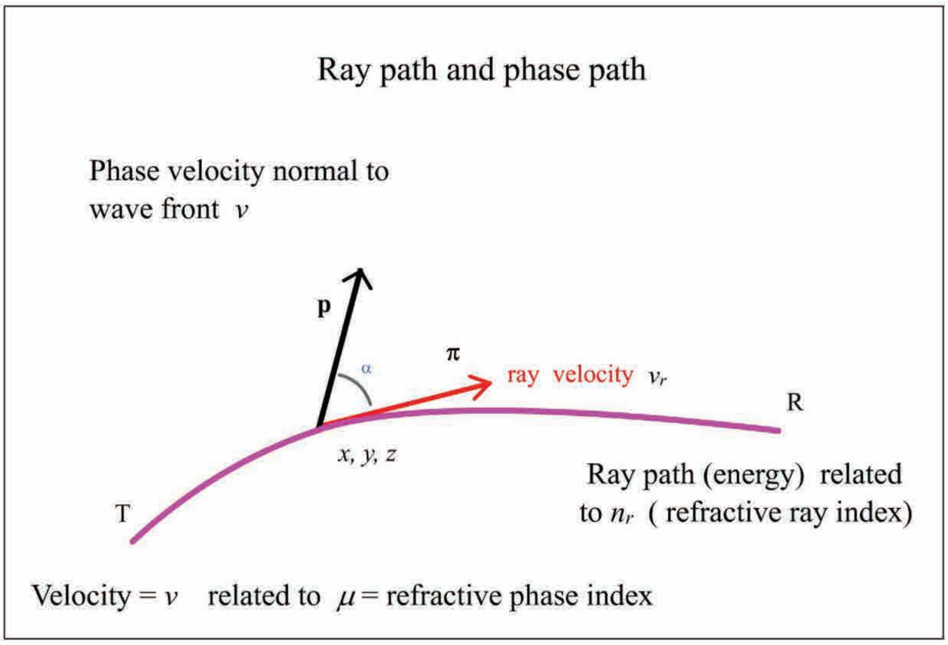

Figure 2. The ray velocity $\vec{v}_r$ is always tangent to the path followed by the wave along the line TR. The phase velocity vector forms an angle α relative to the propagation direction of the velocity $\vec{v}_r$.
Figura 2. La velocità del raggio d'onda $\vec{v}_r$ è sempre tangente al percorso seguito dall'onda lungo la curva TR. Il vettore della velocità di fase forma un angolo α rispetto alla direzione di propagazione della velocità $\vec{v}_r$.

where the Poynting vector $\vec{I}$ is projected in the direction of $\vec{p}$. By virtue of equation (1.3) we have the following:

$$v = \frac{(\vec{I} \cdot \hat{p})v_r}{I} = (\hat{\pi} \cdot \hat{p})v_r = v_r \cos\alpha , \qquad (1.5)$$

with $\hat{\pi}$ and $\hat{p}$ as unit vectors. Equation (1.2) is immediate since $v = c/\mu$ and $v_r = c/n$. The propagation speed of the energy (wavelength range) is always greater than the phase velocity, unless α is not null, when they would be equal.

2. Refractive index surface equation

In an inhomogeneous and isotropic medium, in which the refractive index characterizing the medium does not vary much in relation to the wavelength, a special space is introduced known as the "refractive index space". It is a dimensionless space where the coordinate axes p_x, p_y, p_z can be defined parallel to those of ordinary space described by the coordinate axes x, y, z. A wave front is considered at the point x, y, z, as in figure 3.

The normal to the wave front, i.e. the direction of propagation of the phase, is different from the direction of the ray, which is the energy propagation direction. Now a vector $\vec{p}$ with the modulus $\mu = \sqrt{p_x^2 + p_y^2 + p_z^2}$ can be traced from the origin parallel to the wave normal, with μ the value of the phase refractive index at the point x, y, z.

In this space the cosines of $\vec{p}$ vector are:

$$\frac{p_x}{\sqrt{p_x^2+p_y^2+p_z^2}}, \frac{p_y}{\sqrt{p_x^2+p_y^2+p_z^2}}, \frac{p_z}{\sqrt{p_x^2+p_y^2+p_z^2}}. \qquad (2.1)$$

Moving on the surface of the wavefront, the vector $\vec{p}$ will trace a surface which can be called the "surface of the refractive index" or G, which obviously depends on x, y, z, p_x, p_y, p_z, where the components p_x, p_y, p_z are present only in mutual combination. We can therefore write [Bianchi and Bianchi, 2009]:

$$G(x,y,z,p_x,p_y,p_z) = \frac{\sqrt{p_x^2+p_y^2+p_z^2}}{\mu(x,y,z,p_x,p_y,p_z)} = 1. \qquad (2.2)$$

The surface just described is similar to the equation of the sphere with centre in the origin of the reference system. It can be written as above equation (2.2) and is the locus of points p_x, p_y, p_z, touched by the apex of the vector $\vec{p}$. Now the direction of the ray is normal to the surface of the refractive index with the direction cosines proportional to [Budden, 1988]:

$$\frac{\partial G}{\partial p_x}, \frac{\partial G}{\partial p_y}, \frac{\partial G}{\partial p_z}. \qquad (2.3)$$

The point x, y, z, where the ray intercepts the wave front moves with velocity v_r and it can be represented as:

$$\vec{v}_r = (\dot{x}, \dot{y}, \dot{z}). \qquad (2.4)$$

It can therefore be assumed that every point on the surface has a velocity given by (2.4) and that these components are

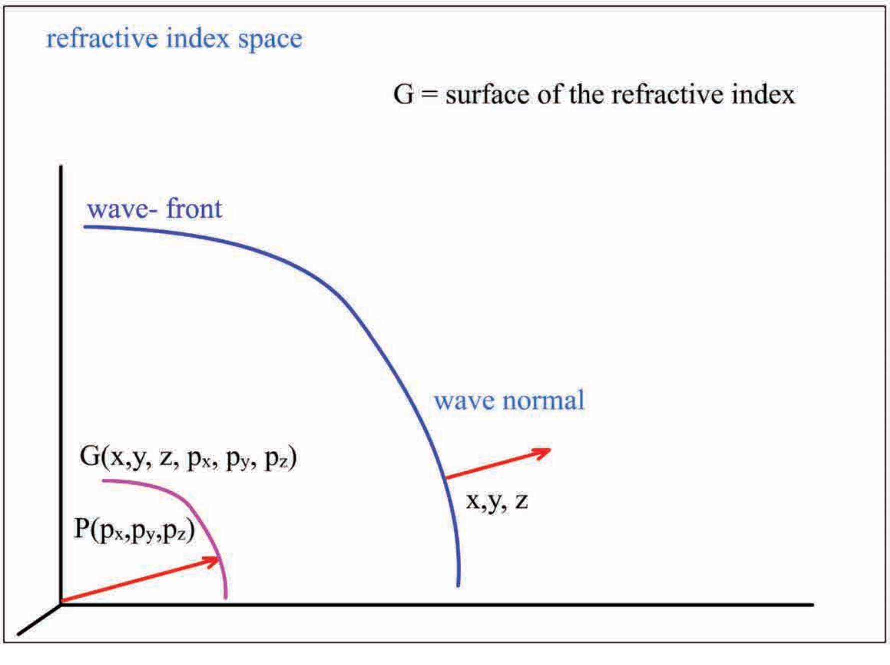

Figure 3. Surface of refractive index. Note that: *P* and *G* superficies show similar shape; in the refractive index space, the *P* axes are parallel to the Cartesian coordinate axes.
Figura 3. Superficie dell'indice di rifrazione. Si noti che: le superfici *P* e *G* mostrano forme simili; nello spazio dell'indice di rifrazione, gli assi di *P* sono paralleli agli assi delle cordinate cartesiane.

proportional to components (2.3). The constant of proportionality between these two is found in the following manner. The x-axis is selected in such a way that p_y, p_z are zero as well as the partial derivatives with respect to $\partial/\partial p_y$ and $\partial/\partial p_z$. The partial derivative of G respect to $\partial/\partial p_x$ i.e. $\partial(p_x/\mu)/\partial p_x$ is:

$$\frac{\partial G}{\partial p_x} = \frac{\partial}{\partial p_x}\left(\frac{p_x}{\mu}\right) = \frac{1}{\mu}, \quad (2.5)$$

as is also the derivative $\partial(1/\mu)/\partial p_x=0$, since there is no variation of μ along the chosen direction p_x. If $\dot{x}$ is used to indicate the velocity component of the ray along x in the direction of the normal wave-front, which is c/μ. Then, according to equation (2.5) it will have:

$$\frac{\partial G}{\partial p_x} = \frac{\dot{x}}{c}. \quad (2.6)$$

Therefore, the constant of proportionality is $1/c$ and, generalizing on the other partial derivatives it will have:

$$\dot{x} = c\frac{\partial G}{\partial p_x}, \dot{y} = c\frac{\partial G}{\partial p_y}, \dot{z} = c\frac{\partial G}{\partial p_z}. \quad (2.7)$$

These relations (2.7) are also valid in the case of an anisotropic medium once the particular refractive index (ordinary or extraordinary) has been selected.

3. Ray surface equation

Similarly to what was done for the surface of the refractive index, a dimensionless space called the "ray space" can be defined. Inside this the coordinate axes $\dot{x}$, $\dot{y}$, $\dot{z}$ are defined for the velocity, parallel to the ordinary space coordinate axes x, y, z.

Considering a wave front taken at the point x, y, z, as shown in figure 4, and following the procedure of the previous paragraph, a vector can be traced from the origin, such that each point on the surface can be associated to a vector of modulus within the ray space:

$$v_r = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}. \quad (3.1)$$

Now the direction cosines of the ray can be expressed as:

$$\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}, \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}, \frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}. \quad (3.2)$$

Considering all the possible directions of the wave front, a surface called the "ray surface" is obtained such that:

$$\frac{c}{n_r} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}. \quad (3.3)$$

It is equivalent to write:

$$F(x,y,z,\dot{x},\dot{y},\dot{z}) = \frac{n_r(x,y,z,\dot{x},\dot{y},\dot{z})}{c}\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = 1, \quad (3.4)$$

with the speed $v_r=c/n_r$ and n_r the refractive index of the ray. The cosines are proportional to:

$$p_x, \ p_y, \ p_z, \quad (3.5)$$

and also proportional to:

$$\frac{\partial F}{\partial \dot{x}}, \frac{\partial F}{\partial \dot{y}}, \frac{\partial F}{\partial \dot{y}}. \quad (3.6)$$

The constant of proportionality between quantities (3.5) and (3.6) is equal to c. Therefore it will have:

$$p_x = c\frac{\partial F}{\partial \dot{x}}, p_y = c\frac{\partial F}{\partial \dot{y}}, p_z = c\frac{\partial F}{\partial \dot{z}}. \quad (3.7)$$

These relationships are reciprocal with equations (2.7). They are referred to as mutual surfaces, since exchanging p_x, p_y, p_z, with the partial derivatives of the velocities gives either relations (2.7), where the function G applies, or relations (3.7) where the function F applies. The constant of proportionality between p_x, p_y, p_z, and the quantities (3.6) are found in the following way. The x-axis is selected in such a way that $\dot{y}$ and $\dot{z}$ are zero, and so that the partial derivatives $\partial/\partial\dot{y}$ and $\partial/\partial\dot{z}$ of the velocity components along y and z do not contribute. The constant of proportionality between p_x, p_y, p_z, and the quantities (3.6) are obtained by the function F, which is equal to $n_r\dot{x}/c$, and by its derivative with respect to $\partial/\partial\dot{x}$, which is:

$$\frac{\partial F}{\partial \dot{x}} = \frac{n_r}{c} = \frac{p_x}{c}, \quad (3.8)$$

where n_r is put in relation to p_x recalling the previous construction of the G surface. Therefore, the proportionality constant is $1/c$. This relationship can be exploited to formulate Fermat's principle when it is combined with the

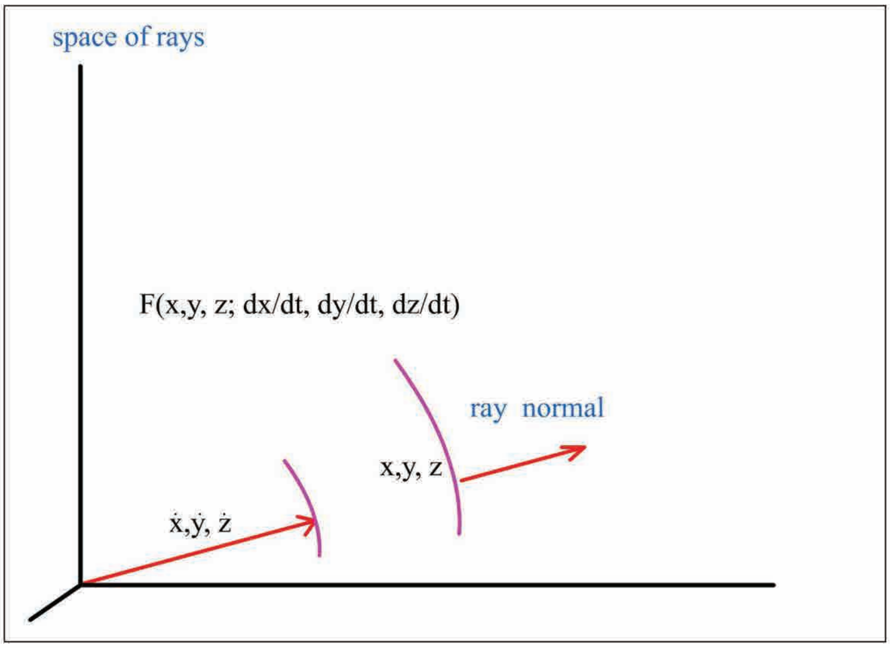

Figure 4. Three dimensional ray space allows definition of a ray surface $F(x,y,z,\dot{x},\dot{y},\dot{z})$ by simple geometrical means. In the ray space the velocity component axes $\dot{x}$, $\dot{y}$, $\dot{z}$ are parallel to the Cartesian coordinate axes.
Figura 4. Lo spazio tridimensionale del raggio d'onda consente la definizione di una superficie del raggio d'onda $F(x,y,z,\dot{x},\dot{y},\dot{z})$ applicando semplici metodi geometrici. Nello spazio del raggio d'onda, gli assi delle componenti in velocità sono paralleli agli assi delle coordinate cartesiane.

relation of the surface refractive index.

4. Phase memory concept

Maxwell's equation $\vec{E}=\vec{E}_0\exp(-j\vec{k}\cdot\vec{r})$ is now considered for a wave which propagates along *r*, chosen in the direction of the normal waveform and having propagation vector module $k=2\lambda/$. In a new set of three coordinates *x, y, z*, this equation can be written $\vec{E}=\vec{E}_0\exp[-jk(xp_x+yp_y+zp_z)]$, where p_x, p_y, p_z are proportional to the cosines of the wave-normal with respect to the new axes and $p_x^2+p_y^2+p_z^2=\mu^2$. These quantities p_x, p_y, p_z play an important role as they take into account the direction of the ray. At this point the homogeneity of the medium is no longer a requirement. In an inhomogeneous medium it is important to set the condition that the refractive index varies slowly [Bianchi et al., 2009]. Adopting this assumption, the concept of phase memory can be applied, in which the phase change can be expressed as $kp_x\delta x$. Along a distance *x* the change can be expressed as $k\int_0^x p_x\,dx$. More generally, the change of phase from a point source to a point *x, y, z*, is given by:

$$k\left(\int_0^x p_x\,dx+\int_0^y p_y\,dy+\int_0^z p_z\,dz\right). \tag{4.1}$$

Therefore it is assumed that there exists a function $\vec{E}=\vec{E}_0\exp(-jS)$ i.e. a solution to Maxwell's equations for which:

$$S(x,y,z,p_x,p_y,p_z)=k\left(\int_0^x p_x\,dx+\int_0^y p_y\,dy+\int_0^z p_z\,dz\right). \tag{4.2}$$

In a homogeneous medium, p_x, p_y, p_z are constant and they can be taken out of the integral. The function *S* is called eikonal [Bianchi et al., 2009]. It can be thought of as the spatial part of the wave phase. It is also evident that:

$$p_x=\frac{1}{k}\frac{\partial S}{\partial x}\ ,\ p_y=\frac{1}{k}\frac{\partial S}{\partial y}\ ,\ p_z=\frac{1}{k}\frac{\partial S}{\partial z}\ , \tag{4.3}$$

then

$$\vec{p}=\frac{1}{k}\nabla S, \tag{4.4}$$

where $\vec{p}$ is a vector of components p_x, p_y, p_z. As the $\vec{p}$ curl is null, i.e. $\nabla\times\vec{p}\equiv 0$, we have:

$$\frac{\partial p_z}{\partial y}=\frac{\partial p_y}{\partial z},\ \frac{\partial p_x}{\partial z}=\frac{\partial p_z}{\partial x},\ \frac{\partial p_y}{\partial x}=\frac{\partial p_x}{\partial y}. \tag{4.5}$$

This relation will be exploited later to derive the canonical equations of the ray.

5. Canonical ray equations

Let $P\,(x, y, z)$ be the point where the ray intercepts the wave front (for example, the coordinate of a wave crest while it travels along the ray). For each point of coordinate x, y, z on this surface, there is a corresponding point p_x, p_y, p_z, on the surface of the refractive index G whose equation, already formulated, for simplicity is recast as follows:

$$G(x,y,z,p_x,p_y,p_z)=1, \tag{5.1}$$

which is generally valid, and must be satisfied for each point p_x, p_y, p_z.

Now considering the functional dependence of refractive index G along the direction x and directly differentiating G relative to x, for both members of equation (5.1), it will result dG/dx=0, which in more explicit form is:

$$\frac{\partial G}{\partial x}+\frac{\partial G}{\partial p_x}\frac{\partial p_x}{\partial x}+\frac{\partial G}{\partial p_y}\frac{\partial p_y}{\partial x}+\frac{\partial G}{\partial p_z}\frac{\partial p_z}{\partial x}=0, \tag{5.2}$$

because p_x, p_y, p_z depend on x. So according to equations (2.7) and (4.5) it will have:

$$\frac{\partial G}{\partial x}+\frac{1}{c}\left(\frac{\partial p_x}{\partial x}\dot{x}+\frac{\partial p_x}{\partial y}\dot{y}+\frac{\partial p_x}{\partial z}\dot{z}\right)=0, \tag{5.3}$$

and therefore, since p_x has null derivative relative to y and z, it will have:

$$\frac{\partial G}{\partial x}=-\frac{1}{c}\frac{dp_x}{dt}, \tag{5.4}$$

where d/dt applies to the arbitrary point x, y, z on the G surface. It can therefore be written:

$$\dot{p}_x=\frac{dp_x}{dt}. \tag{5.5}$$

Generalizing, it follows:

$$\dot{p}_x=-c\frac{\partial G}{\partial x}\ ,\ \dot{p}_y=-c\frac{\partial G}{\partial y}\ ,\ \dot{p}_z=-c\frac{\partial G}{\partial z}\,. \tag{5.6}$$

These equations are very important in this context because they assume a familiar form and are called the canonical ray equations. Together with equation (2.7) they acquire a formal symmetry similar to Hamilton's equations when replacing H to G. Hamilton's equations elegantly synthesize the theory of classical mechanics as shown below by the relation:

$$\frac{\partial H(q_i,p_i)}{\partial p_i}=\frac{dq_i}{dt}, \tag{5.7}$$

$$\frac{\partial H(q_i,p_i)}{\partial q_i}=-\frac{dp_i}{dt}, \tag{5.8}$$

which are comparable with equations (2.7) and (5.6). In the previous equations (5.7) and (5.8), q_i and p_i are respectively the generalized coordinate and generalized moment. For this reason, when dealing with equations (2.7) and (5.6), Hamilton's equations are being discussed, even if the derivation, as seen above, is different from that of Hamilton in classical mechanics. For the applications under discussion here, it is possible to confer more generality to these equations and include cases of time-variant medium and, given the intrinsic dispersivity of the ionospheric plasma, this particular equation even becomes dependent on frequency. Making Hamilton's equation depend on $q_i(t)$ and $p_i(t)$ as well as on time t explicitly and also on frequency ω does not introduce other formal difficulties but only complications in calculation. This dependency can be expressed in Cartesian coordinates and $\vec{k}$ wave vector components as:

$$H(x,y,z,t,k_x,k_y,k_z,\omega). \tag{5.9}$$

This equation is also called the super-Hamiltonian as it is also possible to extract the Doppler shift due to the variation of the refractive index through time. Compared to the function G, the function H differs for the fact that the components of the index of refraction were replaced with components of the vector $\vec{k}$ using the formula:

$$\frac{\omega}{c}n=k, \tag{5.10}$$

with $n=\mu+j\chi$ the complex refractive index. With this position and according to equation (2.2) it is possible to rewrite equation (5.9) as:

$$H(x,y,z,t,k_x,k_y,k_z,\omega)=\frac{c}{\omega}\frac{\sqrt{k_x^2+k_y^2+k_z^2}}{n(x,y,z,t,k_x,k_y,k_z,\omega)}=1. \tag{5.11}$$

The anisotropic properties of the medium depend only on the real part of the refractive index [Bianchi, 1990], while the imaginary part is responsible for the absorption of the waves due to the medium. The present interest is for ray tracing formulation and related implications, and so for the sake of simplicity, from this point onwards the discussion will consider only the real part μ, returning to equations similar to equations (2.2) [Jones and Stephenson, 1975]:

$$H(x,y,z,t,k_x,k_y,k_z,\omega)=\frac{c}{\omega}\frac{\sqrt{k_x^2+k_y^2+k_z^2}}{\mu(x,y,z,t,k_x,k_y,k_z,\omega)}=1. \quad (5.12)$$

These authors used a number of derived forms of the equation (5.12), with spherical coordinates, which will be the subject of the following paragraphs.

6. Application of Hamilton's equations

Assuming that a wave is propagating in an inhomogeneous medium characterized by a phase refractive index $\mu(x,y,z)$, the value of which depends on the position, as long as the wavelength λ is small compared to the spatial variation of the refractive index, the following relations applies:

$$\frac{dx}{d\tau}=\frac{\partial H}{\partial k_x}, \quad (6.1.a)$$

$$\frac{dy}{d\tau}=\frac{\partial H}{\partial k_y}, \quad (6.1.b)$$

$$\frac{dz}{d\tau}=\frac{\partial H}{\partial k_z}, \quad (6.1.c)$$

$$\frac{dk_x}{d\tau}=-\frac{\partial H}{\partial x}, \quad (6.1.d)$$

$$\frac{dk_y}{d\tau}=-\frac{\partial H}{\partial y}, \quad (6.1.e)$$

$$\frac{dk_z}{d\tau}=-\frac{\partial H}{\partial z}. \quad (6.1.f)$$

Even in an anisotropic medium that can present two different values for the refractive index at the same point (as in the ionosphere, with the ordinary and extraordinary refractive index) the above equations are still applicable. The only requirement is to consider the values valid for the ordinary or extraordinary refractive phase indexes separately. The parameter τ varies monotonically along the wave path, and can be considered, for example, $\tau = ct$. Similar equations have also been proposed [Haselgrove, 1955]. Applying the WKB approximation, there are thus six uncoupled differential equations, whose numerical integration provides the wave path or ray tracing. It can also be considered an isotropic medium with particular symmetry as in the case of a flat layered medium. In this case the canonical ray equations (5.6) and (2.7) can be employed in the numerical calculation of ray tracing. They constitute a sort of Snell's law generalization. For example, if we reduce Hamilton's equations making them depend on a single spatial variable, setting $\partial G/\partial x=0$ and $\partial G/\partial y=0$, taking a flat stratified ionosphere where $\mu = \mu(z)$, it is possible to proceed in the following manner. If $p_x=S_1$, $p_y=S_2$, $p_z=q$, which by virtue of equations (5.6) gives:

$$0=\dot{p}_x=\frac{dS_1}{dt}, 0=\dot{p}_y=\frac{dS_2}{dt}. \quad (6.2)$$

Namely S_1, and S_2 are constant along the ray (during the propagation there is no change relative to t). These are nothing more than Snell's law. Initially only equations (2.7) and (5.6) are used, relative to the function G, in a practical case in which $\mu = \mu(z)$ and where they can be exploited.
According to paragraph 2 and in particular equation (2.2), one can write:

$$G(z,p_x,p_z)=\frac{\sqrt{p_x^2+p_z^2}}{\mu(z,p_x,p_z)}=1. \quad (6.3)$$

It is convenient to introduce the angle θ between the wave normal and the vertical

$$p_x=\mu\sin\theta, \quad (6.4.a)$$

$$p_z=\mu\cos\theta, \quad (6.4.b)$$

that provides the following two relations:

$$\frac{p_x}{\sqrt{p_x^2+p_z^2}}=\sin\theta, \quad (6.5.a)$$

$$\frac{p_z}{\sqrt{p_x^2+p_z^2}}=\cos\theta. \quad (6.5.b)$$

That is, the refractive index depends only on p_x and p_z or on z and θ through (6.4) and (6.5). Therefore it can be stated that $\mu = \mu(z, \theta)$. The derivatives of G, given in equation (2.2), compared to p_x and p_z are given by:

$$\frac{\partial G}{\partial p_x} = \frac{p_x}{\mu\sqrt{p_x^2+p_z^2}} - \frac{\sqrt{p_x^2+p_z^2}}{\mu^2}\frac{\partial\mu}{\partial\theta}\frac{\partial\theta}{\partial p_x} = \frac{\sin\theta}{\mu} - \frac{1}{\mu^2\cos\theta}\frac{\partial\mu}{\partial\theta}, \quad (6.6)$$

$$\frac{\partial G}{\partial p_z} = \frac{p_z}{\mu\sqrt{p_x^2+p_z^2}} - \frac{\sqrt{p_x^2+p_z^2}}{\mu^2}\frac{\partial\mu}{\partial\theta}\frac{\partial\theta}{\partial p_z} = \frac{\cos\theta}{\mu} + \frac{1}{\mu^2\sin\theta}\frac{\partial\mu}{\partial\theta}. \quad (6.7)$$

When moving along the magnetic meridian it is seen that $\partial\mu/\partial y=0$, and as a consequence $\partial G/\partial y=0$. Moreover, since $\mu = \mu(z, \theta)$ is constant along x this means $\partial G/\partial x=0$, and because of the first of equations (5.6), p_x will be a constant (Snell's law). Therefore the derivatives of equation (6.4) relative to t will be:

$$\frac{dp_x}{dt} = \sin\theta\frac{d\mu}{dt} + \mu\cos\theta\frac{d\theta}{dt} = 0, \quad (6.8)$$

$$\frac{dp_z}{dt} = \cos\theta\frac{d\mu}{dt} - \mu\sin\theta\frac{d\theta}{dt} = -\frac{d\theta}{dt}(\mu\sin\theta + \mu\frac{\cos^2\theta}{\sin\theta}) = -\frac{\mu}{\sin\theta}\frac{d\theta}{dt}. \quad (6.9)$$

The last step of equation (6.9) can be proved calculating $d\mu/dt$ from equation (6.8). Now taking equation (6.6) and replacing it with $\dot{x} = c\,\partial G/\partial p_x$, the equation (6.7) to $\dot{z} = c\,\partial G/\partial p_z$ the equation (6.9) to $-c\,\partial G/\partial z = \dot{p}_z$, gives:

$$\begin{aligned}\frac{dx}{dt} &= c\frac{\partial G}{\partial p_x} = c(\frac{\sin\theta}{\mu} - \frac{1}{\mu^2\cos\theta}\frac{\partial\mu}{\partial\theta})\\ \frac{dz}{dt} &= c\frac{\partial G}{\partial p_z} = c(\frac{\cos\theta}{\mu} + \frac{1}{\mu^2\sin\theta}\frac{\partial\mu}{\partial\theta})\\ \frac{dp_z}{dt} &= -c\frac{\partial G}{\partial z} = -\frac{\mu}{\sin\theta}\frac{d\theta}{dt},\end{aligned} \quad (6.10)$$

that is to say:

$$\begin{aligned}\frac{dx}{dt} &= \frac{c}{\mu^2}(\mu\sin\theta - \frac{1}{\cos\theta}\frac{\partial\mu}{\partial\theta})\\ \frac{dz}{dt} &= \frac{c}{\mu^2}(\mu\cos\theta + \frac{1}{\sin\theta}\frac{\partial\mu}{\partial\theta})\\ \frac{d\theta}{dt} &= -\frac{c}{\mu^2}\sin\theta\frac{\partial\mu}{\partial z}.\end{aligned} \quad (6.11)$$

A ray path can be obtained integrating this system of equations with the opportune initial conditions.

7. Hamilton's ray equations with spherical coordinates

In OTHR systems, when exactly defining the position of the target, the so called "coordinate registration", and in long range short-wave communication when distances of thousands of km are typically involved, Cartesian coordinates do not provide ideal coverage. In addition, the spherical shell symmetry of the ionosphere and the Earth's curvature makes it more logical to use spherical coordinates. Equations (6.1.a-f) can be written in spherical coordinates r, θ, ϕ (see figure 5).
The transition from Cartesian geometry to spherical geometry is not difficult but rather cumbersome:

$$\frac{\partial H}{\partial k_r} = \frac{dr}{d\tau}, \quad (7.1.a)$$

$$\frac{1}{r}\frac{\partial H}{\partial k_\theta} = \frac{d\theta}{d\tau}, \quad (7.1.b)$$

$$\frac{1}{r\sin\theta}\frac{\partial H}{\partial k_\phi} = \frac{d\phi}{d\tau}, \quad (7.1.c)$$

$$\frac{dk_r}{d\tau} = -\frac{\partial H}{\partial r} + k_\theta\frac{d\theta}{d\tau} + k_\phi\sin\theta\frac{d\phi}{d\tau}, \quad (7.1.d)$$

$$\frac{dk_\theta}{d\tau} = \frac{1}{r}(-\frac{\partial H}{\partial\theta} - k_\theta\frac{dr}{d\tau} + k_\phi r\cos\theta\frac{d\phi}{d\tau}), \quad (7.1.e)$$

$$\frac{dk_\phi}{d\tau} = \frac{1}{r\sin\theta}(-\frac{\partial H}{\partial\phi} - k_\phi\sin\theta\frac{dr}{d\tau} - k_\phi r\cos\theta\frac{d\theta}{d\tau}). \quad (7.1.f)$$

This was achieved by differentiating the canonical ray equations relative to spherical coordinates, with the use of the metric coefficients 1, r and $r\sin\theta$ [Bianchi and Bianchi, 2009]. In particular, the last three equations are obtained by projecting the wave vector $\vec{k} = (k_x, k_y, k_z)$, represented in Cartesian coordinates, on the unit versors $\hat{i}_r, \hat{i}_\theta, \hat{i}_\phi$ and then the wave vector components k_x, k_y, k_z are transformed into spherical coordinates k_r, k_θ, k_ϕ (see figure 6).

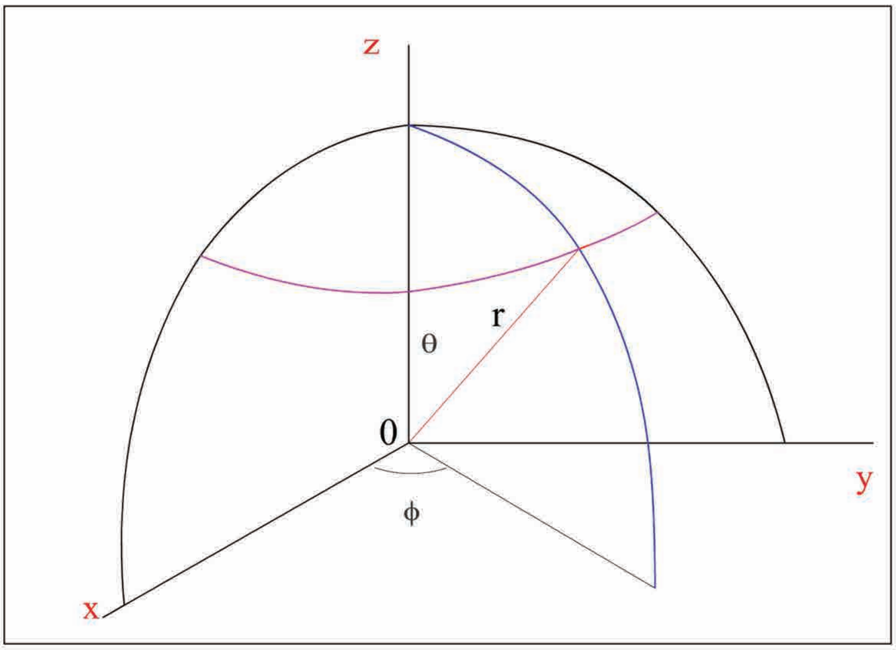

Figure 5. The Cartesian (x, y, z) and spherical (r, θ , ϕ) coordinate systems are represented.
Figura 5. Rappresentazione per i sistemi di coordinate cartesiane (x, y, z) e sferiche (r, θ , ϕ).

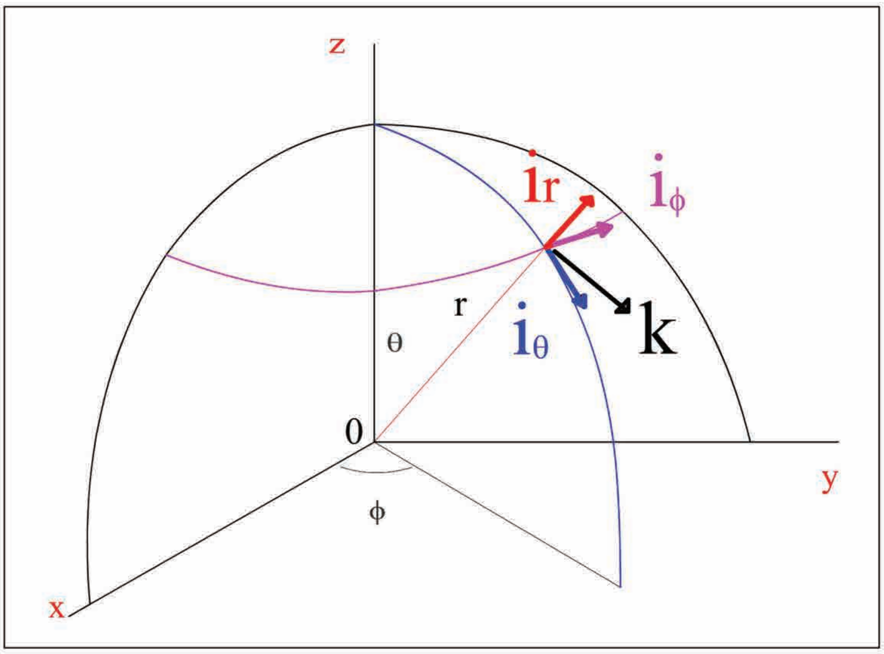

Figure 6. Projection of the wave vector $\vec{k} = (k_r, k_\theta, k_\phi)$ on the directions of the unit versors $\hat{i}_r, \hat{i}_\theta, \hat{i}_\phi$.
Figura 6. Proiezione del vettore d'onda $\vec{k} = (k_r, k_\theta, k_\phi)$ nelle direzioni dei versori unitari $\hat{i}_r, \hat{i}_\theta, \hat{i}_\phi$.

8. Fermat's Principle

Taking into consideration the equation (3.4), which for the sake of simplicity is re-written as:

$$F(x,y,z,\dot{x},\dot{y},\dot{z}) = \frac{n_r(x,y,z,\dot{x},\dot{y},\dot{z})}{c}\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} = 1, \quad (8.1)$$

derivative with respect to x, both members quickly give $dF/dx=0$ i.e.:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial x} + \frac{\partial F}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial x} + \frac{\partial F}{\partial \dot{z}}\frac{\partial \dot{z}}{\partial x} = 0, \quad (8.2)$$

$$\frac{\partial F}{\partial x} = -\frac{1}{c}(p_x\frac{\partial \dot{x}}{\partial x} + p_y\frac{\partial \dot{y}}{\partial x} + p_z\frac{\partial \dot{z}}{\partial x}), \quad (8.3)$$

having taken into account relationships (3.7). Summing the latter equation (8.3) with equation (5.3) gives:

$$\frac{\partial F}{\partial x} + \frac{\partial G}{\partial x} = -\frac{1}{c}\frac{\partial}{\partial x}(\dot{x}p_x + \dot{y}p_y + \dot{z}p_z) = 0, \quad (8.4)$$

with the quantity inside the brackets equal to c and $v = c/\mu$, and equating the second member to zero by a quick visual examination. Similar results are obtained performing derivatives for y and z i.e. $dF/dy+dG/dy=0$, $dF/dz+dG/dz=0$. Starting from the last three and recalling the relations $\dot{p}_x = -c\,\partial G/\partial x$ and $p_x=c\partial F/\partial x$ it is possible to write:

$$\begin{aligned}\frac{\partial F}{\partial x} &= -c\frac{dp_x}{dt} = -\frac{d}{dt}(\frac{\partial F}{\partial \dot{x}})\\ \frac{\partial F}{\partial y} &= -c\frac{dp_y}{dt} = -\frac{d}{dt}(\frac{\partial F}{\partial \dot{y}})\\ \frac{\partial F}{\partial z} &= -c\frac{dp_z}{dt} = -\frac{d}{dt}(\frac{\partial F}{\partial \dot{z}}).\end{aligned} \quad (8.5)$$

These are the three differential equations of Euler which can be expressed in a more compact form or in variational formulation as:

$$\delta\int_A^B F\,dt = 0, \quad (8.6)$$

with the variation between the extremes A and B equal to zero. This implies that among the possible paths only the effective ray-path takes the minimum time (Fermat's principle). A and B are the extreme points of the ray-path s, along the path the surface F is crossed by the ray (see figure 4) normal to F at the point x, y, z. Hence, if s is the curved path of the ray the following equation can be defined:

$$\frac{ds}{dt} = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}. \quad (8.7)$$

Combining the first equation (3.4) with the latter equation (8.7) and inserting the equation (8.6) gives:

$$\delta\int_A^B n_r\,ds = 0, \quad (8.8)$$

which is again Fermat's principle. This principle is applicable only along the effective path (i.e. along the small intervals ds when the surface F is itself described by the two relations (8.6) and (8.8) to the ray's refractive index n_r). Fermat's principle confirms this. Conversely, all other possible paths do no satisfy all the relations from (8.5) to (8.8).

9. General ray theory for a time-varying medium

The complex eikonal method [Weinberg, 1962] regards wave ray propagation in a magneto-plasma with refractive index $n(\vec{r})$ for a fixed angular frequency ω. According to this formulation acting in Cartesian coordinates (not generalized) it is possible to define a group of homogeneous linear differential equations generated by a matrix operator $M(t,\vec{r},\partial/\partial t,\nabla)$ N x N dimensions, operating on a wave function $\vec{\Psi}(t,\vec{r})$ with N components as follows:

$$M(t,\vec{r},\partial/\partial t,\nabla)\vec{\Psi}(t,\vec{r}) \equiv 0. \quad (9.1)$$

The matrix operator $M(t,\vec{r},\partial/\partial t,\nabla)$ tensor quantity expressing the properties of medium, since a weak dependence can be sustained on the time t (time variant) and spatial coordinate $\vec{r}$. Furthermore, it is assumed that the wave function $\vec{\Psi}(t,\vec{r})$ is a vector quantity that depends both on the time t and spatial coordinate $\vec{r}$. Assuming an approximate solution of the wave function $\vec{\Psi}(t,\vec{r})$ for a medium where the refractive index $n(\vec{r})$ slightly varies, all the spatial and temporal dependence of the wave function $\vec{\Psi}(t,\vec{r})$ is transferred to the phase term $\Phi(t,\vec{r})$ according to a function of the type:

$$\vec{\Psi}(t,\vec{r}) = \vec{\Psi}_0(\vec{r})\exp[j\xi\Phi(t,\vec{r})], \quad (9.2)$$

where ξ is only a parameter. The wave function $\vec{\Psi}(t,\vec{r})$ of amplitude $\vec{\Psi}_0(\vec{r})$ satisfies the partial differential equations of the first order with the phase term $\Phi(t,\vec{r})$, the time-space dependence of which has already been specified above. The wave function (9.2) substituted into equation (9.1) gives:

$$M(t,\vec{r},j\xi\frac{\partial\Phi}{\partial t},j\xi\nabla\Phi)\vec{\Psi}_0(\vec{r})\equiv 0. \qquad (9.3)$$

Since in a homogeneous medium $\xi\Phi(t,\vec{r})=\vec{k}\cdot\vec{r}-\omega t$, where $\vec{k}$ and ω are respectively a vector and a scalar constant, it can similarly be assumed that:

$$\omega' = -\frac{\partial\Phi}{\partial t}, \qquad (9.4)$$

$$\vec{k}' = \nabla\Phi, \qquad (9.5)$$

and that the quantities ω' and $\vec{k}'$ play a role of local quantities weighed by the parameter ξ [Felsen and Marcuvitz, 1994]. In equation (9.3), given that the amplitude $\vec{\Psi}_0(\vec{r})$ cannot be null to avoid the non-trivial solution, the determinant of the matrix operator $M(t,\vec{r},\partial/\partial t,\nabla)$ must be equal to zero, i.e.:

$$\left|M(t,\vec{r},j\xi\frac{\partial\Phi}{\partial t},j\xi\nabla\Phi)\right| = 0 = H(t,\vec{r},\omega',\vec{k}'), \qquad (9.6)$$

with the quantities ω' and $\vec{k}'$, given by equations (9.4) and (9.5) respectively, taken as the local angular frequency and wave vector. The values of the quantities ω' and $\vec{k}'$ that satisfy the equation (9.6) are assumed. Here the term $H(t,\vec{r},\omega',\vec{k}')$ was introduced to recall the presence of a Hamiltonian, and the fact that the wave vector $\vec{k}$ is non rotational for equation (9.5) and satisfies the equation (9.6) should be noted.

If the matrix operator $M(t,\vec{r},\partial/\partial t,\nabla)$ is independent on the time *t*, it can be assumed that the wave function $\vec{\Psi}(t,\vec{r})$ is a vector that goes with exp(*-jωt*), and that the matrix operator $M(t,\vec{r},\partial/\partial t,\nabla)$ depends on the time *t* only through the angular frequency ω and spatial coordinate $\vec{r}$. In the zero order approximation it is assumed that the matrix operator $M(t,\vec{r},\partial/\partial t,\nabla)$ weakly depends on the spatial coordinate $\vec{r}$ and that all the spatial dependence of the wave function $\vec{\Psi}(t,\vec{r})$ is in the exponential:

$$\vec{\Psi}(\vec{r}) = \vec{\Psi}_0(\vec{r})\exp[j\xi S(\vec{r})], \qquad (9.7)$$

$$M(\vec{r},j\omega\xi,\vec{k}\xi)\vec{\Psi}_0(\vec{r})\equiv 0. \qquad (9.8)$$

Repeating a procedure similar to that proposed by [Bianchi et al., 2009] and with an approximation to the lowest order of the parameter ξ obtains:

$$\omega = -\frac{\partial S}{\partial t}, \qquad (9.9)$$

$$\vec{k} = \nabla S. \qquad (9.10)$$

The above equation (9.10) is the equation of eikonal [Bianchi et al., 2009]. The solution of equation (9.8) determines the values of the angular frequency ω and wave vector $\vec{k}$ that satisfy the equation from the start. In other words, the matrix operator $M(t,\vec{r},\omega,\vec{k})$ generates a relationship $\omega=\omega(\vec{k},\vec{r})$ which is eventually a dispersion relation. This can be considered a constant during propagation. If it satisfies the equation from the start, the equation will always be satisfied during propagation. In other words it can be considered as the Hamiltonian, which obviously is a constant along the path of wave. This depends on the dispersion equation of medium since it is also deducible from the implicit form of equation [Gorman, 1985; Gorman, 1986]. Repeating the reasoning that led to equation (9.6) gives:

$$H(\vec{r},\vec{k}) = 0. \qquad (9.11)$$

It is demonstrated that it is possible to formulate the problem of ray tracing even in very general terms. This means determining the "evolution" of the wave vector $\vec{k}$ if it is made to depend on a time parameter τ, such that:

$$\vec{r} = \vec{r}(\tau), \qquad (9.12)$$

$$\vec{k} = \vec{k}(\tau), \qquad (9.13)$$

so that if the initial conditions $\vec{r}_0=\vec{r}(\tau_0);\vec{k}_0=\vec{k}(\tau_0)$ satisfy the following:

$$\frac{d\vec{r}}{d\tau} = \frac{\partial H}{\partial\vec{k}}, \qquad (9.14)$$

$$\frac{d\vec{k}}{d\tau} = -\frac{\partial H}{\partial\vec{r}}, \qquad (9.15)$$

this will also apply for each time τ. In practice the equations (9.14) and (9.15) can be integrated as the equations (6.1) to obtain the path of the ray wavelength.

10. Ray tracing method

In 4-D space Hamilton's formalism can be described by the generalized coordinates and momentum [Weinberg, 1962], i.e. p_i (three spatial coordinates x, y, z and time t) and q_i (three wave vector components k_x, k_y, k_z and angular frequency ω), writing the pair of equations as following:

$$\frac{\partial H(q_i,p_i)}{\partial p_i}=\frac{dq_i}{d\tau}, \quad (10.1.a)$$

$$\frac{\partial H(q_i,p_i)}{\partial q_i}=-\frac{dp_i}{d\tau}, \quad (10.1.b)$$

where $H(x, y, z, t, k_x, k_y, k_z, \omega)$ is the Hamiltonian describing the medium and the independent variable of H. From this system of equations or canonical ray equations, eight equations that determine all the dependent variables along the ray path can be generated. This can be achieved by integrating equation system (10.1) bearing in mind that q_i includes three wave vector components k_x, k_y, k_z and angular frequency ω. The introduction of frequency dispersive and time varying medium ensure the generality of the Hamiltonian formulism where H is a constant quantity along the path. Moreover, H is a complex quantity (real and imaginary parts) because the Hamiltonian $H(x, y, z, t, k_x, k_y, k_z, \omega)$ depends on the complex refractive index $n(x, y, z, t, k_x, k_y, k_z, \omega)$ as in the following example:

$$H(x,y,z,t,k_x,k_y,k_z,\omega)=\frac{c}{\omega}\frac{\sqrt{k_x^2+k_y^2+k_z^2}}{n(x,y,z,t,k_x,k_y,k_z,\omega)}=1, \quad (10.2)$$

when $n\,\omega/c=k$ with $n=\mu+j\chi$. It is possible to deal with the real or imaginary part independently but in most cases, when the wavelength is relatively small compared with the spatial scale in which the wave is absorbed, the imaginary part of H does not affect the dependent variables in equation system (10.1). At this point it is possible to take into account only the real part of the refractive index n and then of Hamiltonian H [Bianchi et al., 2010]. Among the possible Hamiltonians, with quantity constant along the path, it is common to have the following formulation [Jones and Stephenson, 1975] for H:

$$H(x,y,z,t,k_x,k_y,k_z,\omega)=\mathrm{Re}\left\{\frac{1}{2}\left[\frac{c^2}{\omega^2}(k_x^2+k_y^2+k_z^2)-n^2\right]\right\}, \quad (10.3)$$

in Cartesian coordinates. Or, for long distances, spherical coordinates r, θ, ϕ (figure 5) are much more suitable:

$$H(r,\theta,\phi,t,k_r,k_\theta,k_\phi,\omega)=\mathrm{Re}\left\{\frac{1}{2}\left[\frac{c^2}{\omega^2}(k_r^2+k_\theta^2+k_\phi^2)-n^2\right]\right\}. \quad (10.4)$$

In high frequency ionospheric propagation only the real part of the refractive index μ is really useful for ray tracing computation.
Expanding the equation system (10.1) gives:

$$\frac{dx}{d\tau}=\frac{\partial H}{\partial k_x}, \quad (10.5.a)$$

$$\frac{dy}{d\tau}=\frac{\partial H}{\partial k_y}, \quad (10.5.b)$$

$$\frac{dz}{d\tau}=\frac{\partial H}{\partial k_z}, \quad (10.5.c)$$

$$\frac{dt}{d\tau}=-\frac{\partial H}{\partial \omega}, \quad (10.5.d)$$

$$\frac{dk_x}{d\tau}=-\frac{\partial H}{\partial x}, \quad (10.5.e)$$

$$\frac{dk_y}{d\tau}=-\frac{\partial H}{\partial y}, \quad (10.5.f)$$

$$\frac{dk_z}{d\tau}=-\frac{\partial H}{\partial z}, \quad (10.5.g)$$

$$\frac{d\omega}{d\tau}=\frac{\partial H}{\partial t}, \quad (10.5.h)$$

with τ a time parameter that varies monotonically along the path. Similar equations in spherical coordinates can be written as:

$$\frac{dr}{d\tau}=\frac{\partial H}{\partial k_r}, \quad (10.6.a)$$

$$\frac{d\theta}{d\tau}=\frac{1}{r}\frac{\partial H}{\partial k_\theta}, \quad (10.6.b)$$

$$\frac{d\phi}{d\tau}=\frac{1}{r\sin\theta}\frac{\partial H}{\partial k_\phi}, \quad (10.6.c)$$

$$\frac{dt}{d\tau} = -\frac{\partial H}{\partial \omega}, \tag{10.6.d}$$

$$\frac{dk_r}{d\tau} = -\frac{\partial H}{\partial r} + k_\theta \frac{d\theta}{d\tau} + k_\phi \sin\theta \frac{d\phi}{d\tau}, \tag{10.6.e}$$

$$\frac{dk_\theta}{d\tau} = \frac{1}{r}\left(-\frac{\partial H}{\partial \theta} - k_\theta \frac{dr}{d\tau} + k_\phi r\cos\theta \frac{d\phi}{d\tau}\right), \tag{10.6.f}$$

$$\frac{dk_\phi}{d\tau} = \frac{1}{r\sin\theta}\left(-\frac{\partial H}{\partial \phi} - k_\phi \sin\theta \frac{dr}{d\tau} - k_r r\cos\theta \frac{d\theta}{d\tau}\right), \tag{10.6.g}$$

$$\frac{d\omega}{d\tau} = \frac{\partial H}{\partial t}. \tag{10.6.h}$$

Figure 6 shows the spherical coordinates r, θ, ϕ and wave vector components k_r, k_θ, k_ϕ, in a geocentric computational framework.

A practical ray-tracing computational program, in order to integrate the equation system (10.6), requires another independent variable that allows the computational algorithm to be simplified. As an independent variable τ it is convenient to assume the group path P' rather than the time parameter τ. The group path P' is equal to ct with c the velocity of light in vacuum, i.e. $\tau = ct = P'$. In this context equation system (10.6) applying the derivative chain rule for partial derivative relative to P' gives:

$$\frac{dr}{dP'} = -\frac{1}{c}\frac{\frac{\partial H}{\partial k_r}}{\frac{\partial H}{\partial \omega}}, \tag{10.7.a}$$

$$\frac{d\theta}{dP'} = -\frac{1}{rc}\frac{\frac{\partial H}{\partial k_\theta}}{\frac{\partial H}{\partial \omega}}, \tag{10.7.b}$$

$$\frac{d\phi}{dP'} = -\frac{1}{rc\sin\theta}\frac{\frac{\partial H}{\partial k_\phi}}{\frac{\partial H}{\partial \omega}}, \tag{10.7.c}$$

$$\frac{dt}{d\tau} = -\frac{\partial H}{\partial \omega}, \tag{10.7.d}$$

$$\frac{dk_r}{dP'} = \frac{1}{c}\frac{\frac{\partial H}{\partial r}}{\frac{\partial H}{\partial \omega}} + k_\theta \frac{d\theta}{dP'} + k_\phi \sin\theta \frac{d\phi}{dP'}, \tag{10.7.e}$$

$$\frac{dk_\theta}{dP'} = \frac{1}{r}\left(-\frac{1}{c}\frac{\frac{\partial H}{\partial \theta}}{\frac{\partial H}{\partial \omega}} - k_\theta \frac{dr}{dP'} + k_\phi r\cos\theta \frac{d\phi}{dP'}\right), \tag{10.7.f}$$

$$\frac{dk_\phi}{dP'} = \frac{1}{r\sin\theta}\left(\frac{1}{c}\frac{\frac{\partial H}{\partial \phi}}{\frac{\partial H}{\partial \omega}} - k_\phi \sin\theta \frac{dr}{dP'} - k_r r\cos\theta \frac{d\theta}{dP'}\right), \tag{10.7.g}$$

$$\frac{d\omega}{d\tau} = \frac{\partial H}{\partial t}. \tag{10.7.h}$$

This system has eight equations with eight independent variables that assume the values able to satisfy the system for a certain P'. The ray tracing algorithm is performed while P' is monotonically increased step by step. In other words, starting in a fixed point where the Hamiltonian H has physical significance, in the present case in ionospheric magneto-plasma with refractive index n, the choice of H as in equation (10.4) directly gives the following equations:

$$\frac{\partial H}{\partial r} = -n\frac{\partial n}{\partial r}, \tag{10.8.a}$$

$$\frac{\partial H}{\partial \theta} = -n\frac{\partial n}{\partial \theta}, \tag{10.8.b}$$

$$\frac{\partial H}{\partial \phi} = -n\frac{\partial n}{\partial \phi}, \tag{10.8.c}$$

$$\frac{\partial H}{\partial t} = -n\frac{\partial n}{\partial t}, \tag{10.8.d}$$

$$\frac{\partial H}{\partial k_r} = \frac{c^2}{\omega^2}k_r - \frac{c}{\omega}n\frac{\partial n}{\partial V_r}, \tag{10.8.e}$$

$$\frac{\partial H}{\partial k_\theta} = \frac{c^2}{\omega^2}k_\theta - \frac{c}{\omega}n\frac{\partial n}{\partial V_\theta}, \tag{10.8.f}$$

$$\frac{\partial H}{\partial k_\phi} = \frac{c^2}{\omega^2}k_\phi - \frac{c}{\omega}n\frac{\partial n}{\partial V_\phi}, \tag{10.8.g}$$

$$\frac{\partial H}{\partial \omega} = -n\frac{n'}{\omega}, \tag{10.8.h}$$

$$\vec{k}\cdot\frac{\partial H}{\partial \vec{k}} = k_r\frac{\partial H}{\partial k_r} + k_\theta\frac{\partial H}{\partial k_\theta} + k_\phi\frac{\partial H}{\partial k_\phi}, \tag{10.8.i}$$

where:

$$n' = n + f\frac{dn}{df} = n + \omega\frac{dn}{d\omega}, \tag{10.9}$$

$$V_r^2 + V_\theta^2 + V_\phi^2 = \mathrm{Re}(n^2). \tag{10.10}$$

The refractive index n as in the Appleton-Hartree formula is:

$$n^2 = 1 - \frac{X}{1 - jZ - \frac{Y_T^2}{2(1-X-jZ)} \pm \sqrt{\frac{Y_T^4}{4(1-X-jZ)^2} + Y_L^2}}, \tag{10.11}$$

where X, Y, and Z are a-dimensional parameters. These are well known quantities in magneto-ionic theory [Bianchi, 1990] and refer respectively to:

$$X = \frac{f_N^2}{f^2} = \frac{\omega_N^2}{\omega^2}, \tag{10.12.a}$$

$$Y = \frac{f_B}{f} = \frac{\omega_B}{\omega}, \tag{10.12.b}$$

$$Z = \frac{\nu}{f}, \tag{10.12.c}$$

where f, f_N, f_B and ν are respectively the wave, plasma, cyclotron and collision frequency, while the subscript terms L and T indicate the longitudinal and transversal projection of Y along the direction of the wave vector $\vec{k}$.

Conclusions

This work deals with ray theory and ray tracing formulation. As well known the problems of radio propagation in the ionosphere constitute a challenge since mathematical formulation are often cumbersome especially for high frequencies (HF) and lower frequencies that interact heavily with this ionospheric plasma. To face this problem the theory of classical geometrical optics was described, and applications to ionospheric propagation are detailed in the last paragraph. We feel that this work can give useful theoretical background to scientists involved in research fields like optics, electromagnetism, radio propagation especially in inhomogeneous anisotropic media, as well as other research branches that use Hamiltonian formalism. The analogies with a mechanical system are relevant too, since certain principles can be easily applied in their evolution to determine the trajectory of a material point. In the radio wave theory many of these concepts and methods apply perfectly. Application in HF radio propagation, radio communication, over-the-horizon-radar (OTHR) coordinate registration and related homing techniques, HF wave direction finding, all rely on ray tracing computational algorithm.

The theory is written for a time independent medium and is then extended to more general cases of an anisotropic, dispersive and time-variant medium (ionospheric magnetoplasma). Hence, the theory leads to the formulation of the canonical, or Hamiltonian, equations related to the wave ray in a continuous, inhomogeneous and anisotropic medium with negligible gradient. When the medium discontinuities are high, the calculation of the ray path using this technique is no longer valid. The six Hamilton's equations are written in Cartesian and spherical coordinates in the simplest way found in literature. These are achieved by introducing the equations of the refractive index surface and the ray surface equations in an appropriate dimensionless space. By the combination of these equations, even the Fermat's principle has been derived and this gives more generality to the formulation of ray theory. In fact based on this principle only a particular path where the ray takes the minimum time is permitted, while the ray does not propagate along any of the other possible paths. In this work we have shown how the canonical equations are dependent on constant quantity H

and the coordinates and how wave vector direction along the whole path can be calculated. These quantities generically indicated as $r_i(\tau)$, $p_i(\tau)$ are dependent on the parameter, that must increase monotonically along the path. Effectively, the procedure described above is the ray tracing formulation. In ray tracing computational techniques, the most convenient Hamiltonian describing the medium can be adopted and the simplest ways to choose *H* will be discussed. Finally, a system of equations, which can be numerically solved, is generated.

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