Soliton Solutions of Noncommutative Integrable Systems

by

Craig M. Sooman

A thesis submitted to the

Faculty of Information and Mathematical Sciences

at the University of Glasgow

for the degree of

Doctor of Philosophy

September 2009

This thesis is concerned with solutions of noncommutative integrable systems where the noncommutativity arises through the dependent variables in either the hierarchy or Lax pair generating the equation.

Both Chapters 1 and 2 are entirely made up of background material and contain no new material. Furthermore, these chapters are concerned with commutative equations.

Chapter 1 outlines some of the basic concepts of integrable systems including historical attempts at finding solutions of the KdV equation, the Lax method and Hirota's direct method for finding multi-soliton solutions of an integrable system. Chapter 2 extends the ideas in Chapter 1 from equations of one spatial dimension to equations of two spatial dimensions, namely the KP and mKP equations. Chapter 2 also covers the concepts of hierarchies and Darboux transformations. The Darboux transformations are iterated to give multi-soliton solutions of the KP and mKP equations. Furthermore, this chapter shows that multi-soliton solutions can be expressed as two types of determinant: the Wronskian and the Grammian. These determinantal solutions are then verified directly.

In Chapter 3, the ideas detailed in the preceding chapters are extended to the noncommutative setting. We begin by outlining some known material on quasideterminants, a noncommutative KP hierarchy containing a noncommutative KP equation, and also two families of solutions. The two families of solutions are obtained from Darboux transformations and can be expressed as quasideterminants. One family of solutions is termed "quasiwronskian" and the other "quasigrammian" as both reduce to Wronskian and Grammian determinants when their entries commute. Both families of solutions are then verified directly. The remainder of Chapter 3 is original material, based on joint work with Claire Gilson and Jon Nimmo. Building on some known results, the solutions obtained from the Darboux transformations are specified as matrices. These solutions have interesting interaction properties not found in the commutative setting. We therefore show various plots of the solutions illustrating these properties.

In Chapter 4, we repeat all of the work of Chapter 3 for a noncommutative mKP equation. The material in this chapter is again based on joint work with Claire Gilson and Jon Nimmo and is mainly original.

The original material in Chapters 3 and 4 appears in [20] and in [21].

Chapter 5 builds on the work of Chapters 3 and 4 and is concerned with exponentially localised structures called dromions, which are obtained by taking the determinant of the matrix solutions of the noncommutative KP and mKP equations. For both equations, we look at a three-dromion structure from which we then perform a detailed asymptotic analysis. This aymptotic forms show interesting interaction properties which are demonstrated by various plots. This chapter is entirely the author's own work.

Chapter 6 presents a summary and conclusions of the thesis.

Firstly, I would like to thank my supervisors, Dr Claire Gilson and Dr Jon Nimmo, for their support and encouragement shown to me throughout the development of this thesis. Thanks are also due to the Engineering and Physical Sciences Research Council for the financial support which enabled me to undertake the research required to compile this thesis.

There are several people in the department of mathematics at the University of Glasgow who made the department a pleasant and enjoyable place in which to learn. The list of such people is exhaustive, but I would particularly like to express my gratefulness to the janitors, fellow postgraduates, academic staff, secretarial staff and several undergraduates whom I enjoyed tutoring.

I would also like to thank both my mother and grandmother for their constant support and assistance during my time as an undergraduate and postgraduate at the University of Glasgow.

Finally, I must thank Monica, who for the last four and a half years, has been a continuous source of encouragement, support and friendship.

Craig M. Sooman

Edinburgh to York

10th January 2010

Contents

Chapter 1

Introduction

1.1 Sir John Scott Russell's observation of a solitary wave

The solitary wave can be traced back to 1834 when Sir John Scott Russell observed what he called the "great wave of translation" on the Union Canal in Scotland. Reporting his observation in [47], he described the physical characteristics of a mass of water put in motion by the sudden stoppage of a horse-drawn boat. According to Russell, the mass of water

...rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed.

Russell then attempted to recreate this phenomenon in laboratory experiments where he created solitary waves by dropping weights at one end of a water channel. He then deduced that the speed of a solitary wave, c, is given by

$$c = \sqrt{g(h+a)}, \tag{1.1}$$

where a denotes amplitude, h is the undisturbed depth of the water and g is the acceleration of gravity. From equation (1.1) it is clear that taller waves travel faster. Continuing on a mathematical theme, Boussinesq (1871) and Lord Rayleigh (1876), used (1.1) in order to find an expression for the wave profile. They showed that the wave profile $z = \zeta(x,t)$ is given by

$$\zeta(x,t) = a\,\mathrm{sech}^2\left(\beta(x-ct)\right), \tag{1.2}$$

where

$$\beta = \sqrt{\frac{3a}{4h^2(h+a)}}$$

for $a > 0$ and $\frac{a}{h} << 1$. Equation (1.2) represents a right-travelling wave with amplitude a, wavelength $\frac{1}{\beta}$ and speed c.

1.2 The Korteweg-de Vries equation

Despite being able to derive (1.2), neither Boussinesq or Lord Rayleigh managed to find an equation governing this wave profile. However, in 1895, Korteweg and de Vries were successful in finding a mathematical schematization to describe Russell's observation. They developed a nonlinear partial differential equation which governs one-dimensional waves. The celebrated Korteweg-de Vries [6] (normally abbreviated to KdV) equation is

$$u_t + \frac{\alpha\beta}{\gamma} u_{xxx} + \frac{\beta}{\gamma^3} u u_x = 0, \tag{1.3}$$

where the subscripts denote partial differentiation and α, β and γ are constants. Equation (1.3) is a general form of the KdV equation. In this thesis, we will use the following version of the KdV equation:

$$u_t + u_{xxx} + 6uu_x = 0. \tag{1.4}$$

To find a solitary-wave solution of equation (1.4), we let $u = l(x - ct) = l(\xi)$ for some constant $c > 0$ so that our solution is a right-travelling wave. Therefore l must satisfy

$$-cl' + 6ll' + l''' = 0, \tag{1.5}$$

where $' \equiv \frac{d}{d\xi}$. Integrating equation (1.5) once gives

$$-cl + 3l^2 + l'' = C_1, \tag{1.6}$$

where C_1 is an arbitrary constant. Multiplying both sides of equation (1.7) by l' and integrating again yields

$$-\frac{c}{2}l^2 + l^3 + \frac{1}{2}(l')^2 = C_1 l + C_2, \tag{1.7}$$

where C_2 is an arbitrary constant. By setting $C_1 = C_2 = 0$ so that we have a wave which has $l \to 0, l' \to 0$ and $l'' \to 0$ as $\xi \to \pm\infty$, we have the first-order ordinary differential equation

$$l' = \pm l\sqrt{c - 2l}. \tag{1.8}$$

By making the substitution $l = \frac{1}{2}\operatorname{sech}^2\theta$, equation (1.8) leads to the solution

$$u(x,t) = \frac{c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x - ct - x_0)\right), \tag{1.9}$$

where the choice $\pm$ has been eliminated since the solution is an even function, and x_0 is an arbitrary constant. In fact, x_0 plays an important role in the behaviour of the solution: it is the phase constant – the position of the peak of the wave at $t = 0$.

Some time passed before the KdV equation was shown to possess multi-soliton solutions. In 1965, Zabusky and Kruskal [33] considered the initial-value problem for a version of the KdV equation

$$u_t + uu_x + \delta^2 u_{xxx} = 0. \tag{1.10}$$

They solved this equation with $u(x,0) = \cos(\pi x), 0 \leq x \leq 2$ and u, u_x, u_{xxx} periodic on $[0,2]$. Their results showed that the initial profile seperated into eight sech^2-like functions propagating around the system with different speeds. The sech^2-like functions collided but emerged from interaction with all of their characteristics preserved. This is as a result of the balancing of the nonlinear and dispersive terms in the equation. Owing to these particle-like properties, Zabusky and Kruskal termed the solitary-wave solution a *soliton*, where the suffix -on indicates a particle. Whilst no precise mathematical definition of the soliton exists, Drazin and Johnson define solitons in [9] as any solution of a nonlinear equation (or system) which:

1. represents a wave of permanent form;
2. is localised, so that it decays or approaches a constant at infinity;
3. can interact strongly with other solitons and maintain its identity.

1.3 The Lax method

In 1968, Lax presented a method [35] which represents nonlinear evolution equations with differential operators that are linear in x. The work of Lax requires two operators, L and M, which operate on elements of $L^2(\mathbb{R})$, the space of integrable functions on the real line, endowed with an inner product

$$\langle \phi, \psi \rangle = \int_{-\infty}^{+\infty} \phi\, \psi\, dx. \tag{1.11}$$

Both L and M are self-adjoint so that $\langle L[\phi], \psi\rangle = \langle \phi, L[\psi]\rangle$ and $\langle M[\phi], \psi\rangle = \langle \phi, M[\psi]\rangle$ $\forall\, \phi, \psi \in L^2(\mathbb{R})$. In the spirit of finding exact solutions via inverse scattering, one has the spectral problem

$$L[\psi] = \lambda\psi, \tag{1.12}$$

so that ψ is an eigenfunction for L with eigenvalue λ. In addition, the eigenfunction ψ evolves in time according to

$$\psi_t = M[\psi]. \tag{1.13}$$

Lax showed that if equations (1.12) and (1.13) both hold, then the operators L and M satisfy the relation

$$L_t + [L, M] = 0, \tag{1.14}$$

where $[L, M] = LM - ML$ denotes the commutator of L and M. To see this, we differentiate both sides of (1.12) with respect to t and then substitute equation (1.13) into the resulting equation. Doing so gives

$$\begin{aligned} \lambda_t\psi &= L_t[\psi] + L[\psi_t] - \lambda\psi_t \\ &= L_t[\psi] + LM[\psi] - M[\lambda\psi] \\ &= L_t[\psi] + LM[\psi] - ML[\psi] \\ &= (L_t + [L, M])[\psi]. \end{aligned} \tag{1.15}$$

Solving equation (1.15) for nontrivial eigenfunction ψ and choosing $\lambda_t = 0$ gives equation (1.14). Since λ_t vanishes, every eigenvalue of L is a constant. Throughout this thesis, we use a more convenient but equivalent form of (1.14), by incorporating ∂_t into the operator M.

The KdV equation (1.3) provides us with a prototypical example of the Lax representation. If we choose the operators

$$\begin{aligned} L_{\mathrm{KdV}} &= \partial_x^2 + u, \\ M_{\mathrm{KdV}} &= 4\partial_x^3 + 6u\partial_x + 3u_x + \partial_t, \end{aligned} \tag{1.16}$$

then

$$[L_{\mathrm{KdV}}, M_{\mathrm{KdV}}][\psi] = (u_t + u_{xxx} + 6uu_x)\psi. \tag{1.17}$$

Therefore, we must have that $[L_{\mathrm{KdV}}, M_{\mathrm{KdV}}] = 0$ if and only if u is a solution of the KdV equation. When a nonlinear evolution equation can be represented in this way, it is said to have a *Lax representation* and the two operators used are referred to as a *Lax pair*.

1.4 The modified KdV equation

By making a simple modification to the nonlinear term in (1.3), we obtain the modified KdV (abbreviated to mKdV) equation

$$w_t + w_{xxx} - 6w^2 w_x = 0, \tag{1.18}$$

which will play an important role in what follows. In 1968, Miura [39] showed that if w satisfies equation (1.18), then u, defined by

$$u = -(w^2 + w_x), \tag{1.19}$$

satisfies the KdV equation (1.3). Substituting (1.19) into (1.3) gives

$$(2w + \partial_x)(w_t + w_{xxx} - 6w^2 w_x) = 0. \tag{1.20}$$

Therefore, every solution of the KdV equation (1.3) can be obtained from a solution of the mKdV equation (1.18). However, the converse of this statement is false.

Equation (1.18) also has a Lax representation: it can be thought of as the compatibility condition of the operators

$$\begin{aligned} L_{\text{mKdV}} &= \partial_x^2 + 2w\partial_x, \\ M_{\text{mKdV}} &= 4\partial_x^3 + 12w\partial_x^2 + 6(w_x + w^2)\partial_x + \partial_t. \end{aligned} \tag{1.21}$$

1.5 Hirota's direct method

Hirota proposed the direct method [28] in 1971. Hirota's method transforms an evolution equation into a type of bilinear differential equation via a transformation of the dependent variable. From this platform we can find exact solutions. In devising this method, Hirota introduced a new differential operator, the D-operator:

$$D_x^l D_t^m (a \cdot b) := \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^l \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^m a(x,t)b(x',t') \Bigg|_{x=x',\, t=t'}, \tag{1.22}$$

for nonnegative integers l and m. For example

$$\begin{aligned}
&D_xD_t(a \cdot b)\\
&= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right) a(x,t)b(x',t')\Big|_{x=x',\,t=t'}\\
&= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)(a_tb - ab_{t'})\Big|_{x=x',\,t=t'}\\
&= a_{xt}b - a_xb_{t'} - a_tb_{x'} + ab_{x't'}\big|_{x=x',\,t=t'}\\
&= a_{xt}b - a_xb_t - a_tb_x + ab_{xt}.
\end{aligned}$$

There are many properties of the D-operator that can be called upon to assist in solving differential equations. In what follows, the following three properties will be utilised:

$$D_xD_t\left(a \cdot 1\right) = a_{xt} = D_xD_t\left(1 \cdot a\right), \tag{1.23}$$

$$D_x^4\left(a \cdot 1\right) = a_{xxxx} = D_x^4\left(1 \cdot a\right), \tag{1.24}$$

$$D_x^mD_t^n \exp\Lambda_1 \cdot \exp\Lambda_2 = (\lambda_1 - \lambda_2)^m\left(\lambda_2^3 - \lambda_1^3\right)^n \exp\left(\Lambda_1 + \Lambda_2\right), \tag{1.25}$$

where $\Lambda_i = \lambda_i(x - \lambda_i^2t) + \lambda_{i_0}$, $i = 1, 2$ and λ_{i_0} is the phase-constant.

Exact solutions of the KdV equation can be found using Hirota's direct method. The first step is to make the dependent variable transformation

$$u = 2(\log\tau)_{xx}. \tag{1.26}$$

Substituting directly into the KdV equation (1.4) gives an equation involving τ:

$$\tau_{xt}\tau - \tau_x\tau_t + \tau_{xxxx}\tau - 4\tau_{xxx}\tau_x + 3\tau_{xx}^2 = 0. \tag{1.27}$$

Using the D-operator, we can express equation (1.27) as

$$(D_xD_t + D_x^4)\tau \cdot \tau = 0. \tag{1.28}$$

To find the solution τ, we expand it as a power series in $\epsilon << 1$:

$$\tau = 1 + \epsilon\tau_1 + \epsilon^2\tau_2 + \epsilon^3\tau_3 + \cdots \tag{1.29}$$

Substituting the above equation into (1.28) and collecting terms in each order of ϵ gives

$$\epsilon : D_x(D_t + D_x^3)(\tau_1 \cdot 1 + 1 \cdot \tau_1) = 0, \tag{1.30}$$

$$\epsilon^2 : D_x(D_t + D_x^3)(\tau_2 \cdot 1 + \tau_1 \cdot \tau_1 + 1 \cdot \tau_2) = 0, \tag{1.31}$$

$$\epsilon^3 : D_x(D_t + D_x^3)(\tau_3 \cdot 1 + \tau_2 \cdot \tau_1 + \tau_1 \cdot \tau_2 + 1 \cdot \tau_3) = 0, \tag{1.32}$$

$$\cdots$$

Properties (1.23) and (1.24) of the D-operator imply that the coefficient of ϵ (1.30) is equivalent to

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial x^3}\right)\tau_1 = 0. \tag{1.33}$$

One solution of the above equation is

$$\tau_1 = e^{\Lambda_1}. \tag{1.34}$$

Using properties (1.23) and (1.24) of the D-operator, (1.31), the coefficient of ϵ^2, can be written as

$$2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial x^3}\right)\tau_2 = -D_x(D_t + D_x^3)\tau_1 \cdot \tau_1. \tag{1.35}$$

Upon substitution of (1.34) into (1.35), and using property (1.25) of the D-operator, we obtain

$$2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial x^3}\right)\tau_2 = 0. \tag{1.36}$$

We may choose the solution of equation (1.36) to be $\tau_2 = 0$. Similar calculations apply for τ_n and we may choose $\tau_n = 0$, $n = 2, 3, \ldots$, for all x,t. The expansion of τ can therefore be truncated at $\tau = 1 + \epsilon\tau_1$. Substituting this expression for τ into (1.26) gives

$$\begin{aligned} u &= 2\left(\log\left(1+e^{\Lambda_1}\right)\right)_{xx}, \\ &= \frac{1}{2}\lambda_1^2 \operatorname{sech}^2\left(\frac{1}{2}\Lambda_1\right), \end{aligned} \tag{1.37}$$

in which ϵ has been absorbed into the phase-constant λ_{1_0}. Equation (1.37) represents a travelling wave solution.

Hirota's method can also be used to find the two-soliton solution of the KdV equation. Consider equation (1.33). Since this equation is linear in τ_1, we may use the linear superposition principle and choose the solution

$$\tau_1 = e^{\Lambda_1} + e^{\Lambda_2}.$$

Upon substitution of this choice of τ into the coefficient of ϵ^2 and using property (1.25), we obtain

$$2\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial x^3}\right)\tau_2 = -2(\lambda_1-\lambda_2)^4 e^{(\Lambda_1+\Lambda_2)}, \tag{1.38}$$

which has solution $\tau_2 = \left(\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}\right)^2 e^{(\Lambda_1+\Lambda_2)}$. Using (1.23), (1.24) and substituting τ_1 and τ_2 into the coefficient of ϵ^3 gives

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+\frac{\partial^3}{\partial x^3}\right)\tau_3 = 0. \tag{1.39}$$

We choose $\tau_3 = 0$ as a solution of (1.39) and may then choose $\tau_n = 0$, $n = 3, 4, \ldots$, for all x, t. The expansion of τ can therefore be truncated at $1 + \epsilon\tau_1 + \epsilon^2\tau_2$. So we have

$$\tau = 1 + \epsilon\left(e^{\Lambda_1} + e^{\Lambda_2}\right) + \epsilon^2\left(\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}\right)^2 e^{(\Lambda_1+\Lambda_2)}. \tag{1.40}$$

By substituting (1.40) into (1.26) and absorbing ϵ into the phase-constants, the two-soliton solution can be written as

$$u = 2\left(\log\left(1 + e^{\Lambda_1} + e^{\Lambda_2} + a_{12}e^{(\Lambda_1+\Lambda_2)}\right)\right)_{xx}, \tag{1.41}$$

in which

$$a_{12} = \left(\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}\right)^2.$$

Hirota showed that if

$$a_{ij} = \left(\frac{\lambda_i-\lambda_j}{\lambda_i+\lambda_j}\right)^2,$$

then by writing $a_{ij} = e^{A_{ij}}$, the n-soliton solution can be expressed as

$$\tau = \sum \exp\left\{\sum_{n=1}^{i}\omega_i\Lambda_i + \sum_{i<j}^{(n)} A_{ij}\omega_i\omega_j\right\},$$

where $\sum$ is the summation over all possible combinations of $\omega_1 = 0,1$, $\omega_2 = 0,1$, $\ldots$, $\omega_n = 0,1$ and $\sum_{i<j}^{(n)}$ is the summation over all possible pairs (i,j) where $i,j \in \{1,2,\ldots,n\}$ and $i < j$.

1.6 Summary

In this chapter, an historical account of the soliton and its association with the KdV equation was discussed, as well as some elementary ideas from integrable systems which help lay the foundations for the material in this thesis. An outline of the construction of soliton solutions obtained from Hirota's method was given and will be referred to in the next chapter for equations in two spatial dimensions. Later in this thesis, we shall see that Hirota's method cannot be used to find soliton solutions of noncommutative integrable systems. It was shown that the KdV equation has a Lax representation and possesses multi-soliton solutions. The idea of a Lax representation is very important for future chapters as the Lax pair used is an essential ingredient needed to generate noncommutative equations. Furthermore, Lax pairs are heavily used in Darboux transformations, introduced in Chapter 2, which we later use as an alternative to Hirota's method for obtaining soliton solutions of noncommutative equations. We also introduced the mKdV equation and it was shown that the Miura transformation mapped solutions of the mKdV equation to solutions of the KdV equation. Most of the original work in this thesis centres around a noncommutative mKP equation, which can be thought of as a generalisation to two spatial dimensions of a noncommutative mKdV equation. In this work, in addition to finding soliton solutions, we also replicate the Miura transformation to map solutions of a noncommutative mKP equation to solutions of a noncommutative KP equation.

Chapter 2

The KP and mKP equations

In this chapter, we are concerned with generalisations to two spatial dimensions of the KdV and mKdV equations, which are the Kadomstev Petviashvili (KP) and modified KP (mKP) equations respectively. Both of these equations are known to have multi-soliton solutions which can be obtained from Hirota's direct method. Alternatively, they may be found from Darboux transformations, which we shall visit later in this chapter. The solutions can be expressed compactly as determinants and then verified directly. The main purpose of this chapter is to demonstrate these methods for both the KP and mKP equations, as the results that we will obtain can be related to noncommutative results in later chapters. Let us begin with the KP equation, which serves as a prototypical example.

2.1 The Kadomtsev-Petviashvili equation

The KP equation is

$$(u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0, \tag{2.1}$$

which can also be written in potential form

$$(v_t + v_{xxx} + 3v_x^2)_x + 3v_{yy} = 0, \tag{2.2}$$

where $u = v_x$. Kadomtsev and Petviashvili [31] derived the equation in 1971 and it was subsequently named after them. By neglecting the y-derivative term in (2.1), we recover the KdV equation (1.4). The Lax pair for the KP equation is

$$L_{\mathrm{KP}} = \partial_x^2 + v_x - \partial_y, \tag{2.3}$$

$$M_{\mathrm{KP}} = 4\partial_x^3 + 6v_x\partial_x + 3v_{xx} + 3v_y + \partial_t, \tag{2.4}$$

whose compatibility condition $[L_{\mathrm{KP}}, M_{\mathrm{KP}}] = 0$ gives (2.2).

2.1.1 The KP hierarchy

The KP hierarchy is an infinite set of nonlinear evolution equations in infinitely many functions $u, u_1, u_2, \ldots$ of the infinitely many variables $x_1, x_2, x_3, \ldots$. There are various approaches to constructing this hierarchy. In this thesis, we use the method of Gelfand and Dickii [15] and Sato [5, 48].

To construct the hierarchy, we need the following extended version of the Leibnitz rule:

$$\partial_x^i u = \sum_{j \geq 0} \binom{i}{j} \frac{\partial^j u}{\partial x^j} \, \partial_x^{i-j}, \tag{2.5}$$

for $i \in \mathbb{Z}$. We define the binomial coefficients in (2.5) to be

$$\binom{i}{j} = \begin{cases} 1 & (j = 0) \\ \frac{i(i-1)\cdots(i-j+1)}{j(j-1)\cdots 1} & (j \neq 0) \end{cases}.$$

For example,

$$\begin{aligned} \partial_x^{-1} u &= \binom{-1}{0} u \partial_x^{-1} + \binom{-1}{1} u_x \partial_x^{-2} + \binom{-1}{2} u_{xx} \partial_x^{-3} + \ldots \\ &= u \partial_x^{-1} - u_x \partial_x^{-2} + u_{xx} \partial_x^{-3} - \ldots, \\ \partial_x^{-2} u &= u \partial_x^{-2} - 2 u_x \partial_x^{-3} + 3 u_{xx} \partial_x^{-4} - \ldots, \\ \partial_x^{-3} u &= u \partial_x^{-3} - 3 u_x \partial_x^{-4} + 6 u_{xx} \partial_x^{-5} - \ldots . \end{aligned}$$

Construction of the KP hierarchy also requires the use of a *pseudodifferential* operator

$$\mathcal{L} = \sum_{i \in \mathbb{Z}} u_i \partial_x^{a-i},$$

of order $\leq a$. Associated with this pseudodifferential operator we consider natural projections $P_{\geq k}$, such that

$$P_{\geq k}(\mathcal{L}) = \sum_{i \geq k} u_i \partial_x^i.$$

For the KP hierarchy, we use the pseudodifferential operator

$$\mathcal{L}_{\mathrm{KP}} = \partial_x + \frac{1}{2} u \partial_x^{-1} + u_2 \partial_x^{-2} + u_3 \partial_x^{-3} + \ldots . \tag{2.6}$$

Let $\mathcal{L} = \mathcal{L}_{\mathrm{KP}}$. Then the KP hierarchy is defined to be

$$\mathcal{L}_{x_q} = [P_{\geq 0}(\mathcal{L}^q), \mathcal{L}], \quad q = 1, 2, 3, \ldots . \tag{2.7}$$

In general, the operators $P_{\geq 0}(\mathcal{L}^q)$ will be differential operators of order q associated with the fields $u, u_2, \ldots, u_{q-1}$. The first three natural projections $P_{\geq 0}(\mathcal{L}^q)$ are

$$\begin{aligned} P_{\geq 0}(\mathcal{L}) &= \partial_x, \\ P_{\geq 0}(\mathcal{L}^2) &= \partial_x^2 + u, \\ P_{\geq 0}(\mathcal{L}^3) &= \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{2}u_x + 3u_2. \end{aligned}$$

Thus, the evolution equation (2.7) gives

$$\mathcal{L}_{x_1} = [P_{\geq 0}(\mathcal{L}), \mathcal{L}] \Leftrightarrow \begin{cases} u_{x_1} &= u_x, \\ u_{2x_1} &= u_{2x}, \\ u_{3x_1} &= u_{3x}, \\ & \cdots, \end{cases} \tag{2.8}$$

$$\mathcal{L}_{x_2} = [P_{\geq 0}(\mathcal{L}^2), \mathcal{L}] \Leftrightarrow \begin{cases} u_y &= u_{xx} + 4u_{2x}, \\ u_{2y} &= u_{2xx} + 2u_{3x} + \frac{1}{2}uu_x, \\ u_{3y} &= u_{3xx} + 2u_{4x} - \frac{1}{2}uu_{xx} + 2u_2u_x, \\ & \cdots, \end{cases} \tag{2.9}$$

$$\mathcal{L}_{x_3} = [P_{\geq 0}(\mathcal{L}^3), \mathcal{L}] \Leftrightarrow \begin{cases} u_t &= u_{xxx} + 3uu_x + 6u_{2xx} + 6u_{3x}, \\ u_{2t} &= u_{2xxx} + 3(uu_2)_x + 3u_{3xx} + 3u_{4x}, \\ & \cdots, \end{cases} \tag{2.10}$$

where we have set $x_1 = x$, $x_2 = y$ and $x_3 = t$. These are the first three equations of the KP hierarchy. Other equations for $\mathcal{L}_{x_q} = [P_{\geq 0}(\mathcal{L}^q), \mathcal{L}]$ may also be considered for $q = 4, 5, \ldots$.

By integrating with respect to x, we can recursively express the fields $u_2, u_3, \ldots$ in terms of u and its x- and y-derivatives. The fields $u_2, u_3, \ldots$ can then be eliminated through (2.9) allowing us to rewrite the first component in (2.10) in terms of u and its x- and y-derivatives. The equation we obtain is the KP equation (2.1), where the scaling $t \to -4t$ has been made.

2.1.2 Wronskian solutions obtained from Hirota's method

When working in two spatial dimensions, the D-operator is defined by

$$\begin{aligned} & D_x^l D_y^m D_t^n a \cdot b \\ &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n a(x,y,t)b(x',y',t')\bigg|_{x=x',\, y=y',\, t=t'}, \end{aligned} \tag{2.11}$$

for nonnegative integers l, m and n.

The dependent variable transformation for the KP equation is

$$u = 2(\log\tau)_{xx}. \tag{2.12}$$

Substituting this into the KP equation (2.1) gives the bilinear form of the KP equation:

$$\tau\tau_{xt} - \tau_x\tau_t + \tau\tau_{xxxx} + 3\tau_{xx}^2 - 4\tau_x\tau_{xxx} + 3\tau\tau_{yy} - 3\tau_y^2 = 0, \tag{2.13}$$

$$(D_x^4 + D_xD_t + 3D_y^2)\tau\cdot\tau = 0. \tag{2.14}$$

Soliton solutions are obtained by using the same perturbation method outlined in Chapter 1. For the one-soliton solution, we truncate the series at $\tau = 1+\epsilon\tau_1$, so that

$$\tau_1 = 1 + e^{\Lambda_1},$$

where $\Lambda_1 = \eta_1 - \gamma_1 + \Lambda_{1_0}$, $\eta_1 = p_1(x + p_1y - 4p_1^2)t + \eta_{1_0}$, $\gamma_1 = q_1(x + q_1y - 4q_1^2)t + \gamma_{1_0}$, p_1, q_1, η_{1_0}, γ_{1_0} are constants and Λ_{1_0} is the phase-constant. Then we have the one-soliton solution

$$\begin{aligned} u &= 2(\log(1+e^{\Lambda_1}))_{xx}, \\ &= \frac{1}{2}(p_1-q_1)^2\operatorname{sech}^2\left(\frac{1}{2}\Lambda_{1_0}\right), \end{aligned}$$

where ϵ has been absorbed into the phase-constant Λ_{1_0}.

For the two-soliton solution, we obtain

$$\tau = 1 + e^{\Lambda_1} + e^{\Lambda_2} + \frac{(p_1-p_2)(q_1-q_2)}{(p_1-q_2)(q_1-p_2)}e^{(\Lambda_1+\Lambda_2)}. \tag{2.15}$$

Let us now define the functions

$$\theta_i = e^{\eta_i} + e^{\gamma_i},$$

in which $\eta_i = p_i(x + p_iy - 4p_i^2t) + \eta_{1_0}$, $\gamma_i = q_i(x + q_iy - 4q_i^2t) + \gamma_{1_0}$ and p_i, q_i, η_{1_0}, γ_{1_0} are constants, for $i = 1, 2, \ldots n$.

We can then show that the two-soliton solution, as given by (2.12), is equivalent to the Wronskian

$$\mathcal{W}(\theta_1,\theta_2) = \begin{vmatrix} \theta_1 & \theta_2 \\ \theta_{1,x} & \theta_{2,x} \end{vmatrix} \tag{2.16}$$

$$= (q_2-q_1)e^{\gamma_1+\gamma_2}\left(1 + \frac{q_2-p_1}{q_2-q_1}e^{\Lambda_1} + \frac{p_2-q_1}{q_2-q_1}e^{\Lambda_2} + \frac{p_2-p_1}{q_2-q_1}e^{\Lambda_1+\Lambda_2}\right), \tag{2.17}$$

in which $\Lambda_i = \eta_i - \gamma_i + \Lambda_{i_0}$, $i = 1, 2$ and Λ_{i_0} is the phase-constant. Since $u = 2(\log \tau)_{xx}$ is invariant under the transformation $\tau \to (q_2 - q_1)e^{\gamma_1+\gamma_2}\tau$,

$$\mathcal{W}(\theta_1, \theta_2) \equiv 1 + \frac{q_2 - p_1}{q_2 - q_1}e^{\Lambda_1} + \frac{p_2 - q_1}{q_2 - q_1}e^{\Lambda_2} + \frac{p_2 - p_1}{q_2 - q_1}e^{\Lambda_1+\Lambda_2}.$$

By choosing $\Lambda_{1_0} = \log\left(\frac{q_2-p_1}{q_2-q_1}\right)$ and $\Lambda_{2_0} = \log\left(\frac{p_2-q_1}{q_2-q_1}\right)$, the Wronskian $\mathcal{W}(\theta_1, \theta_2)$ may be written as

$$1 + e^{\Lambda_1} + e^{\Lambda_2} + \frac{(p_1 - p_2)(q_1 - q_2)}{(p_1 - q_2)(q_1 - p_2)}e^{(\Lambda_1+\Lambda_2)},$$

which is equal to (2.15). This result has been generalised [12, 45] to express the n-soliton solution of the KP equation as the $n \times n$ Wronskian

$$\tau = \mathcal{W}(\theta_1, \theta_2, \ldots, \theta_n), \tag{2.18}$$

where

$$\mathcal{W}(\theta_1, \theta_2, \ldots, \theta_n) = \begin{vmatrix} \theta_1 & \cdots & \theta_n \\ \theta_1^{(1)} & \cdots & \theta_n^{(1)} \\ \vdots & & \vdots \\ \theta_1^{(n-1)} & \cdots & \theta_n^{(n-1)} \end{vmatrix} \tag{2.19}$$

and $\theta^{(k)} := \dfrac{\partial^k \theta}{\partial x^k}$.

By choosing $p_n > q_n > \cdots > p_1 > q_1$, the Wronskian (2.19) is positive-definite and the n-soliton solution is regular.

Solutions of the KdV equation can be recovered by setting $p_i = -q_i = \lambda_i$. For example, the one-soliton Wronskian solution of the KP equation reduces from

$$u = \frac{1}{2}(p_1 - q_1)^2 \operatorname{sech}^2\left(\frac{1}{2}\Lambda_1\right)$$

to

$$u = 2\lambda_1^2 \operatorname{sech}^2\left(\lambda_1(x - 4\lambda_1^2 t)\right).$$

2.1.3 Wronskian solutions obtained from Darboux transformations

In this section, we introduce an alternative method to Hirota's for finding multi-soliton solutions of a nonlinear evolution equation.

As far back as 1882, the French mathematician Jean Gaston Darboux proved [13] that the Sturm-Louville equation

$$\frac{d^2y}{dx^2} + (\lambda - \upsilon(x))\, y = 0, \tag{2.20}$$

with ψ as a fixed solution, is covariant with respect to the transformation

$$y \to \tilde{y} = \frac{dy}{dx} - \psi_x \psi^{-1} y \quad \text{and} \quad \upsilon \to \tilde{\upsilon} = \upsilon - 2\left(\log(\psi)\right)_x .$$

Darboux's result means that $\tilde{y}$ satisfies the Sturm-Louville equation (2.20) with potential $\tilde{\upsilon}$, so that

$$\frac{d^2\tilde{y}}{dx^2} + (\lambda - \tilde{\upsilon}(x))\,\tilde{y} = 0.$$

Almost a century later, in 1979, Matveev [37] realised that a similar covariance property as Darboux's for the Sturm-Louville equation holds for all equations of the form

$$f_t = \sum_{m=0}^{n} \upsilon_m \frac{\partial^m f}{\partial x^m}, \tag{2.21}$$

where $f = f(x,t)$ and $\upsilon = \upsilon(x,t)$.

Returning to the KP equation, let $\theta = \theta(x,y,t)$ be an eigenfunction for L_{KP} and M_{KP} so that $L_{\text{KP}}[\theta] = M_{\text{KP}}[\theta] = 0$, which imply

$$\theta_{xx} + \upsilon_x\theta - \theta_y = 0, \tag{2.22}$$

$$4\theta_{xxx} + 6\upsilon_x\theta_x + 3\upsilon_{xx}\theta + 3\upsilon_{xx}\theta + 3\upsilon_y + \theta_t = 0. \tag{2.23}$$

Equations (2.22) and (2.23) are of similar form to those proposed by Matveev and are therefore Darboux covariant. To generate a new solution, we consider another pair of operators $\tilde{L}_{\text{KP}} = G_\theta L_{\text{KP}} G_\theta^{-1}$ and $\tilde{M}_{\text{KP}} = G_\theta M_{\text{KP}} G_\theta^{-1}$ in which G_θ is an invertible differential operator. By observing that

$$[\tilde{L}_{\text{KP}}, \tilde{M}_{\text{KP}}] = G L_{\text{KP}} M_{\text{KP}} G^{-1} - G M_{\text{KP}} L_{\text{KP}} G^{-1} = G[L_{\text{KP}}, M_{\text{KP}}]G^{-1} = 0,$$

we can see that $\tilde{L}_{\text{KP}}, \tilde{M}_{\text{KP}}$ are compatible if and only if $L_{\text{KP}}, M_{\text{KP}}$ are compatible.

Definition 1. A Darboux transformation from $L_{\text{KP}}, M_{\text{KP}}$ to $\tilde{L}_{\text{KP}}, \tilde{M}_{\text{KP}}$ is defined by $G_\theta = \theta\partial_x\theta^{-1}$ such that $G_\theta[0] = 0$.

Let ϕ be another eigenfunction for L_{KP}, M_{KP}. Then

$$\tilde{L}_{\text{KP}}[G_\theta[\phi]] = G_\theta L_{\text{KP}} G_\theta^{-1}[G_\theta[\phi]] = G_\theta[L_{\text{KP}}[\phi]] = G_\theta[0] = 0$$

and similarly, $\tilde{M}_{\text{KP}}[G_\theta[\phi]] = 0$. Therefore, $\tilde{\phi} := G_\theta[\phi]$ is an eigenfunction for $\tilde{L}_{\text{KP}}$, $\tilde{M}_{\text{KP}}$. Calculating $\tilde{L}_{\text{KP}}$ gives

$$\begin{aligned}\tilde{L}_{\text{KP}} &= \partial_x^2 + u + 2(\theta_{xx}\theta^{-1} - \theta_x^2\theta^{-2}) - \partial_y \\ &= \partial_x^2 + u + 2(\log\theta)_{xx} - \partial_y \\ &= \partial_x^2 + \tilde{u} - \partial_y.\end{aligned}$$

So the effect of the Darboux transformation is that

$$u \to \tilde{u} = u + 2(\log\theta)_{xx}. \tag{2.24}$$

We may conclude that if u is a solution of the KP equation and if $L[\theta] = 0 = M[\theta]$, then $\tilde{u}$ satisfies the KP equation too, that is:

$$(\tilde{u}_t + 6\tilde{u}\tilde{u}_x + \tilde{u}_{xxx})_x + 3\tilde{u}_{yy} = 0.$$

Repeating the process of determining $\tilde{u}$ using $\tilde{M}_{\mathrm{KP}}$ gives entirely consistent results. If we take the trivial vacuum solution $u = 0$, from $L[\theta] = 0 = M[\theta]$ we obtain

$$\theta_y = \theta_{xx} \quad \text{and} \quad \theta_t = -4\theta_{xxx}. \tag{2.25}$$

We choose the simplest solution of equations (2.25), which is

$$\theta = e^{\eta_1} + e^{\gamma_1}.$$

Here, we are using the same notation γ_i, η_i and Λ_i as in the previous section. Upon substitution of this choice of θ into (2.24), we have the one-soliton solution

$$\tilde{u} = \frac{1}{2}\operatorname{sech}^2\left(\frac{1}{2}\Lambda_1\right).$$

In 1955, Crum [4] considered iterating Darboux's result and showed that the iterated solution could be formulated as the Wronskian determinant of eigenfunctions. He also showed that the Darboux transformation adds an eigenvalue to the spectrum of the Schrödinger operator: for nonlinear evolution equations such as the KP and mKP equations, this means that a soliton is added by each Darboux transformation. The key to this iteration is the transformation of the eigenfunction ϕ. Let θ_i, $i = 1, 2, \ldots, n$ be a particular set of invertible, distinct eigenfunctions. Furthermore, let $\phi = \phi_{[1]}$ be an eigenfunction for $L_{\mathrm{KP}[1]} = L_{\mathrm{KP}}$ and $\theta_{[1]} = \theta_1$. So

$$\begin{aligned}\phi_{[2]} = G_{\theta_{[1]}}[\phi_{[1]}] &= \phi_x - \theta_{1,x}\theta_1^{-1}\phi \\ &= \begin{vmatrix}\theta_1 & \phi \\ \theta_{1,x} & \phi_x\end{vmatrix} \Big/ \theta_1,\end{aligned}$$

is an eigenfunction for $L_{\mathrm{KP}[2]}$. For the second iteration

$$\phi_{[3]} = G_{\theta_{[2]}}[\phi_{[2]}] = \begin{vmatrix}\theta_1 & \theta_2 & \phi \\ \theta_{1,x} & \theta_{2,x} & \phi_x \\ \theta_{1,xx} & \theta_{2,xx} & \phi_{xx}\end{vmatrix} \Big/ \begin{vmatrix}\theta_1 & \theta_2 \\ \theta_{1,x} & \theta_{2,x}\end{vmatrix},$$

in which $\theta_{[2]} = \phi_{[2]}|_{\phi\to\theta_2}$, is an eigenfunction for $L_{\text{KP}[3]}$. Here, we have the linear equations

$$\theta_{i,y} = \theta_{i,xx} \quad \text{and} \quad \theta_{i,t} = -4\theta_{i,xxx},$$

for $i = 1, 2, \ldots, n$ and we choose the solutions $\theta_i = e^{\eta_i} + e^{\gamma_i}$.

After n iterations, for $n \geq 1$, we have

$$\phi_{[n+1]} = \frac{\mathcal{W}(\theta_1, \theta_2, \ldots, \theta_n, \phi)}{\mathcal{W}(\theta_1, \theta_2, \ldots, \theta_n)}, \tag{2.26}$$

where $\theta_{[k]} = \phi_{[k]}|_{\phi\to\theta_k}$. From each $G_{\theta_{[k]}}$ we obtain a new compatible Lax pair $L_{\text{KP}[n+1]}, M_{\text{KP}[n+1]}$, from which we obtain a new solution $u_{[n+1]}$. This class of solutions can be written compactly using the Wronskian determinant. For $n \geq 1$, we have:

$$u_{[n+1]} = u + 2\left(\log \mathcal{W}(\theta_1, \theta_2, \ldots, \theta_n)\right)_{xx}. \tag{2.27}$$

We can also obtain this family of solutions by transforming the pseudo-differential operator $\mathcal{L}_{\text{KP}}$. In [43] the authors show how to obtain the one-soliton solution from the Darboux transformation $\mathcal{L}_{\text{KP}} = G_\theta \mathcal{L}_{\text{KP}} G_\theta^{-1}$.

To see why the Darboux transformation $\mathcal{L}_{\text{KP}} = G_\theta \mathcal{L}_{\text{KP}} G_\theta^{-1}$ works, we need the following lemma [43]:

Lemma 1. Let $\mathcal{L} = \mathcal{L}_{\text{KP}}$. If $\tilde{\mathcal{L}} = G_\theta \mathcal{L} G_\theta^{-1}$ and $\tilde{\phi} = G_\theta[\phi]$, then

$$\begin{aligned}
\tilde{\mathcal{L}}_{x_q} - [P_{\geq 0}(\tilde{\mathcal{L}}^q), \tilde{\mathcal{L}}] &\equiv G_\theta(\mathcal{L}_{x_q} - [P_{\geq 0}(\mathcal{L}^q), \mathcal{L}])G_\theta^{-1} \\
&\quad - [\theta(\theta^{-1}|[\theta_{x_q} - P_{\geq 0}(\mathcal{L}^q)\theta]|)_x \partial^{-1}\theta^{-1}, \tilde{\mathcal{L}}], \\
|[\tilde{\phi}_{x_q} - P_{\geq 0}(\tilde{\mathcal{L}}^q)\tilde{\phi}]| &\equiv -\phi(\theta^{-1}|[\theta_{x_q} - P_{\geq 0}(\mathcal{L}^q)\theta]|)_x + \theta(\theta^{-1}|[\phi_{x_q} - P_{\geq 0}(\mathcal{L}^q)\phi]|)_x.
\end{aligned}$$

The notation $|[b]|$ is used to denote multiplication with the function b.

The eigenfunction for the hierarchy $\mathcal{L}_{x_q} = [P_{\geq 0}(\mathcal{L}^q), \mathcal{L}]$, where $\mathcal{L} = \mathcal{L}_{\text{KP}}$, is the function $\theta = \theta(x, x_q)$ satisfying the linear equations

$$\theta_{x_q} = P_{\geq 0}(\mathcal{L}^q)\theta, \quad q = 1, 2, 3, \ldots.$$

The above equations are compatible and may be considered simultaneously for different q's. With this definition of the eigenfunction θ, we may deduce from Lemma 1 that

$$\tilde{\mathcal{L}}_{x_q} - [P_{\geq 0}(\tilde{\mathcal{L}}^q), \tilde{\mathcal{L}}] \equiv 0 \quad \text{and} \quad |[\tilde{\phi}_{x_q} - P_{\geq 0}(\tilde{\mathcal{L}}^q)\tilde{\phi}]| \equiv 0.$$

So if $\mathcal{L}$ satisfies the KP hierarchy with eigenfunctions θ and ϕ, then $\tilde{\mathcal{L}}_{x_q} = G_\theta \mathcal{L} G_\theta^{-1}$ satisfies the hierarchy $\tilde{\mathcal{L}}_{x_q} = [P_{\geq 0}(\tilde{\mathcal{L}}^q), \tilde{\mathcal{L}}]$. Furthermore, $\tilde{\phi} = G_\theta[\phi]$ is an eigenfunction for $\tilde{\mathcal{L}}_{x_q} = [P_{\geq 0}(\tilde{\mathcal{L}}^q), \tilde{\mathcal{L}}]$, so that $\tilde{\phi}$ satisfies the linear equations

$$\tilde{\phi}_{x_q} = P_{\geq 0}(\tilde{\mathcal{L}}^q)\tilde{\phi}, \quad q = 1, 2, 3, \ldots.$$

Let us now calculate $\tilde{\mathcal{L}}$. We find that

$$\begin{aligned}
\tilde{\mathcal{L}} &= \theta \partial_x \theta^{-1} \left(\partial_x + \frac{1}{2} u \partial_x^{-1} + u_2 \partial_x^{-2} + u_3 \partial_x^{-3} + \ldots \right) \theta \partial_x^{-1} \theta^{-1} \\
&= \partial_x + \frac{1}{2} \left(u + 2(\ln(\theta))_{xx} \right) \partial_x^{-1} + \left(u_2 + \frac{1}{2} u_x - u \theta_{1,x} \theta_x^{-1} - \theta_x \theta_{xx} \theta^{-2} + \theta_x^3 \theta^{-3} \right) \partial_x^{-2} \\
&\quad + \ldots.
\end{aligned}$$

So the effect of the Darboux transformation is that

$$\begin{aligned}
u &\to \tilde{u} = u + 2(\ln(\theta))_{xx}, \\
u_2 &\to \tilde{u}_2 = u_2 + \frac{1}{2} u_x - u \theta_x \theta^{-1} - \theta_x \theta_{xx} \theta^{-2} + \theta_x^3 \theta^{-3}, \\
&\ldots.
\end{aligned}$$

The coefficients of $\tilde{\mathcal{L}}$ will satisfy (2.9) and (2.10). In particular, $\tilde{u}$ will satisfy the KP equation (2.1).

By considering n distinct eigenfunctions θ_i, $i = 1, 2, \ldots n$, the Darboux transformation can be iterated so that, schematically,

$$\mathcal{L}_{\mathrm{KP}[1]} \overset{G_{\theta_{[1]}}}{\to} \mathcal{L}_{\mathrm{KP}[2]} \overset{G_{\theta_{[2]}}}{\to} \cdots \overset{G_{\theta_{[n]}}}{\to} \mathcal{L}_{\mathrm{KP}[n+1]},$$

for $n \geq 1$. By taking the vacuum solution $u = 0$, we again obtain (2.26) and (2.27).

2.1.4 Grammian solutions obtained from Hirota's method

Grammians are determinants of matrices whose entries are in integral form. For the KP equation (2.1), the solution τ can be written as the Grammian $\det(\mathcal{G})$ [29], where

$$\mathcal{G} = \begin{pmatrix} c_{1,1} + \int_{-\infty}^x \theta_1 \rho_1 \, dx & \cdots & c_{i,n} + \int_{-\infty}^x \theta_1 \rho_n \, dx \\ \vdots & \ddots & \vdots \\ c_{n,1} + \int_{-\infty}^x \theta_n \rho_1 \, dx & \cdots & c_{n,n} + \int_{-\infty}^x \theta_n \rho_n \, dx \end{pmatrix},$$

and, for $i, j = 1, 2, \ldots, n$, the $c_{i,j}$ are constants. Both θ_i and ρ_i, $i = 1, 2, \ldots, n$ are functions of x, y and t satisfying the linear equations

$$\begin{aligned}
\theta_{i,y} &= \theta_{i,xx}, \quad \theta_{i,t} = -4\theta_{i,xxx}, \\
\rho_{i,xx} &= -\rho_{i,y}, \quad \rho_{i,t} = -4\rho_{i,xxx}.
\end{aligned}$$

By choosing the solutions of these equations to be $\theta = e^{\eta_i}$ and $\rho = e^{-\gamma_i}$, with $\eta_i = p_i(x + p_i y - 4p_i t)$ and $\gamma_i = q_i(x + q_i y - 4q_i t)$, the one-soliton solution is

$$u = 2\left(\log(\tau)\right)_{xx} = \frac{1}{2}(p-q)^2 \operatorname{sech}^2\left(\frac{1}{2}(\Lambda + \xi)\right), \tag{2.28}$$

in which $\tau = 1 + \frac{1}{p-q}e^{\eta-\gamma}$, $\Lambda = \eta - \gamma$ and $\xi = \log\left(\frac{1}{p-q}\right)$ is the phase-constant.

For the n-soliton solution,

$$u = 2\left(\log(\tau)\right)_{xx}, \tag{2.29}$$

where $\tau = \det \mathcal{G}$ and the entries in $\mathcal{G}$ are of the form

$$\mathcal{G}_{i,j} = \delta_{i,j} + \frac{1}{p_i - q_j} e^{\eta_i - \gamma_j}.$$

By choosing $p_n > q_n > \cdots p_1 > q_1$, $\det(\mathcal{G})$ will be positive-definite and the n-soliton solution will be regular.

By setting $p_i = -q_i = \lambda_i$, we recover solutions of the KdV equation. For example, the one-soliton solution

$$u = \frac{1}{2}(p-q)^2 \operatorname{sech}^2\left(\frac{1}{2}(\Lambda + \xi)\right)$$

reduces to

$$u = 2\lambda^2 \operatorname{sech}^2\left(\lambda\left(x - 4\lambda^2 t + \frac{1}{2\lambda}\log\left(\frac{1}{2\lambda}\right)\right)\right).$$

2.1.5 Grammian solutions obtained from binary Darboux transformations

Grammian solutions of the KP equation can also be found using binary Darboux transformations where there are two eigenfunctions transforming. The additional eigenfunction comes from an adjoint system.

Let $\mathcal{H}$ be a Hilbert space with an inner product $\langle\ ,\ \rangle$, and let $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ be a differential operator. Then there exists a differential operator $\mathcal{A}^\dagger : \mathcal{H} \to \mathcal{H}$ with the property $\langle \mathcal{A}[a], b\rangle = \langle a, \mathcal{A}[b]\rangle$ for all a, $b \in \mathcal{H}$. This gives us the properties

1. $(\mathcal{A}\mathcal{B})^\dagger = \mathcal{B}^\dagger\mathcal{A}^\dagger$,

2. $\mathcal{A}^{\dagger\dagger} = \mathcal{A}$,

3. $(\mathcal{A} + \mathcal{B})^\dagger = \mathcal{A}^\dagger + \mathcal{B}^\dagger$,

4. If $\mathcal{A}$ is invertible, so is $\mathcal{A}^\dagger$ and then $(\mathcal{A}^\dagger)^{-1} = (\mathcal{A}^{-1})^\dagger = \mathcal{A}^{-\dagger}$.

For (matrix) differential operators acting on complex vectors,

$$\langle a, b\rangle = \int_{-\infty}^{+\infty} b^\dagger a \, dx.$$

Then integration by parts gives us $(u\partial_x^i)^\dagger = (-1)^i \partial_x^i u^\dagger$, which we can use for all partial derivatives in differential operators.

Binary Darboux transformations for the KP hierarchy have been discussed in [44]. However, here we shall only give the construction in terms of the Lax pair.

We now calculate the adjoint of the Lax pair $L_{\mathrm{KP}}, M_{\mathrm{KP}}$, which is

$$L_{\mathrm{KP}}^\dagger = \partial_x^2 + v_x^\dagger + \partial_y, \tag{2.30}$$

$$M_{\mathrm{KP}}^\dagger = -4\partial_x^3 - 6v_x^\dagger\partial_x - 3v_{xx}^\dagger + 3v_y^\dagger - \partial_t. \tag{2.31}$$

If $[L_{\mathrm{KP}}, M_{\mathrm{KP}}] = 0$, then $[L_{\mathrm{KP}}^\dagger, M_{\mathrm{KP}}^\dagger] = 0$ and the compatibility condition $[L_{\mathrm{KP}}^\dagger, M_{\mathrm{KP}}^\dagger] = 0$ gives

$$(u_t^\dagger + u_{xxx}^\dagger + 6u^\dagger u_x^\dagger)_x + 3u_{yy}^\dagger = 0,$$

which is the adjoint of (2.1), the KP equation.

We have seen that $\tilde{L}_{\mathrm{KP}} = G_\theta L_{\mathrm{KP}} G_\theta^{-1}$ with $G_\theta = \theta\partial_x\theta^{-1}$. The adjoint of $\tilde{L}_{\mathrm{KP}}$ is $\tilde{L}_{\mathrm{KP}}^\dagger = G_\theta^\dagger L_{\mathrm{KP}}^\dagger G_\theta^{-\dagger}$ which can be rearranged to give $L_{\mathrm{KP}}^\dagger = G_\theta^{-\dagger}\tilde{L}_{\mathrm{KP}}^\dagger G_\theta^\dagger$. Similarly $M_{\mathrm{KP}}^\dagger = G_\theta^{-\dagger}\tilde{M}_{\mathrm{KP}}^\dagger G_\theta^\dagger$. So the Darboux transformation from $L_{\mathrm{KP}}, M_{\mathrm{KP}}$ to $\tilde{L}_{\mathrm{KP}}, \tilde{M}_{\mathrm{KP}}$ induces an *adjoint* Darboux transformation in the opposite direction from $\tilde{L}_{\mathrm{KP}}^\dagger, \tilde{M}_{\mathrm{KP}}^\dagger$ to $L_{\mathrm{KP}}^\dagger, M_{\mathrm{KP}}^\dagger$.

To describe the general form of the binary Darboux transformation, we consider another Lax pair $\hat{L}_{\mathrm{KP}}, \hat{M}_{\mathrm{KP}}$ with eigenfunction $\hat{\theta}$ such that $G_{\hat{\theta}} : \hat{L}_{\mathrm{KP}}, \hat{M}_{\mathrm{KP}} \to \tilde{L}_{\mathrm{KP}}, \tilde{M}_{\mathrm{KP}}$. Then we have the mapping

$$G_{\hat{\theta}}^{-1} G_\theta : L_{\mathrm{KP}}, M_{\mathrm{KP}} \to \hat{L}_{\mathrm{KP}}, \hat{M}_{\mathrm{KP}}.$$

However, this mapping can only be defined if we can determine $\hat{\theta}$. This can be achieved by first noticing that, from $\ker G_\theta^\dagger$ we obtain some nontrivial solution of the equations $L_{\mathrm{KP}}^\dagger[\theta] = M_{\mathrm{KP}}^\dagger[\theta] = 0$, which we denote by $i(\theta)$. The equation $G_\theta^\dagger[i(\theta)] = 0$ is satisfied by $i(\theta) = \theta^{-\dagger}$. Now, corresponding to $\hat{\theta} \in \ker \hat{L}_{\mathrm{KP}} \cap \ker \hat{M}_{\mathrm{KP}}$, there exists a solution $i(\hat{\theta}) \in \ker \tilde{L}_{\mathrm{KP}}^\dagger \cap \ker \tilde{M}_{\mathrm{KP}}^\dagger$. We can then use the mapping $G_\theta^{-\dagger} : L_{\mathrm{KP}}^\dagger, M_{\mathrm{KP}}^\dagger \to \tilde{L}_{\mathrm{KP}}^\dagger, \tilde{M}_{\mathrm{KP}}^\dagger$ to obtain $\hat{\theta} = i^{-1}(G_\theta^{-\dagger}[\rho])$ for any $\rho \in \ker L_{\mathrm{KP}}^\dagger \cap \ker M_{\mathrm{KP}}^\dagger$. This enables us to define the binary Darboux transformation for the KP equation.

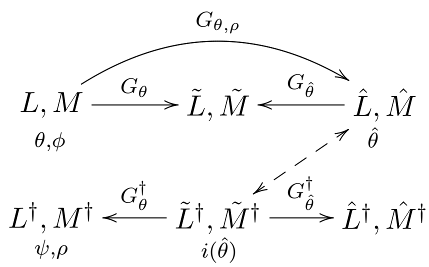


Figure 2.1: Construction of the binary Darboux transformation

Definition 2. For $\rho \in \ker L_{\mathrm{KP}}^\dagger \cap \ker M_{\mathrm{KP}}^\dagger$, we define $G_{\theta,\rho} = G_{\hat{\theta}}^{-1} G_\rho$, where $\hat{\theta} = i^{-1}(G_\theta^{-\dagger}[\rho])$. A binary Darboux transformation from $L_{\mathrm{KP}}, M_{\mathrm{KP}}$ to $\hat{L}_{\mathrm{KP}}, \hat{M}_{\mathrm{KP}}$ is defined by $G_{\theta,\rho}$ such that $G_{\theta,\rho}[0] = 0$.

Figure 2.1 illustrates the construction of the binary Darboux transformation.

To determine the binary Darboux transformation we must calculate $\hat{\theta}$. We have

$$\begin{aligned}\hat{\theta} &= \left(G_\theta^{-\dagger}[\rho]\right)^{-\dagger} \\ &= -\left(\theta^{-\dagger}\partial_x^{-1}\theta^\dagger\rho\right)^{-\dagger} \\ &= -\theta\Omega^{-1},\end{aligned}$$

where

$$\Omega = \partial_x^{-1}[\rho^\dagger\theta]. \tag{2.32}$$

So the binary Darboux transformation is

$$\begin{aligned}G_{\theta,\rho} = G_{\hat{\theta}}^{-1} G_\rho &= \theta\Omega^{-1}\partial_x^{-1}\Omega\partial_x\theta^{-1} \\ &= \theta\Omega^{-1}\partial_x^{-1}\left(\partial\Omega - \Omega_x\right)\theta^{-1} \\ &= 1 - \theta\Omega^{-1}\partial_x^{-1}\rho^\dagger.\end{aligned}$$

A similar calculation gives us

$$G_{\theta,\rho}^{-\dagger} = 1 - \rho\Omega^{-\dagger}\partial_x^{-1}\theta^\dagger.$$

Let ψ be another eigenfunction for $L_{\mathrm{KP}}^\dagger, M_{\mathrm{KP}}^\dagger$. Then

$$\hat{L}_{\mathrm{KP}}^\dagger[G_{\theta,\rho}^{-\dagger}[\psi]] = G_{\theta,\rho}^{-\dagger} L_{\mathrm{KP}}^\dagger G_{\theta,\rho}^\dagger[G_{\theta,\rho}^{-\dagger}[\psi]] = G_{\theta,\rho}^{-\dagger} L_{\mathrm{KP}}^\dagger[\psi] = G_{\theta,\rho}^{-\dagger}[0] = 0,$$

and similarly, $M_{\mathrm{KP}}^{\dagger}[G_{\theta,\rho}^{-\dagger}[\psi]] = 0$. Therefore, $\hat{\psi} := G_{\theta,\rho}^{-\dagger}[\psi]$ is an eigenfunction for $\hat{L}_{\mathrm{KP}}^{\dagger}, \hat{M}_{\mathrm{KP}}^{-\dagger}$. To calculate $\hat{u}$, we use the fact that both $L_{\mathrm{KP}}, M_{\mathrm{KP}}$ and $\hat{L}_{\mathrm{KP}}, \hat{M}_{\mathrm{KP}}$ map to $\tilde{L}_{\mathrm{KP}}, \tilde{M}_{\mathrm{KP}}$. So we have that

$$u + 2\left(\log(\theta)\right)_{xx} = \hat{u} + 2\left(\log(\theta\Omega^{-1}(\theta,\rho))\right)_{xx}.$$

Then isolating $\hat{u}$ gives

$$\hat{u} = u + 2\left(\log(\Omega(\theta,\rho))\right)_{xx}.$$

As was the case with the Darboux transformations, the binary Darboux transformation can be iterated to give an infinite family of solutions of the KP equation. Let θ_i be a particular set of invertible, distinct eigenfunctions of $L_{\mathrm{KP}[i+1]}$ and let ρ_i and ψ_i be a particular set of invertible, distinct eigenfunctions for $L_{\mathrm{KP}[i+1]}^{\dagger}$ for $i = 1, 2, \ldots, n$. Then the formulae for the nth binary Darboux transformations for the eigenfunction ϕ and the binary eigenfunction ψ are:

$$\phi_{[n+1]} = \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \Omega(\phi,\mathrm{P}) \\ \Theta & \phi \end{vmatrix} \Big/ \left|\Omega(\Theta,\mathrm{P})\right|,$$

$$\psi_{[n+1]} = \begin{vmatrix} \Omega(\Theta,\mathrm{P})^{\dagger} & \Omega(\Theta,\psi)^{\dagger} \\ \mathrm{P} & \psi \end{vmatrix} \Big/ \left|\Omega(\Theta,\mathrm{P})^{\dagger}\right|$$

and

$$\Omega(\phi_{[n+1]},\psi_{[n+1]}) = \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \Omega(\phi,\mathrm{P}) \\ \Omega(\Theta,\psi) & \Omega(\phi,\psi) \end{vmatrix} \Big/ \left|\Omega(\Theta,\mathrm{P})\right|.$$

In the above formulae, $\Theta = (\theta_1, \theta_2, \ldots, \theta_n)$, $\mathrm{P} = (\rho_1, \rho_2, \ldots, \rho_n)$ and Ω is defined by (2.32).

The class of solutions $u_{[n+1]}$ can be written compactly using the Grammian. For $n \geq 1$,

$$u_{[n+1]} = u + 2\left(\log\left|\Omega(\Theta,\mathrm{P})\right|\right)_{xx}. \tag{2.33}$$

For soliton solutions, we take the trivial vacuum solution $u = 0$. Then the eigenfunctions θ_i and the binary eigenfunctions ρ_i satisfy

$$\theta_{i,y} = \theta_{i,xx}, \quad \theta_{i,t} = -4\theta_{i,xxx},$$
$$\rho_{i,xx} = -\rho_{i,y}, \quad \rho_{i,t} = -4\rho_{i,xxx}.$$

Finally, we choose the solutions of the above to equations to be $\theta = e^{\eta_i}$ and $\rho = e^{-\gamma_i}$, with $\eta_i = p_i(x + p_i y - 4p_i t)$ and $\gamma_i = q_i(x + q_i y - 4q_i t)$. We then obtain the one-soliton solution (2.28).

2.2 The modified Kadomstsev-Petviashvili equation

The mKP equation is

$$w_{xt} + (w_{xxx} - 6w^2 w_x)_x + 3w_{yy} + 6w_x w_y + 6w_{xx} \int w_y \, dx = 0, \tag{2.34}$$

which can also be written in potential form

$$\mathcal{V}_{tx} + \mathcal{V}_{xxxx} - 6\mathcal{V}_x^2 \mathcal{V}_{xx} + 3\mathcal{V}_{yy} + 6\mathcal{V}_{xx}\mathcal{V}_y = 0, \tag{2.35}$$

where $w = \mathcal{V}_x$. It originated with Dubrovsky and Konopelchenko in [32]. By neglecting the y-derivative term in (2.34), we recover the mKdV equation (1.18). The mKP equation has the Lax pair

$$\begin{aligned} L_{\text{mKP}} &= \partial_x^2 + 2\mathcal{V}_x \partial_x - \partial_y, \\ M_{\text{mKP}} &= 4\partial_x^3 + 12\mathcal{V}_x \partial_x^2 + 6(\mathcal{V}_{xx} + \mathcal{V}_x^2 - \mathcal{V}_y)\partial_x + \partial_t. \end{aligned}$$

2.2.1 The mKP hierarchy

We now construct the mKP hierarchy using a similar analysis to that of the KP hierarchy. Here, we use the pseudodifferential operator

$$\mathcal{L}_{\text{mKP}} = \partial_x + w + w_1 \partial_x^{-1} + w_2 \partial_x^{-2} + \dots. \tag{2.36}$$

Let $\mathcal{L} = \mathcal{L}_{\text{mKP}}$. The mKP hierarchy is defined to be

$$\mathcal{L}_{x_q} = [P_{\geq 1}(\mathcal{L}^q), \mathcal{L}], \quad q = 1, 2, 3, \dots. \tag{2.37}$$

The first four projections are the differential operators:

$$\begin{aligned} P_{\geq 1}(\mathcal{L}) &= \partial_x, \\ P_{\geq 1}(\mathcal{L}^2) &= \partial_x^2 + 2w\partial_x, \\ P_{\geq 1}(\mathcal{L}^3) &= \partial_x^3 + 3w\partial_x^2 + 3(w_x + w^2 + w_1)\partial x, \\ P_{\geq 1}(\mathcal{L}^4) &= \partial_x^4 + 4w\partial_x^3 + (6w_x + 4w_1 + 6w^2)\partial_x^2 \\ &\quad + (4w^3 + 6w_{1x} + 4w_{xx} + 4w_2 + 12ww_x + 12ww_1)\partial_x. \end{aligned} \tag{2.38}$$

The evolution equation (2.37) gives

$$\mathcal{L}_{x_1} = [P_{\geq 1}(\mathcal{L}), \mathcal{L}] \Leftrightarrow \begin{cases} w_{x_1} &= w_x, \\ w_{1x_1} &= w_{1x}, \\ w_{2x_1} &= w_{2x}, \\ \dots & , \end{cases} \tag{2.39}$$

$$\mathcal{L}_{x_2} = [P_{\geq 1}(\mathcal{L}^2), \mathcal{L}] \Leftrightarrow \begin{cases} w_y &= w_{xx} + 2ww_x + 2w_{1x}, \\ w_{1y} &= w_{1xx} + 2(ww_1)_x + 2w_{2x}, \\ w_{2y} &= w_{2xx} - 2w_1 w_{xx} + 2ww_{2x} + 4w_x w_2 + 2w_{3x}, \\ \cdots &, \end{cases} \tag{2.40}$$

$$\mathcal{L}_{x_3} = [P_{\geq 1}(\mathcal{L}^3), \mathcal{L}] \Leftrightarrow \begin{cases} w_t &= w_{xxx} + 3w_{1xx} + 3w_{2x} + 3(ww_x)_x + 3w^2 w_x \\ & \quad +6(ww_1)_x, \\ \cdots &, \end{cases} \tag{2.41}$$

$$\mathcal{L}_{x_4} = [P_{\geq 1}(\mathcal{L}^4), \mathcal{L}] \Leftrightarrow \begin{cases} w_{x_4} &= w_{xxxx} + 6w_{2xx} + 4w_{1xxx} + 4w_{3x} + 4ww_{xxx} \\ & \quad +10w_x w_{xx} + 6w^2 w_{xx} + 12ww_x^2 + 4w^3 w_x \\ & \quad +12ww_{1xx} + 18w_x w_{1x} + 12w_1 w_{1x} \\ & \quad +6w_1 w_{xx} + 12w^2 w_{1x} + 24ww_x w_1 + 12ww_{2x} \\ & \quad +12w_x w_2, \\ \cdots &, \end{cases} \tag{2.42}$$

where again we have set $x_1 = x$, $x_2 = y$, $x_3 = t$. These are the first four equations of the mKP hierarchy. We can eliminate the fields $w_1, w_2, \ldots$ by expressing them in terms of the field w and its x- and y-derivatives. Eliminating w_1 and w_2 via (2.40) allows us to rewrite the first component in (2.41) in terms of w and its x- and y-derivatives. From this we obtain the mKP equation (2.34), where the scaling $t \to -4t$ has been made.

2.2.2 Wronskian solutions obtained from Hirota's method

The dependent variable transformation [30] for the mKP equation is

$$\mathcal{V} = \log\left(\frac{\acute{\tau}}{\tau}\right).$$

Substituting this into the mKP equation (2.34) gives the bilinear form [29, 30]

$$(D_x + D_y)\acute{\tau} \cdot \tau = 0, \tag{2.43}$$

$$(D_x^3 + D_t - 3D_x D_y)\acute{\tau} \cdot \tau = 0. \tag{2.44}$$

The n-soliton solution of the mKP equation [29] can be expressed as

$$w = (-1)^n \log\left(\frac{\tau}{\acute{\tau}}\right),$$

$$\tau = \mathcal{W}(\theta_1, \theta_2, \ldots, \theta_n),$$

$$\acute{\tau} = \mathcal{W}(\theta_{1,x}, \theta_{2,x}, \ldots, \theta_{n,x}),$$

in which $\theta_i = e^{\eta_i} + e^{\gamma_i}$, $\eta_i = p_i(x + p_i y - 4p_i^2 t)$ and $\gamma_i = q_i(x + q_i y - 4q_i^2 t)$ for $i = 1, 2, \ldots, n$.

Both $u = 2(\log \tau)_{xx}$ and $\acute{u} = 2(\log \acute{\tau})_{xx}$ are solutions of the KP equation (2.1).

The one-soliton Wronskian solution of the mKP equation is

$$w = \left(\log\left(\frac{\theta_x}{\theta}\right)\right)_x \tag{2.45}$$

$$= \frac{1}{4}(p_1 - q_1)^2 (p_1 q_1)^{-\frac{1}{2}} \operatorname{sech}\left(\frac{1}{2}(\Lambda_1)\right) \operatorname{sech}\left(\frac{1}{2}(\Lambda_1) + \log\left(\frac{p_1}{q_1}\right)\right), \tag{2.46}$$

where $\Lambda_1 = \eta_1 - \gamma_1$.

2.2.3 Wronskian solutions obtained from Darboux transformations

Darboux transformations can also be used to find a family of Wronskian solutions of the mKP equation. To obtain Wronskian solutions, we use a different differential operator G_θ from the KP equation. For the mKP equation, each of $\mathcal{L}_{\text{mKP}}$, L_{mKP} and M_{mKP}, with eigenfunction θ, is covariant under the Darboux transformation [41, 43]

$$G_\theta = ((\theta^{-1})_x)^{-1}\partial_x \theta^{-1} = 1 - \theta(\theta_x)^{-1}\partial_x.$$

We focus on $\mathcal{L}_{\text{mKP}}$ here, using the following lemma [43]:

Lemma 2. Let $\mathcal{L} = \mathcal{L}_{mKP}$.

1. If $\tilde{L} = \theta^{-1}\mathcal{L}\theta$ and $\tilde{\phi} = \theta^{-1}\phi$ then

$$\tilde{\mathcal{L}}_{x_q} - [P_{\geq 1}(\tilde{\mathcal{L}}^q), \tilde{\mathcal{L}}] \equiv \theta^{-1}(\mathcal{L}_{x_q} - [P_{\geq 1}(\mathcal{L}^q), \mathcal{L}])\theta - [\theta^{-1}|[\theta_{x_q} - P_{\geq 1}(\mathcal{L}^q)\theta]|, \tilde{\mathcal{L}}],$$
$$|[\tilde{\phi}_{x_q} - P_{\geq 1}(\tilde{\mathcal{L}}^q)\tilde{\phi}]| \equiv -\theta^{-2}\phi|[\theta_{x_q} - P_{\geq 1}(\mathcal{L}^q)\theta]| + \theta^{-1}|[\phi_{x_q} - P_{\geq 1}(\mathcal{L}^q)\psi]|. \tag{2.47}$$

2. If $\tilde{\mathcal{L}} = \theta_x^{-1}\partial_x\mathcal{L}\partial_x^{-1}\theta_x$ and $\tilde{\phi} = \theta_x^{-1}\phi_x$ then

$$\tilde{\mathcal{L}}_{x_q} - [P_{\geq 1}(\tilde{\mathcal{L}}^q), \tilde{\mathcal{L}}] \equiv \theta_x^{-1}\partial(\mathcal{L}_{x_q} - [P_{\geq 1}(\mathcal{L}^q), \mathcal{L}])\partial_x^{-1}\theta_x - [\theta_x^{-1}|[\theta_{x_q} - P_{\geq 1}(\mathcal{L}^q)\theta]|_x, \tilde{\mathcal{L}}],$$
$$|[\tilde{\phi}_{x_q} - P_{\geq 1}(\tilde{\mathcal{L}}^q)\tilde{\phi}]| \equiv -\theta_x^2\phi_x|[\theta_{x_q} - P_{\geq 1}(\mathcal{L}^q)\theta]|_x + \theta_x^{-1}|[\phi_{x_q} - P_{\geq 1}(\mathcal{L}^q)\phi]|_x. \tag{2.48}$$

The eigenfunction for the hierarchy $\mathcal{L}_{x_q} = [P_{\geq 1}(\mathcal{L}^q), \mathcal{L}]$, where $\mathcal{L} = \mathcal{L}_{\text{mKP}}$, is the function $\theta = \theta(x, x_q)$ satisfying the linear equations

$$\theta_{x_q} = P_{\geq 1}(\mathcal{L}^q)\theta, \quad q = 1, 2, 3, \ldots$$

The above equations are compatible and may be considered simultaneously for different q's. With this definition of the eigenfunction θ, we may deduce from Lemma 2 that

$$\tilde{\mathcal{L}}_{x_q} - [P_{\geq 1}(\tilde{\mathcal{L}}^q), \tilde{\mathcal{L}}] \equiv 0 \quad \text{and} \quad |[\tilde{\phi}_{x_q} - P_{\geq 1}(\tilde{\mathcal{L}}^q)\tilde{\phi}]| \equiv 0.$$

So if $\mathcal{L}$ satisfies the mKP hierarchy with eigenfunctions θ and ϕ, then $\tilde{\mathcal{L}}_{x_q} = \theta\mathcal{L}\theta^{-1}$ and $\tilde{\mathcal{L}}_{x_q} = \theta_x^{-1}\partial_x\mathcal{L}\partial_x^{-1}\theta_x$ satisfy the hierarchy $\tilde{\mathcal{L}}_{x_q} = [P_{\geq 1}(\tilde{\mathcal{L}}^q), \tilde{\mathcal{L}}]$. However, it is the composition of the two aforementioned transformations that we are interested in here since this will give us Wronskian solutions. In addition to θ and ϕ, the constant 1 is a trivial eigenfunction. Then from part 1 of Lemma 2, $\tilde{\mathcal{L}}_{x_q} = \theta\mathcal{L}\theta^{-1}$ with eigenfunctions $\tilde{\phi} = \theta^{-1}\phi$ and θ^{-1}. Using the eigenfunction θ^{-1} in part 2 of Lemma 2 gives

$$\tilde{\mathcal{L}} \to \tilde{\tilde{\mathcal{L}}} = ((\theta^{-1})_x)^{-1}\partial_x\tilde{\mathcal{L}}\partial_x^{-1}(\theta^{-1})_x = G_\theta\tilde{\mathcal{L}}G_\theta^{-1},$$

with eigenfunction $\tilde{\tilde{\phi}} = \phi - \theta(\theta_x)^{-1}\phi_x$. So if $\mathcal{L}$ satisfies the mKP hierarchy with eigenfunctions θ and ϕ, then $\tilde{\mathcal{L}}_{x_q} = G_\theta\mathcal{L}G_\theta^{-1}$ satisfies the hierarchy $\tilde{\mathcal{L}}_{x_q} = [P_{\geq 1}(\tilde{\mathcal{L}}^q), \tilde{\mathcal{L}}]$. Furthermore, $\tilde{\phi} = G_\theta[\phi]$ is an eigenfunction for $\tilde{\mathcal{L}}_{x_q} = [P_{\geq 1}(\tilde{\mathcal{L}}^q), \tilde{\mathcal{L}}]$, so that $\tilde{\phi}$ satisfies the linear equations

$$\tilde{\phi}_{x_q} = P_{\geq 1}(\tilde{\mathcal{L}}^q)\tilde{\phi}, \quad q = 1, 2, 3, \ldots.$$

Upon calculation of $\tilde{\mathcal{L}}_{\mathrm{mKP}} = G_\theta\mathcal{L}G_\theta^{-1}$, we obtain [43]

$$\tilde{\mathcal{L}}_{\mathrm{mKP}} = \partial_x + (w + \theta_x^{-1}\theta_{xx} - \theta^{-1}\theta_x) + (w_1 + w_x + (\theta^{-1}\theta_x)_x)\partial_x^{-1} + \ldots$$

So the effect of the Darboux transformation is that

$$\begin{aligned} w &\to \tilde{w} = w + \left(\log\left(\frac{\theta_x}{\theta}\right)\right)_x, \\ w_1 &\to \tilde{w_1} = w_1 + w_x + (\theta^{-1}\theta_x)_x, \\ &\ldots \end{aligned}$$

The coefficients of $\tilde{\mathcal{L}}$ will satisfy (2.40), (2.41) and (2.42). In particular, $\tilde{w}$ will satisfy the mKP equation (2.34).

Let θ_i, $i = 1, 2, \ldots, n$ be a particular set of invertible, distinct eigenfunctions. Furthermore, let $\phi = \phi_{[1]}$ be an eigenfunction for $\mathcal{L}_{\mathrm{mKP}[1]} = \mathcal{L}_{\mathrm{mKP}}$ and $\theta_{[1]} = \theta_1$. Then

$$\phi_{[2]} = \phi - \theta_x^{-1}\phi_x = -\begin{vmatrix} \theta_1 & \phi \\ \theta_{1,x} & \phi_x \end{vmatrix} \Big/ \theta_x$$

is an eigenfunction for $\mathcal{L}_{\mathrm{mKP}[2]}$. For the second transformation,

$$\begin{aligned}\phi_{[3]} &= \left(\phi\begin{vmatrix}\theta_{1,x} & \theta_{2,x}\\ \theta_{1,xx} & \theta_{2,xx}\end{vmatrix} - \phi_x\begin{vmatrix}\theta_1 & \theta_2\\ \theta_{1,xx} & \theta_{2,xx}\end{vmatrix} + \phi_{xx}\begin{vmatrix}\theta_1 & \theta_{1,x}\\ \theta_2 & \theta_{2,x}\end{vmatrix}\right)\Big/\begin{vmatrix}\theta_{1,x} & \theta_{2,x}\\ \theta_{1,xx} & \theta_{2,xx}\end{vmatrix}\\ &= \begin{vmatrix}\theta_1 & \theta_2 & \phi\\ \theta_{1,x} & \theta_{2,x} & \phi_x\\ \theta_{1,xx} & \theta_{2,xx} & \phi_{xx}\end{vmatrix}\Big/\begin{vmatrix}\theta_{1,x} & \theta_{2,x}\\ \theta_{1,xx} & \theta_{2,xx}\end{vmatrix}\\ &= \frac{\mathcal{W}(\theta_1,\theta_2,\phi)}{\mathcal{W}(\theta_{1,x},\theta_{2,x})},\end{aligned}$$

in which $\theta_{[2]} = \phi_{[2]}|_{\phi\to\theta_2}$, is an eigenfunction for $\mathcal{L}_{\mathrm{mKP}[3]}$. In general, for $n \geq 1$

$$\phi_{[n+1]} = (-1)^n \frac{\mathcal{W}(\theta_1,\theta_2,\ldots,\theta_n,\phi)}{\mathcal{W}(\theta_{1,x},\theta_{2,x},\ldots,\theta_{n,x})},$$

where $\theta_{[k]} = \phi_{[k]}|_{\phi\to\theta_k}$. From each $G_{\theta_{[k]}}$ we obtain a new covariant $\mathcal{L}_{\mathrm{mKP}[n+1]}$ from which we obtain a new solution $w_{[n+1]}$. This class of solutions can be written compactly using the Wronskian determinant. For $n \geq 1$, we have

$$w_{[n+1]} = w + (-1)^n \left(\frac{\log\left(\mathcal{W}(\theta_1,\theta_2,\ldots,\theta_n)\right)}{\log\left(\mathcal{W}(\theta_{1,x},\theta_{2,x},\ldots,\theta_{n,x})\right)}\right)_x. \tag{2.49}$$

2.2.4 Grammian solutions obtained from Hirota's method

For the mKP equation (2.35), both τ_n and τ_{n+1} can be written as the Grammian determinants $\det(\mathcal{G})$ and $\det(\acute{\mathcal{G}})$ [29], where the entries of $\mathcal{G}$ are of the form

$$\mathcal{G}_{i,j} = \delta_{i,j} - \frac{p_j}{q_i(p_i - q_j)} e^{\eta_i - \gamma_j}$$

and the entries of $\acute{\mathcal{G}}$ are of the form

$$\acute{\mathcal{G}}_{i,j} = \delta_{i,j} + \left(1 - \frac{1}{p_i - q_j}\right) e^{\eta_i - \gamma_j}$$

for $i, j = 1, 2, \ldots, n$.

For example, the one-soliton solution is

$$w = \frac{1}{4}(pq)^{-\frac{1}{2}}(p-q)^2 \operatorname{sech}\left(\frac{\Lambda+\varphi}{2}\right)\operatorname{sech}\left(\frac{\Lambda+\chi}{2}\right), \tag{2.50}$$

where $\varphi = \log\left(\frac{-p}{q(p-q)}\right)$ and $\chi = \log\left(\frac{-1}{p-q}\right)$.

For the n-soliton solution,

$$w = \left(\log\left(\frac{\acute{\tau}}{\tau}\right)\right)_{xx}, \tag{2.51}$$

where $\tau = \det(\mathcal{G})$ and $\acute{\tau} = \det(\acute{\mathcal{G}})$. Both $u = 2\left(\log(\tau)\right)_{xx}$ and $\acute{u} = 2\left(\log(\acute{\tau})\right)_{xx}$ are solutions of the KP equation.

By choosing $p_n > q_n > \cdots p_1 > q_1 > 0$ or $0 > p_n > q_n > \cdots p_1 > q_1$, both $\det(\mathcal{G})$ and $\det(\acute{\mathcal{G}})$ will be positive-definite and the n-soliton solution will be regular.

2.2.5 Grammian solutions obtained from Darboux transformations

Grammian solutions of the mKP equation can also be found from binary Darboux transformations. We use the same construction as for the KP equation as illustrated in Figure 2.1. As a notational convenience, we denote an element of $L^{\dagger}_{\text{mKP}} \cap M^{\dagger}_{\text{mKP}}$ as ρ_x rather than ρ. The equation $G^{\dagger}_{\theta}[i(\theta)]$ is satisfied by $i(\theta) = (\theta^{-\dagger})_x$. To determine $\hat{\theta}$, we need the integrals

$$\Omega = \partial_x^{-1}[\rho^{\dagger}\theta_x] \quad \text{and} \quad \Omega^{'} = \partial_x^{-1}[\rho_x^{\dagger}\theta],$$

where

$$\Omega + \Omega^{'} = \rho^{\dagger}\theta.$$

We have that

$$\begin{aligned} i(\hat{\theta}) &= (\hat{\theta}^{-\dagger})_x \\ &= G_{\theta}^{-\dagger}[\rho_x] \\ &= -(\theta^{-\dagger})_x \partial_x^{-1}\theta^{\dagger}\rho_x \end{aligned}$$

and therefore

$$\begin{aligned} (\hat{\theta}^{-1})_x &= -\partial_x^{-1}(\rho_x^{\dagger}\theta)(\theta^{-1})_x \\ &= -\Omega^{'}(\theta^{-1})_x. \end{aligned}$$

We can then isolate $\hat{\theta}$ by integrating by parts and then taking inverses. Doing so gives

$$\begin{aligned} \hat{\theta} &= (-\Omega^{'}\theta + \partial_x^{-1}(\Omega_x^{'}\theta^{-1}))^{-1} \\ &= (\rho^{\dagger} - \Omega^{'}\theta^{-1})^{-1} \\ &= \theta\Omega^{-1}. \end{aligned}$$

Now that we have determined $\hat{\theta}$, we may obtain

$$\begin{aligned} G_{\theta,\rho_x} &= \hat{\theta}\partial_x^{-1}(\hat{\theta}^{-1})_x(\theta^{-1})_x^{-1}\partial_x\theta^{-1} \\ &= -\theta\Omega^{-1}\partial_x^{-1}\Omega^{'}\partial_x\theta^{-1} \\ &= 1 - \theta\Omega^{-1}\partial_x^{-1}\rho^{\dagger}\partial_x. \end{aligned}$$

A similar calculation gives

$$G^{-\dagger}_{\theta,\rho_x} = 1 - \partial_x \rho \Omega'^{-\dagger} \partial_x^{-1} \theta^\dagger.$$

Let ψ be another eigenfunction for $L^\dagger_{\mathrm{mKP}}, M^\dagger_{\mathrm{mKP}}$. Then

$$\hat{L}^\dagger_{\mathrm{mKP}}[G^{-\dagger}_{\theta,\rho}[\psi]] = G^{-\dagger}_{\theta,\rho} L^\dagger_{\mathrm{mKP}} G^\dagger_{\theta,\rho}[G^{-\dagger}_{\theta,\rho}[\psi]] = G^{-\dagger}_{\theta,\rho} L^\dagger_{\mathrm{mKP}}[\psi] = G^{-\dagger}_{\theta,\rho}[0] = 0,$$

and similarly, $M^\dagger_{\mathrm{mKP}}[G^{-\dagger}_{\theta,\rho}[\psi]] = 0$. Therefore, $\hat{\psi} := G^{-\dagger}_{\theta,\rho}[\psi]$ is an eigenfunction for $\hat{L}^\dagger_{\mathrm{mKP}}, \hat{M}^\dagger_{\mathrm{mKP}}$. To calculate $\hat{w}$, we use the fact that both $L_{\mathrm{mKP}}, M_{\mathrm{mKP}}$ and $\hat{L}_{\mathrm{mKP}}, \hat{M}_{\mathrm{mKP}}$ map to $\tilde{L}_{\mathrm{mKP}}, \tilde{M}_{\mathrm{mKP}}$. So we have that

$$w + \left(\log\left(\frac{\theta_x}{\theta}\right)\right)_x = \hat{w} + \left(\log\left(\frac{\hat{\theta}_x}{\hat{\theta}}\right)\right)_x.$$

Then isolating $\hat{w}$ gives the one-soliton solution

$$\begin{aligned}\hat{w} &= w + \left(\log(1 - \theta\rho^\dagger\Omega^{-1}(\theta,\rho))\right)_x \\ &= w + \left(\log\left(\begin{vmatrix}\Omega(\theta,\rho) & \rho^\dagger \\ \theta & 1\end{vmatrix}\right) \Big/ \Omega(\theta,\rho)\right)_x.\end{aligned}$$

The binary Darboux transformation can be iterated to give an infinite family of solutions of the mKP equation. Let θ_i be a particular set of invertible, distinct eigenfunctions of $L_{\mathrm{mKP}[i+1]}$ and let ρ_i and ψ_i be a particular set of invertible, distinct eigenfunctions for $L_{\mathrm{mKP}[i+1]}$ for $i = 1, 2, \ldots, n$. Furthermore, let $\Theta = (\theta_1, \theta_2, \ldots, \theta_n)$ and $\mathrm{P} = (\rho_1, \rho_2, \ldots, \rho_n)$. Then the formulae for the nth binary Darboux transformations for the eigenfunction ϕ and the binary eigenfunction ψ are:

$$\phi_{[n+1]} = \begin{vmatrix}\Omega(\Theta,\mathrm{P}) & \Omega(\phi,\mathrm{P}) \\ \Theta & \phi\end{vmatrix} \Big/ \left|\Omega(\Theta,\mathrm{P})\right|,$$

$$\psi_{[n+1]} = \begin{vmatrix}\Omega(\Theta,\mathrm{P})^\dagger & \Omega(\Theta,\psi)^\dagger \\ \mathrm{P} & \psi\end{vmatrix} \Big/ \left|\Omega(\Theta,\mathrm{P})^\dagger\right|$$

and

$$\Omega(\phi_{[n+1]}, \psi_{[n+1]}) = \begin{vmatrix}\Omega(\Theta,\mathrm{P}) & \Omega(\phi,\mathrm{P}) \\ \Omega(\Theta,\psi) & \Omega(\phi,\psi)\end{vmatrix} \Big/ \left|\Omega(\Theta,\mathrm{P})\right|.$$

The class of solutions $w_{[n+1]}$ can be written compactly using the Grammian. For $n \geq 1$,

$$w_{[n+1]} = w + \left(\log\left(\begin{vmatrix}\Omega(\Theta,\mathrm{P}) & \mathrm{P}^\dagger \\ \Theta & I\end{vmatrix}\right) \Big/ \Omega(\Theta,\mathrm{P})\right)_x. \qquad (2.52)$$

For soliton solutions, we take the trivial vacuum solution $w = 0$. Then the eigenfunctions θ_i and the binary eigenfunctions ρ_i satisfy

$$\theta_{i,y} = \theta_{i,xx}, \quad \theta_{i,t} = -4\theta_{i,xxx},$$

$$\rho_{i,xx} = -\rho_{i,y}, \quad \rho_{i,t} = -4\rho_{i,xxx}.$$

Finally, we choose the solutions of the above equations to be $\theta = e^{\eta_i}$ and $\rho = e^{-\gamma_i}$, with $\eta_i = p_i(x + p_i y - 4p_i t)$ and $\gamma_i = q_i(x + q_i y - 4q_i t)$. We then obtain the one-soliton solution (2.50).

2.3 Direct verification of the solutions

2.3.1 Derivatives of Wronskian determinants

Consider the n-vector $\Theta = (\theta_1, \theta_2, \ldots, \theta_n)^T$ depending on $x_1 = x$ and possibly other variables $x_2, x_3, \ldots$. Using the notation $\Theta^{(i)}$ to denote the ith x-derivative of Θ with respect to x, we can define the Wronskian determinant

$$\tau = |\Theta^{(0)}, \Theta^{(1)}, \ldots, \Theta^{(n-1)}| = \det\left(\frac{\partial^{j-1}\theta_i}{\partial x^{j-1}}\right), \quad 1 \le i, j \le n.$$

The derivatives of τ with respect to x_j can be calculated from the basic result

$$\tau_{x_j} = \sum_{i=0}^{n-1} |\Theta^{(0)}, \ldots, \Theta^{(i-1)}, \Theta^{(i)}_{x_j}, \Theta^{(i+1)}, \ldots, \Theta^{(n-1)}|.$$

For example, since a determinant with two equal rows or columns vanishes,

$$\tau_x = |\Theta^{(0)}, \Theta^{(1)}, \ldots, \Theta^{(n-2)}, \Theta^{(n)}|.$$

Differentiating with respect to x again gives

$$\tau_{xx} = |\Theta^{(0)}, \Theta^{(1)}, \ldots, \Theta^{(n-2)}, \Theta^{(n+1)}| + |\Theta^{(0)}, \Theta^{(1)}, \ldots, \Theta^{(n-3)}, \Theta^{(n-1)}, \Theta^{(n)}|.$$

We can use a partition notation to denote derivatives of the Wronskian determinant τ. If we take a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_p)$, a sequence of positive integers, where $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_p$, then we can write derivatives of τ as

$$\tau_\lambda := \tau_{x_{\lambda_1} x_{\lambda_2} \ldots x_{\lambda_p}}. \tag{2.53}$$

For example,

$$\tau_{x_2 x_1} = \tau_{(21)} \quad \text{and} \quad \tau_{x_1 x_1} = \tau_{(1^2)}.$$

For the Wronskian determinant (2.19) this notation can be used to denote "shifts" in the index of the columns. For example,

$$\mathcal{W}_{(1)} := \left| \begin{array}{cccccc} \Theta^{(0)} & \Theta^{(1)} & \ldots & \Theta^{(n-2)} & \Theta^{(n)} \end{array} \right|$$
$$\mathcal{W}_{(2)} := \left| \begin{array}{cccccc} \Theta^{(0)} & \Theta^{(1)} & \ldots & \Theta^{(n-2)} & \Theta^{(n+1)} \end{array} \right|,$$

and in general,

$$\mathcal{W}_{\lambda} := \left| \begin{array}{ccccccc} \Theta^{(0)} & \ldots & \Theta^{(n-p-1)} & \Theta^{(n-p+\lambda_p)} & \ldots & \Theta^{(n-1+\lambda_1)} \end{array} \right|. \tag{2.54}$$

If Θ satisfies the linear equations

$$\Theta_{x_j} = \Theta^{(j)}, \quad j = 1, 2, 3, \ldots.$$

then we can write down the relationship between the derivatives τ_λ and the determinants $\mathcal{W}_\lambda$;

$$\tau_\lambda = \sum_\mu \zeta_\lambda^\mu \mathcal{W}_\mu, \tag{2.55}$$

where the sum is over all partitions μ and the matrices ζ_λ^μ are the character tables for the symmetric group S_n.

Some useful derivatives of τ are

$$\begin{aligned}
\begin{bmatrix} \tau_{(1)} \end{bmatrix} &= [1] \begin{bmatrix} \mathcal{W}_{(1)} \end{bmatrix}, \\
\begin{bmatrix} \tau_{(2)} \\ \tau_{(1^2)} \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{W}_{(2)} \\ \mathcal{W}_{(1^2)} \end{bmatrix}, \\
\begin{bmatrix} \tau_{(3)} \\ \tau_{(21)} \\ \tau_{(1^3)} \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{W}_{(3)} \\ \mathcal{W}_{(21)} \\ \mathcal{W}_{(1^3)} \end{bmatrix}, \\
\begin{bmatrix} \tau_{(4)} \\ \tau_{(31)} \\ \tau_{(2^2)} \\ \tau_{(21^2)} \\ \tau_{(1^4)} \end{bmatrix} &= \begin{bmatrix} 1 & -1 & 0 & 1 & -1 \\ 1 & 0 & -1 & 0 & 1 \\ 1 & -1 & 2 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & 3 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{W}_{(4)} \\ \mathcal{W}_{(31)} \\ \mathcal{W}_{(22)} \\ \mathcal{W}_{(21^2)} \\ \mathcal{W}_{(1^4)} \end{bmatrix}.
\end{aligned} \tag{2.56}$$

The relations (2.56) can be inverted using the formula

$$\mathcal{W} = \sum_\lambda l_\lambda^{-1} \zeta_\lambda^\mu \tau_\lambda,$$

where for a partition $\lambda = (r^{m_r}, \ldots, 2^{m_2}, 1^{m_1})$, $l_\lambda := \prod_{i=1}^{r} i^{m_i} m_i!$. Thus

$$\begin{aligned}
\begin{bmatrix} \mathcal{W}_{(1)} \end{bmatrix} &= [1] \begin{bmatrix} \tau_{(1)} \end{bmatrix}, \\
\begin{bmatrix} \mathcal{W}_{(2)} \\ \mathcal{W}_{(1^2)} \end{bmatrix} &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \tau_{(2)} \\ \tau_{(1^2)} \end{bmatrix}, \\
\begin{bmatrix} \mathcal{W}_{(3)} \\ \mathcal{W}_{(21)} \\ \mathcal{W}_{(1^3)} \end{bmatrix} &= \frac{1}{6} \begin{bmatrix} 2 & 3 & 1 \\ -2 & 0 & 2 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} \tau_{(3)} \\ \tau_{(21)} \\ \tau_{(1^3)} \end{bmatrix}, \\
\begin{bmatrix} \mathcal{W}_{(4)} \\ \mathcal{W}_{(31)} \\ \mathcal{W}_{(2^2)} \\ \mathcal{W}_{(21^2)} \\ \mathcal{W}_{(1^4)} \end{bmatrix} &= \frac{1}{24} \begin{bmatrix} 6 & 8 & 3 & 6 & 1 \\ -6 & 0 & -3 & 6 & 3 \\ 0 & -8 & 6 & 0 & 2 \\ 6 & 0 & -3 & -6 & 3 \\ -6 & 8 & 3 & -6 & 1 \end{bmatrix} \begin{bmatrix} \tau_{(4)} \\ \tau_{(31)} \\ \tau_{(22)} \\ \tau_{(21^2)} \\ \tau_{(1^4)} \end{bmatrix}.
\end{aligned} \tag{2.57}$$

2.3.2 Laplace expansion of determinants

A Laplace expansion is an expression of an nth-order determinant as a sum of products of rth- and $(n-r)$th-order determinants.

Consider an $n \times n$ matrix A with determinant $\det(A)$. We use $\Upsilon_{1,\cdots,m}^{j_1,\cdots,j_m}$ to denote the $m \times m$ determinant taken from $\det(A)$, where $1, \cdots, m$ and $j_1, \cdots, j_m$ $(m = n/2)$ are the rows and columns respectively of $\det(A)$. Furthermore, $\Xi_{1,\cdots,m}^{j_1,\cdots,j_m}$ denotes the $m \times m$ determinant obtained by deleting the $1, \cdots, m$ rows and $j_1, \cdots, j_m$ columns of $\det(A)$. We can now define A_l, the Laplace expansion of $\det(A)$, as

$$A_l = \sum_{j_1 < j_2 < \ldots < j_m} (-1)^{(1+\cdots+m)+(j_1+\cdots+j_m)} \Upsilon_{1,2,\cdots,m}^{j_1,\cdots,j_m} \Xi_{1,2,\cdots,m}^{j_1,\cdots,j_m}. \tag{2.58}$$

For example, if $n = 4$, then

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix},$$

$$\Upsilon_{1,2}^{1,2} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \qquad \Xi_{1,2}^{1,2} = \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} \quad \text{etc., and}$$

$$\begin{aligned} A_l = & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} \begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} \\ & + \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \begin{vmatrix} a_{31} & a_{34} \\ a_{41} & a_{44} \end{vmatrix} - \begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{33} \\ a_{41} & a_{43} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix} \begin{vmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}. \end{aligned}$$

2.3.3 Plücker relations

The term Plücker relation will be used to describe a type of quadratic identity amongst determinants $\mathcal{W}_\mu$. Consider the $2n \times 2n$ determinant

$$\Delta = \left| \begin{array}{ccc|cccccccc} \Theta^{(0)} & \dots & \Theta^{(n-3)} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \dots & \mathbf{0} & \Theta^{(0)} & \dots & \Theta^{(n-3)} & \Theta^{(n-2)} & \Theta^{(n-1)} & \Theta^{(n)} & \Theta^{(n+1)} \end{array} \right|, \tag{2.59}$$

where $\mathbf{0}$ denotes the zero n-vector.

Applying (2.58), the Laplace expansion of (2.59) results in $\Delta = 0$. This is the simplest case of a Plücker relation. We can now rearrange the right-hand side of (2.59) by adding the $(n+k)$th row to the kth row ($k = 1, 2, \dots, n$) and subtracting the kth column from the $(n-2-k)$th column ($k = 1, 2, \dots, n-2$). This gives

$$\Delta = \left| \begin{array}{ccc|cccccccc} \Theta^{(0)} & \dots & \Theta^{(n-3)} & \mathbf{0} & \dots & \mathbf{0} & \Theta^{(n-2)} & \Theta^{(n-1)} & \Theta^{(n)} & \Theta^{(n+1)} \\ \hline \mathbf{0} & \dots & \mathbf{0} & \Theta^{(0)} & \dots & \Theta^{(n-3)} & \Theta^{(n-2)} & \Theta^{(n-1)} & \Theta^{(n)} & \Theta^{(n+1)} \end{array} \right|. \tag{2.60}$$

From the Laplace expansion of (2.60) we obtain the identity

$$\begin{aligned} & \left| \Theta^{(0)} \; \dots \; \Theta^{(n-3)} \; \Theta^{(n)} \; \Theta^{(n+1)} \right| \left| \Theta^{(0)} \; \dots \; \Theta^{(n-1)} \right| \\ & \quad - \left| \Theta^{(0)} \; \dots \; \Theta^{(n-3)} \; \Theta^{(n-1)} \; \Theta^{(n+1)} \right| \left| \Theta^{(0)} \; \dots \; \Theta^{(n-2)} \; \Theta^{(n)} \right| \\ & \qquad + \left| \Theta^{(0)} \; \dots \; \Theta^{(n-3)} \; \Theta^{(n-1)} \; \Theta^{(n)} \right| \left| \Theta^{(0)} \; \dots \; \Theta^{(n-2)} \; \Theta^{(n+1)} \right| = 0, \end{aligned}$$

which in our partition notation is

$$\mathcal{W}_{(2^2)}\mathcal{W} - \mathcal{W}_{(21)}\mathcal{W}_{(1)} + \mathcal{W}_{(2)}\mathcal{W}_{(1^2)} = 0. \tag{2.61}$$

To verify the Wronksian solution of the KP equation, we need to show that the left hand side of (2.13) with $t \to -4t$, that is

$$(\tau_{(1^4)} + 3\tau_{(2^2)} - \tau_{(31)})\tau - 4(\tau_{(1^3)} - \tau_{(3)})\tau_{(1)} + 3(\tau_{(1^2)}^2 - \tau_{(2)}^2),$$

vanishes when $\tau = \mathcal{W}$. Using the tables (2.56), this condition becomes

$$\mathcal{W}_{(2^2)}\mathcal{W} - \mathcal{W}_{(21)}\mathcal{W}_{(1)} + \mathcal{W}_{(2)}\mathcal{W}_{(1^2)} = 0,$$

which is the Plücker relation (2.61). Hence the solution is verified.

For the mKP equation, the left hand sides of the coupled system (2.43)-(2.44) can be written in partition notation as

$$(\acute{\tau}_{(1^2)} + \acute{\tau}_{(2)})\tau - 2\tau_{(1)}\acute{\tau}_{(1)} + (\tau_{(1^2)} - \tau_{(2)})\acute{\tau}, \tag{2.62}$$

$$(\acute{\tau}_{(1^3)} + \acute{\tau}_{(3)} - 3\acute{\tau}_{(12)})\tau + (3\tau_{(12)} - \tau_{(1^4)} - \tau_{(3)})\acute{\tau} - 3\acute{\tau}_{(1^2)}\tau_{(1)} + 3\acute{\tau}_{(1)}\tau_{(1^2)} + 3\acute{\tau}_{(2)}\tau_{(1)} + 3\acute{\tau}_{(1)}\tau_{(2)}, \tag{2.63}$$

which are again Plücker relations (see for example [29]) and identically zero. Hence the solution is verified.

2.3.4 Derivatives of Grammian determinants

In addition to the vector Θ previously introduced, consider another vector $\mathrm{P} = (\rho_1, \rho_2, \ldots, \rho_n)^T$ also depending on $x_1, x_2, x_3, \ldots$. For any $n \times n$ matrix A whose entries a_{ij} satisfy $\frac{\partial}{\partial x}a_{ij} = \alpha_i\beta_j$, the derivative of its determinant can be written as

$$\begin{aligned}\frac{\partial}{\partial x}\det(A) &= \sum_{i,j=1}^{n}(-1)^{i+j}\alpha_i\beta_j A^i_j \\ &= -\begin{vmatrix} & & & \alpha_1 \\ & A & & \vdots \\ & & & \alpha_n \\ \beta_1 & \ldots & \beta_n & 0 \end{vmatrix},\end{aligned}$$

where A^i_j is the (i,j)th minor of A. So for the Grammian τ, its derivative with respect to x is the bordered determinant

$$\tau_x = -\begin{vmatrix} \mathcal{G} & \Theta \\ \mathrm{P}^T & \mathbf{0} \end{vmatrix}.$$

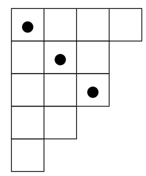
Furthermore, if we assume that Θ and P satisfy

$$\Theta_{x_j} = \Theta^{(j)} \quad \text{and} \quad \mathrm{P}_{x_k} = (-1)^k\mathrm{P}^{(k)}, \quad j,k = 1,2,3,\ldots,$$

then it follows that

$$\frac{\partial \mathcal{G}_{ij}}{\partial x_k} = \sum_{m=0}^{k-1}(-1)^m\frac{\partial^{k-1-m}\theta_i}{\partial x^{k-1-m}}\frac{\partial^m\rho_j}{\partial x^m}.$$

For derivatives of Grammians, we must use a Frobenius notation for partitions. Consider the Young diagram associated with a partition λ and let α_i denote the number of boxes to the right of the diagonal in the ith row and β_i the number of boxes below the diagonal in the ith column. The two sets of nonnegative integers, α_i and β_i, determine the partition which in Frobenius notation we denote as $\lambda = (\alpha_1, \cdots, \alpha_p | \beta_1, \cdots, \beta_p)$. For example, the Young diagram for the partition $\lambda = (310|420)$ would be



where we have used a $\bullet$ to mark the diagonal entries. So, in Frobenius notation, $(43^2 21)$ is $(310|420)$.

As with the Wronskian, we can use a notation to relate the derivatives of the Grammian τ and its original form. For any partition $\lambda = (\alpha_1 \cdots \alpha_p | \beta_1 \cdots \beta_p)$ we define

$$\mathcal{G}_\lambda = \mathcal{G}_{(\alpha_1 \cdots \alpha_p | \beta_1 \cdots \beta_p)} = (-1)^p \begin{vmatrix} \mathcal{G} & \Theta^{(\alpha_1)} & \cdots & \Theta^{(\alpha_p)} \\ \mathrm{P}^{(\beta_1)\dagger} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{P}^{(\beta_p)\dagger} & 0 & \cdots & 0 \end{vmatrix}. \tag{2.64}$$

Then, for example,

$$\mathcal{G}_{(0|1)} = -\begin{vmatrix} \mathcal{G} & \Theta^{(0)} \\ \mathrm{P}^{(1)\dagger} & 0 \end{vmatrix} \quad \text{and} \quad \mathcal{G}_{(10|10)} = \begin{vmatrix} \mathcal{G} & \Theta^{(1)} & \Theta^{(0)} \\ \mathrm{P}^{(1)\dagger} & 0 & 0 \\ \mathrm{P}^{(0)\dagger} & 0 & 0 \end{vmatrix}.$$

The relationship between the derivatives τ_λ and $\mathcal{G}_\mu$ is the same as in (2.55) but with W replaced by $\mathcal{G}$, i.e.

$$\tau_\lambda = \sum_\mu \zeta_\lambda^\mu \mathcal{G}_\mu$$

and the matrices ζ_λ^μ are as given in (2.56).

2.3.5 Jacobi identity for determinants

The Grammian equivalent of a Laplace expansion is Jacobi's identity [1], which relates different size determinants. Here we give the basic Jacobi identity. Let A be an $m \times m$

matrix. We denote by $A^{i,\ldots,j}_{k,\ldots,l}$, the minor obtained by omitting the ith, ..., jth rows and the kth, ..., lth columns. With this notation, the formula

$$\begin{aligned}\det(A)A^{i,j}_{k,l} &= \begin{vmatrix} A^i_k & A^j_k \\ A^i_l & A^j_l \end{vmatrix} \\ &= A^i_k A^j_l - A^j_k A^i_l\end{aligned}$$

is a general case of the Jacobi identity. In this identity, if $\det(A)$ is identified with, for example, $\mathcal{G}_{(10|10)}$, with i, j as the last two rows and k, l as the last two columns then

$$A^i_k = \mathcal{G}_{(1|1)}, A^i_l = \mathcal{G}_{(0|1)}, A^j_k = \mathcal{G}_{(1|0)}, A^j_l = \mathcal{G}_{(0|0)},$$

and $A^{i,j}_{k,l} = \mathcal{G}$. Thus Jacobi's identity gives

$$\mathcal{G}_{(10|10)}\mathcal{G} - \mathcal{G}_{(1|1)}\mathcal{G}_{(0|0)} + \mathcal{G}_{(1|0)}\mathcal{G}_{(0|1)} = 0. \tag{2.65}$$

Rewriting this in partition notation gives

$$\mathcal{G}_{(2^2)}\mathcal{G} - \mathcal{G}_{(21)}\mathcal{G}_{(1)} + \mathcal{G}_{(2)}\mathcal{G}_{(1^2)} = 0, \tag{2.66}$$

which is the same form as the simplest Plücker relation (2.61) for Wronskians.

To verify the Grammian solution, we need to show that the left hand side of (2.1) vanishes when $\tau = \mathcal{G}$. Using equivalent tables to (2.56), we obtain

$$\mathcal{G}_{(10|10)}\mathcal{G} - \mathcal{G}_{(1|1)}\mathcal{G}_{(0|0)} + \mathcal{G}_{(1|0)}\mathcal{G}_{(0|1)} = 0, \tag{2.67}$$

which is the Plücker relation (2.65). Hence the solution is verified.

For the mKP equation, it can again be shown that (see for example [29]) equations (2.62) and (2.63) with $\tau = \mathcal{G}$ and $\tau = \acute{\mathcal{G}}$ are both Plücker relations and identically zero, hence the solution is verified.

2.4 The Miura transformation

A Miura transformation between the KP and mKP heirarchies can be obtained from the following theorem [43]:

Theorem 1. *Let $\mathcal{L}$ satisfy the KP hierarchy* (2.7)*. Then $\tilde{\mathcal{L}} = \theta^{-1}\mathcal{L}\theta$ satisfies the hierarchy $\tilde{\mathcal{L}}_{x_q} = [P_{\geq 1}(\tilde{\mathcal{L}}^q), \tilde{\mathcal{L}}]$ with $G[\phi] = \tilde{\phi} = \theta^{-1}\phi$ being an eigenfunction for $\tilde{\mathcal{L}}$, that is $\phi_{x_q} = P_{\geq 1}(\mathcal{L}^q)[\phi]$.*

We shall now demonstrate this by implementing the gauge transformation $\tilde{\mathcal{L}} = \theta^{-1}\mathcal{L}\theta$. This gives

$$\tilde{\mathcal{L}} = \partial + \theta^{-1}\theta_x + \frac{1}{2}u\partial^{-1} + (u_2 - \frac{1}{2}u\theta^{-1}\theta_x)\partial^{-2} + (\cdots)\partial^{-3} + \ldots \tag{2.68}$$

Comparing this with the operator

$$\mathcal{L}_{mKP} = \partial_x + w + w_1\partial_x^{-1} + w_2\partial_x^{-2} + w_3\partial_x^{-3} + \ldots \tag{2.69}$$

and equating coefficients gives

$$w = \theta^{-1}\theta_x, \tag{2.70}$$

$$w_1 = \frac{1}{2}u, \tag{2.71}$$

$$w_2 = u_2 - \frac{1}{2}u\theta^{-1}\theta_x. \tag{2.72}$$

$$\ldots$$

These coefficients will satisfy (2.40), (2.41) and (2.42) of the mKP hierarchy (2.37). Upon substitution of (2.71) into the first term of (2.41), we obtain

$$u = \mathcal{V}_y - \mathcal{V}_{xx} - \mathcal{V}_x^2. \tag{2.73}$$

Equation (2.73) is the Miura transformation between the KP and mKP equations. Direct substitution of (2.73) into the KP equation (2.1) yields

$$(u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = (\partial_y - \partial_x^2 - 2\mathcal{V}_x\partial_x)(\mathcal{V}_{tx} + \mathcal{V}_{xxxx} - 6\mathcal{V}_x^2\mathcal{V}_{xx} + 3\mathcal{V}_{yy} + 6\mathcal{V}_{xx}\mathcal{V}_y).$$

Therefore, if $\mathcal{V}_x$ is a solution of the mKP equation (2.34), then the Miura transformation (2.73) defines a new solution of the KP equation (2.1).

Chapter 3

A noncommutative KP equation

In this chapter, we look at an example of a noncommuative (nc) integrable system. Different approaches to this noncommutativity exist. For example, the noncommutativity may arise through an underlying nc space defined by the noncommutativity of the coordinates:

$$[x^j, x^k]_\star = i\theta^{jk},$$

where $i = \sqrt{-1}$ and θ^{jk} are real constants called the nc parameters. For Euclidean spaces, the star-product is explicitly given by

$$f \star g(x) := \exp\left(\exp \frac{i}{2}\theta^{ij}\frac{\partial}{\partial x'^i}\frac{\partial}{\partial x''^j}\right) f(x^{'})g(x^{''})|_{x=x'=x''}, \tag{3.1}$$

$$= f(x)g(x) + \frac{i}{2}\theta^{ij}\frac{\partial}{\partial_x^i}f(x)\frac{\partial}{\partial_x^j}g(x) + \mathcal{O}(\theta^2), \tag{3.2}$$

which is known as the Groenewold-Moyal product [24,40]. The star-product is associative, that is $f \star (g \star h) = (f \star g) \star h$, and returns the ordinary product in the commutative limit $\theta^{jk} \to 0$. The star-product makes the ordinary spatial coordinate noncommutative, in that $[x^j, x^k]_\star = x^j \star x^k - x^k \star x^j = i\theta^{jk}$. In this case, an ncKdV equation would have space-time noncommutativity, that is $[t, x]_\star = i\theta$. An ncKP equation could have space-space noncommutativity in that $[x, y]_\star = i\theta$ or space-time non-commutativity, in that either $[t, x] = i\theta$ or $[t, y] = i\theta$. Hamanaka and Toda [25–27] have extensively considered the cases where the Lax method and the Gelfand-Dickii hierarchies give nc equations defined on the nc space.

Lax [35], and later Goncharenko and Veselov [22,23], have considered a matrix version of the KdV equation and Gelfand and Etingof [10] have considered a quaternionic version of the KP equation.

By dropping the assumption that the coefficients in the Lax pair or Gelfand-Dickii hierarchy commute, the results should be valid for all cases. There are three levels of noncommutativity that could arise in the variables in an nc integrable system; the variables could be:

- Matrices with entries that commute,
- Matrices with entries that do not commute, for example a partitioned matrix,
- Noncommutative and not finite-dimensional matrices, for example, in [51], where the commutation relations $[x_j, x_k] = \delta_{jk} i\theta^{jk}$ are the Heisenberg algerba.

Where we have a noncommutative integrable system, we may still have associativity.

In the last chapter, we saw that multi-soliton solutions for the KP and mKP equations could be written as a logarithmic transformation, such as $u = 2(\log \tau)_{xx}$, where τ could be the Wronskian or Grammian determinant. We cannot use Hirota's bilinear method in the nc case as it does not make sense, for example, to take the derivative of the logarithm of a matrix. This can be seen more clearly when attempting to differentiate the power series of $\log(a)$ for some function $a = a(x)$:

$$\log(a) = -(1-a) - \frac{(1-a)^2}{2} - \frac{(1-a)^3}{3} - \frac{(1-a)^4}{4} + \cdots . \tag{3.3}$$

If we were to differentiate (3.3) with respect to x we would obtain terms such as $aa_x + a_x a \neq 2aa_x$, so we have to discard making logarithmic transformations on dependent variables. In addition, we cannot define the determinant of a matrix when its entries do not commute. When this is the case, the natural replacement for a determinant is the quasideterminant.

3.1 Quasideterminants

The concept of quasideterminants originated with Gelfand and Retakh in [16]. An $n \times n$ matrix over a not necessarily commutative unital ring $\mathcal{R}$ has, in general, n^2 quasideterminants. We denote each quasideterminant by $|Z|_{ij}, 1 \leq i, j \leq n$. Let Z^{ij}denote the matrix obtained from Z by deleting the ith row and jth column. Let r_k^j be the row vector obtained from the kth row of Z by deleting the jth entry and let s_l^i be the column vector obtained from the lth row of Z by deleting the ith entry. If Z^{ij} is invertible, then $|Z|_{ij}$

exists and

$$|Z|_{ij} = z_{ij} - r_i^j (Z^{ij})^{-1} s_j^i. \tag{3.4}$$

For example, if $n = 2$ and $Z = (z)_{ij}$, then there are 4 quasideterminants, one of which is

$$|Z|_{22} = \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}_{22} = z_{22} - z_{21} z_{11}^{-1} z_{12}.$$

We shall henceforth adopt an alternative notation for quasideterminants by boxing the leading element. For example,

$$|Z|_{12} = \begin{vmatrix} z_{11} & \boxed{z_{12}} \\ z_{21} & z_{22} \end{vmatrix} = z_{12} - z_{11} z_{21}^{-1} z_{22}.$$

Quasideterminants of Z can also be defined via the inverse of Z. Suppose the matrix Z is invertible with inverse $B = (b_{ij})$. If $|Z|_{ij}$ exists then

$$|Z|_{ij} = b_{ji}^{-1}.$$

The theory of quasideterminants has been greatly developed over the years following their introduction, resulting in several properties and identities which were published by Gelfand, Gelfand, Retakh and Wilson in [14]. Here, we recall some of the main results which we shall use in what follows.

3.1.1 The 2×2 matrix inverse

The 2×2 matrix inverse is given by

$$\begin{aligned}
\begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix}^{-1} &= \begin{bmatrix} \begin{vmatrix} \boxed{z_{11}} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}^{-1} & \begin{vmatrix} z_{11} & z_{12} \\ \boxed{z_{21}} & z_{22} \end{vmatrix}^{-1} \\ \begin{vmatrix} z_{11} & \boxed{z_{12}} \\ z_{21} & z_{22} \end{vmatrix}^{-1} & \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & \boxed{z_{22}} \end{vmatrix}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} (z_{11} - z_{12} z_{22}^{-1} z_{21})^{-1} & (z_{21} - z_{22} z_{12}^{-1} z_{11})^{-1} \\ (z_{12} - z_{11} z_{21}^{-1} z_{22})^{-1} & (z_{22} - z_{21} z_{11}^{-1} z_{12})^{-1} \end{bmatrix},
\end{aligned}$$

provided that all inverses above exist. We can see this in the quasideterminant when $n = 3$. We have

$$Z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix}$$

and there are 9 quasideterminants. For example:

$$
\begin{aligned}
|Z|_{13} &= \begin{vmatrix} z_{11} & z_{12} & \boxed{z_{13}} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \\
&= z_{13} - \begin{pmatrix} z_{11} & z_{12} \end{pmatrix} \begin{pmatrix} z_{21} & z_{22} \\ z_{31} & z_{32} \end{pmatrix}^{-1} \begin{pmatrix} z_{23} \\ z_{33} \end{pmatrix} \\
&= z_{13} - \begin{pmatrix} z_{11} & z_{12} \end{pmatrix} \begin{pmatrix} (z_{21} - z_{22}z_{32}^{-1}z_{31})^{-1} & (z_{31} - z_{32}z_{22}^{-1}z_{21})^{-1} \\ (z_{22} - z_{21}z_{31}^{-1}z_{32})^{-1} & (z_{32} - z_{31}z_{21}^{-1}z_{22})^{-1} \end{pmatrix} \begin{pmatrix} z_{23} \\ z_{33} \end{pmatrix} \\
&= z_{13} - z_{11}(z_{21} - z_{22}z_{32}^{-1}z_{31})^{-1}z_{23} - z_{12}(z_{22} - z_{21}z_{31}^{-1}z_{32})^{-1}z_{23} - z_{11}(z_{31} - z_{32}z_{22}^{-1}z_{21})^{-1}z_{33} \\
&\quad - z_{12}(z_{32} - z_{31}z_{21}^{-1}z_{22})^{-1}z_{33}.
\end{aligned}
$$

3.1.2 Noncommutative Jacobi identity

Quasideterminants can be used to construct a noncommutative version of the Jacobi identity for determinants (also called the noncommutative Sylvester's theorem in [14]). One such case of this is given by

$$
\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}, \tag{3.5}
$$

where A is a square matrix, B, C are column vectors, D, E are row vectors and f, g, h, i are single entries all of compatible length. From these identities, we obtain the following row homology and column homology relations [14]:

$$
\begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}^{-1} = \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix}^{-1} - \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \tag{3.6}
$$

$$
= \begin{vmatrix} A & B & C \\ D & f & g \\ E & \boxed{h} & i \end{vmatrix} \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \tag{3.7}
$$

and

$$\begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix}^{-1} \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} = \begin{vmatrix} A & B \\ E & \boxed{h} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ E & \boxed{i} \end{vmatrix} - \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & C \\ D & \boxed{g} \end{vmatrix} \tag{3.8}$$

$$= \begin{vmatrix} A & B \\ D & \boxed{f} \end{vmatrix}^{-1} \begin{vmatrix} A & B & C \\ D & f & \boxed{g} \\ E & h & i \end{vmatrix}. \tag{3.9}$$

3.1.3 Elementary row and column operations

Replacing the expansion row by a left multiple has the effect of left multiplying the quasideterminant by that factor. For example,

$$\begin{vmatrix} \begin{pmatrix} E & 0 \\ F & g \end{pmatrix} \begin{pmatrix} A & B \\ C & d \end{pmatrix} \end{vmatrix}_{nn} = \begin{vmatrix} EA & EB \\ FA + gC & FB + gd \end{vmatrix}_{nn}$$

$$= g\left(d - CA^{-1}B\right) = g \begin{vmatrix} A & B \\ C & d \end{vmatrix}_{nn}. \tag{3.10}$$

This is again true if we replace row with column operations and left- with right-multiplication. All other row and column operations have no effect.

3.1.4 Comparison with commutative determinants

If Z is an $n \times n$ matrix over a commutative ring R, then $|Z|_{ij}$ is related to $\det Z$. If Z is invertible, then the (j,i)th entry of $Z^{-1} = (-1)^{i+j} \frac{\det Z^{i,j}}{\det Z}$. Recall that in the noncommutative case, if $Z^{-1} = B$, then $|Z|_{ij} = b_{ji}^{-1}$. So

$$|Z|_{ij} \stackrel{c}{=} (-1)^{i+j} \frac{\det Z}{\det Z^{i,j}}, \tag{3.11}$$

where the notation $\stackrel{c}{=}$ has been introduced to denote that the right-hand side of the equation is commutative. For example, if $n = 2$ then

$$\begin{aligned} |Z|_{11} &\stackrel{c}{=} (-1)^{i+j} \frac{\det Z}{\det Z^{i,j}} \\ &\stackrel{c}{=} \frac{z_{11}z_{22} - z_{12}z_{21}}{z_{22}} \\ &\stackrel{c}{=} z_{11} - z_{12}z_{22}^{-1}z_{21}. \end{aligned}$$

3.1.5 Quasi-Plücker coordinates

Given an $(n+k)\times n$ matrix A, denote the ith row of A by A^i, the submatrix of A having rows with indices in a subset I of $\{1,2,\dots,n+k\}$ by A^I. Given $i,j\in\{1,2,\dots,n+k\}$, and I such that $\#I=n-1$ and $j\notin I$, the right quasi-Plücker coordinates are defined by

$$r_{ij}^I = r_{ij}^I(A) := \begin{vmatrix} A^I \\ A^i \end{vmatrix}_{ns} \begin{vmatrix} A^I \\ A^j \end{vmatrix}_{ns}^{-1} = -\begin{vmatrix} A^I & 0 \\ A^i & \boxed{0} \\ A^j & 1 \end{vmatrix}, \tag{3.12}$$

for any $s\in\{1,\dots n\}$. The second equality comes from Jacobi's identity and proves that the definition is independent of s. Therefore we could also write the definition as

$$r_{ij}^I(A) = \begin{vmatrix} A^I \\ A^i \end{vmatrix} \begin{vmatrix} A^I \\ A^j \end{vmatrix}^{-1}.$$

The left quasi-Plücker coordinates are defined in an analogous way. For an $n\times(n+k)$ matrix B,

$$l_{ij}^I(B) = \begin{vmatrix} B^I & \boxed{B^j} \end{vmatrix}^{-1} \begin{vmatrix} B^I & \boxed{B^i} \end{vmatrix} = -\begin{vmatrix} B^I & B^i & B^j \\ 0 & \boxed{0} & 1 \end{vmatrix},$$

in which we here denote the ith column of B by B^i, and the submatrix of B having columns with indices in a subset I of $\{1,2,\dots,n+k\}$ by B^I.

The row homology and column homology relations can now be written in terms of quasi-Plücker coordinates giving the following identities:

$$\begin{vmatrix} A & B & C \\ D & f & g \\ E & \boxed{h} & i \end{vmatrix} = \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ 0 & \boxed{0} & 1 \end{vmatrix} \tag{3.13}$$

and

$$\begin{vmatrix} A & B & C \\ D & f & \boxed{g} \\ E & h & i \end{vmatrix} = \begin{vmatrix} A & B & 0 \\ D & f & \boxed{0} \\ E & h & 1 \end{vmatrix} \begin{vmatrix} A & B & C \\ D & f & g \\ E & h & \boxed{i} \end{vmatrix}. \tag{3.14}$$

3.2 A noncommutative KP hierarchy

Gelfand-Dickii hierarchies in the nc setting have been considered in [7, 25, 27, 51].

In the nc case, with $\mathcal{L} = \mathcal{L}_{\mathrm{KP}}$, we have the differential operators

$$\begin{aligned} P_{\geq 0}(\mathcal{L}) &= \partial_x, \\ P_{\geq 0}(\mathcal{L}^2) &= \partial_x^2 + u, \\ P_{\geq 0}(\mathcal{L}^3) &= \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{2}u_x + 3u_2, \end{aligned} \tag{3.15}$$

which, via the evolution equation (2.7), give the ncKP hierarchy:

$$\mathcal{L}_{t_1} = [P_{\geq 0}(\mathcal{L}), \mathcal{L}] \Leftrightarrow \begin{cases} u_{t_1} &= u_x, \\ u_{2t_1} &= u_{2x}, \\ u_{3t_1} &= u_{3x}, \\ \cdots &, \end{cases} \tag{3.16}$$

$$\mathcal{L}_{t_2} = [P_{\geq 0}(\mathcal{L}^2), \mathcal{L}] \Leftrightarrow \begin{cases} u_y &= u_{xx} + 4u_{2x}, \\ u_{2y} &= u_{2xx} + 2u_{3x} + \frac{1}{2}uu_x + [u, u_2], \\ u_{3y} &= u_{3xx} + 2u_{4x} - \frac{1}{2}uu_{xx} + 2u_2u_x + [u, u_3], \\ \cdots &, \end{cases} \tag{3.17}$$

$$\mathcal{L}_{t_3} = [P_{\geq 0}(\mathcal{L}^3), \mathcal{L}] \Leftrightarrow \begin{cases} u_t &= u_{xxx} + 6u_{2xx} + 6u_{3x} + \frac{3}{2}uu_x + \frac{3}{2}u_xu, \\ u_{2t} &= u_{2xxx} + 3u_{3xx} + 3u_{4x} + 3uu_2 \\ & \quad + \frac{3}{2}(u_xu_2 + u_2u_x) + 3[u, u_3], \\ \cdots &. \end{cases} \tag{3.18}$$

Eliminating u_2 and u_3 via (3.17), and making the scaling $t \to -4t$ allows us to rewrite the first component in (3.18) as

$$(v_t + 3v_xv_x + v_{xxx})_x + 3v_{yy} + 3[v_x, v_y] = 0, \tag{3.19}$$

where again $u = v_x$. Equation (3.19) is the ncKP equation (ncKP) [46] in potential form. When the variables in (3.19) do commute, we recover (2.1).

Equation (3.19) could also be obtained from the Lax pair

$$\begin{aligned} L_{\mathrm{KP}} &= \partial_x^2 + v_x - \partial_y, \\ M_{\mathrm{KP}} &= 4\partial_x^3 + 6v_x\partial_x + 3v_{xx} + 3v_y + \partial_t, \end{aligned}$$

whose compatibility condition $[L_{\mathrm{KP}}, M_{\mathrm{KP}}] = 0$ gives (3.19).

3.3 Quasiwronskian solutions obtained from Darboux transformations

Given that we can attempt to derive a noncommutative integrable system through the Lax method or the Gelfand-Dickii hierarchy, Darboux transformations appear to be the natural choice for finding multi-soliton solutions.

Both L_{KP} and M_{KP} are covariant under the Darboux transformation

$$G_\theta = \theta\partial_x\theta^{-1} = \partial_x - \theta_x\theta^{-1}.$$

Let $\theta_i, i = 1, \ldots, n$, be a particular set of eigenfunctions. It is assumed that, like the dependent variable u, the eigenfunction θ and its derivatives do not commute. Introduce the notation $\Theta = (\theta_1, \ldots, \theta_n)$ and $\hat{\Theta} = (\theta_j^{(i-1)})_{i,j=1,\ldots,n}$, the $n \times n$ Wronskian matrix of $\theta_1, \ldots, \theta_n$.

Let $\phi = \phi_{[1]}$ be an eigenfunction of $L_{\mathrm{KP}[1]} = L_{\mathrm{KP}}$ and $\theta_{[1]} = \theta_1$. Then $\phi_{[2]} := G_{\theta_{[1]}}[\phi_{[1]}]$ and $\theta_{[2]} = \phi_{[2]}|_{\phi\to\theta_2}$ are eigenfunctions for $L_{\mathrm{KP}[2]} = G_{\theta_{[1]}}LG^{-1}_{\theta_{[1]}}$. In general, for $n \geq 1$ define the nth Darboux transform of ϕ by

$$\phi_{[n+1]} = \phi^{(1)}_{[n]} - \theta^{(1)}_{[n]}\theta^{-1}_{[n]}\phi_{[n]},$$

in which

$$\theta_{[k]} = \phi_{[k]}|_{\phi\to\theta_k}.$$

It has been shown in [14] and in [23] that $\phi_{[n+1]}$ can be expressed as

$$\begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \boxed{\phi^{(n)}} \end{vmatrix}.$$

The effect of

$$\tilde{L}_{\mathrm{KP}} = G_\theta L_{\mathrm{KP}} G_\theta^{-1}, \quad \tilde{M}_{\mathrm{KP}} = G_\theta M_{\mathrm{KP}} G_\theta^{-1}$$

is that

$$\tilde{v} = v + 2\theta_x\theta^{-1}.$$

After n Darboux transformations we have

$$v_{[n+1]} = v + 2\sum_{i=1}^{n} \theta_{[i],x}\theta_{[i]}^{-1}$$
$$= v - 2\begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n-2)} & 0 \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}. \tag{3.20}$$

We call this type of quasideterminant in (3.20) a *quasiwronskian.*

3.4 Quasigrammian solutions obtained from binary Darboux transformations

A new family of solutions of ncKP, obtained by binary Darboux transformations and expressible as quasideterminants, was introduced by Gilson and Nimmo in [19]. The adjoint Lax pair for ncKP is

$$L_{\mathrm{KP}}^{\dagger} = \partial_x^2 + v_x^{\dagger} + \partial_y,$$
$$M_{\mathrm{KP}}^{\dagger} = -4\partial_x^3 - 6v_x^{\dagger}\partial_x - 3v_{xx}^{\dagger} + 3v_y^{\dagger} - \partial_t.$$

Here, the notion of the adjoint has been extended from the well-known matrix case to any unital ring $\mathcal{R}$, as considered by Matveev in [38]: suppose that for each $a \in \mathcal{R}$, there exists $a^{\dagger} \in \mathcal{R}$, and for a derivative ∂ acting on $\mathcal{R}$, $\partial_x^{\dagger} = -\partial_x$ and for a product $\mathcal{AB}$ of elements of $\mathcal{R}$, or operators on $\mathcal{R}$, $(\mathcal{AB})^{\dagger} = \mathcal{B}^{\dagger}\mathcal{A}^{\dagger}$.

Analogous to the commutative case, a potential $\Omega(\phi,\psi)$ is introduced, satisfying

$$\Omega(\phi,\psi)_x = \psi^{\dagger}\phi, \quad \Omega(\phi,\psi)_y = \psi^{\dagger}\phi_x - \psi_x^{\dagger}\phi, \quad \Omega(\phi,\psi)_t = -4(\psi^{\dagger}\phi_{xx} - \psi_x^{\dagger}\phi_x + \psi_{xx}^{\dagger}\phi) - 6\psi^{\dagger}v_x\phi.$$

A binary Darboux transformation is defined by

$$\phi_{[n+1]} = \phi_{[n]} - \theta_{[n]}\Omega(\theta_{[n]},\rho_{[n]})^{-1}\Omega(\phi_{[n]},\rho_{[n]})$$

and

$$\psi_{[n+1]} = \psi_{[n]} - \rho_{[n]}\Omega(\theta_{[n]},\rho_{[n]})^{-\dagger}\Omega(\theta_{[n]},\psi_{[n]})^{\dagger},$$

in which

$$\theta_{[n]} = \phi_{[n]}|_{\phi\to\theta_n}, \quad \rho_{[n]} = \psi_{[n]}|_{\psi\to\rho_n}.$$

Using the notation $\Theta = (\theta_1, \dots \theta_n)$ and $\mathrm{P} = (\rho_1, \dots, \rho_n)$ we have, for $n \geq 1$

$$\phi_{[n+1]} = \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \Omega(\phi, \mathrm{P}) \\ \Theta & \boxed{\phi} \end{vmatrix},$$

$$\psi_{[n+1]} = \begin{vmatrix} \Omega(\Theta, \mathrm{P})^\dagger & \Omega(\Theta, \psi)^\dagger \\ \mathrm{P} & \boxed{\psi} \end{vmatrix}$$

and

$$\Omega(\phi_{[n+1]}, \psi_{[n+1]}) = \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \Omega(\phi, \mathrm{P}) \\ \Omega(\Theta, \psi) & \boxed{\Omega(\phi, \psi)} \end{vmatrix}.$$

The effect of

$$\hat{L}_{\mathrm{KP}} = G_{\theta,\phi} L_{\mathrm{KP}} G_{\theta,\phi}^{-1}, \quad \hat{M}_{\mathrm{KP}} = G_{\theta,\phi} M_{\mathrm{KP}} G_{\theta,\phi}^{-1}$$

is that

$$\hat{v} = v + 2\theta\Omega(\theta, \rho)^{-1}\rho^\dagger.$$

After n binary Darboux transformations we have

$$\begin{aligned} v_{[n+1]} &= v + 2\sum_{k=1}^{n} \theta_{[k]}\Omega(\theta_{[k]}, \rho_{[k]})^{-1}\rho_{[k]}^\dagger \\ &= v - 2\begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \mathrm{P}^\dagger \\ \Theta & \boxed{0} \end{vmatrix}. \end{aligned} \tag{3.21}$$

We call this type of quasideterminant in (3.21) a *quasigrammian.*

3.5 Reduction to commutative Wronskian and Grammian solutions

All of the quasideterminants expressing the Darboux-transformed eigenfunctions and potentials $v_{[n+1]}$ of ncKP, should reduce to the corresponding commutative results in Chapter 2. Using (3.11), in the commutative case, we have:

- The transformed eigenfunction

$$\phi_{[n+1]} = \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \boxed{\phi^{(n)}} \end{vmatrix} \stackrel{c}{=} \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \phi^{(n)} \end{vmatrix} \Bigg/ \left|\hat{\Theta}\right|,$$

- The transformed potential

$$v_{[n+1]} = v - 2 \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n-2)} & 0 \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix} \stackrel{c}{=} v - 2 \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n-2)} & 0 \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & 0 \end{vmatrix} \Big/ \left|\hat{\Theta}\right| \stackrel{c}{=} v + 2\left(\log\left|\hat{\Theta}\right|\right)_x,$$

- The transformed binary eigenfunction

$$\phi_{[n+1]} = \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \Omega(\phi, \mathrm{P}) \\ \Theta & \boxed{\phi} \end{vmatrix} \stackrel{c}{=} \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \Omega(\phi, \mathrm{P}) \\ \Theta & \phi \end{vmatrix} \Big/ \left|\Omega(\Theta, \mathrm{P})\right|,$$

- The transformed adjoint eigenfunction

$$\psi_{[n+1]} = \begin{vmatrix} \Omega(\Theta, \mathrm{P})^\dagger & \Omega(\Theta, \psi)^\dagger \\ \mathrm{P} & \boxed{\psi} \end{vmatrix} \stackrel{c}{=} \begin{vmatrix} \Omega(\Theta, \mathrm{P})^\dagger & \Omega(\Theta, \psi)^\dagger \\ \mathrm{P} & \psi \end{vmatrix} \Big/ \left|\Omega(\Theta, \mathrm{P})^\dagger\right|,$$

- The transformed binary potential

$$\begin{aligned} v_{[n+1]} &= v - 2 \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \mathrm{P}^\dagger \\ \Theta & \boxed{0} \end{vmatrix} \stackrel{c}{=} v - 2 \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \mathrm{P}^\dagger \\ \Theta & 0 \end{vmatrix} \Big/ \left|\Omega(\Theta, \mathrm{P})\right| \\ &\stackrel{c}{=} v + 2\left(\log\left|\Omega(\Theta, \mathrm{P})\right|\right)_x. \end{aligned}$$

We therefore recover all of the commutative solutions given in Chapter 2.

3.6 Direct verification of the solutions

3.6.1 Derivatives of quasideterminants

Formulae for derivatives of quasideterminants were considered in [19]. The authors consider differentiating the quasideterminant

$$\begin{vmatrix} A & B \\ C & \boxed{\mathrm{d}} \end{vmatrix}. \tag{3.22}$$

Here, A is an $n \times n$ matrix, d is a single entry, C is a row vector and B a column vector. Differentiating both sides of (3.22) gives

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = d' - C'A^{-1}B + CA^{-1}A'A^{-1}B - CA^{-1}B'. \tag{3.23}$$

The third term on right-hand side of (3.23) can be split into two cases. Firstly, if A has the grammian-like structure, such as $\Omega(\Theta, \mathrm{P})$, then its derivative is the tensor product

$$A' = \sum_{i=1}^{k} E_i F_i,$$

where E_i is a column vector and F_i is a row vector, both of appropriate length. Therefore, the third term on the right-hand side of (3.23) can be written as a product of quasideterminants, giving

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = d' - C'A^{-1}B + \sum_{i=1}^{k}(CA^{-1}E_i)(F_iA^{-1}B) - CA^{-1}B' \tag{3.24}$$

$$= \begin{vmatrix} A & B \\ C' & \boxed{d'} \end{vmatrix} + \begin{vmatrix} A & B \\ C' & \boxed{0} \end{vmatrix} + \sum_{i=1}^{k} \begin{vmatrix} A & E_i \\ C & \boxed{0} \end{vmatrix} \begin{vmatrix} A & B \\ F_i & \boxed{0} \end{vmatrix}. \tag{3.25}$$

If A' does not have the grammian-like structure, like the Wronskian, then it may be factorised by inserting the identity matrix expressed in the form

$$I = \sum_{i=1}^{k} e_k e_k^T,$$

where e_k is the column vector (δ_{ij}), which has a 1 in the kth row and a zero elsewhere. If we let $\hat{Z}^k$ be the kth row and $\hat{Z}_k$ be the kth column of the matrix $\hat{Z}$, we have

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = d' - C'A^{-1}B + \sum_{i=1}^{k}(CA^{-1}e_k)(e_k^T A' A^{-1}B) - \sum_{i=1}^{k}(CA^{-1}e_k)(e_k^T B').$$

This gives

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = \begin{vmatrix} A & B \\ C' & \boxed{d'} \end{vmatrix} + \sum_{i=1}^{k} \begin{vmatrix} A & e_k \\ C & \boxed{0} \end{vmatrix} \begin{vmatrix} A & B \\ (A^k)' & \boxed{(B^k)'} \end{vmatrix}, \tag{3.26}$$

or equivalently

$$\begin{vmatrix} A & B \\ C & \boxed{d} \end{vmatrix}' = \begin{vmatrix} A & B \\ C & \boxed{d'} \end{vmatrix} + \sum_{i=1}^{k} \begin{vmatrix} A & (A^k)' \\ C & \boxed{(C^k)'} \end{vmatrix} \begin{vmatrix} A & B \\ e^{kT} & \boxed{0} \end{vmatrix}. \tag{3.27}$$

The authors of [19] then go on to show how to differentiate quasiwronskians and quasigrammians.

3.6.2 Derivatives of quasiwronskians

Let $\hat{\Theta} = \left(\theta_j^{(i-1)}\right)_{i,j=1,\dots,n}$ be the $n \times n$ wronskian matrix of $\theta_1, \dots, \theta_n$, where $^{(k)}$ denotes the kth derivative, and let e_k be the n-vector (δ_{ik}) (i.e. a column vector with 1 in the kth row and 0 elsewhere). We will calculate derivatives of the form

$$Q(i,j) = \begin{vmatrix} \hat{\Theta} & e_{n-j} \\ \Theta^{(n+i)} & \boxed{0} \end{vmatrix}.$$

In this definition, i and j are allowed to take any integer values, subject to the convention that if $n-j$ lies outside the range $1, 2, \dots, n$, then $e_{n-j} = 0$ and so $Q(i,j) = 0$. There is an important special case: when $n+i = n-j-1 \in [0, n-1]$, (i.e. $i+j+1 = 0$ and $-n \le i < 0$) we have

$$Q(i,j) = \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n+i)} & 1 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n+i)} & \boxed{0} \end{vmatrix} = \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n+i)} & 1 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ 0 & \boxed{-1} \end{vmatrix} = -1.$$

Using the same argument for $n+i \in [0, n-1]$ but $n+i \neq n-j-1$, we see that $Q(i,j) = 0$. Assuming n is arbitrarily large, we may summarise these properties of $Q(i,j)$ as

$$Q(i,j) = \begin{cases} -1 & i+j+1 = 0 \\ 0 & (i < 0 \text{ or } j < 0) \text{ and } i+j+1 \neq 0 \end{cases}. \tag{3.28}$$

If we relabel and rescale the variables so that $x_1 = x$, $x_2 = y$, $x_3 = -4t$, Θ satisfies the linear equations

$$\Theta_{x_2} = \Theta_{xx},$$
$$\Theta_{x_3} = \Theta_{xxx}.$$

We may allow Θ to depend on higher variables x_k and impose the natural dependence $\Theta_{x_k} = \Theta_{\underbrace{x \cdots x}_{k}}$.

Now, for any m, using the linear equations for Θ, we have

$$\begin{aligned} \frac{\partial}{\partial x_m} Q(i,j) &= \begin{vmatrix} \hat{\Theta} & e_{n-j} \\ \Theta^{(n+i+m)} & \boxed{0} \end{vmatrix} + \sum_{k=1}^{n} \begin{vmatrix} \hat{\Theta} & e_k \\ \Theta^{(n+i)} & \boxed{0} \end{vmatrix} \begin{vmatrix} \hat{\Theta} & e_{n-j} \\ \Theta^{(k-1+m)} & \boxed{0} \end{vmatrix} \\ &= Q(i+m, j) + \sum_{k=0}^{n-1} Q(i,k) Q(m-1-k, j). \end{aligned} \tag{3.29}$$

Using the conditions (3.28), the above simplifies considerably and we obtain

$$\frac{\partial}{\partial x_m}Q(i,j) = Q(i+m,j) - Q(i,j+m) + \sum_{k=0}^{m-1} Q(i,k)Q(m-k-1,j). \tag{3.30}$$

In particular

$$\begin{aligned}\frac{\partial}{\partial x}Q(i,j) &= Q(i+1,j) - Q(i,j+1) + Q(i,0)Q(0,j),\\ \frac{\partial}{\partial x_2}Q(i,j) &= Q(i+2,j) - Q(i,j+2) + Q(i,1)Q(0,j) + Q(i,0)Q(1,j),\\ \frac{\partial}{\partial x_3}Q(i,j) &= Q(i+3,j) - Q(i,j+3) + Q(i,2)Q(0,j) + Q(i,1)Q(1,j) + Q(i,0)Q(2,j).\end{aligned}$$

Note that these simplified formulae (3.30) are only valid for sufficiently large n. For smaller n we should use (3.29) directly.

In addition to $Q(i,j)$ we can define a shifted version, which we will call $\hat{Q}(i,j)$:

$$\hat{Q}(i,j) = \begin{vmatrix} \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-j)} & 1 \\ \vdots & \vdots \\ \Theta^{(n)} & 0 \\ \Theta^{(n+i+1)} & \boxed{0} \end{vmatrix}.$$

This satisfies equations similar to (3.30).

3.6.3 Derivatives of quasigrammians

To express the derivatives of a quasigrammian, we define

$$R(i,j) = (-1)^j \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \mathrm{P}^{\dagger(j)} \\ \Theta^{(i)} & \boxed{0} \end{vmatrix}.$$

As we have seen in (3.21), solutions obtained by binary Darboux transformations are of the form $v = v_0 - 2R(0,0)$. As we did in the case of the quasiwronskian type of solutions we choose $v_0 = 0$ for simplicity. Hence Θ satisfies the same linear equations as before and P, the vector of adjoint eigenfunctions, satisfies

$$\mathrm{P}_{x_2} = -\mathrm{P}_{xx}, \qquad \mathrm{P}_{x_3} = \mathrm{P}_{xxx}.$$

Note that choice of the trivial vacuum is inessential and direct verification can be completed for arbitrary vacuum.

Using (3.24), derivatives with respect to the x_m can be calculated:

$$\begin{aligned}\partial_{x_m}R(i,j) &= (-1)^j \begin{vmatrix} \Omega & \mathrm{P}^{\dagger(j)} \\ \Theta^{(i+m)} & \boxed{0} \end{vmatrix} + (-1)^{m+j-1} \begin{vmatrix} \Omega & \mathrm{P}^{\dagger(j+m)} \\ \Theta^{(i)} & \boxed{0} \end{vmatrix} \\ &\quad + \sum_{k=0}^{m-1} \begin{vmatrix} (-1)^{j+k}\Omega & \mathrm{P}^{\dagger(k)} \\ \Theta^{(i)} & \boxed{0} \end{vmatrix} \begin{vmatrix} \Omega & \mathrm{P}^{\dagger(j)} \\ \Theta^{(m-1-k)} & \boxed{0} \end{vmatrix} \\ &= R(i+m,j) - R(i,j+m) + \sum_{k=0}^{m-1} R(i,k)R(m-k-1,j).\end{aligned}$$

This final form for a derivative of a quasigrammian corresponds precisely with the formula for the quasiwronskian (3.30). Thus subsequent calculations carried out for the quasiwronskian solutions will be equally valid for the quasigrammian solutions.

To verify the quasigrammian and quasiwronskian solutions directly, we need to show that both $v = -2Q(0,0)$ and $v = -2R(0,0)$ are solutions of ncKP. In [19], the authors list some of the derivatives of v that can be substituted into ncKP directly. For example:

$$\begin{aligned} v_x &= -2Q(0,0)_x = -2\left[Q(1,0) - Q(0,1) + Q(0,0)Q(0,0)\right], \\ v_y &= -2Q(0,0)_y = -2\left[Q(2,0) - Q(0,2) + Q(0,0)Q(1,0) + Q(0,1)Q(0,0)\right], \\ v_t &= -2Q(0,0)_t = -2\left[Q(3,0) - Q(0,3) - Q(0,0)Q(2,0) + Q(0,1)Q(1,0) + Q(0,2)Q(0,0)\right]. \end{aligned}$$

From here, using (3.30),we could easily calculate higher order derivatives. For example

$$\begin{aligned} v_{xx} = -2[&Q(0,2) - 2Q(1,1) + Q(2,0) - 2Q(0,0)Q(0,1) + Q(0,0)Q(1,0) - Q(0,1)Q(0,0) \\ &+ 2Q(1,0)Q(0,0) + 2Q(0,0)Q(0,0)Q(0,0)]. \end{aligned}$$

Upon substitution of v and its derivatives into (3.19), all the terms cancel and the solution is therefore verified.

3.7 Matrix solutions

In this section, we derive matrix solutions of ncKP (3.19), which are new results. This work, much of which is outlined in [21], builds on that given in [22, 23].

In [22] it was shown that a matrix version of the KdV equation

$$U_t - 3UU_x - 3U_xU + U_{xxx} = 0, \tag{3.31}$$

where $U = U(x,t)$ is a $d \times d$ matrix, possessed multi-soliton solutions obtainable from Darboux transformations. For example, the one-soliton solution is

$$U = -2\lambda^2 P \operatorname{sech}^2\left(\lambda(\varsigma - \upsilon)\right), \tag{3.32}$$

where $\varsigma = \frac{1}{2\lambda}\log\left(\frac{r}{2\lambda}\right)$ and $\upsilon = x - 4\lambda^2 t$. The solution U, as given by (3.32), represents a $d \times d$ one-soliton matrix solution. The *matrix amplitude* of the solution is $-2\lambda^2 P$, where $P = P^2$ is a projection operator sometimes referred to as the polarization of the soliton. Each soliton in the matrix has phase-constant $\phi = \frac{1}{2\lambda}\log\left(\frac{r}{2\lambda}\right)$ for some constant r. Multi-soliton solutions of a matrix sine-Gordon equation were found in [36]. The authors showed that this equation has quasigrammian solutions, into which they introduced projection matrices.

Using the quasigrammian solutions of ncKP, we now follow this approach and investigate the one-, two- and three-soliton matrix solutions. Taking the trivial vacuum solution $v = 0$ gives

$$v_{[n+1]} = -2\begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \mathrm{P}^\dagger \\ \Theta & \boxed{0} \end{vmatrix}. \tag{3.33}$$

The eigenfunctions θ_i and the binary eigenfunctions ρ_i satisfy

$$\theta_{i,xx} = \theta_{i,y}, \quad \theta_{i,t} = -4\theta_{i,xxx}$$

and

$$\rho_{i,xx} = -\rho_{i,y}, \quad \rho_{i,t} = -4\rho_{i,xxx}$$

respectively. The simplest nontrivial solutions of these equations are

$$\theta_j = A_j e^{\eta_j}, \quad \rho_i = B_i e^{-\gamma_i},$$

where $\eta_j = p_j(x + p_j y - 4p_j^2 t), \gamma_i = q_i(x + q_i y - 4q_i^2 t)$ and A_j, B_i are $d \times m$ matrices. At this stage the general structure of $v_{[n+1]}$ is

$$\begin{aligned} v_{[n+1]} &= -2\begin{vmatrix} \Omega(\Theta, \mathrm{P})_{m\times m} & \mathrm{P}^\dagger_{d\times m} \\ \Theta_{d\times m} & \boxed{0_{d\times d}} \end{vmatrix} \\ &= -2\left(0_{d\times d} - \Theta_{d\times m}\Omega(\Theta, \mathrm{P})^{-1}_{m\times m}\mathrm{P}_{m\times d}\right). \end{aligned} \tag{3.34}$$

With this choice of θ_j and ρ_i,

$$\Omega(\theta_j, \rho_i)_x = B_i^T A_j e^{\eta_j - \gamma_i}.$$

Integrating with respect to x gives

$$\Omega(\theta_j, \rho_i) = \frac{B_i^T A_j}{p_j - q_i} e^{\eta_j - \gamma_i} + \delta_{i,j} I,$$

where $\delta_{i,j}I$ is the constant of integration. We take $A_j = r_j P_j$, where r_j is a scalar and P_j is a projection operator. With A_j chosen in this way, we must have $m = d$ and therefore (3.34) becomes

$$v_{[n+1]} = -2\left(0_{d\times d} - \Theta_{d\times d}\Omega(\Theta, \mathrm{P})^{-1}_{d\times d}\mathrm{P}_{d\times d}\right). \tag{3.35}$$

We choose $B_i = I$ and the solution u will be a $d \times d$ matrix.

In the case $n = 1$, we obtain a one-soliton matrix solution. Expanding (3.35) gives

$$v_{[2]} = 2rPe^{\Lambda}\left(I + \frac{r}{p-q}e^{\Lambda}P\right)^{-1},$$

where $\Lambda_i = \eta_i - \gamma_i$. When taking the inverse of Ω, we make use of the formula

$$\begin{aligned}(I - aP)^{-1} &= I + aP + a^2P^2 + a^3P^3 + \dots \\ &= I + aP + a^2P + a^3P + \dots \\ &= I + aP(1 + a + a^2 + \dots) \\ &= I + aP(1-a)^{-1},\end{aligned}$$

where $a \neq 1$ is a scalar and P is any projection matrix. This useful identity is subsequently used throughout. We now have

$$v_{[2]} = \frac{2rP}{e^{-\Lambda} + \frac{r}{p-q}}. \tag{3.36}$$

Consequently, the one-soliton matrix solution is

$$u = v_{[2],x} = \frac{1}{2}(p-q)^2 P \operatorname{sech}^2\left(\frac{1}{2}(\Lambda + \xi)\right), \tag{3.37}$$

where $\xi = \log\left(\frac{r}{p-q}\right)$. Each plane wave in the matrix u is travelling with speed $4\left(\frac{p^3-q^3}{p-q}\right)t$ in the x-direction and $4\left(\frac{p^3-q^3}{p^2-q^2}\right)t$ in the y-direction. A regular solution requires that $\det(\Omega) \neq 0$ for all x,y and t, and $\frac{r}{p-q} > 0$. We have

$$\det(\Omega) = 1 + \frac{r}{p-q}e^{\Lambda}. \tag{3.38}$$

If $r > 0$ and $p > q$ or alternatively, if $r < 0$ and $q > p$, then $\frac{r}{p-q} > 0$ and $\det(\Omega)$ is positive-definite. In each case, we would obtain the same solution (3.37) since sech^2 is an even function.

In the case $n = d = 2$, we obtain a two-soliton 2×2 matrix solution. By expanding (3.35) we get

$$\begin{aligned} v_{[3]} &= 2 \begin{pmatrix} A_1 e^{\eta_1} & A_2 e^{\eta_2} \end{pmatrix} \left(A_j \tfrac{e^{\eta_j - \gamma_i}}{p_j - q_i} + \delta_{i,j} I \right)^{-1}_{2\times 2} \begin{pmatrix} I e^{-\gamma_1} \\ I e^{-\gamma_2} \end{pmatrix} \\ &= 2 \begin{pmatrix} K_1 e^{\gamma_1} & K_2 e^{\gamma_2} \end{pmatrix} \begin{pmatrix} I e^{-\gamma_1} \\ I e^{-\gamma_2} \end{pmatrix}, \quad \text{say} \\ &= 2(K_1 + K_2), \end{aligned}$$

where K_1 and K_2 satisfy

$$\begin{aligned} K_1 \left(I + \frac{r_1 e^{\Lambda_1}}{p_1 - q_1} P_1 \right) &= e^{\Lambda_1} A_1 - \frac{e^{\Lambda_1}}{p_1 - q_2} K_2 A_1, \\ K_2 \left(I + \frac{r_2 e^{\Lambda_2}}{p_2 - q_2} P_2 \right) &= e^{\Lambda_2} A_2 - \frac{e^{\Lambda_2}}{p_2 - q_1} K_1 A_2. \end{aligned}$$

We assume that the P_j are the rank-1 projection matrices

$$P_j = \frac{\mu_j \otimes \nu_j}{(\mu_j, \nu_j)} = \frac{\mu_j \nu_j^T}{\mu_j^T \nu_j},$$

where the d-vectors μ_j, ν_j satisfy the condition $(\mu_j, \nu_j) \neq 0$. Solving for K_1 and K_2 gives

$$K_1 = \frac{p_2 - q_1}{h} \left(h_2 (p_1 - q_2) I - A_2 \right) A_1, \tag{3.39}$$

$$K_2 = \frac{p_1 - q_2}{h} \left(h_1 (p_2 - q_1) I - A_1 \right) A_2, \tag{3.40}$$

where

$$\begin{aligned} h &= h_1 h_2 (p_1 - q_2)(p_2 - q_1) - \alpha r_1 r_2, \\ h_i &= e^{-\Lambda_i} + \frac{r_i}{p_i - q_i}, \quad \text{and} \quad \alpha = \frac{(\mu_1, \nu_2)(\mu_2, \nu_1)}{(\mu_1, \nu_1)(\mu_2, \nu_2)} = Tr(P_1 P_2). \end{aligned}$$

For the solution to be regular, we need $\det(\Omega) \neq 0$ for all x, y and t. Upon expanding $\det(\Omega)$, the result simplifies greatly since the trace of a projection matrix is equal to its rank and the determinant of any projection matrix is zero, and we obtain

$$\det(\Omega) = 1 + \kappa_1 e^{\Lambda_1} + \kappa_2 e^{\Lambda_2} + \kappa_1 \kappa_2 \beta e^{\Lambda_1 + \Lambda_2},$$

in which $\kappa_i = \frac{r_i}{p_i - q_i}$, $i = 1, 2$ and $\beta = 1 - \frac{\alpha (p_1 - q_1)(p_2 - q_2)}{(p_1 - q_2)(p_2 - q_1)}$. By ordering the spectral parameters $p_2 > q_2 > p_1 > q_1$ and choosing $r_i > 0$, for $i = 1, 2$, we ensure that $\kappa_i > 0$. Furthermore, if we insist that $\alpha > 0$, we have $\beta > 0$ and $\det(\Omega)$ is therefore positive-definite.

We now investigate the behaviour of $v_{[3]}$ as $t \to \pm\infty$. This will demonstrate that each soliton emerges from interaction undergoing a phase shift and that the amplitude of each soliton may also change due to the interaction. For each soliton, we need to show that the asymptotic form of $v_{[3]}$ is the same as (3.36). We first fix Λ_1 by making the change of variables (a similar change of variables involving y would also fix Λ_1)

$$x = \hat{x} + 4\left(\frac{q_1^3 - p_1^3}{q_1 - p_1}\right)t.$$

This gives

$$\Lambda_1 = (q_1 - p_1)\hat{x} + (q_1^2 - p_1^2)y,$$
$$\Lambda_2 = (q_2 - p_2)\hat{x} + (q_2^2 - p_2^2)y - 4\left(q_2^3 - p_2^3 - \frac{(q_2 - p_2)(q_1^3 - p_1^3)}{q_1 - p_1}\right)t.$$

Since Λ_1 is now independent of t, soliton 1 is at rest. We may assume without loss of generality that $0 > p_2 > q_2 > p_1 > q_1$. Then as $t \to -\infty$,

$$v_{[3]} \sim 2\frac{r_1 P_1}{h_1}$$

and therefore

$$u \sim \frac{1}{2}(p_1 - q_1)^2 P_1 \operatorname{sech}^2\left(\frac{1}{2}\left(\Lambda_1 + \xi_1^-\right)\right),$$

where $\xi_1^- = \log\left(\frac{r_1}{p_1 - q_1}\right)$.

Note that $u = v_x$ is invariant under the transformation $v \to v + C$, where C is a constant matrix. As $t \to +\infty$ we get

$$\begin{aligned}
v_{[3]} &\sim 2\frac{(r_2(p_1 - q_2) - (p_2 - q_2)A_2)(p_2 - q_1)A_1 + (\alpha r_1(p_2 - q_2) - (p_1 - q_2)A_1)(p_2 - q_2)A_2}{h_1 r_2(p_1 - q_2)(p_2 - q_1) - \alpha r_1 r_2 (p_2 - q_2)} \\
&\quad + 2\frac{(p_2 - q_2)A_2}{r_2} \\
&\sim 2\frac{(r_2(p_1 - q_2) - (p_2 - q_2)A_2)(p_2 - q_1)A_1 + (\alpha r_1(p_2 - q_2) - (p_1 - q_2)A_1)(p_2 - q_2)A_2}{r_2(p_1 - q_2)(p_2 - q_1)\left(h_1 - \frac{\alpha r_1(p_2 - q_2)}{(p_1 - q_2)(p_2 - q_1)}\right)} \\
&\sim 2\frac{\hat{r}_1 \hat{P}_1}{e^{-\Lambda_1} + \frac{\hat{r}_1}{p_1 - q_1}},
\end{aligned}$$

where

$$\hat{r}_1 = \frac{r_1(\hat{\mu}_1, \hat{\nu}_1)}{(\mu_1, \nu_1)}, \quad \hat{P}_1 = \frac{\hat{\mu}_1 \otimes \hat{\nu}_1}{(\hat{\mu}_1, \hat{\nu}_1)},$$
$$\hat{\mu}_1 = \mu_1 - \frac{(p_2 - q_2)(\mu_1, \nu_2)\mu_2}{(p_1 - q_2)(\mu_2, \nu_2)}, \quad \text{and} \quad \hat{\nu}_1 = \nu_1 - \frac{(p_2 - q_2)(\mu_2, \nu_1)\nu_2}{(p_2 - q_1)(\mu_2, \nu_2)}.$$

Therefore

$$u \sim \frac{1}{2}(p_1 - q_1)^2 \hat{P}_1 \operatorname{sech}^2\left(\frac{1}{2}\left(\Lambda_1 + \xi_1^+\right)\right) \quad \text{as} \quad t \to -\infty,$$

where $\xi_1^+ = \log\left(\frac{\hat{r}_1}{p_1 - q_1}\right)$.

Similarly, fixing Λ_2 gives

$$u \sim \frac{1}{2}(p_2 - q_2)^2 \hat{P}_2 \operatorname{sech}^2\left(\frac{1}{2}\left(\Lambda_2 + \xi_2^-\right)\right) \quad \text{as} \quad t \to -\infty,$$
$$u \sim \frac{1}{2}(p_2 - q_2)^2 P_2 \operatorname{sech}^2\left(\frac{1}{2}\left(\Lambda_2 + \xi_2^+\right)\right) \quad \text{as} \quad t \to +\infty,$$

where

$$\xi_2^- = \log\left(\frac{\hat{r}_2}{p_2 - q_2}\right), \quad \xi_2^+ = \log\left(\frac{r_2}{p_2 - q_2}\right), \quad \hat{r}_2 = \frac{r_2(\hat{\mu}_2, \hat{\nu}_2)}{(\mu_2, \nu_2)}, \quad \hat{P}_2 = \frac{\hat{\mu}_2 \otimes \hat{\nu}_2}{(\hat{\mu}_2, \hat{\nu}_2)},$$
$$\hat{\mu}_2 = \mu_2 - \frac{(p_1 - q_1)(\mu_2, \nu_1)\mu_1}{(p_2 - q_1)(\mu_1, \nu_1)}, \quad \text{and} \quad \hat{\nu}_2 = \nu_2 - \frac{(p_1 - q_1)(\mu_1, \nu_2)\nu_1}{(p_1 - q_2)(\mu_1, \nu_1)}.$$

Note that $\frac{(\hat{\mu}_1, \hat{\nu}_1)}{(\mu_1, \nu_1)} = \frac{(\hat{\mu}_2, \hat{\nu}_2)}{(\mu_2, \nu_2)} = 1 - \frac{\alpha(p_1 - q_1)(p_2 - q_2)}{(p_1 - q_2)(p_2 - q_1)} = \beta$. The soliton phase shifts $\Delta_j = \xi_j^+ - \xi_j^-$ are

$$\Delta_1 = \log\left(\frac{\hat{r}_1}{r_1}\right) = \log\beta, \quad \Delta_2 = \log\left(\frac{r_2}{\hat{r}_2}\right) = -\log\beta.$$

We may now summarise the characteristics of the two-soliton matrix solution as follows:

- The matrix amplitude of the first soliton changes from $\frac{1}{2}(p_1 - q_1)^2 P_1$ to $\frac{1}{2}(p_1 - q_1)^2 \hat{P}_1$ and the matrix amplitude of the second soliton changes from $\frac{1}{2}(p_2 - q_2)^2 \hat{P}_2$ to $\frac{1}{2}(p_2 - q_2)^2 P_2$ as t changes from $-\infty$ to $+\infty$.

- If $(\mu_1, \nu_2) = 0\,(P_2 P_1 = 0)$ or $(\mu_2, \nu_1) = 0\,(P_1 P_2 = 0)$ then $\alpha = 0$ and therefore $\beta = 1$, so there is no phase shift but the matrix amplitudes may still change.

- If $(\mu_1, \nu_2) = 0$ and $(\mu_2, \nu_1) = 0$ (giving $P_1 P_2 = P_2 P_1 = 0$) there is no phase shift or change in amplitude and so the solitons have trivial interaction.

In general, for $n \geq 1$, expanding (3.35) gives

$$v_{[n+1]} = 2\begin{pmatrix} A_1 e^{\eta_1} & A_2 e^{\eta_2} & \dots & A_n e^{\eta_n} \end{pmatrix} \left(A_j \frac{e^{\eta_j - \gamma_i}}{p_j - q_i} + \delta_{i,j} I \right)^{-1}_{n\times n} \begin{pmatrix} I e^{-\gamma_1} \\ I e^{-\gamma_2} \\ \vdots \\ I e^{-\gamma_n} \end{pmatrix} \tag{3.41}$$

$$= 2\begin{pmatrix} K_1 e^{\eta_1} & K_2 e^{\eta_2} & \dots & K_n e^{\eta_n} \end{pmatrix} \begin{pmatrix} I e^{-\gamma_1} \\ I e^{-\gamma_2} \\ \vdots \\ I e^{-\gamma_n} \end{pmatrix}, \quad \text{say,} \tag{3.42}$$

$$= \sum_{i=1}^{n} K_i. \tag{3.43}$$

However, for $n > 3$, it is very difficult to isolate each K_i. So we will now only investigate the three-soliton solution. When $n = 3$, (3.43) gives

$$v_{[4]} = 2\left(K_1 + K_2 + K_3\right). \tag{3.44}$$

From (3.41 – 3.43) we must have that

$$K_1 = \frac{1}{h_1}\left(A_1 - \frac{K_2 A_1}{p_1 - q_2} - \frac{K_3 A_1}{p_1 - q_3} \right), \tag{3.45}$$

$$K_2 = \frac{1}{h_2}\left(A_2 - \frac{K_1 A_2}{p_2 - q_1} - \frac{K_3 A_2}{p_2 - q_3} \right), \tag{3.46}$$

$$K_3 = \frac{1}{h_3}\left(A_3 - \frac{K_1 A_3}{p_3 - q_1} - \frac{K_2 A_3}{p_3 - q_2} \right), \tag{3.47}$$

where $h_i = e^{-\Lambda_i} + \frac{r_i}{p_i - q_i}$, for $i = 1, 2, 3$. Solving for K_1, K_2 and K_3 gives

$$K_1 = \frac{h(2,3)}{h(1,2,3)(p_2 - q_3)(p_3 - q_2)}\left(A_1 - \frac{K'_{2,3} A_1}{p_1 - q_2} - \frac{K'_{3,2} A_1}{p_1 - q_3} \right), \tag{3.48}$$

$$K_2 = \frac{h(1,3)}{h(1,2,3)(p_1 - q_3)(p_3 - q_1)}\left(A_2 - \frac{K'_{1,3} A_2}{p_2 - q_1} - \frac{K'_{3,1} A_2}{p_2 - q_3} \right), \tag{3.49}$$

$$K_3 = \frac{h(1,2)}{h(1,2,3)(p_1 - q_2)(p_2 - q_1)}\left(A_3 - \frac{K'_{1,2} A_3}{p_3 - q_1} - \frac{K'_{2,1} A_3}{p_3 - q_2} \right), \tag{3.50}$$

in which

$$\begin{aligned}
h(i,j) &= (p_i - q_j)(p_j - q_i) h_i h_j - r_i r_j \alpha_{i,j}, \\
h(1,2,3) &= h_1 h_2 h_3 - \frac{r_2 r_3 \alpha_{2,3} h_1}{(p_2 - q_3)(p_3 - q_2)} - \frac{r_1 r_3 \alpha_{1,3} h_2}{(p_1 - q_3)(p_3 - q_1)} - \frac{r_1 r_2 \alpha_{1,2} h_3}{(p_1 - q_2)(p_2 - q_1)} \\
&\quad + r_1 r_2 r_3 \left(\frac{\alpha_{1,2,3}}{(p_2 - q_1)(p_1 - q_3)(p_3 - q_2)} + \frac{\alpha_{1,3,2}}{(p_1 - q_2)(p_2 - q_3)(p_3 - q_1)} \right), \\
K'_{i,j} &= \frac{p_j - q_i}{h(i,j)} \left(h_j (p_i - q_j) I - A_j \right) A_i,
\end{aligned}$$

for $i, j \in \{1, 2, 3\}$ and $i \neq j$. Here, the notation from the two-soliton matrix solution has been extended to include the trace of all permutations of products of P_j, $j = 1, 2, 3$, so that

$$\alpha_{i,j} = Tr(P_iP_j) = \frac{(\mu_j, \nu_i)(\mu_i, \nu_j)}{(\mu_i, \nu_i)(\mu_j, \nu_j)} = Tr(P_jP_i),$$
$$\alpha_{1,2,3} = Tr(P_1P_2P_3) = \frac{(\mu_1, \nu_3)(\mu_3, \nu_2)(\mu_2, \nu_1)}{(\mu_1, \nu_1)(\mu_2, \nu_2)(\mu_3, \nu_3)} = Tr(P_2P_3P_1) = Tr(P_3P_1P_2),$$
$$\alpha_{1,3,2} = Tr(P_1P_3P_2) = \frac{(\mu_3, \nu_1)(\mu_2, \nu_3)(\mu_1, \nu_2)}{(\mu_1, \nu_1)(\mu_2, \nu_2)(\mu_3, \nu_3)} = Tr(P_2P_1P_3) = Tr(P_3P_2P_1).$$

Substituting (3.48 - 3.50) into (3.44) gives

$$v_{[4]} = \frac{2}{h(1,2,3)} \left(b_{2,3}A_1 + b_{1,3}A_2 + b_{1,2}A_3 + b_{1,3,2}A_1A_2 + b_{1,2,3}A_1A_3 \right.$$
$$\left. + b_{2,3,1}A_2A_1 + b_{2,1,3}A_2A_3 + b_{3,2,1}A_3A_1 + b_{3,1,2}A_3A_2\right), \qquad (3.51)$$

in which

$$b_{i,j} := \frac{h(i,j)}{(p_i - q_j)(p_j - q_i)} \quad \text{and}$$
$$b_{i,j,k} := \frac{r_j \alpha_{i,j,k}}{(p_k - q_j)(p_j - q_i)\alpha_{i,k}} - \frac{h_j}{p_k - q_i},$$

if $\alpha_{i,k} \neq 0$. For the solution to be regular, we need $\det(\Omega) \neq 0$ for all x, y and t. Here we have

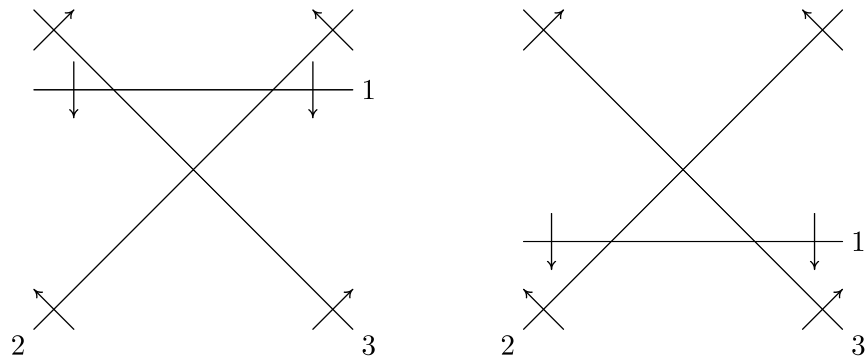
$$\Omega = \left(A_j \tfrac{e^{\eta_j - \gamma_i}}{p_j - q_i} + \delta_{i,j} I\right)_{3\times 3}.$$

Using the fact that $Tr(A_j) = r_j$ and $\det(A_j) = 0$, for $j = 1, 2, 3$, expanding $\det(\Omega)$ gives

$$\det(\Omega) = 1 + \kappa_1 e^{\Lambda_1} + \kappa_2 e^{\Lambda_2} + \kappa_3 e^{\Lambda_3} + \kappa_1\kappa_2\beta_{1,2} e^{\Lambda_1 + \Lambda_2} + \kappa_2\kappa_3\beta_{2,3} e^{\Lambda_2 + \Lambda_3}$$
$$+ \kappa_1\kappa_3\beta_{1,3} e^{\Lambda_1 + \Lambda_3} + \kappa_1\kappa_2\kappa_3\beta_{1,2,3} e^{\Lambda_1 + \Lambda_2 + \Lambda_3}, \qquad (3.52)$$

where

$$\beta_{1,2,3} = 1 - \frac{\alpha_{1,2}(p_1 - q_1)(p_2 - q_2)}{(p_1 - q_2)(p_2 - q_1)} - \frac{\alpha_{2,3}(p_2 - q_2)(p_3 - q_3)}{(p_2 - q_3)(p_3 - q_2)} - \frac{\alpha_{1,3}(p_1 - q_1)(p_3 - q_3)}{(p_1 - q_3)(p_3 - q_1)}$$
$$+ (p_1 - q_1)(p_2 - q_2)(p_3 - q_3)\left(\frac{\alpha_{1,2,3}}{(p_2 - q_1)(p_3 - q_2)(p_1 - q_3)} + \frac{\alpha_{1,3,2}}{(p_1 - q_2)(p_2 - q_3)(p_3 - q_1)}\right),$$
$$\beta_{i,j} = 1 - \frac{\alpha_{i,j}(p_i - q_i)(p_j - q_j)}{(p_i - q_j)(p_j - q_i)} \quad \text{for } i, j \in \{1, 2, 3\},\ i \neq j,$$
$$\kappa_i = \frac{r_i}{p_i - q_i} \quad \text{for } i = 1, 2, 3.$$



(a) $t << 0$ (b) $t >> 0$

Figure 3.1: The chosen configuration of the three-soliton solution

Ordering the spectral parameters $p_3 > q_3 > p_2 > q_2 > p_1 > q_1$ guarantees that $\beta_{i,j} > 0$ and $\kappa_i > 0$. From (3.52), $\det(\Omega)$ will be positive-definite if

$$\frac{\alpha_{1,2,3}}{(p_2-q_1)(p_1-q_3)(p_3-q_2)} + \frac{\alpha_{1,3,2}}{(p_1-q_2)(p_2-q_3)(p_3-q_1)} > 0.$$

To determine the asymptotic forms of each matrix soliton, we fix each Λ_i, $i = 1, 2, 3$ and assume without loss of generality that $0 > p_3 > q_3 > p_2 > q_2 > p_1 > q_1$. In doing so, the solution has the configuration detailed in Figure 3.1. For soliton 1, as $t \to -\infty$, $h_i \to +\infty$ for $i = 2, 3$. Then we can see from (3.45–3.47) that $K_i \to 0$ for $i = 2, 3$. This may be compared with similar expressions in the two-soliton matrix solution, giving

$$v_{[4]} \sim \frac{2r_1^- P_1^-}{h_1},$$

where $r_1^- = r_1$, $P_1^- = \frac{\mu_1^- \otimes \nu_1^-}{(\mu_1^-, \nu_1^-)}$, $\mu_1^- = \mu_1$ and $\nu_1^- = \nu_1$. So as $t \to -\infty$, we have

$$u \sim \frac{1}{2}(p_1-q_1)^2 P_1^- \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_1 + \xi_1^-)\right),$$

where $\xi_1^- = \log\left(\frac{r_1^-}{p_1-q_1}\right)$.

As $t \to +\infty$, $h_i \to \frac{r_i}{p_i-q_i}$, for $i = 2, 3$. Using the fact that u is invariant under the transformation $v_{[4]} \to v_{[4]} + C$, where C is a constant matrix, we have that

$$v_{[4]} \sim \frac{2r_1^+ P_1^+}{e^{-\Lambda_1} + \frac{r_1^+}{p_1-q_1}},$$

where

$$r_1^+ = \frac{r_1(\mu_1^+,\nu_1^+)}{(\mu_1,\nu_1)} = \frac{r_1\beta_{1,2,3}}{\beta_{2,3}}, \quad P_1^+ = \frac{\mu_1^+ \otimes \nu_1^+}{(\mu_1^+,\nu_1^+)},$$

$$\begin{aligned}
\mu_1^+ &= \mu_1 + \frac{(p_2-q_2)(p_3-q_3)}{(\mu_2,\nu_2)\beta_{2,3}}\left(\frac{(\mu_1,\nu_3)(\mu_3,\nu_2)}{(\mu_3,\nu_3)(p_1-q_3)(p_3-q_2)} - \frac{(\mu_1,\nu_2)}{(p_1-q_2)(p_3-q_3)}\right)\mu_2 \\
&\quad + \frac{(p_2-q_3)(p_3-q_2)}{(\mu_3,\nu_3)\beta_{2,3}}\left(\frac{(\mu_1,\nu_2)(\mu_2,\nu_3)}{(\mu_2,\nu_2)(p_1-q_2)(p_2-q_3)} - \frac{(\mu_1,\nu_3)}{(p_1-q_3)(p_2-q_2)}\right)\mu_3, \\
\nu_1^+ &= \nu_1 + \frac{(p_2-q_2)(p_3-q_3)}{(\mu_2,\nu_2)\beta_{2,3}}\left(\frac{(\mu_3,\nu_1)(\mu_2,\nu_3)}{(\mu_3,\nu_3)(p_2-q_3)(p_3-q_1)} - \frac{(\mu_2,\nu_1)}{(p_2-q_1)(p_3-q_3)}\right)\nu_2 \\
&\quad + \frac{(p_2-q_3)(p_3-q_2)}{(\mu_3,\nu_3)\beta_{2,3}}\left(\frac{(\mu_2,\nu_1)(\mu_3,\nu_2)}{(\mu_2,\nu_2)(p_3-q_2)(p_2-q_1)} - \frac{(\mu_3,\nu_1)}{(p_3-q_1)(p_2-q_2)}\right)\nu_3.
\end{aligned}$$

So we have

$$u \sim \frac{1}{2}(p_1-q_1)^2 P_1^+ \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_1+\xi_1^+)\right) \quad \text{as} \quad t \to +\infty,$$

where $\xi_1^+ = \log\left(\frac{r_1^+}{p_1-q_1}\right)$.

Fixing Λ_2 brings soliton 2 to rest. As $t \to -\infty$, $h_1 \to \frac{r_1}{p_1-q_1}$ and $h_3 \to +\infty$. This gives

$$v_{[4]} \sim 2(K'_{1,2} + K'_{2,1}) \sim \frac{2r_2^- P_2^-}{e^{-\Lambda_2} + \frac{r_2^-}{p_2-q_2}},$$

where

$$r_2^- = \frac{r_2(\mu_2^-,\nu_2^-)}{(\mu_2,\nu_2)} = r_2\beta_{1,2}, \quad P_2^- = \frac{\mu_2^- \otimes \nu_2^-}{(\mu_2^-,\nu_2^-)},$$

$$\mu_2^- = \mu_2 - \frac{(p_1-q_1)(\mu_2,\nu_1)}{(p_2-q_1)(\mu_1,\nu_1)}\mu_1, \quad \text{and} \quad \nu_2^- = \nu_2 - \frac{(p_1-q_1)(\mu_1,\nu_2)}{(p_1-q_2)(\mu_1,\nu_1)}\nu_1.$$

As $t \to +\infty$, $h_1 \to +\infty$ and $h_3 \to \frac{r_3}{(p_3-q_3)}$. This gives

$$v_{[4]} \sim 2(K'_{2,3} + K'_{3,2}) \sim \frac{2r_2^+ P_2^+}{e^{-\Lambda_2} + \frac{r_2^+}{p_2-q_2}},$$

where

$$r_2^+ = \frac{r_2(\mu_2^+,\nu_2^+)}{(\mu_2,\nu_2)} = r_2\beta_{2,3}, \quad P_2^+ = \frac{\mu_2^+ \otimes \nu_2^+}{(\mu_2^+,\nu_2^+)},$$

$$\mu_2^+ = \mu_2 - \frac{(p_3-q_3)(\mu_2,\nu_3)}{(p_2-q_3)(\mu_3,\nu_3)}\mu_3, \quad \text{and} \quad \nu_2^+ = \nu_2 - \frac{(p_3-q_3)(\mu_3,\nu_2)}{(p_3-q_2)(\mu_3,\nu_3)}\nu_3.$$

So the asymptotic forms for soliton 2 are

$$\begin{aligned}
u &\sim \frac{1}{2}(p_2-q_2)^2 P_2^- \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_2+\xi_2^-)\right) \quad \text{as} \quad t \to -\infty, \\
u &\sim \frac{1}{2}(p_2-q_2)^2 P_2^+ \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_2+\xi_2^+)\right) \quad \text{as} \quad t \to +\infty,
\end{aligned}$$

where $\xi_2^- = \log\left(\frac{r_2^-}{p_2-q_2}\right)$ and $\xi_2^+ = \log\left(\frac{r_2^+}{p_2-q_2}\right)$.

With Λ_3 fixed, soliton 3 is a rest. As $t \to -\infty$, $h_i \to \frac{r_i}{p_i-q_i}$ for $i = 1, 2$. This gives

$$v_{[4]} \sim \frac{2r_3^- P_3^-}{e^{-\Lambda_3} + \frac{r_3^-}{p_3-q_3}},$$

where

$$\begin{aligned}
r_3^- &= \frac{r_3(\mu_3^-, \nu_3^-)}{(\mu_3, \nu_3)} = \frac{r_3\beta_{1,2,3}}{\beta_{1,2}}, \quad P_3^- = \frac{\mu_3^- \otimes \nu_3^-}{(\mu_3^-, \nu_3^-)},\\
\mu_3^- &= \mu_3 + \frac{(p_2-q_2)(p_1-q_1)}{(\mu_2,\nu_2)\beta_{1,2}}\left(\frac{(\mu_3,\nu_1)(\mu_1,\nu_2)}{(\mu_1,\nu_1)(p_3-q_1)(p_1-q_2)} - \frac{(\mu_3,\nu_2)}{(p_3-q_2)(p_1-q_1)}\right)\mu_2\\
&\quad + \frac{(p_2-q_2)(p_1-q_1)}{(\mu_1,\nu_1)\beta_{1,2}}\left(\frac{(\mu_3,\nu_2)(\mu_2,\nu_1)}{(\mu_2,\nu_2)(p_3-q_2)(p_2-q_1)} - \frac{(\mu_3,\nu_1)}{(p_3-q_1)(p_2-q_2)}\right)\mu_1,\\
\nu_3^- &= \nu_3 + \frac{(p_2-q_2)(p_1-q_1)}{(\mu_2,\nu_2)\beta_{1,2}}\left(\frac{(\mu_1,\nu_3)(\mu_2,\nu_1)}{(\mu_1,\nu_1)(p_2-q_1)(p_1-q_3)} - \frac{(\mu_2,\nu_3)}{(p_2-q_3)(p_1-q_1)}\right)\nu_2\\
&\quad + \frac{(p_2-q_2)(p_1-q_1)}{(\mu_1,\nu_1)\beta_{1,2}}\left(\frac{(\mu_2,\nu_3)(\mu_1,\nu_2)}{(\mu_2,\nu_2)(p_1-q_2)(p_2-q_3)} - \frac{(\mu_1,\nu_3)}{(p_1-q_3)(p_2-q_2)}\right)\nu_1.
\end{aligned}$$

As $t \to +\infty$, $h_i \to +\infty$, for $i = 1, 2$. This gives

$$v_{[4]} \sim \frac{2r_3^+ P_3^+}{e^{-\Lambda_3} + \frac{r_3^+}{p_3-q_3}},$$

where $r_3^+ = r_3$, $P_3^+ = \frac{\mu_3^+ \otimes \nu_3^+}{(\mu_3^+, \nu_3^+)}$, $\mu_3^+ = \mu_3$ and $\nu_3^+ = \nu_3$.

So the asymptotic forms for soliton 3 are

$$\begin{aligned}
u &\sim \frac{1}{2}(p_3-q_3)^2 P_3^- \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_3 + \xi_3^-)\right) \quad \text{as} \quad t \to -\infty,\\
u &\sim \frac{1}{2}(p_3-q_3)^2 P_3^+ \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_3 + \xi_3^+)\right) \quad \text{as} \quad t \to +\infty,
\end{aligned}$$

where $\xi_3^- = \log\left(\frac{r_3^-}{p_3-q_3}\right)$ and $\xi_3^+ = \log\left(\frac{r_3^+}{p_3-q_3}\right)$.

The soliton phase shifts $\Delta_j = \xi_j^+ - \xi_j^-$ are

$$\begin{aligned}
\Delta_1 &= \log\left(\frac{(\mu_1^+, \nu_1^+)}{(\mu_1^-, \nu_1^-)}\right) = \log\left(\frac{\beta_{1,2,3}}{\beta_{2,3}}\right),\\
\Delta_2 &= \log\left(\frac{(\mu_2^+, \nu_2^+)}{(\mu_2^-, \nu_2^-)}\right) = \log\left(\frac{\beta_{2,3}}{\beta_{1,2}}\right),\\
\Delta_3 &= \log\left(\frac{(\mu_3^+, \nu_3^+)}{(\mu_3^-, \nu_3^-)}\right) = \log\left(\frac{\beta_{1,2}}{\beta_{1,2,3}}\right).
\end{aligned}$$

3.8 Plots of the matrix solutions

In this section, we demonstrate the interaction properties of the two-soliton matrix solution of ncKP with various plots.

Figure 3.2 shows a plot of the generic two-soliton matrix solution $u = (u_{ij})$, $i, j = 1, 2$, where

$$\begin{aligned}
r_1 = 2 &\longrightarrow \hat{r}_1 = 40.763,\\
\hat{r}_2 = 20.381 &\longrightarrow r_2 = 1,\\
\mu_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} &\longrightarrow \hat{\mu}_1 = \begin{pmatrix} 10.521 & -14.876 \end{pmatrix},\\
\mu_2 = \begin{pmatrix} 0.8 & -1.25 \end{pmatrix} &\longrightarrow \hat{\mu}_2 = \begin{pmatrix} -0.829 & -1.25 \end{pmatrix},\\
\hat{\nu}_1 = \begin{pmatrix} -0.177 & -0.582 \end{pmatrix} &\longrightarrow \nu_1 = \begin{pmatrix} 0.333 & -0.667 \end{pmatrix},\\
\hat{\nu}_2 = \begin{pmatrix} -27.333 & 51 \end{pmatrix} &\longrightarrow \nu_2 = \begin{pmatrix} -2 & 0.333 \end{pmatrix},\\
P_1 = \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} &\longrightarrow \hat{P}_1 = \begin{pmatrix} -0.274 & -0.901 \\ 0.387 & 1.274 \end{pmatrix},\\
\hat{P}_2 = \begin{pmatrix} -0.551 & 1.028 \\ -0.831 & 1.551 \end{pmatrix} &\longrightarrow P_2 = \begin{pmatrix} 0.793 & -0.132 \\ -1.3 & 0.207 \end{pmatrix},
\end{aligned}$$

as t changes from $-\infty$ to $+\infty$. This plot shows both a change in matrix amplitude and a phase-shift upon interaction.

Figure 3.3 shows a plot of the two-soliton matrix solution $u = (u_{ij})$, $i, j = 1, 2$, where

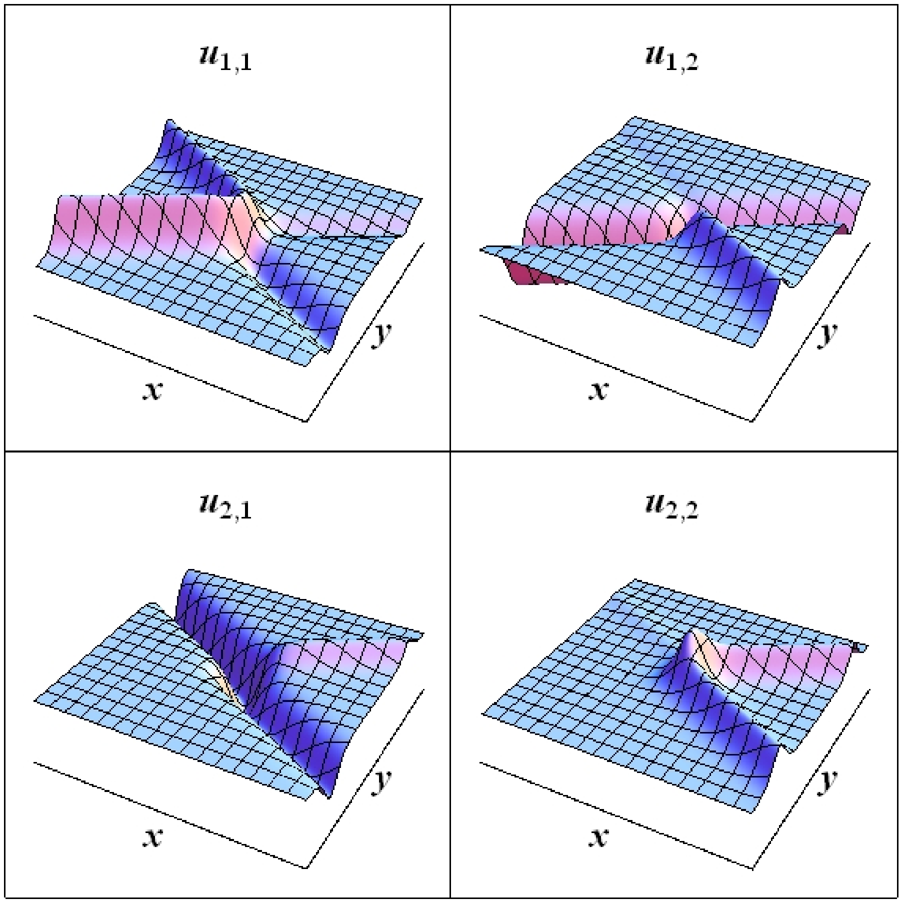
$$\begin{aligned}
r_1 = 2 &\longrightarrow \hat{r}_1 = r_1,\\
\hat{r}_2 = 1 &\longrightarrow r_2 = \hat{r}_2,\\
\mu_1 = \begin{pmatrix} 1 & 3 \end{pmatrix} &\longrightarrow \hat{\mu}_1 = \mu_1,\\
\mu_2 = \begin{pmatrix} 0.8 & -1.25 \end{pmatrix} &\longrightarrow \hat{\mu}_2 = \begin{pmatrix} 1.238 & 0.064 \end{pmatrix},\\
\hat{\nu}_1 = \begin{pmatrix} -0.177 & -0.582 \end{pmatrix} &\longrightarrow \nu_1 = \begin{pmatrix} 0.333 & -0.667 \end{pmatrix},\\
\hat{\nu}_2 = \begin{pmatrix} 1 & 0.333 \end{pmatrix} &\longrightarrow \nu_2 = \hat{\nu}_2,\\
P_1 = \begin{pmatrix} -0.2 & 0.4 \\ -0.6 & 1.2 \end{pmatrix} &\longrightarrow \hat{P}_1 = \begin{pmatrix} -0.032 & 0.344 \\ -0.095 & 1.032 \end{pmatrix},\\
\hat{P}_2 = \begin{pmatrix} 1.018 & -0.339 \\ 0.053 & -0.018 \end{pmatrix} &\longrightarrow P_2 = \begin{pmatrix} 0.658 & -0.219 \\ -1.027 & 0.342 \end{pmatrix},
\end{aligned}$$

as t changes from $-\infty$ to $+\infty$. This plot shows a change in matrix amplitude but no phase-shift because $(\mu_1, \nu_2) = 0$.

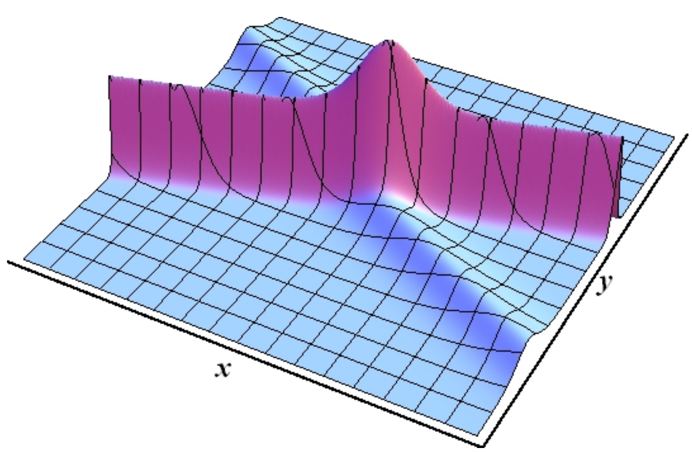
Figure 3.4 shows a plot of the two-soliton matrix solution $u = (u_{ij})$, $i, j = 1, 2$, where

$$\begin{aligned}
r_1 = 2 &\quad\longrightarrow\quad \hat{r}_1 = r_1, \\
\hat{r}_2 = 1 &\quad\longrightarrow\quad r_2 = \hat{r}_2, \\
\mu_1 = \begin{pmatrix} 1 & 3 \end{pmatrix} &\quad\longrightarrow\quad \hat{\mu}_1 = \mu_1, \\
\mu_2 = \begin{pmatrix} 4 & 1.5 \end{pmatrix} &\quad\longrightarrow\quad \hat{\mu}_2 = \mu_2, \\
\hat{\nu}_1 = \begin{pmatrix} 0.25 & -0.667 \end{pmatrix} &\quad\longrightarrow\quad \nu_1 = \hat{\nu}_1, \\
\hat{\nu}_2 = \begin{pmatrix} -1 & 0.333 \end{pmatrix} &\quad\longrightarrow\quad \nu_2 = \hat{\nu}_2, \\
P_1 = \begin{pmatrix} -0.143 & 0.381 \\ -0.429 & 1.143 \end{pmatrix} &\quad\longrightarrow\quad \hat{P}_1 = P_1, \\
\hat{P}_2 = \begin{pmatrix} 1.143 & -0.381 \\ 0.429 & -0.143 \end{pmatrix} &\quad\longrightarrow\quad P_2 = P_2,
\end{aligned}$$

as t changes from $-\infty$ to $+\infty$. This plot shows no change in matrix amplitude and no phase-shift because $(\mu_1, \nu_2) = 0 = (\mu_1, \nu_2)$.



(a)



(b)

Figure 3.2: (a) Plot of matrix KP two-soliton interaction at $t = 0$ with parameters given by $p_1 = -\frac{1}{4}$, $p_2 = \frac{19}{2}$, $q_1 = -\frac{39}{2}$ and $q_2 = \frac{1}{2}$. (b) Plot of the corresponding scalar KP two-soliton interaction with $P_j = r_j = 1$ for $j = 1, 2$.

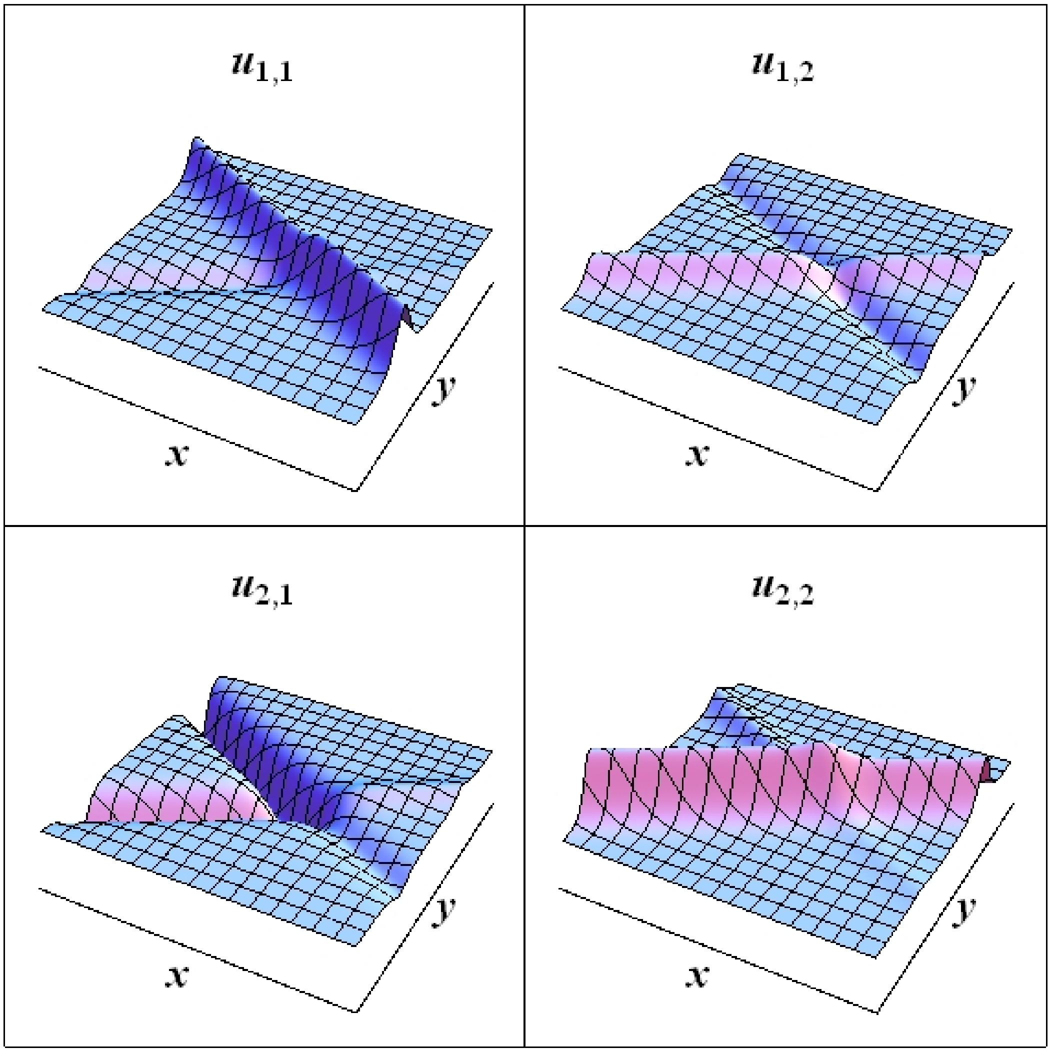


Figure 3.3: Plot of matrix KP two-soliton interaction at $t = 0$ with parameters given by $p_1 = -\frac{1}{4}$, $p_2 = \frac{19}{2}$, $q_1 = -\frac{39}{2}$ and $q_2 = \frac{1}{2}$.

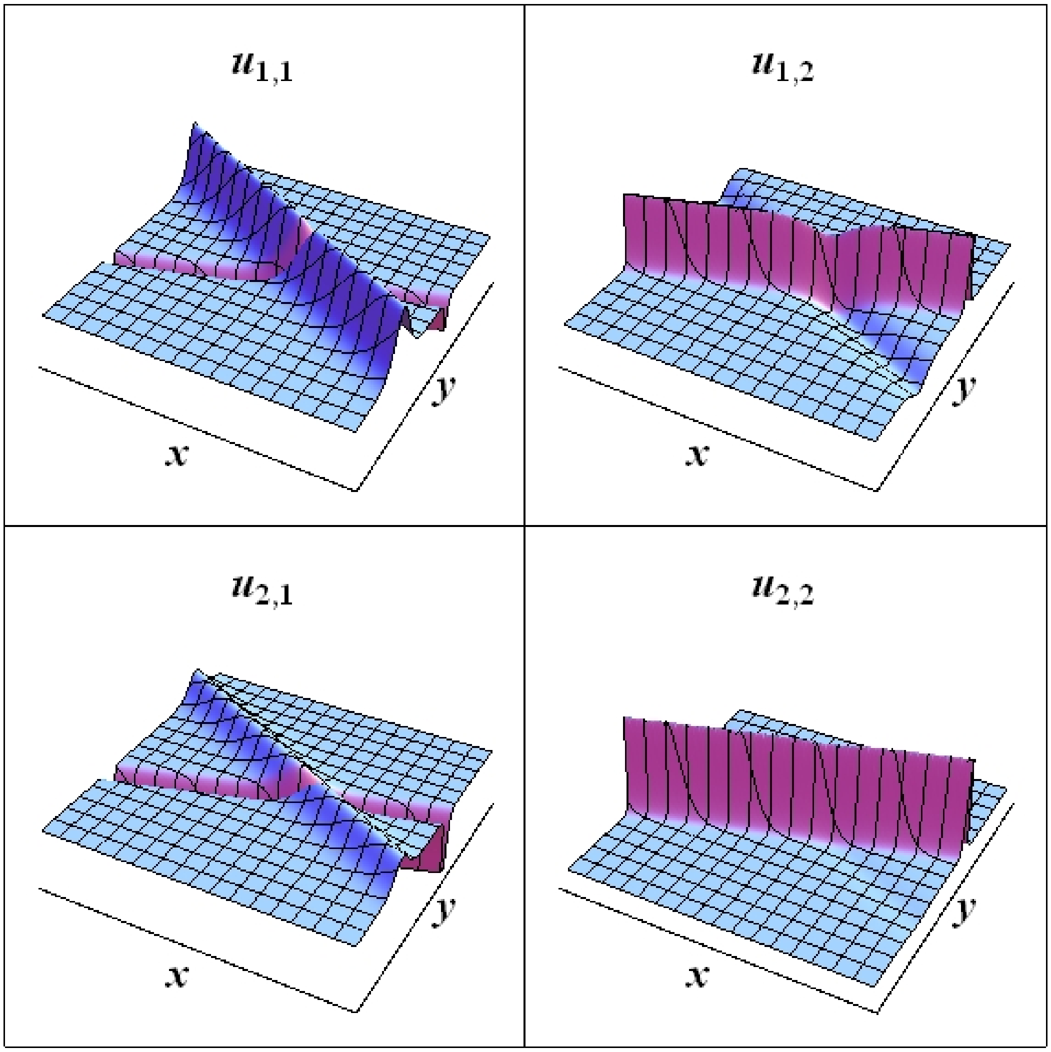


Figure 3.4: Plot of matrix KP two-soliton interaction at $t = 0$ with parameters given by $p_1 = -\frac{1}{4}$, $p_2 = \frac{19}{2}$, $q_1 = -\frac{39}{2}$ and $q_2 = \frac{1}{2}$.

3.9 Reduction to matrix KdV

To make the reduction from matrix KP solutions to matrix KdV solutions, we set $p_i = -q_i = \lambda_i$. For the one-soliton solution, this gives

$$\Lambda = 2\lambda(x - 4\lambda^2 t).$$

So

$$v_{[2]} = \frac{2rP}{e^{-2\lambda(x-4\lambda^2 t)} + \frac{r}{2\lambda}},$$

giving

$$u = 2\lambda^2 P \operatorname{sech}^2\left(\lambda(\upsilon + \varsigma)\right),$$

where $\upsilon = x - 4\lambda^2 t$ and $\varsigma = \frac{1}{2\lambda}\log\left(\frac{r}{2\lambda}\right)$.

For the two-soliton matrix solution, we have

$$h_i = e^{2\lambda_i \upsilon_i + \frac{r_i}{2\lambda_i}}$$

for $i = 1, 2$. The asymptotic forms for soliton 1 are

$$u \sim 2\lambda_1^2 P_1 \operatorname{sech}^2\left(\lambda_1(\upsilon_1 + \varsigma_1^-)\right) \quad \text{as} \quad t \to -\infty,$$
$$u \sim 2\lambda_1^2 \hat{P}_1 \operatorname{sech}^2\left(\lambda_1(\upsilon_1 + \varsigma_1^+)\right) \quad \text{as} \quad t \to +\infty,$$

in which

$$\upsilon_1 = x - 4\lambda_1^2 t, \quad \varsigma_1^- = \log\left(\frac{r_1}{2\lambda_1}\right), \quad \varsigma_1^+ = \log\left(\frac{\hat{r}_1}{2\lambda_1}\right),$$
$$\hat{r}_1 = \frac{r_1(\hat{\mu}_1, \hat{\nu}_1)}{(\mu_1, \nu_1)}, \quad P_1 = \frac{\mu_1 \otimes \nu_1}{(\mu_1, \nu_1)}, \quad \hat{P}_1 = \frac{\hat{\mu}_1 \otimes \hat{\nu}_1}{(\hat{\mu}_1, \hat{\nu}_1)},$$
$$\hat{\mu}_1 = \mu_1 - \frac{2\lambda_2(\mu_1, \nu_2)\mu_2}{(\lambda_1 + \lambda_2)(\mu_2, \nu_2)}, \quad \text{and} \quad \hat{\nu}_1 = \nu_1 - \frac{2\lambda_2(\mu_2, \nu_1)\nu_2}{(\lambda_1 + \lambda_2)(\mu_2, \nu_2)}.$$

The asymptotic forms for soliton 2 are

$$u \sim 2\lambda_2^2 \hat{P}_2 \operatorname{sech}^2\left(\lambda_2(\upsilon_2 + \varsigma_2^-)\right) \quad \text{as} \quad t \to -\infty,$$
$$u \sim 2\lambda_2^2 P_2 \operatorname{sech}^2\left(\lambda_2(\upsilon_2 + \varsigma_2^+)\right) \quad \text{as} \quad t \to +\infty,$$

in which

$$\upsilon_2 = x - 4\lambda_2^2 t, \quad \varsigma_2^- = \log\left(\frac{\hat{r}_2}{2\lambda_2}\right), \quad \varsigma_2^+ = \log\left(\frac{r_2}{2\lambda_2}\right),$$
$$\hat{r}_2 = \frac{r_2(\hat{\mu}_1, \hat{\nu}_1)}{(\mu_1, \nu_1)}, \quad P_1 = \frac{\mu_2 \otimes \nu_2}{(\mu_2, \nu_2)}, \quad \hat{P}_2 = \frac{\hat{\mu}_2 \otimes \hat{\nu}_2}{(\hat{\mu}_2, \hat{\nu}_2)},$$
$$\hat{\mu}_2 = \mu_2 - \frac{2\lambda_2(\mu_2, \nu_1)\mu_1}{(\lambda_1 + \lambda_2)(\mu_1, \nu_1)}, \quad \text{and} \quad \hat{\nu}_2 = \nu_2 - \frac{2\lambda_1(\mu_1, \nu_2)\nu_1}{(\lambda_1 + \lambda_2)(\mu_1, \nu_1)}.$$

These results from the reduction match up with those given in [22].

Chapter 4

A noncommutative mKP equation

Noncommutative mKP equations have been considered by Hamanaka and Toda [27], and Wang and Wadati [51] in the case where the noncommutativity arises through the independent variables. This suggests that we may be able to follow the methods of the previous chapter and try to: derive an ncmKP equation, two families of solutions of ncmKP that can be expressed as quasiwronskians and quasigrammians, directly verify the solutions and investigate matrix solutions. We will also look at an nc Miura transformation which maps solutions of ncmKP to solutions of ncKP.

4.1 A noncommutative mKP hierarchy

A noncommutative mKP hierarchy has been developed by Kupershmidt in [34], but in a more algebraic setting. In [51], a ncmKP hierarchy is given that uses analytic Hirota bilinear identities to give the hierarchy in a condensed form. Here, we construct the ncmKP hierarchy in the spirit of Sato theory.

With $\mathcal{L} = \mathcal{L}_{mKP}$ as defined by (2.36), we obtain the differential operators

$$\begin{aligned}
P_{\geq 1}(\mathcal{L}) &= \partial_x, \\
P_{\geq 1}(\mathcal{L}^2) &= \partial_x^2 + 2w\partial_x, \\
P_{\geq 1}(\mathcal{L}^3) &= \partial_x^3 + 3w\partial_x^2 + 3(w_x + w^2 + w_1)\partial x, \\
P_{\geq 1}(\mathcal{L}^4) &= \partial_x^4 + 4w\partial_x^3 + (6w_x + 4w_1 + 6w^2)\partial_x^2 \\
&\quad + (4w^3 + 6w_{1x} + 4w_{xx} + 4w_2 + 8ww_x + 4w_xw + 6ww_1 + 6w_1w)\partial_x,
\end{aligned} \tag{4.1}$$

which, via the evolution equation (2.37), give the ncmKP hierarchy:

$$\mathcal{L}_{x_1} = [P_{\geq 1}(\mathcal{L}), \mathcal{L}] \Leftrightarrow \begin{cases} w_{x_1} &= w_x, \\ w_{1x_1} &= w_{1x}, \\ w_{2x_1} &= w_{2x}, \\ \dots &, \end{cases} \tag{4.2}$$

$$\mathcal{L}_{x_2} = [P_{\geq 1}(\mathcal{L}^2), \mathcal{L}] \Leftrightarrow \begin{cases} w_y &= w_{xx} + 2w_{1x} + 2ww_x + 2[w, w_1], \\ w_{1y} &= w_{1xx} + 2w_{2x} + 2w_1w_x + 2ww_{1x} + 2[w, w_2], \\ w_{2y} &= w_{2xx} + 2w_{3x} + 2ww_{2x} + 4w_2w_x - 2w_1w_{xx} \\ &\quad + 2[w, w_3], \\ w_{3y} &= w_{3xx} + 2w_{4x} + 2ww_{3x} + 6w_3w_x - 2w_1w_{xxx} \\ &\quad - 6w_2w_{xx} + 2[w, w_4], \\ \dots &, \end{cases} \tag{4.3}$$

$$\mathcal{L}_{x_3} = [P_{\geq 1}(\mathcal{L}^3), \mathcal{L}] \Leftrightarrow \begin{cases} w_t &= w_{xxx} + 3w_{1xx} + 3w_{2x} + 6ww_{1x} + 3w_1w_x \\ &\quad + 3w_xw_1 + 3ww_{xx} + 3w_x^2 + 3w^2w_x + 3[w^2, w_1] \\ &\quad + 3[w, w_2], \\ \dots &, \end{cases} \tag{4.4}$$

$$\mathcal{L}_{x_4} = [P_{\geq 1}(\mathcal{L}^4), \mathcal{L}] \Leftrightarrow \begin{cases} w_{x_4} &= w_{xxxx} + 6w_{2xx} + 4w_{1xxx} + 4w_{3x} + 4ww_{xxx} \\ &\quad +6w_xw_{xx} + 4w_{xx}w_x + 4w_xww_x + 6w^2w_{xx} \\ &\quad +8ww_x^2 + 4w^3w_x + 12ww_{1xx} + 12w_xw_{1x} \\ &\quad +6w_{1x}w_x + 6w_{1x}w_1 + 6w_1w_{1x} + 2w_1w_{xx} \\ &\quad +4w_{xx}w_1 + 12w^2w_{1x} + 8ww_xw_1 + 6ww_1w_x \\ &\quad +4w_xww_1 + 4w_1ww_x + 2w_1w_xw + 12ww_{2x} \\ &\quad +6w_2w_x + 6w_xw_2 + 6[w, w_1^2] + 6[w^2, w_2] \\ &\quad +4[w^3, w_1] + 4[w, w_3], \\ \dots & \end{cases} \tag{4.5}$$

The term $2[w, w_1]$ in the first component of (4.3) prevents us from recursively expressing the fields $w_1, w_2, \dots$ in terms of w and its x- and y-derivatives. However, using the second component of (4.3) and the first component of (4.4), we obtain

$$\begin{aligned} 0 = 2w_t - 2w_{xxx} - 3w_{1xx} - 6ww_{1x} - 3w_{1y} - 6w_xw_1 - 6ww_{xx} - 6w_x^2 - 6w^2w_x \\ - 6[w^2, w_1]. \end{aligned} \tag{4.6}$$

To eliminate the field w_1, we make the change of variables $w_1 = -\frac{1}{2}(w_x + w^2 - W)$. Thus, from the first component of (4.3), and from (4.6), we obtain the following equations:

$$w_t + w_{xxx} - 6ww_xw + 3W_y + 3[w_x, W]_+ - 3[w_{xx}, w] - 3[W, w^2] = 0, \tag{4.7}$$

$$W_x - w_y + [w, W] = 0, \tag{4.8}$$

where the scaling $t \to -4t$ has been made.

Equations (4.7) and (4.8) could also be obtained through the Lax pair

$$L_{\mathrm{mKP}} = \partial_x^2 + 2w\partial_x - \partial_y,$$
$$M_{\mathrm{mKP}} = 4\partial_x^3 + 12w\partial_x^2 + 6(w_x + w^2 + W)\partial_x + \partial_t.$$

The compatibility condition $[L_{\mathrm{mKP}}, M_{\mathrm{mKP}}] = 0$ gives (4.7) and (4.8), which represent the ncmKP equation in a slightly different but equivalent form to that of Wang and Wadati in [51]. Unlike the commutative mKP equation, equation (4.8) is not satisfied by introducing a potential. Instead, we follow the approach in [51] by letting $w = -f_xf^{-1}$, and $W = -f_yf^{-1}$, where $f = f(x, x_q)$ is a differentiable function with an inverse. It is assumed that the function f and its x- and x_q-derivatives do not, in general, commute. These choices of w and W satisfy equation (4.8). It is important to reiterate here that

$$w \neq -(\log f)_x \quad \text{and} \quad W \neq -(\log f)_y.$$

Now that we have $w = -f_xf^{-1}$ and $W = -f_yf^{-1}$, we can attempt to eliminate $w_1, w_2, \ldots$ from the hierarchy. When $w, w_1, w_2, w_3, \ldots$ and their x- and x_q-derivatives commute, we have seen that

$$w_y = (w_x + w^2 + 2w_1)_x,$$
$$w_t = (w_{xx} + 3w_{1x} + 3w_2 + 3ww_x + w^3 + 6ww_1)_x,$$
$$w_{x_4} = (w_{xxx} + 6w_{2x} + 4w_{1xx} + 4w_3 + w^4 + 12ww_2 + 6w^2w_x + 12w^2w_1 + 6w_1^2 + 6w_xw_1 + 12ww_{1x} + 3w_x^2 + 4ww_{xx})_x.$$

Since $w = \mathcal{V}_x$, for the first three fields, say, we can write

$$\mathcal{V}_{xy} = (w_x + w^2 + 2w_1)_x,$$
$$\mathcal{V}_{xt} = (w_{xx} + 3w_{1x} + 3w_2 + 3ww_x + w^3 + 6ww_1)_x,$$
$$\mathcal{V}_{xx_4} = (w_{xxx} + 6w_{2x} + 4w_{1xx} + 4w_3 + w^4 + 12ww_2 + 6w^2w_x + 12w^2w_1 + 6w_1^2 + 6w_xw_1 + 12ww_{1x} + 3w_x^2 + 4ww_{xx})_x.$$

Integrating both sides of each of these equations with respect to x and isolating w_1, w_2, w_3 gives

$$\begin{aligned}
w_1 &= \frac{1}{2}\mathcal{V}_y - \frac{1}{2}w_x - \frac{1}{2}w^2,\\
w_2 &= \frac{1}{3}\mathcal{V}_t - \frac{1}{3}w_{xx} - w_{1x} - ww_x - \frac{1}{3}w^3 - 2ww_1,\\
w_3 &= \frac{1}{4}\mathcal{V}_{x_4} - \frac{1}{4}w_{xxx} - \frac{3}{2}w_{2x} - w_{1xx} - \frac{1}{4}w^4 - 3ww_2 - \frac{3}{2}w^2w_x - 3w^2w_1\\
&\quad - \frac{3}{2}w_1^2 - \frac{3}{2}w_xw_1 - 3ww_{1x} - \frac{3}{4}w_x^2 - ww_{xx}.
\end{aligned}$$

Furthermore, we have that

$$-f_xf^{-1} \to -(\log f)_x \to \mathcal{V}_x \quad \text{and} - f_yf^{-1} \to -(\log f)_y \to \mathcal{V}_y.$$

However, in the noncommutative case we have $w = -f_xf^{-1}$. We may therefore take $w_1, w_2, w_3, \ldots$ to be of the form

$$\begin{aligned}
w_1 &= a_1w_x + a_2w^2 + a_3f_yf^{-1},\\
w_2 &= b_1w_{xx} + b_2w_{1x} + b_3ww_x + b_4w_xw + b_5w^3 + b_6ww_1 + b_7w_1w + b_8f_tf^{-1},\\
w_3 &= c_1w_{xxx} + c_2w_{2x} + c_3w_{1xx} + c_4w^4 + c_5ww_2 + c_6w_2w + c_7w^2w_x + c_8w_xw^2 + c_9ww_xw\\
&\quad + c_{10}w^2w_1 + c_{11}w_1w^2 + c_{12}ww_1w + c_{13}w_1^2 + c_{14}w_xw_1 + c_{15}w_1w_x + c_{16}ww_{1x}\\
&\quad + c_{17}w_{1x}w + c_{18}w_x^2 + c_{19}ww_{xx} + c_{20}w_{xx}w + c_{21}f_{x_4}f^{-1},\\
&\cdots,
\end{aligned}$$

where $a_n, b_n, c_n, \ldots, n = 1, 2, 3, \ldots$, are constants to be chosen such that the resulting noncommutative fields $w_1, w_2, w_3 \ldots$ will then satisfy the ncmKP hierarchy.

We know that, for $n = 1, 2, 3, \ldots$

$$\begin{aligned}
w_{x_n} &= -(f_xf^{-1})_{x_n}\\
&= -f_{xx_n}f^{-1} + f_xf^{-1}f_{x_n}f^{-1}\\
&= -f_{xx_n}f^{-1} - wf_{x_n}f^{-1}.
\end{aligned} \tag{4.9}$$

Using the terms $w_y, w_{1y}, w_{2y}, \ldots$ in (4.3), we can calculate $w_1, w_2, w_3, \ldots$. For example,

$$w_y = (1 + 2a_1)w_{xx} + (2 + 2a_1 + 2a_2)ww_x + (2a_2 - 2a_1)w_xw + 2a_3wf_yf^{-1} + 2a_3f_{xy}f^{-1}.$$

For (4.9) to be satisfied by w_1 and with $w = -f_xf^{-1}$, we require that

$$a_1 = -\frac{1}{2}, \quad a_2 = -\frac{1}{2} \quad \text{and} \quad a_3 = -\frac{1}{2}.$$

Using the same approach for w_2 and w_3, we require that

$$\begin{aligned}
&b_1 = -\frac{1}{3}, \quad b_2 = -1, \quad b_3 = -\frac{2}{3}, \quad b_4 = -\frac{1}{3}, \quad b_5 = -\frac{1}{3}, \quad b_6 = -1, \quad b_7 = -1,\\
&b_8 = -\frac{1}{3}, \quad c_1 = -\frac{1}{4}, \quad c_2 = -\frac{3}{2}, \quad c_3 = -1, \quad c_4 = -\frac{1}{4}, \quad c_5 = -\frac{3}{2}, \quad c_6 = -\frac{3}{2},\\
&c_7 = -\frac{3}{4}, \quad c_8 = -\frac{1}{4}, \quad c_9 = -\frac{1}{2}, \quad c_{10} = -1, \quad c_{11} = -1, \quad c_{12} = -1, \quad c_{13} = -\frac{3}{2},\\
&c_{14} = -1, \quad c_{15} = -\frac{1}{2}, \quad c_{16} = -2, \quad c_{17} = -1, \quad c_{18} = -\frac{3}{4}, \quad c_{19} = -\frac{3}{4}, \quad c_{20} = -\frac{1}{4},\\
&c_{21} = -\frac{1}{4}.
\end{aligned}$$

Therefore, the fields w_1, w_2, w_3 are

$$\begin{aligned}
w_1 &= -\frac{1}{2}w_x - \frac{1}{2}w^2 - \frac{1}{2}f_y f^{-1},\\
w_2 &= -\frac{1}{3}w_{xx} - w_{1x} - \frac{2}{3}ww_x - \frac{1}{3}w_x w - \frac{1}{3}w^3 - ww_1 - w_1 w - \frac{1}{3}f_t f^{-1},\\
w_3 &= -\frac{1}{4}w_{xxx} - \frac{3}{2}w_{2x} - w_{1xx} - \frac{1}{4}w^4 - \frac{3}{2}ww_2 - \frac{3}{2}w_2 w - \frac{3}{4}w^2 w_x - \frac{1}{4}w_x w^2 - \frac{1}{2}ww_x w\\
&\quad - w^2 w_1 - w_1 w^2 - ww_1 w - \frac{3}{2}w_1^2 - w_x w_1 - \frac{1}{2}w_1 w_x - 2ww_{1x} - w_{1x}w - \frac{3}{4}w_x^2\\
&\quad - \frac{3}{4}ww_{xx} - \frac{1}{4}w_{xx}w - \frac{1}{4}f_{x_4} f^{-1}.
\end{aligned}$$

Although tedious, this procedure could easily be used to obtain $w_4, w_5, \ldots$. Rewriting these fields in terms of f gives

$$w_1 = \frac{1}{2}f_{xx}f^{-1} - f_x f^{-1} f_x f^{-1} - \frac{1}{2}f_y f^{-1}, \tag{4.10}$$

$$\begin{aligned}
w_2 = &-\frac{1}{6}f_{xxx}f^{-1} - 2f_x f^{-1} f_x f^{-1} f_x f^{-1} + \frac{1}{2}f_x f^{-1} f_{xx} f^{-1} + f_{xx} f^{-1} f_x f^{-1}\\
&+ \frac{1}{2}f_{xy}f^{-1} - f_y f^{-1} f_x f^{-1} - \frac{1}{2}f_x f^{-1} f_y f^{-1} - \frac{1}{3}f_t f^{-1},
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
w_3 = &-\frac{1}{4}f_{x_4}f^{-1} + \frac{1}{2}f_{xt}f^{-1} - \frac{1}{4}f_{xxy}f^{-1} - f_t f^{-1} f_x f^{-1} - \frac{1}{2}f_x f^{-1} f_t f^{-1} + \frac{1}{2}f_x f^{-1} f_{xy} f^{-1}\\
&- \frac{3}{8}f_y f^{-1} f_y f^{-1} + \frac{9}{8}f_y f^{-1} f_{xx} f^{-1} + \frac{5}{8}f_{xx} f^{-1} f_y f^{-1} - \frac{7}{8}f_{xx} f^{-1} f_{xx} f^{-1}\\
&+ \frac{3}{2}f_{xy} f^{-1} f_x f^{-1} - \frac{1}{2}f_{xxx} f^{-1} f_x f^{-1} - \frac{5}{4}f_x f^{-1} f_x f^{-1} f_y f^{-1} + \frac{7}{4}f_x f^{-1} f_x f^{-1} f_{xx} f^{-1}\\
&- \frac{3}{2}f_x f^{-1} f_y f^{-1} f_x f^{-1} + \frac{3}{2}f_x f^{-1} f_{xx} f^{-1} f_x f^{-1} - 3f_y f^{-1} f_x f^{-1} f_x f^{-1}\\
&+ 3f_{xx} f^{-1} f_x f^{-1} f_x f^{-1} - 6f_x f^{-1} f_x f^{-1} f_x f^{-1} f_x f^{-1}.
\end{aligned} \tag{4.12}$$

This hierarchy, in terms of f, was derived from a different perspective by Dimakis and Müller-Hoissen in [8] in what they refer to as a functional representation of the hierarchy.

4.2 Quasiwronskian solutions obtained from Darboux transformations

In this section, we look at quasiwronskian solutions of ncmKP obtained from the Darboux tranformation $G_\theta = ((\theta^{-1})_x)^{-1}\partial_x\theta^{-1} = 1 - \theta(\theta_x)^{-1}\partial_x$. We will use the pseudodifferential operator $\mathcal{L}_{\mathrm{mKP}}$. Let $\theta_i, i = 1, \ldots, n$ be a particular set of eigenfunctions and introduce the notation $\Theta = (\theta_1, \ldots, \theta_n)$. It is again assumed that the eigenfunction and its derivatives do not commute.

Let $\phi = \phi_{[1]}$ be an eigenfunction of $\mathcal{L}_{\mathrm{mKP}[1]} = \mathcal{L}_{\mathrm{mKP}}$ and $\theta_{[1]} = \theta_1$. Then $\phi_{[2]} := G_{\theta_{[1]}}[\phi_{[1]}]$ and $\theta_{[2]} = \phi_{[2]}|_{\phi\to\theta_2}$ are eigenfunctions for $\mathcal{L}_{\mathrm{mKP}[2]} = G_{\theta_{[1]}}\mathcal{L}_{\mathrm{mKP}}G^{-1}_{\theta_{[1]}}$. In general, for $n \geq 1$ we define the nth Darboux transform of ϕ by

$$\phi_{[n+1]} = \phi_{[n]} - \theta_{[n]}(\theta_{[n]x})^{-1}\phi_{[n]x}, \tag{4.13}$$

in which

$$\theta_{[k]} = \phi_{[k]}|_{\phi\to\theta_k}.$$

It can be shown by induction that $\phi_{[n+1]}$ as given by (4.13) can be expressed as

$$\phi_{[n+1]} = \begin{vmatrix} \Theta & \boxed{\phi} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \phi^{(n)} \end{vmatrix}. \tag{4.14}$$

In the initial case $n = 1$, (4.13) gives

$$\phi_{[2]} = \phi - \theta_1\theta_{1,x}^{-1}\phi_x = \begin{vmatrix} \theta_1 & \boxed{\phi} \\ \theta_{1,x} & \phi_x \end{vmatrix}.$$

So the result is true for $n = 1$. Substituting $n + 1$ for n in (4.13) gives

$$\phi_{[n+2]} = \phi_{[n+1]} - \theta_{[n+1]}(\theta_{[n+1]x})^{-1}\phi_{[n+1]x}. \tag{4.15}$$

Using (3.26) and (3.10), we have

$$\theta^{(1)}_{[n+1]} = \begin{vmatrix} \Theta^{(1)} & \boxed{\theta^{(1)}_{n+1}} \\ \Theta^{(1)} & \theta^{(1)}_{n+1} \\ \vdots & \vdots \\ \Theta^{(n)} & \theta^{(n)}_{n+1} \end{vmatrix} + \sum_{k=1}^{n} \begin{vmatrix} \Theta & \boxed{0} \\ \Theta^{(1)} & \\ \vdots & e_k \\ \Theta^{(n)} & \end{vmatrix} \begin{vmatrix} \Theta^{(k+1)} & \boxed{\theta^{(k+1)}_{n+1}} \\ \Theta^{(1)} & \theta^{(1)}_{n+1} \\ \vdots & \vdots \\ \Theta^{(n)} & \theta^{(n)}_{n+1} \end{vmatrix}$$

$$= \begin{vmatrix} \Theta & \boxed{0} \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & 1 \end{vmatrix} \begin{vmatrix} \Theta^{(n+1)} & \boxed{\theta^{(n+1)}_{n+1}} \\ \Theta^{(1)} & \theta^{(1)}_{n+1} \\ \vdots & \vdots \\ \Theta^{(n)} & \theta^{(n)}_{n+1} \end{vmatrix}. \tag{4.16}$$

Similarly

$$\phi^{(1)}_{[n+1]} = \begin{vmatrix} \Theta & \boxed{0} \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & 1 \end{vmatrix} \begin{vmatrix} \Theta^{(n+1)} & \boxed{\phi^{(n+1)}} \\ \Theta^{(1)} & \phi^{(1)} \\ \vdots & \vdots \\ \Theta^{(n)} & \phi^{(n)} \end{vmatrix}. \tag{4.17}$$

Substituting (4.16) and (4.17) into (4.15) and using the nc Jacobi identity (3.5) gives

$$\phi_{[n+2]} = \begin{vmatrix} \Theta & \boxed{\phi} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \phi^{(n)} \end{vmatrix} - \begin{vmatrix} \Theta & \boxed{\theta_{n+1}} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta^{(n-1)}_{n+1} \\ \Theta^{(n)} & \theta^{(n)}_{n+1} \end{vmatrix} \begin{vmatrix} \Theta^{(1)} & \theta^{(1)}_{n+1} \\ \vdots & \vdots \\ \Theta^{(n)} & \theta^{(n)}_{n+1} \\ \Theta^{(n+1)} & \boxed{\theta^{(n+1)}_{n+1}} \end{vmatrix}^{-1} \begin{vmatrix} \Theta^{(1)} & \phi^{(1)} \\ \vdots & \vdots \\ \Theta^{(n)} & \phi^{(n)} \\ \Theta^{(n+1)} & \boxed{\phi^{(n+1)}} \end{vmatrix} \tag{4.18}$$

$$= \begin{vmatrix} \Theta & \theta_{n+1} & \boxed{\phi} \\ \vdots & \vdots & \vdots \\ \Theta^{(n)} & \theta^{(n)}_{n+1} & \phi^{(n)} \\ \Theta^{(n+1)} & \theta^{(n+1)}_{n+1} & \phi^{(n+1)} \end{vmatrix}. \tag{4.19}$$

This proves the inductive step, completing the proof of (4.14).

We can determine the Darboux-transformed fields $\tilde{w}, \tilde{w}_1, \tilde{w}_2, \dots$ by calculating

$$\begin{aligned}\tilde{\mathcal{L}}_{\text{mKP}} &= (\theta^{-1})_x^{-1}\partial_x\theta^{-1}L\theta\partial_x^{-1}(\theta^{-1})_x \\ &= \partial_x - (-\theta\theta_x^{-1}f)_x(-\theta\theta_x^{-1}f)^{-1} + \bigg(\frac{1}{2}(-\theta\theta_x^{-1}f)_{xx}(-\theta\theta_x^{-1}f)^{-1} \\ &\quad -(-\theta\theta_x^{-1}f)_x(-\theta\theta_x^{-1}f)^{-1}(-\theta\theta_x^{-1}f)_x(-\theta\theta_x^{-1}f)^{-1} - \frac{1}{2}((-\theta\theta_x^{-1}f)^{-1})_y(-\theta\theta_x^{-1}f)^{-1}\bigg)\partial_x^{-1} \\ &\quad + \dots,\end{aligned}$$

which preserves the structure of the ncmKP hierarchy. The coefficients

$$\begin{aligned}\tilde{w} &= -(-\theta\theta_x^{-1}f)_x(-\theta\theta_x^{-1}f)^{-1}, \\ \tilde{w}_1 &= \frac{1}{2}(-\theta\theta_x^{-1}f)_{xx}(-\theta\theta_x^{-1}f)^{-1} - (-\theta\theta_x^{-1}f)_x(-\theta\theta_x^{-1}f)^{-1}(-\theta\theta_x^{-1}f)_x(-\theta\theta_x^{-1}f)^{-1} \\ &\quad - \frac{1}{2}((-\theta\theta_x^{-1}f)^{-1})_y(-\theta\theta_x^{-1}f)^{-1}, \\ &\dots\end{aligned}$$

will satisfy (4.3) and (4.4). In particular, $\tilde{w}$, which could also be obtained from $\tilde{L}_{\text{mKP}} = G_\theta L_{\text{mKP}} G_\theta^{-1}$ or $\tilde{M}_{\text{mKP}} = G_\theta M_{\text{mKP}} G_\theta^{-1}$, will satisfy the ncmKP equation. Using the fact that $\tilde{w}$ is of the form $-\tilde{f}_x\tilde{f}^{-1}$, we obtain

$$\tilde{f} = -\theta\theta_x^{-1}f = \begin{vmatrix} \theta & \boxed{0} \\ \theta_x & 1 \end{vmatrix} f. \tag{4.20}$$

It can be proved by induction that after n Darboux transformations, we have, for $n \geq 1$

$$f_{[n+1]} = \begin{vmatrix} \Theta & \boxed{0} \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & 1 \end{vmatrix} f. \tag{4.21}$$

When $n = 1$, the result (4.21) follows directly from (4.20). Upon substituting $n + 1$

for n, using (4.16), the nc Jacobi identity (3.5) and the homology relations (3.6) we have

$$
\begin{aligned}
f_{[n+2]} &= -\theta_{[n+1]}(\theta_{[n+1],x})^{-1} f \\
&= -\begin{vmatrix} \Theta & \boxed{\theta_{n+1}} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} \\ \Theta^{(n)} & \theta_{n+1}^{(n)} \end{vmatrix}
\begin{vmatrix} \Theta^{(n+1)} & \boxed{\theta_{n+1}^{(n+1)}} \\ \Theta^{(1)} & \theta^{(}_{n+1}1) \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} \\ \Theta^{(n)} & \theta_{n+1}^{(n)} \end{vmatrix}^{-1}
\begin{vmatrix} \Theta & \boxed{0} \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & 1 \end{vmatrix} f \\
&= -\begin{vmatrix} \Theta & \boxed{\theta_{n+1}} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} \\ \Theta^{(n)} & \theta_{n+1}^{(n)} \end{vmatrix}
\begin{vmatrix} \Theta^{(n+1)} & \theta_{n+1}^{(n+1)} \\ \Theta^{(1)} & \theta_{n+1}^{(1)} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \theta_{n+1}^{(n-1)} \\ \Theta^{(n)} & \boxed{\theta_{n+1}^{(n)}} \end{vmatrix}^{-1} f.
\end{aligned}
$$

Now if we use the quasi-Plücker coordinate formula (3.12), we get

$$
f_{[n+2]} = \begin{vmatrix} \Theta & \theta_{n+1} & \boxed{0} \\ \vdots & \vdots & \vdots \\ \Theta^{(n)} & \theta_{n+1}^{(n)} & 0 \\ \Theta^{(n+1)} & \theta_{n+1}^{(n+1)} & 1 \end{vmatrix} f.
$$

This completes the proof of (4.21).

An analogous transformation can be made on f^{-1}. Let $g = f^{-1}$. This gives

$$
w = -(g^{-1})_x g = g^{-1} g_x.
$$

The effect of

$$
\tilde{\mathcal{L}}_{\mathrm{mKP}} = G_\theta \mathcal{L}_{\mathrm{mKP}} G_\theta^{-1}, \quad \tilde{L}_{\mathrm{mKP}} = G_\theta L_{\mathrm{mKP}} G_\theta^{-1} \quad \text{or} \quad \tilde{M}_{\mathrm{mKP}} = G_\theta M_{\mathrm{mKP}} G_\theta^{-1}
$$

is that

$$
\tilde{w} = \tilde{g}^{-1}\tilde{g}_x = (-g\theta_x\theta^{-1})^{-1}(-g\theta_x\theta^{-1})_x,
$$

giving

$$
\tilde{g} = -g\theta_x\theta^{-1} = g\begin{vmatrix} \theta & 1 \\ \theta_x & \boxed{0} \end{vmatrix}.
$$

After n Darboux transformations we have

$$g_{[n+1]} = g \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}. \tag{4.22}$$

We can see that (4.22) is consistent with (4.21) since

$$\begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}^{-1} = \begin{vmatrix} \Theta & \boxed{0} \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & 1 \end{vmatrix}.$$

4.3 Quasigrammian solutions obtained from binary Darboux transformations

In the same way as for ncKP, we may extend the notion of the adjoint to obtain the adjoint Lax pair

$$\begin{aligned} L^{\dagger}_{\mathrm{mKP}} &= \partial_x^2 - 2w^{\dagger}_x - 2w^{\dagger}\partial_x + \partial_y, \\ M^{\dagger}_{\mathrm{mKP}} &= -4\partial_x^3 + 12w^{\dagger}\partial_x^2 + 6(3w^{\dagger}_x - {w^{\dagger}}^2 - W^{\dagger})\partial_x + 6(w^{\dagger}_{xx} - w^{\dagger}_x w^{\dagger} - w^{\dagger}w^{\dagger}_x - W^{\dagger}_x) - \partial_t. \end{aligned}$$

The compatibility condition $[L^{\dagger}_{\mathrm{mKP}}, M^{\dagger}_{\mathrm{mKP}}]$ gives

$$w^{\dagger}_t + w^{\dagger}_{xxx} - 6w^{\dagger}w^{\dagger}_x w^{\dagger} + 3W^{\dagger}_y + 3[w^{\dagger}_x, W^{\dagger}]_+ - 3[w^{\dagger}_{xx}, w^{\dagger}] - 3[W^{\dagger}, {w^{\dagger}}^2] = 0, \tag{4.23}$$

$$W^{\dagger}_x - w^{\dagger}_y + [w^{\dagger}, W^{\dagger}] = 0, \tag{4.24}$$

which is the adjoint of (4.7) and (4.8).

In order to be able to define a binary Darboux transformatiom, we need to introduce a potential $\Omega(\phi, \psi)$ satisfying

$$\begin{aligned} &\Omega(\phi,\psi)_x = \psi^{\dagger}\phi_x, \quad \Omega(\phi,\psi)_y = 2\psi^{\dagger}w\phi_x + \psi^{\dagger}\phi_{xx} - \psi^{\dagger}_x\phi_x, \\ &\Omega(\phi,\psi)_t = 2(-2\psi^{\dagger}_{xx}\phi_x - 2\psi^{\dagger}\phi_{xxx} + 2\psi^{\dagger}_x\phi_{xx} - 3\psi^{\dagger}w^2\phi_x - 3\psi^{\dagger}W\phi_x - 3\psi^{\dagger}w_x\phi_x \\ &\qquad + 6\psi^{\dagger}_x w\phi_x - 6\psi^{\dagger}w\phi_{xx}). \end{aligned}$$

The parts of this definition are compatible when $L_{\mathrm{mKP}}[\phi] = M_{\mathrm{mKP}}[\phi] = 0$ and $L^{\dagger}_{\mathrm{mKP}}[\psi] = M^{\dagger}_{\mathrm{mKP}}[\psi] = 0$. In addition, we may define $\Omega(\Phi, \Psi)$ for any row vectors Φ and Ψ such that $L_{\mathrm{mKP}}[\Phi] = M_{\mathrm{mKP}}[\Phi] = 0$ and $L^{\dagger}_{\mathrm{mKP}}[\Psi] = M^{\dagger}_{\mathrm{mKP}}[\Psi] = 0$. If Φ is an n-vector and Ψ is an m-vector then $\Omega(\Phi, \Psi)$ is an $m \times n$ matrix.

Let θ_i, $i = 1, 2, \ldots, n$ be the eigenfunctions defined in the previous section and let ρ_i, $i = 1, 2, \ldots, n$ be adjoint eigenfunctions. For Lax operators with matrix coefficients, a binary Darboux transformation was defined in [41] and is

$$\phi_{[n+1]} = \phi_{[n]} - \theta_{[n]}\Omega(\rho_{[n]}, \theta_{[n]})^{-1}\Omega(\rho_{[n]}, \phi_{[n]})$$

and

$$\psi_{[n+1]} = \psi_{[n]} - \rho_{[n]}\Omega(\rho_{[n]}, \theta_{[n]})^{\dagger^{-1}}\Omega(\psi_{[n]}, \theta_{[n]})^{\dagger}$$

in which

$$\theta_{[n]} = \phi_{[n]}|_{\phi \to \theta_n}, \quad \rho_{[n]} = \psi_{[n]}|_{\psi \to \rho_n}.$$

Using the notation $\Theta = (\theta_1, \ldots \theta_n)$ and $\mathrm{P} = (\rho_1, \ldots, \rho_n)$ we have, for $n \geq 1$

$$\phi_{[n+1]} = \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \Omega(\phi, \mathrm{P}) \\ \Theta & \boxed{\phi} \end{vmatrix}, \tag{4.25}$$

$$\psi_{[n+1]} = \begin{vmatrix} \Omega(\Theta, \mathrm{P})^{\dagger} & \Omega(\Theta, \psi)^{\dagger} \\ \mathrm{P} & \boxed{\psi} \end{vmatrix}. \tag{4.26}$$

If the above binary Darboux transformation holds then, using (3.25), we have

$$\Omega(\phi_{[n+1]}, \psi_{[n+1]}) = \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \Omega(\phi, \mathrm{P}) \\ \Omega(\Theta, \psi) & \boxed{\Omega(\phi, \psi)} \end{vmatrix}.$$

To prove (4.25), first observe that when $n = 1$,

$$\begin{aligned} \phi_{[2]} &= \phi - \theta_1\Omega(\rho_1, \theta_1)^{-1}\Omega(\rho_1, \phi) \\ &= \begin{vmatrix} \Omega(\rho_1, \theta_1) & \Omega(\phi, \rho_1) \\ \theta_1 & \boxed{\phi} \end{vmatrix}. \end{aligned}$$

Using the nc Jacobi identity (3.5) and the row homological relations (3.6), we have

$$\begin{aligned}
\phi_{[n+2]} &= \phi_{[n+1]} - \theta_{[n+1]}\Omega(\rho_{[n+1]},\theta_{[n+1]})^{-1}\Omega(\rho_{[n+1]},\phi_{[n+1]}) \\
&= \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \Omega(\phi,\mathrm{P}) \\ \Theta & \boxed{\phi} \end{vmatrix} \\
&\quad - \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \Omega(\theta_{n+1},\mathrm{P}) \\ \Theta & \boxed{\theta_{n+1}} \end{vmatrix} \begin{vmatrix} \Omega(\theta,\mathrm{P}) & \Omega(\phi,\mathrm{P}) \\ \Omega(\Theta,\rho) & \boxed{\Omega(\theta,\rho)} \end{vmatrix}^{-1} \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \Omega(\phi,\mathrm{P}) \\ \Omega(\Theta,\rho) & \boxed{\Omega(\phi,\rho)} \end{vmatrix} \\
&= \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \Omega(\phi,\mathrm{P}) \\ \Theta & \boxed{\phi} \end{vmatrix} \\
&\quad - \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \Omega(\theta_{n+1},\mathrm{P}) \\ \Theta & \boxed{\theta_{n+1}} \end{vmatrix} \begin{vmatrix} \Omega(\theta,\mathrm{P}) & \boxed{\Omega(\phi,\mathrm{P})} \\ \Omega(\Theta,\rho) & \Omega(\theta,\rho) \end{vmatrix}^{-1} \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \boxed{\Omega(\phi,\mathrm{P})} \\ \Omega(\Theta,\rho) & \Omega(\phi,\rho) \end{vmatrix} \\
&= \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \Omega(\theta_{n+1},\mathrm{P}) & \Omega(\phi,\mathrm{P}) \\ \Omega(\Theta,\rho_{n+1}) & \Omega(\theta_{n+1},\rho_{n+1}) & \Omega(\phi,\rho_{n+1}) \\ \Theta & \theta_{n+1} & \boxed{\phi} \end{vmatrix}.
\end{aligned}$$

This proves the inductive step and completes the proof. The proof of (4.26) is very similar.

The effect of

$$\hat{\mathcal{L}}_{\mathrm{mKP}} = G_{\theta,\phi_x}\mathcal{L}_{\mathrm{mKP}}G^{-1}_{\theta,\phi_x}, \quad \hat{L}_{\mathrm{mKP}} = G_{\theta,\phi_x}L_{\mathrm{mKP}}G^{-1}_{\theta,\phi_x} \quad \text{or} \quad \hat{M}_{\mathrm{mKP}} = G_{\theta,\phi_x}M_{\mathrm{mKP}}G^{-1}_{\theta,\phi_x}$$

is that

$$\hat{f} = (1-\theta\Omega^{-1}\rho^{\dagger})f = \begin{vmatrix} \Omega & \rho^{\dagger} \\ \theta & \boxed{1} \end{vmatrix} f. \tag{4.27}$$

After n Darboux transformations we have, for $n \geq 1$

$$f_{[n+1]} = \begin{vmatrix} \Omega(\mathrm{P},\Theta) & \mathrm{P}^{\dagger} \\ \Theta & \boxed{I} \end{vmatrix} f. \tag{4.28}$$

Proof of (4.28) is again by induction. For $n=1$, the result clearly follows from (4.27). Next, replacing n with $n+1$ gives

$$\begin{aligned}
f_{[n+2]} &= \left(I - \Theta_{[n+1]}\Omega(\Theta_{[n+1]},\mathrm{P}_{[n+1]})^{-1}\mathrm{P}_{[n+1]}\right) f_{[n+1]} \\
&= \left(I - \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \Omega(\theta_{n+1},\mathrm{P}) \\ \Theta & \boxed{\theta_{n+1}} \end{vmatrix} \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \Omega(\theta_{n+1},\mathrm{P}) \\ \Omega(\Theta,\rho_{n+1}) & \boxed{\Omega(\theta_{n+1},\rho_{n+1})} \end{vmatrix}^{-1} \right. \\
&\qquad \left. \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \mathrm{P}^{\dagger} \\ \Omega(\Theta,\rho_{n+1}) & \boxed{\rho^{\dagger}_{n+1}} \end{vmatrix} \right) \begin{vmatrix} \Omega(\mathrm{P},\Theta) & \mathrm{P}^{\dagger} \\ \Theta & \boxed{I} \end{vmatrix} f.
\end{aligned}$$

By observing that

$$\begin{aligned}
&\begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \mathrm{P}^\dagger \\ \Omega(\Theta,\rho_{n+1}) & \boxed{\rho_{n+1}^\dagger} \end{vmatrix} \begin{vmatrix} \Omega(\mathrm{P},\Theta) & \mathrm{P}^\dagger \\ \Theta & \boxed{I} \end{vmatrix} \\
&= \left(\rho_{n+1}^\dagger - \Omega(\Theta,\rho_{n+1})\Omega(\Theta,\mathrm{P})^{-1}\mathrm{P}^\dagger\right)\left(I - \Theta\Omega(\mathrm{P},\Theta)^{-1}\mathrm{P}^\dagger\right) \\
&= \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \mathrm{P}^\dagger \\ \Omega(\Theta,\rho_{n+1}) & \boxed{\rho_{n+1}^\dagger} \end{vmatrix}
\end{aligned}$$

and using the nc Jacobi identity (3.5) we have

$$\begin{aligned}
f_{[n+2]} &= \left(\begin{vmatrix} \Omega(\mathrm{P},\Theta) & \mathrm{P}^\dagger \\ \Theta & \boxed{I} \end{vmatrix} - \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \Omega(\theta_{n+1},\mathrm{P}) \\ \Theta & \boxed{\theta_{n+1}} \end{vmatrix} \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \Omega(\theta_{n+1},\mathrm{P}) \\ \Omega(\Theta,\rho_{n+1}) & \boxed{\Omega(\theta_{n+1},\rho_{n+1})} \end{vmatrix}^{-1} \right. \\
&\qquad \left. \begin{vmatrix} \Omega(\Theta,\mathrm{P}) & \mathrm{P}^\dagger \\ \Omega(\Theta,\rho_{n+1}) & \boxed{\rho_{n+1}^\dagger} \end{vmatrix} \right) f \\
&= \begin{vmatrix} \Omega(\mathrm{P},\Theta) & \Omega(\mathrm{P},\theta_{n+1}) & \mathrm{P}^\dagger \\ \Omega(\rho_{n+1},\Theta) & \Omega(\rho_{n+1},\theta_{n+1}) & \rho_{n+1}^\dagger \\ \Theta & \theta_{n+1} & \boxed{I} \end{vmatrix} f,
\end{aligned}$$

which proves the inductive step and the proof is now complete.

There appears to be no way of inverting (4.28). Consequently, an analogous transformation on $f_{[n+1]}^{-1}$ is not made in the quasigrammian case.

4.4 Reduction to commutative Wronskian and Grammian solutions

As we saw with ncKP, all of the quasideterminants expressing the Darboux-transformed eigenfunctions and variables $f_{[n+1]}$ should reduce to the corresponding commutative results in Chapter 2. Using (3.11), in the commutative case we have:

- The transformed eigenfunction

$$\phi_{[n+1]} = \begin{vmatrix} \Theta & \boxed{\phi} \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \phi^{(n)} \end{vmatrix} \overset{c}{=} (-1)^{n+1} \begin{vmatrix} \Theta & \phi \\ \vdots & \vdots \\ \Theta^{(n-1)} & \phi^{(n-1)} \\ \Theta^{(n)} & \phi^{(n)} \end{vmatrix} \Big/ \left|\hat{\Theta}\right|,$$

- The transformed variable

$$f_{[n+1]} = \begin{vmatrix} \Theta & \boxed{0} \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & 1 \end{vmatrix} f \overset{c}{=} (-1)^{n+1} e^{-\mathcal{V}} \begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} \Theta^{(1)} \\ \vdots \\ \Theta^{((n))} \end{vmatrix}.$$

  This gives

$$\mathcal{V}_{[n+1]} \overset{c}{=} -\log(f_{[n+1]}) \overset{c}{=} \mathcal{V} + (-1)^n \log \left(\begin{vmatrix} \Theta & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} \Theta^{(1)} \\ \vdots \\ \Theta^{((n))} \end{vmatrix} \right)$$

$$\overset{c}{=} \mathcal{V} + (-1)^n \log \left(\hat{\Theta} \Big/ \begin{vmatrix} \Theta^{(1)} \\ \vdots \\ \Theta^{((n))} \end{vmatrix} \right).$$

- The transformed binary eigenfunction

$$\phi_{[n+1]} = \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \Omega(\phi, \mathrm{P}) \\ \Theta & \boxed{\phi} \end{vmatrix} \overset{c}{=} \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \Omega(\phi, \mathrm{P}) \\ \Theta & \phi \end{vmatrix} \Bigg/ \left| \Omega(\Theta, \mathrm{P}) \right|,$$

- The transformed adjoint eigenfunction

$$\psi_{[n+1]} = \begin{vmatrix} \Omega(\Theta, \mathrm{P})^\dagger & \Omega(\Theta, \psi)^\dagger \\ \mathrm{P} & \boxed{\psi} \end{vmatrix} \overset{c}{=} \begin{vmatrix} \Omega(\Theta, \mathrm{P})^\dagger & \Omega(\Theta, \psi)^\dagger \\ \mathrm{P} & \psi \end{vmatrix} \Bigg/ \left| \Omega(\Theta, \mathrm{P})^\dagger \right|,$$

- The transformed binary variable

$$f_{[n+1]} = \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \mathrm{P}^\dagger \\ \Theta & \boxed{I} \end{vmatrix} f \overset{c}{=} e^{-\mathcal{V}} \begin{vmatrix} \Omega(\Theta, \mathrm{P}) & \mathrm{P}^\dagger \\ \Theta & I \end{vmatrix} \Bigg/ \left| \Omega(\Theta, \mathrm{P}) \right|.$$

  Therefore

$$\begin{aligned} \mathcal{V}_{[n+1]} &\overset{c}{=} -\log(f_{[n+1]}) \\ &\overset{c}{=} \mathcal{V} + \log \left(\left| \Omega(\Theta, \mathrm{P}) \right| \right)_x . \end{aligned}$$

We therefore recover all of the commutative solutions given in Chapter 2.

4.5 Direct verification of the solutions

Since we did not derive an expression for $\hat{f}^{-1}_{[n+1]}$ when finding quasigrammian solutions, we only prove the quasiwronskian solutions in this section.

The Lax pairs of KP and mKP are the same when the vacuum solutions are trivial. Let Θ be a common eigenfunction for these two (trivial vacuum) Lax pairs. For ncmKP, the trivial vacuum solution, obtained from $f = 1$, which gives $w = 0 = W$, is

$$F = \begin{vmatrix} \Theta & \boxed{0} \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & 1 \end{vmatrix}. \tag{4.29}$$

To verify directly that (4.29) is a solution, we also use the solutions $v = -2Q$ and $\hat{v} = -2\hat{Q}$ of ncKP (3.19). Here, for convenience this is written in potential form

$$Q = Q(0,0) = \begin{vmatrix} \Theta & 0 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}, \quad \hat{Q} = \hat{Q}(0,0) = \begin{vmatrix} \Theta^{(1)} & 0 \\ \Theta^{(2)} & 0 \\ \vdots & \vdots \\ \Theta^{(n)} & 1 \\ \Theta^{(n+1)} & \boxed{0} \end{vmatrix}.$$

Note that $\hat{Q}$ is only a solution if the vacuum is zero.

In a similar way we define

$$F(j) = \begin{vmatrix} \Theta & \boxed{0} \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-j)} & 1 \\ \vdots & \vdots \\ \Theta^{(n)} & 0 \end{vmatrix}, \tag{4.30}$$

so that $F = F(0)$. Using (3.14), we have homological relations expressed as the identities

$$FQ(0,j) = F(j+1). \tag{4.31}$$

From the quasi-Plücker coordinates (3.12), the inverse of F can be obtained from the expression for F by swapping the boxed entry and the 1 in the last column of F. Thus we define

$$G = F^{-1} = \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix} \tag{4.32}$$

and additionally

$$G(j) = \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n+j)} & \boxed{0} \end{vmatrix}.$$

Then

$$\hat{Q}(j,0)G = -G(j+1). \tag{4.33}$$

Now consider the derivatives of $F(j)$: using (4.30) and (4.31),

$$F(j)_x = F\hat{Q}(0,j) - F(j+1) = F(\hat{Q}(0,j) - Q(0,j)). \tag{4.34}$$

More generally, if we assume that Θ satisfies the linear equations $\Theta_{x_k} = \Theta_{\underbrace{x \cdots x}_{k}}$, we have

$$F(j)_{x_{k+1}} = \sum_{i=0}^{k} F(i)\hat{Q}(k-i,j) - F(k+j+1) \tag{4.35}$$

$$= F\left(\hat{Q}(k,j) + \sum_{i=1}^{k} Q(0,i-1)\hat{Q}(k-i,j) - Q(0,k+j)\right). \tag{4.36}$$

Thus, using (3.30),

$$F_x = F\hat{Q} - F(1) = F(\hat{Q} - Q), \tag{4.37}$$

$$F_{xx} = F\big((\hat{Q} - Q)^2 + \hat{Q}_x - Q_x\big), \tag{4.38}$$

and

$$F_y = F\hat{Q}(1,0) + F(1)\hat{Q} - F(2),$$

and so

$$F_{xx} + F_y = 2F\hat{Q}_x. \tag{4.39}$$

Using the nc Jacobi identity and (4.33) we can show that

$$\hat{Q}(0,1) = \begin{vmatrix} \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 1 \\ \Theta^{(n)} & 0 \\ \Theta^{(n+1)} & \boxed{0} \end{vmatrix} = \begin{vmatrix} \Theta & 1 & 0 \\ \Theta^{(1)} & 0 & 0 \\ \vdots & \vdots & \vdots \\ \Theta^{(n-1)} & 0 & 1 \\ \Theta^{(n)} & 0 & 0 \\ \Theta^{(n+1)} & 0 & \boxed{0} \end{vmatrix} = Q(1,0) - G(1)F^{-1}Q = Q(1,0) + \hat{Q}Q.$$

This is the noncommutative version of the first bilinear identity in the ncmKP hierarchy. It can be generalized to get to the other members of the hierarchy:

$$\hat{Q}(i,j) = Q(i+1,j-1) + \hat{Q}(i,0)Q(0,j-1). \tag{4.40}$$

This follows immediately from considering $\hat{Q}(i,j)$ written as

$$\begin{vmatrix} \Theta & 1 & 0 \\ \Theta^{(1)} & 0 & 0 \\ \vdots & \vdots & \vdots \\ \Theta^{(n-j)} & 0 & 1 \\ \vdots & \vdots & \vdots \\ \Theta^{(n-1)} & 0 & 0 \\ \Theta^{(n)} & 0 & 0 \\ \Theta^{(n+1+i)} & 0 & \boxed{0} \end{vmatrix} = \begin{vmatrix} \Theta & 0 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-j)} & 1 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n+1+i)} & \boxed{0} \end{vmatrix} - \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-j)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n+1+i)} & \boxed{0} \end{vmatrix} \begin{vmatrix} \Theta & 1 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-j)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}^{-1} \begin{vmatrix} \Theta & 0 \\ \Theta^{(1)} & 0 \\ \vdots & \vdots \\ \Theta^{(n-j)} & 1 \\ \vdots & \vdots \\ \Theta^{(n-1)} & 0 \\ \Theta^{(n)} & \boxed{0} \end{vmatrix}$$

$$= Q(i+1,j-1) - G(i+1)F^{-1}Q(0,j-1)$$

and then using (4.33).

On substituting F and its derivatives into (4.7), all the terms cancel and the solution is therefore verified.

Using (4.37) and (4.38) in (4.39), we can isolate $u = -2Q_x$ to give

$$u = -F^{-1}w_xF - F^{-1}w^2F - F^{-1}F_y, \tag{4.41}$$

which is the noncommutative Miura transformation between the KP and mKP equations.

4.6 Matrix solutions

We take trivial the vacuum solution $f = 1$ so that $w = W = 0$. This gives

$$f_{[n+1]} = \begin{vmatrix} \Omega(\mathrm{P}, \Theta) & \mathrm{P}^{\dagger} \\ \Theta & \boxed{I} \end{vmatrix}. \tag{4.42}$$

The eigenfunctions θ_i and the binary eigenfunctions ρ_i satisfy

$$\theta_{i,xx} = \theta_{i,y}, \quad \theta_{i,t} = -4\theta_{i,xxx}$$

and

$$\rho_{i,xx} = -\rho_{i,y}, \quad \rho_{i,t} = -4\rho_{i,xxx}.$$

The simplest nontrivial solutions of these equations are

$$\theta_j = A_j e^{\eta_j}, \quad \rho_i = B_i e^{-\gamma_i},$$

where $\eta_j = p_j(x + p_j y - 4p_j^2 t), \gamma_i = q_i(x + q_i y - 4q_i^2 t)$ and A_j, B_i are $d \times m$ matrices. With this, we have

$$\Omega(\theta, \rho) = \delta_{i,j} I - \frac{p_j B_i^T A_j}{q_i(p_j - q_i)} e^{\eta_j - \gamma_i}.$$

We take $A_j = r_j P_j$, where r_j is a scalar and P_j is a projection matrix, and we take $B_i = I$.

In the case $n = 1$, expanding (4.42) gives

$$f_{[2]} = I + \frac{\frac{r}{q} P}{e^{-\Lambda} - \frac{rp}{q(p-q)}}.$$

If $r > 0$ and either $q > p > 0$ or $0 > q > p$, or alternatively, if $r < 0$ and either $p > q > 0$ or $0 > p > q$ then

$$w = -f_{[2],x} f_{[2]}^{-1} = \frac{1}{4}(pq)^{-\frac{1}{2}}(p-q)^2 P \operatorname{sech}\left(\frac{\Lambda + \varphi}{2}\right) \operatorname{sech}\left(\frac{\Lambda + \chi}{2}\right),$$
$$W = -f_{[2],y} f_{[2]}^{-1} = (p+q)w,$$

where $\varphi = \log\left(\frac{-pr}{q(p-q)}\right)$ and $\chi = \log\left(\frac{-r}{p-q}\right)$. Both w and W have a unique maximum where

$$\Lambda = -\log\left(\frac{-(pq^{-1})^{\frac{1}{2}} r}{p-q}\right) = \xi.$$

In the case $n = 2$, expanding (4.42) gives

$$\begin{aligned} f_{[3]} &= I + \begin{pmatrix} A_1 e^{\eta_1} & A_2 e^{\eta_2} \end{pmatrix} \left(\delta_{i,j} I - \tfrac{p_j A_j}{q_i (p_j - q_i)} e^{\eta_j - \gamma_i} \right)^{-1}_{2\times 2} \begin{pmatrix} I \frac{e^{-\gamma_1}}{q_1} \\ I \frac{e^{-\gamma_2}}{q_2} \end{pmatrix} \\ &= I + \begin{pmatrix} L_1 e^{\gamma_1} & L_2 e^{\gamma_2} \end{pmatrix} \begin{pmatrix} I \frac{e^{-\gamma_1}}{q_1} \\ I \frac{e^{-\gamma_2}}{q_2} \end{pmatrix}, \quad \text{say} \\ &= I + \frac{1}{q_1} L_1 + \frac{1}{q_2} L_2, \end{aligned}$$

where L_1 and L_2 satisfy

$$\begin{aligned} L_1 \left(I - \frac{p_1 r_1 e^{\Lambda_1}}{(p_1 - q_1) q_1} P_1 \right) &= e^{\Lambda_1} A_1 + \frac{p_1 e^{\Lambda_1}}{(p_1 - q_2) q_2} L_2 A_1, \\ L_2 \left(I - \frac{p_2 r_2 e^{\Lambda_2}}{(p_2 - q_2) q_2} P_2 \right) &= e^{\Lambda_2} A_2 + \frac{p_2 e^{\Lambda_2}}{(p_2 - q_1) q_1} L_1 A_2. \end{aligned}$$

Solving for L_1 and L_2 gives

$$\begin{aligned} L_1 &= \frac{(p_2 - q_1) q_1}{h} ((p_1 - q_2) q_2 h_2 I + p_1 A_2) A_1, \\ L_2 &= \frac{(p_1 - q_2) q_2}{h} ((p_2 - q_1) q_1 h_1 I + p_2 A_1) A_2, \end{aligned}$$

where $h_i = e^{-\Lambda_i} - \frac{p_i r_i}{(p_i - q_i) q_i}$ and $h = h_1 h_2 q_1 q_2 (p_1 - q_2)(p_2 - q_1) - \alpha p_1 p_2 r_1 r_2$.

We now investigate the behaviour of $f_{[3]}$ as $t \to \pm\infty$. We first fix Λ_1 and assume without loss of generality that $0 > p_2 > q_2 > p_1 > q_1$. Then, as $t \to -\infty$,

$$f_{[3]} \sim I + \frac{\frac{r_1}{q_1} P_1}{h_1}$$

and therefore

$$w \sim \frac{1}{4} (p_1 q_1)^{-\frac{1}{2}} (p_1 - q_1)^2 P_1 \operatorname{sech}\left(\frac{\Lambda_1 + \varphi_1^-}{2} \right) \operatorname{sech}\left(\frac{\Lambda_1 + \chi_1^-}{2} \right), \tag{4.43}$$

where $\varphi_1^- = \log\left(\frac{-p_1 r_1}{q_1 (p_1 - q_1)} \right)$ and $\chi_1^- = \log\left(\frac{-r_1}{p_1 - q_1} \right)$. We also have the phase-constant $\xi_1^- = -\log\left(\frac{-(p_1 q_1^{-1})^{\frac{1}{2}} r_1}{p_1 - q_1} \right)$.

Note that $w = -f_{[3],x} f_{[3]}^{-1}$ and $W = -f_{[3],y} f_{[3]}^{-1}$ are invariant under the transformation

$f_{[3]} \to f_{[3]}C$ where C is a non-singular constant matrix. As $t \to +\infty$, we get

$$\begin{aligned} f_{[3]} &\sim \left(I + \left(\frac{M}{h_1 r_2 p_2 q_1 (p_1 - q_2)(p_2 - q_1) + \alpha p_1 p_2 r_1 r_2 (p_2 - q_2)}\right) \right.\\ &\quad \times \left.\left(I + \frac{(p_2 - q_2)A_2}{q_2 r_2}\right)\right)\left(I - \frac{(p_2 - q_2)A_2}{p_2 r_2}\right) \\ &\sim \left(I + \left(\frac{M}{h_1 r_2 p_2 q_1 (p_1 - q_2)(p_2 - q_1) + \alpha p_1 p_2 r_1 r_2 (p_2 - q_2)}\right)\right.\\ &\quad \times \left.\left(I + \frac{(p_2 - q_2)A_2}{q_2 r_2}\right)\right) \\ &\sim I + \frac{\frac{\hat{r}_1}{q_1}\hat{P}_1}{e^{\gamma_1 - \eta_1} - \frac{p_1 \hat{r}_1}{q_1(p_1 - q_1)}}, \end{aligned}$$

where

$$\begin{aligned} M &= r_2 p_2 (p_1 - q_2)(p_2 - q_1)A_1 - (p_2 - q_2)(p_1(p_2 - q_1)A_2 A_1 \\ &\quad + p_2(p_1 - q_2)A_1 A_2 + \alpha p_1 r_1 (p_2 - q_2))A_2, \\ \hat{r}_1 &= r_1\left(1 - \frac{\alpha(p_1 - q_1)(p_2 - q_2)}{(p_1 - q_2)(p_2 - q_1)}\right) = \frac{r_1(\hat{\mu}_1, \hat{\nu}_1)}{(\mu_1, \nu_1)}, \\ \hat{\mu}_1 &= \mu_1 - \frac{p_1(p_2 - q_2)(\mu_1, \nu_2)\mu_2}{p_2(p_1 - q_2)(\mu_2, \nu_2)}, \\ \hat{\nu}_1 &= \nu_1 - \frac{q_1(p_2 - q_2)(\mu_2, \nu_1)\nu_2}{q_2(p_2 - q_1)(\mu_2, \nu_2)}, \\ \hat{P}_1 &= \frac{\hat{\mu}_1 \otimes \hat{\nu}_1}{(\hat{\mu}_1, \hat{\nu}_1)}. \end{aligned}$$

Therefore

$$w \sim \frac{1}{4}(p_1 q_1)^{-\frac{1}{2}}(p_1 - q_1)^2 \hat{P}_1 \operatorname{sech}\left(\frac{\Lambda_1 + \varphi_1^+}{2}\right) \operatorname{sech}\left(\frac{\Lambda_1 + \chi_1^+}{2}\right),$$

where $\varphi_1^+ = \log\left(\frac{-p_1 \hat{r}_1}{q_1(p_1 - q_1)}\right)$, $\chi_1^+ = \log\left(\frac{-\hat{r}_1}{p_1 - q_1}\right)$ and $\xi_1^+ = -\log\left(\frac{-(p_1 q_!^{-1})^{\frac{1}{2}}\hat{r}_1}{p_1 - q_1}\right)$.

Similarly, fixing Λ_2 gives

$$\begin{aligned} w &\sim \frac{1}{4}(p_2 q_2)^{-\frac{1}{2}}(p_2 - q_2)^2 \hat{P}_2 \operatorname{sech}\left(\frac{\Lambda_2 + \varphi_2^-}{2}\right) \operatorname{sech}\left(\frac{\Lambda_2 + \chi_2^-}{2}\right) \quad \text{as} \quad t \to -\infty, \\ w &\sim \frac{1}{4}(p_2 q_2)^{-\frac{1}{2}}(p_2 - q_2)^2 P_2 \operatorname{sech}\left(\frac{\Lambda_2 + \varphi_2^+}{2}\right) \operatorname{sech}\left(\frac{\Lambda_2 + \chi_2^+}{2}\right) \quad \text{as} \quad t \to +\infty, \end{aligned}$$

where $\hat{\mu}_2 = \mu_2 - \frac{p_2(p_1 - q_1)(\mu_2, \nu_1)\mu_1}{p_1(p_2 - q_1)(\mu_1, \nu_1)}$, $\hat{\nu}_2 = \nu_2 - \frac{q_2(p_1 - q_1)(\mu_1, \nu_2)\nu_1}{q_1(p_1 - q_2)(\mu_1, \nu_1)}$, $\hat{P}_2 = \frac{\hat{\mu}_2 \otimes \hat{\nu}_2}{(\hat{\mu}_2, \hat{\nu}_2)}$, $\varphi_2^- = \log\left(\frac{-p_2 \hat{r}_2}{q_2(p_2 - q_2)}\right)$, $\chi_2^- = \log\left(\frac{-\hat{r}_2}{p_2 - q_2}\right)$, $\hat{r}_2 = r_2\left(1 - \frac{\alpha(p_1 - q_1)(p_2 - q_2)}{(p_1 - q_2)(p_2 - q_1)}\right) = \frac{r_2(\hat{\mu}_2, \hat{\nu}_2)}{(\mu_2, \nu_2)}$, $\varphi_2^+ = \log\left(\frac{-p_2 r_2}{q_2(p_2 - q_2)}\right)$ and $\chi_2^+ = \log\left(\frac{-r_2}{p_2 - q_2}\right)$. The soliton phase-constants are: $\xi_1^- = -\log\left(\frac{-(pq^{-1})^{\frac{1}{2}} r_1}{p_1 - q_1}\right)$, $\xi_1^+ = -\log\left(\frac{-(pq^{-1})^{\frac{1}{2}} \hat{r}_1}{p - q}\right)$, $\xi_2^- = -\log\left(\frac{-(p_2 q_2^{-1})^{\frac{1}{2}} \hat{r}_2}{p_2 - q_2}\right)$ and $\xi_2^+ = -\log\left(\frac{-(p_2 q_2^{-1})^{\frac{1}{2}} r_2}{p_2 - q_2}\right)$.

The soliton phase shifts $\Delta_i = \xi_i^+ - \xi_i^-$ are

$$\Delta_1 = \log\left(\frac{r_1}{\hat{r}_1}\right) = -\log\beta, \quad \Delta_2 = \log\left(\frac{\hat{r}_2}{r_2}\right) = \log\beta.$$

The characteristics of the two-soliton solution may be summarised in the same way as the matrix solutions of ncKP:

- The matrix amplitude of the first soliton changes from $\frac{1}{4}(p_1q_1)^{-\frac{1}{2}}(p_1-q_1)^2P_1$ to $\frac{1}{4}(p_1q_1)^{-\frac{1}{2}}(p_1-q_1)^2\hat{P}_1$ and the matrix amplitude of the second soliton changes from $\frac{1}{4}(p_2q_2)^{-\frac{1}{2}}(p_2-q_2)^2\hat{P}_2$ to $\frac{1}{4}(p_2q_2)^{-\frac{1}{2}}(p_2-q_2)^2P_2$ as t changes from $-\infty$ to $+\infty$.
- If $(\mu_1,\nu_2)=0\,(P_2P_1=0)$ or $(\mu_2,\nu_1)=0\,(P_1P_2=0)$ then $\alpha=0$ and therefore $\beta=1$, so there is no phase shift but the matrix amplitudes may still change.
- If $(\mu_1,\nu_2)=0$ and $(\mu_2,\nu_1)=0$ (giving $P_1P_2=P_2P_1=0$) there is no phase shift or change in amplitude and so the solitons have trivial interaction.

In general, for $n \geq 1$, expanding (4.42) gives

$$f_{[n+1]} = I + \begin{pmatrix} A_1e^{\eta_1} & A_2e^{\eta_2} & \dots & A_ne^{\eta_n}\end{pmatrix}\left(\delta_{i,j}I - A_j\frac{p_je^{(\eta_j-\gamma_i)}}{(p_j-q_i)q_i}\right)^{-1}_{n\times n}\begin{pmatrix} Ie^{-\gamma_1} \\ Ie^{-\gamma_2} \\ \vdots \\ Ie^{-\gamma_n}\end{pmatrix} \tag{4.44}$$

$$= I + \begin{pmatrix} L_1e^{\gamma_1} & L_2e^{\gamma_2} & \dots & L_ne^{\gamma_n}\end{pmatrix}\begin{pmatrix} I\frac{e^{-\gamma_1}}{q_1} \\ I\frac{e^{-\gamma_2}}{q_2} \\ \vdots \\ I\frac{e^{-\gamma_n}}{q_n}\end{pmatrix}, \quad \text{say,} \tag{4.45}$$

$$= I + \sum_{i=1}^{n}\frac{1}{q_i}L_i. \tag{4.46}$$

In a similar way to that of matrix solutions of ncKP, for $n > 3$, it is very difficult to isolate each L_i. So we will now only investigate the three-soliton solution. When $n=3$, (4.46) gives

$$f_{[4]} = I + \frac{1}{q_1}L_1 + \frac{1}{q_2}L_2 + \frac{1}{q_3}L_3. \tag{4.47}$$

From (4.44 – 4.46) we must have that

$$L_1 = \frac{1}{h_1}\left(A_1 + \frac{p_1 L_2 A_1}{q_2(p_1-q_2)} + \frac{p_1 L_3 A_1}{q_3(p_1-q_3)}\right), \tag{4.48}$$

$$L_2 = \frac{1}{h_2}\left(A_2 + \frac{p_2 L_1 A_2}{q_1(p_2-q_1)} + \frac{p_2 L_3 A_2}{q_3(p_2-q_3)}\right), \tag{4.49}$$

$$L_3 = \frac{1}{h_3}\left(A_3 + \frac{p_3 L_1 A_3}{q_1(p_3-q_1)} + \frac{p_3 L_2 A_3}{q_2(p_3-q_2)}\right), \tag{4.50}$$

where $h_i = e^{-\Lambda_i} - \frac{r_i p_i}{(p_i-q_i)q_i}$, for $i = 1, 2, 3$. Solving for L_1, L_2 and L_3 gives

$$L_1 = \frac{h(2,3)}{h(1,2,3)q_2q_3(p_2-q_3)(p_3-q_2)}\left(A_1 + \frac{p_1 L'_{2,3} A_1}{q_2(p_1-q_2)} + \frac{p_1 L'_{3,2} A_1}{q_3(p_1-q_3)}\right), \tag{4.51}$$

$$L_2 = \frac{h(1,3)}{h(1,2,3)q_1q_3(p_1-q_3)(p_3-q_1)}\left(A_2 + \frac{p_2 L'_{1,3} A_2}{q_1(p_2-q_1)} + \frac{p_2 L'_{3,1} A_2}{q_3(p_2-q_3)}\right), \tag{4.52}$$

$$L_3 = \frac{h(1,2)}{h(1,2,3)q_1q_2(p_1-q_2)(p_2-q_1)}\left(A_3 + \frac{p_3 L'_{1,2} A_3}{q_1(p_3-q_1)} + \frac{p_3 L'_{2,1} A_3}{q_2(p_3-q_2)}\right), \tag{4.53}$$

in which

$$\begin{aligned}
h(i,j) &= (p_i-q_j)(p_j-q_i)q_iq_jh_ih_j - p_ip_jr_ir_j\alpha_{i,j},\\
h(1,2,3) &= -\frac{p_2p_3r_2r_3\alpha_{2,3}h_1}{q_2q_3(p_2-q_3)(p_3-q_2)} - \frac{p_1p_3r_1r_3\alpha_{1,3}h_2}{q_1q_3(p_1-q_3)(p_3-q_1)} - \frac{p_1p_2r_1r_2\alpha_{1,2}h_3}{q_1q_2(p_1-q_2)(p_2-q_1)}\\
&\quad - \frac{p_1p_2p_3r_1r_2r_3}{q_1q_2q_3}\left(\frac{\alpha_{1,2,3}}{(p_2-q_1)(p_1-q_3)(p_3-q_2)} + \frac{\alpha_{1,3,2}}{(p_1-q_2)(p_2-q_3)(p_3-q_1)}\right)\\
&\quad + h_1h_2h_3,\\
L'_{i,j} &= \frac{q_i(p_j-q_i)}{h(i,j)}\left(h_jq_j(p_i-q_j)I + p_iA_j\right)A_i,
\end{aligned}$$

for $i, j \in \{1, 2, 3\}$ and $i \neq j$.

Substituting (4.51 - 4.53) into (4.47) gives

$$\begin{aligned}
f_{[4]} = I + \frac{1}{h(1,2,3)q_1q_2q_3}&\left(b_{2,3}A_1 + b_{1,3}A_2 + b_{1,2}A_3 + b_{1,3,2}A_1A_2 + b_{1,2,3}A_1A_3\right.\\
&\left.+ b_{2,3,1}A_2A_1 + b_{2,1,3}A_2A_3 + b_{3,2,1}A_3A_1 + b_{3,1,2}A_3A_2\right),
\end{aligned} \tag{4.54}$$

in which

$$\begin{aligned}
b_{i,j} &= \frac{h(i,j)}{(p_i-q_j)(p_j-q_i)},\\
b_{i,j,k} &= p_k\left(\frac{p_jr_j\alpha_{i,j,k}}{(p_k-q_j)(p_j-q_i)\alpha_{i,k}} + \frac{h_jq_j}{p_k-q_i}\right).
\end{aligned}$$

Using the fact that $Tr(A_j) = r_j$ and $\det(A_j) = 0$, for $j = 1, 2, 3$, expanding $\det(\Omega)$ gives

$$\begin{aligned}
\det(\Omega) = 1 &- \frac{p_1}{q_1}\kappa_1e^{\Lambda_1} - \frac{p_2}{q_2}\kappa_2e^{\Lambda_2} - \frac{p_3}{q_3}\kappa_3e^{\Lambda_3} + \frac{p_1p_2}{q_1q_2}\kappa_1\kappa_2\beta_{1,2}e^{\Lambda_1+\Lambda_2} + \frac{p_2p_3}{q_2q_3}\kappa_2\kappa_3\beta_{2,3}e^{\Lambda_2+\Lambda_3}\\
&+ \frac{p_1p_3}{q_1q_3}\kappa_1\kappa_3\beta_{1,3}e^{\Lambda_1+\Lambda_3} - \frac{p_1p_2p_3}{q_1q_2q_3}\kappa_1\kappa_2\kappa_3\beta_{1,2,3}e^{\Lambda_1+\Lambda_2+\Lambda_3}.
\end{aligned} \tag{4.55}$$

If $r > 0$ and $q_3 > p_3 > q_2 > p_2 > q_1 > p_1 > 0$ or $0 > q_3 > p_3 > q_2 > p_2 > q_1 > p_1$, or alternatively, if $r < 0$ and $p_3 > q_3 > p_2 > q_2 > p_1 > q_1 > 0$ or $0 > p_3 > q_3 > p_2 > q_2 > p_1 > q_1$ then, $\beta_{i,j} > 0$ and $\kappa_i > 0$. From (4.55), $\det(\Omega)$ will be positive-definite if

$$\frac{\alpha_{1,2,3}}{(p_2 - q_1)(p_1 - q_3)(p_3 - q_2)} + \frac{\alpha_{1,3,2}}{(p_1 - q_2)(p_2 - q_3)(p_3 - q_1)} < 0.$$

The asymptotic forms of each soliton can be determined by following the methods in Chapter 3. It can again be assumed, without loss of generality, that $0 > p_3 > q_3 > p_2 > q_2 > p_1 > q_1$. For soliton 1, as $t \to -\infty$, $h_i \to -\infty$ for $i = 2, 3$. Then we can see from (4.48 - 4.50) that $L_i \to 0$ for $i = 2, 3$. The resulting solution may be compared with similar expressions in the two-soliton matrix solution, giving

$$f_{[4]} \sim I + \frac{\frac{r_1^-}{q_1} P_1^-}{e^{\gamma_1 - \eta_1} - \frac{p_1 r_1^-}{q_1(p_1 - q_1)}},$$

where $r_1^- = r_1$ and $P_1^- = P_1$. Therefore

$$w \sim \frac{1}{4}(p_1 q_1)^{-\frac{1}{2}}(p_1 - q_1)^2 P_1^- \operatorname{sech}\left(\frac{\Lambda_1 + \varphi_1^-}{2}\right) \operatorname{sech}\left(\frac{\Lambda_1 + \chi_1^-}{2}\right),$$

where

$$\varphi_1^- = \log\left(-\frac{p_1 r_1^-}{q_1(p_1 - q_1)}\right), \quad \chi_1^- = \log\left(-\frac{r_1^-}{p_1 - q_1}\right).$$

The phase-constant of this soliton is $\xi_1^- = -\log\left(\frac{-(p_1 q_1^{-1})^{\frac{1}{2}} r_1^-}{p_1 - q_1}\right)$.

As $t \to +\infty$, $h_i \to -\frac{p_i r_i}{q_i(p_i - q_i)}$, for $i = 2, 3$. Using the fact that w and W are invariant under the transformation $f_{[4]} \to f_{[4]} C$, where C is a constant matrix, we have

$$f_{[4]} \sim I + \frac{\frac{r_1^+}{q_1} P_1^+}{e^{\gamma_1 - \eta_1} - \frac{p_1 r_1^+}{q_1(p_1 - q_1)}},$$

where

$$\begin{aligned}
r_1^+ &= \frac{r_1(\mu_1^+, \nu_1^+)}{(\mu_1, \nu_1)}, \quad P_1^+ = \frac{\mu_1^+ \otimes \nu_1^+}{(\mu_1^+, \nu_1^+)}, \\
\mu_1^+ &= \mu_1 + \frac{p_1(p_2 - q_2)(p_3 - q_3)}{p_2(\mu_2, \nu_2)\beta_{2,3}} \left(\frac{(\mu_1, \nu_3)(\mu_3, \nu_2)}{(\mu_3, \nu_3)(p_1 - q_3)(p_3 - q_2)} - \frac{(\mu_1, \nu_2)}{(p_1 - q_2)(p_3 - q_3)}\right) \mu_2 \\
&\quad + \frac{p_1(p_2 - q_3)(p_3 - q_2)}{p_3(\mu_3, \nu_3)\beta_{2,3}} \left(\frac{(\mu_1, \nu_2)(\mu_2, \nu_3)}{(\mu_2, \nu_2)(p_1 - q_2)(p_2 - q_3)} - \frac{(\mu_1, \nu_3)}{(p_1 - q_3)(p_2 - q_2)}\right) \mu_3, \\
\nu_1^+ &= \nu_1 + \frac{q_1(p_2 - q_2)(p_3 - q_3)}{q_2(\mu_2, \nu_2)\beta_{2,3}} \left(\frac{(\mu_3, \nu_1)(\mu_2, \nu_3)}{(\mu_3, \nu_3)(p_2 - q_3)(p_3 - q_1)} - \frac{(\mu_2, \nu_1)}{(p_2 - q_1)(p_3 - q_3)}\right) \nu_2 \\
&\quad + \frac{q_1(p_2 - q_3)(p_3 - q_2)}{q_3(\mu_3, \nu_3)\beta_{2,3}} \left(\frac{(\mu_2, \nu_1)(\mu_3, \nu_2)}{(\mu_2, \nu_2)(p_3 - q_2)(p_2 - q_1)} - \frac{(\mu_3, \nu_1)}{(p_3 - q_1)(p_2 - q_2)}\right) \nu_3.
\end{aligned}$$

So we have

$$w \sim \frac{1}{4}(p_1q_1)^{-\frac{1}{2}}(p_1-q_1)^2P_1^+\operatorname{sech}\left(\frac{\Lambda_1+\varphi_1^+}{2}\right)\operatorname{sech}\left(\frac{\Lambda_1+\chi_1^+}{2}\right),$$

where

$$\varphi_1^+ = \log\left(-\frac{p_1r_1^+}{q_1(p_1-q_1)}\right), \quad \chi_1^+ = \log\left(-\frac{r_1^+}{p_1-q_1}\right).$$

The phase-constant of this soliton is $\xi_1^+ = -\log\left(\frac{-(p_1q_1^{-1})^{\frac{1}{2}}r_1^+}{p_1-q_1}\right)$.

Fixing Λ_2 brings soliton 2 to rest. As $t \to -\infty$, $h_1 \to \frac{r_1}{p_1-q_1}$ and $h_3 \to +\infty$. This gives

$$\begin{aligned} f_{[4]} &\sim I + \frac{L'_{1,2}}{q_1} + \frac{L'_{2,1}}{q_2} \\ &\sim I + \frac{\frac{r_2^-}{q_2}P_2^-}{e^{-\Lambda_2} - \frac{p_2r_2^-}{q_2(p_2-q_2)}}, \end{aligned}$$

where

$$r_2^- = \frac{r_2(\mu_2^-,\nu_2^-)}{(\mu_2,\nu_2)} = r_2\beta_{1,2}, \quad P_2^- = \frac{\mu_2^-\otimes\nu_2^-}{(\mu_2^-,\nu_2^-)},$$
$$\mu_2^- = \mu_2 - \frac{p_2(p_1-q_1)(\mu_2,\nu_1)}{p_1(p_2-q_1)(\mu_1,\nu_1)}\mu_1, \quad \nu_2^- = \nu_2 - \frac{q_2(p_1-q_1)(\mu_1,\nu_2)}{q_1(p_1-q_2)(\mu_1,\nu_1)}\nu_1.$$

As $t \to +\infty$, $h_1 \to +\infty$ and $h_3 \to \frac{r_3}{(p_3-q_3)}$. This gives

$$\begin{aligned} f_{[4]} &\sim I + \frac{L'_{2,3}}{q_1} + \frac{L'_{3,2}}{q_2} \\ &\sim I + \frac{\frac{r_2^-}{q_2}P_2^-}{e^{-\Lambda_2} - \frac{p_2r_2^-}{q_2(p_2-q_2)}}, \end{aligned}$$

where

$$r_2^+ = \frac{r_2(\mu_2^+,\nu_2^+)}{(\mu_2,\nu_2)} = r_2\beta_{2,3}, \quad P_2^+ = \frac{\mu_2^+\otimes\nu_2^+}{(\mu_2^+,\nu_2^+)},$$
$$\mu_2^+ = \mu_2 - \frac{p_2(p_3-q_3)(\mu_2,\nu_3)}{p_3(p_2-q_3)(\mu_3,\nu_3)}\mu_3, \quad \nu_2^+ = \nu_2 - \frac{q_2(p_3-q_3)(\mu_3,\nu_2)}{q_3(p_3-q_2)(\mu_3,\nu_3)}\nu_3.$$

So the asymptotic forms for soliton 2 are

$$w \sim \frac{1}{4}(p_2q_2)^{-\frac{1}{2}}(p_2-q_2)^2P_2^-\operatorname{sech}\left(\frac{\Lambda_2+\varphi_2^-}{2}\right)\operatorname{sech}\left(\frac{\Lambda_2+\chi_2^-}{2}\right) \quad \text{as} \quad t\to-\infty,$$
$$w \sim \frac{1}{4}(p_2q_2)^{-\frac{1}{2}}(p_2-q_2)^2P_2^+\operatorname{sech}\left(\frac{\Lambda_2+\varphi_2^+}{2}\right)\operatorname{sech}\left(\frac{\Lambda_2+\chi_2^+}{2}\right) \quad \text{as} \quad t\to+\infty,$$

where $\varphi_2^- = \log\left(\frac{-p_2r_2^-}{q_2(p_2-q_2)}\right)$, $\varphi_2^+ = \log\left(\frac{-p_2r_2^+}{q_2(p_2-q_2)}\right)$, $\chi_2^- = \log\left(\frac{-r_2^-}{p_2-q_2}\right)$ and $\chi_2^+ = \log\left(\frac{-r_2^+}{p_2-q_2}\right)$.

We also have the soliton phase-constants

$$\xi_2^- = -\log\left(\frac{-(p_2q_2^{-1})^{\frac{1}{2}}r_2^-}{p_2-q_2}\right), \quad \xi_2^+ = -\log\left(\frac{-(p_2q_2^{-1})^{\frac{1}{2}}r_2^+}{p_2-q_2}\right).$$

With Λ_3 fixed, soliton 3 is a rest. As $t \to -\infty$, $h_i \to \frac{-p_i r_i}{q_i(p_i-q_i)}$ for $i = 1, 2$. This gives

$$f_{[4]} \sim I + \frac{\frac{r_3^-}{q_3} P_3^-}{e^{-\Lambda_3} - \frac{p_3 r_3^-}{q_3(p_3-q_3)}},$$

where

$$\begin{aligned}
r_3^- &= \frac{r_3(\mu_3^-, \nu_3^-)}{(\mu_3, \nu_3)} = \frac{r_3 \beta_{1,2,3}}{\beta_{1,2}}, \quad P_3^- = \frac{\mu_3^- \otimes \nu_3^-}{(\mu_3^-, \nu_3^-)}, \\
\mu_3^- &= \mu_3 + \frac{p_3(p_2-q_2)(p_1-q_1)}{p_2(\mu_2,\nu_2)\beta_{1,2}} \left(\frac{(\mu_3,\nu_1)(\mu_1,\nu_2)}{(\mu_1,\nu_1)(p_3-q_1)(p_1-q_2)} - \frac{(\mu_3,\nu_2)}{(p_3-q_2)(p_1-q_1)} \right) \mu_2 \\
&\quad + \frac{p_3(p_2-q_2)(p_1-q_1)}{p_1(\mu_1,\nu_1)\beta_{1,2}} \left(\frac{(\mu_3,\nu_2)(\mu_2,\nu_1)}{(\mu_2,\nu_2)(p_3-q_2)(p_2-q_1)} - \frac{(\mu_3,\nu_1)}{(p_3-q_1)(p_2-q_2)} \right) \mu_1, \\
\nu_3^- &= \nu_3 + \frac{q_3(p_2-q_2)(p_1-q_1)}{q_2(\mu_2,\nu_2)\beta_{1,2}} \left(\frac{(\mu_1,\nu_3)(\mu_2,\nu_1)}{(\mu_1,\nu_1)(p_2-q_1)(p_1-q_3)} - \frac{(\mu_2,\nu_3)}{(p_2-q_3)(p_1-q_1)} \right) \nu_2 \\
&\quad + \frac{q_3(p_2-q_2)(p_1-q_1)}{q_1(\mu_1,\nu_1)\beta_{1,2}} \left(\frac{(\mu_2,\nu_3)(\mu_1,\nu_2)}{(\mu_2,\nu_2)(p_1-q_2)(p_2-q_3)} - \frac{(\mu_1,\nu_3)}{(p_1-q_3)(p_2-q_2)} \right) \nu_1.
\end{aligned}$$

As $t \to +\infty$, $h_i \to +\infty$, for $i = 1, 2$. This gives

$$f_{[4]} \sim I + \frac{\frac{r_3^+}{q_3} P_3^+}{e^{-\Lambda_3} - \frac{p_3 r_3^+}{q_3(p_3-q_3)}},$$

where $r_3^+ = r_3$, $P_3^+ = \frac{\mu_3^+ \otimes \nu_3^+}{(\mu_3^+, \nu_3^+)}$, $\mu_3^+ = \mu_3$ and $\nu_3^+ = \nu_3$.

So the asymptotic forms for soliton 3 are

$$\begin{aligned}
w &\sim \frac{1}{4}(p_3 q_3)^{-\frac{1}{2}} (p_3 - q_3)^2 P_3^- \operatorname{sech}\left(\frac{\Lambda_3 + \varphi_3^-}{2}\right) \operatorname{sech}\left(\frac{\Lambda_3 + \chi_3^-}{2}\right) \quad \text{as} \quad t \to -\infty, \\
w &\sim \frac{1}{4}(p_3 q_3)^{-\frac{1}{2}} (p_3 - q_3)^2 P_3^+ \operatorname{sech}\left(\frac{\Lambda_3 + \varphi_3^+}{2}\right) \operatorname{sech}\left(\frac{\Lambda_3 + \chi_3^+}{2}\right) \quad \text{as} \quad t \to +\infty,
\end{aligned}$$

where $\varphi_3^- = \log\left(\frac{-p_3 r_3^-}{q_3(p_3-q_3)}\right)$, $\varphi_3^+ = \log\left(\frac{-p_3 r_3^+}{q_3(p_3-q_3)}\right)$, $\chi_3^- = \log\left(\frac{-r_3^-}{p_3-q_3}\right)$ and $\chi_3^+ = \log\left(\frac{-r_3^+}{p_3-q_3}\right)$. In addition, we have the soliton phase-constants

$$\xi_3^- = -\log\left(\frac{-(p_3 q_3^{-1})^{\frac{1}{2}} r_3^-}{p_3 - q_3}\right), \quad \xi_3^+ = -\log\left(\frac{-(p_3 q_3^{-1})^{\frac{1}{2}} r_3^+}{p_3 - q_3}\right).$$

The soliton phase shifts $\Delta_j = \xi_j^+ - \xi_j^-$ are

$$\begin{aligned}
\Delta_1 &= \log\left(\frac{(\mu_1^+, \nu_1^+)}{(\mu_1^-, \nu_1^-)}\right) = \log\left(\frac{\beta_{1,2,3}}{\beta_{2,3}}\right), \\
\Delta_2 &= \log\left(\frac{(\mu_2^+, \nu_2^+)}{(\mu_2^-, \nu_2^-)}\right) = \log\left(\frac{\beta_{2,3}}{\beta_{1,2}}\right), \\
\Delta_3 &= \log\left(\frac{(\mu_3^+, \nu_3^+)}{(\mu_3^-, \nu_3^-)}\right) = \log\left(\frac{\beta_{1,2}}{\beta_{1,2,3}}\right).
\end{aligned}$$

4.7 Plots of the matrix solutions

In this section, we demonstrate the interaction properties of the two- and three-soliton matrix solution of ncmKP with various plots.

Figure 4.1 shows a plot of the generic two-soliton matrix solution $w = (w_{ij})$, $i, j = 1, 2$, where

$$\begin{aligned}
r_1 = 1 \quad &\longrightarrow \quad \hat{r}_1 = 0.846, \\
\hat{r}_2 = -0.846 \quad &\longrightarrow \quad r_2 = -1, \\
\mu_1 = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad &\longrightarrow \quad \hat{\mu}_1 = \begin{pmatrix} 1.615 & 0.308 \end{pmatrix}, \\
\mu_2 = \begin{pmatrix} -1 & -0.5 \end{pmatrix} \quad &\longrightarrow \quad \hat{\mu}_2 = \begin{pmatrix} -1.25 & -0.5 \end{pmatrix}, \\
\hat{\nu}_1 = \begin{pmatrix} 0.692 & -0.885 \end{pmatrix} \quad &\longrightarrow \quad \nu_1 = \begin{pmatrix} 1 & -1 \end{pmatrix}, \\
\hat{\nu}_2 = \begin{pmatrix} -0.5 & -0.125 \end{pmatrix} \quad &\longrightarrow \quad \nu_2 = \begin{pmatrix} -1 & 0.375 \end{pmatrix}, \\
P_1 = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad &\longrightarrow \quad \hat{P}_1 = \begin{pmatrix} 1.321 & -1.689 \\ 0.252 & -0.321 \end{pmatrix}, \\
\hat{P}_2 = \begin{pmatrix} 0.909 & 0.227 \\ 0.364 & 0.091 \end{pmatrix} \quad &\longrightarrow \quad P_2 = \begin{pmatrix} 1.231 & -0.462 \\ 0.615 & -0.231 \end{pmatrix},
\end{aligned}$$

as t changes from $-\infty$ to $+\infty$. This plot shows both a change in matrix amplitude and a phase-shift upon interaction.

Figure 4.2 shows a plot of the generic three-soliton matrix solution $w = (w_{ij})$, $i, j = 1, 2$, where

$$
\begin{aligned}
r_1^- = 1 &\quad\longrightarrow\quad r_1^+ = 0.711,\\
r_2^- = 2 &\quad\longrightarrow\quad r_2^+ = 1.135,\\
r_3^- = 1.21 &\quad\longrightarrow\quad r_3^+ = 1,\\
\mu_1^- = \begin{pmatrix}1 & 0.333\end{pmatrix} &\quad\longrightarrow\quad \mu_1^+ = \begin{pmatrix}0.857 & 0.205\end{pmatrix},\\
\nu_1^- = \begin{pmatrix}1 & 2\end{pmatrix} &\quad\longrightarrow\quad \nu_1^+ = \begin{pmatrix}2.549 & 6.656\end{pmatrix},\\
\mu_2^- = \begin{pmatrix}26 & 11\end{pmatrix} &\quad\longrightarrow\quad \mu_1^+ = \begin{pmatrix}0.427 & 3.315\end{pmatrix},\\
\nu_2^- = \begin{pmatrix}0.125 & 2.25\end{pmatrix} &\quad\longrightarrow\quad \nu_2^+ = \begin{pmatrix}1.455 & 4.606\end{pmatrix},\\
\mu_3^- = \begin{pmatrix}2.362 & 0.328\end{pmatrix} &\quad\longrightarrow\quad \mu_3^+ = \begin{pmatrix}1 & -0.2\end{pmatrix},\\
\nu_3^- = \begin{pmatrix}1.377 & -1.806\end{pmatrix} &\quad\longrightarrow\quad \nu_3^+ = \begin{pmatrix}3 & 4\end{pmatrix},\\
P_1^- = \begin{pmatrix}0.6 & 1.2\\ 0.2 & 0.4\end{pmatrix} &\quad\longrightarrow\quad P_1^+ = \begin{pmatrix}0.615 & 1.606\\ 0.147 & 0.385\end{pmatrix},\\
P_2^- = \begin{pmatrix}0.116 & 2.089\\ 0.049 & 0.884\end{pmatrix} &\quad\longrightarrow\quad P_2^+ = \begin{pmatrix}0.039 & 0.124\\ 0.303 & 0.961\end{pmatrix},\\
P_3^- = \begin{pmatrix}1.222 & -1.604\\ 0.17 & -0.222\end{pmatrix} &\quad\longrightarrow\quad P_3^+ = \begin{pmatrix}1.364 & 1.818\\ -0.273 & -0.364\end{pmatrix},
\end{aligned}
$$

as t changes from $-\infty$ to $+\infty$. This plot shows both a change in matrix amplitude and a phase-shift upon interaction.

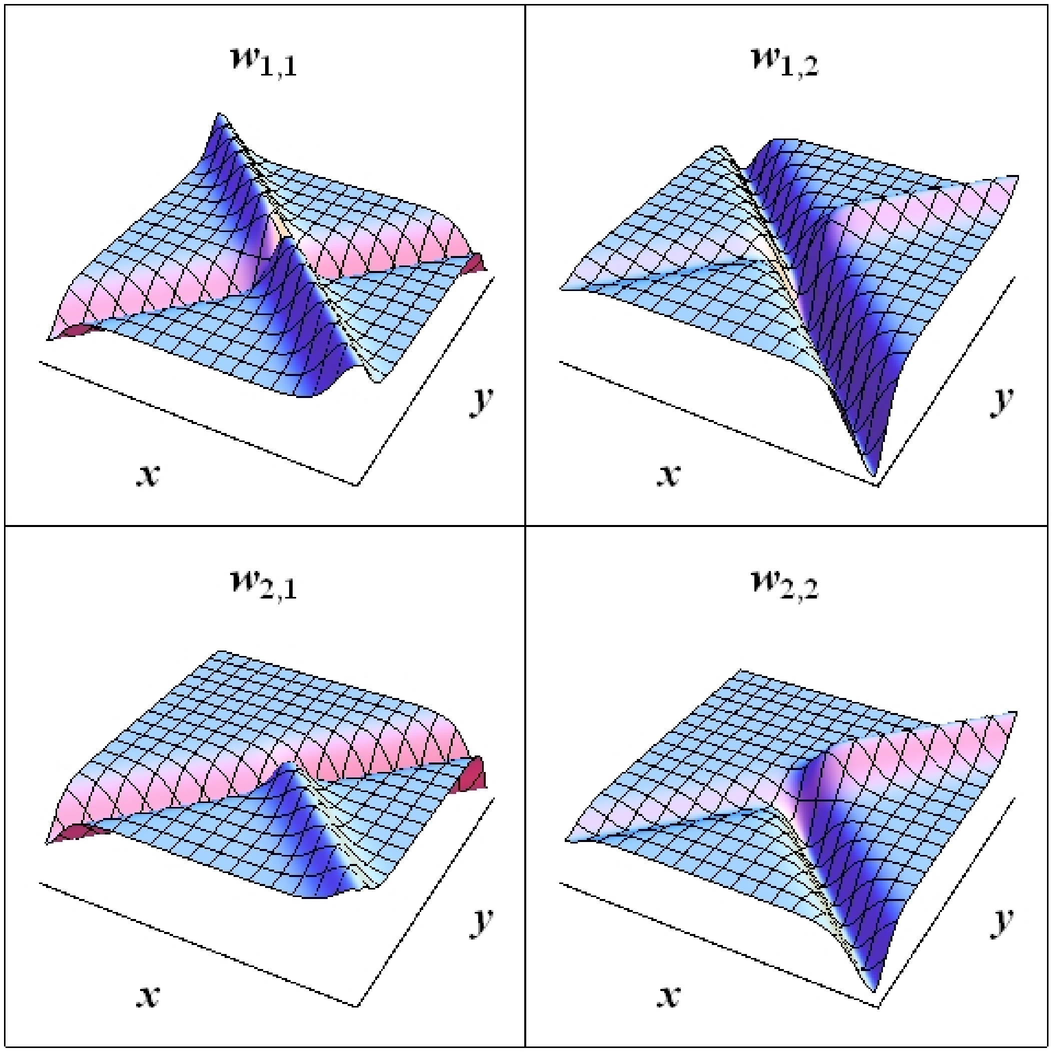


Figure 4.1: Plot of matrix mKP two-soliton solution at $t = 0$ with parameters $p_1 = \frac{1}{4}$, $p_2 = \frac{-1}{4}$, $q_1 = \frac{3}{4}$, $q_2 = \frac{-3}{4}$, $r_1 = 1$ and $r_2 = -1$.

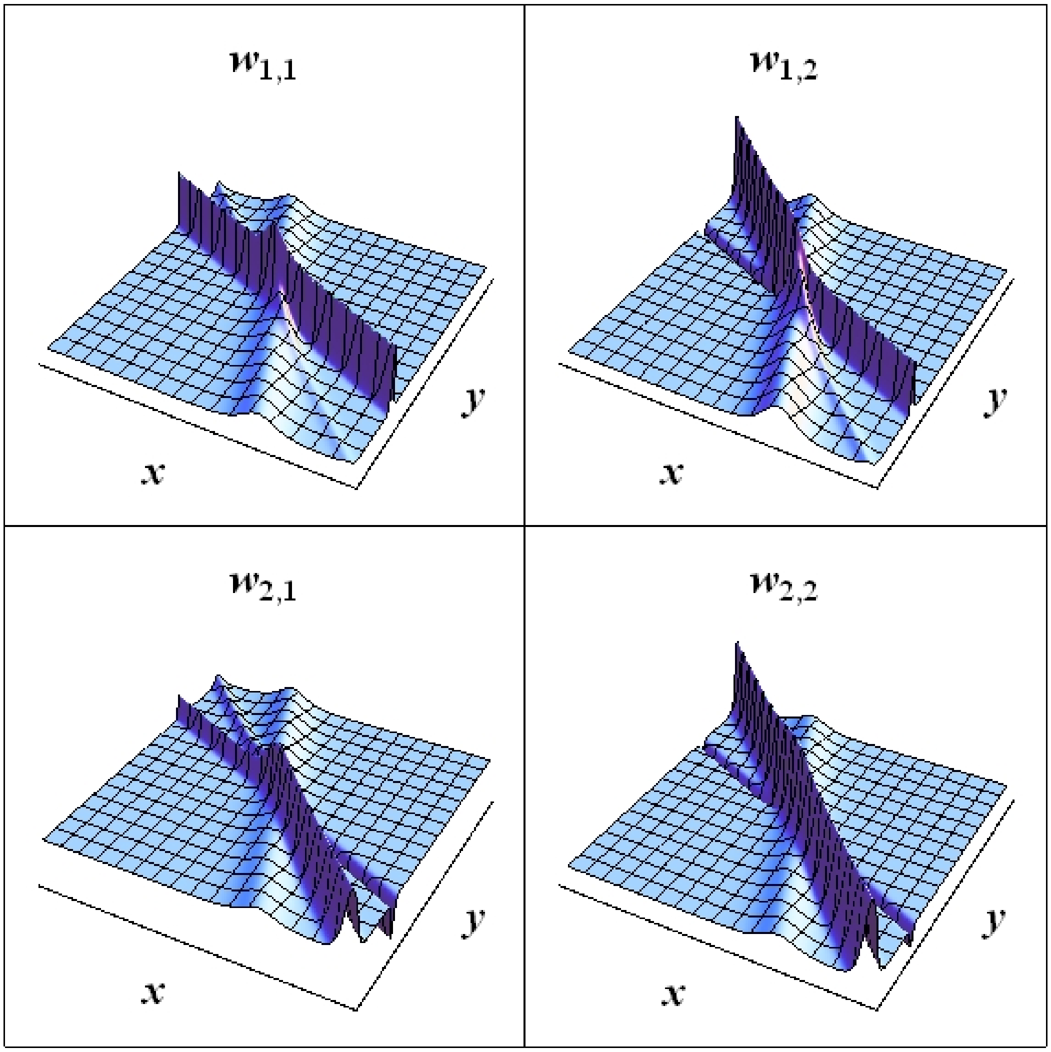


Figure 4.2: Plot of matrix mKP three-soliton solution at $t = 0$ with parameters $p_1 = 1$, $p_2 = \frac{5}{2}$, $p_3 = 6$, $q_1 = 2$, $q_2 = 5$, $q_3 = 9$ and $r_1 = r_2 = r_3 = 1$.

4.8 The noncommutative Miura transformation

Using the same method as in the commutative case, we can construct a noncommutative Miura transformation between the ncKP and ncmKP equations. Upon calculation of the transformed pseudodifferential operator $\tilde{\mathcal{L}}_{\text{KP}} = \theta^{-1}\mathcal{L}_{\text{KP}}\theta$, we have

$$\mathcal{L}_{\text{KP}} = \partial + \theta^{-1}\theta_x + \frac{1}{2}\theta^{-1}u\theta\partial^{-1} + (\theta^{-1}u_2\theta - \frac{1}{2}\theta^{-1}u\theta_x)\partial^{-2} + \ldots.$$

Comparing this with the operator

$$\mathcal{L}_{mKP} = \partial_x + w + w_1\partial_x^{-1} + w_2\partial_x^{-2} + w_3\partial_x^{-3} + \ldots. \qquad (4.56)$$

and equating coefficients gives

$$w = \theta^{-1}\theta_x, \tag{4.57}$$

$$w_1 = \frac{1}{2}\theta^{-1}u\theta, \tag{4.58}$$

$$w_2 = \theta^{-1}u_2\theta - \frac{1}{2}\theta^{-1}u\theta_x. \tag{4.59}$$

$$\dots$$

These coefficients will satisfy (4.3), (4.4) and (4.5). Isolating u in (4.58) gives

$$u = 2\theta w_1\theta^{-1}. \tag{4.60}$$

Given that $w = \theta^{-1}\theta_x$ and $w = -f_x f^{-1}$, we can conclude that $f = \theta^{-1}$. Therefore, we can rewrite (4.60) as

$$u = 2f^{-1}w_1 f, \tag{4.61}$$

which was also derived in [8]. Upon substitution of (4.10) in (4.61) we obtain the noncommutative Miura transformation between the ncKP and ncmKP equations:

$$u = -f^{-1}w_x f - f^{-1}w^2 f - f^{-1}f_y \tag{4.62}$$

$$= f^{-1}f_{xx} - 2f_x f^{-1}f_x f^{-1} - f^{-1}f_y. \tag{4.63}$$

Direct substitution of (4.63) into ncKP (3.19) leads to the left-hand side of (3.19) being identically zero. Therefore, (4.63) defines a new solution of ncKP (3.19). Furthermore, (4.62) is consistent with the nc Miura transformation (4.41).

Chapter 5

Dromions of the matrix equations

The aim of this chapter is to find exponentially localized structures, obtained from the matrix versions of the nc KP and nc mKP equations. The commutative KP equation [3] and the commutative Davey-Stewartson (DS) equations [49] are known to have localized lump solutions, which have algebraic decay at infinity. However, when each lump collides, the interaction is completely trivial. Equations such as the DSI equations [2, 18] and the Nizhnik-Veselov-Novikov (NVN) [42] equations are known to have localized solutions which have exponential decay at infinity. In this case, the solutions have interesting interaction properties such as changes in amplitude and trajectory. For the NVN equations, it was shown, in [50], that an exponentially localized solution may be thought of as a two-soliton solution made out of two intersecting "ghost" solitons. In [18], the authors show, by means of direct methods, how to derive the characteristics of exponentially localized solutions of the DSI equations. This has been extended to the noncommutative setting in [17]. In both the DSI and NVN equations, the underlying solitons which interact to create the localized solution are perpendicular. These localized solutions are called *dromions* which comes from the Greek word dromos meaning track. This term was coined in [11] because the dromions are located at soliton interactions which can be thought of as forming tracks.

5.1 Matrix KP single dromion

Since the one-soliton matrix solutions of ncKP and ncmKP have projection matrices governing their amplitude, their determinant will equal zero. Therefore, the natural thing to investigate when looking for dromions of these solutions, is the determinant of the two-soltion matrix solution. We begin by investigating the two-soliton matrix solution of ncKP.

Recall from Chapter 3 that the two-soliton matrix solution of ncKP is $u = v_x$ where

$$v = 2(K_1 + K_2),$$

in which

$$K_1 = \frac{p_2 - q_1}{h}\left(h_2(p_1 - q_2)I - A_2\right)A_1,$$
$$K_2 = \frac{p_1 - q_2}{h}\left(h_1(p_2 - q_1)I - A_1\right)A_2,$$

and

$$h = h_1 h_2 (p_1 - q_2)(p_2 - q_1) - \alpha r_1 r_2,$$
$$h_i = e^{-\Lambda_i} + \frac{r_i}{p_i - q_i} \quad \text{and} \quad \alpha = \frac{(\mu_1, \nu_2)(\mu_2, \nu_1)}{(\mu_1, \nu_1)(\mu_2, \nu_2)} = Tr(P_1 P_2).$$

Since the determinant of a projection matrix is zero and the trace of a projection matrix is equal to its rank, $\det(u)$ can easily be expanded along any row or column. In doing so, the result greatly simplifies and we obtain

$$\begin{aligned}\det(u) &= 4r_1 r_2 (p_1 - q_2)(p_2 - q_1)(1 - \alpha) \\ &\quad \times \left((p_1 - q_2)(p_2 - q_1)\left(\frac{h_1}{h}\right)_x \left(\frac{h_2}{h}\right)_x - r_1 r_2 \alpha \left(\frac{1}{h}\right)_x^2\right) \\ &= 4r_1 r_2 (p_1 - q_2)^2 (p_2 - q_1)^2 (1 - \alpha) h_{1,x} h_{2,x} h^{-2}. \end{aligned} \tag{5.1}$$

The single dromion (5.1) can be rewritten as

$$\det(u) = \frac{4r_1 r_2 (p_1 - q_1)(p_2 - q_2)(1 - \alpha)e^{-(\Lambda_1 + \Lambda_2)}}{\left(e^{-(\Lambda_1 + \Lambda_2)} + \kappa_1 e^{-\Lambda_2} + \kappa_2 e^{-\Lambda_1} + \kappa\right)^2} \tag{5.2}$$

$$= \frac{4r_1 r_2 (p_1 - q_1)(p_2 - q_2)(1 - \alpha)}{\left(e^{-\frac{1}{2}(\Lambda_1 + \Lambda_2)} + \kappa_1 e^{\frac{1}{2}(\Lambda_1 - \Lambda_2)} + \kappa_2 e^{\frac{1}{2}(\Lambda_2 - \Lambda_1)} + \kappa e^{\frac{1}{2}(\Lambda_1 + \Lambda_2)}\right)^2}. \tag{5.3}$$

where $\kappa_i = \frac{r_i}{(p_i - q_i)}$, for $i = 1, 2$ and $\kappa = \kappa_1 \kappa_2 \beta$, where $\beta = 1 - \frac{\alpha(p_1 - q_1)(p_2 - q_2)}{(p_1 - q_2)(p_2 - q_1)}$.

Figure 5.2 shows a plot of the dromion (5.3). The method of describing the characteristics of this dromion is in the spirit of that in [18] and [50] with one main difference being that the solitons governing the dromion are not necessarily perpendicular to one another. The characteristics of $\det(u)$ as given by equation (5.3) may be summarised by the following theorem:

Theorem 2. *If* $\det(\Omega)$ *is positive-definite and if* $\alpha \neq 1$*, then* $\det(u)$ *has the following properties:*

1. $\det(u)$ *decays to zero exponentially as* $(x, y) \to \infty$ *in any direction and has a unique maximum or minimum value*

$$\det(u)_{max/min} = \frac{(1-\alpha)(p_1 - q_1)^2(p_2 - q_2)^2}{\left(\sqrt{\beta}+1\right)^2}. \tag{5.4}$$

The dromion will have negative, zero or positive amplitude. The amplitude is

- *negative if* $\alpha > 1$,
- *zero if* $\alpha = 1$,
- *positive if* $\alpha < 1$.

2. *At time* t *this maximum or minimum is located at*

$$(x, y) = \frac{1}{2l_{1,2}} \Big(l_2^{(2)}(\xi_1^- + \xi_1^+) - l_1^{(2)}(\xi_2^- + \xi_2^+) + 8l_{2,3}t\ , \\ l_1^{(1)}(\xi_2^- + \xi_2^+) - l_2^{(1)}(\xi_1^- + \xi_1^+) + 8l_{1,3}t \Big)\,, \tag{5.5}$$

where $l_{i,j} = l_i^{(i)} l_j^{(j)} - l_i^{(j)} l_j^{(i)}$ *and* $l_i^{(j)} = q_i^j - p_i^j$. *This result implies that the dromion is located symmetrically between the solitons in the two-soliton matrix solution as illustrated by Figure 5.1.*

3. *The trajectory of the dromion is the straight line*

$$y = \left(\frac{l_{1,3}}{l_{2,3}}\right) x + \frac{\left(l_1^{(1)} l_{2,3} + l_1^{(2)} l_{1,3}\right)(\xi_2^- + \xi_2^+) - \left(l_2^{(1)} l_{2,3} + l_2^{(2)} l_{1,3}\right)(\xi_1^- + \xi_1^+)}{2l_{1,2}l_{2,3}}. \tag{5.6}$$

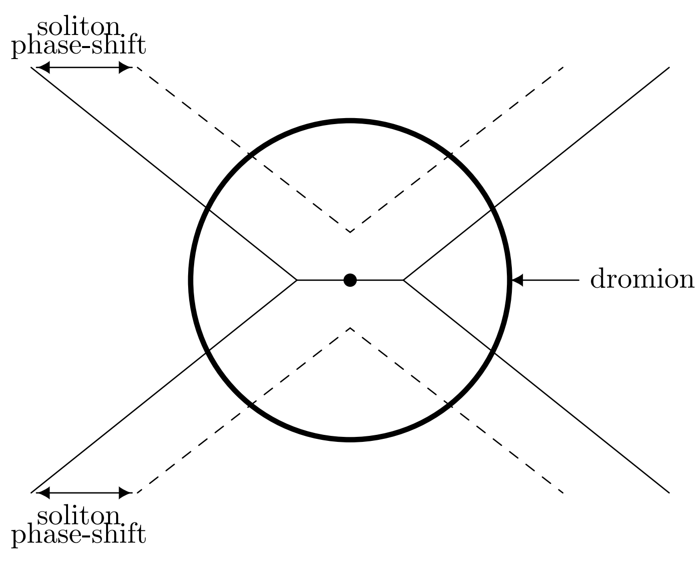


Figure 5.1: Phase-shifts in the solitons and the location of the dromion.

Proof. From (5.3), we see that $\det(u)$ decays to zero exponentially as $(x,y) \to \infty$ in any direction since, along any ray in the (x,y)-plane, at least one of the exponentials in the denominator is unbounded as (x,y) approaches infinity. To see this, let $y = kx$, where $k \in \mathbb{R}$, be a ray in any direction. Substituting this into (5.3) gives

$$e^{-\frac{1}{2}\left(\left(l_1^{(1)}+l_2^{(1)}+\left(l_1^{(2)}+l_2^{(2)}\right)k\right)x-4\left(l_1^{(3)}+l_2^{(3)}\right)t\right)}+\kappa_1 e^{\frac{1}{2}\left(\left(l_1^{(1)}-l_2^{(1)}+\left(l_1^{(2)}-l_2^{(2)}\right)k\right)x-4\left(l_1^{(3)}-l_2^{(3)}\right)t\right)}$$
$$+\kappa_2 e^{\frac{1}{2}\left(\left(l_2^{(1)}-l_1^{(1)}+\left(l_2^{(2)}-l_1^{(2)}\right)k\right)x-4\left(l_2^{(3)}-l_1^{(3)}\right)t\right)}+\kappa e^{\frac{1}{2}\left(\left(l_1^{(1)}+l_2^{(1)}+\left(l_1^{(2)}+l_2^{(2)}\right)k\right)x-4\left(l_1^{(3)}+l_2^{(3)}\right)t\right)}$$

on the denominator. This expression must tend to infinity for any values of k and $l_i^{(j)}$, $i, j = 1, 2$ as $x \to \pm\infty$.

Since $\det(u)$ is exponentially localized, a unique critical point must be either a maximum or a minimum. If we consider the conditions that $\alpha \neq 1$ and $(\det(u))_x$ and $(\det(u))_y$ vanish simultaneously, we get

$$\left(l_1^{(1)}+l_2^{(1)}\right)e^{-\frac{1}{2}(\Lambda_1+\Lambda_2)}+\kappa_1\left(-l_1^{(1)}+l_2^{(1)}\right)e^{\frac{1}{2}(\Lambda_1-\Lambda_2)}$$
$$+\kappa_2\left(l_1^{(1)}-l_2^{(1)}\right)e^{\frac{1}{2}(\Lambda_2-\Lambda_1)}+\kappa\left(-l_1^{(1)}-l_2^{(1)}\right)e^{\frac{1}{2}(\Lambda_1+\Lambda_2)}=0$$

and

$$\left(l_1^{(2)}+l_2^{(2)}\right)e^{-\frac{1}{2}(\Lambda_1+\Lambda_2)}+\kappa_1\left(-l_1^{(2)}+l_2^{(2)}\right)e^{\frac{1}{2}(\Lambda_1-\Lambda_2)}$$
$$+\kappa_2\left(l_1^{(2)}-l_2^{(2)}\right)e^{\frac{1}{2}(\Lambda_2-\Lambda_1)}+\kappa\left(-l_1^{(2)}-l_2^{(2)}\right)e^{\frac{1}{2}(\Lambda_1+\Lambda_2)}=0$$

which imply that

$$e^{-(\Lambda_1+\Lambda_2)}=\kappa \quad \text{and} \quad e^{(-\Lambda_1+\Lambda_2)}=\frac{\kappa_1}{\kappa_2}$$

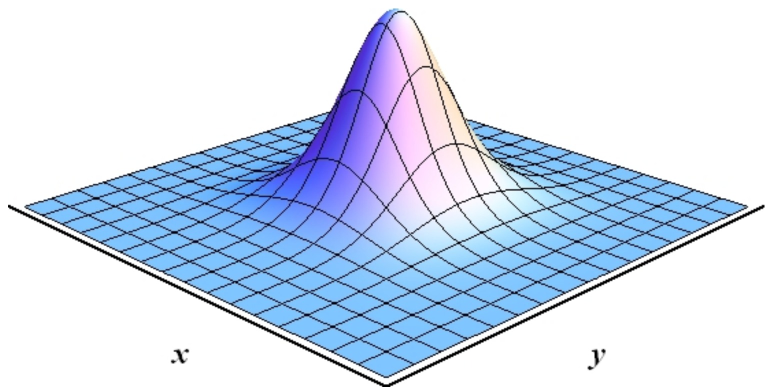
and so

$$e^{-\Lambda_1}=\sqrt{\frac{\kappa_1\kappa}{\kappa_2}}=\kappa_1\sqrt{\beta} \quad \text{and} \quad e^{-\Lambda_2}=\sqrt{\frac{\kappa_2\kappa}{\kappa_1}}=\kappa_2\sqrt{\beta}. \tag{5.7}$$

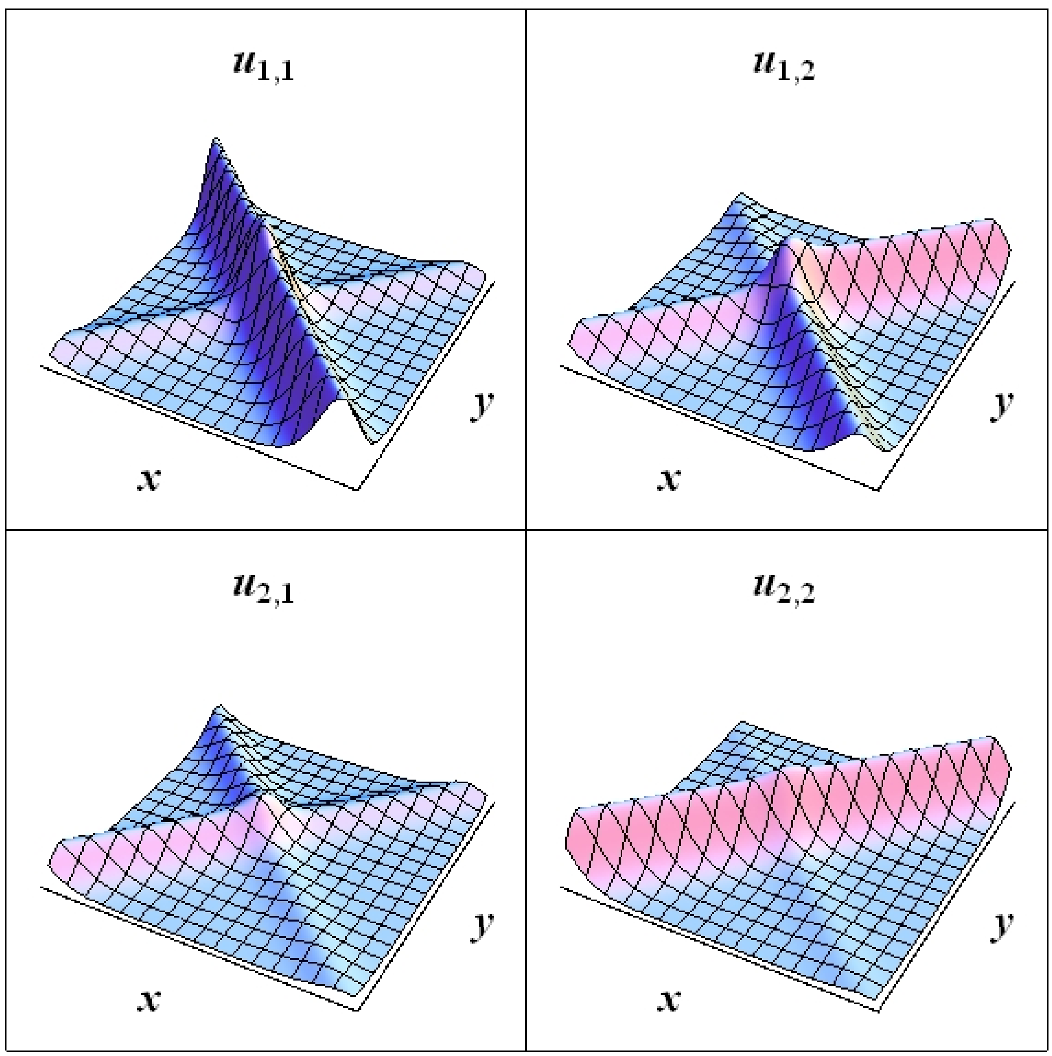
Substituting (5.7) into (5.2) gives the maximum or minimum of $\det(u)$.

Solving (5.7) for x and y gives (5.5), the location of the dromion. Eliminating t in (5.5) gives the trajectory of the dromion.

The dromion as given by (5.3) is still prevalent when there is no phase-shift and when there is both no phase-shift and no change in amplitude. This is different from the DSI and NVN equations where the dromion vanishes when there is no phase-shift.



(a)



(b)

Figure 5.2: (a) Plot of a single dromion at $t = 0$ with parameters $r_1 = 2$, $r_2 = 1$, $p_1 = -\frac{1}{4}$, $q_1 = -\frac{3}{4}$, $p_2 = \frac{3}{4}$, $q_2 = \frac{1}{4}$, $\mu_1^T = (1\ 2)$, $\mu_2^T = \left(\frac{4}{5}\ \frac{1}{4}\right)$, $\nu_1^T = \left(\frac{1}{3}\ \frac{2}{3}\right)$ and $\nu_2^T = \left(2\ \frac{1}{3}\right)$. (b) Plot of the corresponding ncKP two-soliton matrix solution.

5.1.1 A three-dromion example

The determinant of the three-soliton matrix solution gives a three-dromion structure. In this case, the three dromions will always collide as they are effectively sharing the same origin. This is illustrated in Figure 5.3. Adding more solitons to the solution would give a much more complicated dromion scattering scheme in which all dromions may not simultaneously collide. Therefore, we will concentrate on the three-dromion case here and perform a detailed asymptotic analysis.

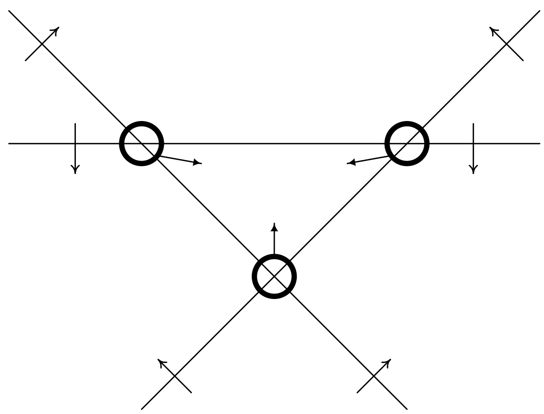


Figure 5.3: Schematic form of the dromion scattering.

Upon expansion of $\det(u) = \det(v_{[4],x})$, where u is the three-soliton matrix solution of ncKP, by using the fact that the trace of a projection matrix is equal to its rank and its determinant is equal to zero we get

$$\det(u) = \frac{h_{1,x}h_{2,x}m_{1,2,3} + h_{1,x}h_{3,x}m_{1,3,2} + h_{2,x}h_{3,x}m_{2,3,1}}{h^2(1,2,3)}, \tag{5.8}$$

where we have the quadratic equations in h_k:

$$\begin{aligned} m_{i,j,k} = r_i r_j \Bigg(& h_k^2(1-\alpha_{i,j}) + h_k r_k \bigg(\frac{\alpha_{i,j,k}(\ell_{i,k}+\ell_{k,j})}{\ell_{i,k}\ell_{k,j}} - \frac{\alpha_{i,k,j}(\ell_{j,k}+\ell_{k,i})}{\ell_{j,k}\ell_{k,i}} - \frac{\alpha_{j,k}(\ell_{j,k}+\ell_{k,j})}{\ell_{j,k}\ell_{k,j}} \\ & - \frac{\alpha_{i,k}(\ell_{i,k}+\ell_{k,i})}{\ell_{i,k}\ell_{k,i}} \bigg) - r_k^2 \bigg(\frac{\alpha_{i,j,k}}{\ell_{i,j}\ell_{k,j}} + \frac{\alpha_{i,k,j}}{\ell_{j,k}\ell_{k,i}} + \frac{\alpha_{i,k}\alpha_{j,k}(\ell_{j,k}-\ell_{k,i})(\ell_{i,k}-\ell_{k,j})}{\ell_{i,k}\ell_{k,i}\ell_{j,k}\ell_{k,j}} \\ & + \frac{\alpha_{j,k}}{\ell_{j,k}\ell_{k,j}} + \frac{\alpha_{i,k}}{\ell_{i,k}\ell_{k,i}} \bigg) \Bigg), \end{aligned}$$

in which $\ell_{i,j} = p_i - q_j$ for $i,j,k \in \{1,2,3\}$ and $i \neq j \neq k$. To investigate the behaviour of each dromion as $t \to \pm\infty$, we fix attention on the dromion arising from the interaction of the ith and jth solitons, which we term $d(i,j)$. In addition, the corresponding two-soliton interaction matrix potential will be termed $v_{i,j}$. We consider $\det(u)$ as given by (5.8) in a frame moving with the (i,j)th dromion by rewriting it in terms of

$$x = \hat{x} + 4\left(\frac{(q_j^3 - p_j^3)(q_i^2 - p_i^2) - (q_i^3 - p_i^3)(q_j^2 - p_j^2)}{(q_j - p_j)(q_i^2 - p_i^2) - (q_j^2 - p_j^2)(q_i - p_i)}\right)t,$$
$$y = \hat{y} + 4\left(\frac{(q_i^3 - p_i^3)(q_j - p_j) - (q_j^3 - p_j^3)(q_i - p_i)}{(q_j - p_j)(q_i^2 - p_i^2) - (q_j^2 - p_j^2)(q_i - p_i)}\right)t,$$

from which we obtain

$$\Lambda_i = (q_i - p_i)\hat{x} + (q_i^2 - p_i^2)\hat{y}, \quad \Lambda_j = (q_j - p_j)\hat{x} + (q_j^2 - p_j^2)\hat{y},$$

for $i, j = 1, 2, 3$.

In accordance with the three-soliton matrix solution, we will assume, without loss of generality, that $0 > p_3 > q_3 > p_2 > q_2 > p_1 > q_1$. Let us begin with the frame moving with $d(1,2)$. To obtain the characteristics of this dromion, we let soliton 3 pass through solitons 1 and 2, which are stationary, as $t \to \pm\infty$. We then study the asymptotic behaviour of the resulting two-soliton interaction. With solitons 1 and 2 fixed, h_1 and h_2 are are also fixed and we study the asymptotic behaviour of h_3 as $t \to \pm\infty$. We have that

$$h_3 \to \begin{cases} \frac{r_3}{p_3 - q_3} & \text{as} \quad t \to -\infty, \\ +\infty & \text{as} \quad t \to +\infty. \end{cases}$$

When $h_3 \to r_3/(p_3 - q_3)$, equation (3.51) gives

$$\begin{aligned} v_{1,2} = \frac{2(p_3 - q_3)}{\tilde{h}(1,2) r_3} &\left(b_{1,2}A_3 + b_{1,2,3}A_1A_3 + b_{3,2,1}A_3A_1 + b_{2,1,3}A_2A_3 + b_{3,1,2}A_3A_2\right. \\ &+ \left(\frac{r_3 h_2}{p_3 - q_3} - \frac{r_2 r_3 \alpha_{2,3}}{(p_2 - q_3)(p_3 - q_2)}\right) A_1 + \left(\frac{r_3 h_1}{p_3 - q_3} - \frac{r_1 r_3 \alpha_{1,3}}{(p_1 - q_3)(p_3 - q_1)}\right) A_2 \\ &+ \left(\frac{r_3 \alpha_{1,3,2}}{(p_2 - q_3)(p_3 - q_1)\alpha_{1,2}} - \frac{r_3}{(p_2 - q_1)(p_3 - q_3)}\right) A_1 A_2 \\ &+ \left.\left(\frac{r_3 \alpha_{1,2,3}}{(p_1 - q_3)(p_3 - q_2)\alpha_{1,2}} - \frac{r_3}{(p_1 - q_2)(p_3 - q_3)}\right) A_2 A_1\right), \end{aligned} \tag{5.9}$$

where

$$\begin{aligned} \tilde{h}(1,2) = &\left(h_1 h_2 - \frac{r_2 \alpha_{2,3}(p_3 - q_3) h_1}{(p_2 - q_3)(p_3 - q_2)} - \frac{r_1 \alpha_{1,3}(p_3 - q_3) h_2}{(p_1 - q_3)(p_3 - q_1)} - \frac{r_1 r_2 \alpha_{1,2}}{(p_1 - q_2)(p_2 - q_1)}\right. \\ &\left. + r_1 r_2 (p_3 - q_3)\left(\frac{\alpha_{1,2,3}}{(p_2 - q_1)(p_1 - q_3)(p_3 - q_2)} + \frac{\alpha_{1,3,2}}{(p_1 - q_2)(p_2 - q_3)(p_3 - q_1)}\right)\right). \end{aligned}$$

To obtain the characteristics of the dromion $d(1,2)$, we find the asymptotic forms of $\tilde{u} := (v_{1,2})_x$. Firstly, let us fix Λ_1. Since $\tilde{u}$ is invariant under the transformation $v_{1,2} \to v_{1,2} + C$, where C is a constant matrix, we have

$$v_{1,2}^{-} = \frac{2r_1(1,2)^- P_1(1,2)^-}{e^{-\Lambda_1} + \frac{r_1(1,2)^-}{p_1 - q_1}} \quad \text{as} \quad t \to -\infty,$$
$$v_{1,2}^{+} = \frac{2r_1(1,2)^+ P_1(1,2)^+}{e^{-\Lambda_1} + \frac{r_1(1,2)^+}{p_1 - q_1}} \quad \text{as} \quad t \to +\infty,$$

where

$$P_1(1,2)^- = \frac{\mu_1(1,2)^- \nu_1(1,2)^{-^T}}{(\mu_1(1,2)^-, \nu_1(1,2)^-)}, \quad P_1(1,2)^+ = \frac{\mu_1(1,2)^+ \nu_1(1,2)^{+^T}}{(\mu_1(1,2)^+, \nu_1(1,2)^+)},$$
$$\mu_1(1,2)^- = \mu_1 - \frac{(p_3 - q_3)(\mu_1, \nu_3)\mu_3}{(p_1 - q_3)(\mu_3, \nu_3)}, \quad \nu_1(1,2)^- = \nu_1 - \frac{(p_3 - q_3)(\mu_3, \nu_1)\nu_3}{(p_3 - q_1)(\mu_3, \nu_3)},$$
$$\mu_1(1,2)^+ = \mu_1 + \frac{(p_2 - q_2)(p_3 - q_3)}{(\mu_2, \nu_2)\beta_{2,3}} \left(\frac{(\mu_1, \nu_3)(\mu_3, \nu_2)}{(\mu_3, \nu_3)(p_1 - q_3)(p_3 - q_2)} - \frac{(\mu_1, \nu_2)}{(p_1 - q_2)(p_3 - q_3)} \right) \mu_2$$
$$+ \frac{(p_2 - q_3)(p_3 - q_2)}{(\mu_3, \nu_3)\beta_{2,3}} \left(\frac{(\mu_1, \nu_2)(\mu_2, \nu_3)}{(\mu_2, \nu_2)(p_1 - q_2)(p_2 - q_3)} - \frac{(\mu_1, \nu_3)}{(p_1 - q_3)(p_2 - q_2)} \right) \mu_3,$$
$$\nu_1(1,2)^+ = \nu_1 + \frac{(p_2 - q_2)(p_3 - q_3)}{(\mu_2, \nu_2)\beta_{2,3}} \left(\frac{(\mu_3, \nu_1)(\mu_2, \nu_3)}{(\mu_3, \nu_3)(p_2 - q_3)(p_3 - q_1)} - \frac{(\mu_2, \nu_1)}{(p_2 - q_1)(p_3 - q_3)} \right) \nu_2$$
$$+ \frac{(p_2 - q_3)(p_3 - q_2)}{(\mu_3, \nu_3)\beta_{2,3}} \left(\frac{(\mu_2, \nu_1)(\mu_3, \nu_2)}{(\mu_2, \nu_2)(p_3 - q_2)(p_2 - q_1)} - \frac{(\mu_3, \nu_1)}{(p_3 - q_1)(p_2 - q_2)} \right) \nu_3,$$
$$r_1(1,2)^- = \frac{r_1(\mu_1(1,2)^-, \nu_1(1,2)^-)}{(\mu_1, \nu_1)} = r_1\beta_{1,3} \quad \text{and}$$
$$r_1(1,2)^+ = \frac{r_1(\mu_1(1,2)^+, \nu_1(1,2)^+)}{(\mu_1, \nu_1)} = r_1\frac{\beta_{1,2,3}}{\beta_{2,3}}.$$

These asymptotic expressions for $v_{1,2}$ are of the same form as the one-soliton matrix potential $v_{[2]}$ discussed in Chapter 3. Therefore, the asymptotic forms for $\tilde{u}$ are

$$\tilde{u} \sim \frac{1}{2}(p_1 - q_1)^2 P_1(1,2)^- \operatorname{sech}^2 \left(\frac{1}{2}(\Lambda_1 + \xi_1(1,2)^-) \right) \quad \text{as} \quad t \to -\infty,$$
$$\tilde{u} \sim \frac{1}{2}(p_1 - q_1)^2 P_1(1,2)^+ \operatorname{sech}^2 \left(\frac{1}{2}(\Lambda_1 + \xi_1(1,2)^+) \right) \quad \text{as} \quad t \to +\infty,$$

with phase-constants $\xi_1(1,2)^- = \log\left(\frac{r_1(1,2)^-}{p_1 - q_1}\right)$ and $\xi_1(1,2)^+ = \log\left(\frac{r_1(1,2)^+}{p_1 - q_1}\right)$.

Next we fix Λ_2. Since $\tilde{u}$ is invariant under the transformation $v_{1,2} \to v_{1,2} + C$, where C is a constant matrix, we have

$$v_{1,2}^{-} = \frac{2r_2(1,2)^- P_2(1,2)^-}{e^{-\Lambda_2} + \frac{r_2(1,2)^-}{p_2 - q_2}} \quad \text{as} \quad t \to -\infty,$$
$$v_{1,2}^{+} = \frac{2r_2(1,2)^+ P_2(1,2)^+}{e^{-\Lambda_2} + \frac{r_2(1,2)^+}{p_2 - q_2}} \quad \text{as} \quad t \to +\infty,$$

in which

$$P_2(1,2)^- = \frac{\mu_2(1,2)^- \nu_2(1,2)^{-^T}}{(\mu_2(1,2)^-, \nu_2(1,2)^-)} \quad P_2(1,2)^+ = \frac{\mu_2(1,2)^+ \nu_2(1,2)^{+^T}}{(\mu_2(1,2)^+, \nu_2(1,2)^+)},$$

$$\mu_2(1,2)^+ = \mu_2 - \frac{(p_3-q_3)(\mu_2,\nu_3)\mu_3}{(p_2-q_3)(\mu_3,\nu_3)}, \quad \nu_2(1,2)^+ = \nu_2 - \frac{(p_3-q_3)(\mu_3,\nu_2)\nu_3}{(p_3-q_2)(\mu_3,\nu_3)},$$

$$\mu_2(1,2)^- = \mu_2 + \frac{(p_1-q_1)(p_3-q_3)}{(\mu_1,\nu_1)\beta_{1,3}} \left(\frac{(\mu_2,\nu_3)(\mu_3,\nu_1)}{(\mu_3,\nu_3)(p_2-q_3)(p_3-q_1)} - \frac{(\mu_2,\nu_1)}{(p_2-q_1)(p_3-q_3)} \right) \mu_1$$
$$+ \frac{(p_1-q_1)(p_3-q_3)}{(\mu_3,\nu_3)\beta_{1,3}} \left(\frac{(\mu_2,\nu_1)(\mu_1,\nu_3)}{(\mu_1,\nu_1)(p_2-q_1)(p_1-q_3)} - \frac{(\mu_2,\nu_3)}{(p_2-q_3)(p_1-q_1)} \right) \mu_3,$$

$$\nu_2(1,2)^- = \nu_2 + \frac{(p_1-q_1)(p_3-q_3)}{(\mu_1,\nu_1)\beta_{1,3}} \left(\frac{(\mu_3,\nu_2)(\mu_1,\nu_3)}{(\mu_3,\nu_3)(p_1-q_3)(p_3-q_2)} - \frac{(\mu_1,\nu_2)}{(p_1-q_2)(p_3-q_3)} \right) \nu_1$$
$$+ \frac{(p_1-q_1)(p_3-q_3)}{(\mu_3,\nu_3)\beta_{1,3}} \left(\frac{(\mu_1,\nu_2)(\mu_3,\nu_1)}{(\mu_1,\nu_1)(p_3-q_1)(p_1-q_2)} - \frac{(\mu_3,\nu_2)}{(p_3-q_2)(p_1-q_1)} \right) \nu_3,$$

$$r_2(1,2)^- = \frac{r_2(\mu_2(1,2)^-, \nu_2(1,2)^-)}{(\mu_2,\nu_2)} = r_2 \frac{\beta_{1,2,3}}{\beta_{1,3}} \quad \text{and}$$

$$r_2(1,2)^+ = \frac{r_2(\mu_2(1,2)^+, \nu_2(1,2)^+)}{(\mu_2,\nu_2)} = r_2 \beta_{2,3}.$$

So the asymptotic forms for $\tilde{u}$ are

$$\tilde{u} \sim \frac{1}{2}(p_2-q_2)^2 P_2(1,2)^- \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_2 + \xi_2(1,2)^-)\right) \quad \text{as} \quad t \to -\infty,$$
$$\tilde{u} \sim \frac{1}{2}(p_2-q_2)^2 P_2(1,2)^+ \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_2 + \xi_2(1,2)^+)\right) \quad \text{as} \quad t \to +\infty.$$

The phase-constants are: $\xi_2(1,2)^- = \log\left(\frac{r_2(1,2)^-}{p_2-q_2}\right)$ and $\xi_2(1,2)^+ = \log\left(\frac{r_2(1,2)^+}{p_2-q_2}\right)$. Furthermore, the soliton phase-shifts $\Delta_j(1,2) = \xi_j(1,2)^+ - \xi_j(1,2)^-$, for $j = 1,2$ are

$$\Delta_1(1,2) = \log\left(\beta_{1,2}^-\right) \quad \text{and} \quad \Delta_2(1,2) = -\log\left(\beta_{1,2}^-\right), \quad \text{in which} \quad \beta_{1,2}^- = \frac{\beta_{1,2,3}}{\beta_{1,3}\beta_{2,3}}.$$

The asymptotic expressions for $\tilde{u}$ can now be used to describe the dromion $d(1,2)$ as $t \to -\infty$. When $h_3 \to r_3/(p_3-q_3)$, equation (5.8) gives

$$d(1,2) \sim \frac{4r_1(1,2)^- r_2(1,2)^+ (p_1-q_1)(p_2-q_2)(1-\alpha_{1,2}^-)}{\left(e^{-\frac{1}{2}(\Lambda_1+\Lambda_2)} + \kappa_1(1,2)^- e^{\frac{1}{2}(\Lambda_1-\Lambda_2)} + \kappa_2(1,2)^- e^{\frac{1}{2}(\Lambda_2-\Lambda_1)} + \kappa_{1,2}^- e^{\frac{1}{2}(\Lambda_1+\Lambda_2)}\right)^2}, \quad (5.10)$$

where

$$\alpha_{1,2}^- = Tr(P_1(1,2)^- P_2(1,2)^+), \quad \kappa_1(1,2)^- = \frac{r_1(1,2)^-}{p_1-q_1}, \quad \kappa_2(1,2)^- = \frac{r_2(1,2)^+}{p_2-q_2} \quad \text{and}$$

$$\kappa_{1,2}^- = \kappa_1^- \kappa_2^- \beta_{1,2}^-.$$

In Chapter 3, we had that

$$K'_{i,j} = \frac{p_j - q_i}{h(i,j)} \left(h_j(p_i-q_j)I - A_j\right) A_i,$$

for $i, j \in \{1, 2, 3\}$ and $i \neq j$. When $h_3 \to +\infty$, from (3.47) we have that $K_3 \to 0$ and therefore

$$v_{1,2} \sim 2(K'_{1,2} + K'_{2,1}). \tag{5.11}$$

The asymptotic expression (5.11) is of the same form as the two-soliton matrix potential $v_{[3]}$ and the resulting dromion is therefore of the same form as the single dromion as given by (5.3). Therefore, when Λ_1 is fixed, the asymptotic forms for $\hat{\hat{u}} := (v_{1,2})_x$ are

$$\hat{\hat{u}} \sim \frac{1}{2}(p_1 - q_1)^2 \hat{P}_1(1,2)^- \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_1 + \hat{\xi}_1(1,2)^-)\right) \quad \text{as} \quad t \to -\infty,$$
$$\hat{\hat{u}} \sim \frac{1}{2}(p_1 - q_1)^2 \hat{P}_1(1,2)^+ \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_1 + \hat{\xi}_1(1,2)^+)\right) \quad \text{as} \quad t \to +\infty,$$

where

$$\hat{P}_1(1,2)^- = \frac{\hat{\mu}_1(1,2)^- \hat{\nu}_1(1,2)^{-^T}}{(\hat{\mu}_1(1,2)^-, \hat{\nu}_1(1,2)^-)}, \quad \hat{P}_1(1,2)^+ = \frac{\hat{\mu}_1(1,2)^+ \hat{\nu}_1(1,2)^{+^T}}{(\hat{\mu}_1(1,2)^+, \hat{\nu}_1(1,2)^+)},$$
$$\hat{\mu}_1(1,2)^- = \mu_1, \quad \hat{\nu}_1(1,2)^- = \nu_1, \quad \hat{\mu}_1(1,2)^+ = \mu_1 - \frac{(p_2 - q_2)(\mu_1, \nu_2)\mu_2}{(p_1 - q_2)(\mu_2, \nu_2)},$$
$$\hat{\nu}_1(1,2)^+ = \nu_1 - \frac{(p_2 - q_2)(\mu_2, \nu_1)\nu_2}{(p_2 - q_1)(\mu_2, \nu_2)}, \quad \hat{r}_1(1,2)^- = r_1, \quad \hat{r}_1(1,2)^+ = \frac{r_1(\hat{\mu}_1, \hat{\nu}_1)}{(\mu_1, \nu_1)} = r_1 \beta_{1,2},$$
$$\hat{\xi}_1(1,2)^- = \log\left(\frac{\hat{r}_1(1,2)^-}{p_1 - q_1}\right) \quad \text{and} \quad \hat{\xi}_1(1,2)^+ = \log\left(\frac{\hat{r}_1(1,2)^+}{p_1 - q_1}\right).$$

When Λ_2 is fixed, the asymptotic forms for $\hat{\hat{u}}$ are

$$\hat{\hat{u}} \sim \frac{1}{2}(p_2 - q_2)^2 \hat{P}_2(1,2)^- \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_2 + \hat{\xi}_2(1,2)^-)\right) \quad \text{as} \quad t \to -\infty,$$
$$\hat{\hat{u}} \sim \frac{1}{2}(p_2 - q_2)^2 \hat{P}_2(1,2)^+ \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_2 + \hat{\xi}_2(1,2)^+)\right) \quad \text{as} \quad t \to +\infty,$$

in which

$$\hat{P}_2(1,2)^- = \frac{\hat{\mu}_2(1,2)^- \hat{\nu}_2(1,2)^{-^T}}{(\hat{\mu}_2(1,2)^-, \hat{\nu}_2(1,2)^-)}, \quad \hat{P}_2(1,2)^+ = \frac{\hat{\mu}_2(1,2)^+ \hat{\nu}_2(1,2)^{+^T}}{(\hat{\mu}_2(1,2)^+, \hat{\nu}_2(1,2)^+)},$$
$$\hat{\mu}_2(1,2)^+ = \mu_2, \quad \hat{\nu}_2(1,2)^+ = \nu_2, \quad \hat{\mu}_2(1,2)^- = \mu_2 - \frac{(p_1 - q_1)(\mu_2, \nu_1)\mu_1}{(p_2 - q_1)(\mu_1, \nu_1)},$$
$$\hat{\nu}_2(1,2)^- = \nu_2 - \frac{(p_1 - q_1)(\mu_1, \nu_2)\nu_1}{(p_1 - q_2)(\mu_1, \nu_1)}, \quad \hat{r}_2(1,2)^- = \frac{r_2(\hat{\mu}_2, \hat{\hat{\nu}}_2)}{(\mu_2, \nu_2)} = r_2 \beta_{1,2}, \quad \hat{r}_2(1,2)^+ = r_2,$$
$$\hat{\xi}_2(1,2)^- = \log\left(\frac{\hat{r}_2^-}{p_2 - q_2}\right) \quad \text{and} \quad \hat{\xi}_2(1,2)^+ = \log\left(\frac{\hat{r}_2(1,2)^+}{p_2 - q_2}\right).$$

The soliton phase-shifts $\hat{\Delta}_j(1,2) = \hat{\xi}_j(1,2)^+ - \hat{\xi}_j(1,2)^-$, for $j = 1, 2$ are

$$\hat{\Delta}_1(1,2) = \log\left(\beta^+_{1,2}\right) \quad \text{and} \quad \hat{\Delta}_2(1,2) = -\log\left(\beta^+_{1,2}\right), \quad \text{where} \quad \beta^+_{1,2} = \beta_{1,2}.$$

The dromion $d(1,2)$ as $t \to +\infty$ can now be written as

$$d(1,2) \sim \frac{4\hat{r}_1(1,2)^-\hat{r}_2(1,2)^+(p_1-q_1)(p_2-q_2)(1-\alpha^+_{1,2})}{\left(e^{-\frac{1}{2}(\Lambda_1+\Lambda_2)} + \kappa_1(1,2)^+e^{\frac{1}{2}(\Lambda_1-\Lambda_2)} + \kappa_2(1,2)^+e^{\frac{1}{2}(\Lambda_2-\Lambda_1)} + \kappa^+_{1,2}e^{\frac{1}{2}(\Lambda_1+\Lambda_2)}\right)^2}, \quad (5.12)$$

where $\alpha^+_{1,2} = Tr(\hat{P}_1(1,2)^-\hat{P}_2(1,2)^+)$, $\kappa_1(1,2)^+ = \kappa_1$, $\kappa_2(1,2)^+ = \kappa_2$ and $\kappa^+_{1,2} = \kappa_1(1,2)^+\kappa_2(1,2)^+\beta^+_{1,2}$.

When in a frame moving with the dromion $d(2,3)$, the asymptotic analysis is very similar to that of $d(1,2)$. In this case, h_2 and h_3 are fixed and

$$h_1 \to \begin{cases} \frac{r_1}{p_1-q_1} & \text{as} \quad t \to -\infty, \\ +\infty & \text{as} \quad t \to +\infty. \end{cases}$$

When $h_1 \to \frac{r_1}{p_1-q_1}$, fixing Λ_2 gives the asymptotic forms for $\tilde{u}$:

$$\tilde{u} \sim \frac{1}{2}(p_2-q_2)^2P_2(2,3)^- \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_2+\xi_2(2,3)^-)\right) \quad \text{as} \quad t \to -\infty,$$
$$\tilde{u} \sim \frac{1}{2}(p_2-q_2)^2P_2(2,3)^+ \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_2+\xi_2(2,3)^+)\right) \quad \text{as} \quad t \to +\infty,$$

with phase-constants $\xi_2(2,3)^- = \log\left(\frac{r_2(2,3)^-}{p_2-q_2}\right)$ and $\xi_2(2,3)^+ = \log\left(\frac{r_2(2,3)^+}{p_2-q_2}\right)$ and

$$P_2(2,3)^- = \frac{\mu_2(2,3)^-\nu_2(2,3)^{-T}}{(\mu_2(2,3)^-,\nu_2(2,3)^-)}, \quad P_2(2,3)^+ = \frac{\mu_2(2,3)^+\nu_2(2,3)^{+T}}{(\mu_2(2,3)^+,\nu_2(2,3)^+)},$$
$$\mu_2(2,3)^- = \mu_2 - \frac{(p_1-q_1)(\mu_2,\nu_1)\mu_1}{(p_2-q_1)(\mu_1,\nu_1)}, \quad \nu_2(2,3)^- = \nu_2 - \frac{(p_1-q_1)(\mu_1,\nu_2)\nu_1}{(p_1-q_2)(\mu_1,\nu_1)},$$
$$\mu_2(2,3)^+ = \mu_2 + \frac{(p_1-q_1)(p_3-q_3)}{(\mu_1,\nu_1)\beta_{1,3}}\left(\frac{(\mu_2,\nu_3)(\mu_3,\nu_1)}{(\mu_3,\nu_3)(p_2-q_3)(p_3-q_1)} - \frac{(\mu_2,\nu_1)}{(p_2-q_1)(p_3-q_3)}\right)\mu_1$$
$$+ \frac{(p_1-q_1)(p_3-q_3)}{(\mu_3,\nu_3)\beta_{1,3}}\left(\frac{(\mu_2,\nu_1)(\mu_1,\nu_3)}{(\mu_1,\nu_1)(p_2-q_1)(p_1-q_3)} - \frac{(\mu_2,\nu_3)}{(p_2-q_3)(p_1-q_1)}\right)\mu_3,$$
$$\nu_2(2,3)^+ = \nu_2 + \frac{(p_1-q_1)(p_3-q_3)}{(\mu_1,\nu_1)\beta_{1,3}}\left(\frac{(\mu_3,\nu_2)(\mu_1,\nu_3)}{(\mu_3,\nu_3)(p_1-q_3)(p_3-q_2)} - \frac{(\mu_1,\nu_2)}{(p_1-q_2)(p_3-q_3)}\right)\nu_1$$
$$+ \frac{(p_1-q_1)(p_3-q_3)}{(\mu_3,\nu_3)\beta_{1,3}}\left(\frac{(\mu_1,\nu_2)(\mu_3,\nu_1)}{(\mu_1,\nu_1)(p_3-q_1)(p_1-q_2)} - \frac{(\mu_3,\nu_2)}{(p_3-q_2)(p_1-q_1)}\right)\nu_3,$$
$$r_2(2,3)^- = \frac{r_2(\mu_2(2,3)^-,\nu_2(2,3)^-)}{(\mu_2,\nu_2)} = r_2\beta_{1,2} \quad \text{and}$$
$$r_2(2,3)^+ = \frac{r_2(\mu_2(2,3)^+,\nu_2(2,3)^+)}{(\mu_2,\nu_2)} = r_2\frac{\beta_{1,2,3}}{\beta_{1,3}}.$$

Fixing Λ_3 gives the asymptotic forms for $\tilde{u}$:

$$\tilde{u} \sim \frac{1}{2}(p_3-q_3)^2P_3(2,3)^- \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_3+\xi_3(2,3)^-)\right) \quad \text{as} \quad t \to -\infty,$$
$$\tilde{u} \sim \frac{1}{2}(p_3-q_3)^2P_3(2,3)^+ \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_3+\xi_3(2,3)^+)\right) \quad \text{as} \quad t \to +\infty,$$

where $\xi_3(2,3)^- = \log\left(\frac{r_3(2,3)^-}{p_3-q_3}\right)$ and $\xi_3(2,3)^+ = \log\left(\frac{r_3(2,3)^+}{p_3-q_3}\right)$ and

$$\begin{aligned}
P_3(2,3)^- &= \frac{\mu_3(2,3)^-\nu_3(2,3)^{-T}}{(\mu_3^-,\nu_3^-)}, \quad P_3(2,3)^+ = \frac{\mu_3(2,3)^+\nu_3(2,3)^{+T}}{(\mu_3(2,3)^+,\nu_3(2,3)^+)},\\
\mu_3(2,3)^+ &= \mu_2 - \frac{(p_3-q_3)(\mu_2,\nu_3)\mu_3}{(p_2-q_3)(\mu_3,\nu_3)}, \quad \nu_3(2,3)^+ = \mu_2 - \frac{(p_3-q_3)(\mu_2,\nu_3)\mu_3}{(p_2-q_3)(\mu_3,\nu_3)},\\
\mu_3(2,3)^- &= \mu_3 + \frac{(p_2-q_2)(p_1-q_1)}{(\mu_2,\nu_2)\beta_{1,2}}\left(\frac{(\mu_3,\nu_1)(\mu_1,\nu_2)}{(\mu_1,\nu_1)(p_3-q_1)(p_1-q_2)} - \frac{(\mu_3,\nu_2)}{(p_3-q_2)(p_1-q_1)}\right)\mu_2\\
&\quad + \frac{(p_2-q_2)(p_1-q_1)}{(\mu_1,\nu_1)\beta_{1,2}}\left(\frac{(\mu_3,\nu_2)(\mu_2,\nu_1)}{(\mu_2,\nu_2)(p_3-q_2)(p_2-q_1)} - \frac{(\mu_3,\nu_1)}{(p_3-q_1)(p_2-q_2)}\right)\mu_1,\\
\tilde{\nu}_3^- &= \nu_3 + \frac{(p_2-q_2)(p_1-q_1)}{(\mu_2,\nu_2)\beta_{1,2}}\left(\frac{(\mu_1,\nu_3)(\mu_2,\nu_1)}{(\mu_1,\nu_1)(p_2-q_1)(p_1-q_3)} - \frac{(\mu_2,\nu_3)}{(p_2-q_3)(p_1-q_1)}\right)\nu_2\\
&\quad + \frac{(p_2-q_2)(p_1-q_1)}{(\mu_1,\nu_1)\beta_{1,2}}\left(\frac{(\mu_2,\nu_3)(\mu_1,\nu_2)}{(\mu_2,\nu_2)(p_1-q_2)(p_2-q_3)} - \frac{(\mu_1,\nu_3)}{(p_1-q_3)(p_2-q_2)}\right)\nu_1,\\
r_3(2,3)^- &= \frac{r_3(\mu_3(2,3)^-,\nu_3(2,3)^-)}{(\mu_3,\nu_3)} = r_3\frac{\beta_{1,2,3}}{\beta_{1,2}} \quad \text{and}\\
r_3(2,3)^+ &= \frac{r_3(\mu_3(2,3)^+,\nu_3(2,3)^+)}{(\mu_3,\nu_3)} = r_3\beta_{2,3}.
\end{aligned}$$

The soliton phase-shifts $\Delta_j(2,3) = \xi_j(2,3)^+ - \xi_j(2,3)^-$, for $j = 2,3$ are

$$\Delta_2(2,3) = \log\left(\beta_{2,3}^-\right) \quad \text{and} \quad \Delta_3(2,3) = -\log\left(\beta_{2,3}^-\right), \quad \text{in which} \quad \beta_{2,3}^- = \frac{\beta_{1,2,3}}{\beta_{1,2}\beta_{1,3}}.$$

The asymptotic expressions for $\tilde{u}$ can be used to describe the dromion $d(2,3)$ as $t \to -\infty$. When $h_1 \to r_1/(p_1-q_1)$, equation (5.8) gives

$$d(2,3) \sim \frac{4r_2(2,3)^-r_3(2,3)^+(p_2-q_2)(p_3-q_3)(1-\alpha_{2,3}^-)}{\left(e^{-\frac{1}{2}(\Lambda_2+\Lambda_3)} + \kappa_2(2,3)^-e^{\frac{1}{2}(\Lambda_2-\Lambda_3)} + \kappa_3(2,3)^-e^{\frac{1}{2}(\Lambda_3-\Lambda_2)} + \kappa_{2,3}^-e^{\frac{1}{2}(\Lambda_2+\Lambda_3)}\right)^2}, \quad (5.13)$$

where

$$\alpha_{2,3}^- = Tr(P_2(2,3)^-P_3(2,3)^+), \quad \kappa_2(2,3)^- = \frac{r_2(2,3)^-}{p_2-q_2}, \quad \kappa_3(2,3)^- = \frac{r_3(2,3)^-}{p_3-q_3}, \quad \text{and}$$
$$\kappa_{2,3}^- = \kappa_2(2,3)^-\kappa_3(2,3)^-\beta_{2,3}^-.$$

In the case that $h_1 \to +\infty$, when Λ_2 is fixed, the asymptotic forms for $\hat{\tilde{u}} := (v_{2,3})_x$ are

$$\begin{aligned}
\hat{\tilde{u}} &\sim \frac{1}{2}(p_2-q_2)^2\hat{P}_2(2,3)^-\,\mathrm{sech}^2\left(\frac{1}{2}(\Lambda_2+\hat{\xi}_2(2,3)^-)\right) \quad \text{as} \quad t \to -\infty,\\
\hat{\tilde{u}} &\sim \frac{1}{2}(p_2-q_2)^2\hat{P}_2(2,3)^+\,\mathrm{sech}^2\left(\frac{1}{2}(\Lambda_2+\hat{\xi}_2(2,3)^+)\right) \quad \text{as} \quad t \to +\infty,
\end{aligned}$$

in which

$$\begin{aligned}
&\hat{P}_2(2,3)^- = \frac{\hat{\mu}_2(2,3)^-\hat{\nu}_2(2,3)^{-T}}{(\hat{\mu}_2(2,3)^-,\hat{\nu}_2(2,3)^-)}, \quad \hat{P}_2(2,3)^+ = \frac{\hat{\mu}_2(2,3)^+\hat{\nu}_2(2,3)^{+T}}{(\hat{\mu}_2(2,3)^+,\hat{\nu}_2(2,3)^+)},\\
&\hat{\mu}_2(2,3)^- = \mu_2, \quad \hat{\nu}_2(2,3)^- = \nu_2, \quad \hat{\mu}_2(2,3)^+ = \mu_2 - \frac{(p_3-q_3)(\mu_2,\nu_3)\mu_3}{(p_2-q_3)(\mu_3,\nu_3)}\\
&\hat{\nu}_2(2,3)^+ = \nu_2 - \frac{(p_3-q_3)(\mu_3,\nu_2)\nu_3}{(p_3-q_2)(\mu_3,\nu_3)} \quad \hat{r}_2(2,3)^- = r_2,\\
&\hat{r}_2(2,3)^+ = \frac{r_2(\hat{\mu}_2(2,3),\hat{\nu}_2(2,3))}{(\mu_2,\nu_2)} = r_2\beta_{2,3}, \quad \hat{\xi}_2(2,3)^- = \log\left(\frac{\hat{r}_2(2,3)^-}{p_2-q_2}\right) \quad \text{and}\\
&\hat{\xi}_2(2,3)^+ = \log\left(\frac{\hat{r}_2(2,3)^+}{p_2-q_2}\right).
\end{aligned}$$

When Λ_3 is fixed, the asymptotic forms for $\hat{\hat{u}}$ are

$$\begin{aligned}
\hat{\hat{u}} &\sim \frac{1}{2}(p_3-q_3)^2\hat{P}_3(2,3)^-\operatorname{sech}^2\left(\frac{1}{2}(\Lambda_3+\hat{\xi}_3(2,3)^-)\right) \quad \text{as} \quad t\to-\infty,\\
\hat{\hat{u}} &\sim \frac{1}{2}(p_3-q_3)^2\hat{P}_3(2,3)^+\operatorname{sech}^2\left(\frac{1}{2}(\Lambda_3+\hat{\xi}_3(2,3)^+)\right) \quad \text{as} \quad t\to+\infty,
\end{aligned}$$

where

$$\begin{aligned}
&\hat{P}_3(2,3)^- = \frac{\hat{\mu}_3(2,3)^-\hat{\nu}_3(2,3)^{-T}}{(\hat{\mu}_3(2,3)^-,\hat{\nu}_3(2,3)^-)}, \quad \hat{P}_3(2,3)^+ = \frac{\hat{\mu}_3(2,3)^+\hat{\nu}_3(2,3)^{+T}}{(\hat{\mu}_3(2,3)^+,\hat{\nu}_3(2,3)^+)},\\
&\hat{\mu}_3(2,3)^+ = \mu_3, \quad \hat{\nu}_3(2,3)^+ = \nu_3, \quad \hat{\mu}_3(2,3)^- = \mu_3 - \frac{(p_2-q_2)(\mu_3,\nu_2)\mu_2}{(p_3-q_2)(\mu_2,\nu_2)},\\
&\hat{\nu}_3(2,3)^- = \nu_3 - \frac{(p_2-q_2)(\mu_2,\nu_3)\nu_2}{(p_2-q_3)(\mu_2,\nu_2)}, \quad \hat{r}_3(2,3)^- = \frac{r_3(\hat{\mu}_3(2,3),\hat{\nu}_3(2,3))}{(\mu_3,\nu_3)} = r_3\beta_{2,3},\\
&\hat{r}_3(2,3)^+ = r_3, \quad \hat{\xi}_3(2,3)^- = \log\left(\frac{\hat{r}_3(2,3)^-}{p_3-q_3}\right) \quad \text{and}\\
&\hat{\xi}_3(2,3)^+ = \log\left(\frac{\hat{r}_3(2,3)^+}{p_3-q_3}\right).
\end{aligned}$$

Furthermore, the soliton phase-shifts are $\hat{\Delta}_j(2,3) = \hat{\xi}_j(2,3)^+ - \hat{\xi}_j(2,3)^-$, for $j = 2,3$ are

$$\hat{\Delta}_2 = \log\left(\beta^+_{2,3}\right) \quad \text{and} \quad \hat{\Delta}_3 = -\log\left(\beta^+_{2,3}\right), \quad \text{where} \quad \beta^+_{2,3} = \beta_{2,3}.$$

The dromion $d(2,3)$ as $t\to+\infty$ can now be written as

$$d(2,3) \sim \frac{4\hat{r}_2(2,3)^-\hat{r}_3(2,3)^+(p_2-q_2)(p_3-q_3)(1-\alpha^+_{2,3})}{\left(e^{-\frac{1}{2}(\Lambda_2+\Lambda_3)}+\kappa_2(2,3)^+e^{\frac{1}{2}(\Lambda_2-\Lambda_3)}+\kappa_3(2,3)^+e^{\frac{1}{2}(\Lambda_3-\Lambda_2)}+\kappa^+_{2,3}e^{\frac{1}{2}(\Lambda_2+\Lambda_3)}\right)^2}, \quad (5.14)$$

where $Tr(\hat{P}_2(2,3)^-\hat{P}_3(2,3)^+)$, $\kappa_2(2,3)^+ = \kappa_2$, $\kappa_3(2,3)^+ = \kappa_3$ and $\kappa^+_{2,3} = \kappa_2(2,3)^+\kappa_3(2,3)^+\beta^+_{2,3}$.

Finally, we move to a frame moving with $d(1,3)$. In this case h_1 and h_3 are fixed and

$$h_2 \to \begin{cases} +\infty & \text{as} \quad t\to-\infty,\\ \frac{r_2}{p_2-q_2} & \text{as} \quad t\to+\infty. \end{cases}$$

When $h_2 \to +\infty$, fixing Λ_1 gives the asymptotic forms for $\tilde{u} := (v_{1,3})_x$:

$$\tilde{u} \sim \frac{1}{2}(p_1 - q_1)^2 P_1(1,3)^- \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_1 + \xi_1(1,3)^-)\right) \quad \text{as} \quad t \to -\infty,$$
$$\tilde{u} \sim \frac{1}{2}(p_1 - q_1)^2 P_1(1,3)^+ \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_1 + \xi_1(1,3)^+)\right) \quad \text{as} \quad t \to +\infty,$$

where

$$P_1(1,3)^- = \frac{\mu_1(1,3)^- \nu_1(1,3)^{-T}}{(\mu_1(1,3)^-, \nu_1(1,3)^-)}, \quad P_1(1,3)^+ = \frac{\mu_1(1,3)^+ \nu_1(1,3)^{+T}}{(\mu_1(1,3)^+, \nu_1(1,3)^+)},$$
$$\mu_1(1,3)^- = \mu_1, \quad \nu_1(1,3)^- = \nu_1, \quad \mu_1(1,3)^+ = \mu_1 - \frac{(p_3 - q_3)(\mu_1, \nu_3)\mu_3}{(p_1 - q_3)(\mu_3, \nu_3)}$$
$$\nu_1(1,3)^+ = \nu_1 - \frac{(p_3 - q_3)(\mu_3, \nu_1)\nu_3}{(p_3 - q_1)(\mu_3, \nu_3)} \quad r_1(1,3)^- = r_1, \quad r_1(1,3)^+ = \frac{r_1(\mu_1, \nu_1)}{(\mu_1, \nu_1)} = r_1\beta_{1,3},$$
$$\xi_1(1,3)^- = \log\left(\frac{r_1(1,3)^-}{p_1 - q_1}\right) \quad \text{and} \quad \xi_1(1,3)^+ = \log\left(\frac{r_1(1,3)^+}{p_1 - q_1}\right).$$

When Λ_3 is fixed, the asymptotic forms for $\tilde{u}$ are

$$\tilde{u} \sim \frac{1}{2}(p_3 - q_3)^2 P_3(1,3)^- \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_3 + \xi_3(1,3)^-)\right) \quad \text{as} \quad t \to -\infty,$$
$$\tilde{u} \sim \frac{1}{2}(p_3 - q_3)^2 P_3(1,3)^+ \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_3 + \xi_3(1,3)^+)\right) \quad \text{as} \quad t \to +\infty,$$

where

$$P_3(1,3)^- = \frac{\mu_3(1,3)^- \nu_3(1,3)^{-T}}{(\mu_3(1,3)^-, \nu_3(1,3)^-)}, \quad P_3(1,3)^+ = \frac{\mu_3(1,3)^+ \nu_3(1,3)^{+T}}{(\mu_3(1,3)^+, \nu_3(1,3)^+)},$$
$$\mu_3(1,3)^+ = \mu_3, \quad \nu_3(1,3)^+ = \nu_3, \quad \mu_3(1,3)^- = \mu_3 - \frac{(p_1 - q_1)(\mu_3, \nu_1)\mu_1}{(p_3 - q_1)(\mu_1, \nu_1)},$$
$$\nu_3(1,3)^- = \nu_3 - \frac{(p_1 - q_1)(\mu_1, \nu_3)\nu_1}{(p_1 - q_3)(\mu_1, \nu_1)}, \quad r_3(1,3)^- = \frac{r_3(\mu_3(1,3), \nu_3(1,3))}{(\mu_3, \nu_3)} = r_3\beta_{1,3},$$
$$r_3(1,3)^+ = r_3, \quad \xi_3(1,3)^- = \log\left(\frac{r_3(1,3)^-}{p_3 - q_3}\right) \quad \text{and}$$
$$\xi_3(1,3)^+ = \log\left(\frac{r_3(1,3)^+}{p_3 - q_3}\right).$$

In addition, we have the soliton phase-shifts $\Delta_j(1,3) = \xi_j(1,3)^+ - \xi_j(1,3)^-$, for $j = 1, 3$, which are

$$\Delta_1(1,3) = \log\left(\beta_{1,3}^-\right) \quad \text{and} \quad \Delta_3(1,3) = -\log\left(\beta_{1,3}^-\right), \quad \text{in which} \quad \beta_{1,3}^- = \beta_{1,3}.$$

The dromion $d(1,3)$ as $t \to -\infty$ can now be written as

$$d(1,3) \sim \frac{4r_1(1,3)^- r_3(1,3)^+ (p_1 - q_1)(p_3 - q_3)(1 - \alpha_{1,3}^-)}{\left(e^{-\frac{1}{2}(\Lambda_1 + \Lambda_3)} + \kappa_1(1,3)^- e^{\frac{1}{2}(\Lambda_1 - \Lambda_3)} + \kappa_3(1,3)^- e^{\frac{1}{2}(\Lambda_3 - \Lambda_1)} + \kappa_{1,3}^- e^{\frac{1}{2}(\Lambda_1 + \Lambda_3)}\right)^2}, \quad (5.15)$$

where $\alpha_{1,3}^- = Tr(P_1(1,3)^- P_3(1,3)^+)$, $\kappa_1(1,3)^- = \kappa_1$, $\kappa_3(1,3)^- = \kappa_3$, $\kappa_{1,3}^- = \kappa_1(1,3)^- \kappa_3(1,3)^- \beta_{1,3}^-$ and $\beta_{1,3}^- = \beta_{1,3}$.

When $h_1 \to r_1/(p_1 - q_1)$, fixing Λ_1 gives the asymptotic forms for $\hat{\hat{u}}$:

$$\hat{\hat{u}} \sim \frac{1}{2}(p_1 - q_1)^2 \hat{P}_1(1,3)^- \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_1 + \hat{\xi}_1(1,3)^-)\right) \quad \text{as} \quad t \to -\infty,$$
$$\hat{\hat{u}} \sim \frac{1}{2}(p_1 - q_1)^2 \hat{P}_1(1,3)^+ \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_1 + \hat{\xi}_1(1,3)^+)\right) \quad \text{as} \quad t \to +\infty,$$

with phase-constants $\hat{\xi}_1(1,3)^- = \log\left(\frac{\hat{r}_1(1,3)^-}{p_1 - q_1}\right)$ and $\hat{\xi}_1(1,3)^+ = \log\left(\frac{\hat{r}_1(1,3)^+}{p_1 - q_1}\right)$ and

$$\hat{P}_1(1,3)^- = \frac{\hat{\mu}_1(1,3)^- \hat{\nu}_1(1,3)^{-T}}{(\hat{\mu}_1(1,3)^-, \hat{\nu}_1(1,3)^-)}, \quad \hat{P}_1(1,3)^+ = \frac{\hat{\mu}_1(1,3)^+ \hat{\nu}_1(1,3)^{+T}}{(\hat{\mu}_1(1,3)^+, \hat{\nu}_1(1,3)^+)},$$
$$\hat{\mu}_1(1,3)^- = \mu_1 - \frac{(p_2 - q_2)(\mu_1, \nu_2)\mu_2}{(p_1 - q_2)(\mu_2, \nu_2)}, \quad \hat{\nu}_1(1,3)^- = \nu_1 - \frac{(p_2 - q_2)(\mu_2, \nu_1)\nu_2}{(p_2 - q_1)(\mu_2, \nu_2)},$$
$$\hat{\mu}_1(1,3)^+ = \mu_1 + \frac{(p_2 - q_2)(p_3 - q_3)}{(\mu_2, \nu_2)\beta_{2,3}} \left(\frac{(\mu_1, \nu_3)(\mu_3, \nu_2)}{(\mu_3, \nu_3)(p_1 - q_3)(p_3 - q_2)} - \frac{(\mu_1, \nu_2)}{(p_1 - q_2)(p_3 - q_3)}\right)\mu_2$$
$$+ \frac{(p_2 - q_3)(p_3 - q_2)}{(\mu_3, \nu_3)\beta_{2,3}} \left(\frac{(\mu_1, \nu_2)(\mu_2, \nu_3)}{(\mu_2, \nu_2)(p_1 - q_2)(p_2 - q_3)} - \frac{(\mu_1, \nu_3)}{(p_1 - q_3)(p_2 - q_2)}\right)\mu_3,$$
$$\hat{\nu}_1(1,3)^+ = \nu_1 + \frac{(p_2 - q_2)(p_3 - q_3)}{(\mu_2, \nu_2)\beta_{2,3}} \left(\frac{(\mu_3, \nu_1)(\mu_2, \nu_3)}{(\mu_3, \nu_3)(p_2 - q_3)(p_3 - q_1)} - \frac{(\mu_2, \nu_1)}{(p_2 - q_1)(p_3 - q_3)}\right)\nu_2$$
$$+ \frac{(p_2 - q_3)(p_3 - q_2)}{(\mu_3, \nu_3)\beta_{2,3}} \left(\frac{(\mu_2, \nu_1)(\mu_3, \nu_2)}{(\mu_2, \nu_2)(p_3 - q_2)(p_2 - q_1)} - \frac{(\mu_3, \nu_1)}{(p_3 - q_1)(p_2 - q_2)}\right)\nu_3,$$
$$\hat{r}_1(1,3)^- = \frac{r_1(\hat{\mu}_1^-, \hat{\nu}_1^-)}{(\mu_1, \nu_1)} = r_1\beta_{1,2} \quad \text{and} \quad \hat{\hat{r}}_1^+ = \frac{r_1(\hat{\mu}_1(1,3)^+, \hat{\nu}_1(1,3)^+)}{(\mu_1, \nu_1)} = r_1\frac{\beta_{1,2,3}}{\beta_{2,3}}.$$

Next we fix Λ_3. Then the asymptotic forms for $\hat{\hat{u}}$ are

$$\hat{\hat{u}} \sim \frac{1}{2}(p_3 - q_3)^2 \hat{P}_3(1,3)^- \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_3 + \hat{\xi}_3(1,3)^-)\right) \quad \text{as} \quad t \to -\infty,$$
$$\hat{\hat{u}} \sim \frac{1}{2}(p_3 - q_3)^2 \hat{P}_3(1,3)^+ \operatorname{sech}^2\left(\frac{1}{2}(\Lambda_3 + \hat{\xi}_3(1,3)^+)\right) \quad \text{as} \quad t \to +\infty.$$

The phase-constants are: $\hat{\xi}_3^- = \log\left(\frac{\hat{r}_3^-}{p_3-q_3}\right)$ and $\hat{\xi}_3^+ = \log\left(\frac{\hat{r}_3^+}{p_3-q_3}\right)$ and we also have

$$\hat{P}_3(1,3)^- = \frac{\hat{\mu}_3(1,3)^-\hat{\nu}_3(1,3)^{-T}}{(\hat{\mu}_3(1,3)^-,\hat{\nu}_3(1,3)^-)}, \quad \hat{P}_3(1,3)^+ = \frac{\hat{\mu}_3(1,3)^+\hat{\nu}_3(1,3)^{+T}}{(\hat{\mu}_3(1,3)^+,\hat{\nu}_3(1,3)^+)},$$

$$\hat{\mu}_3(1,3)^+ = \mu_3 - \frac{(p_2-q_2)(\mu_3,\nu_2)\mu_2}{(p_3-q_2)(\mu_2,\nu_2)}, \quad \hat{\nu}_3(1,3)^+ = \nu_3 - \frac{(p_2-q_2)(\mu_2,\nu_3)\nu_3}{(p_2-q_3)(\mu_2,\nu_2)},$$

$$\hat{\mu}_3(1,3)^- = \mu_3 + \frac{(p_2-q_2)(p_1-q_1)}{(\mu_2,\nu_2)\beta_{1,2}}\left(\frac{(\mu_3,\nu_1)(\mu_1,\nu_2)}{(\mu_1,\nu_1)(p_3-q_1)(p_1-q_2)} - \frac{(\mu_3,\nu_2)}{(p_3-q_2)(p_1-q_1)}\right)\mu_2$$
$$+ \frac{(p_2-q_2)(p_1-q_1)}{(\mu_1,\nu_1)\beta_{1,2}}\left(\frac{(\mu_3,\nu_2)(\mu_2,\nu_1)}{(\mu_2,\nu_2)(p_3-q_2)(p_2-q_1)} - \frac{(\mu_3,\nu_1)}{(p_3-q_1)(p_2-q_2)}\right)\mu_1,$$

$$\hat{\nu}_3(1,3)^- = \nu_3 + \frac{(p_2-q_2)(p_1-q_1)}{(\mu_2,\nu_2)\beta_{1,2}}\left(\frac{(\mu_1,\nu_3)(\mu_2,\nu_1)}{(\mu_1,\nu_1)(p_2-q_1)(p_1-q_3)} - \frac{(\mu_2,\nu_3)}{(p_2-q_3)(p_1-q_1)}\right)\nu_2$$
$$+ \frac{(p_2-q_2)(p_1-q_1)}{(\mu_1,\nu_1)\beta_{1,2}}\left(\frac{(\mu_2,\nu_3)(\mu_1,\nu_2)}{(\mu_2,\nu_2)(p_1-q_2)(p_2-q_3)} - \frac{(\mu_1,\nu_3)}{(p_1-q_3)(p_2-q_2)}\right)\nu_1,$$

$$\hat{r}_3(1,3)^- = \frac{r_3(\hat{\mu}_3(1,3)^-,\hat{\nu}_3(1,3)^-)}{(\mu_3,\nu_3)} = \frac{r_3\beta_{1,2,3}}{\beta_{1,2}} \quad \text{and}$$

$$\hat{r}_3(1,3)^+ = \frac{r_3(\hat{\mu}_3(1,3)^+,\hat{\nu}_3(1,3)^+)}{(\mu_3,\nu_3)} = r_3\beta_{2,3}.$$

Furthermore, the soliton phase-shifts $\hat{\Delta}_j(1,3) = \hat{\xi}_j(1,3)^+ - \hat{\xi}_j(1,3)^-$, for $j = 1,3$ are

$$\hat{\Delta}_1(1,3) = \log\left(\beta_{1,3}^+\right) \quad \text{and} \quad \hat{\Delta}_3(1,3) = -\log\left(\beta_{1,3}^+\right), \quad \text{where} \quad \beta_{1,3}^+ = \frac{\beta_{1,2,3}}{\beta_{1,2}\beta_{2,3}}.$$

The asymptotic expressions for $\hat{\hat{u}}$ can again be used to describe the dromion $d(1,3)$ as $t \to +\infty$. When $h_2 \to r_2/(p_2-q_2)$, equation (5.8) gives

$$d(1,3) \sim \frac{4\hat{r}_1(1,3)^-\hat{r}_3(1,3)^+(p_1-q_1)(p_3-q_3)(1-\alpha_{1,3}^+)}{\left(e^{-\frac{1}{2}(\Lambda_1+\Lambda_3)} + \kappa_1(1,3)^+e^{\frac{1}{2}(\Lambda_1-\Lambda_3)} + \kappa_3(1,3)^+e^{\frac{1}{2}(\Lambda_3-\Lambda_1)} + \kappa_{1,3}^+e^{\frac{1}{2}(\Lambda_1+\Lambda_3)}\right)^2}, \quad (5.16)$$

where

$$\alpha_{1,3}^+ = Tr(\hat{P}_1(1,3)^-\hat{P}_3(1,3)^+), \quad \kappa_1(1,3)^+ = \frac{\hat{r}_1(1,3)^-}{p_1-q_1}, \quad \kappa_3(1,3)^+ = \frac{\hat{r}_3(1,3)^+}{p_3-q_3} \quad \text{and}$$

$$\kappa_{1,3}^+ = \kappa_1(1,3)^+\kappa_3(1,3)^+\beta_{1,3}^+.$$

5.1.2 Summary of the three-dromion structure

The asymptotic expressions (5.10), (5.12), (5.13), (5.14), (5.15) and (5.16) all have the same form as the dromion structure given by (5.3). Therefore, we have shown that the three-dromion structure $\det(u) = \det(v_{[4],x})$ decomposes asymptotically into six dromions:

$$d(i,j) \sim \begin{cases} \dfrac{4r_i(i,j)^-r_j(i,j)^+(p_i-q_i)(p_j-q_j)(1-\alpha_{i,j}^-)}{\left(e^{-\frac{1}{2}(\Lambda_i+\Lambda_j)}+\kappa_i(i,j)^-e^{\frac{1}{2}(\Lambda_i-\Lambda_j)}+\kappa_j(i,j)^-e^{\frac{1}{2}(\Lambda_j-\Lambda_i)}+\kappa_{i,j}^-e^{\frac{1}{2}(\Lambda_i+\Lambda_j)}\right)^2}, & t \to -\infty, \\ \dfrac{4\hat{r}_i(i,j)^-\hat{r}_j(i,j)^+(p_i-q_i)(p_j-q_j)(1-\alpha_{i,j}^+)}{\left(e^{-\frac{1}{2}(\Lambda_i+\Lambda_j)}+\kappa_i(i,j)^+e^{\frac{1}{2}(\Lambda_i-\Lambda_j)}+\kappa_j(i,j)^+e^{\frac{1}{2}(\Lambda_j-\Lambda_i)}+\kappa_{i,j}^+e^{\frac{1}{2}(\Lambda_i+\Lambda_j)}\right)^2}, & t \to +\infty, \end{cases}$$

for $i,j \in \{1,2,3\}$ and $i \neq j$, giving the following generalisation of Theorem 2:

Theorem 3. *If* $\det(\Omega)$ *is positive-definite, then* $\det(u)$*, as given by* (5.8)*, has the following properties:*

1. $\det(u)$ *decomposes asymptotically into six dromions as described in Theorem 2. Each* $d(i,j)$ *decays to zero exponentially as* $(x,y) \to \infty$ *in any direction.*

2. *The amplitude of* $d(i,j)$ *is*

$$\mathcal{A}^- := \frac{(1-\alpha_{i,j}^-)(p_i - q_i)^2(p_j - q_j)^2}{\left(\sqrt{\beta_{i,j}^-}+1\right)^2} \quad as \quad t \to -\infty,$$
$$\mathcal{A}^+ := \frac{(1-\alpha_{i,j}^+)(p_i - q_i)^2(p_j - q_j)^2}{\left(\sqrt{\beta_{i,j}^+}+1\right)^2} \quad as \quad t \to +\infty.$$

The amplitude is

- *negative*

  (a) *as* $t \to -\infty$*: if* $\alpha_{i,j}^- > 1$,

  (b) *as* $t \to +\infty$*: if* $\alpha_{i,j}^+ > 1$,

- *zero*

  (a) *as* $t \to -\infty$*: if* $\alpha_{i,j}^- = 1$,

  (b) *as* $t \to +\infty$*: if* $\alpha_{i,j}^+ = 1$,

- *positive*

  (a) *as* $t \to -\infty$*: if* $\alpha_{i,j}^- < 1$,

  (b) *as* $t \to +\infty$*: if* $\alpha_{i,j}^+ < 1$,

3. *At time* t *the location of* $d(i,j)$ *moves from*

$$(x,y) = \frac{1}{2l_{i,j}}\Big(l_j^{(2)}(\xi_i(i,j)^- + \xi_i(i,j)^+) - l_i^{(2)}(\xi_j(i,j)^- + \xi_j(i,j)^+) + 8l_{j,k}t\ ,$$
$$l_i^{(1)}(\xi_j(i,j)^- + \xi_j(i,j)^+) - l_j^{(1)}(\xi_i(i,j)^- + \xi_i(i,j)^+) + 8l_{i,k}t\Big),$$

as $t \to -\infty$ *to*

$$(x,y) = \frac{1}{2l_{i,j}}\Big(l_j^{(2)}(\hat{\xi}_i(i,j)^- + \hat{\xi}_i(i,j)^+) - l_i^{(2)}(\hat{\xi}_j(i,j)^- + \hat{\xi}_j(i,j)^+) + 8l_{j,k}t\ ,$$
$$l_i^{(1)}(\hat{\xi}_j(i,j)^- + \hat{\xi}_j(i,j)^+) - l_j^{(1)}(\hat{\xi}_i(i,j)^- + \hat{\xi}_i(i,j)^+) + 8l_{i,k}t\Big),$$

as $t \to +\infty$.

4. *The trajectory of* $d(i,j)$ *changes from*

$$y = \left(\frac{l_{i,k}}{l_{j,k}}\right) x + \frac{\left(l_i^{(1)} l_{j,k} + l_i^{(2)} l_{i,k}\right)\left(\xi_j(i,j)^- + \xi_j(i,j)^+\right) - \left(l_j^{(1)} l_{j,k} + l_j^{(2)} l_{i,k}\right)\left(\xi_i(i,j)^- + \xi_i(i,j)^+\right)}{2 l_{i,j} l_{j,k}},$$

as $t \to -\infty$ *to*

$$y = \left(\frac{l_{i,k}}{l_{j,k}}\right) x + \frac{\left(l_i^{(1)} l_{j,k} + l_1^{(2)} l_{i,k}\right)\left(\hat{\xi}_j(i,j)^- + \hat{\xi}_j(i,j)^+\right) - \left(l_j^{(1)} l_{j,k} + l_j^{(2)} l_{i,k}\right)\left(\hat{\xi}_i(i,j)^- + \hat{\xi}_i(i,j)^+\right)}{2 l_{i,j} l_{j,k}},$$

as $t \to +\infty$.

5.1.3 Plots of dromions

In this section, the interaction properties of the three-dromion structure are highlighted with various plots. It is interesting to see under what conditions each dromion vanishes before and after undergoing interaction. Let us label the elements of the 2-vectors as

$$\mu(i,j)^- = \begin{pmatrix} a^-_{ij,1} \\ a^-_{ij,2} \end{pmatrix}, \quad \nu(i,j)^- = \begin{pmatrix} b^-_{ij,1} \\ b^-_{ij,2} \end{pmatrix}, \quad \mu(i,j)^+ = \begin{pmatrix} a^+_{ij,1} \\ a^+_{ij,2} \end{pmatrix}, \quad \nu(i,j)^+ = \begin{pmatrix} b^+_{ij,1} \\ b^+_{ij,2} \end{pmatrix},$$

$$\hat{\mu}(i,j)^- = \begin{pmatrix} c^-_{ij,1} \\ c^-_{ij,2} \end{pmatrix}, \quad \hat{\nu}(i,j)^- = \begin{pmatrix} d^-_{ij,1} \\ d^-_{ij,2} \end{pmatrix}, \quad \hat{\mu}(i,j)^+ = \begin{pmatrix} c^+_{ij,1} \\ c^+_{ij,2} \end{pmatrix}, \quad \hat{\nu}(i,j)^+ = \begin{pmatrix} d^+_{ij,1} \\ d^+_{ij,2} \end{pmatrix},$$

for $j = 1,2,3$ and $i \neq j$.

As $t \to -\infty$, $d(i,j)$ vanishes

$$\begin{aligned} &\Leftrightarrow Tr(P_i(i,j)^- P_j(i,j)^+) = 1 \\ &\Leftrightarrow (\mu(i,j)^-, \nu(i,j)^+)(\mu(i,j)^+, \nu(i,j)^-) = (\mu(i,j)^-, \nu(i,j)^-)(\mu(i,j)^+, \nu(i,j)^+) \\ &\Leftrightarrow a^-_{ij,2} a^+_{ij,1} = a^-_{ij,1} a^+_{ij,2} \text{ or } b^-_{ij,2} b^+_{ij,1} = b^-_{ij,1} b^+_{ij,2}. \end{aligned}$$

As $t \to +\infty$, $d(i,j)$ vanishes

$$\begin{aligned} &\Leftrightarrow Tr(\hat{P}_i(i,j)^- \hat{P}_j(i,j)^+) = 1 \\ &\Leftrightarrow (\hat{\mu}(i,j)^-, \hat{\nu}(i,j)^+)(\hat{\mu}(i,j)^+, \hat{\nu}(i,j)^-) = (\hat{\mu}(i,j)^-, \hat{\nu}(i,j)^-)(\hat{\mu}(i,j)^+, \hat{\nu}(i,j)^+) \\ &\Leftrightarrow c^-_{ij,2} c^+_{ij,1} = c^-_{ij,1} c^+_{ij,2} \text{ or } d^-_{ij,2} d^+_{ij,1} = d^-_{ij,1} d^+_{ij,2}. \end{aligned}$$

There is enough freedom in the parameters of the three-dromion structure to set up a situation where there are three dromions where one or two dromions vanish as $t \to \pm\infty$.

Figure 5.4 shows a plot of the three-dromion structure with

$$\mu_1^T = \begin{pmatrix} 1 & -\frac{1}{2} \end{pmatrix}, \quad \mu_2^T = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \end{pmatrix}, \quad \mu_3^T = \begin{pmatrix} 1 & -1 \end{pmatrix},$$
$$\nu_1^T = \begin{pmatrix} -\frac{3}{7} & \frac{1}{2} \end{pmatrix}, \quad \nu_2^T = \begin{pmatrix} -1 & -1 \end{pmatrix} \quad \text{and} \quad \nu_3^T = \begin{pmatrix} \frac{5}{8} & \frac{7}{8} \end{pmatrix},$$

so that no dromions vanish as $t \to \pm\infty$.

Figure 5.5 shows a plot of the three-dromion structure with

$$\mu_1^T = \begin{pmatrix} 1 & 3 \end{pmatrix}, \quad \mu_2^T = \begin{pmatrix} 1 & 3 \end{pmatrix}, \quad \mu_3^T = \begin{pmatrix} -\frac{1}{4} & 6 \end{pmatrix},$$
$$\nu_1^T = \begin{pmatrix} 1 & -3 \end{pmatrix}, \quad \nu_2^T = \begin{pmatrix} 5 & -2 \end{pmatrix} \quad \text{and} \quad \nu_3^T = \begin{pmatrix} -\frac{5}{2} & 1 \end{pmatrix},$$

so that $d(1,2)$ and $d(2,3)$ vanish as $t \to +\infty$.

Figure 5.6 shows a plot of the three-dromion structure with

$$\mu_1^T = \begin{pmatrix} 1 & -\frac{1}{2} \end{pmatrix}, \quad \mu_2^T = \begin{pmatrix} \frac{1}{2} & \frac{3}{4} \end{pmatrix}, \quad \mu_3^T = \begin{pmatrix} 1 & -\frac{1}{2} \end{pmatrix},$$
$$\nu_1^T = \begin{pmatrix} -\frac{3}{7} & \frac{1}{2} \end{pmatrix}, \quad \nu_2^T = \begin{pmatrix} -1 & -1 \end{pmatrix} \quad \text{and} \quad \nu_3^T = \begin{pmatrix} \frac{5}{8} & \frac{7}{8} \end{pmatrix},$$

so that $d(1,3)$ vanishes as $t \to -\infty$. Note that this structure has a dromion with negative amplitude.

Figure 5.7 shows the details of the interaction in Figure 5.6.

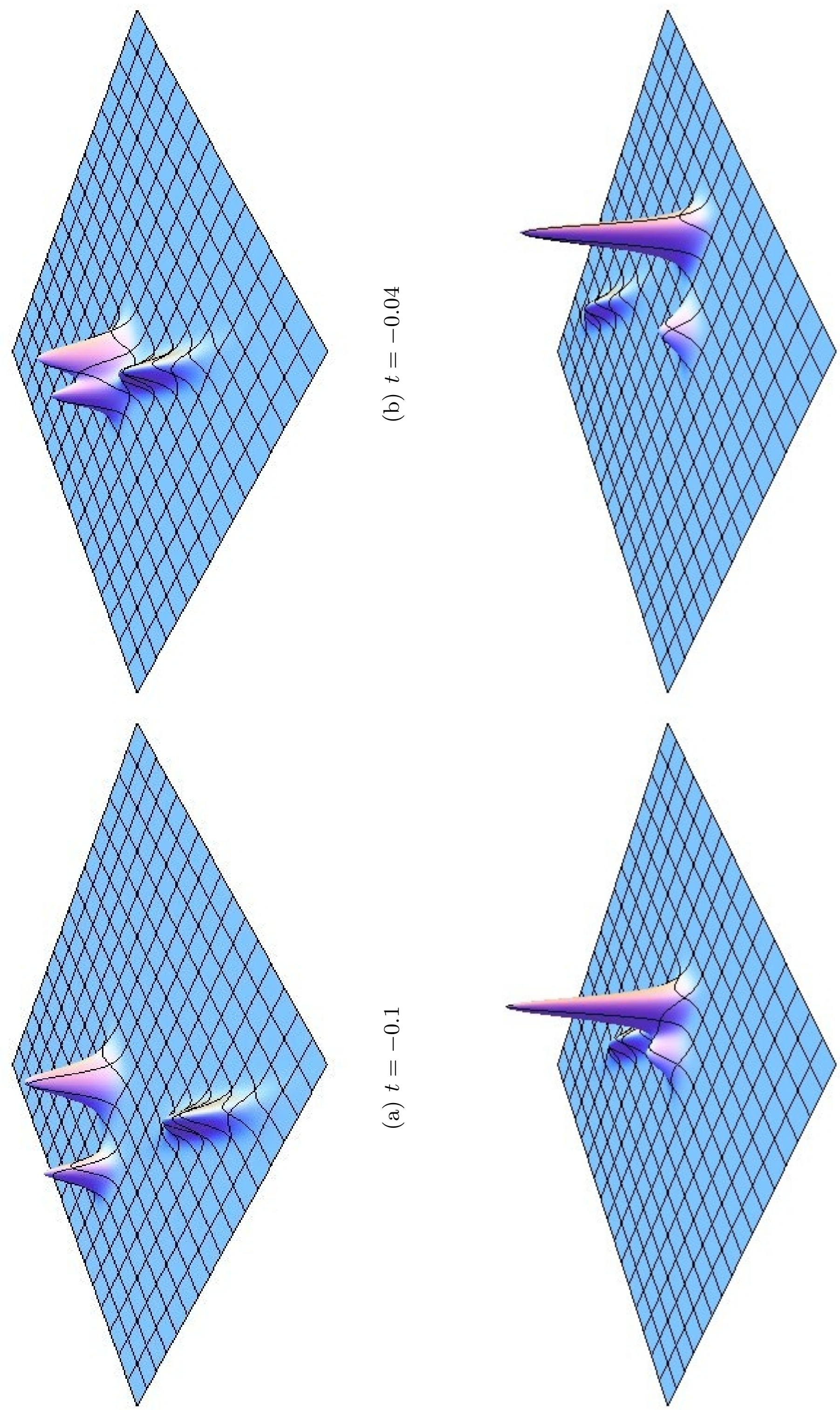


Figure 5.4: Plot of the three-dromion structure with parameters $p_1 = -2$, $p_2 = 2$, $p_3 = 4$, $q_1 = -3$, $q_2 = 1$, $q_3 = 3$, $r_1 = 2$ and $r_2 = 1 = r_3$.

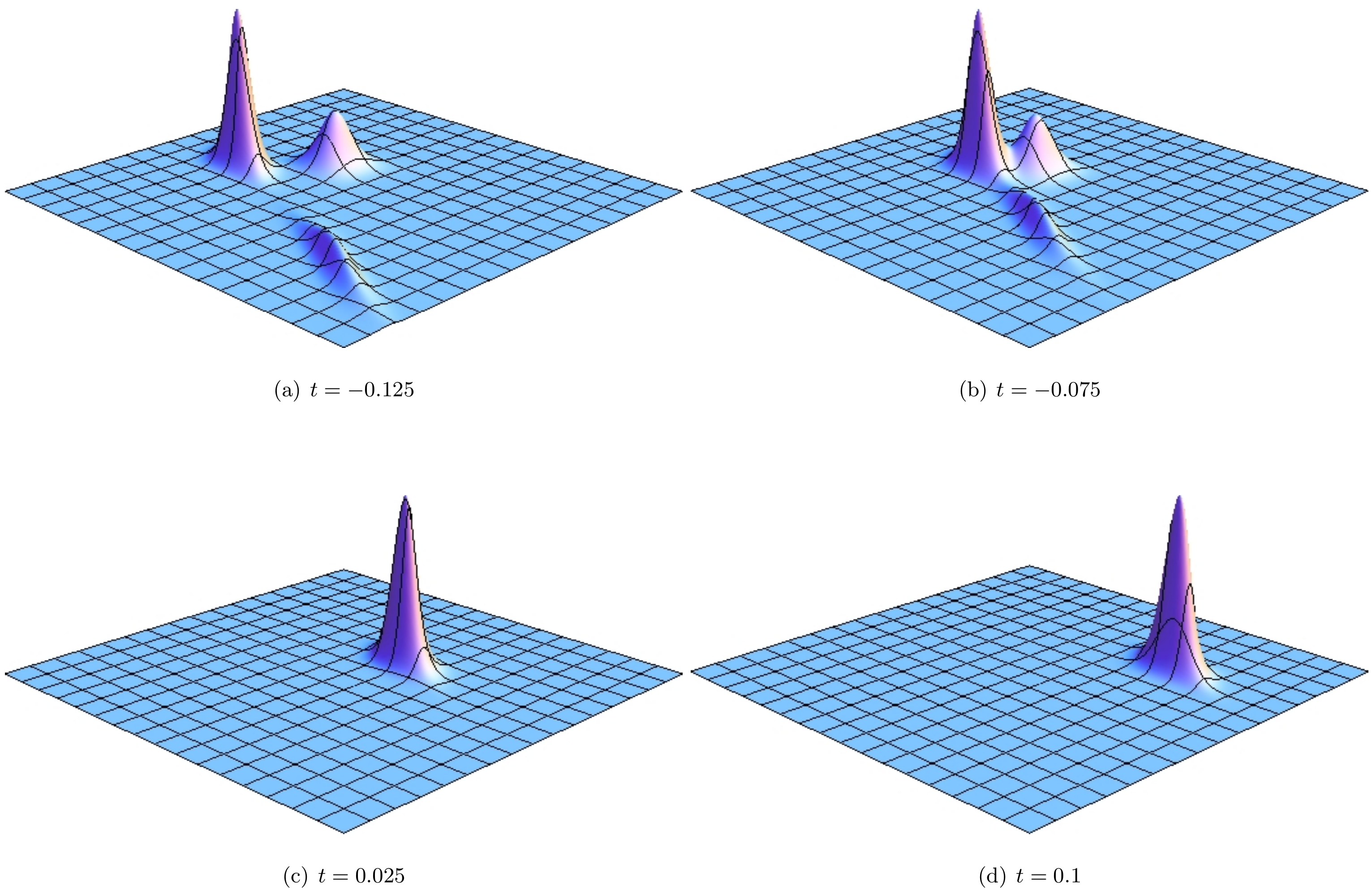
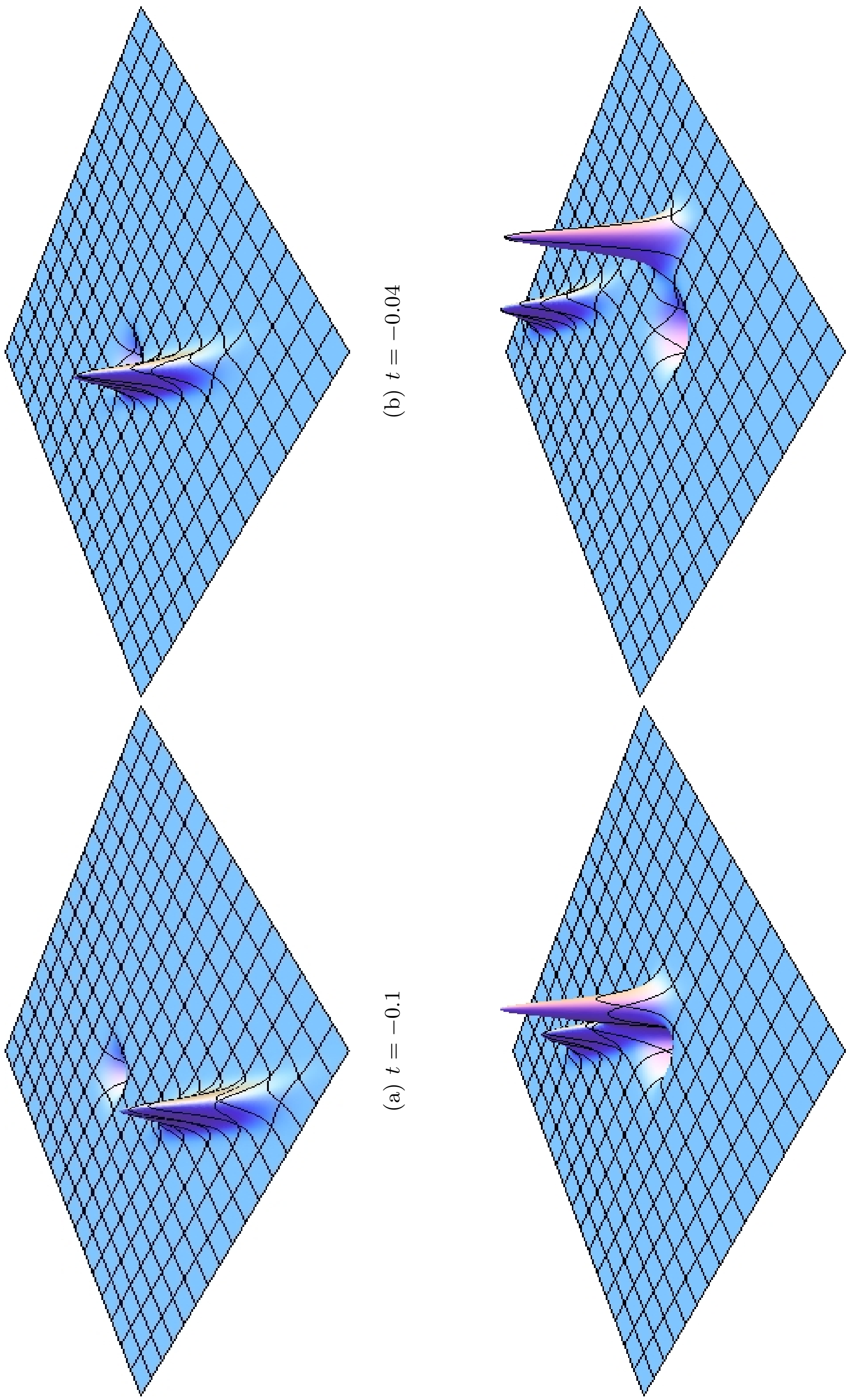


Figure 5.5: Plot of the three-dromion structure with parameters $p_1 = -1$, $p_2 = 2$, $p_3 = 4$, $q_1 = -2$, $q_2 = 1$, $q_3 = 3$ and $r_1 = r_2 = r_3 = 1$.



(a) $t = -0.1$

(b) $t = -0.04$

(c) $t = 0.04$

(d) $t = 0.1$

Figure 5.6: Plot of the three-dromion structure with parameters $p_1 = -2$, $p_2 = 2$, $p_3 = 4$, $q_1 = -3$, $q_2 = 1$, $q_3 = 3$, $r_1 = 2$ and $r_2 = 1 = r_3$.

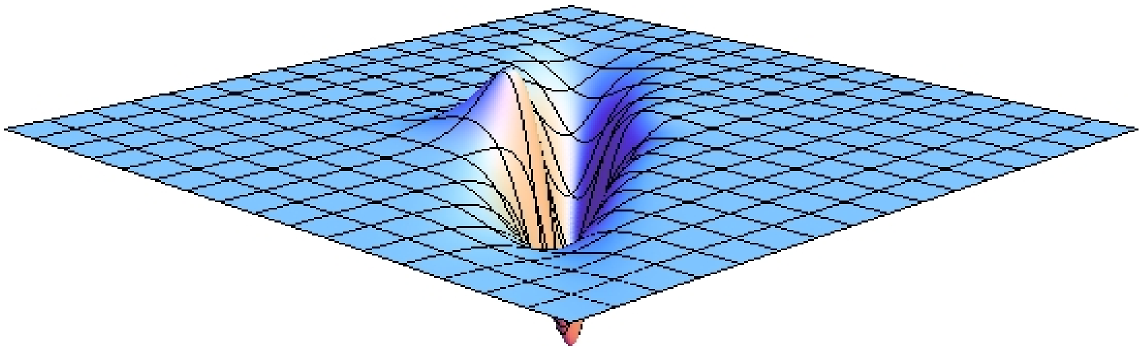


Figure 5.7: Details of the interaction shown in Figure 5.6

.

5.2 Matrix mKP single dromion

As we were able to find dromions of the matrix version of the ncKP equation, the matrix mKP equation should also possess dromions. Again, the simplest case is the single dromion which appears from the two-soliton matrix solution obtained when $n = 2$. Most of the results obtained in this section bear close resemblance to those of the matrix KP solutions.

Recall from Chapter 4 that the two-soliton matrix solution of ncmKP can be written in terms of

$$f_{[3]} = I + \frac{1}{q_1}L_1 + \frac{1}{q_2}L_2,$$

in which

$$L_1 = \frac{(p_2 - q_1)q_1}{h}((p_1 - q_2)q_2h_2I + p_1A_2)A_1,$$
$$L_2 = \frac{(p_1 - q_2)q_2}{h}((p_2 - q_1)q_1h_1I + p_2A_1)A_2,$$

and

$$h = h_1h_2q_1q_2(p_1 - q_2)(p_2 - q_1) - \alpha p_1p_2r_1r_2$$
$$h_i = e^{-\Lambda_i} - \frac{p_ir_i}{(p_i - q_i)q_i} \quad \text{and} \quad \alpha = \frac{(\mu_1, \nu_2)(\mu_2, \nu_1)}{(\mu_1, \nu_1)(\mu_2, \nu_2)} = Tr(P_1P_2).$$

Expanding $\det((f_{[3]})_x)$ and $\det(f_{[3]})$ and using the fact that the determinant of a projection matrix is zero and its trace is equal to its rank, we get

$$\det((f_{[3]})_x) = q_1q_2r_1r_2(p_1 - q_2)^2(p_2 - q_1)^2(1 - \alpha)h_{1,x}h_{2,x}h^{-2} \quad \text{and}$$
$$\det(f_{[3]}) = 1 + h^{-1}\left(h_1(p_1 - q_2)(p_2 - q_1)q_1r_2 + h_2(p_1 - q_2)(p_2 - q_1)q_2r_1 \right.$$
$$\left. +(p_1 - q_2)(p_2 - q_1)r_1r_2 + r_1r_2\alpha(p_1p_2 - q_1q_2)\right).$$

We may now calculate the determinant of w, which is

$$\det(w) = \frac{\det((f_{[3]})_x)}{\det(f_{[3]})} = \frac{r_1 r_2 (p_1 - q_1)(p_2 - q_2)(1-\alpha)}{q_1 q_2 S_\kappa S_\iota}, \tag{5.17}$$

where

$$S_\kappa = \left(e^{-\frac{1}{2}(\Lambda_1+\Lambda_2)} - \kappa_1 e^{\frac{1}{2}(\Lambda_1\Lambda_2)} - \kappa_2 e^{\frac{1}{2}(\Lambda_2-\Lambda_1)} + \kappa e^{\frac{1}{2}(\Lambda_1+\Lambda_2)}\right),$$
$$S_\iota = \left(e^{-\frac{1}{2}(\Lambda_1+\Lambda_2)} - \iota_1 e^{\frac{1}{2}(\Lambda_1\Lambda_2)} - \iota_2 e^{\frac{1}{2}(\Lambda_2-\Lambda_1)} + \iota e^{\frac{1}{2}(\Lambda_1+\Lambda_2)}\right),$$

$\iota_i = \frac{p_i}{q_i}\kappa_i$, $i = 1, 2$, $\kappa = \kappa_1\kappa_2\beta$ and $\iota = \iota_1\iota_2\beta$.

The characteristics of $\det(w)$, as given by equation (5.17), may be summarised by the following theorem:

Theorem 4. *If* $\det(\Omega)$ *is positive-definite and if* $\alpha \neq 1$, *then* $\det(w)$ *has the following properties:*

1. $\det(w)$ *decays to zero exponentially as* $(x, y) \to \infty$ *in any direction and has a unique maximum or minimum value*

$$\det(w)_{max/min} = \frac{(1-\alpha)p_1 p_2 (p_1 - q_1)^2 (p_2 - q_2)^2}{\left(\sqrt{\beta}\left(\sqrt{\frac{p_1 p_2}{q_1 q_2}} + \frac{p_1 p_2}{q_1 q_2}\right) + \frac{p_1}{q_1}\sqrt{\frac{p_2}{q_2}} + \frac{p_2}{q_2}\sqrt{\frac{p_1}{q_1}}\right)^2 q_1^2 q_2^2}.$$

The dromion will have negative, zero or positive amplitude. The amplitude is

- *negative if* $\alpha > 1$,
- *zero if* $\alpha = 1$,
- *positive if* $\alpha < 1$.

2. *At time* t *this maximum or minimum as located at*

$$(x, y) = \frac{-1}{2l_{1,2}}\left(l_2^{(2)}(\xi_1^- + \xi_1^+) - l_1^{(2)}(\xi_2^- + \xi_2^+) + 8l_{2,3}t\,,\right.$$
$$\left. l_1^{(1)}(\xi_2^- + \xi_2^+) - l_2^{(1)}(\xi_1^- + \xi_1^+) + 8l_{1,3}t\right), \tag{5.18}$$

where $l_{i,j} = l_i^{(i)} l_j^{(j)} - l_i^{(j)} l_j^{(i)}$ *and* $l_i^{(j)} = q_i^j - p_i^j$. *This result implies that the dromion is located symmetrically between the solitons in the two-soliton matrix solution.*

3. *The trajectory of the dromion is the straight line*

$$y = \left(\frac{l_{1,3}}{l_{2,3}}\right)x + \frac{\left(l_2^{(1)} l_{2,3} + l_2^{(2)} l_{1,3}\right)(\xi_1^- + \xi_1^+) - \left(l_1^{(1)} l_{2,3} + l_1^{(2)} l_{1,3}\right)(\xi_2^- + \xi_2^+)}{2l_{1,2}l_{2,3}}. \tag{5.19}$$

Proof. From (5.17), we see that $\det(w)$ decays to zero exponentially as $(x, y) \to \infty$ in any direction since, along any ray in the (x, y)-plane, at least one of the exponentials in the denominator is unbounded as (x, y) approaches infinity. To see this, we use the same technique as for the matrix KP single dromion. Let $y = kx$, where $k \in \mathbb{R}$, be a ray in any direction. Substituting this into (5.17) gives

$$\begin{aligned} S_\kappa = {} & e^{-\frac{1}{2}\left(\left(l_1^{(1)}+l_2^{(1)}+\left(l_1^{(2)}+l_2^{(2)}\right)k\right)x-4\left(l_1^{(3)}+l_2^{(3)}\right)t\right)} - \kappa_1 e^{\frac{1}{2}\left(\left(l_1^{(1)}-l_2^{(1)}+\left(l_1^{(2)}-l_2^{(2)}\right)k\right)x-4\left(l_1^{(3)}-l_2^{(3)}\right)t\right)} \\ & - \kappa_2 e^{\frac{1}{2}\left(\left(l_2^{(1)}-l_1^{(1)}+\left(l_2^{(2)}-l_1^{(2)}\right)k\right)x-4\left(l_2^{(3)}-l_1^{(3)}\right)t\right)} + \kappa e^{\frac{1}{2}\left(\left(l_1^{(1)}+l_2^{(1)}+\left(l_1^{(2)}+l_2^{(2)}\right)k\right)x-4\left(l_1^{(3)}+l_2^{(3)}\right)t\right)}, \\ S_\iota = {} & e^{-\frac{1}{2}\left(\left(l_1^{(1)}+l_2^{(1)}+\left(l_1^{(2)}+l_2^{(2)}\right)k\right)x-4\left(l_1^{(3)}+l_2^{(3)}\right)t\right)} - \iota_1 e^{\frac{1}{2}\left(\left(l_1^{(1)}-l_2^{(1)}+\left(l_1^{(2)}-l_2^{(2)}\right)k\right)x-4\left(l_1^{(3)}-l_2^{(3)}\right)t\right)} \\ & - \iota_2 e^{\frac{1}{2}\left(\left(l_2^{(1)}-l_1^{(1)}+\left(l_2^{(2)}-l_1^{(2)}\right)k\right)x-4\left(l_2^{(3)}-l_1^{(3)}\right)t\right)} + \iota e^{\frac{1}{2}\left(\left(l_1^{(1)}+l_2^{(1)}+\left(l_1^{(2)}+l_2^{(2)}\right)k\right)x-4\left(l_1^{(3)}+l_2^{(3)}\right)t\right)} \end{aligned}$$

on the denominator. This expression must tend to infinity for any values of k and $l_i^{(j)}$, $i, j = 1, 2$ as $x \to \pm\infty$.

Since $\det(w)$ is exponentially localized, a unique critical point must be either a maximum or minimum. If we consider the conditions that $\alpha \neq 1$ and $(\det(w))_x$ and $\det((w))_y$ vanish simultaneously, we get

$$X^2Y^2 - \kappa_1\iota_1 Y^2 + \kappa_2\iota^2 X^2 - \kappa_1\kappa_2\iota_1\iota_2\beta^2 - (\kappa_2 + \iota_2)X^2Y + (\kappa_2 + \iota_2)\kappa_1\iota_1\beta Y = 0$$

and

$$X^2Y^2 + \kappa_1\iota_1 Y^2 - \kappa_2\iota^2 X^2 - \kappa_1\kappa_2\iota_1\iota_2\beta^2 - (\kappa_1 + \iota_1)XY^2 + (\kappa_1 + \iota_1)\kappa_2\iota_2\beta X = 0,$$

in which

$$X = e^{-\Lambda_1} \quad \text{and} \quad Y = e^{-\Lambda_2}.$$

Solving this equation for X and Y gives only one pair of positive roots which is

$$e^{-\Lambda_1} = \sqrt{\kappa_1\iota_1\beta} \quad \text{and} \quad e^{-\Lambda_2} = \sqrt{\kappa_2\iota_2\beta}. \tag{5.20}$$

Substituting (5.20) into (5.17) gives the maximum or minimum of $\det(w)$.

Solving (5.20) for x and y gives (5.18), the location of the dromion. Eliminating t in (5.18) gives the trajectory of the dromion.

5.2.1 A three-dromion example

For the matrix mKP solution, the determinant of the three-soliton matrix solution again gives a three-dromion structure. The schematic form of the dromion scattering will be in accordance with the dromions of the matrix KP solutions as illustrated in Figure 5.3.

When $n = 3$, expanding $\det((f_{[4]})_x)$ and $\det(f_{[4]})$, the expressions simplify greatly and we obtain

$$\det((f_{[4]})_x) = \frac{h_{1,x}h_{2,x}m_{1,2,3} + h_{1,x}h_{3,x}m_{1,3,2} + h_{2,x}h_{3,x}m_{2,3,1}}{h^2(1,2,3)}, \tag{5.21}$$

$$\begin{aligned}\det(f_{[4]}) = 1 + &\frac{1}{q_1q_2q_3h(1,2,3)}\Big(r_1r_2r_3\mathcal{M} + q_1h_1\varrho_{2,3} + q_2h_2\varrho_{1,3} + q_3h_3\varrho_{1,2} \\ &+ \frac{r_3h(1,2)}{(p_1-q_2)(p_2-q_1)} + \frac{r_2h(1,3)}{(p_1-q_3)(p_3-q_1)} + \frac{r_1h(2,3)}{(p_2-q_3)(p_3-q_3)}\Big),\end{aligned} \tag{5.22}$$

where

$$\begin{aligned}
\varrho_{i,j} &= r_ir_j\left(1 - \frac{\alpha_{i,j}(q_iq_j - p_ip_j)}{(p_i-q_j)(p_j-q_i)}\right),\\
\mathcal{M} &= \alpha_{1,2,3}\frac{p_1(p_3q_1 + (p_2-q_1)q_2) + q_3(p_2(p_3-q_2) - p_3q_1)}{(p_1-q_3)(p_3-q_2)(p_2-q_1)}\\
&\quad \alpha_{1,3,2}\frac{q_2((p_3-q_1)p_2 - p_3q_3) + p_1(p_2q_1 + q_3(p_3-q_1))}{(p_1-q_2)(p_2-q_3)(p_3-q_1)}\\
&\quad + \alpha_{1,2}\left(\frac{p_2(p_1-q_2) + p_1(p_2-q_1)}{(p_1-q_2)(p_2-q_1)}\right) + \alpha_{1,3}\left(\frac{p_3(p_1-q_3) + p_1(p_3-q_1)}{(p_1-q_3)(p_3-q_1)}\right)\\
&\quad + \alpha_{2,3}\left(\frac{p_3(p_2-q_3) + p_2(p_3-q_2)}{(p_2-q_3)(p_3-q_2)}\right),\\
m_{i,j,k} &= q_iq_jr_ir_j\Bigg(q_k^2h_k^2(1-\alpha_{i,j}) + h_kq_kr_k\Bigg(-\frac{\alpha_{i,j,k}(p_k\ell_{i,k} + p_i\ell_{k,j})}{\ell_{i,k}\ell_{k,j}}\\
&\quad -\frac{\alpha_{i,k,j}(p_j\ell_{j,k} + p_k\ell_{k,i})}{\ell_{j,k}\ell_{k,i}} + \frac{\alpha_{j,k}(p_k\ell_{j,k} + p_j\ell_{k,j})}{\ell_{j,k}\ell_{k,j}} + \frac{\alpha_{i,k}(p_k\ell_{i,k} + p_i\ell_{k,i})}{\ell_{i,k}\ell_{k,i}}\Bigg)\\
&\quad -p_kr_k^2\left(-\frac{p_i\alpha_{i,j,k}}{\ell_{i,j}\ell_{k,j}} - \frac{p_j\alpha_{i,k,j}}{\ell_{j,k}\ell_{k,i}} + \frac{\alpha_{i,k}\alpha_{j,k}q_k(p_i-p_j)(q_i-q_j)}{\ell_{i,k}\ell_{k,i}\ell_{j,k}\ell_{k,j}} + \frac{p_j\alpha_{j,k}}{\ell_{j,k}\ell_{k,j}} + \frac{p_i\alpha_{i,k}}{\ell_{i,k}\ell_{k,i}}\right)\Bigg),
\end{aligned}$$

for $i, j, k \in \{1, 2, 3\}$ and $i \neq j \neq k$. As in the previous section, we have $\ell_{i,j} = p_i - q_j$.

To investigate the behaviour of each dromion as $t \to \pm\infty$, we fix attention on the dromion arising from the interaction of the ith and jth solitons, which will again be termed $d(i,j)$. Furthermore, we will call the corresponding two-soliton interaction matrix variable $f_{i,j}$. We consider $\det(w) = \det(-(f_{[4]})_x)/\det(f_{[4]})$ as given by (5.21) and (5.22) in a frame moving with the (i,j)th dromion by rewriting it in terms of

$$x = \hat{x} + 4\left(\frac{(q_j^3-p_j^3)(q_i^2-p_i^2) - (q_i^3-p_i^3)(q_j^2-p_j^2)}{(q_j-p_j)(q_i^2-p_i^2) - (q_j^2-p_j^2)(q_i-p_i)}\right)t,$$
$$y = \hat{y} + 4\left(\frac{(q_i^3-p_i^3)(q_j-p_j) - (q_j^3-p_j^3)(q_i-p_i)}{(q_j-p_j)(q_i^2-p_i^2) - (q_j^2-p_j^2)(q_i-p_i)}\right)t,$$

from which we obtain

$$\Lambda_i = (q_i-p_i)\hat{x} + (q_i^2-p_i^2)\hat{y}, \quad \Lambda_j = (q_j-p_j)\hat{x} + (q_j^2-p_j^2)\hat{y},$$

for $i, j = 1, 2, 3$.

In accordance with the three-soliton matrix solution, we will assume, without loss of generality, that $0 > p_3 > q_3 > p_2 > q_2 > p_1 > q_1$. Let us begin by fixing $d(1,2)$. With solitons 1 and 2 fixed, h_1 and h_2 are fixed and we study the asymptotic behaviour of h_3 as $t \to \pm\infty$. We have that

$$h_3 \to \begin{cases} -\frac{p_3 r_3}{q_3(p_3-q_3)}, & t \to -\infty, \\ +\infty, & t \to +\infty. \end{cases}$$

When $h_3 \to \frac{-p_3 r_3}{q_3(p_3-q_3)}$, equation (4.54) gives

$$\begin{aligned} f_{1,2} = & -\Big(b_{1,2}A_3 + b_{1,2,3}A_1A_3 + b_{3,2,1}A_3A_1 + b_{2,1,3}A_2A_3 + b_{3,1,2}A_3A_2 \\ & - \left(\frac{p_3q_2r_3h_2}{p_3-q_3} + \frac{p_2p_3r_2r_3\alpha_{2,3}}{(p_2-q_3)(p_3-q_2)}\right)A_1 - \left(\frac{p_3q_1r_3h_1}{p_3-q_3} + \frac{p_1p_3r_1r_3\alpha_{1,3}}{(p_1-q_3)(p_3-q_1)}\right)A_2 \\ & + \left(\frac{p_2p_3r_3\alpha_{1,3,2}}{(p_2-q_3)(p_3-q_1)\alpha_{1,2}} - \frac{p_2p_3r_3}{(p_2-q_1)(p_3-q_3)}\right)A_1A_2 \\ & + \left(\frac{p_1p_3r_3\alpha_{1,2,3}}{(p_1-q_3)(p_3-q_2)\alpha_{1,2}} - \frac{p_1p_3r_3}{(p_1-q_2)(p_3-q_3)}\right)A_2A_1\Big) \frac{(p_3-q_3)}{\tilde{h}(1,2)p_3q_1q_2r_3} + I, \end{aligned} \tag{5.23}$$

where

$$\begin{aligned} \tilde{h}(1,2) = & \Bigg(h_1h_2 - \frac{p_2r_2\alpha_{2,3}(p_3-q_3)h_1}{q_2(p_2-q_3)(p_3-q_2)} - \frac{p_1r_1\alpha_{1,3}(p_3-q_3)h_2}{q_1(p_1-q_3)(p_3-q_1)} - \frac{p_1p_2r_1r_2\alpha_{1,2}}{q_1q_2(p_1-q_2)(p_2-q_1)} \\ & + \frac{p_1p_2r_1r_2(p_3-q_3)}{q_1q_2}\left(\frac{\alpha_{1,2,3}}{(p_2-q_1)(p_1-q_3)(p_3-q_2)} + \frac{\alpha_{1,3,2}}{(p_1-q_2)(p_2-q_3)(p_3-q_1)}\right)\Bigg). \end{aligned}$$

To obtain the characteristics of the dromion $d(1,2)$, we find the asymptotic forms of $\tilde{w} := (-f_{1,2})_x f_{1,2}^{-1}$. Firstly, let us fix Λ_1. Since $\tilde{w}$ is invariant under the transformation $f_{1,2} \to f_{1,2}C$, where C is a constant matrix, we have

$$f_{1,2}^- = I + \frac{\frac{r_1(1,2)^-}{q_1}P_1(1,2)^-}{e^{-\Lambda_1} + \frac{p_1r_1(1,2)^-}{q_1(p_1-q_1)}} \quad \text{as} \quad t \to -\infty,$$

$$f_{1,2}^+ = I + \frac{\frac{r_1(1,2)^+}{q_1}P_1(1,2)^+}{e^{-\Lambda_1} + \frac{p_1r_1(1,2)^+}{q_1(p_1-q_1)}} \quad \text{as} \quad t \to +\infty,$$

where

$$P_1(1,2)^- = \frac{\mu_1(1,2)^-\nu_1(1,2)^{-T}}{(\mu_1(1,2)^-,\nu_1(1,2)^-)}, \quad P_1(1,2)^+ = \frac{\mu_1(1,2)^+\nu_1(1,2)^{+T}}{(\mu_1(1,2)^+,\nu_1(1,2)^+)},$$

$$\mu_1(1,2)^- = \mu_1 - \frac{p_1(p_3-q_3)(\mu_1,\nu_3)\mu_3}{p_3(p_1-q_3)(\mu_3,\nu_3)}, \quad \nu_1(1,2)^- = \nu_1 - \frac{q_1(p_3-q_3)(\mu_3,\nu_1)\nu_3}{q_3(p_3-q_1)(\mu_3,\nu_3)}$$

$$\mu_1(1,2)^+ = \mu_1 + \frac{p_1(p_2-q_2)(p_3-q_3)}{p_2(\mu_2,\nu_2)\beta_{2,3}}\left(\frac{(\mu_1,\nu_3)(\mu_3,\nu_2)}{(\mu_3,\nu_3)(p_1-q_3)(p_3-q_2)} - \frac{(\mu_1,\nu_2)}{(p_1-q_2)(p_3-q_3)}\right)\mu_2$$
$$+ \frac{p_1(p_2-q_3)(p_3-q_2)}{p_3(\mu_3,\nu_3)\beta_{2,3}}\left(\frac{(\mu_1,\nu_2)(\mu_2,\nu_3)}{(\mu_2,\nu_2)(p_1-q_2)(p_2-q_3)} - \frac{(\mu_1,\nu_3)}{(p_1-q_3)(p_2-q_2)}\right)\mu_3,$$

$$\nu_1(1,2)^+ = \nu_1 + \frac{q_1(p_2-q_2)(p_3-q_3)}{q_2(\mu_2,\nu_2)\beta_{2,3}}\left(\frac{(\mu_3,\nu_1)(\mu_2,\nu_3)}{(\mu_3,\nu_3)(p_2-q_3)(p_3-q_1)} - \frac{(\mu_2,\nu_1)}{(p_2-q_1)(p_3-q_3)}\right)\nu_2$$
$$+ \frac{q_1(p_2-q_3)(p_3-q_2)}{q_3(\mu_3,\nu_3)\beta_{2,3}}\left(\frac{(\mu_2,\nu_1)(\mu_3,\nu_2)}{(\mu_2,\nu_2)(p_3-q_2)(p_2-q_1)} - \frac{(\mu_3,\nu_1)}{(p_3-q_1)(p_2-q_2)}\right)\nu_3,$$

$$r_1(1,2)^- = \frac{r_1(\mu_1(1,2)^-,\nu_1(1,2)^-)}{(\mu_1,\nu_1)} = r_1\beta_{1,3} \quad \text{and}$$

$$r_1(1,2)^+ = \frac{r_1(\mu_1(1,2)^+,\nu_1(1,2)^+)}{(\mu_1,\nu_1)} = r_1\frac{\beta_{1,2,3}}{\beta_{2,3}}.$$

These asymptotic expressions for $f_{1,2}$ are of the same form as the one-soliton matrix variable $f_{[2]}$ discussed in Chapter 4. So the asymptotic forms for $\tilde{w}$ are

$$\tilde{w} \sim \frac{(p_1-q_1)^2}{4(p_1q_1)^{\frac{1}{2}}}P_1(1,2)^-\,\mathrm{sech}\left(\frac{\Lambda_1+\varphi_1(1,2)^-}{2}\right)\mathrm{sech}\left(\frac{\Lambda_1+\chi_1(1,2)^-}{2}\right) \quad \text{as} \quad t\to-\infty,$$
$$\tilde{w} \sim \frac{(p_1-q_1)^2}{4(p_1q_1)^{\frac{1}{2}}}P_1(1,2)^+\,\mathrm{sech}\left(\frac{\Lambda_1+\varphi_1(1,2)^+}{2}\right)\mathrm{sech}\left(\frac{\Lambda_1+\chi_1(1,2)^+}{2}\right) \quad \text{as} \quad t\to+\infty,$$

in which $\varphi_1(1,2)^- = \log\left(-\frac{p_1r_1(1,2)^-}{q_1(p_1-q_1)}\right)$, $\varphi_1(1,2)^+ = \log\left(-\frac{p_1r_1(1,2)^+}{q_1(p_1-q_1)}\right)$, $\chi_1(1,2)^- = \log\left(-\frac{r_1(1,2)^-}{p_1-q_1}\right)$, $\chi_1(1,2)^+ = \log\left(-\frac{r_1(1,2)^+}{p_1-q_1}\right)$ and the phase-constants are:

$$\xi_1(1,2)^- = -\log\left(\frac{-(p_1q_1^{-1})^{\frac{1}{2}}r_1(1,2)^-}{p_1-q_1}\right) \quad \text{and} \quad \xi_1(1,2)^+ = -\log\left(\frac{-(p_1q_1^{-1})^{\frac{1}{2}}r_1(1,2)^+}{p_1-q_1}\right).$$

Next we fix Λ_2. Since $\tilde{w}$ is invariant under the transformation $f_{1,2}\to f_{1,2}C$, where C is a constant matrix, we have

$$f_{1,2}^- = I + \frac{\frac{r_2(1,2)^-}{q_2}P_2(1,2)^-}{e^{-\Lambda_2}+\frac{p_2r_2(1,2)^-}{q_2(p_2-q_2)}} \quad \text{as} \quad t\to-\infty$$
$$f_{1,2}^+ = I + \frac{\frac{r_2(1,2)^+}{q_2}P_2(1,2)^+}{e^{-\Lambda_2}+\frac{p_2r_2(1,2)^+}{q_2(p_2-q_2)}} \quad \text{as} \quad t\to+\infty,$$

in which

$$P_2(1,2)^- = \frac{\mu_2(1,2)^-\nu_2(1,2)^{-T}}{(\mu_2^-,\nu_2^-)} \quad P_2(1,2)^+ = \frac{\mu_2(1,2)^+\nu_2(1,2)^{+T}}{(\mu_2(1,2)^+,\nu_2(1,2)^+)},$$

$$\mu_2(1,2)^+ = \mu_2 - \frac{p_2(p_3-q_3)(\mu_2,\nu_3)\mu_3}{p_3(p_2-q_3)(\mu_3,\nu_3)}, \quad \nu_2(1,2)^+ = \nu_2 - \frac{q_2(p_3-q_3)(\mu_3,\nu_2)\nu_3}{q_3(p_3-q_2)(\mu_3,\nu_3)}$$

$$\begin{aligned}\mu_2(1,2)^- = \mu_2 &+ \frac{p_2\,(p_1-q_1)\,(p_3-q_3)}{p_1(\mu_1,\nu_1)\beta_{1,3}}\left(\frac{(\mu_2,\nu_3)(\mu_3,\nu_1)}{(\mu_3,\nu_3)(p_2-q_3)(p_3-q_1)} - \frac{(\mu_2,\nu_1)}{(p_2-q_1)(p_3-q_3)}\right)\mu_1 \\ &+ \frac{p_2(p_1-q_1)(p_3-q_3)}{p_3(\mu_3,\nu_3)\beta_{1,3}}\left(\frac{(\mu_2,\nu_1)(\mu_1,\nu_3)}{(\mu_1,\nu_1)(p_2-q_1)(p_1-q_3)} - \frac{(\mu_2,\nu_3)}{(p_2-q_3)(p_1-q_1)}\right)\mu_3,\end{aligned}$$

$$\begin{aligned}\nu_2(1,2)^- = \nu_2 &+ \frac{q_2(p_1-q_1)(p_3-q_3)}{q_1(\mu_1,\nu_1)\beta_{1,3}}\left(\frac{(\mu_3,\nu_2)(\mu_1,\nu_3)}{(\mu_3,\nu_3)(p_1-q_3)(p_3-q_2)} - \frac{(\mu_1,\nu_2)}{(p_1-q_2)(p_3-q_3)}\right)\nu_1 \\ &+ \frac{q_2(p_1-q_1)(p_3-q_3)}{q_3(\mu_3,\nu_3)\beta_{1,3}}\left(\frac{(\mu_1,\nu_2)(\mu_3,\nu_1)}{(\mu_1,\nu_1)(p_3-q_1)(p_1-q_2)} - \frac{(\mu_3,\nu_2)}{(p_3-q_2)(p_1-q_1)}\right)\nu_3,\end{aligned}$$

$$r_2(1,2)^- = \frac{r_2(\mu_2(1,2)^-,\nu_2(1,2)^-)}{(\mu_2,\nu_2)} = r_2\frac{\beta_{1,2,3}}{\beta_{1,3}} \quad \text{and}$$

$$r_2(1,2)^+ = \frac{r_2(\mu_2(1,2)^+,\nu_2(1,2^+)}{(\mu_2,\nu_2)} = r_2\beta_{2,3}.$$

So the asymptotic forms for $\tilde{w}$ are

$$\tilde{w} \sim \frac{(p_2-q_2)^2}{4(p_2q_2)^{\frac{1}{2}}}P_2(1,2)^-\operatorname{sech}\left(\frac{\Lambda_2+\varphi_2(1,2)^-}{2}\right)\operatorname{sech}\left(\frac{\Lambda_2+\chi_2(1,2)^-}{2}\right) \quad \text{as} \quad t\to-\infty,$$

$$\tilde{w} \sim \frac{(p_2-q_2)^2}{4(p_2q_2)^{-\frac{1}{2}}}P_2(1,2)^+\operatorname{sech}\left(\frac{\Lambda_2+\varphi_2(1,2)^+}{2}\right)\operatorname{sech}\left(\frac{\Lambda_2+\chi_2(1,2)^+}{2}\right) \quad \text{as} \quad t\to+\infty,$$

in which $\varphi_2(1,2)^- = \log\left(-\frac{p_2r_2(1,2)^-}{q_2(p_2-q_2)}\right)$, $\varphi_2(1,2)^+ = \log\left(-\frac{p_2r_2(1,2)^+}{q_2(p_2-q_2)}\right)$, $\chi_2(1,2)^- = \log\left(-\frac{r_2(1,2)^-}{p_2-q_2}\right)$, $\chi_2(1,2)^+ = \log\left(-\frac{r_2(1,2)^+}{p_2-q_2}\right)$ and the phase-constants are:

$$\xi_2(1,2)^- = -\log\left(\frac{-(p_2q_2^{-1})^{\frac{1}{2}}r_2(1,2)^-}{p_2-q_2}\right) \quad \text{and} \quad \xi_2(1,2)^+ = -\log\left(\frac{-(p_2q_2^{-1})^{\frac{1}{2}}r_2(1,2)^+}{p_2-q_2}\right).$$

Furthermore, the soliton phase-shifts $\Delta_j(1,2) = \xi_j(1,2)^+ - \xi_j(1,2)^-$, for $j=1,2$ are

$$\Delta_1(1,2) = -\log\left(\beta_{1,2}^-\right) \quad \text{and} \quad \Delta_2(1,2) = \log\left(\beta_{1,2}^-\right), \quad \text{in which} \quad \beta_{1,2}^- = \frac{\beta_{1,2,3}}{\beta_{1,3}\beta_{2,3}}.$$

The asymptotic expressions for $\tilde{w}$ can now be used to describe the dromion $d(1,2)$ as $t\to-\infty$. When $h_3 \to \frac{-p_3r_3}{q_3(p_3-q_3)}$, equation (5.21) and (5.22) give

$$d(1,2) \sim \frac{r_1(1,2)^-r_2(1,2)^+(p_1-q_1)(p_2-q_2)(1-\alpha_{1,2}^-)}{S_\kappa^-(1,2)S_\iota^-(1,2)}, \tag{5.24}$$

where

$$
\begin{aligned}
S^-_\kappa(1,2) &= e^{-\frac{1}{2}(\Lambda_1+\Lambda_2)} - \kappa_1(1,2)^- e^{\frac{1}{2}(\Lambda_1-\Lambda_2)} - \kappa_2(1,2)^- e^{\frac{1}{2}(\Lambda_2-\Lambda_1)} + \kappa^-_{1,2} e^{\frac{1}{2}(\Lambda_1+\Lambda_2)},\\
S^-_\iota(1,2) &= e^{-\frac{1}{2}(\Lambda_1+\Lambda_2)} - \iota_1(1,2)^- e^{\frac{1}{2}(\Lambda_1-\Lambda_2)} - \iota_2(1,2)^- e^{\frac{1}{2}(\Lambda_2-\Lambda_1)} + \iota^-_{1,2} e^{\frac{1}{2}(\Lambda_1+\Lambda_2)},\\
\alpha^-_{1,2} &= Tr(P_1(1,2)^- P_2(1,2)^+), \quad \kappa_1(1,2)^- = \frac{r_1(1,2)^-}{p_1-q_1}, \quad \kappa_2(1,2)^- = \frac{r_2(1,2)^+}{p_2-q_2},\\
\iota_1(1,2)^- &= \frac{p_1}{q_1}\kappa_1(1,2)^-, \quad \iota^-_2 = \frac{p_2}{q_2}\kappa_2(1,2)^-, \quad \iota^-_{1,2} = \iota_1(1,2)^-\iota_2(1,2)^-\beta^-_{1,2}, \quad \text{and}\\
\kappa^-_{1,2} &= \kappa_1(1,2)^-\kappa_2(1,2)^-\beta^-_{1,2}.
\end{aligned}
$$

In Chapter 4, we had that

$$
L'_{i,j} = \frac{q_i(p_j-q_i)}{h(i,j)}\left(h_j(q_jp_i-q_j)I + p_iA_j\right)A_i,
$$

for $i,j \in \{1,2,3\}$ and $i \neq j$. When $h_3 \to +\infty$, from (4.50) we have that $L_3 \to 0$ and therefore

$$
f_{1,2} \sim I + \frac{L'_{1,2}}{q_1} + \frac{L'_{2,1}}{q_2}, \tag{5.25}
$$

The asymptotic expression (5.25) is of the same form as the two-soliton matrix variable $f_{[3]}$ and the resulting dromion is therefore of the same form as the single dromion as given by (5.17). Therefore, when Λ_1 is fixed, the asymptotic forms $\hat{\tilde{w}} := -(f_{1,2})_x f^{-1}_{1,2}$ are

$$
\begin{aligned}
\hat{\tilde{w}} &\sim \frac{(p_1-q_1)^2}{4(p_1q_1)^{\frac{1}{2}}}\hat{P}_1(1,2)^- \operatorname{sech}\left(\frac{\Lambda_1+\hat{\varphi}_1(1,2)^-}{2}\right)\operatorname{sech}\left(\frac{\Lambda_1+\hat{\chi}_1(1,2)^-}{2}\right) \quad \text{as} \quad t\to-\infty,\\
\hat{\tilde{w}} &\sim \frac{(p_1-q_1)^2}{4(p_1q_1)^{\frac{1}{2}}}\hat{P}_1(1,2)^+ \operatorname{sech}\left(\frac{\Lambda_1+\hat{\varphi}_1(1,2)^+}{2}\right)\operatorname{sech}\left(\frac{\Lambda_1+\hat{\chi}_1(1,2)^+}{2}\right) \quad \text{as} \quad t\to+\infty,
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{P}_1(1,2)^- &= \frac{\hat{\mu}_1(1,2)^-\hat{\nu}_1(1,2)^{-T}}{(\hat{\mu}_1(1,2)^-,\hat{\nu}_1(1,2)^-)}, \quad \hat{P}_1(1,2)^+ = \frac{\hat{\mu}_1(1,2)^+\hat{\nu}_1(1,2)^{+T}}{(\hat{\mu}_1(1,2)^+,\hat{\nu}_1(1,2)^+)},\\
\hat{\mu}_1(1,2)^- &= \mu_1, \quad \hat{\nu}_1(1,2)^- = \nu_1, \quad \hat{\mu}_1(1,2)^+ = \mu_1 - \frac{p_1(p_2-q_2)(\mu_1,\nu_2)\mu_2}{p_2(p_1-q_2)(\mu_2,\nu_2)},\\
\hat{\nu}_1(1,2)^+ &= \nu_1 - \frac{q_1(p_2-q_2)(\mu_2,\nu_1)\nu_2}{q_2(p_2-q_1)(\mu_2,\nu_2)}, \quad \hat{r}_1(1,2)^- = r_1,\\
\hat{r}_1(1,2)^+ &= \frac{r_1(\hat{\mu}_1(1,2),\hat{\nu}_1(1,2))}{(\mu_1,\nu_1)} = r_1\beta_{1,2}, \quad \hat{\varphi}_1(1,2)^- = \log\left(\frac{-p_1\hat{r}_1(1,2)^-}{q_1(p_1-q_1)}\right),\\
\hat{\chi}_1(1,2)^- &= \log\left(\frac{-\hat{r}_1(1,2)^-}{p_1-q_1}\right), \quad \hat{\varphi}_1(1,2)^+ = \log\left(\frac{-p_1\hat{r}_1(1,2)^+}{q_1(p_1-q_1)}\right) \quad \text{and}\\
\hat{\chi}_1(1,2)^+ &= \log\left(\frac{-\hat{r}_1(1,2)^+}{p_1-q_1}\right).
\end{aligned}
$$

The soliton phase-constants are

$$\hat{\xi}_1(1,2)^- = -\log\left(\frac{-(p_1 q_1^{-1})^{\frac{1}{2}}\hat{r}_1(1,2)^-}{p_1 - q_1}\right) \quad \text{and} \quad \hat{\xi}_1(1,2)^+ = -\log\left(\frac{-(p_1 q_1^{-1})^{\frac{1}{2}}\hat{r}_1(1,2)^+}{p_1 - q_1}\right).$$

When Λ_2 is fixed, the asymptotic forms for $\hat{\hat{w}}$ are

$$\hat{\hat{w}} \sim \frac{(p_2 - q_2)^2}{4(p_2 q_2)^{\frac{1}{2}}}\hat{P}_2(1,2)^- \operatorname{sech}\left(\frac{\Lambda_2 + \hat{\varphi}_2(1,2)^-}{2}\right)\operatorname{sech}\left(\frac{\Lambda_2 + \hat{\chi}_2(1,2)^-}{2}\right) \quad \text{as} \quad t \to -\infty,$$
$$\hat{\hat{w}} \sim \frac{(p_2 - q_2)^2}{4(p_2 q_2)^{-\frac{1}{2}}}\hat{P}_2(1,2)^+ \operatorname{sech}\left(\frac{\Lambda_2 + \hat{\varphi}_2(1,2)^+}{2}\right)\operatorname{sech}\left(\frac{\Lambda_2 + \hat{\chi}_2(1,2)^+}{2}\right) \quad \text{as} \quad t \to +\infty,$$

in which

$$\hat{P}_2(1,2)^- = \frac{\hat{\mu}_2(1,2)^-\hat{\nu}_2(1,2)^{-T}}{(\hat{\mu}_2(1,2)^-, \hat{\nu}_2(1,2)^-)}, \quad \hat{P}_2(1,2)^+ = \frac{\hat{\mu}_2(1,2)^+\hat{\nu}_2(1,2)^{+T}}{(\hat{\mu}_2(1,2)^+, \hat{\nu}_2(1,2)^+)},$$
$$\hat{\mu}_2(1,2)^+ = \mu_2, \quad \hat{\nu}_2(1,2)^+ = \nu_2, \quad \hat{\mu}_2(1,2)^- = \mu_2 - \frac{p_2(p_1 - q_1)(\mu_2, \nu_1)\mu_1}{p_1(p_2 - q_1)(\mu_1, \nu_1)},$$
$$\hat{\nu}_2(1,2)^- = \nu_2 - \frac{q_2(p_1 - q_1)(\mu_1, \nu_2)\nu_1}{q_1(p_1 - q_2)(\mu_1, \nu_1)}, \quad \hat{r}_2(1,2)^- = \frac{r_2(\hat{\mu}_2(1,2), \hat{\nu}_2(1,2))}{(\mu_2, \nu_2)} = r_2\beta_{1,2},$$
$$\hat{r}_2(1,2)^+ = r_2, \quad \hat{\varphi}_2(1,2)^- = \log\left(\frac{-p_2\hat{r}_2(1,2)^-}{q_2(p_2 - q_2)}\right), \quad \hat{\chi}_2(1,2)^- = \log\left(\frac{-\hat{r}_2(1,2)^-}{p_2 - q_2}\right),$$
$$\hat{\varphi}_2(1,2)^+ = \log\left(\frac{-p_2\hat{r}_2(1,2)^+}{q_2(p_2 - q_2)}\right) \quad \text{and} \quad \hat{\chi}_2(1,2)^+ = \log\left(\frac{-\hat{r}_2(1,2)^+}{p_2 - q_2}\right).$$

The soliton phase-constants are

$$\hat{\xi}_2(1,2)^- = -\log\left(\frac{-(p_2 q_2^{-1})^{\frac{1}{2}}\hat{r}_2(1,2)^-}{p_2 - q_2}\right) \quad \text{and} \quad \hat{\xi}_2(1,2)^+ = -\log\left(\frac{-(p_2 q_2^{-1})^{\frac{1}{2}}\hat{r}_2(1,2)^+}{p_2 - q_2}\right).$$

Furthermore, the soliton phase-shifts $\hat{\Delta}_j(1,2) = \hat{\xi}_j(1,2)^+ - \hat{\xi}_j(1,2)^-$, for $j = 1, 2$ are

$$\hat{\Delta}_1(1,2) = -\log\left(\beta_{1,2}^+\right) \quad \text{and} \quad \hat{\Delta}_2(1,2) = \log\left(\beta_{1,2}^+\right), \quad \text{where} \quad \beta_{1,2}^+ = \beta_{1,2}.$$

The dromion $d(1,2)$ as $t \to +\infty$ can now be written as

$$d(1,2) \sim \frac{\hat{r}_1(1,2)^-\hat{r}_2(1,2)^+(p_1 - q_1)(p_2 - q_2)(1 - \alpha_{1,2}^+)}{S_\kappa^+(1,2)S_\iota^+(1,2)}, \tag{5.26}$$

where

$$S_\kappa^+(1,2) = e^{-\frac{1}{2}(\Lambda_1+\Lambda_2)} - \kappa_1(1,2)^+e^{\frac{1}{2}(\Lambda_1-\Lambda_2)} - \kappa_2(1,2)^+e^{\frac{1}{2}(\Lambda_2-\Lambda_1)} + \kappa_{1,2}^+e^{\frac{1}{2}(\Lambda_1+\Lambda_2)},$$
$$S_\iota^+(1,2) = e^{-\frac{1}{2}(\Lambda_1+\Lambda_2)} - \iota_1(1,2)^+e^{\frac{1}{2}(\Lambda_1-\Lambda_2)} - \iota_2(1,2)^+e^{\frac{1}{2}(\Lambda_2-\Lambda_1)} + \kappa_{1,2}^+e^{\frac{1}{2}(\Lambda_1+\Lambda_2)},$$
$$\alpha_{1,2}^+ = Tr(\hat{P}_1(1,2)^-\hat{P}_2(1,2)^+), \quad \kappa_1(1,2)^+ = \kappa_1, \quad \kappa_2(1,2)^+ = \kappa_2, \quad \kappa_{1,2}^+ = \kappa_1^+\kappa_2^+\beta_{1,2}^+,$$
$$\iota_1(1,2)^+ = \frac{p_1}{q_1}\kappa_1(1,2)^+ \quad \iota_2(1,2)^+ = \frac{p_2}{q_2}\kappa_2(1,2)^+ \quad \text{and} \quad \iota_{1,2}^+ = \iota_1^+\iota_2^+\beta_{1,2}^+.$$

Similar calculations give expressions for the dromions $d(2,3)$ and $d(1,3)$. As $t \to -\infty$.

$$d(2,3) \sim \frac{r_2(2,3)^- r_3(2,3)^+ (p_2 - q_2)(p_3 - q_3)(1 - \alpha_{2,3}^-)}{S_\kappa^-(2,3) S_\iota^-(2,3)}, \tag{5.27}$$

$$d(1,3) \sim \frac{r_1(1,3)^- r_3(1,3)^+ (p_1 - q_1)(p_3 - q_3)(1 - \alpha_{1,3}^-)}{S_\kappa^-(2,3) S_\iota^-(2,3)}, \tag{5.28}$$

and as $t \to +\infty$ we have

$$d(2,3) \sim \frac{\hat{r}_2(2,3)^- \hat{r}_3(2,3)^+ (p_2 - q_2)(p_3 - q_3)(1 - \alpha_{2,3}^+)}{S_\kappa^+(2,3) S_\iota^+(2,3)}, \tag{5.29}$$

$$d(1,3) \sim \frac{\hat{r}_1(1,3)^- \hat{r}_3(1,3)^+ (p_1 - q_1)(p_3 - q_3)(1 - \alpha_{1,3}^+)}{S_\kappa^+(1,3) S_\iota^+(1,3)}. \tag{5.30}$$

5.2.2 Summary of the three-dromion structure

The asymptotic expressions (5.24), (5.26), (5.27), (5.28), (5.29) and (5.30) all have the same form as the dromion given by (5.17). Therefore, we have shown that the three-dromion structure $\det(w) = -\det((f_{[4]})x)/\det(f_{[4]})$ decomposes asymptotically into six dromions:

$$d(i,j) \sim \begin{cases} \frac{r_i(i,j)^- r_j(i,j)^+ (p_i - q_i)(p_j - q_j)(1-\alpha_{i,j}^-)}{S_\kappa^-(i,j) S_\iota^-(i,j)} & \text{as} \quad t \to -\infty, \\ \frac{\hat{r}_i(i,j)^- \hat{r}_j(i,j)^+ (p_i - q_i)(p_j - q_j)(1-\alpha_{i,j}^+)}{S_\kappa^+(i,j) S_\iota^+(i,j)} & \text{as} \quad t \to +\infty, \end{cases}$$

for $i, j \in \{1,2,3\}$ and $i \neq j$, giving the following generalisation of Theorem 4:

Theorem 5. *If* $\det(\Omega)$ *is positive-definite, then* $\det(w)$*, as given by* (5.21) *and* (5.22)*, has the following properties:*

1. $\det(w)$ *decomposes asymptotically into six dromions as described in Theorem 4. Each* $d(i,j)$ *decays to zero exponentially as* $(x,y) \to \infty$ *in any direction.*

2. *The amplitude of* $d(i,j)$ *is*

$$\mathcal{A}^- := \frac{(1-\alpha_{i,j}^-) p_i p_j (p_i - q_i)^2 (p_j - q_j)^2}{\left(\sqrt{\beta_{i,j}^-}\left(\sqrt{\frac{p_i p_j}{q_i q_j}} + \frac{p_i p_j}{q_i q_j}\right) + \frac{p_i}{q_i}\sqrt{\frac{p_j}{q_j}} + \frac{p_j}{q_j}\sqrt{\frac{p_i}{q_i}}\right)^2 q_i^2 q_j^2}, \quad t \to -\infty$$

$$\mathcal{A}^+ := \frac{(1-\alpha_{i,j}^+) p_i p_j (p_i - q_i)^2 (p_j - q_j)^2}{\left(\sqrt{\beta_{i,j}^+}\left(\sqrt{\frac{p_i p_j}{q_i q_j}} + \frac{p_i p_j}{q_i q_j}\right) + \frac{p_i}{q_i}\sqrt{\frac{p_j}{q_j}} + \frac{p_j}{q_j}\sqrt{\frac{p_i}{q_i}}\right)^2 q_i^2 q_j^2}, \quad t \to +\infty.$$

   The amplitude is

   - *negative*

     (a) *as* $t \to -\infty$*: if* $\alpha_{i,j}^- > 1$*,*

(b) *as* $t \to +\infty$*: if* $\alpha^{+}_{i,j} > 1$,

- *zero*

(a) *as* $t \to -\infty$*: if* $\alpha^{-}_{i,j} = 1$,

(b) *as* $t \to +\infty$*: if* $\alpha^{+}_{i,j} = 1$,

- *positive*

(a) *as* $t \to -\infty$*: if* $\alpha^{-}_{i,j} < 1$,

(b) *as* $t \to +\infty$*: if* $\alpha^{+}_{i,j} < 1$,

3. *At time* t *the location of* $d(i,j)$ *moves from*

$$(x,y) = \frac{-1}{2l_{i,j}}\Big(l_j^{(2)}(\xi_i(i,j)^- + \xi_i(i,j)^+) - l_i^{(2)}(\xi_j(i,j)^- + \xi_j(i,j)^+) + 8l_{j,k}t\ ,\\ l_i^{(1)}(\xi_j(i,j)^- + \xi_j(i,j)^+) - l_j^{(1)}(\xi_i(i,j)^- + \xi_i(i,j)^+) + 8l_{i,k}t\Big)\,,$$

as $t \to -\infty$ *to*

$$(x,y) = \frac{-1}{2l_{i,j}}\Big(l_j^{(2)}(\hat{\xi}_i(i,j)^- + \hat{\xi}_i(i,j)^+) - l_i^{(2)}(\hat{\xi}_j(i,j)^- + \hat{\xi}_j(i,j)^+) + 8l_{j,k}t\ ,\\ l_i^{(1)}(\hat{\xi}_j(i,j)^- + \hat{\xi}_j(i,j)^+) - l_j^{(1)}(\hat{\xi}_i(i,j)^- + \hat{\xi}_i(i,j)^+) + 8l_{i,k}t\Big)\,,$$

as $t \to +\infty$.

4. *The trajectory of* $d(i,j)$ *changes from*

$$y = \left(\frac{l_{i,k}}{l_{j,k}}\right)x + \\ \frac{\left(l_j^{(1)}l_{j,k} + l_j^{(2)}l_{i,k}\right)(\xi_i(i,j)^- + \xi_i(i,j)^+) - \left(l_i^{(1)}l_{j,k} + l_i^{(2)}l_{i,k}\right)(\xi_j(i,j)^- + \xi_j(i,j)^+)}{2l_{i,j}l_{j,k}},$$

as $t \to -\infty$ *to*

$$y = \left(\frac{l_{i,k}}{l_{j,k}}\right)x + \\ \frac{\left(l_j^{(1)}l_{j,k} + l_j^{(2)}l_{i,k}\right)(\hat{\xi}_i(i,j)^- + \hat{\xi}_i(i,j)^+) - \left(l_i^{(1)}l_{j,k} + l_1^{(2)}l_{i,k}\right)(\hat{\xi}_j(i,j)^- + \hat{\xi}_j(i,j)^+)}{2l_{i,j}l_{j,k}},$$

as $t \to +\infty$.

5.2.3 Plots of dromions

Let us again label the elements of the 2-vectors as

$$\mu(i,j)^- = \begin{pmatrix} a^-_{ij,1} \\ a^-_{ij,2} \end{pmatrix}, \quad \nu(i,j)^- = \begin{pmatrix} b^-_{ij,1} \\ b^-_{ij,2} \end{pmatrix}, \quad \mu(i,j)^+ = \begin{pmatrix} a^+_{ij,1} \\ a^+_{ij,2} \end{pmatrix}, \quad \nu(i,j)^+ = \begin{pmatrix} b^+_{ij,1} \\ b^+_{ij,2} \end{pmatrix},$$
$$\hat{\mu}(i,j)^- = \begin{pmatrix} c^-_{ij,1} \\ c^-_{ij,2} \end{pmatrix}, \quad \hat{\nu}(i,j)^- = \begin{pmatrix} d^-_{ij,1} \\ d^-_{ij,2} \end{pmatrix}, \quad \hat{\mu}(i,j)^+ = \begin{pmatrix} c^+_{ij,1} \\ c^+_{ij,2} \end{pmatrix}, \quad \hat{\nu}(i,j)^+ = \begin{pmatrix} d^+_{ij,1} \\ d^+_{ij,2} \end{pmatrix},$$

for $j = 1, 2, 3$ and $i \neq j$.

As $t \to -\infty$, $d(i,j)$ vanishes

$$\Leftrightarrow Tr(P_i(i,j)^- P_j(i,j)^+) = 1$$
$$\Leftrightarrow (\mu(i,j)^-, \nu(i,j)^+)(\mu(i,j)^+, \nu(i,j)^-) = (\mu(i,j)^-, \nu(i,j)^-)(\mu(i,j)^+, \nu(i,j)^+)$$
$$\Leftrightarrow a^-_{ij,2} a^+_{ij,1} = a^-_{ij,1} a^+_{ij,2} \text{ or } b^-_{ij,2} b^+_{ij,1} = b^-_{ij,1} b^+_{ij,2}.$$

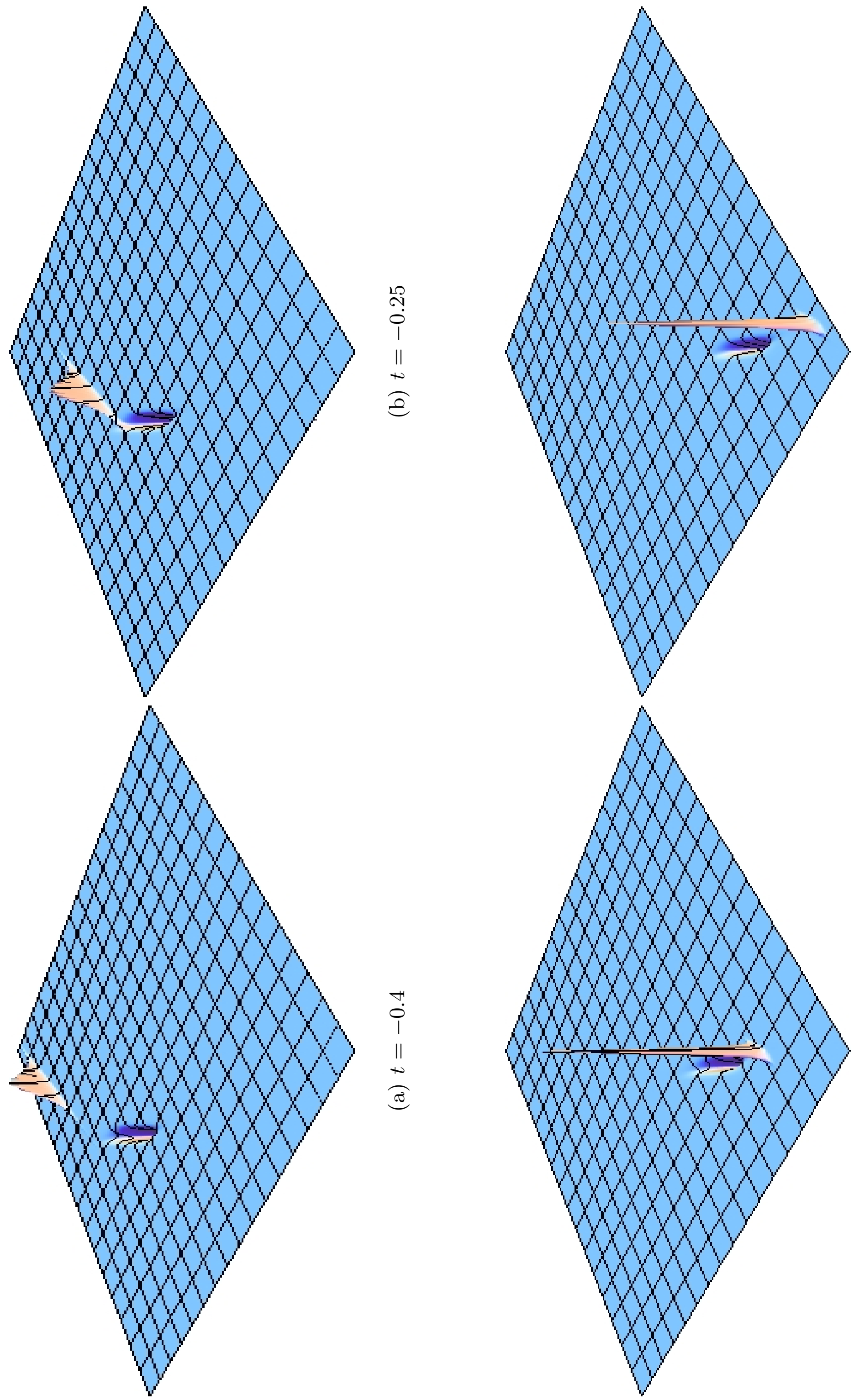
As $t \to +\infty$, $d(i,j)$ vanishes

$$\Leftrightarrow Tr(\hat{P}_i(i,j)^- \hat{P}_j(i,j)^+) = 1$$
$$\Leftrightarrow (\hat{\mu}(i,j)^-, \hat{\nu}(i,j)^+)(\hat{\mu}(i,j)^+, \hat{\nu}(i,j)^-) = (\hat{\mu}(i,j)^-, \hat{\nu}(i,j)^-)(\hat{\mu}(i,j)^+, \hat{\nu}(i,j)^+)$$
$$\Leftrightarrow c^-_{ij,2} c^+_{ij,1} = c^-_{ij,1} c^+_{ij,2} \text{ or } d^-_{ij,2} d^+_{ij,1} = d^-_{ij,1} d^+_{ij,2}.$$

Figure 5.8 shows a plot of the three-dromion structure with

$$\mu_1^T = \begin{pmatrix} 1 & 2 \end{pmatrix}, \quad \mu_2^T = \begin{pmatrix} 3 & 6 \end{pmatrix}, \quad \mu_3^T = \begin{pmatrix} -1 & -3 \end{pmatrix},$$
$$\nu_1^T = \begin{pmatrix} 2 & \frac{2}{3} \end{pmatrix}, \quad \nu_2^T = \begin{pmatrix} -1 & 3 \end{pmatrix} \quad \text{and} \quad \nu_3^T = \begin{pmatrix} 3 & 1 \end{pmatrix},$$

so that $d(1,3)$ vanishes as $t \to -\infty$ and $d(1,2)$, $d(2,3)$ vanish as $t \to +\infty$.



(a) $t = -0.4$

(b) $t = -0.25$

(c) $t = 0.25$

(d) $t = 0.4$

Figure 5.8: Plot of the three-dromion structure with parameters $p_1 = -\frac{9}{2}$, $p_2 = -\frac{5}{2}$, $p_3 = -\frac{1}{3}$, $q_1 = -6$, $q_2 = -3$, $q_3 = -2$, $r_1 = r_2 = r_3 = -1$.

Chapter 6

Summary and conclusions

In Chapter 3, we saw that our noncommutative KP equation had two families of solutions, obtained from Darboux and binary Darboux transformations, which could be expressed as quasiwronskians and quasigrammians. Like the commutative case reviewed in Chapter 2, it was shown that these solutions can be verified directly. In doing so, both types of solution reduced to identities with the same structure, just as the Wronskian and Grammian solutions did. Our noncommutative mKP equation again had a family of quasiwronskian and a family of quasigrammian solutions, obtained from Darboux and binary Darboux transformations. However, when attempting to directly verify these solutions, we were only able to do so for the quasiwronskians as there appears to be no obvious way of inverting the quasigrammian F discussed in Chapter 4. Further work is required to investigate the invertibility of F and the subsequent direct verification of the family of quasigrammian solutions of ncmKP. The process of directly verifying the quasigrammian solutions of ncKP proved to be easier than that of the commutative case outlined in Chapter 2 since no Frobenius partition was needed.

For both ncKP and ncmKP, we saw in Chapters 3 and 4 that all of the quasiwronskians and quasigrammians expressing transformed eigenfunctions and solutions can always be reduced to the corresponding commutative results in Chapter 2 using a known result of a quasideterminant. Quasideterminants therefore appear to be more beneficial as a compact expression for iterated Darboux transformations. However, quasideterminant solutions cannot be obtained from Hirota's method and in this thesis their existence was limited to Darboux transformations.

The nature of the noncommutativity of the dependent variables in ncKP and ncmKP was specified in Chapters 3 and 4 by projection matrices. Though not the most general case of matrices, they provided a richer picture of interaction of soliton solutions of ncKP and ncmKP by adding a change of matrix amplitude in addition to a phase-shift. Examples of other types of noncommutative variables such as vectors could also be investigated in addition to the projection matrix examples discussed in this thesis.

Taking the determinant of the projection matrix solutions allowed us to find a new class of dromion structures. These dromions are unique to this example of noncommutativity and clearly have no commutative analogue. The properties of the single dromion are similar to those of the DSI and NVN equations and are found using the same techniques. After finding the single dromion, interaction properties of these new structures were then examined for three-dromion structures. The key to changes in amplitude lies in the trace of the projection matrices governing the amplitude of the underlying interacting solitons. This is different from the DSI and NVN equations. The changes in location and trajectory are similar to the DSI and NVN dromions in that they are governed by changes in phase-shift of the underlying solitons in the three-soliton structure.

Another different feature of the dromions of ncKP and ncmKP is the orientation of the underlying solitons in the solution. For the DSI and NVN equations, the underlying solitons are perpendicular and/or parallel to one another. This orientation gives a nicer schematic form of dromion scattering allowing for generalisation of the asymptotics to any $n \geq 1$. However, the orientation of the solitons of the matrix versions of both ncKP and ncmKP is not fixed in the same way which prevented us from finding asymptotic expressions for each dromion in the solution beyond $n = 3$. Further work would be required to generalise the result to any $n \geq 1$.

References

[1] A.C. Aitken. *Determinants and Matrices.* Oliver and Boyd, Edinburgh, 1939.

[2] M. Boiti, J. Jp. Leon, L. Martina, and F. Pempinelli. Scattering of localized solitons in the plane. *Phys. Lett. A*, 132(8-9):432–439, 1988.

[3] L. A. Bordag, A. R. Its, S.V. Manakov, V. B. Matveev, and V.E Zakharov. Two dimensional solitons of the Kadomtsev-Petviashvili equation and their interaction. *Phys. Lett. A*, 63:205, 1977.

[4] M. M. Crum. Associated Sturm-Liouville systems. *Quart. J. Math. Oxford Ser. (2)*, 6:121–127, 1955.

[5] E. Date, M. Kashiwara, M. Jimbo, and T. Miwa. Transformation groups for soliton equations. In *Nonlinear integrable systems—classical theory and quantum theory (Kyoto, 1981)*, pages 39–119. World Sci. Publishing, Singapore, 1983.

[6] G. de Vries and D.J. Korteweg. On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves. *Philos. Mag. Ser*, 39(5):422–443, 1895.

[7] A. Dimakis and F. Müller-Hoissen. Extension of noncommutative soliton hierarchies. *J. Phys. A*, 37(13):4069–4084, 2004.

[8] A. Dimakis and F. Müller-Hoissen. Functional representations of integrable hierarchies. *J. Phys. A*, 39(29):9169–9186, 2006.

[9] P. G. Drazin and R. S. Johnson. *Solitons: an introduction.* Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1989.

[10] P. Etingof, I. Gel′fand, and V. Retakh. Factorization of differential operators, quasideterminants, and nonabelian Toda field equations. *Math. Res. Lett.*, 4(2-3):413–425, 1997.

[11] A. S. Fokas and P. M. Santini. Coherent structures in multidimensions. *Phys. Rev. Lett.*, 63(13):1329–1333, 1989.

[12] N. C. Freeman and J. J. C. Nimmo. Soliton solutions of the Korteweg-de Vries and Kadomtsev-Petviashvili equations: the Wronskian technique. *Phys. Lett. A*, 95(1):1–3, 1983.

[13] Darboux G. Sur une proposition relative aux quations linaires. *C. R. Acad. Sci. Paris*, 94:1456–1459, 1882.

[14] I. Gel′fand, S. Gel′fand, V. Retakh, and R. L. Wilson. Quasideterminants. *Adv. Math.*, 193(1):56–141, 2005.

[15] I. M. Gel′fand and L. A. Dickii. Fractional powers of operators and hamiltonian systems. *Funct. Anal. Appl.*, (10):259–273, 1976.

[16] I. M. Gel′fand and V. S. Retakh. Determinants of matrices over noncommutative rings. *Funktsional. Anal. i Prilozhen.*, 25(2):13–25, 96, 1991.

[17] C. Gilson and S. MacFarlane. Dromion solutions of noncommutative Davey-Stewartson equations. *J. Phys. A*, 2009. To appear.

[18] C. R. Gilson and J. J. C. Nimmo. A direct method for dromion solutions of the Davey-Stewartson equations and their asymptotic properties. *Proc. Roy. Soc. London Ser. A*, 435(1894):339–357, 1991.

[19] C. R. Gilson and J. J. C. Nimmo. On a direct approach to quasideterminant solutions of a noncommutative KP equation. *J. Phys. A*, 40(14):3839–3850, 2007.

[20] C. R. Gilson, J. J. C. Nimmo, and C. M. Sooman. On a direct approach to quasideterminant solutions of a noncommutative modified KP equation. *J. Phys. A*, 41(8):085202, 10, 2008.

[21] C. R. Gilson, J. J. C. Nimmo, and C. M. Sooman. Matrix solutions of a noncommutative KP equation an a noncommutative modified KP equation. *Theoretical and Mathematical Physics*, 159(3):448–458, 2009.

[22] V. M. Goncharenko. On multisoliton solutions of the matrix KdV equation. *Teoret. Mat. Fiz.*, 126(1):102–114, 2001.

[23] V. M. Goncharenko and A. P. Veselov. Monodromy of the matrix Schrödinger equations and Darboux transformations. *J. Phys. A*, 31(23):5315–5326, 1998.

[24] H. J. Groenewold. On the principles of elementary quantum mechanics. *Physica*, 12:405–460, 1946.

[25] M. Hamanaka. Noncommutative solitons and integrable systems. In *Noncommutative geometry and physics*, pages 175–198. World Sci. Publ., Hackensack, NJ, 2005.

[26] M. Hamanaka and K. Toda. Towards noncommutative integrable systems. *Phys. Lett. A*, 316(1-2):77–83, 2003.

[27] M. Hamanaka and K. Toda. Towards noncommutative integrable equations. In *Symmetry in nonlinear mathematical physics. Part 1, 2, 3*, volume 2 of *Pr. Inst. Mat. Nats. Akad. Nauk Ukr. Mat. Zastos., 50, Part 1*, pages 404–411. Natsīonal. Akad. Nauk Ukraïni Īnst. Mat., Kiev, 2004.

[28] R. Hirota. Exact solution of the Korteweg-de Vries equation for multiple collisions of solitons. *Phys. Rev. Lett.*, 27:1192–1194, 1971.

[29] R. Hirota. *The direct method in soliton theory*, volume 155 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2004. Translated from the 1992 Japanese original and edited by Atsushi Nagai, Jon Nimmo and Claire Gilson, With a foreword by Jarmo Hietarinta and Nimmo.

[30] M. Jimbo and T. Miwa. Solitons and infinite-dimensional Lie algebras. *Publ. Res. Inst. Math. Sci.*, 19(3):943–1001, 1983.

[31] B. B. Kadomtsev and V. I Petviashvili. On the stability of solitary waves in weakly dispersive media. *Soviet Phys. Dokl*, 15(6):539–541, 1970.

[32] B. G. Konopel′chenko and V. G. Dubrovsky. Some new integrable nonlinear evolution equations in $2+1$ dimensions. *Phys. Lett. A*, 102(1-2):15–17, 1984.

[33] M. D. Kruskal and N. J. Zabusky. Interaction of “solitons” in a collisionless plasma and the recurrence of initial states. *Physical Review Letters*, 15(6):240–243, August 1965.

[34] B.A. Kupershmidt. *KP or mKP noncommutative mathematics of Lagrangian, Hamiltonian, and integrable systems*, volume 78. American Mathematical Society, 2000.

[35] P. D. Lax. Integrals of nonlinear equations of evolution and solitary waves. *Comm. Pure Appl. Math.*, 21:467–490, 1968.

[36] C. X. Li and J. J. C. Nimmo. Quasideterminant solutions of a non-abelian Toda lattice and kink solutions of a matrix sine-Gordon equation. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 464(2092):951–966, 2008.

[37] V. B. Matveev. Darboux transformation and explicit solutions of the Kadomtcev-Petviaschvily equation, depending on functional parameters. *Lett. Math. Phys.*, 3(3):213–216, 1979.

[38] V. B. Matveev. Darboux transformations in associative rings and functional-difference equations. In *The bispectral problem (Montreal, PQ, 1997)*, volume 14 of *CRM Proc. Lecture Notes*, pages 211–226. Amer. Math. Soc., Providence, RI, 1998.

[39] R. M. Miura. Korteweg-de Vries equation and generalizations. I. A remarkable explicit nonlinear transformation. *J. Mathematical Phys.*, 9:1202–1204, 1968.

[40] J. E. Moyal. Quantum mechanics as a statistical theory. *Proc. Cambridge Philos. Soc.*, 45:99–124, 1949.

[41] J. J. C. Nimmo. Darboux transformations from reductions of the KP hierarchy. In *Nonlinear evolution equations & dynamical systems: NEEDS '94 (Los Alamos, NM)*, pages 168–177. World Sci. Publ., River Edge, NJ, 1995.

[42] S. P. Novikov and A. P. Veselov. Two-dimensional Schrödinger operator − inverse scattering transform and evolutional equations. 18:267–273, 1986.

[43] W. Oevel and C. Rogers. Gauge transformations and reciprocal links in $2+1$ dimensions. *Rev. Math. Phys.*, 5(2):299–330, 1993.

[44] W. Oevel and W. Schief. Darboux theorems and the KP hierarchy. In *Applications of analytic and geometric methods to nonlinear differential equations (Exeter, 1992)*, volume 413 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 193–206. Kluwer Acad. Publ., Dordrecht, 1993.

[45] Y. Ohta, J. Satsuma, D. Takahashi, and T. Tokihiro. An elementary introduction to Sato theory. *Progr. Theoret. Phys. Suppl.*, (94):210–241, 1988. Recent developments in soliton theory.

[46] L. D. Paniak. Exact noncommutative KP and KdV multi-solitons. *Preprint*, 2001.

[47] J. S. Russell. Report on waves. *Report of the 14th Meeting of the British Association for the Advancement of Science*, pages 311–390, 1844.

[48] M. Sato and Y. Sato. Soliton equations as dynamical systems on infinite-dimensional Grassmann manifold. In *Nonlinear partial differential equations in applied science (Tokyo, 1982)*, volume 81 of *North-Holland Math. Stud.*, pages 259–271. North-Holland, Amsterdam, 1983.

[49] J. Satsuma and M. J. Ablowitz. Two-dimensional lumps in nonlinear dispersive systems. *J. Math. Phys.*, 20(7):1496–1503, 1979.

[50] M. Unal. *Applications of Pfaffians to Soliton Theory.* PhD thesis, University of Glasgow, 1998.

[51] N. Wang and M. Wadati. Noncommutative KP hierarchy and Hirota triple-product relations. *J. Phys. Soc. Japan*, 73(7):1689–1698, 2004.