GROTHENDIECK RINGS OF THEORIES OF MODULES

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Simon Perera

School of Mathematics

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Simon Perera
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We consider right modules over a ring, as models of a first order theory. We explore the definable sets and the definable bijections between them. We employ the notions of Euler characteristic and Grothendieck ring for a first order structure, introduced by J. Krajiček and T. Scanlon in [24]. The Grothendieck ring is an algebraic structure that captures certain properties of a model and its category of definable sets. If $M \in \text{Mod-}(R_1 \times R_2)$, then M has a decomposition $M = M_1 \oplus M_2$ where $M_i \in \text{Mod-}R_i$ for $i = 1, 2$. Theorem 3.5.1 states that then $K_0(M) = K_0(M_1) \otimes K_0(M_2)$.

Theorem 4.3.1 states that the Grothendieck ring of every infinite module over a field or skew field is isomorphic to $\mathbb{Z}[X]$.

Proposition 5.2.4 states that for an elementary extension $M \preceq N$ of models of any theory, the elementary embedding induces an embedding of rings $K_0(M) \xrightarrow{F} K_0(N)$. Theorem 5.3.1 is that for modules $M \preceq N$ we have the stronger result $K_0(M) \cong K_0(N)$.

We define a model-theoretic Grothendieck ring of the category Mod-R and explore the relationship between $K_0(\text{Mod-}R)$ and the Grothendieck rings of the right R-modules. The category of pp-imaginaries, shown by K. Burke [7] to be equivalent to $(\text{mod-}R, Ab)^{fp}$, provides a functorial approach to studying the generators of the Grothendieck rings of R-modules. It is shown in Theorem 6.3.5 that whenever R and S are Morita equivalent rings, the rings $K_0(\text{Mod-}R)$ and $K_0(\text{Mod-}S)$ are isomorphic.

Combining results from previous chapters, we derive Theorem 7.2.1 saying that the Grothendieck ring of any module over a semisimple ring is isomorphic to a polynomial ring $\mathbb{Z}[X_1, \ldots, X_n]$ for some n.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

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Chapter 1

Introduction

The Euler characteristic, originally defined for the surfaces of polyhedra, is an invariant describing the structure of a topological space. The original formulation is the famous $\chi = V - E + F$, the number of vertices minus the number of edges plus the number of faces. The notion has been generalised to many areas of mathematics; the Euler characteristic of a simplicial complex or CW-complex topology is the alternating sum of the number of cells of each dimension; the Euler characteristic of a chain complex in homology is an alternating sum of the ranks of the homology groups.

Another concept of simple origin, with varied applications in modern mathematics, is that of a Grothendieck group. For any commutative monoid, the Grothendieck group is defined simply to be the completion to a group by adding negative elements defined via an equivalence relation on pairs of elements, as in the completion of $\mathbb{N}$ to $\mathbb{Z}$. Motivated by his study of coherent sheaves on an algebraic variety, which led to the development of K-theory, Alexander Grothendieck defined a group using the isomorphism classes of sheaves as generators, and completing the monoid to a group as described. Variations of the Grothendieck group abound and in some cases, the initial monoid is a semiring and it is completed to a ring, called the Grothendieck ring.

Both of these concepts, in their various guises, capture certain "geometric" information about mathematical objects.

The context of this thesis is model theory, primarily but not exclusively that of right modules over a ring. A right R-module is a structure for the first order language $\mathcal{L}_R = \mathcal{L}\langle +, -, 0, f_r : r \in R\rangle$, as described in Chapter 3. Of interest here are the "combinatorial" properties of definable sets; the definable bijections between them, notions of rank and dimension, the question of when a definable set can properly contain a self-similar definable copy of itself or even infinitely many disjoint such sets. Much of the material deals with classes of definable sets up to definable bijection.

To this end we use the notions of Euler characteristic and Grothendieck ring for a first order structure, introduced by J. Krajiček and T. Scanlon in [24] and detailed in Chapter 2 of this thesis. These notions capture some of the "structure" of the objects under consideration. The Grothendieck ring of M is denoted $K_0(M)$ following the conventions of K-theory. Following [24] we denote by $\mathrm{Def}(M)$ the category with objects the definable sets of M and its powers M^n, and arrows the definable maps between them. We denote by $\widetilde{\mathrm{Def}}(M)$ the set of equivalence classes of elements of $\mathrm{Def}(M)$ up to the equivalence modulo definable bijections.

Chapter 3 contains background material on the model theory of modules, which is covered in great detail in [31]. We also prove Theorem 3.5.1 for modules over a product of rings. If $M \in \text{Mod-}(R_1 \times R_2)$, then M has a decomposition $M = M_1 \oplus M_2$ where $M_1 \in \text{Mod-}R_1$ and $M_2 \in \text{Mod-}R_2$. Theorem 3.5.1 states that for such modules $K_0(M) = K_0(M_1) \otimes K_0(M_2)$.

We calculate the Grothendieck rings of certain particular modules and provide results towards calculating others. Theorem 4.3.1 states that the Grothendieck ring of every infinite module over a field or skew field is isomorphic to $\mathbb{Z}[X]$.

Elementary extensions are considered in Chapter 5, much of which is a departure from the context of modules. Proposition 5.2.4 states that for an elementary extension $M \preceq N$ of models of any theory, the elementary embedding induces an embedding of rings $K_0(M) \xrightarrow{F} K_0(N)$. Theorem 5.3.1 is that for modules $M \preceq N$ we have the stronger result that F is an isomorphism $K_0(M) \cong K_0(N)$. The remainder of Chapter 5 is spent investigating a structure in the language $\mathcal{L}$ with just one equivalence relation and no function symbols. The $\mathcal{L}$-structures M and M_0 we consider appeared in [24] in a section on examples and open questions, separate to the main argument of the paper. Therein they give incorrect values for $K_0(M)$ and $K_0(M_0)$, as is shown in Chapter 5.

In Chapter 6 we define the Grothendieck ring of the category Mod-R to be $K_0(P)$ where $P \in$ Mod-R is a direct sum of one model of each complete theory of R-modules, and prove that this is well defined. We explore the relationship between K_0(Mod-R) and the Grothendieck rings $K_0(M)$ of a general R-module M. The category of pp-imaginaries, shown by K. Burke [7] to be equivalent to (mod-$R, Ab)^{fp}$, provides a functorial approach to studying the generators of the Grothendieck rings of R-modules. It is shown in Theorem 6.3.5 that whenever R and S are Morita equivalent rings, the rings K_0(Mod-R) and K_0(Mod-S) are isomorphic. We investigate the examples of the categories Mod-$\mathbb{Z}_4$ and Mod-$k[\varepsilon]$, where k is a field and ε an indeterminate satisfying $\varepsilon^2 = 0$. In order to study the latter, we give a brief account of Auslander-Reiten theory in Section 6.7. This material was introduced by M. Auslander and I. Reiten in [2],[3] and an account of the material from the perspective of module categories is given in Chapter 15 of [30].

Every semisimple ring is isomorphic to a product of matrix rings over division rings. Combining the results of previous chapters, we derive Theorem 7.2.1 saying that the Grothendieck ring of any module over a semisimple ring is isomorphic to a polynomial ring $\mathbb{Z}[X_1, \ldots, X_n]$ for some n.

A question that remains unanswered is whether a nonzero module can have trivial Grothendieck ring, a property equivalent to $M \models ontoPHP$ by Theorem 3.2 and Corollary 3.4 of [24]. An answer to this question would complete the calculation of $K_0(\mathbb{Z})$ where $\mathbb{Z}$ is regarded as right module over itself. It is also a question that needs to be addressed when calculating the Grothendieck ring of any module. For modules over semisimple rings we have a proof that $K_0(M)$ is nontrivial but it does not generalise. Chapter 8 consists of arguments towards answering this question, particularly in the case when the theory of M has $T = T^{(\aleph_0)}$, which would imply the nontriviality of K_0(Mod-R) for every ring R.

Chapter 2

Grothendieck rings of first order structures

2.1 Background and definitions

A key definition in this work is that of the model-theoretic Grothendieck ring of a first-order structure. This is a ring associated to any structure, founded upon the sets definable in a given first-order language and the definable combinatorics of the structure. Therefore it is dependent on the choice of model-theoretic language as well as the mathematical object in question. For example it is significant whether a ring R is being viewed as a ring in the language of rings or as a module in the language of R-modules. Strictly speaking the Grothendieck ring is defined for the theory in question, and by the Grothendieck ring of an $\mathcal{L}$-structure M we shall mean the Grothendieck ring of $Th(M)$ in a language with constant symbols for every element of M. We denote by $\mathcal{L}(M)$ the language $\mathcal{L}$ enriched with constant symbols for every element of a structure M.

Before defining the Grothendieck ring of a structure M, it is first necessary to recall some simple model theoretic definitions and define the notion of an *Euler characteristic*. The following account is based on [24].

In model theory, a *sort* is a domain or universe which elements of a model may belong to. A *one-sorted structure* has a single universe. All variables and constants are assumed to live in this domain, and all relations and functions are taken over this domain.

A *many-sorted structure* has a number of universes or domains. Each domain X has its own countably infinite set of variables, said to be of sort X. Each constant symbol must have a designated sort and then constants will be interpreted as a fixed elements of that specified universe. Each function or relation symbol of the language must have a specified sort for each its variables and constants. Each n-ary relation will assign a sort to each of the n variables to specify the universe that it lives in. Similarly an n-ary function will have a list of n specified sorts for the variables of the domain of the function and also a sort for the value.

Given any one-sorted structure M, we may also view it as a many-sorted structure by taking the powers M^n as additional sorts, referred to as the *basic sorts*. We may also obtain a related structure known as M^{eq} by including additional sorts of the form S/R, where S is a basic sort of M and R is an 0-definable equivalence relation on S. If M is many-sorted, then we may define M^{eq} similarly, taking the basic sorts to be the finite products of powers of the original sorts of M and defining additional sorts via 0-definable equivalence relations on these basic sorts. The equivalence classes under a definable equivalence relation are called *imaginaries*. These *imaginaries* are elements of the multi-sorted structure M^{eq}.

In this thesis we work in the context of one-sorted structures. The basic sorts are the powers of the home sort or domain of the structure. **Throughout this thesis "definable" will mean definable with parameters, unless otherwise specified.**

Definition 2.1.1. *For a basic sort S, we define $Def^S(M)$ to be the collection of all definable sets whose elements are of sort S.*

Definition 2.1.2. *Let $Def(M) = \bigcup_S Def^S(M)$ where the union is over all basic sorts.*

2.2 The construction of a general Grothendieck ring

Definition 2.2.1. *We say that two definable sets A and B in Def(M) are* definably isomorphic *if there exists a definable bijection between them. This is an equivalence relation on Def(M). The equivalence class of a set A is denoted $\widetilde{[A]}$.*

Let A and B be definable sets over a given $\mathcal{L}$-structure M. Then there are $\mathcal{L}$-formulas $\alpha(\overline{u},\overline{c}),\beta(\overline{v},\overline{d})$ with constant symbols $\overline{c}$ and $\overline{d}$ from M, such that $A = \alpha(M) = \{\overline{x} \in M^m : M \models \alpha(\overline{x},\overline{c})\}$ and $B = \beta(M) = \{\overline{y} \in M^n : M \models \beta(\overline{y},\overline{d})\}$. We say that A and B are in *definable bijection* if there is an $\mathcal{L}$-formula $\rho(\overline{u},\overline{v},\overline{c},\overline{d})$ satisfying:

$$\begin{aligned} M \models\ & \forall \overline{u},\overline{v}\ (\rho(\overline{u},\overline{v},\overline{c},\overline{d}) \to (\alpha(\overline{u},\overline{c}) \wedge \beta(\overline{v},\overline{d})) \wedge \\ & (\alpha(\overline{u},\overline{c}) \to \exists! \overline{v}\, \rho(\overline{u},\overline{v},\overline{c},\overline{d})) \wedge (\beta(\overline{v},\overline{d}) \to \exists! \overline{u}\, \rho(\overline{u},\overline{v},\overline{c},\overline{d}))) \end{aligned}$$

Definition 2.2.2. *Let $\widetilde{Def}(M)$ be the quotient set of Def(M) under this equivalence relation. The quotient map $[-] : Def(M) \to \widetilde{Def}(M)$ takes a set A to its equivalence class $\widetilde{[A]}$.*

Let $\mathcal{L}_{\text{ring}} := \mathcal{L}(+,\cdot,0,1)$ be the language of rings. Then we can interpret $\widetilde{\text{Def}}(M)$ as an $\mathcal{L}_{\text{ring}}$-structure by defining $+,\cdot,0$ and 1 as follows:

$$\begin{aligned} 0 &:= [\emptyset] \\ 1 &:= [\{*\}] \quad \text{where } \{*\} \text{ is any singleton subset of } M. \\ [A]+[B] &:= [A' \cup B'] \quad \text{where } [A'] = [A], [B'] = [B] \text{ and } A' \cap B' = \emptyset \\ [A]\cdot[B] &:= [A \times B] \end{aligned}$$

Observe that every singleton set is definable as we have parameters in our language for all elements of the universe and furthermore is definably isomorphic to any other singleton set. The addition function symbol in $\mathcal{L}_{\text{ring}}$ is interpreted as the map $\widetilde{\text{Def}}(M) \times \widetilde{Def}(M) \to \widetilde{\text{Def}}(M)$ taking two equivalence classes of definable sets to the equivalence class of the disjoint union of representatives of each of the classes. We can always find disjoint representatives since for definable sets A and B and constant symbols c_1 and c_2 for distinct elements of M, we will have $[\{c_1\} \times A] = [A]$, $[\{c_2\} \times B] = [B]$ and $(\{c_1\} \times A) \cap (\{c_2\} \times B) = \emptyset$.

Observe that $\widetilde{\mathrm{Def}}(M)$ is an $\mathcal{L}_{\mathrm{ring}}$-structure but $\widetilde{\mathrm{Def}}(M)$ is not a ring since it does not have additive inverses. A *semiring* is an algebraic structure similar to a ring but without the necessary existence of additive inverses for each element. Formally a semiring is a set S with two binary operations $\cdot$ and $+$ called multiplication and addition with identity elements 1 and 0 respectively, such that:

- $(S, \cdot)$ is a monoid
- multiplication is associative
- $1 \cdot s = s \cdot 1 = s$, $\forall s \in S$
- $0 \cdot s = s \cdot 0 = 0$, $\forall s \in S$
- $(S, +)$ is a monoid
- addition is associative and commutative
- $s + 0 = 0 + s = s$, $\forall s \in S$
- multiplication is distributive over addition

Since $\widetilde{\mathrm{Def}}(M)$ is an $\mathcal{L}_{\mathrm{ring}}$-structure, given any commutative ring R with unity, there may be $\mathcal{L}_{\mathrm{ring}}$-homomorphisms from $\widetilde{\mathrm{Def}}(M)$ to R. For instance there will always be the trivial map to the commutative ring $R_0 := \{0\}$ wherein $0 = 1$, and this will be a $\mathcal{L}_{\mathrm{ring}}$-homomorphism.

Definition 2.2.3. *A* (weak) Euler characteristic *on a structure M taking values in a commutative ring with unity R is a map $\chi = \tilde{\chi} \circ [-] : Def(M) \to R$ where $\tilde{\chi}$ is an $\mathcal{L}_{ring}$-homomorphism from $\widetilde{Def}(M)$ to R.*

The notation $\chi = \chi_R$ is sometimes used to denote the fact that the Euler characteristic takes values in the ring R. It is always possible to construct a weak Euler characteristic χ_R, for some commutative ring with unity R, on a structure M. Define an equivalence relation on $\widetilde{\mathrm{Def}}(M)$ by $[A] \sim [B]$ iff $[A]+[C]=[B]+[C]$ for some $[C] \in \widetilde{\mathrm{Def}}(M)$. Factoring by this equivalence relation yields a semiring $\widetilde{\mathrm{Def}}(M)/\sim$.

Claim. For any cancellative semiring S, there is a unique ring R into which S embeds that is minimal such. *Proof.* Define an equivalence relation $\equiv$ on the set $S \times S$ by $(x_1, x_2) \equiv (y_1, y_2)$ iff $x_1 + y_2 = x_2 + y_1$. Then $R := (S \times S)/\equiv$ is a ring and there is an embedding $e : S \to R$ given by $s \mapsto [(s, 0)]$, the equivalence class of the element $(s, 0)$. The additive inverse of $[(s, 0)]$ is $[(0, s)]$. Furthermore any embedding of S into a ring R' must factor through e. R is the ring generated by the semiring S.

Definition 2.2.4. *Let M be an $\mathcal{L}$-structure. The* Grothendieck ring *of the theory of M in the language $\mathcal{L}(M)$, or for brevity the Grothendieck ring of M, denoted $K_0(M)$ is the ring generated as above by the semiring $\widetilde{Def}(M)/\sim$.*

The weak Euler characteristic taking values in $K_0(M)$ is called the *universal weak Euler characteristic* χ_0 because if $\delta : \mathrm{Def}(M) \to R'$ is any weak Euler characteristic on M, then δ must factor through χ_0.

The following well known lemma is included here, together with a simple proof, for reference in the sequel.

Lemma 2.2.5. *Let M be any finite structure. The Grothendieck ring of M is $K_0(M) = \mathbb{Z}$.*

Proof. Every definable set $\phi(M)$ must be finite since if the formula defining it is an n-ary formula $\phi(v_1, \ldots, v_n, \overline{c})$ say, with constants $\overline{c}$ then we must have $\phi(M) \subseteq M^n$ and hence it is contained in a finite set. The sets in bijection with a given finite set are exactly those sets of equal size. Since we allow parameters in our formulas, all such bijections are definable. For example a bijection between $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ is given by the formula $\phi(v_1, \ldots, v_n, w_1, \ldots, w_n) := (\bigwedge_{i=1}^n (v_i = a_i \wedge w_i = b_i))$.

So each natural number n gives rise to an equivalence class of definable sets of size n, which for simplicity we denote $[\mathbf{n}]$. There will be no further identification due to the cancelation relation $[\mathbf{n}] \sim [\mathbf{m}]$ if $\exists\, l \in \mathbb{N}$ such that $[\mathbf{n}] + [\mathbf{l}] = [\mathbf{m}] + [\mathbf{l}]$.

This equivalence involves a bijection between disjoint unions of pairs of finite sets and hence $[\mathbf{n}] + [\mathbf{l}] = [\mathbf{m}] + [\mathbf{l}] \Rightarrow n + l = m + l \Rightarrow n = m$. Therefore $\widetilde{\mathrm{Def}}(M)/\sim = \mathbb{N}$ and $K_0(M) = \mathbb{Z}$. □

A brief survey of some known Grothendieck rings. The Grothendieck rings of first order structures have been studied in a number of contexts, mostly structures for the language of rings $\mathcal{L}_{\text{rings}}$ or extensions of it. J. Krajiček and T. Scanlon prove in [24] that the real closed field $\mathbb{R}$ has $K_0(\mathbb{R}) = \mathbb{Z}$ using the dimension theory and cell decomposition of o-minimality, showing that all cells of positive dimension are sent to 0 in $K_0(\mathbb{R})$ by the Euler characteristic, but the finite sets are not. Krajiček shows in ([23], Theorem 5.6) that the Grothendieck ring of any pseudofinite field F, regarded as an $\mathcal{L}_{\text{rings}}$-structure, admits quotient rings isomorphic to each finite field $\mathbb{F}_p$. R. Cluckers and D. Haskell prove that the fields of p-adic numbers and $\mathbb{F}_q((t))$, the field of formal Laurent series, both have trivial Grothendieck rings, by constructing definable bijections from a set to the same set minus a point. They also show that $\mathbb{Z}$-valued fields over certain extensions $\mathcal{L}^+$ of $\mathcal{L}_{\text{rings}}$ have trivial Grothendieck rings. Cluckers shows in [11] that certain fields of formal Laurent series have trivial Grothendieck ring by defining bijections from a set to itself minus a point. Also J. Denef and F. Loeser have found that the field of complex numbers regarded as an $\mathcal{L}_{\text{rings}}$-structure has $K_0(\mathbb{C})$ admitting the ring $\mathbb{Z}[X, Y]$ as a quotient. Krajiček and Scanlon have strengthened this result and shown that $K_0(\mathbb{C})$ contains an algebraically independent set of size *continuum*, and hence the ring $\mathbb{Z}[X_i : i \in \mathfrak{c}]$ embeds into $K_0(\mathbb{C})$. The ring of p-adic integers is known to have trivial Grothendieck ring, as shown independently by D. Marker and L. van den Dries. M. Fujita and M. Kageyama prove in [17] that every o-minimal expansion of an ordered abelian group has either $K_0(M) \cong \mathbb{Z}$ or $K_0(M) \cong \mathbb{Z}[X]/\langle X^2 + X\rangle$.

It is sometimes useful to consider combinatorial conditions on the definable sets and maps in a first order structure.

Definition 2.2.6. *We say that an infinite structure M satisfies the* pigeonhole principle *if it has the property that any definable injective map f from any definable set*

A to itself must be surjective. In this case we write $M \models PHP$.

Definition 2.2.7. *We say that an infinite structure M satisfies the* onto pigeonhole principle *if it has the property that there exists no definable set A, element $a \in A$ and definable injective map f with domain A and image $A \setminus a$. In this case we write $M \models ontoPHP$.*

In addition to the conditions PHP and $ontoPHP$ defined in [24] and studied in this thesis, there have been other combinatorial conditions defined in the literature and referred to as 'principles' that may be satisfied or not in a given structure. There are *weak pigeonhole principles*; $WPHP^{2n}$ stating that no definable set can contain two disjoint subsets that are in definable bijection with the original, and another $WPHP^{n^2}$ that no definable set A with $|A| > 1$ can contain a subset in definable bijection with A^2. There is a counting principle for each n saying that a definable set cannot be partitioned into subsets each of size n and also partitioned into one set B with $1 \leq |B| < n$ and other sets all of size n.

In this thesis the Grothendieck rings, and the combinatorics of definable sets, are considered for modules in the natural language, defined in the sequel. The 'pp-elimination' in the theory of modules dictates which sets and functions will be definable in a model.

Chapter 3

The model theory of modules

3.1 Background and definitions

This chapter opens with well known material on the model theory of modules. A detailed exposition can be found in [31].

The usual language for a right R-module M is $\mathcal{L} = \langle 0, +, -, f_r : r \in R \rangle$, where each f_r is a unary function symbol representing the action of multiplication by r. For brevity we usually write mr as shorthand for $f_r(m)$. When it is necessary to specify the ring in question we denote this with a subscript, for example $\mathcal{L}_R$ or $\mathcal{L}_{\mathbb{Z}}$.

Definition 3.1.1. *A parameter free* positive primitive formula *or pp-formula is (or is equivalent to) one of the form*

$$\phi(v_1, \ldots, v_n) = \exists w_1, \ldots, w_m \bigwedge_{i=1}^{t} \Big(\sum_{j=1}^{n} v_j r_{ij} + \sum_{k=1}^{m} w_k s_{ik} = 0 \Big)$$

where r_{ij} and s_{ik} are elements of the ring R.

Here we use the convention of writing $v_j r_{ij}$ as shorthand for $f_{r_{ij}}(v_j)$. Sometimes we may wish to include parameters and for this we require constant symbols to represent elements of a model. In this case we take an expansion of the language by adding constant symbols $\mathcal{L} = \langle 0, +, -, f_r, c_h : r \in R, h \in H \rangle$ for some index set H. In this expansion of $\mathcal{L}_R$, the formulas may include terms in the constants, so we will assume that the set of constants $\{c_h : h \in H\}$ in our expansion is always closed under terms. By *closed under terms* we mean that this expanded language contains enough constant symbols such that for every well-formed term in $\mathcal{L}_R$ consisting of function symbols and constant symbols, there is a constant symbol whose interpretation is equal to that of the term.

Then a general pp-formula may contain constant symbols c_i, and is equivalent to one of the form

$$\phi(v_1,\ldots,v_n) = \exists w_1,\ldots,w_m \bigwedge_{i=1}^{t}\Big(\sum_{j=1}^{n} v_j r_{ij} + \sum_{k=1}^{m} w_k s_{ik} + c_i = 0\Big)$$

Lemma 3.1.2. *Let $\phi(v_1,\ldots,v_n)$ be any parameter-free pp-formula in n variables. Then*

$$\phi(M) = \{\overline{m} \in M^n \mid M \models \phi(\overline{m})\}$$

is a subgroup of the additive abelian group M^n.

Proof. Referring to the definition of a pp-formula above, suppose $\phi(v_1,\ldots,v_n) = \exists w_1,\ldots,w_m \bigwedge_{i=1}^{t}\Big(\sum_{j=1}^{n} v_j r_{ij} + \sum_{k=1}^{m} w_k s_{ik} = 0\Big)$. By taking $w_1,\ldots,w_m$ to equal $0 \in M$, we see that $M \models \phi(\overline{0})$ for any pp-formula ϕ. Now suppose $\overline{x} = (x_1,\ldots,x_n)$ and $\overline{y} = (y_1,\ldots,y_n)$ are in $\phi(M)$. Then there exist some $a_1,\ldots,a_m,b_1,\ldots,b_m \in M$ such that $M \models \bigwedge_{i=1}^{t}\Big(\sum_{j=1}^{n} x_j r_{ij} + \sum_{k=1}^{m} a_k s_{ik} = 0\Big)$ and $M \models \bigwedge_{i=1}^{t}\Big(\sum_{j=1}^{n} y_j r_{ij} + \sum_{k=1}^{m} b_k s_{ik} = 0\Big)$. Therefore $M \models \bigwedge_{i=1}^{t}\Big(\sum_{j=1}^{n}(x_j - y_j) r_{ij} + \sum_{k=1}^{m}(a_k - b_k) s_{ik} = 0\Big)$ and $\overline{x} - \overline{y} \in \phi(M)$. □

If we allow parameters in the pp-formula ϕ, then the set it defines will be either the empty set or a coset of the additive subgroup defined by the same formula with the parameters all replaced by 0. Let

$$\phi(\overline{v},\overline{c}) = \phi(v_1,\ldots,v_n,c_1,\ldots,c_t) = \exists w_1,\ldots,w_m \bigwedge_{i=1}^{t}\Big(\sum_{j=1}^{n} v_j r_{ij} + \sum_{k=1}^{m} w_k s_{ik} + c_i = 0\Big)$$

and let

$$\phi(\overline{v},\overline{0}) = \phi(v_1,\ldots,v_n) = \exists w_1,\ldots,w_m \bigwedge_{i=1}^{t}\Big(\sum_{j=1}^{n} v_j r_{ij} + \sum_{k=1}^{m} w_k s_{ik} = 0\Big)$$

Lemma 3.1.3. *For any R-module M and pp-formula $\phi(\overline{v})$, $\phi(M,\overline{c})$ is either the empty set or a coset of the additive subgroup $\phi(M,\overline{0}) \leq M^n$.*

Proof. Suppose $\phi(M, \overline{c}) \neq \emptyset$ and let $\overline{m}_1$ and $\overline{m}_2$ be elements of $\phi(M, \overline{c})$. Then since $\phi(v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+t})$ defines an additive subgroup of M^{n+t}, we have that

$$M \models \phi(\overline{m}_1, \overline{c}) \wedge \phi(\overline{m_2}, \overline{c}) \Rightarrow M \models \phi(\overline{m}_1 - \overline{m}_2, \overline{0})$$

and for any $\overline{m} \in M^n$,

$$M \models \phi(\overline{m}, \overline{0}) \wedge \phi(\overline{m}_2, \overline{c}) \Rightarrow M \models \phi(\overline{m} + \overline{m}_2, \overline{c})$$

Therefore $\phi(M, \overline{c}) = \overline{m}_2 + \phi(M, \overline{0})$. □

Let M be an R-module and let ϕ and ψ be parameter free pp-formulas of equal length l in $\mathcal{L}_R$. Since parameter free pp-formulas define additive subgroups, ϕ and ψ will define additive subgroups of M^l and $(\phi \wedge \psi)(M) = \phi(M) \cap \psi(M)$ will be a subgroup of $\phi(M)$.

Definition 3.1.4. *For a triple M, ϕ, ψ as above we define the* invariant $Inv(M, \phi, \psi) = |\phi(M)/\phi(M) \cap \psi(M)|$, *the index of the subgroup.*

An *invariants condition* is a statement that a given invariant is equal to or greater than or less than a certain natural number. These invariants conditions are elementary properties, i.e. they can expressed as sentences in $\mathcal{L}$. For $k \in \mathbb{N}$, we have

$$Inv(M, \phi, \psi) > k \Leftrightarrow M \models \forall \overline{v}_1, \ldots, \overline{v}_k \bigwedge_{i=1}^{k} \phi(\overline{v}_i) \rightarrow \exists \overline{u} \, (\phi(\overline{u}) \wedge \bigwedge_{i=1}^{k} \neg\psi(\overline{u} - \overline{v}_i))$$

The property $Inv(M, \phi, \psi) < k$ is expressed by the formula $\neg(Inv(M, \phi, \psi) > k - 1)$ and the property $Inv(M, \phi, \psi) = k$ is expressed by the formula $(Inv(M, \phi, \psi) > k - 1) \wedge \neg(Inv(M, \phi, \psi) > k)$. An *invariants statement* is a boolean combination of invariant conditions.

In the model theory of modules, the following partial quantifier elimination result, due to Baur and Monk, says that every formula is equivalent to a boolean combination of pp-formulas and invariants conditions.

Theorem 3.1.5. *[5] Let T be the theory of R-modules and ϕ an arbitrary $\mathcal{L}_R$-formula. Then we have:*

$$T \models \forall \overline{v}(\phi(\overline{v}) \leftrightarrow \bigvee_{i=1}^{n} (\phi_i \wedge \bigwedge_{j=1}^{m} \neg\psi_{ij})(\overline{v}) \wedge I)$$

where ϕ_i and ψ_{ij} are pp-formulas and I is an invariants statement.

Remark. In a complete theory such as $\mathrm{Th}(M)$, the invariants statements will vanish and so the definable sets will be the solution sets of a boolean combination of pp-formulas.

If $\phi(v_1, \dots, v_n)$ is an n-ary formula then $\phi(M) = \{\overline{m} \in M^n \mid M \models \phi(\overline{m})\}$. A *definable function* in the model M is a function whose graph is the solution set of some $\mathcal{L}_R(M)$-formula.

Lemma 3.1.6. ***Neumann's Lemma*** *([31] Theorem 2.12)*
If H and G_i are subgroups of some group K and a coset of H is covered by a finite union of cosets of the G_i, then this coset of H is in fact covered by the union of just those cosets where G_i is of finite index in H, i.e. when $[H : G_i] := |H/H \cap G_i|$ is finite.

$$c + H \subseteq \bigcup_{i \in I} c_i + G_i \quad \Rightarrow \quad c + H \subseteq \bigcup_{i \in I_0} c_i + G_i$$

where $I_0 = \{i \in I \mid [H : G_i] < \infty\}$.

Given an R-module M, the endomorphism ring $\mathrm{End}_R(M)$ is the collection of R-module homomorphisms from M to itself, equipped with a ring structure where the multiplication of two endomorphisms is defined to be composition and the addition is defined as the endomorphism $(f + g)(m) = f(m) + g(m)$, $\forall m \in M$. We may omit the subscript when there is no confusion as to the ring in question.

There is a left action of $\mathrm{End}_R(M)$ on M^n by $f \cdot (m_1, \dots, m_n) = (f(m_1), \dots, f(m_n))$ for every $f \in \mathrm{End}_R(M)$ and every $\overline{m} = (m_1, \dots, m_n) \in M^n$. Thus every power of M is a left $\mathrm{End}_R(M)$-module.

Lemma 3.1.7. *For any parameter-free n-ary pp-formula $\phi(v_1, \dots, v_n)$, $\phi(M)$ is an End(M)-module. It is an End(M)-submodule of M^n.*

Proof. It is shown above that $\phi(M)$ is an additive abelian subgroup of M^n. We show that it is also closed under the left action of the endomorphism ring of M. Since $\phi(M) \subseteq M^n$, the action of $\mathrm{End}_R(M)$ is inherited and it is sufficient to show that the solution set of the pp-formula is closed under this action.

$$\forall f \, \forall \overline{m} \in \phi(M) \quad f \cdot \overline{m} \in \phi(M)$$

Suppose that $M \models \phi(\overline{m})$ and let $f \in \mathrm{End}_R(M)$. Then we have

$$
\begin{aligned}
& M \models \phi(m_1, \ldots, m_n) \\
\Rightarrow\quad & M \models \exists w_1, \ldots, w_m \bigwedge_{i=1}^{t} \Big(\sum_{j=1}^{n} m_j r_{ij} + \sum_{k=1}^{m} w_k s_{ik} = 0 \Big) \\
\Rightarrow\quad & M \models \bigwedge_{i=1}^{t} \Big(\sum_{j=1}^{n} m_j r_{ij} + \sum_{k=1}^{m} x_k s_{ik} = 0 \Big), \text{ for some } x_1, \ldots, x_m \in M \\
\Rightarrow\quad & M \models \bigwedge_{i=1}^{t} \Big(f\Big(\sum_{j=1}^{n} m_j r_{ij} + \sum_{k=1}^{m} x_k s_{ik} \Big) = f(0) \Big) \\
\Rightarrow\quad & M \models \bigwedge_{i=1}^{t} \Big(\sum_{j=1}^{n} f(m_j) r_{ij} + \sum_{k=1}^{m} f(x_k) s_{ik} = 0 \Big) \\
\Rightarrow\quad & M \models \exists w_1, \ldots, w_m \bigwedge_{i=1}^{t} \Big(\sum_{j=1}^{n} f(m_j) r_{ij} + \sum_{k=1}^{m} w_k s_{ik} = 0 \Big) \\
\Rightarrow\quad & M \models \phi(f(\overline{m}))
\end{aligned}
$$

Hence the pp-subgroup is a left $\mathrm{End}_R(M)$-submodule. □

Remark. In general $\phi(M)$ will not be an R-module, but it will be if R is a commutative ring. This is because there is a ring morphism taking any commutative ring R into $\mathrm{End}_R(M)$, given by $r \mapsto f_r$, $\forall r \in R$ where $f_r : M \to M$ is the endomorphism $m \mapsto m \cdot r$, $\forall m \in M$ i.e. right multiplication by r. So any left $\mathrm{End}_R(M)$-module will also be a right R-module via the ring homomorphism $R \to \mathrm{End}_R(M)$.

Let $R := M_2(\mathbb{Q})$ be the ring of 2×2 matrices with entries from the rational numbers. R is an example of a noncommutative ring and it is possible to find a pp-definable subgroup of an R-module that is not an R-module itself. Consider the module $R_R \in \text{Mod-}R$ and the pp-formula $\phi(v)$ given by $\exists w\ w \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = v$. Then

$$\phi(R) = \left\{ \begin{pmatrix} a & a \\ b & b \end{pmatrix} : a, b \in \mathbb{Q} \right\}$$

can easily be seen to be an additive subgroup of R but it is not an R-submodule of

R. It is not closed under the action of right multiplication by R, since for example:

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix} \notin \phi(R)$$

3.2 A useful lemma

The pp-elimination result of Baur and Monk implies that any formula, in the language of R-modules, is logically equivalent modulo $Th(M)$, the theory of $M \in$ Mod-R, to one of the form $\bigvee_{j=1}^{n}(\phi_j \wedge \bigwedge_{i=1}^{k} \neg\psi_{ji})(v_1, \ldots, v_m)$, for some pp-formulas ϕ_j, ψ_{ji}. We may always assume that $M \models \forall \overline{v}\ (\psi_{ji}(\overline{v}) \rightarrow \phi_j(\overline{v}))$. Clearly then, any definable set can be expressed in the form

$$(\bigvee_{j=1}^{n}(\phi_j \wedge \bigwedge_{i=1}^{k_j} \neg\psi_{ji}))(M) = \bigcup_{j=1}^{n}(\phi_j \wedge \bigwedge_{i=1}^{k_j} \neg\psi_{ji})(M)$$

For various calculations in this thesis, it will be helpful to work with disjoint unions of sets. The notation $X \sqcup Y$ and $\bigsqcup_i X_i$ is sometimes employed to highlight the fact that a union is disjoint.

Lemma 3.2.1. *For any right module M, it is always possible to write any set $A \in$ Def(M) as the disjoint union of sets of the form $(\mu \wedge \bigwedge_{i=1}^{k} \neg\nu_i)(M)$ where $\mu, \nu_1, \ldots, \nu_k$ are pp-formulas, allowing any parameters from M.*

Proof. Let A be an arbitrary definable set. Then we prove our claim by induction on n. If $n = 1$ then our union is only over one term so it cannot fail to be a disjoint union. For the inductive step we may write wlog $A := \bigcup_{j=1}^{n}(\phi_j \wedge \bigwedge_{i=1}^{k_j} \neg\psi_{ji})(M)$ and we assume the inductive hypothesis for unions of less than n terms. Thus there are pp-formulas μ_i, ν_{ij} such that

$$A = (\phi_n \wedge \bigwedge_{t=1}^{k_n} \neg\psi_{nt})(M) \cup \bigsqcup_{i=1}^{m}(\mu_i \wedge \bigwedge_{j=1}^{k_i} \neg\nu_{ij})(M)$$

There follows some intermediate steps to obtain an expression for A in the desired form. For brevity we define $B_i := (\mu_i \wedge \bigwedge_{j=1}^{k_i} \neg\nu_{ij})(M)$ for $1 \leq i \leq m$. Then observe

$$A = \left((\phi_n \wedge \bigwedge_{t=1}^{k_n} \neg\psi_{nt})(M) \setminus \bigsqcup_{i}^{m} B_i \right) \sqcup \bigsqcup_{i}^{m} B_i$$

Therefore it suffices to partition the first term here into disjoint sets defined by formulas of the required form. We will argue by induction on m within this inductive step of the larger proof. We establish the base case, when $m = 1$, by noting:

$$\begin{aligned}
& (\phi_n \wedge \bigwedge_{t=1}^{k_n} \neg\psi_{nt})(M) \setminus (\mu_1 \wedge \bigwedge_{j=1}^{k_1} \neg\nu_{1j})(M) \\
= \quad & (\phi_n \wedge \bigwedge_{t=1}^{k_n} \neg\psi_{nt} \wedge \neg\mu_1)(M) \sqcup (\phi_n \wedge \bigwedge_{t=1}^{k_n} \neg\psi_{nt} \wedge \mu_1 \wedge \nu_{11})(M) \\
& \sqcup(\phi_n \wedge \bigwedge_{t=1}^{k_n} \neg\psi_{nt} \wedge \mu_1 \wedge \nu_{12} \wedge \neg\nu_{11})(M) \sqcup \\
\ldots \quad & \sqcup(\phi_n \wedge \bigwedge_{t=1}^{k_n} \neg\psi_{nt} \wedge \mu_1 \wedge \nu_{1k} \wedge \bigwedge_{j<k} \neg\nu_{1j})(M)
\end{aligned}$$

For the inductive step for this 'sub-induction' on m, we first observe that

$$\begin{aligned}
A &= \left((\phi_n \wedge \bigwedge_{t=1}^{k_n} \neg\psi_{nt})(M) \setminus \bigsqcup_{i}^{m} B_i \right) \sqcup \bigsqcup_{i}^{m} B_i \\
&= \bigsqcup_{i}^{m} B_i \sqcup \left((\phi_n \wedge \bigwedge_{t=1}^{k_n} \neg\psi_{nt})(M) \setminus \bigsqcup_{i}^{m-1} B_i \right) \setminus B_m
\end{aligned}$$

By the inductive hypothesis and introducing the shorthand $\sigma_s(\overline{v})$ for the formula $(\phi'_s \wedge \bigwedge_{i=1}^{k_s} \neg\psi'_{si})(\overline{v})$, where the ϕ'_s and ψ'_{si} are pp-formulas, we may find an expression

$$A = \bigsqcup_{i}^{m} B_i \sqcup ((\bigsqcup_{s=1}^{n} \sigma_s(M)) \setminus B_m) = \bigsqcup_{i}^{m} B_i \sqcup (\bigsqcup_{s=1}^{n} (\sigma_s(M) \setminus B_m))$$

Recall $B_m = (\mu_m \wedge \bigwedge_{j=1}^{k_m} \neg\nu_{mj})(M)$ and so $(\sigma_s(M) \setminus B_m)) = (\sigma_s \wedge \neg\mu_m)(M) \sqcup (\sigma_s \wedge \mu_m \wedge \nu_{m1})(M) \sqcup \ldots \sqcup (\sigma_s \wedge \mu_m \wedge \nu_{mj} \wedge \bigwedge_{l<j} \neg\nu_{ml})(M) \sqcup \ldots \sqcup (\sigma_s \wedge \mu_m \wedge \nu_{mk_m} \wedge \bigwedge_{l<k_m} \neg\nu_{ml})(M)$.

Thus we complete the 'sub-induction' on m within the inductive step for n and this in turn completes the proof that our arbitrary set A can be given by a formula which naturally partitions it into disjoint sets, each of which is defined by a conjunction of pp-formulas and negations of pp-formulas in $\mathcal{L}_R(M)$. $\square$

3.3 Epimorphisms of rings

Let $f : R \to S$ be a morphism of rings. Then every right S-module M_S can be viewed as a right R-module M_R by restriction of scalars as follows. We define an R-action on M by $m \cdot r := m \cdot f(r)$, $\forall m \in M \, \forall r \in R$. The resulting R-module we denote M_R.

Definition 3.3.1. *The process described above gives us a functor ρ : Mod-$S \to$ Mod-R, defined on the objects by the mapping $M_S \mapsto M_R$ and sending each S-module morphism to itself. The functor ρ is called* restriction of scalars (via f)*.*

Definition 3.3.2. *A morphism of rings $f : R \to S$ is called an* epimorphism of rings *if for all rings Q and morphisms $g, h : S \to Q$, $gf = hf \Rightarrow g = h$.*

Recall that a subcategory $\mathcal{C}'$ of a category $\mathcal{C}$ is said to be *full* if for every two objects $A, B \in \mathcal{C}'$, $Hom_{\mathcal{C}'}(A, B) = Hom_{\mathcal{C}}(A, B)$. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be full if the image of F is a full subcategory of $\mathcal{D}$.

It is known (Prop 1.1 [39]) that the morphism f is an epimorphism iff the restriction of scalars functor is full iff the canonical map $S \otimes_R S \to S$ is an isomorphism. M. Prest shows, in [32], that if $f : R \to S$ is an epimorphism of rings, then there is an *interpretation* of Mod-S in Mod-R. Informally this can be read as saying the model theory of S-modules is contained in the model theory of R-modules. Also if there is an epimorphism of rings $R \to S$, then Mod-S is a definable subcategory of Mod-R. The following is a summary of selected material from [32].

Definition 3.3.3. *Let M be a right R-module. A* definable scalar *of M is a map $f : M \to M$ whose action can be given by a parameter free pp-formula $\rho(v, w)$ in $\mathcal{L}_R$, i.e. f is a total function from M to itself and for all $x, y \in M$, $f(x) = y$ if and only if $M \models \rho(x, y)$.*

The definable scalars of a module form a ring with the multiplication operation being composition.

Definition 3.3.4. *If a parameter free pp-formula defines a scalar for every model of a theory T, then it is called a* definable scalar for T*.*

The definable scalars for T form a ring with the multiplication operation being composition.

Definition 3.3.5. *If M is a right R-module and S is its endomorphism ring, $S := End_R(M)$, then M is a left S-module. A* biendomorphism *of $M \in Mod\text{-}R$ an endomorphism of ${}_SM$, i.e. an endomorphism of M regarded as a left S-module.*

Lemma 3.3.6. *[8] Let $M \in Mod\text{-}R$ and let ρ be a definable scalar of M. Then ρ is a biendomorphism of M_R.*

This is easily seen to follow from results in the opening section of this chapter. Every parameter-free pp-formula defines an additive group, and hence a map defined by a pp-formula must be an additive function. Furthermore the set $\rho(M)$ is an End(M)-module so the function given by ρ commutes with every endomorphism of M and is itself a biendomorphism.

Let R, S be arbitrary rings. Let T be a consistent set of $\mathcal{L}_R$-sentences, i.e. a theory, extending $Th(\text{Mod-}R)$. A *pp-interpretation* of S-modules into R-modules *via the theory* T is given if there are n-ary pp-formulas ϕ and ψ and $2n$-ary pp-formulas $\{\rho_s \mid s \in S\}$ in $\mathcal{L}_R$ such that:

- For every $M \models T$, each formula ρ_s defines an additive function on $\phi(M)/\psi(M)$.
- For every $M \models T$, $\phi(M)/\psi(M)$ is an S-module where the action of S is defined by each ρ_s giving the multiplication by the corresponding ring element s.
- Every S-module is isomorphic to one obtained in this fashion from an R-module satisfying T.

If we have some theory of R-modules T as above then we can replace it with T' containing T and the set of $\mathcal{L}_R$-sentences saying that all the ρ_s are functional and total on ϕ/ψ and that the functions defined by the ρ_s satisfy all the same addition and multiplication equations as the corresponding elements s in S.

Let R, S be rings and let $f : R \to S$ be an epimorphism of rings. Then there exists a theory T' of right R-modules such that there is a pp-interpretation of S-modules into R-modules via T' as described above and T' axiomatises the class of all R-modules that are the restriction to R of some S-module. This is Theorem 1 of [32].

A *pp-interpretation* of S-modules *into the home sort* of R-modules *via the theory* T is given if we have binary pp-formulas $\{\rho_s : s \in S\}$ in $\mathcal{L}_R$ such that:

- For every $M_R \models T$ and for every $m \in M$, $M \models \exists y \, \rho_s(m, y)$ and $M \models \rho_s(0, m) \to m = 0$.
- For every $M_R \models T$, the same M can be regarded as a module with the actions defined by the scalars ρ_s and thus becomes an S-module with $M \models \rho_s(m, ms)$ for every $s \in S$ and every $m \in M$.
- Every S-module is isomorphic to one obtained in this fashion from an R-module satisfying T.

Theorem 3.3.7. *(Theorem 7, [32]) If $f : R \to S$ is an epimorphism of rings, then there is a pp-interpretation of right S-modules into the home sort of right R-modules via some theory T.*

Conversely, if $g : R \to S$ is a morphism of rings and g induces a pp-interpretation of right S-modules into the home sort of right R-modules via some theory T, then g is an epimorphism.

Given parameter free pp-formulas δ and θ in $\mathcal{L}_R$, the language of right R-modules (respectively in ${}_R\mathcal{L}$ the language of left R-modules), there are pp-formulas $D\delta$ and $D\theta$ in ${}_R\mathcal{L}$, the language of left R-modules (respectively in $\mathcal{L}_R$), such that $\delta \to \theta$ iff $D\theta \to D\delta$, and $DD\delta \leftrightarrow \delta$. The pp-formula $D\delta$ is called the *dual* of δ. Prest introduced the notion of the dual of a pp-formula in [33].

We recall from [31] that when the class of models of T, a theory of right R-modules, is closed under direct products and direct summands then it may be axiomatised by some collection of sentences of the form $|\delta/\theta| = 1$ where δ and θ are pp-formulas such that $\theta \to \delta$. I. Herzog defines, in [19], the *dual of a theory* T of right R-modules axiomatised in this way to be DT, the theory of left R-modules axiomatised by the sentences $|D\theta/D\delta| = 1$ for the same pp-pairs.

Let f be an epimorphism of rings $f : R \to S$ and let T' be the theory of restrictions of right S-modules to R via f. Then the dual theory of T', DT' is the theory of restrictions of left S-modules to left R-modules via the epimorphism $f : R \to S$. For each pp-formula θ in $\mathcal{L}_S$, there is a pp-formula θ_R in $\mathcal{L}_R$ such that for every $\overline{m} \in M_S$, we have $M_S \models \theta(\overline{m})$ iff $M_R \models \theta_R(\overline{m})$. There is a similar translation of formulas for left modules, a pp-formula in ${}_R\mathcal{L}$ which we denote ${}_R\theta$. For any pp-formula θ in $\mathcal{L}_S$, we have that $D(\theta_R)$ and ${}_R(D\theta)$ are equivalent formulas in ${}_R\mathcal{L}$.

Remark. Of the examples of right modules that appear in this thesis, many are over rings R with obvious epimorphisms $R \to S$. For example there are epimorphisms from the ring $\mathbb{Z}$ to each of $\mathbb{Q}, \mathbb{Z}_{(p)}$ and $\mathbb{Z}_n$. The interpretation of Mod-S in Mod-R will yield extra results for M_S or $Th(\text{Mod-}S)$ as corollaries of results for M_R or $Th(\text{Mod-}R)$.

In particular the inclusion of rings $\mathbb{Z} \hookrightarrow \mathbb{Q}$ and the surjection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}_4$ are both epimorphisms of rings. The ring $\mathbb{Q}$ is a field and thus its right modules are $\mathbb{Q}$-vector spaces. The Grothendieck rings of every vector space $M_{\mathbb{Q}}$ over the field of rational numbers, and also the Grothendieck ring of the module category Mod-$\mathbb{Q}$ are computed in Chapter 4. The Grothendieck rings of every right module $M_{\mathbb{Z}_4}$ over the ring of integers modulo 4, and also the Grothendieck ring of the module category Mod-$\mathbb{Z}_4$ are computed in Chapter 6.6. The categories Mod-$\mathbb{Q}$ and Mod-$\mathbb{Z}_4$ are interpretable in Mod-$\mathbb{Z}$.

3.4 Decomposition of modules

Definition 3.4.1. *A right R-module P is said to be* injective *if for every monomorphism of right R-modules $g : X \hookrightarrow Y$ and arbitrary homomorphism $f : X \to P$, there exists a homomorphism $h : Y \to P$ such that $f = hg$.*

$$\begin{tikzcd} X \arrow[r, hook, "g"] \arrow[d, "f"'] & Y \arrow[dl, dashed, "\exists h"] \\ P & \end{tikzcd}$$

Definition 3.4.2. *A submodule $X \subseteq Y$ is said to be a* pure submodule *if $\phi(X) = X^n \cap \phi(Y)$ for every n-ary pp-formula ϕ in $\mathcal{L}_R$.*

Definition 3.4.3. *A monomorphism $e : X \to Y$ of right R-modules is said to be a* pure embedding *if $e(X)$ is a pure submodule of Y.*

Direct summands are always pure submodules. The collection of pure embeddings of right R-modules is closed under composition, direct limits and direct products. These basic properties of pure embeddings are all proved succinctly in ([30], Lemma 2.1.2).

Definition 3.4.4. *A right R-module P is said to be* pure injective*, or algebraically compact, if it is injective over all pure embeddings $e : X \to Y$.*

$$\begin{tikzcd} X \arrow[r, hook, "e", "\text{pure}"'] \arrow[d, "f"'] & Y \arrow[dl, dashed, "\exists h"] \\ P & \end{tikzcd}$$

Every right R-module is elementarily equivalent to a direct sum of indecomposable pure-injective right R-modules, as shown in ([40], 6.8,6.9). Let $M \in$ Mod-R and suppose $M \equiv \bigoplus_{i \in I} P_i$, where the P_i are indecomposable pure-injective modules. We wish to find the relation between the category of pp-pairs for the module M and the categories of pp-pairs for the P_i. The modules considered here are all $\mathcal{L}_R$-structures and will have the same parameter-free formulas. For each right R-module N, we define an equivalence relation on the pairs of positive primitive $\mathcal{L}_R$-formulas by setting $\phi_1/\psi_1 \sim_N \phi_2/\psi_2$ if there is some pp-formula ρ which is a pp-definable bijection between the pp-pairs in N. If ψ_1 and ψ_2 are equivalent to $\overline{v} = \overline{0}$ then we write $\phi_1 \sim_N \phi_2$ as shorthand for $\phi_1/\psi_1 \sim_N \phi_2/\psi_2$.

Lemma 3.4.5. *Let the parameter free pp-formula $\phi(v_1, \ldots, v_n)$ be*

$$\exists w_1, \ldots, w_m \bigwedge_{l=1}^{t} (\sum_{j=1}^{n} v_j r_{lj} + \sum_{k=1}^{m} w_k s_{lk} = 0)$$

where v_j, w_k are variables and r_{lj}, s_{lk} are function symbols for the action of ring elements. Let P_i for $i \in I$ be any right R-modules. Then we have

$$\phi(\bigoplus_{i\in I} P_i) = \bigoplus_{i \in I} \phi(P_i)$$

Proof. LHS $\subseteq$ RHS. Let $\overline{x}_j = (x_{ij} : i \in I)$ with $x_{ij} \in P_i$ for all $1 \le j \le n, i \in I$ and suppose $\bigoplus_{i\in I} P_i \models \phi(\overline{x}_1, \ldots, \overline{x}_n)$. Then there exist elements in $\bigoplus_i P_i$:

$$\overline{y}_1 = (y_{i1} : i \in I), \ldots, \overline{y}_m = (y_{im} : i \in I)$$

such that

$$\bigoplus_i P_i \models \bigwedge_{l=1}^{t} (\sum_{j=1}^{n} \overline{x}_j r_{lj} + \sum_{k=1}^{m} \overline{y}_k s_{lk} = 0)$$

It follows from the basic properties of addition in a direct sum that for each $i \in I$ we have $P_i \models \bigwedge_{l=1}^{t}(\sum_{j=1}^{n} x_{ij} r_{lj} + \sum_{k=1}^{m} y_{ik} s_{lk} = 0)$.

Hence $\overline{x}_j \in \phi(\bigoplus_{i\in I} P_i) \Rightarrow \forall i \in I \ x_{ij} \in \phi(P_i)$.

RHS $\subseteq$ LHS. This follows from a similar argument. Let $(x_{i1}, \ldots, x_{in}) \in \phi(P_i)$ for each $i \in I$. Then

$$P_i \models \exists w_{i1}, \ldots, w_{im} \bigwedge_{l=1}^{t} (\sum_{j=1}^{n} x_{ij} r_{lj} + \sum_{k=1}^{m} w_{ik} s_{lk} = 0)$$

for each $i \in I$. Then there are witnesses $\{y_{ij} : i \in I, 1 \le j \le n\}$ with $y_{ij} \in P_i$ for the existence conditions of ϕ. If we form $\overline{y}_j = (y_{ij} : i \in I)$ for $j = 1, \ldots, n$, then the definition of addition in a direct sum ensures that

$$\bigoplus_i P_i \models \bigwedge_{l=1}^{t} (\sum_{j=1}^{n} \overline{x}_j r_{lj} + \sum_{k=1}^{m} \overline{y}_k s_{lk} = 0)$$

and hence

$$\bigoplus_i P_i \models \exists \overline{w}_1, \ldots, \overline{w}_m \bigwedge_{l=1}^{t} (\sum_{j=1}^{n} \overline{x}_j r_{lj} + \sum_{k=1}^{m} \overline{w}_k s_{lk} = 0)$$

Therefore we have $\bigoplus_i P_i \models \phi(\overline{x}_1, \ldots, \overline{x}_n)$ as required.

$\square$

The following simple result makes use of the above lemma. It is used in Chapters 6 and 8.

Lemma 3.4.6. *If $M \in$ Mod-R has theory $T = T^{(\aleph_0)}$, so $M \equiv M^{(\aleph_0)}$, then every pp-pair $\psi \to \phi$ in $\mathcal{L}_R$ must have $Inv(M, \phi, \psi)$ either infinite or equal to 1.*

Proof. Recall that for any pp-formula $\phi \in \mathcal{L}_R$ and any modules $M_i \in$ Mod-R,

$$\phi(\bigoplus_i M_i) = \bigoplus_i \phi(M_i)$$

If $T \models \exists v, w\ \phi(v) \wedge \phi(w) \wedge \neg\psi(v-w)$, then $Inv(M, \phi, \psi) \geq 2$, i.e there are at least two cosets of $\psi(M)$ in $\phi(M)$. Therefore there are infinitely many cosets of $\psi(M^{(\aleph_0)})$ in $\phi(M^{(\aleph_0)})$. Thus it follows from $T = T^{(\aleph_0)}$ that $Inv(M, \phi, \psi) > n$ for any integer. □

Proposition 3.4.7. *Let $\phi_1(\overline{v})$ and $\phi_2(\overline{w})$ be pp-formulas of lengths n and m respectively such that $\phi_1 \sim_M \phi_2$. So there exists a pp-formula $\rho(\overline{v}, \overline{w})$ of length $n+m$ such that $\rho(M)$ is the graph of a bijection between $\phi_1(M)$ and $\phi_2(M)$. Suppose we have $M \equiv \bigoplus_{i \in I} P_i$ as above. Then it follows that for each $i \in I$, $\rho(P_i)$ is the graph of a bijection between $\phi_1(P_i)$ and $\phi_2(P_i)$, and hence $\phi_1 \sim_{P_i} \phi_2$.*

Proof. Our assumption on ρ gives us:

$$\begin{aligned} M \models \quad & \forall\overline{v}(\phi_1(\overline{v}) \to \exists\overline{w}(\rho(\overline{v}, \overline{w}) \wedge \phi_2(\overline{w}) \wedge \forall\overline{u}(\rho(\overline{v}, \overline{u}) \to \overline{u} = \overline{w}))) \\ & \wedge \forall\overline{w}(\phi_2(\overline{w}) \to \exists\overline{v}(\rho(\overline{v}, \overline{w}) \wedge \phi_1(\overline{v}) \wedge \forall\overline{u}(\rho(\overline{u}, \overline{w}) \to \overline{u} = \overline{v}))) \end{aligned}$$

Hence by elementary equivalence, we have:

$$\begin{aligned} \bigoplus_{i \in I} P_i \models \quad & \forall\overline{v}(\phi_1(\overline{v}) \to \exists\overline{w}(\rho(\overline{v}, \overline{w}) \wedge \phi_2(\overline{w}) \wedge \forall\overline{u}(\rho(\overline{v}, \overline{u}) \to \overline{u} = \overline{w}))) \\ & \wedge \forall\overline{w}(\phi_2(\overline{w}) \to \exists\overline{v}(\rho(\overline{v}, \overline{w}) \wedge \phi_1(\overline{v}) \wedge \forall\overline{u}(\rho(\overline{u}, \overline{w}) \to \overline{u} = \overline{v}))) \end{aligned}$$

Since $\rho(\overline{v}, \overline{w})$ is a pp-formula, Lemma 3.4.6 yields $\rho(\bigoplus_i P_i) = \bigoplus_i \rho(P_i)$. We fix $i_0 \in I$ and show that $\rho(P_{i_0})$ is the graph of a bijection and witnesses the equivalence $\phi_1 \sim_{P_{i_0}} \phi_2$.

Let $\overline{a}$ be an n-tuple from P_{i_0} and $(\overline{x}_i : i \in I)$ the n-tuple in $\bigoplus_i P_i$ with $\overline{x}_{i_0} = \overline{a}$ and $\overline{x}_i = \overline{0}$ for every $i \neq i_0$. Now if $\bigoplus_i P_i \models \phi_1((\overline{x}_i : i \in I))$ then there exists a unique m-tuple $(\overline{y}_i : i \in I) \in \phi_2(\bigoplus_i P_i)$ such that $((\overline{x}_i, \overline{y}_i) : i \in I) \in \rho(\bigoplus_i P_i)$. Hence $P_i \models \rho(\overline{x}_i, \overline{y}_i)$ for every $i \in I$.

Let $\overline{b}$ be the entry $\overline{y}_{i_0}$. For $i \neq i_0$, $\overline{x}_i = \overline{0}$ and since ρ is a pp-formula, we must have $P_i \models \rho(\overline{0}, \overline{0})$. Therefore the uniqueness condition implies that the m-tuples $\overline{y}_i$ for each $i \neq i_0$ will all be the zero m-tuples from the respective modules P_i. We also have $P_{i_0} \models \rho(\overline{a}, \overline{b}) \wedge \phi_2(\overline{b})$. Furthermore this $\overline{b}$ is the unique m-tuple from P_{i_0} satisfying the formula $\rho(\overline{a}, \overline{w})$ with free variables $\overline{w}$.

For if we assume that the m-tuple $\overline{c}$ from P_{i_0} also satisfies the formula, we can deduce that $\overline{b} = \overline{c}$. Define $(\overline{z}_i : i \in I)$ to be the m-tuple from $\bigoplus_i P_i$ given by $\overline{z}_{i_0} = \overline{c}$ and $\overline{z}_i = \overline{0}$ for every $i \in I \setminus \{i_0\}$. Then since $((\overline{x}_i, \overline{y}_i) : i \in I) \in \rho(\bigoplus_i P_i)$ and $\rho(\bigoplus_i P_i) = \bigoplus_i \rho(P_i)$ and this set is the graph of a bijection, we have $((\overline{x}_i, \overline{z}_i) : i \in I) \in \rho(\bigoplus_i P_i)$ implies that $\overline{z}_i = \overline{y}_i$ for every $i \in I$ and in particular i_0. Hence $\overline{b} = \overline{c}$ is unique and $\rho(P_{i_0})$ is the graph of a well defined function.

By a symmetrical argument to the preceding one, again using the facts that $\rho(\bigoplus_i P_i)$ is the direct sum of the pp-sets $\rho(P_i)$ and that $\rho(\bigoplus_i P_i)$ is the graph of a bijection, it follows that $\rho(P_{i_0})$ is the graph of a one-to-one function.

□

The converse will not hold in general. For $M \equiv \bigoplus_{i \in I} P_i$ and pp-formulas ϕ_1 and ϕ_2 such that $\forall i \in I$, $\phi_1 \sim_{P_i} \phi_2$, it may not be the case that $\phi_1 \sim_M \phi_2$. There follows an example where this does not hold.

Definition 3.4.8. *The* Prüfer p-group $\mathbb{Z}_{p^\infty}$ *is the direct limit of the abelian groups* $\mathbb{Z}_{p^n}$ *ordered by inclusion.*

$$\mathbb{Z}_p \subset \mathbb{Z}_{p^2} \subset \ldots \subset \mathbb{Z}_{p^\infty}$$

The Prüfer p-group is a module over various rings, including the ring of integers $\mathbb{Z}$, its localisation at the prime ideal (p), $\mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} : p \nmid b\}$ and its completion $\overline{\mathbb{Z}_{(p)}}$. The only proper additive subgroups of $\mathbb{Z}_{p^\infty}$ are the $\mathbb{Z}_{p^n}$ and so in particular the only infinite subgroup is $\mathbb{Z}_{p^\infty}$ itself. It follows immediately from Baur's pp-elimination that the only definable subsets of $\mathbb{Z}_{p^\infty}$ will be finite or cofinite but this does not hold for subsets of higher powers.

For the remainder of this section, we set M to be the right $\mathbb{Z}_{(p)}$-module $\mathbb{Z}_{(p)} \oplus \mathbb{Z}_{p^\infty}$. We take $\phi_1(v)$ and $\phi_2(u)$ to be the unary pp-formulas $v = v$ and $\exists w\ w \cdot p = u$ respectively. Then $\phi_1(\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}$ and $\phi_2(\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)} \cdot p \subset \mathbb{Z}_{(p)}$ and the pp-formula $v \cdot p = u$ is a bijection between them. Also $\phi_1(\mathbb{Z}_{p^\infty}) = \mathbb{Z}_{p^\infty}$ and $\phi_2(\mathbb{Z}_{p^\infty}) = \mathbb{Z}_{p^\infty}$ and the pp-formula $v = u$ is a bijection between them.

Proposition 3.4.9. *Writing $A := \mathbb{Z}_{(p)}$ and $B := \mathbb{Z}_{p^\infty}$ for brevity, we have $M = A \oplus B$, $\phi_1 \sim_A \phi_2$ and $\phi_1 \sim_B \phi_2$ as shown immediately above. However $\phi_1 \nsim_M \phi_2$.*

Proof. We assume for contradiction that $\rho(v, u)$ is a pp-formula and $\rho(M)$ is the graph of a bijection from $\phi_1(M) = M = A \oplus B$ to $\phi_2(M) = (A \cdot p) \oplus B$.

Given $N \in \text{Mod-}R$, we write $\langle N \rangle$ for the definable subcategory of Mod-R generated by N. A *definable subcategory* of Mod-R is a subcategory closed under direct limits, direct products and pure submodules. Then an R-module N' is in $\langle N \rangle$ iff every pp-pair closed on N is closed on N'. Definable subcategories of a module category are closed under pure submodules and hence direct summands.

Corollary 6.1.5 of [30] says that if X is a closed subset of Zg_R, the Ziegler spectrum of R, and R_R belongs to the corresponding definable subcategory of Mod-R, then the ring of definable scalars of X is R.

Our module M has the ring $R = \mathbb{Z}_{(p)}$ as a direct summand and hence $R_R \in \langle M \rangle$. Therefore R is the ring of definable scalars of $\langle M \rangle$ by the result quoted above. Hence if ρ is a pp-formula defining a bijection between $\phi_1(M)$ and $\phi_2(M)$, the map given by ρ is equivalent to the action of some scalar $r \in R = \mathbb{Z}_{(p)}$.

Clearly the action of a definable scalar, being in this case the action of a scalar element of the ring, on a direct sum of modules projects to the action of the same ring element in each of the summands. If the definable scalar given by $\rho(v, w)$ acts as multiplication by r, then $\phi_1(\mathbb{Z}_{(p)} \oplus \mathbb{Z}_{p^\infty})$ is mapped bijectively to $\phi_2(\mathbb{Z}_{(p)} \oplus \mathbb{Z}_{p^\infty})$ via the map $v \mapsto vr$. Writing v_1 and v_2 for the projections of the variables onto their $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_{p^\infty}$ arguments respectively, we have that $v_1 \mapsto v_1 r$ is a bijection from $\phi_1(\mathbb{Z}_{(p)})$ to $\phi_2(\mathbb{Z}_{(p)})$ and $v_2 \mapsto v_2 r$ is a bijection from $\phi_1(\mathbb{Z}_{p^\infty})$ to $\phi_2(\mathbb{Z}_{p^\infty})$.

Recalling the formulas in question, $\phi_1(v)$ is $v = v$ and $\phi_2(u)$ is $p \mid u$. Therefore

$\phi_1(\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}$, $\phi_1(\mathbb{Z}_{p^\infty}) = \mathbb{Z}_{p^\infty}$, $\phi_2(\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)} \cdot p$ and $\phi_2(\mathbb{Z}_{p^\infty)} = \mathbb{Z}_{p^\infty}$. It remains to observe that multiplication by no element of the ring $\mathbb{Z}_{(p)}$ is simultaneously a bijection from $\mathbb{Z}_{p^\infty}$ to itself and from $\mathbb{Z}_{(p)}$ to $\mathbb{Z}_{(p)} \cdot p$. Let $r \in \mathbb{Z}_{(p)}$. Then $v \mapsto vr$ is a bijection from $\mathbb{Z}_{(p)}$ to $\mathbb{Z}_{(p)} \cdot p$ if and only if $p \mid r$ and $p^2 \nmid r$. But multiplication by any such element cannot be a bijection on the Prüfer p-group $\mathbb{Z}_{p^\infty}$ as this operation is not one-to-one on $\mathbb{Z}_{p^\infty}$. Therefore in this example $\phi_1 \sim_A \phi_2$ and $\phi_1 \sim_B \phi_2$ but $\phi_1 \not\sim_M \phi_2$.

□

Remark. The example above demonstrates that for $A, B \in$ Mod-R, $M = A \oplus B$ and pp-formulas $\theta_1, \theta_2 \in \mathcal{L}_R$, the conditions $\theta_1 \sim_A \theta_2$ and $\theta_1 \sim_B \theta_2$ together do not imply $\theta_1 \sim_M \theta_2$.

There is an epimorphism of rings from $\mathbb{Z}$ to its localisation at the ideal (p). Thus the ring of definable scalars of $\mathbb{Z}_{(p)}$ regarded as a $\mathbb{Z}$-module via the ring epimorphism from $\mathbb{Z} \to \mathbb{Z}_{(p)}$, is the ring itself, $\mathbb{Z}_{(p)}$. Thus if we set $C = \mathbb{Z}_{(p)}, D = \mathbb{Z}_{p^\infty}$ and $N = C \oplus D$ all in Mod-$\mathbb{Z}$, we have the conditions $\phi_1 \sim_C \phi_2$, $\phi_1 \sim_D \phi_2$ and $\phi_1 \not\sim_N \phi_2$ because they held over $\mathbb{Z}_{(p)}$, with the same ring of definable scalars. By abuse of notation, the $\mathcal{L}_{\mathbb{Z}_{(p)}}$-formulas ϕ_1 and ϕ_2 from the preceding discussion; the unary pp-formulas $v = v$ and $\exists w \; w \cdot p = u$ respectively, are identified with their copies in $\mathcal{L}_{\mathbb{Z}}$.

The ring $\mathbb{Z}_{(p)}$ and its completion $\overline{\mathbb{Z}_{(p)}}$ are elementarily equivalent $\mathcal{L}_{\mathbb{Z}}$-structures. Hence, writing $E = \overline{\mathbb{Z}_{(p)}}$, $F = \mathbb{Z}_{p^\infty}$, $L = E \oplus F \in$ Mod-$\mathbb{Z}$, for brevity, we have $\phi_1 \sim_C \phi_2 \Rightarrow \phi_1 \sim_E \phi_2$. This is because there is some pp-bijection θ witnessing the equivalence relation $\phi_1 \sim_C \phi_2$ and there is a sentence of $\mathcal{L}_{\mathbb{Z}}$ that is interpreted in any $\mathcal{L}_{\mathbb{Z}}$-structure as saying "the solution set of θ is the graph of a bijection between the solution sets of ϕ_1 and ϕ_2". This sentence holds in $C = \mathbb{Z}_{(p)}$ and $E = \overline{\mathbb{Z}_{(p)}} \equiv \mathbb{Z}_{(p)}$. Therefore $\phi_1 \sim_E \phi_2$ and $\phi_1 \sim_F \phi_2$ but $\phi_1 \not\sim_L \phi_2$.

The module L here is a direct sum of indecomposable pure-injective $\mathbb{Z}$-modules, E and F. Recall that every module is elementarily equivalent to a direct sum of indecomposable pure-injectives. The above discussion shows that given pp-formulas that are in the same equivalence classes (modulo positive primitive bijections) for each indecomposable pure-injective in the sum, they are not necessarily equivalent in the direct sum.

Definition 3.4.10. *For any ring R, the category $(\text{mod-}R, Ab)^{fp}$ has as objects the finitely presented functors from the finitely presented right R-modules to Ab, the category of abelian groups, and is called fun-R. The morphisms of the category fun-R are the natural transformations between the functors.*

Definition 3.4.11. *The category $\mathbb{L}_R^{eq+}$ of positive primitive sorts (pp-sorts) has as its objects the (parameter free) pp-pairs and its morphisms are the pp-definable maps between them in the language of R-modules. $\mathbb{L}_R^{eq+}$ is also called the category of pp imaginaries.*

The terminology of a pp-sort comes from the fact that a parameter free pp-formula defines a subgroup of a power of the home sort for every R-module M (the power being the number of free variables), and a pp-pair ϕ/ψ defines a quotient set of the cosets of $(\phi \wedge \psi)(M)$ in $\phi(M)$. The elements of this quotient set can be regarded as imaginary elements of a multi-sorted structure. This thesis concerns Grothendieck rings of one-sorted structures, but the category of pp-sorts plays an important rôle in the later chapters.

The pp-pairs may be regarded as functors from Mod-R to *Ab*. The evaluation of these functors at R-modules are the factor groups $\phi(M)/\psi(M)$ and a pp-definable map between pp-sorts in $\mathbb{L}_R^{eq+}$ yields a group homomorphism between the corresponding factor groups. Recall that every module is elementarily equivalent to a direct sum of indecomposable pure-injective modules. Our example above demonstrates that pp-pairs that are isomorphic in each indecomposable pure-injective summand of a module M are not necessarily isomorphic in M itself, although the converse does hold. Hence if we are interested in the isomorphisms between pp-sorts (equivalently functors in fun-R), over each module of Mod-R, it is not sufficient to calculate the equivalence classes for just the indecomposable pure-injective modules, disregard the particular pp-definable isomorphisms, and compute them for general modules by considering the decomposition.

3.5 Modules over a direct product of rings

If M is a right R-module for a ring $R = R_1 \times R_2$ then there is a natural decomposition induced of the form $M = M_1 \oplus M_2$, where M_i is a module over R_i and $M \cdot (1,0) = M_1$, $M \cdot (0,1) = M_2$. We fix $R = R_1 \times R_2$ and $M = M_1 \oplus M_2$ throughout this section. The remainder of this section is devoted to proving the following theorem.

Theorem 3.5.1. *For a module M as above, the Grothendieck ring is isomorphic to the tensor product over $\mathbb{Z}$ of the Grothendieck rings of the summands,*
$K_0(M) \cong K_0(M_1) \otimes_{\mathbb{Z}} K_0(M_2)$.

Definition 3.5.2. *Let $\phi(\overline{v})$ be an $\mathcal{L}_{R_1}(M_1)$-formula. Define $\phi_R(\overline{v})$ to be the $\mathcal{L}_R(M)$-formula obtained from ϕ by regarding the variables $\overline{v}$ in $\mathcal{L}_{R_1}$ now as variables in $\mathcal{L}_R$ and each occurrence of a function symbol for a ring element r replaced by the function symbol for $(r,0)$ in $\mathcal{L}_R$ and each constant symbol $c \in M_1$ to be replaced by the constant $(c,0) \in M$.*

Analogously, for an $\mathcal{L}_{R_2}(M_2)$-formula ψ we define $\psi_R(\overline{v})$ to be the $\mathcal{L}_R(M)$-formula obtained from ψ by regarding the variables $\overline{v}$ in $\mathcal{L}_{R_2}$ now as variables in $\mathcal{L}_R$ and each occurrence of a function symbol for a ring element r replaced by the function symbol for $(0,r)$ in $\mathcal{L}_R$ and each constant symbol $c \in M_2$ to be replaced by the constant $(0,c) \in M$.

Definition 3.5.3. *With the above notation, define the $\mathcal{L}_R(M)$-formula ϕ' to be $\phi_R(\overline{v}) \wedge \overline{v} \cdot (0,1) = \overline{0}$.*

Analogously, for an $\mathcal{L}_{R_2}$-formula ψ we define the $\mathcal{L}_R(M)$-formula ψ' to be $\psi_R(\overline{v}) \wedge \overline{v} \cdot (1,0) = \overline{0}$.

In the above definitions the action of a ring element on a tuple of elements of a module is taken to be the diagonal action, e.g. $(v_1, \ldots, v_n) \cdot r = (v_1 r, \ldots, v_n r)$.

Definition 3.5.4. *Given an arbitrary $\mathcal{L}_R(M)$-formula ψ, we obtain the $\mathcal{L}_{R_i}(M_i)$-formula ψ_{R_i}, for $i = 1, 2$ by regarding the variables as living in the new language and replacing every function symbol for the ring element (r_1, r_2) with the function symbol for r_i and every constant (c_1, c_2) from M with the constant $c_i \in M_i$.*

For the remainder of this section all languages are assumed to contain constant symbols for every element of the appropriate model $M, M_1, M2$ and all formulas may contain parameters.

Observe that for any $\mathcal{L}_{R_1}$ formula $\theta(v_1, \ldots, v_n)$ and any $x_1, \ldots, x_n \in M_1$, we have $M_1 \models \theta(v_1, \ldots, v_n) \Leftrightarrow M \models \theta_R((v_1, 0), \ldots, (v_n, 0)) \Leftrightarrow M \models \theta'(v_1, \ldots, v_n)$. Also, for any formula $\phi(\overline{v})$ in $\mathcal{L}_{R_i}$, the formula $(\phi')_{R_i}(\overline{v})$ is simply $\phi(\overline{v}) \wedge \overline{v} \cdot 0 = \overline{0}$. This is trivially equivalent to the original formula ϕ.

Recall that $[\phi]$ denotes the class in $\widetilde{\mathrm{Def}}(M_1)$ of an $\mathcal{L}_{R_1}$-formula ϕ, and similarly for $\widetilde{\mathrm{Def}}(M_2)$ and $\widetilde{\mathrm{Def}}(M)$. That the map $\phi \mapsto \phi'$ establishes an embedding from $\mathrm{Def}(M_1)$ to $\mathrm{Def}(M)$ is immediate since clearly $\phi'(M) = \phi(M_1)$, regarding M_1 as a subset of M via the canonical embedding.

Proposition 3.5.5. *This embedding from Def(M_1) to Def(M) induces a well defined map from $\widetilde{Def}(M_1)$ to $\widetilde{Def}(M)$, i.e. the equivalence relation of being in definable bijection is preserved and reflected.*

Proof. Let $\phi_1, \phi_2 \in \mathcal{L}_{R_1}$ be such that $[\phi_1] = [\phi_2]$. We will show that it follows that $[\phi_1'] = [\phi_2'] \in \widetilde{\mathrm{Def}}(M)$. For clarity of presentation, we prove the result for unary formulas ϕ_1, ϕ_2 although there is no extra work for the more general proof. Firstly let $(x, y) \in M^2$ and suppose $M \models \rho'(x, y)$ for some $\mathcal{L}_{R_1}$-formula ρ. Then $(x, y) \cdot (0, 1) = (0, 0)$ where $(0, 1)$ in the equation is the element of $R = R_1 \times R_2$ and $(0, 0)$ is the zero element of M^2. Writing elements of $M = M_1 \oplus M_2$ in two argument form, this equation is
$((x_1, x_2), (y_1, y_2)) \cdot (0, 1) = ((0, x_2), (0, y_2)) = ((0, 0), (0, 0))$ and thus $x_2 = y_2 = 0 \in M_2$.

Let the $\mathcal{L}_{R_1}$-formula $\rho(\overline{v}, \overline{w})$ define (the graph of) a bijection between $\phi_1(M_1)$ and $\phi_2(M_1)$. It suffices to show $\rho'(\overline{v}, \overline{w})$ is the desired bijection in $\mathrm{Def}(M)$. Now $M \models \rho'(\overline{x}, \overline{y})$ implies that $(\overline{x}, \overline{y}) \cdot (0, 1) = \overline{0}$ and also $M \models \rho_R(\overline{x}, \overline{y})$. From the first implication, we have $\rho'(M) \subseteq (M_1 \oplus 0)^l$.

We will show that $M \models \exists w\, \rho'(x, w)$ iff $x \in \phi_1'(M)$. Assuming the left hand side, there is some y such that $(x, y) \in \rho'(M)$. So $M \models \rho_R(x, y) \wedge (x, y) \cdot (0, 1) = (0, 0)$.

Therefore writing $(x, y) = ((x_1, x_2), (y_1, y_2))$ we have $x_2 = y_2 = 0$ and $M \models \rho_R((x_1, 0), (y_1, 0))$, which implies $M_1 \models \rho(x_1, y_1)$. Since $\rho(M_1)$ is the graph of a bijection from $\phi_1(M_1)$ to $\phi_2(M_1)$, this implies that $x_1 \in \phi_1(M_1)$ and $x = (x_1, 0) \in \phi'_1(M)$ as required.

Conversely assuming the right hand side, $x \in \phi'_1(M)$ implies $x = (x_1, 0)$ and $M \models (\phi_1)_R(x_1, 0)$. Hence $M_1 \models \phi_1(x_1)$ and therefore there is some (unique) $y_1 \in M_1$ such that $M_1 \models \rho(x_1, y_1)$. Therefore $M \models \rho'((x_1, 0), (y_1, 0))$ and we have $M \models \exists w\, \rho'(x, w)$ as required.

A symmetrical argument yields $M \models \exists w\, \rho'(w, y)$ iff $y \in \phi'_2(M)$. It remains to show that the set $\rho'(M)$ is the graph of a bijection. If $M \models \rho'(x, y) \wedge \rho'(z, y)$, then $x = (x_1, 0), y = (y_1, 0)$ and $z = (z_1, 0)$ and we have $M \models \rho_R((x_1, 0), (y_1, 0)) \wedge \rho_R((z_1, 0), (y_1, 0))$. Hence $M_1 \models \rho(x_1, y_1) \wedge \rho(z_1, y_1)$ and since ρ defines a bijection over M_1, this yields $x_1 = z_1$ and $x = z$. By a symmetrical argument

$M \models \rho'(x, y) \wedge \rho'(x, z) \quad \Rightarrow \quad y = z.$ □

Proposition 3.5.6. *The converse to the previous proposition holds. If ϕ_1, ϕ_2 in $\mathcal{L}_{R_1}$ are such that $[\phi'_1] = [\phi'_2] \in \widetilde{Def}(M)$, then $[\phi_1] = [\phi_2] \in \widetilde{Def}(M_1)$.*

Proof. For clarity of presentation, we again work with unary formulas ϕ_1, ϕ_2. Suppose $\phi_1(v), \phi_2(v) \in \mathcal{L}_{R_1}$ and $[\phi'_1] = [\phi'_2] \in \widetilde{\mathrm{Def}}(M)$. Then there is some $\mathcal{L}_R$-formula $\theta(v, w)$ whose solution set $\theta(M) \subset M^2$ is the graph of a bijection from $\phi'_1(M)$ to $\phi'_2(M)$. For any $x, y \in M$, if $M \models \theta(x, y)$ then $x \in \phi'_1(M)$ and $y \in \phi'_2(M)$. If follows that $M \models x \cdot (0, 1) = 0 \wedge y \cdot (0, 1) = 0$ and we may write $x = (x_1, 0), y = (y_1, 0) \in M_1 \oplus M_2$.

The set $\theta(M)$ is contained in $(M_1 \oplus 0)^2$ and we have $M \models \forall v, w(\theta(v, w) \rightarrow (v(0, 1) = 0 \wedge w(0, 1) = 0))$. Thus if we set $\hat{\theta}(v, w)$ to be the $\mathcal{L}_R$-formula obtained by replacing all the function symbols in $\theta(v, w)$ for every ring element (r_1, r_2) with that for $(r_1, 0)$ and every constant (c_1, c_2) from M with the constant $(c_1, 0)$, we have $\theta(M) = \hat{\theta}(M)$.

We define this formula $\hat{\theta}$, which is equivalent to θ, so that we can now simply observe that $(\hat{\theta}_{R_1})' = \hat{\theta}$. Therefore $\hat{\theta}_{R_1}(v, w)$ is the formula for a definable bijection in $\mathrm{Def}(M_1)$ between $\phi_1(M_1)$ and $\phi_2(M_1)$.

□

The embeddings from $\mathrm{Def}(M_i)$ to $\mathrm{Def}(M)$, for $i = 1, 2$, thereby induce embeddings $\widetilde{\mathrm{Def}}(M_i) \hookrightarrow \widetilde{\mathrm{Def}}(M)$. These in turn each induce a map of rings from $K_0(M_i)$ to $K_0(M)$.

We introduce the notation π_{M_1} for the projection of any element or tuple from M onto its M_1 argument, i.e. $\pi_{M_1}((x_{11}, x_{12}), \ldots, (x_{n1}, x_{n2})) = (x_{11}, \ldots, x_{n1}) \in {M_1}^n$.

Claim. Given any sets $A, B \in \mathrm{Def}(M)$, there are sets $A', B' \in \mathrm{Def}(M)$ such that $[A] = [A']$ and $[B] = [B'] \in \widetilde{\mathrm{Def}}(M)$ and $\pi_{M_i}(A') \cap \pi_{M_i}(B') = \emptyset$ for $i = 1, 2$. We may for example set $A' := A \times \{(x_1, x_2)\}$ and $B' := B \times \{(y_1, y_2)\}$ where $x_i \neq y_i$.

The equivalence classes $[\phi] \in \widetilde{\mathrm{Def}}(M)$ of the pp-formulas $\phi \in \mathcal{L}_R$ form a generating set for $K_0(M)$, as seen in Chapter 3. Similarly, for $i = 1, 2$, the classes of positive primitive $\mathcal{L}_{R_i}$-formulas in $\widetilde{\mathrm{Def}}(M_i)$ generate the Grothendieck ring $K_0(M_i)$.

The Grothendieck rings $K_0(M_1)$ and $K_0(M_2)$ are commutative rings and their tensor product over $\mathbb{Z}$ is denoted $K_0(M_1) \otimes_{\mathbb{Z}} K_0(M_2)$ or simply $K_0(M_1) \otimes K_0(M_2)$. It is the $\mathbb{Z}$-module characterised by the universal property of tensor products; every bilinear $\mathbb{Z}$-module homomorphism $f : K_0(M_1) \times K_0(M_2) \to S$ lifts to a unique $\mathbb{Z}$-module homomorphism $\tilde{f} : K_0(M_1) \otimes_{\mathbb{Z}} K_0(M_2) \to S$ such that $f((x, y)) = \tilde{f}(x \otimes y)$.

$$\begin{tikzcd}
K_0(M_1) \times K_0(M_2) \arrow[r, "\otimes"] \arrow[dr, "f"] & K_0(M_1) \otimes K_0(M_2) \arrow[d, dashed, "\tilde{f}"] \\
 & S
\end{tikzcd}$$

We define a map T from $K_0(M_1) \times K_0(M_2)$ to $K_0(M)$ and show that it is a bilinear $\mathbb{Z}$-module homomorphism and that it satisfies the universal property of tensor products. Then the image of T, which is all of $K_0(M)$, must be isomorphic to $K_0(M_1) \otimes K_0(M_2)$. It is sufficient to define T on a generating set for $K_0(M_1) \times K_0(M_2)$, namely the elements corresponding to a pp-formula from each language, and check that the function is bilinear.

Let (x, y) be an element of $K_0(M_1) \times K_0(M_2)$ such that $x = [X]$ is the equivalence class of some set $X \in \mathrm{Def}(M_1)$ and $y = [Y]$ for some $Y \in \mathrm{Def}(M_2)$. Such elements generate the whole of $K_0(M_1) \times K_0(M_2)$. Choose formulas α, β from $\mathcal{L}_{R_1}(M_1)$ and $\mathcal{L}_{R_2}(M_2)$ respectively such that $X = \alpha(M_1)$ and $Y = \beta(M_2)$. Then there are $\mathcal{L}_R(M)$-formulas α', β' as shown above such that $X = \alpha'(M)$ and $Y = \beta'(M)$. We set $T((x, y))$ to be the equivalence class of the set $\alpha'(M) \times \beta'(M)$ in $K_0(M)$. This is well defined by Proposition 3.5.5.

Claim. The map T is bilinear on $K_0(M_1) \times K_0(M_2)$, i.e. $T(x, y_0 + y_1) = T((x, y_0)) + T((x, y_1))$ and $T((x_0 + x_1, y)) = T((x_0, y)) + T((x_1, y))$.

Proof. By symmetry, we may demonstrate only the first equality. Again it is sufficient to prove this for generators. Let $x = [X]$, $y_0 = [Y_0]$ and $y_1 = [Y_1]$ for $X \in \mathrm{Def}(M_1)$ and $Y_0, Y_1 \in \mathrm{Def}(M_2)$. The element "$y_0 + y_1$" of $K_0(M_2)$ is then equal to $[Y_0] + [Y_1] = [\{c\} \times Y_0 \cup \{c'\} \times Y_1]$ for parameters c, c' from M_2. So $T((x, y_0 + y_1)) = T(([X], [\{c\} \times Y_0 \cup \{c'\} \times Y_1])) = [X \times (\{c\} \times Y_0 \cup \{c'\} \times Y_1)] = [(X \times \{c\} \times Y_0) \cup (X \times \{c'\} \times Y_1)] = [X \times Y_0] + [X \times Y_1] = T((x, y_0)) + T((x, y_1))$ as desired. Using the fact that all sets in $\mathrm{Def}(M_1)$ and $\mathrm{Def}(M_2)$ are present in $\mathrm{Def}(M)$, as explained above. This establishes the bilinearity of T.

□

The element "ny" of $K_0(M_2)$ is equal to $n \cdot [Y]$ or $[\{c_1, \ldots, c_n\} \times Y]$ for any parameters $c_1, \ldots, c_n$. So $T((x, ny)) = T([X], [\{c_1, \ldots, c_n\} \times Y]) = [X \times \{c_1, \ldots, c_n\} \times Y] = n \cdot [X \times Y]$. Observe that this implies $T((nx, y)) = T((x, ny)) = nT((x, y))$ where n is a natural number.

Lemma 3.5.7. *For $i = 1, 2$, the sets in Def(M_i) that are defined by pp-formulas are* exactly *the projections onto the M_i components of pp-sets in Def(M).*

Proof. We prove the result for $i = 1$. Given a set in $\mathrm{Def}(M_1)$ defined by a pp-formula $\phi(\overline{v}, \overline{c}) \in \mathcal{L}_{R_1}$, the set $\phi(M_1) \oplus \{\overline{0}\}$ in $\mathrm{Def}(M)$ will be defined by a pp-formula of $\mathcal{L}_R$.

The set $\phi(M_1) \oplus \{\overline{0}\}$ is the solution set to the formula $\phi_R(\overline{v}) \wedge \overline{v} \cdot (\overline{0}, \overline{1}) = 0$, which is clearly positive primitive.

Conversely, the projection of a pp-set $\psi(M)$ onto its M_1 component will be definable by some pp-formula from $\mathcal{L}_{R_1}(M_1)$. Let $\psi((v_1, w_1), \ldots, (v_l, w_l))$ be an l-ary pp-formula in $\mathcal{L}_R$. The variables in the formula ψ live in M and abusing notation, we write $(\overline{v}, \overline{w})$ for the tuple $((v_1, w_1), \ldots, (v_l, w_l)) \in (M_1 \oplus M_2)^l$. Similarly, we write $\overline{c} = (\overline{c}_1, \overline{c}_2)$ to display the M_1 and M_2 components of each parameter in the formula.

Let $\overline{v}$ be an l-tuple from M_1. Then $M_1 \models \psi_{R_1}(\overline{v}, \overline{c}_1)$ iff there is some l-tuple $\overline{w}$ from M_2 such that $M \models \psi((\overline{v}, \overline{w}), (\overline{c}_1, \overline{c}_2))$.

Suppose $M \models \psi(\overline{u}, \overline{c})$. The parameter free formula $\psi(\overline{u}, \overline{0})$ defines an additive subgroup of M^l, where l is the number of free variables in the formula ψ. The additive group $\psi(M, \overline{0}) \subseteq M^l$ is a direct sum of additive groups $\psi(M, \overline{0}) = \psi_{R_1}(M_1, \overline{0}) \oplus \psi_{R_2}(M_2, \overline{0})$. Replacing the 0 constants in the formula with the constants $\overline{c}$ produces a coset $\psi(M, \overline{c})$ of the additive group $\psi(M, \overline{0})$. This is then a direct sum $\psi(M, \overline{c}) = \psi_{R_1}(M_1, \overline{c}_1) \oplus \psi_{R_2}(M_2, \overline{c}_2)$ of cosets of the groups $\psi_{R_1}(M_1, \overline{0})$ and $\psi_{R_2}(M_2, \overline{0})$ respectively. Hence $\psi_1(M_1, \overline{c}_1)$ is the projection onto the M_1 component of the pp-set $\psi(M, \overline{c}) \in \mathrm{Def}(M)$. $\square$

It should be noted that for an arbitrary $\mathcal{L}_R$-formula θ that is not positive primitive, it does not generally hold that $\theta(M) = \theta_{R_1}(M_1) \oplus \theta_{R_2}(M_2)$.

Proposition 3.5.8. *Given a general formula $\theta(\overline{v}) \in \mathcal{L}_R(M)$, we may find formulas $\alpha_1, \ldots, \alpha_n \in \mathcal{L}_{R_1}(M_1)$ and $\beta_1, \ldots, \beta_n \in \mathcal{L}_{R_2}(M_2)$ such that $\theta(M) = \bigcup_{i=1}^n \alpha_i(M_1) \times \beta_i(M_2)$.*

Proof. This follows from the Baur-Monk elimination in $\mathcal{L}_R(M)$ and the fact proven in the claim above that the pp-sets in $\mathrm{Def}(M_1)$ and $\mathrm{Def}(M_2)$ are exactly the projections of the pp-sets in $\mathrm{Def}(M_1 \oplus M_2)$. Since we are working in a language of right modules, there exist pp-formulas $\phi_i(\overline{v}), \psi_{ij}(\overline{v})$ in $\mathcal{L}_R$ such that

$$M \models \forall \overline{v} \left(\theta(\overline{v}) \leftrightarrow (\bigvee_{i=1}^{m} \phi_i(\overline{v}) \wedge \bigwedge_{j=1}^{n_i} \neg \psi_{ij}(\overline{v})) \right)$$

We will prove the proposition by induction on complexity of θ and we will prove a number of minor claims in the process. If $\theta(\overline{v})$ is logically equivalent to a pp-formula $\phi(\overline{v})$ then $\theta(M) = \phi(M)$ and the latter is equal to $\phi_{R_1}(M_1) \oplus \phi_{R_2}(M_2)$ as shown above.

Claim. The proposition holds when $\theta(\overline{v})$ is logically equivalent to the formula $\phi \wedge \bigwedge_{j=1}^{n} \neg\psi_j(\overline{v})$, where $\phi, \psi_1, \ldots, \psi_n$ are pp-formulas of $\mathcal{L}_R(M)$. We prove this claim by induction on n.

Proof of claim. For $n = 0$ the result is the positive primitive case above. For the inductive step, we assume the result holds for $n = N$. Let

$$\theta(M) = (\phi \wedge \bigwedge_{j=1}^{N+1} \neg\psi_j)(M) = (\phi \wedge \bigwedge_{j=1}^{N} \neg\psi_j(M)) \setminus \psi_{N+1}(M)$$

and by the inductive hypothesis $(\phi \wedge \bigwedge_{j=1}^{N} \neg\psi_j)(M) = \bigcup_{i=1}^{t} \alpha_i(M_1) \times \beta_i(M_2)$. Hence

$$\theta(M) = (\bigcup_{i=1}^{t} \alpha_i(M_1) \times \beta_i(M_2)) \setminus ((\psi_{N+1})_{R_1}(M_1) \times {M_2}^L \cup {M_1}^L \times (\psi_{N+1})_{R_2}(M_2))$$

where L is the number of free variables, the length of $\overline{v}$. Now we observe

$$\left(\bigcup_{i=1}^{t} \alpha_i(M_1) \times \beta_i(M_2)\right) \setminus ((\psi_{N+1})_{R_1}(M_1) \times {M_2}^L \cup {M_1}^L \times (\psi_{N+1})_{R_2}(M_2))$$

$$= \bigcup_{i=1}^{t} (\alpha_i(M_1) \times (\beta_i \wedge \neg(\psi_{N+1})_{R_2})(M_2) \cup (\alpha_i \wedge \neg(\psi_{N+1})_{R_1})(M_1) \times \beta_i(M_2))$$

This establishes the inductive step and hence the claim.

Continuing the induction on the complexity of θ, the most general formula in $\mathcal{L}_R(M)$ is logically equivalent to a finite disjunction of formulas of the form just considered. Furthermore this disjunction can be taken to be a disjunction of mutually exclusive formulas by Lemma 3.2.1. Assume that we have

$$M \models \forall\overline{v}\left(\theta(\overline{v}) \leftrightarrow (\bigvee_{i=1}^{m} \phi_i(\overline{v}) \wedge \bigwedge_{j=1}^{n_i} \neg\psi_{ij}(\overline{v}))\right)$$

and that the sets $(\phi_i \wedge \bigwedge_{j=1}^{n_i} \neg\psi_{ij})(M)$ for $i = 1, \ldots, m$ are disjoint. We have demonstrated that the sets $(\phi_i \wedge \bigwedge_{j=1}^{n_i} \neg\psi_{ij})(M)$ are each a finite union of sets of the form

$\alpha_k(M_1) \times \beta_k(M_2)$. Hence $\theta(M) = \bigcup_{i=1}^{m}(\phi_i \wedge \bigwedge_{j=1}^{n_i} \neg\psi_{ij})(M)$ is a finite union of a finite unions of these product sets. This completes the induction and the proof of Proposition 3.5.8.

□

Now we show that the function $T : K_0(M_1) \times K_0(M_2) \to K_0(M)$ satisfies the universal property of tensor products. Let S be any $\mathbb{Z}$-module and let $f : K_0(M_1) \times K_0(M_2) \to S$ be a bilinear homomorphism. Then we may define a homomorphism $\tilde{f}$ on $K_0(M)$ such that $f = \tilde{f} \circ T$. It is sufficient to consider generators for the Grothendieck ring as all the maps involved are $\mathbb{Z}$-linear. Let $z \in K_0(M)$ and suppose z is the equivalence class of the definable set $Z = \theta(M)$. By the above remark, Z may be expressed in the form $\bigcup_{i=1}^{n} \alpha_i(M_1) \times \beta_i(M_2)$, with the union a disjoint one. Set $\tilde{f}(z)$ to be $\sum_{i=1}^{n} f([\alpha_i(M_1)], [\beta_i(M_2)])$. Then $\tilde{f}$ is $\mathbb{Z}$-linear precisely because f is bilinear. Let $(x, y) \in K_0(M_1) \times K_0(M_2)$ and suppose $x = [\alpha(M_1)]$ and $y = [\beta(M_2)]$. Then $T((x, y)) = [\alpha'(M) \times \beta'(M)]$ and $\tilde{f} \circ T((x, y)) = \tilde{f}([\alpha'(M) \times \beta'(M)]) = \tilde{f}([\alpha(M_1) \times \beta(M_2)]) = f((x, y))$ as desired.

It remains to show that this map $\tilde{f}$ is well defined; suppose that our set $Z \in \mathrm{Def}(M)$, with representative $z \in K_0(M)$, can be expressed as $\bigcup_{i=1}^{n} \alpha_i(M_1) \times \beta_i(M_2)$. Suppose also that Z or another set in the same class of $\widetilde{\mathrm{Def}}(M)$ can be expressed as $\bigcup_{j=1}^{m} \gamma_j(M_1) \times \delta_j(M_2)$. Then we have defined $\tilde{f}(z)$ to be $\sum_{i=1}^{n} f([\alpha_i(M_1)], [\beta_i(M_2)])$ and also $\sum_{j=1}^{m} f([\gamma_j(M_1)], [\delta_j(M_2)])$, so we must demonstrate that these are in fact the same element of S. The map f is $\mathbb{Z}$-linear so

$$\sum_{i=1}^{n} f([\alpha_i(M_1)], [\beta_i(M_2)]) = f(\sum_{i=1}^{n}([\alpha_i(M_1)], [\beta_i(M_2)]))$$

and

$$\sum_{j=1}^{m} f([\gamma_j(M_1)], [\delta_j(M_2)]) = f(\sum_{j=1}^{m}([\gamma_j(M_1)], [\delta_j(M_2)]))$$

Let $\rho(\overline{v}, \overline{w})$ be an $\mathcal{L}_R$-formula with solution set over M equal to the graph of a bijection from the set $\bigcup_{i=1}^{n} \alpha_i(M_1) \times \beta_i(M_2)$ to the set $\bigcup_{j=1}^{m} \gamma_j(M_1) \times \delta_j(M_2)$. We do not treat the case where these are two decompositions of the same set Z, separately since it follows from the general case. By Proposition 3.5.8, we may also express the set $\rho(M)$ in the form $\bigcup_{k=1}^{t} \sigma_k(M_1) \times \tau_k(M_2)$. Clearly, we may take this union to be disjoint, by repeated use of this result of naïve set theory: for any sets A_1, A_2, B_1, B_2, we have $A_1 \times B_1 \cup A_2 \times B_2 = (A_1 \times B_1) \cup (A_2 \setminus A_1 \times B_2) \cup (A_1 \cap A_2 \times B_2 \setminus B_1)$.

Now the sets $\sigma_k(M_1) \times \tau_k(M_2)$ are subsets of $\rho(M)$, the graph of a bijection. So they must the graphs of bijections themselves, namely restrictions of the original.

Claim. It follows that each $\sigma_k(M_1)$ and $\tau_k(M_2)$ must be functional and injective. For example, if $M_1 \models \sigma_k(\overline{x}, \overline{y}) \wedge \sigma_k(\overline{x}', \overline{y})$, then taking any $(\overline{a}, \overline{b}) \in \tau_k(M_2)$, the tuples $((\overline{x}, \overline{a}), (\overline{y}, \overline{b}))$ and $((\overline{x}', \overline{a}), (\overline{y}, \overline{b}))$ are both contained in $\rho(M)$ and hence $\overline{x} = \overline{x}'$. Clearly this argument can be adapted to prove that σ_k is functional and that τ_k is both functional and injective.

For each $k = 1, \ldots, t$, observe that since $\sigma_k(\overline{v}_1, \overline{w}_1)$ is the formula of a bijection, its solution set is in definable bijection with its domain and its range. Therefore for each $k = 1, \ldots, t$ the sets defined by the formulas $\sigma_k(\overline{v}_1, \overline{w}_1)$, $\exists \overline{v}_1\, \sigma_k(\overline{v}_1, \overline{w}_1)$ and $\exists \overline{w}_1 \sigma_k(\overline{v}_1, \overline{w}_1)$ are all in the same definable bijection class and thus have the same representative in $\widetilde{\mathrm{Def}}(M_1)$ and in $K_0(M_1)$.

We will prove that

$$\sum_{i=1}^{n}([\alpha_i(M_1)], [\beta_i(M_2)]) = \sum_{j=1}^{m}([\gamma_j(M_1)], [\delta_j(M_2)]) \in K_0(M_1) \times K_0(M_2)$$

by showing that both sides of the equation are equal to the same particular sum of elements of $K_0(M_1) \times K_0(M_2)$, which are defined in the sequel. First observe that:

$$M_1 \models \forall \overline{v}_1 (\alpha_i(\overline{v}_1) \leftrightarrow (\bigvee_{k=1}^{t} \alpha_i(\overline{v}_1) \wedge \exists \overline{w}_1\, \sigma_k(\overline{v}_1, \overline{w}_1)) \leftrightarrow$$
$$(\bigvee_{k=1}^{t} \bigvee_{j=1}^{m} \alpha_i(\overline{v}_1) \wedge \exists \overline{w}_1\, (\sigma_k(\overline{v}_1, \overline{w}_1) \wedge \gamma_j(\overline{w}_1))))$$

This implies that in the Grothendieck ring, the following equation holds for each $i = 1, \ldots, n$:

$$[\alpha_i(M_1)] = \sum_{k=1}^{t}[\exists \overline{w}_1(\sigma_k \wedge \alpha_i)(M_1)] = \sum_{k=1}^{t}\sum_{j=1}^{m}[\exists \overline{w}_1(\sigma_k \wedge \alpha_i \wedge \gamma_j)(M_1)]$$

Similarly, for each $j = 1, \ldots, m$:

$$[\gamma_j(M_1)] = \sum_{k=1}^{t} [\exists \overline{v}_1(\sigma_k \wedge \gamma_j)(M_1)] = \sum_{k=1}^{t} \sum_{i=1}^{n} [\exists \overline{v}_1(\sigma_k \wedge \alpha_i \wedge \gamma_j)(M_1)]$$

For each $k = 1, \ldots, t$, $\sigma_k(\overline{v}_1, \overline{w}_1)$ is the formula of a bijection, and hence its solution set is in definable bijection with its domain and its range. Also any definable subset of $\sigma_k(M_1)$ is the graph of some restriction of the bijection. In particular for any $1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq t$ the sets defined by the formula $\alpha_i(\overline{v}_1) \wedge \sigma_k(\overline{v}_1, \overline{w}_1) \wedge \gamma_j(\overline{w}_1)$ and the two projection formulas $\exists \overline{v}_1(\alpha_i(\overline{v}_1) \wedge \sigma_k(\overline{v}_1, \overline{w}_1) \wedge \gamma_j(\overline{w}_1))$ and $\exists \overline{w}_1(\alpha_i(\overline{v}_1) \wedge \sigma_k(\overline{v}_1, \overline{w}_1) \wedge \gamma_j(\overline{w}_1))$ are in definable bijection. Hence the following equation holds in $K_0(M_1)$:

$$[\exists \overline{w}_1(\sigma_k \wedge \alpha_i \wedge \gamma_j)(M_1)] = [\exists \overline{v}_1(\sigma_k \wedge \alpha_i \wedge \gamma_j)(M_1)] = [(\sigma_k \wedge \alpha_i \wedge \gamma_j)(M_1)]$$

Let X_{ijk} denote this element of $K_0(M_1)$ in the sequel.

All that has been shown to be true of $\sigma_k(\overline{v}_1, \overline{w}_1)$ in $\widetilde{\mathrm{Def}}(M_1)$ also holds for $\tau_k(\overline{v}_2, \overline{w}_2)$ in $\widetilde{\mathrm{Def}}(M_2)$. Therefore, translating the above results to the corresponding results in M_2 we obtain:

$$[\beta_i(M_2)] = \sum_{k=1}^{t} [\exists \overline{w}_2(\tau_k \wedge \beta_i)(M_2)] = \sum_{k=1}^{t} \sum_{j=1}^{m} [\exists \overline{w}_2(\tau_k \wedge \beta_i \wedge \delta_j)(M_2)]$$

$$[\delta_j(M_2)] = \sum_{k=1}^{t} [\exists \overline{v}_2(\tau_k \wedge \delta_j)(M_2)] = \sum_{k=1}^{t} \sum_{i=1}^{n} [\exists \overline{v}_2(\tau_k \wedge \beta_i \wedge \delta_j)(M_2)]$$

Also the following equation holds in $K_0(M_2)$:

$$[\exists \overline{w}_2(\tau_k \wedge \beta_i \wedge \delta_j)(M_2)] = [\exists \overline{v}_2(\tau_k \wedge \beta_i \wedge \delta_j)(M_2)] = [(\tau_k \wedge \beta_i \wedge \delta_j)(M_2)]$$

Let Y_{ijk} denote this element of $K_0(M_2)$ in the sequel. Now in terms of the elements X_{ijk} of $K_0(M_1)$ and Y_{ijk} of $K_0(M_2)$ we have:

$$\sum_{i=1}^{n} ([\alpha_i(M_1)], [\beta_i(M_2)]) = \sum_{i=1}^{n} \left(\sum_{k=1}^{t} \sum_{j=1}^{m} X_{ijk}, \sum_{k=1}^{t} \sum_{j=1}^{m} Y_{ijk} \right)$$

and

$$\sum_{j=1}^{m} ([\gamma_j(M_1)], [\delta_j(M_2)]) = \sum_{j=1}^{m} \left(\sum_{k=1}^{t} \sum_{i=1}^{n} X_{ijk}, \sum_{k=1}^{t} \sum_{i=1}^{n} Y_{ijk} \right)$$

Hence by the $\mathbb{Z}$-bilinearity of f, we have as required:

$$f(\sum_{i=1}^{n}([\alpha_i(M_1)],[\beta_i(M_2)])) = f(\sum_{j=1}^{m}([\gamma_j(M_1)],[\delta_j(M_2)]))$$

Therefore the map $\tilde{f}$ is well defined, the map $T : K_0(M_1) \times K_0(M_2) \to K_0(M)$ is the tensor product map and $K_0(M) \cong K_0(M_1) \otimes K_0(M_2)$.

Remark. The results of this section will not generally apply to arbitrary modules of the form $M = M_1 \oplus M_2$ for $M_1, M_2 \in$ Mod-R. This material relies on the decomposition into a direct sum of modules being induced by the ring R being a product of rings $R_1 \times R_2$.

3.6 Morita equivalent rings

The key result of this section is that if R and S are Morita equivalent rings, then the categories of pp-sorts for R- and S-modules are equivalent categories. This is Proposition 3.6.7. In Section 6.3, we make use of this equivalence to prove Theorem 6.3.5. The theorem says that for Morita equivalent rings $R \equiv_M S$, there is an isomorphism of rings $K_0(\text{Mod-}R) \cong K_0(\text{Mod-}S)$. The rest of this section is a presentation of material, not due to the author, that is needed to prove Proposition 3.6.7.

Definition 3.6.1. *Two rings, R and S, are* Morita equivalent *if there is an additive equivalence between the categories Mod-R and Mod-S. The equivalence is a pair of additive functors F : Mod-R $\to$ Mod-S and G : Mod-S $\to$ Mod-R such that there exists an (R,S)-bimodule P which is a finitely generated projective generator for S-Mod and Mod-R and natural isomorphisms $F \cong (- \otimes_R P)$ and $G \cong Hom_R(-, P)$. An equivalent definition is that R and S are Morita equivalent if there exist bimodules X and Y such that:*

i) X is a left R-module and a right S-module

ii) Y is a right R-module and a left S-module

iii) $X \otimes_S Y = R$

iv) $Y \otimes_R X = S$

v) $X \cong Hom(Y_R, R_R)$

We write $R \equiv_M S$ when R and S are Morita equivalent rings. Morita equivalent rings R and S have equivalent categories of right modules (and equivalent categories of left modules). The proof of Proposition 3.6.7 will require a few lemmas.

Lemma 3.6.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be equivalent categories. Then the full subcategories of finitely presented objects $\mathcal{C}^{\mathrm{fp}}$ and $\mathcal{D}^{\mathrm{fp}}$ are also equivalent.*

Proof. Suppose the equivalence of $\mathcal{C}$ and $\mathcal{D}$ is given by a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$. Let M be a finitely presented object in C. Then $FM \in Ob(\mathcal{D})$ and we prove that it is finitely presented. It is sufficient to show that given any directed system $\{D_\lambda \mid \lambda \in \Lambda\}$ with a limit $D = \varinjlim D_\lambda$ and any $\mathcal{D}$-arrow $f : FM \to D$, the arrow f must factor through some D_λ, because this is one characterisation of finitely presented objects.

Let $\{D_\lambda \mid \lambda \in \Lambda\}$ be a directed system in $\mathcal{D}$ and consider its image under the functor G. The $\{GD_\lambda \mid \lambda \in \Lambda\}$ form a directed system in $\mathcal{C}$ and we put $C := \varinjlim GD_\lambda$. F and G form an equivalence of categories and hence M is isomorphic to GFM, via i say. M is finitely presented so the composite arrow $j := Gf \circ i$ from M to GD must factor through some GD_λ, say GD_{λ_0}. Since C is the direct limit of the GD_λ there must be a $\mathcal{C}$-arrow, c say, from C to GD such that the maps from each of the GD_λ to GD factor through c. The map $j : M \to GD$ then factors through c since it factors through GD_{λ_0}. All of this is presented in the following diagrams.

$$\begin{tikzcd} & FM \arrow[dd, "f"] \\ D_\mu \arrow[dr, dashed] \arrow[d, dotted, no head] & \\ {} \arrow[r, dashed] \arrow[d, dotted, no head] & D \\ D_\nu \arrow[ur, dashed] & \end{tikzcd}
\quad \overset{\text{functor } G}{\rightsquigarrow} \quad
\begin{tikzcd} & M \arrow[d, "i"', "\simeq"] \\ & GFM \arrow[dd, "Gf"] \\ GD_\mu \arrow[dr, dashed] \arrow[d, dotted, no head] & \\ {} \arrow[r, dashed] \arrow[d, dotted, no head] & GD \\ GD_\nu \arrow[ur, dashed] & \end{tikzcd}$$

$$
\begin{tikzcd}
& M \arrow[ddl, bend right] \arrow[dr, "\simeq", "i"'] \arrow[dddr, "j"'] & \\
& & GFM \arrow[dd, "Gf"] \\
GD_{\lambda_0} \arrow[drr, dashed] \arrow[d, dotted, no head] & & \\
{} \arrow[rr, dashed] \arrow[d, dotted, no head] & & GD \\
GD_\mu \arrow[urr, dashed] & &
\end{tikzcd}
\quad
\underset{\text{through } C}{\overset{\text{maps factor}}{\rightsquigarrow}}
\quad
\begin{tikzcd}
& M \arrow[ddl, bend right] \arrow[dddrr, "j"] & & \\
& & & \\
GD_{\lambda_0} \arrow[dr, dashed] \arrow[d, dotted, no head] & & & \\
{} \arrow[r, dashed] \arrow[d, dotted, no head] & C \arrow[rr, "c"] & & GD \\
GD_\mu \arrow[ur, dashed] & & &
\end{tikzcd}
$$

Consider the image of the righthand diagram above under the functor F. There will be a directed system $\{FGD_\lambda \mid \lambda \in \Lambda\}$ and the arrows from each FGD_λ to FC must factor through the direct limit of the directed system $\varinjlim FGD_\lambda$. Furthermore, since $FG \simeq \mathrm{id}_{\mathcal{D}}$, we have that FG will commute with direct limits so $\varinjlim FGD_\lambda \simeq FG(\varinjlim D_\lambda) = FGD$. The equivalence FG gives isomorphisms on the individual objects, so $D_\lambda \simeq FGD_\lambda$ for each $\lambda \in \Lambda$ and $D \simeq FGD$.

$$
\begin{tikzcd}
D_\nu \simeq FGD_\nu \arrow[dr, dashed] \arrow[d, dotted, no head] & & & \\
{} \arrow[r, dashed] \arrow[d, dotted, no head] & \varinjlim FGD_\lambda \arrow[r] \arrow[rr, bend left, "\simeq"] & FC \arrow[r, "Fc"] & FGD \simeq D \\
D_\mu \simeq FGD_\mu \arrow[ur, dashed] & & &
\end{tikzcd}
$$

Since j factors through GD_{λ_0}, its image $Fj : FM \to FGD$ factors through FGD_{λ_0}. The isomorphism $i : M \to GFM$ has an inverse, and the image of this inverse under F is $Fi^{-1} : FGFM \to FM$. The diagram below shows the action of FG on the original diagram. Recall that $FG \simeq id_{\mathcal{D}}$ and so we have the module isomorphisms as shown on the righthand diagram.

$$
\begin{tikzcd}
 & FM \arrow[dd, "f"] \\
D_\mu \arrow[dr, dashed] \arrow[d, dotted, no head] & \\
\cdot \arrow[r, dashed] \arrow[d, dotted, no head] & D \\
D_\nu \arrow[ur, dashed] &
\end{tikzcd}
\quad \overset{\rightsquigarrow}{\text{functor } FG} \quad
\begin{tikzcd}
 & FM \arrow[dl, bend right] \arrow[dd, "Fj"] & FGFM \arrow[l, "Fi^{-1}"', "\simeq"] \arrow[ddl, "FGf"] \\
D_{\lambda_0} \simeq FGD_{\lambda_0} \arrow[dr, dashed] \arrow[d, dotted, no head] & & \\
D_\mu \simeq FGD_\mu \arrow[r, dashed] \arrow[d, dotted, no head] & FGD \arrow[r, "\simeq"] & D \\
\cdot \arrow[ur, dashed] \arrow[d, dotted, no head] & & \\
D_\nu \simeq FGD_\nu \arrow[uur, dashed] & &
\end{tikzcd}
$$

Also since $FG \simeq id_{\mathcal{D}}$ we have that the existence of maps in the image under FG implies the existence of corresponding maps in the original (lefthand) diagram and furthermore all of the commuting will be preserved from left to right and vice versa.

Therefore the original map $f : FM \to D$ necessarily factors through some D_{λ_0}. Thus we have that FM is a finitely presented object of $\mathcal{D}$. Clearly this situation is symmetric and so for all $N \in \mathcal{D}^{\text{fp}}$ we will have $GN \in \mathcal{C}^{\text{fp}}$. The subcategory of finitely presented objects is always a full subcategory, so all of the commuting diagrams exhibiting the equivalence of $\mathcal{C}$ and $\mathcal{D}$ that involve only finitely presented objects and morphisms between them are preserved. Therefore appropriate restrictions of the functors F and G will satisfy the definition of an equivalence of categories between $\mathcal{C}^{\text{fp}}$ and $\mathcal{D}^{\text{fp}}$, namely the compositions are naturally equivalent to the identities on the objects of $\mathcal{C}^{\text{fp}}$ and $\mathcal{D}^{\text{fp}}$. □

A simple consequence of this lemma is:

$$\text{Mod-}R \equiv \text{Mod-}S \quad \Rightarrow \quad \text{mod-}R \equiv \text{mod-}S$$

Recall that a category is said to be *preadditive* if for any two objects A, B the hom-set $Hom(A, B)$ is an abelian group and composition is bilinear where defined.

Definition 3.6.3. *An* additive functor F *between two preadditive categories* $F : \mathcal{C} \to \mathcal{D}$ *is a functor such that for all objects* $A, B \in \mathcal{C}$*, the function* $F : Hom(A, B) \to Hom(F(A), F(B))$ *is a homomorphism of abelian groups.*

Definition 3.6.4. *An* additive equivalence *between two preadditive categories is a pair of additive functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $GF \to 1_{\mathcal{C}}$ and $FG \to 1_{\mathcal{D}}$.*

If $R \equiv_M S$ then there are additive functors F : Mod-$R \to$ Mod-S and G : Mod-$S \to$ Mod-R such that the pair form an additive equivalence.

Lemma 3.6.5. *If mod-R and mod-S are equivalent categories, then the functor categories (mod-R, Ab) and (mod-S, Ab) are equivalent.*

Proof. Suppose mod-$R \equiv$ mod-S and the equivalence is given by the functors mod-$R \underset{G}{\overset{F}{\rightleftarrows}}$ mod-S . Then we have a pair of functors

$$(\text{mod-}R, Ab) \underset{(-\circ F)}{\overset{(-\circ G)}{\rightleftarrows}} (\text{mod-}S, Ab)$$

and we prove that these also form an equivalence of categories. Due to the symmetry, it is sufficient to show that $(- \circ F) \circ (- \circ G)$ is naturally equivalent to the identity on (mod-R, Ab). We have the pair of functors

$$(\text{mod-}R, Ab) \underset{(-\circ F)\circ(-\circ G)}{\overset{id}{\rightrightarrows}} (\text{mod-}R, Ab)$$

and we prove that $(- \circ (GF))$ is a natural equivalence between them. We have $GF \simeq \mathrm{id}_{\text{mod-}R}$ and $(- \circ (GF)) \simeq \mathrm{id}_{(\text{mod-}R,Ab)}$, so for every $H_i \in$ (mod-R, Ab), there is a natural equivalence $H_i \simeq H_i \circ (GF)$. A morphism τ in the category (mod-R, Ab), $\tau : H_1 \to H_2$ is a collection of maps $\{\tau_M \mid M \in \text{mod-}R\}$ such that $\tau_M : H_1(M) \to H_2(M)$ is a homomorphism of abelian groups and for every morphism $f : M \to N$ in mod-R the following diagram commutes:

$$\begin{tikzcd}
H_1(M) \arrow[r, "\tau_M"] \arrow[d, "H_1(f)"'] & H_2(M) \arrow[d, "H_2(f)"] \\
H_1(N) \arrow[r, "\tau_N"'] & H_2(N)
\end{tikzcd}$$

Now $H_1 \circ GF$ and $H_2 \circ GF$ are objects of the category (mod-R, Ab) and $(- \circ GF)(\tau)$ is the collection of maps $\{(- \circ GF)(\tau)_M \mid M \in \text{mod-}R\}$ where $(- \circ GF)(\tau)_M = \tau_{GF(M)}$.

So the following diagram commutes:

$$\begin{tikzcd} H_1 \arrow[r, "(-\circ GF)_{H_1}"] \arrow[d, "\tau"'] & H_1 \circ GF \arrow[d, "(-\circ GF)(\tau)"] \\ H_2 \arrow[r, "(-\circ GF)_{H_2}"'] & H_2 \circ GF \end{tikzcd}$$

Thus $(- \circ GF)$ is a natural equivalence between the two natural transformations id and $(- \circ F) \circ (- \circ G)$ from (mod-R, Ab) to (mod-R, Ab). □

Theorem 3.6.6. *([7], Theorem 3.2.5) For any ring R, the category $(mod\text{-}R, Ab)^{fp}$ is equivalent to the category of $\mathbb{L}_R^{eq+}$ of positive primitive imaginaries for R-modules.*

Proposition 3.6.7. *If R and S are Morita equivalent rings, then there is an equivalence of categories $\mathbb{L}_R^{eq+} \simeq \mathbb{L}_S^{eq+}$ between the categories of positive primitive imaginaries for R-modules and S-modules.*

Proof. Suppose $R \equiv_M S$. Then the categories of right modules over the two rings are equivalent and Lemmas 3.6.2 and 3.6.5 together yield (mod-R, Ab) $\simeq$ (mod-S, Ab) as seen above. Applying Lemma 3.6.2 again, we have an equivalence of categories:

$$R \equiv_M S \Rightarrow (\text{mod-}R, Ab)^{\text{fp}} \simeq (\text{mod-}S, Ab)^{\text{fp}}$$

By Theorem 3.6.6, we then have that Morita equivalent rings will have equivalent categories of pp-imaginaries:

$$R \equiv_M S \quad \Rightarrow \quad \mathbb{L}_R^{eq+} \simeq \mathbb{L}_S^{eq+}$$

□

3.7 Some example modules

We examine the combinatorics of definable sets in some example modules.

The module $\mathbb{Z}_\mathbb{Z}$

The ring of integers $\mathbb{Z}$ forms a module over itself. By Theorem 3.1.5, the pp-elimination in theories of modules, an arbitrary definable set, where we allow parameters, is a boolean combination of cosets of pp-definable subgroups.

The formula $v = w \cdot 2$ defines a bijection between $\mathbb{Z}$ and $\mathbb{Z} \cdot 2$ and the formula $u = v+1$ defines a bijection between $\mathbb{Z}\cdot 2$ and $1+\mathbb{Z}\cdot 2$. Therefore these three definable sets are all definably isomorphic and we have $[\mathbb{Z}] = [\mathbb{Z} \cdot 2] = [1 + \mathbb{Z} \cdot 2]$ in $\mathrm{Def}(\mathbb{Z})$. Observe that $\mathbb{Z} = \mathbb{Z} \cdot 2 \sqcup (1 + \mathbb{Z} \cdot 2)$. Hence $[\mathbb{Z}] = [\mathbb{Z}] + [\mathbb{Z}]$ and this implies that in the Grothendieck ring we have $[\mathbb{Z}] = [\emptyset] =: 0 \in K_0(\mathbb{Z})$.

Remark. Let $M := \mathbb{Z}_{\mathbb{Z}}$. Observe that by the above we have $M \not\models PHP$ and $M \not\models WPHP^{2n}$, but no counterexample to $WPHP^{n^2}$ has been found.

Every formula in $\mathcal{L}_{\mathbb{Z}}(\mathbb{Z})$ is equivalent modulo $Th(\mathbb{Z}_{\mathbb{Z}})$ to a boolean combination of pp-formulas. Every pp-formula defines a coset of an additive subgroup of some $\mathbb{Z}^n$. These cosets are either singletons (cosets of $\overline{0} \in \mathbb{Z}^n$ for some n) or in definable bijection with some finite power of $\mathbb{Z}$. For $n \geq 1$ we have $[\mathbb{Z}^n] = [\mathbb{Z}\times\ldots\times\mathbb{Z}] = [\mathbb{Z}]\times\ldots\times[\mathbb{Z}] = 0$. Observe that $\{0\} \sqcup (\mathbb{Z} \setminus \{0\}) = \mathbb{Z}$ and hence in the Grothendieck ring, the additive inverse of 1 is $-[\{0\}] = [\mathbb{Z} \setminus \{0\}]$. Similarly for any finite subset of $\mathbb{Z}$, we have $-[\{a_1, \ldots, a_k\}] = [\mathbb{Z} \setminus \{a_1, \ldots, a_k\}]$.

Therefore the boolean combinations of pp-formulas define unions, intersections and complements of singletons and sets which have representative 0 in $K_0(\mathbb{Z})$. Therefore every definable set has image in $K_0(\mathbb{Z})$ equal to that of some boolean combination (of finite unions, intersections and complements) of finite sets. For brevity, let $[\mathbf{n}]$ denote the class $[\{1, \ldots, n\}]$ for each $n \in \mathbb{N}$. Since we have all parameters in our language we can, as noted earlier, define a bijection between any two finite sets of equal size.

It is easy to see that the map $n \mapsto [\mathbf{n}]$ from natural numbers to the representatives in $K_0(\mathbb{Z})$ of the finite sets, preserves the arithmetic operations $+, -, \times$. All of the required definable bijections between the representative sets are easily constructed using the parameters for the elements. If $n > m > 0$ then $[\mathbf{n}] - [\mathbf{m}] = [\mathbf{n}] + 0 - [\mathbf{m}] = [\{1, \ldots, n\}] + [\mathbb{Z}] - [\mathbf{m}] = [\{1, \ldots, m\}] + [\{m + 1, \ldots, n\}] + [\mathbb{Z} \setminus \{1, \ldots, m\}] = [\{m + 1, \ldots, n\}] + [(\mathbb{Z}\setminus\{1, \ldots, m\})\cup\{1, \ldots, m\}] = [\{1, \ldots, n-m\}] + [\mathbb{Z}] = [\mathbf{n\text{-}m}] + 0 = [\mathbf{n\text{-}m}]$.

Remark. For some other modules considered in this thesis, such as modules over semisimple rings which are discussed in Chapter 7, the Grothendieck rings will be ordered or partially ordered and the image of $\mathrm{Def}(M)$ under the Euler characteristic $\chi_0 : \mathrm{Def}(M) \to K_0(M)$ will be nonnegative w.r.t. the order. For these modules, the negative elements of $K_0(M)$ are defined when the semiring $\widetilde{\mathrm{Def}}(M)$ is completed to a ring. But for the module $\mathbb{Z}_\mathbb{Z}$, the additive inverse in the Grothendieck ring of $\chi_0(A)$, the image of a definable set A under the Euler characteristic, is itself the image of a definable set, i.e. an element of $\chi_0(\mathrm{Def}(\mathbb{Z}))$.

Conjecture. No two finite sets of different size have the same representative in $K_0(\mathbb{Z})$ and thus $K_0(\mathbb{Z}) \cong \mathbb{Z}$.

It remains to show that there are no further relations $[\mathbf{m}] = [\mathbf{n}]$. Since we are dealing with a ring, $[\mathbf{m}] = [\mathbf{n}] \Rightarrow [\mathbf{m} - \mathbf{n}] = 0$, and hence it is sufficient to show that no non-empty finite set will be represented in the Grothendieck ring by $0 = [\emptyset]$. A proof for this is still required; some investigation into the problem is discussed in Chapter 8.

If the images of the finite sets under χ_0 are not identified in the Grothendieck ring, we would have $K_0(\mathbb{Z}) \cong \mathbb{Z}$. The other possibilities are; if $[\mathbf{1}] = 0$ in $K_0(\mathbb{Z})$, then $K_0(\mathbb{Z}) = 0$, and if some $n > 0$ is the least natural number with $[\mathbf{n}] = 0$, then $K_0(\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$.

The ring of the p-adic integers as a module over itself

Definition 3.7.1. *The ring of p-adic integers for a prime p is the inverse limit of the system $\ldots \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z} \twoheadrightarrow \ldots \twoheadrightarrow \mathbb{Z}/p^2\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p\mathbb{Z} \twoheadrightarrow 0$. Each member of the ring can be given in the form $\sum_{i=0}^{\infty} a_i p^i$, where $a_i \in \{0, 1, \ldots, p-1\}$.*

The ring operations are defined as follows:

- $\sum_{i=0}^{\infty} a_i p^i + \sum_{i=0}^{\infty} b_i p^i = \sum_{i=0}^{\infty} c_i p^i$,
  where the $c_i \in \{0, 1, \ldots, p-1\}$ and are uniquely determined by
  $c_n p^n = (\sum_{i=0}^{n} a_i p^i + \sum_{i=0}^{n} b_i p^i) \pmod{p^{n+1}} - (\sum_{i=0}^{n} a_i p^i + \sum_{i=0}^{n} b_i p^i) \pmod{p^{n-1}}$
- $\sum_{i=0}^{\infty} a_i p^i \times \sum_{i=0}^{\infty} b_i p^i = \sum_{i=0}^{\infty} d_i p^i$,
  where the $d_i \in \{0, 1, \ldots, p-1\}$ and are uniquely determined by

$\sum_{i=0}^{n} d_i p^i = (\sum_{i=0}^{n} a_i p^i \times \sum_{i=0}^{n} b_i p^i) \pmod{p^{n+1}}$.

An alternative definition is that a p-adic integer is an infinite sequence of natural numbers $(s_n)_{n\geq 0}$ where $s_{n+1} \equiv s_n (\text{mod } p^n)$ for all n. This definition can be seen to be equivalent to the original one above, since setting $s_n := \sum_{i=0}^{n-1} a_i p^i$ sends $\sum_{i=0}^{\infty} a_i p^i$ to $(s_n)_{n\geq 0}$ and the arithmetic is preserved.

The ring of p-adic integers is written $\overline{\mathbb{Z}_{(p)}}$ to denote the fact that it is the completion of $\mathbb{Z}_{(p)}$. The p-adic integers form a module over various rings including $\mathbb{Z}$, $\mathbb{Z}_{(p)}$ and $\overline{\mathbb{Z}_{(p)}}$.

Definition 3.7.2. *A ring R is said to be* relatively divisible *or* RD *if for every embedding of R-modules $A \subseteq B$, if $Ar = A \cap Br$, $\forall r \in R$ then the embedding is pure.*

For R a relatively divisible ring, we have the following elimination result in $Th(\text{Mod-}R)$.

Lemma 3.7.3. *([30], 2.4.10) Every pp-formula is equivalent modulo $Th(\text{Mod-}R)$ to a finite conjunction of formulas of the form $s|(\sum_{i=1}^{n} v_i r_i)$ with $s, r_1, \ldots, r_n \in R$.*

In the ring $R = \overline{\mathbb{Z}_{(p)}}$ and also the ring $\mathbb{Z}_{(p)}$, we have $x|y$ if and only if for every $k \geq 1$, $p^k|x \Rightarrow p^k|y$. Thus in the module $\overline{\mathbb{Z}_{(p)}}$ over the ring R, and also over $\mathbb{Z}_{(p)}$, the divisibility condition $s|(\sum_{i=1}^{n} v_i r_i)$ is simplified further to $p^a|(\sum_i v_i r_i)$, where a is the greatest power of p dividing s.

Lemma 3.7.4. *([10], 3.19) Let R be a principal ideal domain (PID). Then any finitely generated R-module is free iff it is torsion free.*

Lemma 3.7.5. *([10], 3.21) Every submodule N of a free module $M \cong R^n$ over a principal ideal domain (PID) R, is a free module; $N \cong R^m$ with $m \leq n$.*

Let $R = \overline{\mathbb{Z}_{(p)}}$, the p-adic integers, and regard R as a right module over itself. For the remainder of the section, we set $M := R_R$ to make it clear that we are regarding it as a module, and the domain of an $\mathcal{L}_R$-structure.

Lemma 3.7.6. *In M, the solution sets of pp-formulas are all isomorphic to powers of M.*

Proof. Every pp-formula $\phi(v_1, \ldots, v_n)$ defines a subgroup $\phi(M)$ of M^n and since the ring R is commutative, $\phi(M)$ will be an submodule of M^n. The module M is torsion free, as is every power of M. Hence the submodule $\phi(M)$ must be torsion free. Now $M = R_R$, and hence M^n is a free module over R, and $\phi(M)$ is a submodule of this free module for every n-ary pp-formula ϕ. The ring of p-adic integers is a PID since the only nonzero proper ideals of R are Rp^n, for $n \geq 1$. Thus Lemma 3.7.5 implies that $\phi(M) \cong M^m$ for some $m \leq n$. □

The isomorphism $\phi(M) \cong M^m$ is an R-linear map and thus it will be definable in $\mathcal{L}_R(M)$. Therefore in the Grothendieck ring, we have $[\phi(M)] = [M^m] = [M]^m$.

Now $M = \overline{\mathbb{Z}_{(p)}}$ has no finite subgroups apart from the trivial zero subgroup. The definable infinite subgroups are $\overline{\mathbb{Z}_{(p)}} \cdot p^n$ for every $n \in \mathbb{N}$. The formula $v = w \cdot p$ defines a bijection between $\overline{\mathbb{Z}_{(p)}} \cdot p$ and $\overline{\mathbb{Z}_{(p)}}$. Hence in the Grothendieck ring $K_0(M)$, we have the identity $[\overline{\mathbb{Z}_{(p)}} \cdot p] = [\overline{\mathbb{Z}_{(p)}}]$. The subgroup $\overline{\mathbb{Z}_{(p)}} \cdot p$ has index p in the module $\overline{\mathbb{Z}_{(p)}}$, since:

$$\overline{\mathbb{Z}_{(p)}} = \overline{\mathbb{Z}_{(p)}} \cdot p \ \sqcup \ 1 + \overline{\mathbb{Z}_{(p)}} \cdot p \ \sqcup \ldots \sqcup \ p - 1 + \overline{\mathbb{Z}_{(p)}} \cdot p = \bigsqcup_{i=0}^{p-1} i + \overline{\mathbb{Z}_{(p)}} \cdot p$$

Combining these two observations, we deduce that in the Grothendieck ring, $[\overline{\mathbb{Z}_{(p)}}] = p \times [\overline{\mathbb{Z}_{(p)}}]$ and hence that $(p-1)[M] = 0$.

Remark. Let $M := \overline{\mathbb{Z}_{(p)}}$ a right module over itself. Observe that by the above we have $M \not\models PHP$, but no counterexample to $WPHP^{n^2}$ or $WPHP^{2n}$, has been found.

The pp-definable subgroups of M^1 are $0, M$ and $M \cdot p^n$ for every $n \geq 1$. There are infinitely many pp-subgroups of M, but due to the definable bijections given by right multiplication by p^n, $[M] = [M \cdot p^n]$ for every n, so there are only two isomorphism classes of pp-subgroups in $\widetilde{\mathrm{Def}}(M)$, those of M and 0.

The infinite pp-sets in this example all have representative $[\phi(M)] = [M]^m$, for some m, in the Grothendieck ring. As always in the model theory of modules, an arbitrary definable set is a boolean combination of pp-sets. Thus the Grothendieck ring is generated by the classes of M and of the singleton set $\{0\}$. From the construction of a Grothendieck ring, the disjoint unions, products and removed subsets in Def(M), correspond to sums, products and subtractions of the elements in the ring $K_0(M)$ that are the images of the corresponding sets under χ_0. This implies that there is a ring homomorphism from $\mathbb{Z}[X]$ to $K_0(M)$ given by $1 \mapsto 1$ and $[M] \mapsto X$. However this ring homomorphism is not injective as we have the added relation $p \times [M] = [M]$ found above. Therefore $K_0(M)$ admits the quotient ideal $\langle (p-1)X \rangle$.

Conjecture. The author believes that $K_0[\overline{\mathbb{Z}_{(p)}}] \cong \mathbb{Z}[X]/\langle (p-1)X \rangle$.

It remains to show that there is no further cancellation, in particular that the finite sets have nontrivial representatives in $K_0(M)$ and that $[M] \neq [\emptyset]$. Since the set $\overline{\mathbb{Z}_{(p)}}$ is not in (definable) bijection with the empty set, this would only occur if there were definable sets S and A with $[A] = [M]$ and a definable bijection $f : S \to S \sqcup A$.

Remark. The pp-definable sets in $\overline{\mathbb{Z}_{(p)}}$ are the same whether it is regarded as a $\mathbb{Z}$-module or a module over itself. The ring $\mathbb{Z}$ is commutative and hence the pp-subgroups are again submodules. The ring $\mathbb{Z}$ is also a PID and so by Lemma 3.7.5 again we have that the pp-definable sets are again free modules and hence isomorphic to powers of the module.

Chapter 4

The Grothendieck rings of modules over division rings

4.1 Background and definitions

Definition 4.1.1. *A* division ring *(or* skew field*) D is a ring in which every nonzero element a has a two-sided multiplicative inverse.*

Remark. All fields are division rings but the converse does not hold as multiplication in a division ring need not be commutative. Hence all results of this chapter which are stated for infinite modules (also referred to as vector spaces) over division rings are true in particular for infinite modules over fields.

The language $\mathcal{L}_D$ will have a function symbol for the scalar action of each element of the division ring D. If the division ring D is finite then it is necessarily a finite field. In this case, if the dimension of the vector space M is finite, then M is finite and $K_0(M) = \mathbb{Z}$ by Lemma 2.2.5, which holds for any finite structure in any first order language. The theory of infinite modules over a division ring is complete and has elimination of quantifiers, as shown in ([31], 16.16).

For a general D-module M, regarded as an $\mathcal{L}_D$-structure, the definable sets are boolean combinations (i.e. finite unions, intersections and complements) of the solution sets of pp-formulas. These solutions sets are subgroups of the powers of M in the case of parameter free formulas, and their cosets in the general case. Not every subgroup of a power of M will be definable. When D is a field, these subgroups and their cosets are vector subspaces and affine spaces.

Claim. No proper left D-submodule of M can be defined by a pp-formula, the only pp-definable submodules of M are M and 0. *Proof.* Recall that every pp-definable subset of a module M is an $\mathrm{End}_D(M)$-module and $D \cong \mathrm{End}(D)$. For any $x, y \in M$ with $x \neq 0$, there is a D-endomorphism of M, in other words a D-linear map $f : M \to M$, such that $f(x) = y$. Hence if $\phi(v)$ is a pp-formula and $x \in \phi(M)$ then $y = f(x) \in \phi(M)$. The only pp-definable subsets of M are M and $\{0\}$ and so M is what we call *strongly minimal.*

Definition 4.1.2. *A first order structure M is said to be* ω-saturated *if for every finite subset of the domain $A \subseteq M$, M realises every complete type over A.*

Definition 4.1.3. *An ω-saturated first order structure M is said to be* strongly minimal *if the only definable subsets of M^1 are finite or cofinite.*

The pp-definable sets are among the cosets of $\mathrm{End}(M)$-submodules (or affine subspaces) of M^n where n is the length of the pp-formula in question. Let $\phi(v_1, \ldots, v_n)$ be a pp-formula in the language $\mathcal{L}_D$, say

$$\phi(v_1, \ldots, v_n) = \exists w_1, \ldots, w_m \bigwedge_{i=1}^{t} \Big(\sum_{j=1}^{n} v_j r_{ij} + \sum_{k=1}^{m} w_k s_{ik} = c_i \Big)$$

where the r_{ij} and s_{ik} are function symbols for multiplication by elements of D and the c_i are constant symbols. Since the theory of modules over a division ring has complete elimination of quantifiers, the most general pp-formula is equivalent modulo $Th(M)$, or $Th(\text{Mod-}D)$, to a system of linear equations. Hence we may assume without loss of generality that the pp-formula $\phi(v_1, \ldots, v_n)$ is $\bigwedge_{i=1}^{t}(\sum_{j=1}^{n} v_j r_{ij} = c_i)$, a system of simultaneous D-linear equations, and hence its solution set will be a D-module if $c_i = 0$ for every $i \in I$ and a coset of one otherwise.

We saw that the pp-definable subsets, allowing parameters, of M^1 were simply singletons or the whole of M. For pp-definable subsets of higher powers of M, say M^n, they will also be $\mathrm{End}_D(M)$-modules. They are $\mathrm{End}(M)$-submodules of M^n, which is a $\mathrm{End}_D(M)$-module under the diagonal action $f : (v_1, \ldots, v_n) \mapsto (f(v_1), \ldots, f(v_n))$. Each projection of a pp-definable subgroup of M^n is itself a pp-definable subgroup of M^l for some $l \leq n$. Every pp-definable subset is an $\mathrm{End}_D(M)$-module and hence is closed under D-linear maps. Given $\overline{v} = (v_1, \ldots, v_n) \in M^n$, let $S_{\overline{v}} := \{f(\overline{v}) = (f(v_1), \ldots, f(v_n)) : f \in \mathrm{End}_D(M)\}$. So for any $\overline{v}$ in $\phi(M)$, the set $S_{\overline{v}}$ is contained in $\phi(M)$. Therefore $\phi(M)$ contains $S_{\overline{v}}$ for each $\overline{v}$ in M.

Lemma 4.1.4. *Let $\phi(v_1, \ldots, v_n, \overline{c})$ be a pp-formula in $\mathcal{L}_D(M)$. Then the set $\phi(M)$ is isomorphic as a left D-module to M^k for some $k \geq 1$.*

Proof. Recall $\phi(v_1, \ldots, v_n, \overline{c})$ is a system of simultaneous D-linear equations

$$\bigwedge_{i=1}^{t}(\sum_{j=1}^{n} v_j r_{ij} = c_i)$$

with $r_{ij} \in D$ and $c_i \in M$. Hence the parameter free version of ϕ, denoted $\phi_0(\overline{v})$, is $\bigwedge_{i=1}^{t}(\sum_{j=1}^{n} v_j r_{ij} = 0)$. Now $\phi_0(M)$ is a left $\mathrm{End}(D)$-module by Lemma 3.1.7 and for any division ring we have $D \cong \mathrm{End}(D)$. Therefore $\phi_0(M)$ is a left D-module. Regarded as a left D-module, M is isomorphic to $D^{(I)}$ where I is the (possibly infinite) dimension of M. Therefore $\phi(M) \cong \phi_0(M) = \phi_0(D^{(I)}) = (\phi_0(D))^{(I)}$. Now $\phi_0(D)$ is a subspace of D^n and hence is isomorphic to D^k for some $k \leq n$. So we have $(\phi_0(D))^{(I)} \cong (D^k)^{(I)} = (D^{(I)})^k \cong M^k$. □

Remark. In the sequel, when we refer to the *dimension* of a pp-set $\phi(M)$, we will mean the value k such that $\phi_0(M) \cong M^k$ as left D-modules.

Conjecture. The author believes that any strong minimal module will satisfy the pigeonhole principle, $M \models \mathrm{PHP}$, but has not encountered this result in the literature.

4.2 A survey of related material

In this section, we describe and expand on material from Section 4 of [24].

Theorem 4.2.1. *(Thm 4.5, [24]) If M is an infinite structure that satisfies the* pigeonhole principle*, then $K_0(M)$ has a subring isomorphic to $\mathbb{Z}[X]$, the polynomial ring in one indeterminate over the ring of integers.*

We first need the following lemmas in order to prove the theorem.

Remark. The lemma below is implied by Theorem 4.3 of [24], but the authors give just an indication of proof.

Lemma 4.2.2. *If $M \models PHP$ then the elements 0 and 1 in $K_0(M)$ are distinct.*

Proof. Assume $M \models PHP$ and recall that in $K_0(M)$ we have $0 = \chi([\emptyset])$ and $1 = \chi([\{*\}])$, where χ is the weak universal Euler characteristic and $[-]$ denotes the class of a set in $\widetilde{\mathrm{Def}}(M)$. Following Krajiček and Scanlon, we define the relation $\leq$ on $\widetilde{\mathrm{Def}}(M)$ by writing $[A] \leq [B]$ if there exist disjoint sets $A', B', X \in \mathrm{Def}(M)$ with $[A'] = [A], [B'] = [B]$ and a definable injection $f : A' \cup X \hookrightarrow B' \cup X$.

Let X be definable set and c any element of $M \setminus X$. Then $X \subset X \cup \{c\}$ and hence $0 = [\emptyset] \leq [\{c\}] = 1$. To see that $0 \lneq 1$, we assume for a contradiction that $1 \leq 0$. Then there exists some definable set B, an element d of $M \setminus B$ and a definable injection $h : B \cup \{d\} \hookrightarrow B$. But then $h(B) = B \setminus \{h(d)\}$ and $h \restriction_B : B \to B \setminus \{h(d)\}$ is a definable injection from a definable set to a proper subset of itself, contradicting our assumption $M \models PHP$.

□

The next lemma is stated with just an indication of proof in [24].

Lemma 4.2.3. *(Thm 4.3, [24]) Let M be any structure. $M \models$ PHP if and only if $K_0(M)$ is partially ordered, with $0 < 1$ in the order, and the image of the weak Euler characteristic $\chi : Def(M) \to K_0(M)$ is nonnegative in the partial order, i.e. $\chi(Def(M)) \subseteq \{r \in K_0(M) \mid r \geq 0\}$.*

Proof. Assume that M satisfies the pigeonhole principle. We define the partial ordering on $K_0(M)$ by putting $[A] \leq [B]$ iff there exist pairwise disjoint sets A', B' and X in $\mathrm{Def}(M)$ such that $[A] = [A']$, $[B] = [B']$ in $\widetilde{\mathrm{Def}}(M)$ and there is a definable injection from $A' \cup X$ into $B' \cup X$. We need to prove that this relation is a partial ordering. The relation $\leq$ will be reflexive; for any set A we can take X to be the empty set and the injection $A \hookrightarrow A$ to be the identity.

Let $A, B, C \in \mathrm{Def}(M)$ and suppose $[A] \leq [B]$ and $[B] \leq [C]$. It follows that $[A] \leq [C]$ and therefore $\leq$ is transitive. Our assumptions imply that there exist $A_0, B_0, B_1, C_1, X, Y \in \mathrm{Def}(M)$ such that $[A_0] = [A], [B_0] = [B] = [B_1]$ and $[C_1] = [C]$ and $X \cap A_0 = X \cap B_0 = \emptyset$ and $Y \cap B_1 = Y \cap C_1 = \emptyset$ and there exist definable injections $f : A_0 \cup X \hookrightarrow B_0 \cup X$ and $g : B_1 \cup Y \hookrightarrow C_1 \cup Y$. Since we allow parameters in our formulas, it is always possible to take definable disjoint copies of any definable sets by taking products with singleton sets consisting of distinct constant symbols. So we may find pairwise disjoint $\hat{A}, \hat{B}, \hat{C}, \hat{X}$ and $\hat{Y}$ such that $[\hat{S}] = S$ for each set $S \in \{A, B, C, X, Y\}$ and each of them is also disjoint from $A, B, C, A_0, B_0, B_1, C_1, X$ and Y. Now we can construct a definable injection from $\hat{A} \cup \hat{X} \cup \hat{Y}$ to $\hat{C} \cup \hat{X} \cup \hat{Y}$ and this yields that $[A] = [\hat{A}] \leq [\hat{C}] = [C]$.

Define f' taking $(A_0 \cup X) \cup \hat{Y}$ to $(B_0 \cup X) \cup \hat{Y}$ by $f' \restriction \hat{Y} = \mathrm{id}_{\hat{Y}}$ and $f' \restriction (A_0 \cup X) = f$. Then f' is injective because f is injective and $\mathrm{id}_{\hat{Y}}$ is obviously a bijection. Similarly we can define an injective map $g' : (B_1 \cup Y) \cup \hat{X} \to (C_1 \cup Y) \cup \hat{X}$ given by $g' \restriction \hat{X} = \mathrm{id}_{\hat{X}}$ and $g' \restriction (B_1 \cup Y) = g$. Observe that f' and g' are definable functions.

In the diagram below we can define the maps p, q, r and s to be the obvious bijections. Each has as its domain a disjoint union of three previously defined sets and has restrictions to these sets given by either the identity or a bijection we already know to exist by the hypothesis. For example r acts as the identity on $\hat{X}$ and $r \restriction \hat{B} : \hat{B} \to B_1$ is a bijection whose existence follows from the fact that $[\hat{B}] = [B_1] \in \widetilde{\mathrm{Def}}(M)$ and $r \restriction \hat{Y} : \hat{Y} \to Y$ is a bijection whose existence follows from the fact that $[\hat{Y}] = [Y] \in \widetilde{\mathrm{Def}}(M)$. The bijections p, q and s on the diagram are similarly defined.

$$\begin{tikzcd}
(\hat{A} \cup \hat{X}) \cup \hat{Y} \arrow[r, "\hat{f}"] \arrow[d, "p"] & (\hat{B} \cup \hat{X}) \cup \hat{Y} \arrow[r, equal] & (\hat{B} \cup \hat{Y}) \cup \hat{X} \arrow[r, "\hat{g}"] \arrow[d, "r"] & (\hat{C} \cup \hat{Y}) \cup \hat{X} \\
(A_0 \cup X) \cup \hat{Y} \arrow[r, "f'"'] & (B_0 \cup X) \cup \hat{Y} \arrow[u, "q"'] & (B_1 \cup Y) \cup \hat{X} \arrow[r, "g'"'] & (C_1 \cup Y) \cup \hat{X} \arrow[u, "s"']
\end{tikzcd}$$

Finally we put $\hat{f} := qf'p$ and $\hat{g} := sg'r$. As compositions of injective maps, these will be injective and the composition $\hat{g} \circ \hat{f} : (\hat{A} \cup \hat{X}) \cup \hat{Y} \hookrightarrow (\hat{C} \cup \hat{X}) \cup \hat{Y}$ is the desired injection exhibiting that $[\hat{A}] \leq [\hat{C}]$.

The remaining criterion for $\leq$ to be a partial order is that for any $[A], [B] \in \widetilde{\mathrm{Def}}(M)$, if $[A] \leq [B] \wedge [B] \leq [A]$ then $[A] = [B]$. Using the argument immediately above for transitivity with $C = A$, we may find injections $\hat{f} : \hat{A} \sqcup Z \to \hat{B} \sqcup Z$ and $\hat{g} : \hat{B} \sqcup Z \to \hat{A} \sqcup Z$. The construction may be via some other $\breve{A}$ with $[\breve{A}] = [\hat{A}] = [A]$ and some $\breve{Z}$ with $[\breve{Z}] = [Z]$, but we can remove this complication by composing with the appropriate definable bijections. Then the composition $\hat{g} \circ \hat{f}$ is an injective map from $\hat{A} \sqcup Z$ to itself and hence is a bijection because by assumption $M \models \mathrm{PHP}$. Therefore $\hat{g} : \hat{B} \sqcup Z \to \hat{A} \sqcup Z$ is surjective and injective, as is its restriction to $\hat{B}$. Now in $K_0(M)$ we have

$$\begin{aligned} \chi(B) = \chi(\hat{B}) &= \tilde{\chi}([\hat{B}]) \\ = \tilde{\chi}([(\hat{A} \sqcup Z) \setminus \hat{g}(Z)]) &= \tilde{\chi}([\hat{A}]) + \tilde{\chi}[Z] - \tilde{\chi}[\hat{g}(Z)] \\ = \chi(A) + \chi(Z) - \chi(Z) &= \chi(A) \end{aligned}$$

Observe that $\chi(\mathrm{Def}(M)) \subseteq \{r \in K_0(M) \mid r \geq 0\}$ with respect to this partial ordering. For every definable set $A \in \mathrm{Def}(M)$ we have $\emptyset \subset A$ trivially definable so $\chi(A) \geq 0 \in K_0(M)$.

To prove the converse, we assume that $\widetilde{\mathrm{Def}}(M)$ is partially ordered by $\leq$, that $0 \lneq 1$ in the order, and the image of χ is nonnegative in the induced partial order on $K_0(M)$. Let $B \subsetneq A$ be a strict inclusion of the sets $A, B \in \mathrm{Def}(M)$. Then $A \setminus B \neq \emptyset$ and $[A \setminus B] \geq 0$. Let $a \in A \setminus B$. Then $[A \setminus B] \geq [\{a\}] = 1 \gneq 0$. Therefore $[A] = [B] + [A \setminus B] \Rightarrow [A] \gneq [B]$ and there can be no definable injection from A into B, as this would imply $[A] \leq [B]$. Thus we have $M \models PHP$.

□

We are now in a position to prove Theorem 4.2.1.

Proof of Theorem 4.2.1. This argument is essentially the same as Theorem 4.5 of [24]. It is included here for completeness. Since $M \models \mathrm{PHP}$, Lemma 4.2.3 implies that $K_0(M)$ has a partial ordering $\leq$ as described and the image of the weak Euler characteristic $\chi : \mathrm{Def}(M) \to K_0(M)$ is nonnegative in this ordering. Define a new relation in $K_0(M)$ by putting $r \ll s$ if there exists $k \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ we have $nr < ks$. Let $X := \chi(M)$. For any polynomial $P(x) \in \mathbb{Z}[x]$ we claim that if the degree of $P(x)$ is strictly less than d, then $P(X) \ll X^d$ in $K_0(M)$. We prove the claim by induction on d, the bound on the polynomial degree.

If $d = 0$ then P is identically zero and $n0 = 0 < 1 = X^0$. If $d = 1$ then P is a constant, $a \in \mathbb{Z}$ say and $P(X)$ is the class of some (equivalently any) finite set of size a. For any $n \in \mathbb{N}$, $nP(x) = na \in \mathbb{Z}$. If $a \leq 0$, then $na \leq 0 < X^1$. Observe that $0 < X = \chi(M)$ since we have an injection $\{*\} \hookrightarrow M$ and hence $1 = \chi(\{*\}) \leq \chi(M)$ and $0 < 1$ by Lemma 4.2.2. If $a > 0$ then the infinite set M will contain a proper subset of size na. Therefore $na < M$. In both cases we can take $k = 1$ to witness $P(X) = a \ll X^1$.

For the inductive step, suppose the claim holds for degree d. Let $P(x)$ be a polynomial of degree less than $d+1$. Then we can write $P(X)$ in the form $a+X\cdot Q(X)$ with $a \in \mathbb{Z}$. By the induction hypothesis there is some integer k such that $\forall n \in \mathbb{Z}\ nQ(X) < kX^d$, and hence $nP(X) = na+nX\cdot Q(X) < X+X\cdot kX^d \leq (1+k)X^{d+1}$. This concludes the proof of our claim.

We show that the Grothendieck ring of M has as a subring the polynomial ring in one variable over the integers by demonstrating that the map $\mathbb{Z}[x] \to K_0(M)$ given by $P(x) \mapsto P(X) := P(\chi(M))$ is an embedding of rings. It is a ring homomorphism so it suffices to show that the kernel is 0. Let $P(x)$ be a nonzero polynomial and write $P(x) = ax^d+Q(x)$ where $0 \neq a \in \mathbb{Z}$ and $Q(x)$ is of degree less than d. We may assume $a > 0$ since $P(X) = 0$ iff $-P(X) = 0$. By the above claim, $Q(X) \ll X^d \leq aX^d$ and therefore $Q(X) \neq -aX^d$ and $P(X) \neq 0$.

□

4.3 The Grothendieck ring of a vector space

Theorem 4.3.1. *Let M be an infinite module over a division ring D. Then the Grothendieck ring $K_0(M)$ of M (regarded as an $\mathcal{L}_D(M)$-structure) is isomorphic to*

$\mathbb{Z}[X]$*, the polynomial ring in one indeterminate over the ring of integers.*

We prove this theorem via a series of intermediate results. We first define a family of sets in $\mathrm{Def}(M)$, indexed by the polynomials over $\mathbb{Z}$ with positive leading coefficient. We show that every set in $\mathrm{Def}(M)$ is definably isomorphic to one of these *representative sets*. We show that the images of these representative sets in $K_0(M)$ satisfy the same relationships in the ring operations $+, \times$ that their corresponding polynomials do in $\mathbb{Z}[X]$. We show that no two of these representative sets for distinct polynomials are in definable bijection, i.e. they have distinct classes in $\widetilde{\mathrm{Def}}(M)$, and moreover that they are not identified in $K_0(M)$ under the equivalence relation $\sim$ on $\widetilde{\mathrm{Def}}(M)$. These lemmas combined then yield a proof of Theorem 4.3.1, which is given at the end of this chapter.

4.4 Representative sets

Let D be a division ring. Let M be an infinite D-module and hence an $\mathcal{L}$-structure for $\mathcal{L}$ the language of right D-modules. M is then strongly minimal. For each polynomial in $\mathbb{Z}[X]$ with positive leading coefficient, we choose a unique canonical *representative set* in $\mathrm{Def}(M)$. We allow parameters in our formulas. Since M is infinite we may choose countably many distinct elements $a_0, a_1, a_2, \ldots$ of M, and fix these throughout this section. These are used to construct finite sets of any desired size and also to ensure that we can construct as many mutually disjoint sets as we wish.

- The polynomials of degree zero and positive leading coefficient are the natural numbers. For $f(X) = n \in \mathbb{Z}[X]$, we set $S_n = S_{f(X)} = \{a_1, \ldots, a_n\} \times \{a_0\}$.
- For each monomial $f(X) = cX^n$, we set $S_{cX^n} = \{a_1, \ldots, a_c\} \times M^n \times \{a_n\}$
- Given a typical polynomial $f(x) = \sum_{i=0}^{n} c_i X^i$ with $c_n > 0$, we define $S_{f(X)}$ by starting with $S_f^n := S_{c_n X^n}$ and adding each term in order of descending power as follows for $n-1 \geq k \geq 0$:

1. If $c_k > 0$ then we set $S_f^k := S_f^{k+1} \cup (\{a_1 \ldots, a_{c_k}\} \times M^k \times \{(a_k, \ldots, a_k)\})$ where the final term is $\{a_k\}^{n+1-k}$. In the special case of $k = 0$, the M^k term disappears altogether.

2. If $c_k = 0$ then we set $S_f^k := S_f^{k+1}$.

3. If $c_k < 0$ then we set

$$S_f^k := S_f^{k+1} \setminus (\{a_1\} \times \{a_1, \ldots, a_{-c_k}\} \times M^k \times \{(a_k, \ldots, a_k)\} \times \{a_n\})$$

   where the penultimate term is $\{a_k\}^{n-1-k}$. This is the required length to ensure we remove a subset of S_f^{k+1}. In the special case of $k = 0$, the M^k term disappears altogether.

4. Finally we set $S_{f(X)} = S_f^0$ and this is the canonical representative set of a typical polynomial.

Observe that in this procedure the terms of negative coefficient in the polynomial are manifested by sets being removed from the set corresponding to the leading term, and that these removed sets are, by construction, disjoint for different k. Observe also that the different positive terms of the polynomial will always contribute disjoint sets to the representative set $S_{f(X)}$.

A special case requiring an alteration

There is a special case for which the above construction requires an alteration. The technical reason and a solution are given here, but it does not break the argument (in character) at any stage. The subsequent calculations and proofs overlook this special case, but it is clear that they are all valid, up to trivial technical adjustments.

If $f(X) = \Sigma_{i=0}^n c_i X^i$ and $c_{n-1} < 0$, then the set $S_{f(X)}$ described above will have $\{a_1\} \times \{a_1, \ldots, a_{-c_{n-1}}\} \times M^{n-1} \times \{a_n\}$ removed from the set $S_{c_n X^n}$. But this deleted set will not be disjoint from any other deleted set that we remove on account of other negative terms in $f(X)$. The construction given above relies upon the 'labeling tuples' $\{(a_k, \ldots, a_k)\}$ to keep the removed sets disjoint but as this tuple is of length $n - 1 - k$, it is absent for $k = n - 1$.

One inelegant but valid way to avoid this problem is to define our representative sets as above with one difference. If $c_{n-1} < 0$, define $t := \max\{|c_i| : 0 \leq i \leq n\}$, the maximum of the moduli of the coefficients in the polynomial.

Then set S_f^{n-1} to be $S_f^n \setminus (\{a_1\} \times \{a_{t+1}, \ldots, a_{t-c_{n-1}}\} \times M^{n-1} \times \{a_n\})$. Then we can proceed as before until we reach $S_f^0 = S_{f(X)}$.

Definition 4.4.1. *We have defined a family of sets*

$$\{S_{f(X)} \in \mathit{Def}(M) : f(X) \in \mathbb{Z}[X],\ f(X) \text{ has positive leading coefficient}\}$$

These sets have definable isomorphism classes in $\widetilde{\mathit{Def}}(M)$ *and the images of these under the universal weak Euler characteristic are the elements* $[S_{f(X)}]$ *in the Grothendieck ring* $K_0(M)$*, by definition. For polynomials* $g(X) \in \mathbb{Z}[X]$ *with negative leading coefficient, set* $[S_{g(X)}] := -[S_{-g(X)}] \in K_0(M)$*.*

Remark. This is a slight abuse of notation as we have not defined a set $S_{g(X)}$ among our representatives and in fact, there is no definable set A such that $[S_{g(X)}] = [A]$. $[S_{g(X)}]$ is defined as the negative in $K_0(M)$ of the element $[S_{-g(X)}]$, and the set $S_{-g(X)}$ is among the representative sets defined above.

4.5 Addition in $\mathbb{Z}[X]$

Lemma 4.5.1. *Let* $f(X), g(X) \in \mathbb{Z}[X]$ *with positive leading coefficients. Then* $[S_{f(X)}] + [S_{g(X)}] = [S_{f+g}]$ *in* $K_0(M)$*.*

We can explicitly give the definable bijections between these definable sets. Let $f(X) = \sum_{i=0}^{n} c_i X^i$ and $g(X) = \sum_{i=0}^{m} d_i X^i$. Without loss of generality, we may write

$$f(X) = \sum_{i \in I^+} c_i X^i + \sum_{i \in I^-} c_i X^i$$

where $I^+ := \{0 \leq i \leq n : c_i \geq 0\}$ and $I^- := \{0 \leq i < n : c_i < 0\}$ and similarly

$$g(X) = \sum_{i \in J^+} d_i X^i + \sum_{i \in J^-} d_i X^i$$

Then we may express $S_f(M)$ in the form $S_f(M) = ((\{a_1, \ldots, a_{c_n}\} \times M^n \times \{a_n\}) \setminus A) \cup B$, where A is the set $\bigcup_{i \in I^-} (\{a_1\} \times \{a_1, \ldots, a_{-c_i}\} \times M^i \times \{(a_i, \ldots, a_i, a_n)\})$ and B is the set $\bigcup_{i \in I^+} (\{a_1, \ldots, a_{c_i}\} \times M^i \times \{(a_i, \ldots, a_i)\})$.

Proof of Lemma 4.5.1. Assuming wlog that $n \geq m$, observe that $f(X) + g(X) = \sum_{i=0}^n (c_i + d_i) X^i$ where we set any necessary dummy coefficients $d_{m+1}, \ldots, d_n$ equal to 0. Let $K^+ = \{0 \leq i \leq n : c_i + d_i \geq 0\}$ and $K^- = \{0 \leq i < n : c_i + d_i < 0\}$. We note that $S_f(M), S_{f+g}(M) \subset M^{n+2}$.

Observe that $[S_f(M)] + [S_g(M)] = [\{a_0\} \times S_f(M) \cup \{a_1\} \times S_g(M)]$, by the definition of addition in the Grothendieck ring. We define a bijection θ from $\{a_0\} \times S_f(M) \cup \{a_1\} \times S_g(M)$ to $S_{f+g}(M)$ that acts by 'rearranging' the pp-sets, preserving their internal shape but altering the labeling parameters, to give an heuristic description. The formula for θ is essentially instructions for the rearrangement, depending on the signs (+/-) of the coefficients $c_i, d_i, c_i + d_i$.

We define θ to be a disjunction of formulas defining functions acting on different disjoint subsets of $\{a_0\} \times S_f(M) \cup \{a_1\} \times S_g(M)$. For clarity, we give a function on $\{a_0\} \times S_f$ with formula θ_1 and a function on $\{a_1\} \times S_g$ with formula θ_2.

Let θ_1 act on $\{a_0\} \times S_f(M)$ by simply omitting the a_0, that is $\theta_1(\overline{v}, \overline{w}) \Leftrightarrow \overline{v} = (a_0, \overline{w}) \wedge \overline{w} \in S_f(M)$. Then for $i < m$, if $d_i \geq 0$ we define the action of θ_2 on $S_g^{(i)} := \{a_1\} \times \{a_1, \ldots, a_{d_i}\} \times M^i \times \{(a_i, \ldots, a_i)\}$ as follows. If c_i is also nonnegative, then we have $\{a_1, \ldots, a_{c_i}\} \times M^i \times \{(a_i, \ldots, a_i)\}$ in the image of θ_1 already and on $S_g^{(i)}$, $\theta_2(\overline{v}, \overline{w})$ is given by

$$(\overline{v} \in S_g^{(i)}) \wedge \bigvee_{k=1}^{d_i} (\overline{v} = (a_1, a_k, \overline{u}) \wedge \overline{w} = (a_{c_i+k}, \overline{u}))$$

If $c_i < 0$, and $d_i \geq 0$ then we know from the above that the image of $\{a_0\} \times S_f$ under θ_1 includes an n-dimensional pp-set with complement including $\{a_1\} \times \{a_1, \ldots, a_{-c_i}\} \times M^i \times \{(a_i, \ldots, a_i, a_n)\}$. Then if $d_i \geq 0 > d_i + c_i$ we will set θ_2 to act on $S_g^{(i)}$ by changing the parameter-label part of each element to 'fill in' the gap of d_i of the missing pp-sets of dimension i. Explicitly, θ_2 contains a subformula

$\bigvee_{k=1}^{d_i} (\overline{v} = (a_1, a_k, \overline{u}, a_i, \ldots, a_i) \wedge \overline{w} = (a_1, \overline{u}, a_i, \ldots, a_i, a_n))$ for each such i.

Alternatively if $d_i > d_i + c_i > 0$ then we have

$$S_g^{(i)} := \{a_1\} \times \{a_1, \ldots, a_{d_i+c_i}, a_{d_i+c_i+1}, \ldots, a_{d_i}\} \times M^i \times \{(a_i, \ldots, a_i)\} \subset S_g$$

The bijection will take $-c_i$ of these dimension i pp-sets into the dimension i 'removed sets' in $\theta_1(S_f)$, and will take the remaining $d_i + c_i$ to the dimension i part of $S_{f+g}(M)$. The formula $\theta_2(\overline{v}, \overline{w})$ will distinguish between tuples depending on their second component.

$$\begin{aligned} &\bigvee_{1 \le k \le d_i + c_i} (\overline{v} = \overbrace{(a_1, a_k, \overline{u}, a_i, \ldots, a_i)}^{m+3} \wedge \overline{w} = \overbrace{(a_1, a_k, \overline{u}, a_i, \ldots, a_i, a_i)}^{n+2}) \\ \vee &\bigvee_{d_i + c_i < k \le d_i} (\overline{v} = \overbrace{(a_1, a_k, \overline{u}, a_i, \ldots, a_i)}^{m+3} \wedge \overline{w} = \overbrace{(a_1, a_k, \overline{u}, a_i, \ldots, a_i, a_n)}^{n+2}) \end{aligned}$$

If $d_i < 0$ then we have in S_g the contribution from the term $d_i X^i$ of $g(X)$ is that the set $\{a_1\} \times \{a_1, \ldots, a_{-d_i}\} \times M^i \times \{(a_i, \ldots, a_i, a_m)\}$ is removed from the subset of S_g associated to the leading term $d_m X^m$. Recall from our definition of representative sets that all negative terms of $g(X)$ are represented by appropriate sets being removed from $S_g^m = \{a_1, \ldots, a_{d_m}\} \times M^m \times \{a_m\}$ and these removed sets are disjoint subsets of $\{a_1\} \times M^m \times \{a_m\} \subseteq S_g^m$.

The action of θ_2 on these terms depends of the signs of the coefficients c_m and $c_m + d_m$ of X^m in the polynomial $(f+g)(X)$ and also the signs of the coefficients $c_i, c_i + d_i$. Since $d_i < 0$, the restriction of the function given by θ_2 will be acting on a pp-set of dimension m with d_i disjoint subsets that are pp-sets of dimension i removed.

If c_i, d_i are both negative then $0 > c_i, d_i > c_i + d_i$. Now θ_1 as defined above is a function acting on $\{a_0\} \times S_f(M)$, given by $(a_0, \overline{w}) \mapsto \overline{w}$. We define also θ_2 acting on $\{a_1\} \times S_g(M)$ by the action $(a_1, \overline{w}) \mapsto \overline{w}$. Observe that $\theta_1(M) \cup \theta_2(M)$ is a bijection but the image is not equal to $S_{f+g}(M)$ as desired. We show that the image is in definable bijection with the desired image. These maps θ_1, θ_2 act as 'prototypes' for the restrictions $\theta \restriction \{a_0\} \times S_f(M)$ and $\theta \restriction \{a_1\} \times S_g(M)$ in the sense that we first define the action on the domain of θ by partitioning it and taking θ_1, θ_2 to act on the two subsets. Then we alter the function, by changing the action of specified pp-subsets, to obtain the final action of θ, which will have image $S_{f+g}(X)$.

Since $c_i < 0$, the image of θ_1 will have $\{a_1\} \times \{a_1, \ldots, a_{-c_i}\} \times M^i \times \{(a_i, \ldots, a_i, a_n)\}$ removed from $\{a_1\} \times M^n \times \{a_n\}$. We require the image of θ, that is $S_{f+g}(M)$, to have $\{a_1\} \times \{a_1, \ldots, a_{-c_i-d_i}\} \times M^i \times \{(a_i, \ldots, a_i, a_n)\}$ removed from $S^n_{f+g} = \{a_1\} \times M^n \times \{a_n\}$. Observe that the set $\{(a_0, a_1)\} \times \{a_{-c_i+1}, a_{-c_i+2}, \ldots, a_{-c_i-d_i}\} \times M^i \times \{(a_i, \ldots, a_i, a_n)\}$ is a subset of $\{a_0\} \times S_f(M)$. The function θ_1 acts on this set simply by projection, losing the first argument a_0, but we do not want the resulting set in the image of θ as it does not intersect $S_{f+g}(M)$. Thus we construct θ so as to take this set and 'fill' the removed $-d_i$ cosets of dimension i that are missing (on account of the term $d_i X^i$ in the polynomial $g(X)$) from the $\{a_1\} \times M^m \times \{a_m\}$ part of $\theta_2(\{a_1\} \times S_g(M))$.

The remaining cases to consider are when $d_i < 0$ and $c_i \geq 0$. The two cases are *(i)* when $c_i > c_i + d_i \geq 0 > d_i$ and *(ii)* when $c_i \geq 0 > c_i + d_i \geq d_i$. In case *(i)* we have, in the image of θ_1, c_i disjoint cosets of dimension i arising from the term $c_i X^i$ of $f(X)$. But in $S_{f+g}(M)$, the desired image of θ, there are only $c_i + d_i$. In the image of θ_2 there are $-d_i$ cosets of dimension i removed from the first coset of dimension m (i.e. that with labeling parameter a_1). Now $c_i > -d_i > 0$ so we take θ to 'fill' the $-d_i$ missing cosets of dimension i in the image of θ_2 by mapping bijectively into them $-d_i$ of the cosets from $\{a_0\} \times S_f(M)$. Thus for $\{a_0\} \times \{a_1, \ldots, a_{c_i+d_i}\} \times M^i \times \{(a_i, \ldots, a_i, a_i)\}$ the action of θ will be exactly the action of θ_1, but θ will take the set $\{a_0\} \times \{a_{c_i+d_i+1}, \ldots, a_{c_i}\} \times M^i \times \{(a_i, \ldots, a_i, a_i)\}$ to fill the missing cosets in the image of θ_2.

Finally, for case *(ii)*, the i^{th} term of $f(X) + g(X)$ is negative. In the image of θ_1, we have c_i cosets of dimension i, as in the representative set of any polynomial with the term $c_i X^i$ for $c_i > 0$. In the image of θ_2, we have $-d_i$ cosets of dimension i removed from the first coset of dimension m (i.e. that with labeling parameter a_1). Note $-d_i > c_i$. In $S_{f+g}(M)$, the desired image of θ, the first coset of dimension n has

$-c_i - d_i$ disjoint pp-subsets of dimension i removed. Thus we take θ to send the c_i pp-sets of dimension i and also a further $-d_i - c_i$ pp-subsets of dimension i to fill the $-d_i$ disjoint removed pp-sets in the image of θ_2. These additional $-d_i - c_i$ cosets, we take to be $\{(a_0, a_1, a_k)\} \times M^i \times \{(a_i \ldots, a_i, a_n)\} \subset \{a_0\} \times S_f(M)$ for $1 \leq k \leq -d_i - c_i$, and this gives the image of θ the desired form for the dimension i term, namely $-c_i - d_i$ removed pp-subsets from $\{a_1\} \times M^n \times \{a_n\}$ and no additional disjoint pp-sets of dimension i.

In the first cases considered, we explicitly gave the action of θ as a formula. In the remainder of the cases, we described the action of θ in terms of its action on the cosets (and their complements) comprising the domain. On each coset the formula for the restricted action could be easily deduced, as in the earlier cases, but is omitted for brevity. Thus we have given an explicit construction, depending on the signs of the coefficients of the polynomials $f(X)$ and $g(X)$, for the desired bijection

$$\theta : \{a_0\} \times S_f(M) \cup \{a_1\} \times S_g(M) \to S_{f+g}(M)$$

and hence Lemma 4.5.1 holds.

□

Remark. This long formula for θ does nothing more than rearrange the cosets defined by the pp-formulas occurring in $S_{f(X)}$ and $S_{g(X)}$ and in some cases alter the parameters that serve to keep subsets disjoint and to regulate the length of tuples in the representative sets.

4.6 Multiplication in $\mathbb{Z}[X]$

Lemma 4.6.1. *Let $f(X), g(X) \in \mathbb{Z}[X]$ with positive leading coefficients. Then $[S_{f(X)}] \times [S_{g(X)}] = [S_{f \cdot g}]$ in $K_0(M)$.*

Proof. Let $f(X) = \sum_{i=0}^{n} c_i X^i$ and $g(X) = \sum_{i=0}^{m} d_i X^i$. Recalling the construction of $S_{p(X)}$ for polynomials $p(X)$ of negative leading coefficient, the subset removed from $S_{c_n X^n}$ in the construction of $S_{f(X)}$ for each negative term $c_i X^i$, with $c_i < 0$, is in definable bijection with the corresponding monomial $S_{-c_iX^i}$. Then by Lemma 4.5.1 and this observation for the negative coefficients, we have

$$[S_{f(X)}] = \sum_{i=0}^{n} [S_{c_iX^i}] \text{ and } [S_{g(X)}] = \sum_{i=0}^{m} [S_{d_iX^i}]$$

Hence the LHS in the statement of the lemma is

$$[S_{f(X)}] \times [S_{g(X)}] = \sum_{i=0}^{n} [S_{c_iX^i}] \times \sum_{i=0}^{m} [S_{d_iX^i}] = \sum_{(i,j)} [S_{c_iX^i}] \cdot [S_{d_jX^j}]$$

Since

$$f(X) \cdot g(X) = (\sum_{i=0}^{n} c_iX^i)(\sum_{j=0}^{m} d_jX^j) = \sum_{(i,j)} c_id_jX^{i+j}$$

we have by Lemma 4.5.1 again, that the RHS is

$$[S_{f(X)\cdot g(X)}] = \sum_{(i,j)} [S_{c_id_jX^{i+j}}]$$

Therefore to prove the equality, it suffices to show that the elements $[S_{c_id_jX^{i+j}}]$ and $[S_{c_iX^i}] \cdot [S_{d_jX^j}]$ are equal in $K_0(M)$, for every $i, j \in \mathbb{N}$ and every $c_i, d_j \in \mathbb{Z}$. If $c_i, d_j \geq 0$ then we can explicitly give a definable bijection from $S_{c_iX^i} \times S_{d_jX^j}$ to $S_{c_id_jX^{i+j}}$. Recalling the definition of the representative sets and the countable set of parameters $a_0, a_1, \ldots$, we have

$$S_{c_iX^i} = \{a_1, \ldots, a_{c_i}\} \times M^i \times \{a_i\}$$

$$S_{d_jX^j} = \{a_1, \ldots, a_{d_j}\} \times M^j \times \{a_j\}$$

$$S_{c_id_jX^{i+j}} = \{a_1, \ldots, a_{c_id_j}\} \times M^{i+j} \times \{a_{i+j}\}$$

A general element of $S_{c_iX^i}$ is therefore $(a_s, m_1, m_2, \ldots, m_i, a_i)$ for some $1 \leq s \leq c_i$ and a general element of $S_{d_jX^j}$ is therefore $(a_t, n_1, n_2, \ldots, n_j, a_j)$ for some $1 \leq t \leq d_j$. We may define the desired bijection from $S_{c_iX^i} \times S_{d_jX^j}$ to $S_{c_id_jX^{i+j}}$ to be the map $(a_s, m_1, m_2, \ldots, m_i, a_i, a_t, n_1, n_2, \ldots, n_j, a_j) \mapsto (a_{(s-1)d_j+t}, m_1, \ldots, m_i, n_1, \ldots, n_j, a_{i+j})$. Thus we have $[S_{c_id_jX^{i+j}}] = [S_{c_iX^i}] \cdot [S_{d_jX^j}]$ and the proposition holds whenever $c_i, d_j \geq 0$. In fact if either coefficient is equal to 0, both sides of the equation in the proposition are equal to the empty set.

For $c_i < 0$, the element $[S_{c_i X^i}] \in K_0(M)$ is not the image under the universal weak Euler characteristic χ_0 of some definable set. It is the difference between $[A]$ and $[A \setminus B]$ for any $A \in \mathrm{Def}(M)$ of dimension greater than i and any $B \in \mathrm{Def}(M)$ such that $B \subseteq A$ and B is in definable bijection with $S_{-c_i X^i}$. The construction of the Grothendieck ring ensures that for definable sets $B \subseteq A$, we have $[B] + [A \setminus B] = [A]$ and every element has a corresponding negative element such that $[A] + (-[B]) = [A \setminus B]$. So for $c_i < 0$, we have by construction of the representative sets $[S_{c_i X^i}] = -[S_{-c_i X^i}] \in K_0(M)$.

If $c_i, d_j < 0$ we have $c_i d_j > 0$ and the element $[S_{c_i d_j X^{i+j}}]$ is the image of the definable set $S_{c_i d_j X^{i+j}}$ under χ_0. Also

$$[S_{c_i X^i}] \cdot [S_{d_j X^j}] = -[S_{-c_i X^i}] \cdot -[S_{-d_j X^j}] = [S_{-c_i X^i}] \cdot [S_{-d_j X^j}]$$

and this is equal to $[S_{c_i d_i X^{i+j}}]$ by the above argument for nonnegative coefficients.

Finally if only one coefficient is negative, then the product $c_i d_j < 0$ and $[S_{c_i d_j X^{i+j}}]$ $= -[S_{-c_i d_j X^{i+j}}]$. Assuming without loss of generality that $c_i \geq 0 > d_j$ we have $[S_{d_j X^j}] = -[S_{-d_j X^j}]$. So the product $[S_{c_i X^i}] \cdot [S_{d_j X^j}]$ is equal to $[S_{c_i X^i}] \cdot -[S_{-d_j X^j}] = -[S_{-c_i d_j X^{i+j}}]$ as required.

□

4.7 The representative sets are distinct up to isomorphism

Definition 4.7.1. *The* Morley rank *of an $\mathcal{L}$-formula θ, with solution set $S = \theta(M)$ in an $\mathcal{L}$-structure M, is defined inductively as follows:*

- *The Morley rank of θ is at least zero if $S \neq \emptyset$.*
- *Let κ be a successor ordinal (including all integers $n \geq 1$). If there is some elementary extension $N \succeq M$ such that $\theta(N)$ contains countably many disjoint definable subsets of Morley rank at least $\kappa - 1$, then θ has Morley rank at least κ.*

- *Let λ be a limit ordinal. Then θ has Morley rank at least λ if it has Morley rank at least κ for all $\kappa < \lambda$.*

The Morley rank of a definable set S is defined to be the Morley rank of an $\mathcal{L}$-formula defining S, and is denoted $Mrank(S)$.

Definition 4.7.2. *The* Morley degree *of a definable set S of Morley rank κ is the largest integer m such that S may be expressed as the disjoint union of m definable sets of Morley rank κ. We write $Mdeg(S) = m$ to mean that the set S has Morley degree m.*

Lemma 4.7.3. *Definable bijections preserve Morley rank and Morley degree.*

This is well known and the proof is immediate from the definitions. Let f be a definable bijection from A to B. If B has a partition into definable sets, then the preimages under f of the sets in the partition, will be a definable partition of A.

Lemma 4.7.4. *Let $f(X), g(X)$ be polynomials over the ring of integers with positive leading coefficient and suppose the representative sets of these polynomials are definably isomorphic, $S_{f(X)} \cong S_{g(X)}$. Then $f(X) = g(X)$.*

Proof. We argue by induction on the polynomial degree of $f(X)$. Suppose $f(X)$ is a constant, so has polynomial degree 0. Then $S_{f(X)}$ is finite and its Morley rank is zero. It follows immediately from the existence of a bijection that $S_{g(X)}$ is of the same cardinality, say m, and $S_{f(X)} = S_{g(X)} = \{a_1, \ldots, a_m\}$ and $f(X) = g(X) = m$. Bijections preserve Morley rank and Morley degree which correspond to polynomial degree and leading coefficient respectively. Polynomials of degree 0 are uniquely determined by their leading coefficient, in fact they are just a leading coefficient. So the lemma holds for $f(X)$ of degree 0 simply by the remark above. This does not apply for higher degrees.

Polynomials of degree 1. If $f(X)$ is of *polynomial degree* 1 then $S_{f(X)}$ is of Morley rank 1 and we argue by induction on Morley degree. The base case is Morley degree 1. Suppose the bijection between $S_{f(X)}$ and $S_{g(X)}$ is given by the formula $\rho(\overline{v},\overline{w})$. By Baur's quantifier elimination result this is equivalent to $\bigvee_{i=1}^{n}(\phi_i \wedge \bigwedge_{j=1}^{m_i} \neg\psi_{ij})(\overline{v},\overline{w})$ for some pp-formulas ϕ_i, ψ_{ij}. Without loss of generality these pp-formulas can be chosen such that the sets $(\phi_i \wedge \bigwedge_{j=1}^{m_i} \neg\psi_{ij})(M)$ are pairwise disjoint sets by Lemma 3.2.1, and for brevity we call them $\rho_1(M), \ldots, \rho_n(M)$. Since $\rho(\overline{v},\overline{w})$ defines the graph of the bijection and the set $\rho(M)$ is the disjoint union of the sets $\rho_1(M), \ldots, \rho_n(M)$, each $\rho_i(\overline{v},\overline{w})$ must define the graph of a bijection. The set $\rho(M)$ is in bijection with both $S_{f(X)}$ and $S_g(X)$, as it is the graph of a bijection from the former to the latter. For each $i = 1, \ldots, n$, let $\pi_1\rho_i(M)$ and $\pi_2\rho_i(M)$ denote the projections of the set $\rho_i(M)$ onto the domain and image respectively. Note $\pi_1\rho_i(M) \subseteq S_{f(X)}$ and $\pi_2\rho_i(M) \subseteq S_{g(X)}$.

Now by the construction of the representative sets for polynomials, $S_{f(X)} = ((\{a_1\} \times M \times \{a_1\}) \setminus A) \sqcup B$, where A and B are of Morley rank strictly less than 1, i.e. they are finite (and at least one will be empty). Hence for some $p, q \geq 0$:

$$S_{f(X)} = (\{a_1\} \times (M \setminus \{a_1, \ldots, a_p\}) \times \{a_1\}) \sqcup (\{a_1, \ldots, a_q\} \times \{(a_0, a_0)\})$$

In fact at most one of p, q will be non-zero but we treat the two cases simultaneously. There is clearly a definable bijection, F say, between this set and a set obtained from M by either removing a finite subset or adding a disjoint finite set of constant elements as a disjoint set (depending on which of p and q is the larger).

The set M can be partitioned into n definable sets by taking the images under F of the sets $\pi_1\rho_i(M)$, and some straightforward correction to account for the finitely many constant elements. Then since M is strongly minimal, exactly one of these n sets must be infinite. Since bijections preserve Morley rank and degree, and the graph of a bijection is in bijection with the domain (and with the image), exactly one of the $\rho_i(M)$ will be infinite. Reordering if necessary, let the infinite one be $\rho_1(M) = (\phi_1 \wedge \neg\psi_{11} \wedge \ldots \wedge \neg\psi_{1n_1})(M)$. Therefore $\phi_1(M)$ is of Morley rank 1.

Clearly $\phi_1(M)$ must also have Morley degree 1. Assume for a contradiction that A_1 and A_2 are disjoint definable infinite subsets of $\phi_1(M)$. Then the formulas $\rho_1(\overline{x},\overline{y}) \wedge ((\overline{x},\overline{y}) \in A_i)$ for $i = 1, 2$ would define disjoint sets, contradicting the fact that $\rho_1(M)$ has Morley degree 1.

Now the set $\phi_1(M)$ is of Morley rank and degree 1. Since every pp-set is definably isomorphic as a left D-module to M^k for some k by Lemma 4.1.4, this implies that $\phi_1(M) \cong M$. By strong minimality none of the sets $\psi_{1j}(M)$ for $j = 1, \dots, m_1$ can be infinite, and they are all singleton sets as the only finite pp-sets in $\mathrm{Def}(M)$ are singletons. Therefore the number of elements from the projections of $\phi_1(M)$ that are excluded in the projections of $\rho(M)$ is equal for the two projections. Denote this number t. Observe that the projections of the sets $\psi_{11}(M), \dots, \psi_{1m_1}(M)$ onto the domain (for these sets are graphs of functions) must be disjoint because $\phi_1(M)$ is the graph of a bijection and $\psi_{ij} \to \phi_i$ for each i, j. Similarly the projections of these sets onto the codomain must be disjoint.

For $k = 2, \dots, n$ the sets defined by $\rho_k(M) = (\phi_k \wedge \bigwedge_{j=1}^{m_k} \neg\psi_{kj})(M)$ are all finite sets, and for each $k > 1$ the projections $\pi_1(\rho_k) := \exists \overline{w}\, \rho_k(M, \overline{w})$ and $\pi_2(\rho_k) := \exists \overline{v}\, \rho_k(\overline{v}, M)$ are the same size, say $t_k \in \mathbb{N}$. Also the sets $\pi_1(\rho_1), \dots, \pi_1(\rho_n)$ are pairwise disjoint, as are the sets $\pi_2(\rho_1), \dots, \pi_2(\rho_n)$. Therefore the values of q and q' are equal in $S_{f(X)} = (\{a_1\} \times (M \setminus \{a_1, \dots, a_p\}) \times \{a_1\}) \sqcup (\{a_1, \dots, a_q\} \times \{a_0\})$ and in $S_{g(X)} = (\{a_1\} \times (M \setminus \{a_1, \dots, a_p\}) \times \{a_1\}) \sqcup (\{a_1, \dots, a_{q'}\} \times \{a_0\})$.

Now we know $\pi_1(\rho) = \bigcup_k \pi_1(\rho_k) \cong S_{f(X)}$ and $\pi_1(\rho_1) \cong S_{X-t}$ and for each $2 \leq k \leq n$, $\pi_1(\rho_k) \cong S_{t_k}$. Therefore $\pi_1(\rho) = \bigsqcup_k \pi_1(\rho_k) \cong S_{X-t} \sqcup S_{t_2} \sqcup \ldots \sqcup S_{t_n}$. Also $\pi_2(\rho) = \bigcup_k \pi_2(\rho_k) \cong S_{g(X)}$ and $\pi_2(\rho) = \bigsqcup_k \pi_2(\rho_k) \cong S_{X-t} \sqcup S_{t_2} \sqcup \ldots \sqcup S_{t_n}$. Therefore the two projections of $\rho(M)$, namely $S_{f(X)}$ and $S_{g(X)}$ must be equal. This concludes the proof of the base case.

With the base case established, i.e. sets of Morley rank 1 and Morley degree 1, we can prove the lemma for rank-1-sets by induction on Morley degree. Suppose the leading term in $f(X)$ is cX and we have the result for rank-1-sets of degree less than c. Then if we write $f(X) = cX + d$, we may observe that $S_{f(X)} = (\{a_1, \dots, a_c\} \times M \times \{a_1\}) \cup (\{a_1, \dots, a_d\} \times \{(a_0, a_0)\})$ if $d \geq 0$ and $S_{f(X)} = (\{a_1, \dots, a_c\} \times M \times \{a_1\}) \setminus (\{a_1\} \times \{a_1, \dots, a_{-d}\} \times \{a_1\})$ if $d < 0$. In both cases we may split $S_{f(X)}$ into $\{a_c\} \times M \times \{a_1\}$ and $S_{f(X)-X}$, these are disjoint sets both of Morley rank 1 (unless $c = 1$ but that was the base case and in that case the lemma holds by the above).

Suppose we have done likewise for $g(X)$. If we prove that $S_{f(X)-X} \cong S_{g(X)-X}$, then it follows from our inductive hypothesis that $g(X) - X = f(X) - X$ and hence $f(X) = g(X)$.

Case $d > 0$**.** Now $\rho(\overline{v}, \overline{w})$ defines a bijection from $S_{f(X)} = (\{a_1, \ldots, a_c\} \times M \times \{a_1\}) \cup (\{a_1, \ldots, a_d\} \times \{(a_0, a_0)\})$ to $S_{g(X)}$. Abusing notation slightly, we will call the bijection, as well as the formula defining its graph, ρ. So $S_{g(X)} = \rho(\{a_c\} \times M \times \{a_1\}) \cup \rho(S_{f(X)} \setminus (\{a_c\} \times M \times \{a_1\}))$. Following the above argument relying on strong minimality, $\rho(\{a_c\} \times M \times \{a_1\})$ must be equal up to finitely many elements to $(\{a_t\} \times M \times \{a_1\})$ for some $1 \leq t \leq c$. Hence must be equal to $(\{a_t\} \times M \times \{a_1\}) \setminus (\{a_t\} \times \{d_1, \ldots, d_s\} \times \{a_1\}) \sqcup \{\overline{b}_1, \ldots, \overline{b}_s\}$ for some $\overline{b}_1, \ldots, \overline{b}_s \in S_{g(X)}$.

Therefore $S_{f(X)-X} \cong \rho(S_{f(X)} \setminus \{a_c\} \times M \times \{a_1\}) = S_{g(X)} \setminus \rho(\{a_c\} \times M \times \{a_1\})$ $= (\{a_1, \ldots, a_e\} \times \{(a_0, a_0)\}) \cup ((\{a_1, \ldots, a_{t-1}, a_{t+1}, \ldots, a_c\} \times M \times \{a_1\}) \cup (\{a_t\} \times \{d_1, \ldots, d_s\} \times \{a_1\})) \setminus (\{\overline{b}_1, \ldots, \overline{b}_s\})$, where c and e are the coefficients of $g(X) = cX + e$. Then both the left- and right-hand sides are of Morley rank 1 and Morley degree $c - 1$, so we may apply the inductive hypothesis to yield $f(X) - X = g(X) - X$ and then the result follows immediately. This establishes the lemma for all polynomials of degree less than 2.

Polynomials of higher degree. We prove the lemma for general $f(X)$ by induction on the degree of the polynomial, which is the Morley rank of the representative set $S_{f(X)}$. Let the leading term of $f(X)$ be cX^k and suppose the lemma holds for all polynomials of degree less than k *and* polynomials of degree k with leading coefficient less than c. As before we split $S_{f(X)}$ into two disjoint sets, $S_{f(X)-X^k}$ and $\{a_c\} \times M^k \times \{a_k\}$. If $S_{f(X)} \cong S_{g(X)}$, then let $\rho(\overline{v}, \overline{w})$ be a formula defining a bijection between the two sets. As before the bijection and any restriction of it will preserve Morley rank and degree. Also $\rho(\{a_c\} \times M^k \times \{a_k\}) \subseteq S_{g(x)}$ will be equal to $A_1 \sqcup ((\{a_t\} \times M^k \times \{a_k\}) \setminus A_2)$ for some t and some sets A_1, A_2 of Morley rank less than k and $A_2 \subset \{a_t\} \times M^k \times \{a_k\}$. Therefore if $A_1 \cong S_{h_1(X)}$ and $A_2 \cong S_{h_2(X)}$, then we have

$$S_{X^k} \cong \rho(\{a_c\} \times M^k \times \{a_k\}) = A_1 \sqcup (\{a_t\} \times M^k \times \{a_k\}) \setminus A_2 \cong S_{X^k + h_1(X) - h_2(X)}$$

and our inductive hypothesis on Morley degree yields that $h_1(X) = h_2(X)$. Therefore there exists a bijection between A_1 and A_2.

Recall $S_{f(X)} = S_{f(X)-X^k} \cup (\{a_c\} \times M^k \times \{a_k\})$. If we define B by $\rho(S_{f(X)-X^k})$ then $S_{g(X)} = \rho(\{a_c\} \times M^k \times \{a_k\}) \sqcup B$. Also $B = S_{g(X)} \setminus ((A_1 \sqcup (\{a_t\} \times M^k \times \{a_k\})) \setminus A_2)$ and therefore definably isomorphic to $S_{g(X)-X^k-h_1(X)+h_2(X)} = S_{g(X)-X^k}$. Now the inductive hypothesis yields $[S_{f(X)-X^k}] = [B] = [S_{g(X)-X^k}] \Rightarrow f(X) = g(X)$ as required. This completes the proof of Lemma 4.7.4.

□

4.8 General definable sets

Lemma 4.8.1. *Every definable set $A \in Def(M)$ is definably isomorphic to one of the representative sets $S_{f(X)}$ for some $f(X) \in \mathbb{Z}[X]$.*

By the previous section, the representative sets of distinct polynomials are never in definable isomorphism, so this polynomial $f(X)$ will be determined uniquely for A, provided such a polynomial exists.

Proof. We prove the lemma by induction on the complexity of the formula θ defining A. The simplest formulas in the language of R-modules are the positive primitive formulas. If $\phi(\overline{v})$ is a parameter free pp-formula, then $\phi(M)$ is an $\mathrm{End}(M)$-module.

Induction on the complexity of a formula.

If θ is a pp-formula $\phi(v_1, \ldots, v_n)$, then $\phi(M)$ will be definably isomorphic to M^k for some $k \leq n$ by Lemma 4.1.4. Moreover $\phi(M)$ will actually be a coset of some additive group $G \leq M^n$ and this G will be isomorphic *as a group* to M^k.

If $\phi(M) = \overline{c} + G$ where G is a subgroup of M^n, then clearly the formula $\overline{v} = \overline{w} + \overline{c}$ is a bijection between G and $\phi(M)$, so it is sufficient to consider the parameter-free case. The theory of vector spaces, or modules over a division ring, has quantifier elimination and so every parameter free pp-formula is equivalent to a conjunction of linear equations $\bigwedge_{i=1}^{t} \left(\sum_{j=1}^{m} v_j r_{ij} = 0\right)$, where the free variables of the pp-formula are $v_1, \ldots, v_m$ and the r_{ij} are function symbols for multiplication by elements of the field K. The linear equations in the conjunction can be rearranged (by substitution) to an *equivalent* conjunction of linear equations in the original variables where none of the coefficients are zero. *Equivalent* here means modulo $\mathrm{Th}(M)$. Obviously we may need equations of the form $v_j = v_j$ in our conjunction. Then the solution set, which is equal to the original one, is clearly definably isomorphic to M^k where k is the sum, over all the linear equations in the conjunction, of the number of distinct variables in the equation minus one. Hence $\phi(M) \cong S_{X^k}$.

Now let the formula be the negation of a pp-formula, $\neg\phi(v_1, \ldots, v_n)$. Clearly $\neg\phi(M) \cup \phi(M) = M^n$ since ϕ is an n-ary formula, and this union is disjoint. Therefore $[\neg\phi(M)] + [M^k] = [M^n]$ and rearranging this yields $[\neg\phi(M)] = [M^n] - [M^k] = [S_{X^n - X^k}]$.

The conjunction of finitely many formulas all of the above forms, i.e. pp-formulas and their negations, may be simplified to an equivalent formula

$$(\phi \wedge \bigwedge_{i=1}^{k} \neg\psi_i)(v_1, \ldots, v_m)$$

with the $\phi, \psi_1, \ldots, \psi_k$ all positive primitive. We assume wlog that all our ψ_i are in fact $\phi \wedge \psi_i$. We will proceed by induction on $k \in \mathbb{N}$. The cases $k = 0, 1$ are covered above.

Inductive step. Take as our inductive hypothesis that any set defined by a formula $(\mu \wedge \bigwedge_{i=1}^{t} \neg\nu_i)(\overline{v})$ with $\mu(\overline{v}), \nu_i(\overline{v})$ pp-formulas and $t < k$, will be definably isomorphic to the representative set of some polynomial, $S_{p(X)}$. Then observe that $(\phi \wedge \bigwedge_{i=1}^{k} \neg\psi_i)(M) = (\phi \wedge \bigwedge_{i=1}^{k-1} \neg\psi_i)(M) \setminus \psi_k(M)$ and hence $(\phi \wedge \bigwedge_{i=1}^{k} \neg\psi_i)(M) \sqcup (\psi_k \wedge \bigwedge_{i=1}^{k-1} \neg\psi_i)(M) = (\phi \wedge \bigwedge_{i=1}^{k-1} \neg\psi_i)(M)$. Now our inductive hypothesis says that there are polynomials $g(X), h(X) \in \mathbb{Z}[X]$ such that $[(\psi_k \wedge \bigwedge_{i=1}^{k-1} \neg\psi_i)(M)] = [S_{g(X)}]$ and $[(\phi \wedge \bigwedge_{i=1}^{k-1} \neg\psi_i)(M)] = [S_{h(X)}]$. Hence $[(\phi \wedge \bigwedge_{i=1}^{k} \neg\psi_i)(M)] + [S_{g(X)}] = [S_{h(X)}]$ and $[(\phi \wedge \bigwedge_{i=1}^{k} \neg\psi_i)(M)] = [S_{h(X)}] - [S_{g(X)}] = [S_{h(X)-g(X)}]$. This concludes the induction on k in the formula $(\phi \wedge \bigwedge_{i=1}^{k} \neg\psi_i)(v_1, \ldots, v_m)$.

To complete the the proof of the lemma by induction on complexity of $\mathcal{L}_D$-formulas, we finally consider the most general case. Let our formula $\theta(v_1, \dots, v_l)$ be equivalent to a finite disjunction of the formulas of the type considered above, namely

$$\bigvee_{j=1}^{m} (\phi_j \wedge \bigwedge_{i=1}^{n_j} \neg\psi_{ji})(v_1 \dots, v_l)$$

By Lemma 3.2.1, we may assume wlog that for all $1 \leq j < k \leq m$ the sets $(\phi_j \wedge \bigwedge_{i=1}^{n_j} \neg\psi_{ji})(M)$ and $(\phi_k \wedge \bigwedge_{i=1}^{n_k} \neg\psi_{ki})(M)$ are disjoint. Therefore

$$\theta(M) = \bigsqcup_{j=1}^{m} (\phi_j \wedge \bigwedge_{i=1}^{n_j} \neg\psi_{ji})(M)$$

and thus by the construction of the Grothendieck ring, it follows that

$$[\theta(M)] = \sum_{j=1}^{m} [(\phi_j \wedge \bigwedge_{i=1}^{n_j} \neg\psi_{ji})(M)]$$

We have shown above that there must be polynomials $f_1(X), \dots, f_m(X) \in \mathbb{Z}[X]$ such that $[(\phi_j \wedge \bigwedge_{i=1}^{n_j} \neg\psi_{ji})(M)] = [S_{f_j}]$ for $j = 1, \dots, m$. By Lemma 4.5.1, this implies that $[\theta(M)] = \sum_{j=1}^{m} [S_{f_j}] = [S_{f_1 + \dots + f_m}]$. Hence a general definable set is in definable bijection with one of the representable sets indexed by the polynomials as desired.

$\square$

We are now in a position to prove Theorem 4.3.1

Proof of Theorem 4.3.1. Let M be a right D-module. In section 4.4 we defined a family of sets in $\mathrm{Def}(M)$, indexed by the polynomials over $\mathbb{Z}$ with positive leading coefficient. By Lemma 4.8.1, every set in $\mathrm{Def}(M)$ is definably isomorphic to one of these *representative sets*. Thus every element of $\widetilde{\mathrm{Def}}(M)$ is the class of a representative set. Lemmas 4.5.1 and 4.6.1 imply that the images $[S_f], [S_g]$ of these representative sets in $K_0(M)$ satisfy the same relationships in terms of the ring operations $+, \times$ that their corresponding polynomials $f(X), g(X)$ do in the ring $\mathbb{Z}[X]$.

By Lemma 4.7.4, we have that no two of these representative sets for distinct polynomials are in definable bijection. Hence they have distinct classes in $\widetilde{\mathrm{Def}}(M)$.

Moreover there is no extra identification in $K_0(M)$ under the equivalence relation $\sim$ on $\widetilde{\mathrm{Def}}(M)$.

Assume for a contradiction that for some $a \neq b \in \widetilde{\mathrm{Def}}(M)$, there exists $c \in \widetilde{\mathrm{Def}}(M)$ such that $a + c = b + c$. Then by Lemma 4.8.1, there exist $f(X), g(X), h(X) \in \mathbb{Z}[X]$ such that $a = [S_{f(X)}]$, $b = [S_{g(X)}]$, $c = [S_{h(X)}]$. By Lemma 4.5.1, $a + c = [S_{f(X)}] + [S_{h(X)}] = [S_{f(X)+h(X)}]$ and $b + c = [S_{g(X)}] + [S_{h(X)}] = [S_{g(X)+h(X)}]$. Then by Lemma 4.7.4, $a+c = b+c$ implies that $f(X)+h(X) = g(X)+h(X)$ and this can only hold in $\mathbb{Z}[X]$ if $f(X) = g(X)$. Thus $a = b \in \widetilde{\mathrm{Def}}(M)$ and we have the desired contradiction. Hence $K_0(M) \cong \mathbb{Z}[X]$.

$\square$

Remarks. Since every field is a division ring, Theorem 4.3.1 applies in particular to infinite vector spaces over any field K. The Grothendieck rings of every infinite module over a division ring are isomorphic, including those of the monster models considered in Chapter 6, where we define Grothendieck ring of a module category.

Chapter 5

Grothendieck rings of elementary extensions

5.1 Background

The material of this chapter is not specific to theories of modules, except where stated. Recall the following notation from Chapter 2.

- We write $\mathrm{Def}(M)$ for the collection of all definable subsets in M of any basic sort, i.e. any power of M. Alternatively $\mathrm{Def}(M)$ can be viewed as the collection of $\mathcal{L}(M)$-formulas up to the equivalence identifying ϕ_1 and ϕ_2 iff $M \models \forall \overline{v}\ (\phi_1(\overline{v}) \leftrightarrow \phi_2(\overline{v}))$.
- We write $\widetilde{\mathrm{Def}}(M)$ for the collection of equivalence classes in $\mathrm{Def}(M)$, where two formulas are equivalent if there is a definable bijection between their solution sets in M.
- We write $\widetilde{[\phi]}_M$ for the element of $\widetilde{\mathrm{Def}}(M)$ with representative element ϕ.
- We write $K_0(M)$ for the Grothendieck ring of M.
- We write $[\phi]_M$, or simply $[\phi]$, for the element of $K_0(M)$ with representative element ϕ.

In Section 7 of [24], J. Krajiček and T. Scanlon observe that for any $M \preceq N$ there is a natural embedding of $K_0(M)$ in $K_0(N)$. The result is included here as Proposition 5.2.4. It follows from Lemma 3.2 of [23], which says that for structures $M \preceq N$, whenever N admits a weak Euler characteristic over a ring R, M will also. Hence M must admit a weak Euler characteristic to the ring $K_0(N)$ and it must factor through $K_0(M)$. A detailed proof is given below. In the same section of [24], the authors note the stronger result that for any $M \equiv N$, the rings $K_0(M), K_0(N)$ satisfy the same $\exists_1$-sentences of $\mathcal{L}_{\text{rings}}$.

5.2 The induced embedding of Grothendieck rings

Let M, N be $\mathcal{L}$-structures for a first order language $\mathcal{L}$. Then $\mathcal{L}(M)$ denotes the language $\mathcal{L}$ with additional constant symbols for every element of M and likewise for $\mathcal{L}(N)$. Let N be an elementary extension of M via $E : M \preceq N$. We aim to define a map $H : K_0(M) \to K_0(N)$. To this end we will consider the elements of $K_0(M)$ as classes of $\mathcal{L}(M)$-formulas rather than classes of definable sets. By replacing each constant $c \in M$ by its image $E(c) \in N$, we can view each $\mathcal{L}(M)$-formula as an $\mathcal{L}(N)$-formula. In the following diagram, the vertical maps take a formula ϕ to its class $[\phi]$ in the Grothendieck ring. The map e is given by $e : \phi(\overline{v}, \overline{c}) \mapsto \phi(\overline{v}, E(\overline{c}))$ on formulas with free variables $\overline{v}$ and constant symbols $\overline{c} \in M$.

$$\begin{tikzcd} K_0(M) \arrow[r, dashed, "H"] & K_0(N) \\ \mathrm{Fml}(\mathcal{L}(M)) \arrow[u] \arrow[r, "e"] & \mathrm{Fml}(\mathcal{L}(N)) \arrow[u] \end{tikzcd}$$

Lemma 5.2.1. *The map e on formulas induces a well defined map $H : K_0(M) \to K_0(N)$.*

Proof. Let ϕ and ψ be $\mathcal{L}(M)$-formulas such that $[\phi]_M = [\psi]_M \in K_0(M)$. It suffices to prove that then $[\phi]_N = [\psi]_N \in K_0(N)$. Suppose that the equivalence in $\widetilde{\mathrm{Def}}(M)$ is witnessed by an $\mathcal{L}(M)$-formula ρ and that ρ defines a bijection between $(\phi \vee \chi)(M)$

and $(\psi \vee \chi')(M)$ for some $\phi(M) \cap \chi(M) = \emptyset = \psi(M) \cap \chi'(M)$, and that σ is an $\mathcal{L}(M)$-formula defining a bijection between $\chi(M)$ and $\chi'(M)$.

This property is *elementary*, meaning there is an $\mathcal{L}(M)$-sentence saying exactly this. Let $B(\phi, \psi, \rho, \sigma, \chi, \chi')$ be the $\mathcal{L}(M)$-sentence saying "σ defines a bijection between those tuples satisfying χ and those satisfying χ' and there are no common solutions of χ and ϕ, nor of χ' and ψ, and ρ defines a bijection between those tuples satisfying $\phi \vee \chi$ and those satisfying $\psi \vee \chi'$".

Then $M \models B(\phi, \psi, \rho, \sigma, \chi, \chi')$ and since this is an $\mathcal{L}(M)$-sentence, its image under e must hold in the elementary extension N. Thus we have $N \models e(B(\phi, \psi, \rho, \sigma, \chi, \chi'))$. So we have in $\widetilde{\mathrm{Def}}(N)$, the solution sets of the same formulas, with corresponding conditions of empty intersections and one being the graph of a bijection between two of the others. Hence we confirm that $[\phi]_N = [\psi]_N \in K_0(N)$ as desired. Therefore the map $H : [\phi]_M \mapsto [\phi]_N$ is well defined, and the lemma holds. $\square$

Lemma 5.2.2. *The map H defined above is one-to-one.*

Proof. Suppose that $[\phi]_M, [\psi]_M$ are elements of $K_0(M)$ and that their images under $H([\phi]_M) = H([\psi]_M)$ are identified, That is $[\phi]_N = [\psi]_N$. Then in the Grothendieck ring of N the formulas ϕ and ψ are identified so, by definition of $K_0(N)$, we must have an $\mathcal{L}(N)$-formula ρ defining the graph of a bijection from $\phi(N) \sqcup \theta(N)$ to $\psi(N) \sqcup \theta'(N)$ for some $\mathcal{L}(N)$-formulas θ, θ' with $\widetilde{[\theta]}_N = \widetilde{[\theta']}_N \in \widetilde{\mathrm{Def}}(N)$. Now ϕ and ψ are $\mathcal{L}(M)$-formulas but the formulas θ, θ', ρ may contain parameters from $N \setminus M$.

We will construct an $\mathcal{L}(M)$-sentence that holds in M and implies that $[\phi]_M = [\psi]_M$, thus establishing the lemma. We first take the $\mathcal{L}(N)$-sentence corresponding to the sentence B in the previous lemma. Let this sentence also be denoted B. Then we convert this $\mathcal{L}(N)$-sentence into an $\mathcal{L}(M)$-sentence by existentially quantifying out all of the parameters not present in M. The sentence B will only contain finitely many constant symbols from $N \setminus M$, since it contains only finitely many constant symbols in total, say $c_1, \ldots, c_n$ and we may write $B = B(c_1, \ldots, c_n)$. Then $N \models B(c_1, \ldots, c_n)$. Now we can "quantify out" these parameters to obtain an $\mathcal{L}(M)$-sentence that still holds in N:

$$N \models \exists v_1, \ldots, v_n\, B(v_1, \ldots, v_n)$$

Then $M \preceq N \Rightarrow M \models \exists v_1, \ldots, v_n\, B(v_1, \ldots, v_n)$ and we can choose some n-tuple $\overline{a}$ from M that witnesses the existential clause. Then substituting the constant symbols a_i for c_i in $B(c_1, \ldots, c_n)$, we obtain an $\mathcal{L}(M)$-sentence $B(a_1, \ldots, a_n)$ that holds in M. Note that in the above discussion, we combine all the properties we need for our bijection ρ into one sentence B. This is necessary because we must existentially quantify over all these sentences simultaneously (as they all occur within B) to ensure that the same constants are used in the $\mathcal{L}(M)$ sentence we produce. By this approach we find a well defined map H between the two Grothendieck rings.

This sentence now asserts that we have a bijection between the disjoint union of $\phi(M)$ and some set S and the disjoint union of $\psi(M)$ and another set S' with $\widetilde{[S]} = \widetilde{[S']}$ and hence $[S] = [S']$. This implies that $[\phi]_M = [\psi]_M \in K_0(M)$ as desired and we have that H is one-to-one. □

Lemma 5.2.3. *The map $H : K_0(M) \to K_0(N)$ is a ring homomorphism.*

Proof. The formula $(\neg v = v) \in \mathcal{L}$ has empty solution set in M. The image $e(\neg v = v)$ under the embedding $e : \mathcal{L}(M) \hookrightarrow \mathcal{L}(N)$ is $(\neg v = v)$ again and it has empty solution set in N. The map H is induced by the embedding e and hence $H(0) = H([\emptyset]) = 0$. Similarly, for a constant symbol $m \in M$, the formula $v = m$ defines a singleton set in M and $e(v = m)$ is $(v = E(m)) \in \mathcal{L}(N)$, which again defines a singleton set. Hence $H(1) = H([\{m\}]) = [\{E(m)\}] = 1$.

For multiplication in the Grothendieck rings, observe that $H([A][B]) = H([\alpha(M) \times \beta(M)])$. Let $\gamma(\overline{v}, \overline{w})$ be the formula $\alpha(\overline{v}) \wedge \beta(\overline{w})$ where there are no common variables in the two formulas. Then $H([A][B]) = H([\gamma(M)]) = [e(\gamma)(N)] = [e(\alpha)(N) \times e(\beta)(N)] = H([A])H([B])$ as required. For addition, take disjoint representative sets from $[A]$ and $[B]$ with $\mathcal{L}(M)$-formulas α and β of the same arity and set $\delta(\overline{v}) := \alpha(\overline{v}) \vee \beta(\overline{v})$. Then $H([A] + [B]) = H([\alpha \vee \beta(M)]) = [e(\alpha \vee \beta)(N)] = [(e(\alpha) \vee e(\beta))(N)] = [e(\alpha)(N)] + [e(\beta)(N)] = H([A]) + H([B])$.

□

The lemmas in this section, 5.2.1, 5.2.2 and 5.2.3, combined yield the following proposition, which was stated by Krajiček and Scanlon.

Proposition 5.2.4. *([24], Section 7)*

$$M \preceq N \Rightarrow K_0(M) \leq K_0(N)$$

If N is an elementary extension of M then there is an embedding of rings $K_0(M) \xrightarrow{H} K_0(N)$.

5.3 Elementary submodules

If we restrict our attention to R-modules $M \preceq N$, then we find that H as defined above is also a surjection, and hence an isomorphism of the Grothendieck rings. This stronger result is a consequence of the fact that any set in Def(N) definable by an arbitrary $\mathcal{L}(N)$-formula is in definable bijection with a set in Def(N) definable by an $\mathcal{L}(M)$-formula, i.e. a formula without any parameters from $N \setminus M$.

Theorem 5.3.1. *Let $M, N \in$ Mod-R. If $M \preceq N$, the elementary embedding induces an isomorphism of their Grothendieck rings $H : K_0(M) \cong K_0(N)$.*

Proof. Recall, if $\phi(\overline{v}, \overline{c})$ is a pp-formula in $\mathcal{L}(N)$ with constants $\overline{c}$ and variables $\overline{v}$, then the set $\phi(N, \overline{c})$ is either empty or a coset of $\phi(N, \overline{0})$. Let $\theta(\overline{v})$ be an arbitrary $\mathcal{L}(N)$-formula. By Baur's elimination result, $\theta(\overline{v})$ is equivalent (modulo the theory $T = Th(N)$) to a boolean combination of pp-formulas. By Lemma 3.2.1, we may assume without loss of generality that the sets $\left(\alpha_i \wedge \bigwedge_{j=1}^{t_i} \neg\beta_{ij}\right)(N, \overline{c})$ are disjoint. Let $\theta'(\overline{v})$ be the formula $\left(\bigvee_{i=1}^{s}\left(\alpha_i \wedge \bigwedge_{j=1}^{t_i} \neg\beta_{ij}\right)\right)(\overline{v}, \overline{c})$. We may list the pp-formulas α_i, β_{ij} that occur in this boolean combination and define the set

$$\Phi := \{\phi_1(\overline{v}, \overline{c}_1), \ldots, \phi_k(\overline{v}, \overline{c}_k)\}$$

of all the pp-formulas occurring in θ', making no distinction in this notation between the formulas that occur negated and those that occur unnegated. This set Φ is a finite set. Note that here the parameters from different formulas need not all be distinct. Some of the parameters occurring in the constituent pp-formulas $\phi_i \in \Phi$ may not be contained in M but all the constituent pp-sets are cosets of 0-definable pp-subgroups.

The critical information for this proof is which are the non-empty intersections of the pp-sets $\phi_i(N, \overline{c}_i)$. If $\phi_i(N, \overline{c}_i) \cap \phi_j(N, \overline{c}_j)$ is non-empty, then the intersection will be a coset of $(\phi_i \wedge \phi_j)(N, \overline{0})$. Hence there will be a definable bijection between the set $\phi_i(N, \overline{c}_i) \cap \phi_j(N, \overline{c}_j)$ and the set obtained by a change of parameters $c_h \mapsto d_h$ for each $h = 1, \ldots, k$, namely $\phi_i(N, \overline{d}_i) \cap \phi_j(N, \overline{d}_j)$ provided both sets are nonempty. Similarly if we take any subset $I \subseteq \{1, \ldots, k\}$ and the formulas $\phi_i(\overline{v}, \overline{c}_i) \in \Phi$ for $i \in I$, then the intersection $\bigcap_{i \in I} \phi_i(N, \overline{c}_i)$ will be a coset of the pp-set $\bigcap_{i \in I} \phi_i(N, \overline{0}) = \left(\bigwedge_{i \in I} \phi_i\right)(N, \overline{0})$ or else empty.

Form an $\mathcal{L}(N)$-sentence by taking a conjunction, over all pairs $1 \leq i < j \leq k$, of either the condition $\exists \overline{v}(\phi_i(\overline{v}, \overline{c}_i) \wedge \phi_j(\overline{v}, \overline{c}_j))$ or $\neg \exists \overline{v}(\phi_i(\overline{v}, \overline{c}_i) \wedge \phi_j(\overline{v}, \overline{c}_j))$. We take for each pair (i, j) in our conjunction, whichever condition is satisfied in N. We call the resulting sentence $\sigma(\overline{c})$, with $\overline{c}$ the tuple combining $\overline{c}_1, \ldots, \overline{c}_k$, all of the parameters occurring in the formulas of Φ.

Observe that $N \models \sigma(\overline{c})$ implies $N \models \exists \overline{w}\, \sigma(\overline{w})$. Now this latter is a sentence of $\mathcal{L}(M)$ and therefore $M \preceq N \Rightarrow M \models \exists \overline{w}\, \sigma(\overline{w})$. If we take witnesses of the existential quantifier in this sentence in M and denote their constant symbols $\overline{m} \in M \subset N$ then we have $M \models \sigma(\overline{m})$ and hence $N \models \sigma(\overline{m})$. The condition $N \models \sigma(\overline{m})$ implies that for each $i = 1, \ldots, k$ the set $\phi_1(N, \overline{c}_i)$ is definably isomorphic to the set $\phi_1(N, \overline{m}_i)$. In fact the isomorphism is a coset translation. Moreover $N \models \sigma(\overline{m})$ implies that for every $I \subseteq \{1, \ldots, k\}$ and every choice of signs $f : I \to \{+, -\}$ (or more formally $f : I \to \{\neg\neg, \neg\}$), there is an isomorphism

$$\bigcap_{i \in I} f(i)\phi_i(N, \overline{c}_i) \cong \bigcap_{i \in I} f(i)\phi_i(N, \overline{m}_i)$$

given simply by exchanging the parameters $\overline{c}$ for $\overline{m}$. This is implied directly by $N \models \sigma(\overline{c}) \wedge \sigma(\overline{m})$.

Claim. Therefore the set $\theta(N) = \theta'(N, \overline{c})$ is in definable bijection with $\theta'(N, \overline{m})$. To see this we write $\theta'(N, \overline{c}) = \bigsqcup_{i=1}^{s} (\alpha_i \wedge \bigwedge_{j=1}^{t_i} \neg \beta_{ij})(N, \overline{c})$. The condition $N \models \sigma(\overline{m}) \wedge$

$\sigma(\overline{c})$ implies that the sets $(\alpha_i \wedge \bigwedge_{j=1}^{t_i} \neg\beta_{ij})(N, \overline{m})$ must also be disjoint for $i = 1, \ldots, s$. Therefore it is sufficient to prove for a fixed $1 \leq i \leq s$, that $(\alpha_i \wedge \bigwedge_{j=1}^{t_i} \neg\beta_{ij})(N, \overline{m})$ is in definable bijection with $(\alpha_i \wedge \bigwedge_{j=1}^{t_i} \neg\beta_{ij})(N, \overline{c})$. We will prove the result for all pp-formulas from Φ simultaneously and we proceed by induction on t_i.

Base case, $t_i = 0$. If $t_i = 0$ then the claim is simply that $\alpha_i(N, \overline{c}) \cong \alpha_i(N, \overline{m})$ and this follows immediately from $N \models \sigma(\overline{m}) \wedge \sigma(\overline{c})$. Observe that for any pp-formula that is an element, or conjunction of elements, of Φ, the definable bijection resulting from the change of parameters $\overline{c} \mapsto \overline{m}$ is simply a translation of cosets, i.e. addition of some constant tuple.

Inductive step. Assume the claim holds for $t_i = T$. Then

$$(\alpha_i \wedge \bigwedge_{j=1}^{T} \neg\beta_{ij})(N, \overline{m}) \cong (\alpha_i \wedge \bigwedge_{j=1}^{T} \neg\beta_{ij})(N, \overline{c})$$

and also

$$(\beta_{i(T+1)} \wedge \bigwedge_{j=1}^{T} \neg\beta_{ij})(N, \overline{m}) \cong (\beta_{i(T+1)} \wedge \bigwedge_{j=1}^{T} \neg\beta_{ij})(N, \overline{c})$$

by the inductive hypothesis. Recall that these bijections are coset translations. Since $\beta_{i(T+1)} \to \alpha_i$, the bijection will be via addition of the same tuple in both cases. Hence

$$(\alpha_i \wedge \bigwedge_{j=1}^{T+1} \neg\beta_{ij})(N, \overline{m}) = \left((\alpha_i \wedge \bigwedge_{j=1}^{T} \neg\beta_{ij})(N, \overline{m})\right) \setminus \left((\beta_{i(T+1)} \wedge \bigwedge_{j=1}^{T} \neg\beta_{ij})(N, \overline{m})\right)$$

is in definable bijection with

$$(\alpha_i \wedge \bigwedge_{j=1}^{T+1} \neg\beta_{ij})(N, \overline{c}) = \left((\alpha_i \wedge \bigwedge_{j=1}^{T} \neg\beta_{ij})(N, \overline{c})\right) \setminus \left((\beta_{i(T+1)} \wedge \bigwedge_{j=1}^{T} \neg\beta_{ij})(N, \overline{c})\right)$$

This concludes the proof of the claim.

Therefore $[\theta(N)] = [\theta'(N, \overline{c})] = [\theta'(N, \overline{m})] \in K_0(N)$. Noting that $\theta'(\overline{v}, \overline{m})$ is a formula of $\mathcal{L}(M)$ and recalling the definition of the embedding H from the previous section, we have $H([\theta'(M, \overline{m})]) = [\theta'(N, \overline{m})]$. Therefore for an arbitrary set $\theta(N) \in \mathrm{Def}(N)$, we have $[\theta(N)]$ in the image of H. Thus the entirety of $\chi_0(\mathrm{Def}(N))$ will be in the image of H and therefore the induced map $H : K_0(M) \to K_0(N)$ is a surjection. We saw in the previous section on arbitrary elementary extensions, that H is an injection and respects the ring operations for any elementary extension of first order structures. Hence for modules $M, N \in$ Mod-R, if $M \preceq N$ then $K_0(M) \cong K_0(N)$.

□

Corollary 5.3.2. *If M and N are elementarily equivalent R-modules, $M \equiv N$, then there will be an isomorphism of their Grothendieck rings $H : K_0(M) \cong K_0(N)$.*

Proof. Since $M \equiv N$, we can find a large saturated module that is a common elementary extension of the two $M, N \preceq M^*$. This is well known (see for example [26] 4.3.15, 4.3.17). Therefore the proposition yields $K_0(M) \cong K_0(M^*) \cong K_0(N)$. □

Remark. Theorem 5.3.1 and Corollary 5.3.2 do not extend to arbitrary theories of first order structures, as seen in the sequel.

5.4 An example with a stable theory

Consider the first order language $\mathcal{L} = \mathcal{L}\langle E \rangle$ with just one binary relation symbol E (in addition to equality) and no function symbols or constant symbols. Let the theory T consist of axioms saying that E is an equivalence relation and an axiom for each $n \geq 1$ saying that there is an element with exactly n distinct elements in its equivalence class and this is the unique equivalence class of size n.

One model of T is the structure M_0, having exactly one equivalence class of each finite size and no infinite equivalence classes. Another model of T is the structure $M = M_0 \cup C$, having exactly one equivalence class of each finite size and one countably infinite E-class denoted C. This second structure is an elementary extension of the first $M_0 \preceq M$. These structures are considered by J. Krajíček and T. Scanlon as example 7.4 in an exploratory section of examples and open questions in [24]. Their primary motivation for introducing the pair $M_0 \preceq M$ is as an example to demonstrate that whilst the Grothendieck rings of any two elementarily equivalent structures must satisfy the same $\exists_1$-sentences in $\mathcal{L}_{\text{rings}}$, they need not satisfy the same $\forall\exists$-sentences in $\mathcal{L}_{\text{rings}}$.

In addition, they state that $Th(M)$ in the language $\mathcal{L}(M)$ has quantifier elimination and that M_0 is a locally finite structure. However they also claim without proof that $K_0(M_0)$ is isomorphic to $\mathbb{Z}[X]$ where $X = [M_0] = \chi_0(M_0)$, and that $K_0(M) \cong \mathbb{Z}[X, Y]$ where $X = [M_0]$ and $Y = [C]$. However, we demonstrate in Proposition 5.6.15 that neither of these values for the Grothendieck rings is correct. The correct values are calculated in Proposition 5.6.16. It is shown in Section 5.6 that this pair of structures also provide an example that Theorem 5.3.1, which says that an elementary embedding of modules $E : P \preceq Q$ induces an isomorphism of Grothendieck rings $K_0(P) \cong K_0(Q)$, does not generalise to elementary embeddings in arbitrary theories and languages.

5.5 Quantifier elimination for $Th(M_0, M_0)$

The language $\mathcal{L}$ has no function symbols or constant symbols and the only relation symbols are equality and the binary relations symbol $E(-,-)$. Therefore the only atomic formulas of $\mathcal{L}$ are $v = w$ and $E(v, w)$ for each pair of variables in the language. We may extend $\mathcal{L}$ by adding constants for every element of the structure M to obtain $\mathcal{L}(M) := \mathcal{L}\langle E; c_m\rangle_{m\in M}$. The language $\mathcal{L}(M_0)$ is defined analogously to be $\mathcal{L}\langle E; c_m\rangle_{m\in M_0}$.

Definition 5.5.1. *An* atomic type *in a first order language is a maximal, consistent set of atomic formulas and negated atomic formulas.*

Definition 5.5.2. *The* atomic type of an element or tuple $\overline{a}$ *in a model A is the set of atomic formulas and their negations satisfied by $\overline{a}$ in A.*

It follows that the atomic type of an n-tuple $\overline{a} = (a_1, \ldots, a_n)$ in M, is the set of all formulas $\theta(v_1, \ldots, v_n)$ in the set

$$\{\pm v_i = v_j,\ \pm v_i = c_m,\ \pm E(v_i, v_j),\ \pm E(v_i, c_m) : 1 \le i, j \le n\}$$

such that $M \models \theta(a_1, \ldots, a_n)$, where c_m is the constant symbol for $m \in M$ and $\pm\alpha$ denotes the fact that both α and $\neg\alpha$ are in the set. The atomic type of an n-tuple $\overline{b} = (b_1, \ldots, b_n)$ in M_0, is defined analogously with constants from M_0.

In model theory, an *automorphism* of a structure is an injective and surjective map f from the (domain of the) structure to itself such that f preserves constants, f commutes with functions and all relations are invariant under f. Since the language $\mathcal{L}$ in this chapter has no function symbols, an automorphism of M is an injective and surjective map $f : M \to M$ such that for every a_1, a_2, $M \models E(a_1, a_2)$ iff $M \models E(f(a_1), f(a_2))$ and also for any constant symbols c_m, $f(c_m) = c_m$. An automorphism of M_0 is defined in exactly the same manner.

Theorem 5.5.3. *([20], 7.4.1 or [31], 16.1) Let A be a structure over a first order language. If there is an $|A|^+$-saturated elementary extension A' of A, such that whenever $\overline{x}, \overline{y} \in A'$ have the same atomic type, there is an automorphism $f : A' \to A'$ with $f(\overline{x}) = \overline{y}$, then the theory of A will admit complete elimination of quantifiers.*

Since $M_0 \preceq M$, both of the structures model $Th(M_0, M_0)$, the theory of M_0 in the language $\mathcal{L}(M_0) := \mathcal{L}\langle E; c_m \rangle_{m \in M_0}$.

Proposition 5.5.4. *The theory $Th(M_0, M_0)$ of the structure M_0 in the language $\mathcal{L}(M_0)$ has elimination of quantifiers, i.e. every $\mathcal{L}(M_0)$-formula is equivalent modulo this theory to a quantifier free formula in $\mathcal{L}(M_0)$.*

Proof. Let M' be an $\mathcal{L}(M_0)$-structure with one E-class of each finite cardinality and all of its infinite E-classes of equal cardinality. Then M' satisfies the theory $Th(M_0, M_0)$ and M' is an elementary extension of M_0 (and also of M). The cardinality of M_0 is $\aleph_0$ and the structure M' is $|M_0|$-saturated. Therefore it is sufficient to prove that M' satisfies the hypothesis of Theorem 5.5.3 and it will follow that $Th(M_0, M_0)$ has elimination of quantifiers.

Suppose $(x_1, \dots, x_n), (y_1, \dots, y_n) \in (M')^n$ have the same atomic type. Then for every $1 \leq i, j \leq n$, $M' \models x_i = x_j$ iff $M' \models y_i = y_j$ and $M' \models E(x_i, x_j)$ iff $M' \models E(y_i, y_j)$, and for $m \in M_0$ with constant symbol c_m, $M' \models x_i = c_m$ iff $M' \models y_i = c_m$. Every element of M_0 has a constant symbol in $\mathcal{L}(M_0)$. The theory of M_0 includes the axioms that there is a unique E-class of each finite size, and the language includes constant symbols for every element of each of the finite classes.

If an entry x_i of $(x_1, \ldots, x_n)$ is in a finite E-class then it is equal to some $m \in M_0$ and the atomic type of $(x_1, \ldots, x_n)$ contains the formula $v_i = c_m$. By assumption $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ have the same atomic type, hence $M' \models y_i = c_m$ and thus $y_i = x_i$ for any entry in a finite class. If an entry x_i is in an infinite E-class then for every $m \in M_0$, the formulas $\neg v_i = c_m$ with constant symbol c_m are in the atomic type of $(x_1, \ldots, x_n)$. Hence y_i cannot be in a finite E-class. Having determined which entries x_i and y_i are in infinite E-classes, the only information the atomic type will contain on these elements is the conditions of which entries are equal $M' \models x_i = x_j$ exactly when $M' \models y_i = y_j$, and which entries are in the same infinite equivalence class $M' \models E(x_i, x_j)$ exactly when $M' \models E(y_i, y_j)$.

Let the map $f : M' \to M'$ be given by:

- f acts as the identity on $M_0 \subset M'$,
- for any E-class containing none of the elements $x_1, \ldots, x_n, y_1, \ldots, y_n$, f acts as the identity on the class,
- for $1 \leq i \leq n$, f restricted to the class of x_i is a bijection from the class of x_i to the class of y_i, such that $f(x_j) = y_j$ for all x_j with $M' \models E(x_i, x_j)$,
- for $1 \leq i \leq n$, f restricted to the class of y_i is a bijection from the class of y_i to the class of x_i, such that $f(y_j) = x_j$ for all y_j with $M' \models E(y_i, y_j)$.

This map $f : M' \to M'$ exchanges the E-classes of x_i and y_i, in particular exchanging any entries of the tuples therein, and every other class is preserved. Note that the identity map on $M_0 \subset M'$ has the desired effect on all the entries of our tuple that are in finite E-classes. This f is an automorphism of M' and $f(\overline{x}) = \overline{y}$. It is an automorphism because it preserves all constants, i.e. elements of M_0, and all relations $M' \models E(a, b)$ if and only if $M' \models E(f(a), f(b))$.

Therefore the hypothesis of the theorem is satisfied, and the theory $Th(M_0, M_0)$ has elimination of quantifiers. □

Corollary 5.5.5. *The theory $Th(M_0, M_0)$ is stable.*

Proof. We have elimination of quantifiers in this theory and $\mathcal{L}(M_0)$ has no function symbols, countably many constant symbols and only the relation symbol E. Let A be a countable set of parameters. We will show that $S_1(A)$, the set of complete 1-types over A, is countable and therefore $Th(M_0, M_0)$ is ω-stable and hence stable.

There is one 1-type saying that an element is not in a finite E-class and not in the same E-class as any element a of A. The infinite E-classes containing no element of A cannot be distinguished, the elements of such equivalence classes have a common type. There are the complete types of each element a of the set A including the formula $v = a$. For each $n \geq 1$ there is a complete type saying that an element is in the unique E-class of size n and not equal to any $a \in A$ (unless every element of the size n class is in A). There are at most countably many 1-types for elements that are in an infinite E-class that contains one or more elements of A. The number of these types depends on how many distinct infinite E-classes contain elements of A but this is obviously bounded by $|A| = \aleph_0$. These are all the complete 1-types over A.

Therefore there are only countably many complete 1-types over A. □

5.6 The models M and M_0

The theory T in the language $\mathcal{L} = \mathcal{L}\langle E \rangle$ was defined in Section 5.4, as were the models M_0 and M. The theory $Th(M_0, M_0)$ is a complete, stable theory, and $Th(M, M_0) = Th(M_0, M_0)$. The theory $Th(M, M)$ contains an infinite family of sentences that together imply the existence of an infinite E-class. These sentences necessarily use parameters from the infinite E-class, C. Observe that it is impossible to say that there is an infinite E-class with any sentence or collection of sentences in the language $\mathcal{L}(M_0)$. Even with parameters from M, it is not possible with first order sentences to express the fact that M has a unique infinite class. Hence M has elementary extensions over the language $\mathcal{L}(M)$ with more than one infinite equivalence class under the relation $E(-,-)$.

Krajíček and Scanlon claim (example 7.4 of [24]) that the Grothendieck ring $K_0(M)$ is isomorphic to $\mathbb{Z}[Y, X]$ where the indeterminates are the images under χ_0 of

M itself and the infinite E-class $C \subseteq M$. However we prove in Proposition 5.6.15 that there is an algebraically independent set in $K_0(M)$ of cardinality $\aleph_0$. The elements of this set are the images under χ_0 of each of the sets $E_n \subset M^n$, for each $n \in \mathbb{N}$, defined by

$$E_n := \{(v_1, \ldots, v_n) \in M^n : M \models E(v_1, v_2) \wedge \ldots \wedge E(v_1, v_n)\}$$

Remark. It should be explicitly noted that $M = E_1$.

Lemma 5.6.1. *The theory of M in the language $\mathcal{L}(M)$ has elimination of quantifiers.*

Proof. The proof of Proposition 5.5.4 translates to $Th(M, M)$ because again we have that in a saturated model, given any tuples $\overline{x}$ and $\overline{y}$ with the same quantifier free type, there is an automorphism taking $\overline{x}$ to $\overline{y}$. □

We will prove that in the model M the sets C, E_n, E_m with $n \neq m$ are all distinct up to definable bijection i.e. they have distinct classes in $\widetilde{\mathrm{Def}}(M)$. Similarly in the model M_0, the sets E_n, E_m with $n \neq m$ all have distinct classes in $\widetilde{\mathrm{Def}}(M_0)$. We prove that every formula of $\mathcal{L}(M)$ (respectively $\mathcal{L}(M_0)$) defines a set that is in definable bijection with some set formed from a finite sequence of taking disjoint unions, products and set complements of the sets M, C and E_n (without C when we are working over M_0) or their isomorphic copies. We refer to these as 'polynomial' sets as they correspond, heuristically speaking, to polynomials in these generating sets (with $+, \times, -$ being disjoint union, product and set complement respectively).

Then we show that for both M and its elementary substructure M_0, the equivalence relation $\sim$ on $\widetilde{\mathrm{Def}}(M_0)$ is no coarser than the identity. Hence $\widetilde{\mathrm{Def}}(M_0)/{\sim}$ is equivalent to $\widetilde{\mathrm{Def}}(M_0)$ for the structure M_0, and likewise for M. The claims of the above argument are expounded in this section.

The definable bijections. By the quantifier elimination, proved in Section 5.5, a formula $\rho(\overline{v}, \overline{w}, \overline{m})$ can be assumed to be in disjunctive normal form, i.e. there is a logically equivalent formula $\bigvee_{i=1}^{n}(\bigwedge_{j=1}^{m_i} \pm A_{ij})(\overline{v}, \overline{w}, \overline{m})$, where the A_{ij} are atomic formulas. The only atomic formulas in $\mathcal{L}(M)$ and $\mathcal{L}(M_0)$ are $\tau_1 = \tau_2$ or $E(\tau_1, \tau_2)$, and the only terms that can take the place of τ_1, τ_2 are single variables and (unary) constants, since the language $\mathcal{L}$ has no function symbols.

The definable bijections in $Th(M, M)$ and $Th(M_0, M_0)$ are all essentially trivial in the following sense. The inherent restraints of the language force every definable bijection to act piecewise on a partition of its domain as either the identity or a permutation of the arguments on each part with possible exchanging of 'labelling' parameters that have no effect on the 'shape' of the set, on each part. Therefore the definable bijections over M or M_0 are all "of trivial character". The precise meaning of what it means for a bijection to be of *trivial character* is given in the sequel.

Since the following argument is valid for both models M and M_0 of T, we write M_T to mean either model, and the two cases are treated in parallel. Where the two cases require separate treatment, it is made explicit. Let B be a bijection of sets in $\mathrm{Def}(M_T)$ and suppose the graph of B is the solution set of the $\mathcal{L}(M_T)$-formula ρ with parameters $\overline{m}$ from M_T. Since B is a function, for any $\overline{x}$ in the domain of B, there exists a unique tuple $\overline{y}$ such that $M_T \models \rho(\overline{x}, \overline{y}, \overline{m})$ or equivalently

$$M \models \bigvee_{i=1}^{n} (\bigwedge_{j=1}^{m_i} \pm A_{ij})(\overline{x}, \overline{y}, \overline{m})$$

Thus if we write $\rho(\overline{x}, \overline{w}, \overline{m})$ with $\overline{x}$ and $\overline{m}$ tuples of parameters, with $\overline{x}$ in the domain of B, and $\overline{w}$ a tuple of variables, the formula has unique solution $\overline{y}$ in M.

An exhaustive list of the atomic formulas and their negations in $\mathcal{L}(M_T)$ is;

$$E(v, u), E(v, m), \neg E(v, u), \neg E(v, m), v = u, v = m, \neg v = u \text{ and } \neg v = m$$

with constant symbols $m \in M_T$. By considering the solution sets of these formulas, we see that the only values that the w_k may take are the entries of $\overline{x} = B^{-1}\overline{y}$, the entries of $\overline{m}$ or elements of some finite E-class where every other element of the class is among the entries of $\overline{x}$ and the entries of the parameter tuple $\overline{m}$.

As we allow $\overline{x}$ to range over the domain of B, keeping $\overline{m}$ fixed as it is the tuple of parameters, every tuple gives a different solution to the formula

$$\bigvee_{i=1}^{n} (\bigwedge_{j=1}^{m_i} \pm A_{ij})(\overline{x}, \overline{y}, \overline{m})$$

as the function B is one-to-one.

Therefore for each term in the disjunction with $1 \leq i \leq n$, the formula $\rho_i := (\bigwedge_{j=1}^{m_i} \pm A_{ij})(\overline{v}, \overline{w}, \overline{m})$ will imply that each w_k is equal to some variable v_t or a constant. Only finitely many constants are possible; the entries of the parameter tuple $\overline{m}$ or some constant in a finite E-class containing an entry of $\overline{m}$.

The formula ρ is a disjunction of finitely many formulas $\rho_1, \ldots, \rho_n$, each of which is the graph of a function. These functions given by the ρ_i are essentially trivial; differing from the identity function only by permuting the arguments of elements in their domain by some uniform permutation, or exchanging finitely many constants. The bijection B acts on its domain, by locally acting as one of the functions with graph $\rho_i(M_T)$, say B_i. If these B_i have intersecting domains then they must agree on them, and if they have intersecting images then their preimages must agree since B is a bijection.

Hence, if ρ defines a bijection B of infinite sets, then the domain of B partitions into a finite set and finitely many infinite sets such that on each of the infinite sets, ρ acts uniformly as either the identity or a permutation of the arguments $v_1, \ldots, v_n$, up to possible exchanging of constants. This is what we refer to as a bijection of *trivial character*.

Proposition 5.6.2. *In the model M, the set C is not in definable bijection with M or E_n for any $n > 1$.*

Proof. The set C is *minimal* in the model theoretic sense; the only definable subsets of C are finite or have finite complement in C. Observe that for any $a, b \in C$ the types $tp(a/(M \setminus \{a, b\}))$ and $tp(b/(M \setminus \{a, b\}))$ are equal. Note that a and b are non-algebraic since they are contained in the infinite equivalence class C. Therefore only equalities involving parameters will distinguish two elements of C. A formula can only contain finitely many parameters. Hence every unary formula is either satisfied by only finitely many elements of C or by all but finitely many. This proves that C is minimal.

Assume for a contradiction that $\rho(v, w, \overline{m})$ defines a bijection from M to C with parameters $\overline{m}$. Then the formulas $\exists v\, \rho(v, w, \overline{m}) \wedge v \in C$ and $\exists v\, \rho(v, w, \overline{m}) \wedge v \notin C$ define disjoint infinite sets in C, contradicting the minimality. Similarly E_n cannot be definably isomorphic to C since it has disjoint definable infinite subsets C^n and $E_n \setminus C^n$. $\square$

We refer to the sets E_n as *fundamental sets* in M_0 and the sets E_n and C as *fundamental sets* in M.

For each $n \in \mathbb{N}$, let $S_n \subset M_0$ be the unique E-class of size n. Then $M_0 = \bigcup_n S_n$ and $E_m = \bigcup_n {S_n}^m$. Both of these unions are disjoint. In the model M, we have $M = C \cup \bigcup_n S_n$ and $E_m = C^m \cup \bigcup_n {S_n}^m$. Again these unions are disjoint.

Lemma 5.6.3. *Every formula θ of $\mathcal{L}(M_T)$ defines a set that can be constructed from the fundamental sets in M_T, or isomorphic copies thereof, together with finite sets of constants, via some finite sequence of taking products, disjoint unions and set complements (of subsets).*

Proof. By the quantifier elimination, proved in Section 5.5, the formula θ can be expressed in disjunctive normal form (DNF), i.e. there is some logically equivalent formula in DNF. Hence we may assume without loss of generalisation that $\theta(\overline{v})$ is of the form $\bigvee_{i=1}^n (\bigwedge_{j=1}^{m_i} \pm A_{ij})(\overline{v}, \overline{c})$, where the A_{ij} are atomic formulas.

Now $\theta(M_T) = \left(\bigvee_{i=1}^n (\bigwedge_{j=1}^{m_i} \pm A_{ij})\right)(M_T, \overline{c}) = \bigcup_{i=1}^n (\bigwedge_{j=1}^{m_i} \pm A_{ij})(M_T, \overline{c})$. For arbitrary sets P, Q, we have $P \cup Q = P \sqcup (Q \setminus (P \cap Q))$. Hence by induction on n, it is sufficient to prove the hypothesis for conjunctions of atomic formulas and their negations.

Given one of the disjuncts in the formula, $\bigwedge_{j=1}^{m_i} \pm A_{ij}(v_1, \ldots, v_t, \overline{c})$, if we may reorder $j = 1, \ldots, m_i$ such that all the variables occurring in the A_{ij} for $1 \leq j \leq p$ do not occur at all in the A_{ij} for $p + 1 \leq j \leq m_i$, then

$$\bigwedge_{j=1}^{m_i} \pm A_{ij}(M_T, \overline{c}) = \left(\bigwedge_{j=1}^{p} \pm A_{ij}(M_T, \overline{c})\right) \times \left(\bigwedge_{j=p+1}^{m_i} \pm A_{ij}(M_T, \overline{c})\right)$$

Therefore it is sufficient to prove the lemma for conjunctions of interdependent atomic formulas, as now every definable set is formed from taking unions, products and complements of solution sets of such formulas.

Suppose $(\bigwedge_{j=1}^{p} \pm A_{ij}(v_1, \ldots, v_t, \overline{c}))$ is a conjunction of interdependent atomic formulas. We may assume wlog that for some $1 \leq q \leq p$, the formula is

$$(\bigwedge_{j=1}^{q} A_{ij}(v_1, \ldots, v_t, \overline{c})) \wedge (\bigwedge_{j=q+1}^{p} \neg A_{ij}(v_1, \ldots, v_t, \overline{c}))$$

We consider first the solution set of the formula

$$\theta_+(v_1, \ldots, v_t, \overline{c}) := (\bigwedge_{j=1}^{q} A_{ij}(v_1, \ldots, v_t, \overline{c}))$$

and show that this set is of the desired form. Then we show that the property of a formula having a solution set that can be constructed from (isomorphic copies of) the sets E_n (and also C when working in M) and finite sets is preserved when we take a conjunction of said formula with a negated atomic formula. We need to include isomorphic copies of fundamental sets in the hypothesis of the lemma because for example the formula $v_1 = v_2$ has solution set the diagonal in M_T^2, which is isomorphic to M_T, and the formula $E(v_1, v_2) \wedge v_1 = v_3$ has solution set isomorphic to E_2.

Now $\theta_+(M_T, \overline{c}) = (\bigwedge_{j=1}^{q} A_{ij})(M_T, \overline{c}) \subseteq M_T^t$ is clearly a product of fundamental sets. By abuse of notation, isomorphic copies of fundamental sets are identified with the fundamental sets for brevity. There are natural numbers $m, n, k_1, \ldots, k_r, t_1, \ldots, t_r$ and a finite set $F \in \mathrm{Def}(M_T)$ such that $\theta_+(M_T, \overline{c}) = M_T{}^m \times E_{k_1}{}^{t_1} \times \ldots \times E_{k_r}{}^{t_r} \times C^n \times F$. Obviously the term C^n is omitted from the product when working in M_0.

Recall from naïve set theory that for arbitrary sets P, Q, R we have $P \cup Q = P \sqcup (Q \setminus (P \cap Q))$, $(P \cup Q) \setminus R = (P \setminus R) \cup (Q \setminus R)$ and $(P \setminus Q) \setminus R = (P \setminus (R \cap P)) \setminus ((P \cap Q) \setminus (P \cap Q \cap R))$. Note that all of the set complements here are the complements of a subset in a superset. Also observe that for any $m, n \geq 1$ the following inclusions hold; $S_m{}^n \subset E_n \subset M_T{}^n$, $E_{m+n} \subset E_m \times E_n$ and in the model M we also have $C^n \subset E_n$.

We start with the set defined by $\theta_+(v_1, \ldots, v_t, \overline{c}) = (\bigwedge_{j=1}^{q} A_{ij})(v_1, \ldots, v_t, \overline{c})$ and take the conjunction with $(v_i = v_i)$ for any variable that does not occur in θ_+. Then it is clear that we may take the conjunctions with the negated atomic formulas one by one and by using the set theory identities and the inclusions of fundamental sets above, we may construct the set $\theta(M_T, \overline{c})$ by a finite sequence of taking disjoint unions, products and complements of the fundamental sets and finite sets of parameters. $\square$

Definition 5.6.4. *Let N be a structure for a first order language and let $A \subset N^n$. Then the* fibre projections *of A are the sets of the form*

$$\{(x_1, \ldots, x_{t-1}, x_{t+1}, \ldots, x_n) \in N^{n-1} : (x_1, \ldots, x_{t-1}, c, x_{t+1}, \ldots, x_n) \in A\}$$

for each $1 \leq t \leq n$ and for constants $c \in N$ such that the set is non-empty.

Lemma 5.6.5. *For natural numbers $n > m \geq 1$, the sets E_n and E_m are not in definable bijection over M_T.*

Proof. Assume for contradiction that $\rho(\overline{v}, \overline{w})$ defines a bijection from E_n to E_m. Then the formula $\exists w_1, \ldots, w_m \, \rho(v_1, \ldots, v_n, w_1, \ldots, w_m)$ is equivalent modulo $Th(M_T, M_T)$ to the formula $E(v_1, v_2) \wedge \ldots \wedge E(v_1, v_n)$, i.e. it is another formula defining the set E_n. Similarly $\exists v_1, \ldots, v_n \, \rho(v_1, \ldots, v_n, w_1, \ldots, w_m)$ is equivalent to the formula $E(w_1, w_2) \wedge \ldots \wedge E(w_1, w_m)$.

If we replace the variable v_1 with the constant symbol c, for an element of S_k the unique E-class of size $k \geq 1$, then the number of tuples satisfying

$$M_T \models \exists w_1, \ldots, w_m \rho(c, v_2, \ldots, v_n, w_1, \ldots, w_m)$$

for the free variables $v_2, \ldots, v_n$ is k^{n-1}. The same is true for constants taken from each finite E-class in place of c. In the model M there will be infinitely many tuples satisfying the formula when $c \in C$.

If we replace the variable w_1 with the constant symbol c, for an element of the unique E-class of size k, then the number of tuples satisfying

$$M_T \models \exists v_1, \ldots, v_n \rho(v_1, \ldots, v_n, c, w_2, \ldots, w_m)$$

for the free variables $w_2, \ldots, w_m$ is k^{m-1}. Again, this is true for constants taken from each finite E-class in place of c. The solution set is infinite in M when $c \in C$.

Now ρ is a bijection defined by an $\mathcal{L}(M_T)$-formula and hence has the piecewise trivial character. There must be a partition of its domain E_n (or a cofinite subset), such that ρ acts on each part uniformly by simply permuting the arguments, or other trivially isomorphic actions such as taking products with a fixed constant symbol or mapping a set to its image in the diagonal of some power of M_T. Such bijections of trivial character cannot alter the size of the fibre projections. Therefore it cannot map the set E_n where the finite fibre projections take all sizes k^{n-1}, for $k \in \mathbb{N}$, injectively to the set E_m where the finite fibre projections take all sizes k^{m-1}. This is the desired contradiction. □

Therefore no two of the sets $E_1, E_2, E_3, \ldots$ are in definable bijection, and so their equivalence classes in $\widetilde{\mathrm{Def}}(M_T)$ are distinct. In $\widetilde{\mathrm{Def}}(M)$ we also have that the class of C is distinct from those of the other fundamental sets, by Proposition 5.6.2.

By Lemma 5.6.3 we have that every set in $\mathrm{Def}(M_T)$ is in definable bijection with a set formed from a finite union of finite products of the fundamental sets and their isomorphic copies, and the complement of such sets in one another. Hence $\chi_0(\mathrm{Def}(M_T))$ will be generated by the images of these sets.

This is suggestive of the polynomial ring, over $\mathbb{Z}$, with indeterminates the bijection classes of the sets E_n (and C for the model M) in $\widetilde{\mathrm{Def}}(M_T)$. Certainly we have that every set in $\mathrm{Def}(M_T)$ will have the same representative in $K_0(M_T)$ as the 'polynomial construction in the fundamental sets' that it is in definable bijection with. Let $X_n := [E_n]$ denote the definable isomorphism class of the set E_n and let $Y := [C]$ denote the definable isomorphism class of the set C. Then, every definable isomorphism class in $\widetilde{\mathrm{Def}}(M_T)$ is the class of some polynomial in the X_n (and Y) over the ring of integers. We will call a set built from finitely many arithmetical operations (where $+, \times, -$ correspond to disjoint union, product and set complement of a subset respectively) of the fundamental sets, a *polynomial set* for the remainder of this chapter.

Definition 5.6.6. *The group of automorphisms $\alpha : M_T \to M_T$ is denoted Aut_0. The subscript* 0 *is to emphasise the fact that no parameters need be fixed by the automorphism.*

Definition 5.6.7. *The Aut_0-orbit of an element or tuple x is the set $\{\alpha(x) : \alpha \in Aut_0\}$. We denote this orbit $\mathcal{O}_0(x)$.*

Definition 5.6.8. *The group of automorphisms of M_T fixing a parameter set P pointwise is denoted Aut_P.*

Definition 5.6.9. *The Aut_P-orbit of an element or tuple x is the set $\{\alpha(x) : \alpha \in Aut_P\}$. We denote this orbit $\mathcal{O}_P(x)$.*

Lemma 5.6.10. *Let F_E be the map from the set $\{E_n \subset M_T^n : n \geq 1\}$ to the power series ring $\mathbb{Z}[[t]]$ defined by setting*

$$F_E(E_n) = \sum_{i=1}^{\infty} t^{i^n}$$

Then the coefficient of t^j in $F_E(E_n)$ is equal to the number of Aut_0-orbits of size j in E_n.

Proof. $E_n = \bigcup_i S_i{}^n$. Each $S_i{}^n$ is an orbit of size i^n. The coefficients of t_j in $F_E(E_n) = \sum_{i=1}^{\infty} t^{i^n}$ are equal to 0 if j is not an n^{th} power of an integer and 1 if it is. Thus the lemma holds over both M and M_0. □

Definition 5.6.11. *There is an equivalence relation on the power series ring $\mathbb{Z}[[t]]$ given by $f \sim g$ if $f = \sum_{i \geq 1} a_i t^i$ and $g = \sum_{i \geq 1} b_i t^i$ and there exists i_0 such that $a_i = b_i$ for every $i > i_0$. We say that f and g are* asymptotically equal *whenever $f \sim g$.*

Lemma 5.6.12. *The map F_E extends to a map F from the ring of polynomial sets in the E_n to $\mathbb{Z}[[t]]$ via the operations such that:*

$$F(E_n \cup E_m) := F_E(E_n) + F_E(E_m)$$

$$F(E_n \times E_m) := F_E(E_n) \times F_E(E_m)$$

$$F(E_n \times E_m \setminus E_{n+m}) := F_E(E_n) \times F_E(E_m) - F_E(E_{n+m})$$

In the asymptotic behaviour, i.e. for all i greater than some i_0, which depends on the particular polynomial p, we have that the coefficient of t^i in $F(p(E_1, E_2, \ldots))$ is again the number of orbits of size i in the set $p(E_1, E_2, \ldots)$.

The statement about the coefficients is only true for the asymptotic behaviour due to the parameters required to define some of the sets in $p(E_1, E_2, \ldots)$, and the effect of these parameters on the smaller powers of t in the image under F.

Proof. We demonstrate that the coefficient of t^j in $F(E_n \cup E_m)$, $F(E_n \times E_m)$ and $F(E_n \times E_m \setminus E_{n+m})$ is the number of orbits of size j in the respective sets.

$$E_n \cup E_m = \bigcup_i S_i{}^n \cup \bigcup_i S_i{}^m$$

for $E_n, E_m \in \mathrm{Def}(M_0)$, whereas for $E_n, E_m \in \mathrm{Def}(M)$ the union is:

$$E_n \cup E_m = \bigcup_i S_i{}^n \cup C^n \cup \bigcup_i S_i{}^m \cup C^m$$

The number of orbits of size j is equal to; 2 if $j = k^m = l^n$ for some k and l, 1 if $j = k^m$ but j not an n^{th} power, it is also 1 if $j = l^n$ is not an m^{th} power, and 0 if j is neither an n^{th} power or an m^{th} power. Clearly $F_E(E_n) + F_E(E_m) = \sum_{i=1}^{\infty} t^{i^n} + \sum_{k=1}^{\infty} t^{k^m}$ will have the desired coefficient.

$$E_n \times E_m = \bigcup_i S_i{}^n \times \bigcup_i S_i{}^m$$

in $\mathrm{Def}(M_0)$ and in $\mathrm{Def}(M)$ the product is:

$$E_n \times E_m = (C^n \cup \bigcup_i S_i{}^n) \times (C^m \cup \bigcup_i S_i{}^m)$$

The orbit of an element $(x, y) \in E_n \times E_m$, writing x, y for tuples, is the product of their orbits since every $\alpha \in Aut_0$ will permute the elements of E_n and permute the elements of E_m. Hence the orbits in $E_n \times E_m$ in both models are all of size $k^m l^n$ for some integers k, l or are infinite. The number of orbits of size j will equal the number of ways of writing $j = k^m l^n$. The product $F_E(E_n) \times F_E(E_m) = \sum_{i=1}^{\infty} t^{i^n} \times \sum_{k=1}^{\infty} t^{k^m}$ will have coefficient of t^j equal to the sum, over all pairs (i, k) such that $i^n k^m = j$, of the products of coefficients of t^{i^n} in $F_E(E_n)$ and t^{k^m} in $F_E(E_m)$, namely 1×1. This is clearly equal to the number of ways of writing $j = k^m l^n$ as desired.

Finally for subtraction, in $\mathrm{Def}(M_0)$ we have

$$E_n \times E_m \setminus E_{n+m} = (\bigcup_i S_i{}^n \times \bigcup_i S_i{}^m) \setminus \bigcup_i S_i{}^{n+m}$$

and in $\mathrm{Def}(M)$ we have

$$E_n \times E_m \setminus E_{n+m} = ((C^n \cup \bigcup_i S_i{}^n) \times (C^m \cup \bigcup_i S_i{}^m)) \setminus (C^{n+m} \cup \bigcup_i S_i{}^{n+m}$$

All of the orbits over M_0 are finite. As the function F is defined in terms of the finite orbits, we need only consider them for both models. The finite orbits in $E_n \times E_m$ are products of a finite orbit in E_n and one in E_m, that is some ${S_i}^n \times {S_k}^m$. Observe that all the finite orbits of E_{n+m} are actually among these, they are the sets ${S_i}^n \times {S_i}^m$. So in taking the complement of E_{n+m} in $E_m \times E_n$, we actually are removing some whole orbits and leaving the others intact.

Fixing $i = i_0$, look at the orbits of (x, y) in the set $(\bigcup_i {S_i}^n \times \bigcup_k {S_k}^m)$ with $x \in {S_{i_0}}^n$. They are ${S_{i_0}}^n \times {S_k}^m$ and their union is $U_{i_0} := {S_{i_0}}^n \times \bigcup_k {S_k}^m$. Among them the orbit ${S_{i_0}}^n \times {S_{i_0}}^m$ is an orbit of E_{n+m} and the remainder of the union U_{i_0} does not intersect E_{n+m}.

Hence the number of orbits of size j in $E_n \times E_m \setminus E_{n+m}$ is equal to the number in $E_n \times E_m$ minus the number in E_{n+m}. Hence it is equal to the coefficient of t^j in $F_E(E_n) \times F_E(E_m) - F_E(E_{n+m})$.

□

One can see that this map F behaves like an embedding of rings, in the weak sense that it respects the arithmetic operations up to asymptotic equality.

We may extend F to arbitrary definable sets in the following manner. A definable set A is in definable bijection with some polynomial set in the E_n, say $p(E_1, E_2, \ldots)$. Set $F(A)$ to be $F(p(E_1, E_2, \ldots))$. The value of $F(A)$ is independent of the choice, if there is one, of the set $p(E_1, E_2, \ldots)$ because these sets are definably isomorphic and their images under F will be asymptotically equal.

Let A be a definable set. Then there is some integer i_0 such that for $i > i_0$, the coefficient of t^i in the power series $F(A)$ is the number of finite orbits contained in A of size i. A negative coefficient refers to removal of such sets from supersets in the construction of A. Note that for any E_n we may take $i_0 = 0$, as the coefficient of t^{k^n} is one for the orbit ${S_k}^n$ and zero for powers of t that are not equal to some k^n.

Proposition 5.6.13. *There is no identification under the equivalence relation $\sim$ on $\widetilde{Def}(M_T)$ between the X_n.*

Proof. It is sufficient to demonstrate that for $m > n$, X_n and X_m cannot become identified under the relation: $\widetilde{[A]} \sim \widetilde{[B]}$ if there exists $z \in \widetilde{\mathrm{Def}}(M_T)$ such that $\widetilde{[A]}+z = \widetilde{[B]} + z$.

Assume for a contradiction that such a z exists. Then z is the class of some definable set, say $z = \widetilde{[Z]}$. Then $X_m + z = X_n + z$ implies that there exists a definable bijection, f say, between $E_n \sqcup Z$ and $E_m \sqcup Z$.

Let f be given by the formula $\theta(v_1, \ldots, v_l, c_1, \ldots, c_q)$ with parameters $c_1, \ldots, c_q \in M_T$. Note that the set Z is definable using only these parameters, since Z is the domain of f minus the set E_n, and E_n is definable over the empty set.

For $x \in M_T$, the type of x over the parameter set $P := \{c_1, \ldots, c_q\}$ is denoted $tp(x/\overline{c})$ or $tp(x/P)$. Since the bijection f is given by the formula θ with parameters $\overline{c}$, we have that for every $x, y \in M_T$

$$tp(x/\overline{c}) = tp(y/\overline{c}) \Rightarrow tp(f(x)/\overline{c}) = tp(f(y)/\overline{c})$$

The inverse of f is also definable with the same parameters, so the reverse implication holds.

Let the formula $\neg E(v, c_1) \wedge \ldots \wedge \neg E(v, c_q)$, saying that a variable v is not in the same E-class as any of the parameters $c_1, \ldots, c_q$, be denoted $\psi(v, \overline{c})$. For $x \in M_T$, if $M \models \psi(v, \overline{c})$, then clearly the type of x over $\overline{c}$ is determined by its type over the empty set together with the formula $\psi(v, \overline{c})$.

For any $x \in M_T$, $tp(x/\emptyset)$ is determined by its E-class. This is obvious but for details, refer to the proof of quantifier elimination in $\mathcal{L}$, shown in Section 5.5. Thus for any $x, y \in M_T$ we have $tp(x/\emptyset) = tp(y/\emptyset)$ if and only if $M_T \models E(x, y)$. And if $M_T \models E(x, y) \wedge \psi(x, \overline{c})$, i.e. if x and y are in the same E-class and this E-class does not contain any parameter from P, then $tp(x/\overline{c}) = tp(y/\overline{c})$.

Observe that $M_T \models \psi(x, \overline{c})$ holds for all x in M_T except for those in some finite collection of the E-classes S_n (and possibly C for the model M), namely those containing some $c_j \in P$. Thus there is an infinite subset $N \subset \mathbb{N}$ (actually cofinite in $\mathbb{N}$) such that $M_T \models \psi(x, \overline{c})$ for every x in $\bigcup_{n \in N} S_n$.

The automorphisms in Aut_P will permute the elements of each set S_k for $k \in N$. Thus for each x in $\bigcup_{k \in N} S_k$ the orbit $\mathcal{O}_P(x)$ is the class S_k containing x. Similarly for a tuple x in ${S_k}^n$, $\mathcal{O}_P(x) = {S_k}^n$.

We consider the subsets of the domain of f, $E_n \cup Z$, consisting of elements with a common type over P. These are Aut_P-orbits. By the above, we have for every pair of tuples x, y in the domain

$$tp(x/P) = tp(y/P) \Leftrightarrow tp(f(x)/P) = tp(f(y)/P)$$

Therefore, for every $k \in N$, the set ${S_k}^n$ is mapped by f to an Aut_P-orbit of size k^n in $E_m \cup Z$. So for every $k \in N$, the cofinite subset of $\mathbb{N}$, there must be an orbit of size k^n in the image of f, which is $E_m \cup Z$. But observe that for any $h \in N$ and $y \in {S_h}^m \subset E_m$, the orbit $\mathcal{O}_P(y)$ is ${S_h}^m$. There are infinitely many values k for which k^n is not an element of $\{a^m : a \in \mathbb{N}\}$ since $m \neq n$.

For every $k \in N$, let x_k be an element of ${S_k}^n$. Then $\mathcal{O}_P(x_k) = {S_k}^n$ and $\mathcal{O}_P(f(x_k))$ must be of the same size, i.e. an orbit of size k^n. Therefore there are infinitely many $k \in N$ such that $\mathcal{O}_P(f(x_k))$ is not among the ${S_h}^m$ with $h \in N$ which are Aut_P-orbits in E_m. The set $\mathbb{N} \setminus N$ is finite. Hence the union of Aut_P-orbits given by

$$\bigcup_{h' \in \mathbb{N} \setminus N} {S_{h'}}^m$$

must be finite also. Observe that in $\mathrm{Def}(M_0)$

$$E_m = \bigcup_{h \geq 1} {S_h}^m = \bigcup_{h \in N} {S_h}^m \cup \bigcup_{h' \notin N} {S_{h'}}^m$$

and in $\mathrm{Def}(M)$

$$E_m = C^m \cup \bigcup_{h \geq 1} {S_h}^m = C^m \cup \bigcup_{h \in N} {S_h}^m \cup \bigcup_{h' \notin N} {S_{h'}}^m$$

and hence there are infinitely many $k \in N$ such that $\mathcal{O}_P(f(x_k))$ is an orbit of size k^n in Z'. Now Z is a definable set and as such is in definable bijection with some 'polynomial' in the fundamental sets of M_T. Recalling Lemma 5.6.10, we have $F(Z) \in \mathbb{Z}[[t]]$.

Let

$$F(Z) = \sum_{i \geq 1} z_i t^i$$

and recall that

$$F(E_n) = \sum_{i \geq 1} t^{i^n}$$

$$F(E_m) = \sum_{i \geq 1} t^{i^m}$$

By Lemma 5.6.10,

$$F(E_n \cup Z) = \sum_{i \geq 1} (z_i + \delta_i) t^i, \text{ where } \delta_i = 1 \text{ if } i \text{ is an } n^{th} \text{ power and 0 otherwise}$$

$$F(E_m \cup Z) = \sum_{i \geq 1} (z_i + \delta'_i) t^i \text{ where } \delta'_i = 1 \text{ if } i \text{ is an } m^{th} \text{ power and 0 otherwise}$$

But recall that $E_m \cup Z = f(E_n \cup Z)$. Now the definable bijection f can only have a different number of orbits of size h in its image and domain, for finitely many values of $h \in \mathbb{N}$. Thus $F(E_m \cup Z) = F(f(E_n \cup Z))$ and $F(E_n \cup Z)$ can only differ in coefficients of the powers t^i for finitely many values of i. There exists an integer i_0 such that the coefficients agree for all $i > i_0$, they agree asymptotically in the growth of power of t. That is $(z_i + \delta_i) = (z_i + \delta'_i)$ for all $i > i_0$. But this implies that the large n^{th} powers are all large m^{th} powers and vice versa. Hence we have a contradiction. No such set as Z and bijection f can exist.

Therefore the representatives of the sets E_n are distinct in $\widetilde{\mathrm{Def}}(M_T)/\sim$ as desired.

$\square$

Lemma 5.6.14. *For every $n \geq 1$ the sets E_n and C have distinct representatives in $\widetilde{Def}(M)/\sim$.*

Proof. Suppose for a contradiction that there exists a definable bijection, f say, between $E_n \sqcup Z$ and $C \sqcup Z$ for some $[\widetilde{Z}] = z$. Let f be given by the formula $\theta(v_1, \ldots, v_l, c_1, \ldots, c_q)$ with parameters $c_1, \ldots, c_q \in M$. Let P be the parameter set $\{c_1, \ldots, c_q\}$. Note that the set Z is definable using only these parameters, since Z is the domain of f minus the set E_n, and E_n is definable over the empty set.

We may count the finite Aut_P orbits of each size in the domain and image of f. The set C is one orbit by itself if $P \cap C = \emptyset$, and if any of the parameters c_j of θ are elements of C then those elements are singleton orbits and the remainder of C is one infinite orbit. Since $C \cup Z = f(E_n \cup Z)$ and f is a definable bijection, the sets $E_n \cup Z$

and $C \cup Z$ can only have a different number of orbits of size h for finitely many values of $h \in \mathbb{N}$. Thus $F(E_n \cup Z), F(C \cup Z) \in \mathbb{Z}[[t]]$ can only differ in coefficients of the powers t^i for finitely many values of i.

The equalities $F(E_n \cup Z) = F(E_n) + F(Z)$, $F(C \cup Z) = F(C) + F(Z)$ are both true asymptotically, the coefficients agree in each 'equality' for sufficiently large powers of t. But this implies that $F(E_n)$ and $F(C)$ are asymptotically equal in $\mathbb{Z}[[t]]$, which is not true. Hence we have the desired contradiction. $\square$

Claim. There are no definable bijections between distinct monomials in the fundamental sets. *Proof of claim.* Suppose a formula $\rho(\overline{v}, \overline{w}, \overline{c}) \in \mathcal{L}(M_T)$ defines a bijection from $E_{n_1} \times \ldots \times E_{n_s} \times C^k$ to $E_{m_1} \times \ldots \times E_{m_t} \times C^l$ allowing multiplicities greater than one (and with the terms in C absent for $M_T = M_0$). Then the proof of Lemma 5.6.5 generalises to imply that the monomials must be equal. The definable bijection $\rho \in \mathcal{L}(M_T)$ must have the piecewise trivial character and hence the sizes of the fibre projections in the domain and image of the bijection must agree, for each constant we choose to fix in each argument of the formula $\rho(\overline{v}, \overline{w}, \overline{c})$. The domain and the image are the two monomials in the fundamental sets and this condition on the fibre projections is only satisfied when the monomials are equal. This establishes the claim.

Proposition 5.6.15. *The set $\{X_n : n \geq 1\}$ is an algebraically independent set in $K_0(M_0)$ and the set $\{Y, X_n : n \geq 1\}$ is an algebraically independent set in $K_0(M)$.*

Proof. We may treat the two cases $M_T = M_0, M$ simultaneously. Assume for a contradiction that there is a polynomial relation in $K_0(M_T)$ between some elements of the set, namely $p(X_1, \ldots, X_m, Y) = 0$ for some polynomial p over $\mathbb{Z}$. Obviously since $Y \notin K_0(M_0)$, we assume that Y does not occur anywhere in p for the case $M_T = M_0$.

From Lemma 5.6.3 we know that every definable set in $\mathrm{Def}(M_T)$ is in bijection with some polynomial set in the fundamental sets E_n and C. From the construction of a general model-theoretic Grothendieck ring, we have that these polynomial sets will have representatives in $K_0(M_T)$ the corresponding polynomials in the X_n and

Y. Therefore the equation $p(X_1, \ldots, X_m, Y) = 0$ in $K_0(M_T)$ implies that there is a definable bijection $b(\overline{v}, \overline{w}, \overline{c})$ between two distinct polynomials sets in the fundamental sets, with difference (meaning the complement of one in the other) isomorphic to $p(E_1, \ldots, E_m, C)$.

Observe that not every polynomial in the X_n and Y will be the image under χ_0 of some definable set because subtraction is only defined on polynomial sets where a complement of a subset may be taken. Note that for any $n, m \geq 1$ we have $E_{n+m} \subset E_n \times E_m$ and $C^n \subset E_n$, and hence $X_n X_m - X_{n+m}$ and $X_n - Y^n$ are in the image of χ_0. But in general the subtraction in $K_0(M_T)$ is defined only when the semiring is completed to a ring.

By taking unions with monomials in the fundamental sets corresponding to the terms of negative coefficient in the domain and image of the bijection b, we may find distinct polynomials sets $p_1(E_1, \ldots, E_m, C)$ and $p_2(E_1, \ldots, E_m, C)$ such that all the coefficients are nonnegative and $p_1(X_1, \ldots, X_m, Y) = p_2(X_1, \ldots, X_m, Y)$ in $K_0(M_T)$. Therefore there exists some polynomial $q(X_1, \ldots, X_n, Y)$ with all its coefficients nonnegative, and a definable bijection from $p_1(E_1, \ldots, E_m, C) \sqcup q(E_1, \ldots, E_n, C)$ to $p_2(E_1, \ldots, E_m, C) \sqcup q(E_1, \ldots, E_n, C)$. Without loss of generality we may assume $n \geq m$. Let $b'(\overline{v}, \overline{w}, \overline{c}')$ denote the formula of the definable bijection between the sets $(p_1 + q)(E_1, \ldots, E_n, C)$ and $(p_2 + q)(E_1, \ldots, E_n, C)$.

Let P be the set of parameters in the formula $b'(\overline{v}, \overline{w}, \overline{c}')$. Then, following the proof of Proposition 5.6.13 the bijection b' will send elements of the domain that are in the same Aut_P orbit to elements of the image that are in the same Aut_P orbit. Thus the finite Aut_P orbits need to match up in equal quantities of each size in the domain and image of b'.

The domain and image of b' are polynomial sets with nonnegative coefficients, hence they are unions of monomials in the fundamental sets, i.e. sets of the form $\{a\} \times E_{n_1} \times \ldots \times E_{n_s} \times C^k$ where the first parameter a serves simply to keep sets disjoint if necessary and where there may be repeats among the E_{n_i}. There are only finitely many parameters in P and hence only finitely many Aut_P orbits in the domain and the image of b' that contain any elements of P. The other Aut_P orbits are also

Aut_0 orbits. Recall that:

$$E_{n_1} \times \ldots \times E_{n_s} \;=\; (\bigcup_{i\geq 1} S_i^{\,n_1} \cup C^{n_1}) \times \ldots \times (\bigcup_{i\geq 1} S_i^{\,n_s} \cup C^{n_s})$$

(without the terms in C in the case of M_0) and thus the finite Aut_0 orbits in this set will be of sizes ${k_1}^{n_1}{k_2}^{n_2}\ldots{k_s}^{n_s}$ for every $k_1,\ldots,k_s \in \mathbb{N}$. The only way to have Aut_0 orbits of all the required finite sizes, in the correct quantities, in the image of b' which itself is a union of monomials in the fundamental sets, is if this same monomial $E_{n_1} \times \ldots \times E_{n_s}$ is present in the union. This applies to every monomial set in the domain and the argument is symmetrical. Hence the polynomial sets $(p_1+q)(E_1,\ldots,E_n,C)$ and $(p_2+q)(E_1,\ldots,E_n,C)$ are equal and thus $p_1 = p_2$. This is the required contradiction.

The only case not covered by the above argument is if every monomial set in the union of monomials $(p_1+q)(E_1,\ldots,E_n,C)$ has as a factor a nonzero power of C, because then the Aut_0 orbits are all infinite. In this case however Aut_P orbits are still sent to Aut_P orbits by the bijection b' and since P is finite, all but finitely many of these orbits are exactly products of E-classes. By considering the fibre projections in a product of E-classes, it is clear that the only products of E-classes that can be in definable bijection with each other are identical products (up to permuting the factors). The only way to have all the same products of E-classes present in the same number of copies in two positive polynomial sets, i.e. unions of monomials like p_1+q and p_2+q, is if they are the same polynomial sets.

Therefore we have the desired contradiction and the images under χ_0 of the fundamental sets are an algebraically independent set in $K_0(M_T)$. □

Proposition 5.6.16. *The Grothendieck rings for these two models are*

$$K_0(M_0) = \mathbb{Z}[X_1, X_2, \ldots] \;\textit{and}\; K_0(M) = \mathbb{Z}[Y, X_1, X_2, \ldots]$$

Proof. By Lemma 5.6.3 every definable set in M and M_0 is definably isomorphic to a polynomial set in the fundamental sets. Hence the images under χ_0 of the fundamental sets form a generating set over the integers for the Grothendieck rings. Proposition 5.6.13 yields that the elements $X_n = [E_n]$ in the Grothendieck ring of

M_0 or M are all distinct, and Lemma 5.6.14 yields that $Y = [C]$ is also distinct in $K_0(M)$. Finally Proposition 5.6.15 says that they are algebraically independent. □

Remarks. This proves that the statement in [24] that $K_0(M_0)$ and $K_0(M)$ are isomorphic to $\mathbb{Z}[X]$ and $\mathbb{Z}[X, Y]$ respectively is incorrect.

The map H induced by the embedding of $\mathcal{L}(M_0)$ into $\mathcal{L}(M)$, as defined in the opening section of this chapter and shown in the diagram below, is not a surjection because it is not onto $Y = [C] \in K_0(M)$ for example. Hence H is not an isomorphism of rings.

$$\begin{tikzcd}
K_0(M_0) \arrow[r, "H"] & K_0(M) \\
\widetilde{\mathrm{Def}}(M_0) \arrow[u] & \widetilde{\mathrm{Def}}(M) \arrow[u] \\
\mathrm{Fml}(\mathcal{L}(M_0)) \arrow[u] \arrow[r, "e"] & \mathrm{Fml}(\mathcal{L}(M)) \arrow[u]
\end{tikzcd}$$

Thus this pair of structures provide an example demonstrating that the result of Theorem 5.3.1, namely that an elementary embedding of modules $e : M \preceq N$ induces an isomorphism between the Grothendieck rings their theories $H : K_0(M) \cong K_0(N)$, does not generalise to elementary extensions in arbitrary theories.

Chapter 6

The Grothendieck ring of a module category

In this chapter, we define the Grothendieck ring for the category of right modules over a ring R, which we denote $K_0(\text{Mod-}R)$. We do this by taking a *largest theory* of R-modules, the theory of a monster model P.

The Grothendieck rings of elementarily equivalent modules are isomorphic by Corollary 5.3.2. Therefore for modules M, $K_0(M)$ is determined by $Th(M)$; this does not hold for arbitrary first order structures. Any complete theory of modules is determined by its invariants conditions. Thus two modules are elementarily equivalent iff they satisfy the same invariants conditions. Therefore $K_0(M)$ is completely determined by the values of $Inv(M, \phi, \psi)$ for each pp-pair ϕ/ψ.

6.1 Defining K_0(Mod-R)

Given a ring R, the theory $Th(\text{Mod-}R)$ is not a complete theory, but we may take a canonical complete theory extending it as follows. Let P be a direct sum of one model of each complete theory of right R-modules. Then $T^* := Th(P)$ is sometimes referred to as the *largest complete theory of right R-modules*.

Every right R-module is elementarily equivalent to a direct summand of some model of $Th(P)$. And given any model $P' \models Th(P)$, every right R-module is elementarily equivalent to a direct summand of an ultrapower of P'. Let $M \in$ Mod-R. Then there exists some model $P'' \models Th(P)$ and some $M'' \equiv M$ such that $M''|P''$. Since P' and P'' are both models of $Th(P)$, they are elementarily equivalent and (by a theorem of Shelah's [38]) there exists ultrapowers of P' and P'' that are isomorphic to each other. The module M'' is a direct summand of P'' and hence of its ultrapower which was isomorphic to an ultrapower of P'. The theory $Th(P)$ is the starting point for constructing a Grothendieck ring for the category of right R-modules.

Lemma 6.1.1. *For P, a large saturated R-module as above, the invariant $Inv(P, \phi, \psi)$ is equal to 1 if the pp-formulas ϕ and ψ are equivalent modulo the theory of R-modules, and infinite otherwise.*

Proof. By a *proper* pp-pair ϕ/ψ in $\mathcal{L}_R$, we mean one where Mod-$R \models \forall \overline{v}\,(\psi(\overline{v}) \to \phi(\overline{v}))$ but Mod-$R \not\models \forall \overline{v}\,(\psi(\overline{v}) \leftrightarrow \phi(\overline{v}))$.

If Mod-$R \models \forall \overline{v}\,(\phi(\overline{v}) \leftrightarrow \psi(\overline{v}))$, then $Inv(M, \phi, \psi) = 1$ for every R-module M. P is an R-module and hence $Inv(P, \phi, \psi) = 1$. If Mod-$R \models \forall \overline{v}\,(\phi(\overline{v}) \to \psi(\overline{v}))$ and Mod-$R \not\models \forall \overline{v}\,(\psi(\overline{v}) \to \phi(\overline{v}))$, then $Inv(M, \phi, \psi) > 1$ for some module M. This M is elementarily equivalent to a summand of P and so is every finite power of M. Hence if $Inv(M, \phi, \psi) \geq 2$ then Lemma 3.4.5 implies that $Inv(P, \phi, \psi) \geq 2^n$ for every n. Therefore $Inv(P, \phi, \psi)$ is infinite. $\square$

Proposition 6.1.2. *The map G : Rings $\to$ Rings assigning to any ring R, a 'largest complete theory of R-modules' by taking the theory of a model P as described above and then taking the Grothendieck ring of P, is well defined on the category Rings.*

Proof. By Corollary 5.3.2, any pair of elementarily equivalent R-modules will have isomorphic Grothendieck rings. Let both P_1 and P_2 be the direct sum of one model of each complete theory of R-modules. Let $T_1^* = Th(P_1)$ and $T_2^* = Th(P_2)$.

The theory of any R-module M is completely determined by the invariants conditions, i.e. which statements of the form $Inv(M, \phi, \psi) > n$ hold, for integers n and pp-pairs $\psi \to \phi$. Thus if T_1^* and T_2^* satisfy the same invariants conditions, then they are equivalent theories and $P_1 \equiv P_2$.

For every proper pp-pair $\psi \to \phi$ there is some summand of P_1, and likewise of P_2, on which the pp-pair is open, i.e. the inclusion of pp-subgroups is proper. Suppose N is an R-module with $\psi(N) < \phi(N)$ a strict inclusion. Then $Inv(N, \phi, \psi) > 1$. The modules $N \oplus N$, all finite direct sums of copies of N and also $N^{(\aleph_0)}$ are all elementarily equivalent to direct summands of P_1. Let N' and N'' be such that $P_1 = N' \oplus N''$ and $N' \equiv N^{(\aleph_0)}$. Then by Lemma 3.4.5 we have that $\phi(P_1) = \phi(N') \oplus \phi(N'')$ and similarly for ψ. Clearly the same applies to P_2. Hence for every $\psi \to \phi$ with Mod-$R \not\models \psi \leftrightarrow \phi$ the invariant $Inv(P_1, \phi, \psi)$ is infinite by Lemma 6.1.1, as is $Inv(P_2, \phi, \psi)$. This implies that $P_1 \equiv P_2$ as required.

Therefore the map G is well defined on the category of rings, and the Grothendieck ring is independent of our choice of P. $\square$

Definition 6.1.3. *The* Grothendieck ring of the category of R-modules $K_0(\textit{Mod-}R)$ *is defined to be the Grothendieck ring of some (equivalently any) direct sum of one copy of a model of each complete theory of R-modules.*

Proposition 6.1.4. *Let D be a division ring. Then $K_0(\textit{Mod-}D)$ is isomorphic to $\mathbb{Z}[X]$.*

Proof. The Grothendieck ring $K_0(\text{Mod-}D)$ is defined to be $K_0(P)$ where P is the direct sum of one copy of a model of each complete theory of D-modules. Such a P is itself a D-module under the natural diagonal action of D. Therefore $K_0(\text{Mod-}D) = K_0(P)$ is isomorphic to $\mathbb{Z}[X]$ by Theorem 4.3.1. $\square$

Remark. Note that the theory of infinite modules over a division ring D is a complete theory. The theories of D-modules are entirely determined by the cardinalities of their models. Thus $Th(P)$ is equal to $Th(D_D)$ if D is infinite and $Th(D^{(\aleph_0)})$ if D is a finite division ring (and hence necessarily a finite field).

6.2 The rings $K_0(\text{Mod-}R)$ and $K_0(M_R)$

Generators for the Grothendieck rings. For any $M \in \text{Mod-}R$, including a monster R-module $P \models T^*$, the images under the Euler characteristic χ_0 of pp-sets in Def(M) generate the Grothendieck ring of the theory of M in $\mathcal{L}_R(M)$.

From the pp-elimination in the language of right R-modules, due to Baur, the formulas of $\mathcal{L}_R(M)$ are equivalent to boolean combinations of pp-formulas and invariants conditions. In any complete theory, the value of each invariant for every pp-pair is a consequence of the theory, and so the formulas are equivalent, modulo the theory, to boolean combinations of pp-formulas. Therefore the definable sets will be formed from the solution sets of the pp-formulas (with parameters) via finite unions, intersections and complements.

Further we may assume without loss of generality that the disjunctions are always mutually exclusive by Lemma 3.2.1, and that every pp-formula β_i that occurs in the formula within a subformula '$\alpha \wedge \bigwedge_i \neg\beta_i$' has already $\beta_i \leftrightarrow \alpha \wedge \beta_i$.

From the construction of the Grothendieck ring of a theory of modules, detailed in Chapter 3, the image under the Euler characteristic χ_0 of a disjoint union of definable sets $[X \sqcup Y]$ is the sum of their images $[X] + [Y]$, the image under χ_0 of a definable set with a definable subset removed $[X \setminus Z]$ is the difference of their images $[X] - [Z]$, and the image under χ_0 of a cross product of sets $[X \times Y]$ is the product of their images $[X] \cdot [Y]$. Therefore the image of Def(M) under $\chi_0 : \mathrm{Def}(M) \to K_0(M)$ will be generated as ring by the images of the solution sets of pp-formulas.

Isomorphisms of pp-sorts. From the construction it is clear that $K_0(M)$ is generated as a ring by the classes of the pp-formulas up to logical equivalence modulo $Th(M)$ and isomorphism via pp-definable maps. To calculate the Grothendieck ring for specific theories of modules it is necessary to establish which pp-sorts are isomorphic, and which are subsorts of basic sorts. Here isomorphism means a pp-definable bijective correspondence following from the theory in question. To actually calculate the Grothendieck ring, it is also necessary to calculate whether there will be any relations among the generators resulting from general definable bijections and from the additive cancellation.

Let P be a model of T^*. So $K_0(P) = K_0(\text{Mod-}R)$. Suppose ϕ_1/ψ_1 and ϕ_2/ψ_2 are pp-pairs in $\mathbb{L}_R^{eq+}$ and there is a isomorphism, given by a pp-formula ρ, between the factor groups $\phi_1(P)/\psi_1(P)$ and $\phi_2(P)/\psi_2(P)$. Then ρ will be an isomorphism between the factor groups $\phi_1(M)/\psi_1(M)$ and $\phi_2(M)/\psi_2(M)$ for every module $M \in$ Mod-R. This follows immediately from Lemma 3.4.5 and the fact that M is elementarily equivalent to some direct summand of a model of $Th(P)$.

Given a ring R such that the category fun-R has finite type, the calculation of the pp-sorts up to isomorphism is made simpler by the fact that every pp-sort will decompose as a direct sum of indecomposable pp-sorts and hence it is sufficient to identify these. By indecomposable, we mean not expressible as a product of (nontrivial) pp-pairs.

Over some rings, the solution sets of all n-ary pp-formulas will be isomorphic to products of solution sets of unary pp-formulas. Then the indecomposable basic pp-sorts are all unary. This simplifies the task of finding all the subsorts of powers of the home sort in $\mathbb{L}_R^{eq+}$. The characterisation of such rings and an example, $R = \mathbb{Z}_4$, are discussed in the sequel.

However in an arbitrary module $M \in$ Mod-R many of the invariants that are infinite over P may have finite values. There may be extra definable isomorphisms between pp-sets and unions of pp-sets in any given module that are not satisfied in P. These additional definable isomorphisms may cause extra identification in the category $\widetilde{\mathrm{Def}}(M)$ and extra relations between the generators of the Grothendieck ring $K_0(M)$ for a general R-module M.

6.3 Grothendieck rings of module categories over Morita equivalent rings

For Morita equivalent rings, R and S, we know that there is an additive equivalence between the module categories Mod-R and Mod-S. In this section it is shown that moreover the Grothendieck rings of the module categories are isomorphic.

Proposition 6.3.1. *(First appearing in the doctoral thesis of K. Burke [7], also proved in [30], 10.2.14) Let F be a finitely presented functor in $(mod\text{-}R, Ab)$. Then $pdim(F) \leq 2$ and $pdim(F) \leq 1$ iff $F \simeq F_\phi$ for some pp-formula ϕ, and $pdim(F) = 0$*

iff $F \simeq F_\theta$ *for some quantifier free pp-formula* θ*, i.e. a system of equations over* R*.*

We saw in section 3.6 that for Morita equivalent rings $R \equiv_M S$ the categories $\mathbb{L}_R^{eq+}$ and $\mathbb{L}_S^{eq+}$ are equivalent, so there is a full and faithful functor $F : \mathbb{L}_R^{eq+} \simeq \mathbb{L}_S^{eq+}$ such that every pp-sort over S is isomorphic to one of the form $F(\phi/\psi)$.

Lemma 6.3.2. *([18] Introduction) The rings* R *and* S *are Morita equivalent if and only if there exists an integer* $n \geq 1$ *and an idempotent* $e \in M_n(R)$ *such that there is an isomorphism of rings* $i : S \cong eM_n(R)e$*.*

This isomorphism i of rings induces a natural map from formulas in the language of S-modules to formulas in the language of $eM_n(R)e$-modules, and thereby a map to formulas in the language of R-modules where an m-ary formula over S is mapped to an mn-ary formula over R. The equivalence $\mathbb{L}_R^{eq+} \simeq \mathbb{L}_S^{eq+}$ between the categories of pp-sorts can therefore be given explicitly via the transformation of pp-formulas from one language to the other.

There is an equivalence of categories between the categories of pp-sorts over the two rings, $\mathbb{L}_R^{eq+} \simeq \mathbb{L}_S^{eq+}$, by Proposition 3.6.7.

Definition 6.3.3. *Given an object* X *of an abelian category* $\mathcal{C}$*, such as* $M \in$ *Mod-*R *or* $F \in$ *(mod-*R*, Ab), a* projective resolution *of* X *is an exact sequence with* X *as the final nonzero term* $\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0$ *and all the* P_i *in the sequence projective objects of* $\mathcal{C}$*.*

Definition 6.3.4. *The* projective dimension *of an object* X *in an abelian category* $\mathcal{C}$*, written pdim*(X)*, is the least integer* n *such that there is a projective resolution for* X *with* $P_i = 0$ *for all* $i > n$*, or* ∞ *if there is no such* n*.*

Theorem 6.3.5. *Let* R *and* S *be Morita equivalent rings. Then the Grothendieck rings of their respective module categories are isomorphic.*

$$R \equiv_M S \;\Rightarrow\; K_0(\textit{Mod-}R) \cong K_0(\textit{Mod-}S)$$

Proof. By Lemma 6.3.2 there is a natural number n and an idempotent matrix $e \in M_n(R)$ such that $S \cong eM_n(R)e$ and thus we may identify S with this ring. Then, given any right R-module A, we may take the direct product of n copies of A and multiply on the right by the matrix e to form $A^n e$. This is a subgroup of A^n and it is definable by a parameter free pp-formula of $\mathcal{L}_R$. This $A^n e$ is also a right S-module. Therefore the action $A_R \mapsto (A^n e)_S$ defines a functor from Mod-R to Mod-S. This functor is one half of a pair providing the equivalence of categories Mod-$R \simeq$ Mod-S.

Let M be a saturated model of the largest complete theory of right R-modules, $Th(P)$ defined above. Define N to be the right S-module $M^n e$. Recall that the home sort in $\mathbb{L}_R^{eq+}$ is $(x = x/x = 0)$ and it has projective dimension 0. The home sort $(y = y/y = 0)$ in $\mathbb{L}_S^{eq+}$ is equal to the sort $(x = x/x = 0)^n e \in \mathbb{L}_R^{eq+}$. A direct sum of projective objects is projective and hence $(x = x/x = 0)^n$ has projective dimension 0. Also since e is an idempotent, the home sort of S is a direct summand of $(x = x/x = 0)^n$. Direct summands of projective objects are projective. Therefore $pdim((x = x/x = 0)^n e) = 0$.

The basic sorts, or subsorts of powers of the home sort, in $\mathbb{L}_R^{eq+}$ are precisely those sorts with projective dimension no greater than 1. This is known from Proposition 6.3.1. In fact it is shown in ([30], Section 10.2) that a sort is given by a pp-formula (as opposed to a proper pp-pair) iff it had $pdim \leq 1$ iff it is a subsort of a projective (equivalently representable) sort iff it is a basic sort.

Let $\mathcal{C}$ denote the full subcategory of $\mathbb{L}_R^{eq+}$ whose objects are those sorts X with $pdim(X) \leq 1$. These are the subsorts of powers of the home sort $(x = x/x = 0)$. The sorts in $\mathbb{L}_S^{eq+}$ of projective dimension no greater than 1 are the subsorts of powers of $(y = y/y = 0)$, which we recall is equal to the sort $(x = x/x = 0)^n e$ in $\mathbb{L}_R^{eq+}$. Since the sort e defines a pp-subgroup of the sort $(x = x/x = 0)^n$, the sort $(x = x/x = 0)^n e$ is a projective sort. Hence the basic sorts in $\mathbb{L}_S^{eq+}$ are exactly the same as the sorts in $\mathcal{C}$, when we regard S as the ring $eM_n(R)e$.

Let M^{eq+} denote the functor from $\mathbb{L}_R^{eq+}$ to Ab with the action $\phi/\psi \mapsto \phi(M)/\psi(M)$. On each pp-sort, the functor M^{eq+} takes the value the factor group of the pp-pair on M. Then the image of the full subcategory $\mathcal{C}$ under the restriction of M^{eq+} is precisely the category of pp-definable subgroups in powers of M and the pp-definable maps between them. Here 'pp-definable' means without parameters. We denote this category $\mathcal{D}_R$. Let N^{eq+} denote the functor from $\mathbb{L}_R^{eq+}$ to Ab with the action $\phi/\psi \mapsto \phi(N)/\psi(N)$, and let $\mathcal{D}_S$ be the image of $\mathcal{C}$ under the restriction of N^{eq+}. Then $\mathcal{D}_S$ is the category of pp-definable subgroups of powers of $N = M^n e$. By the above, this is exactly $\mathcal{D}_R$. The images of the functors M^{eq+} and N^{eq+} agree.

The Grothendieck ring $K_0(M) = K_0(\text{Mod-}R)$ is computed from $\widetilde{\text{Def}}(M)$, but is completely determined by the category $\mathcal{D}_R$ of pp-definable sets and maps in M. Arbitrary $\mathcal{L}_R$-formulas including parameters from M are equivalent, modulo $Th(M)$, to finite boolean combinations of pp-formulas. The parameter free version of these pp-formulas correspond directly to the objects of $\mathcal{D}_R$. The objects of $\mathcal{D}_R$ are sets in $\text{Def}(M)$ and their images under χ_0 generate $K_0(M)$. The construction of $K_0(M)$ from the 'building blocks' of pp-sets depends on which pp-sets intersect, what inclusions there are between pp-sets, and which pp-formulas define morphisms in $\mathcal{D}_R$. This information is all contained in the category $\mathcal{D}_R$. The $\mathcal{L}_R(M)$-formulas defining bijections in $\text{Def}(M)$ are also characterised by information contained in $\mathcal{D}_R$, namely which parameter free pp-formulas themselves define bijections between pp-sets and which implications hold between conjunctions of pp-formulas. In exactly the same way, $K_0(N)$ can be computed from the same information in $\mathcal{D}_S$, the category of pp-definable sets and maps in N. But the categories $\mathcal{D}_S$ and $\mathcal{D}_R$ are equal, so the Grothendieck rings produced from them must be equal. Therefore $K_0(M) \cong K_0(N)$.

Finally we observe that $K_0(N) = K_0(\text{Mod-}S)$. M is a model of the largest complete theory of R-modules iff every R-module purely embed in an ultrapower of M. Pure embeddings and ultrapowers are properties of the category Mod-R and hence are preserved by the category equivalence $\text{Mod-}R \simeq \text{Mod-}S$ and thus the functor $M \mapsto M^n e = N$. Therefore every S-module purely embeds in an ultrapower of N and N is a model of the largest complete theory of S-modules. Thus $K_0(N) = K_0(\text{Mod-}S)$.

□

Remark. By Proposition 6.1.2, we have that $K_0(\text{Mod-}R)$ is independent of which model we choose for the monster complete theory. Hence image of $\mathcal{C}_R$ under the restriction of the the functor P^{eq+}, for any module $P \equiv M$, would be a category equivalent to $\mathcal{D}_R$.

6.4 A functorial approach

To investigate the isomorphism classes of pp-formulas, and their equivalence classes in particular models, we consider the associated functor categories. The functor category fun-R can be localised at a Serre subcategory determined by the theory of a particular model. The material introduced in this section is used in the sequel to study the isomorphisms of pp-sorts for modules over $\mathbb{Z}_4$ and modules over the ring $k[\varepsilon]$, defined in Section 6.8.

The category $\mathbb{L}_R^{eq+}$ of pp-sorts in the language of right R-modules, has as its objects all the pp-pairs ϕ/ψ of $\mathcal{L}_R$. It is equivalent, as an additive category, to the category fun-R = (mod-$R, Ab)^{fp}$ of finitely presented functors from mod-R, the category of finitely presented right R-modules, to the category of abelian groups Ab. We will see in the sequel that every $M \in$ Mod-R induces a localisation of fun-R denoted fun$\langle M \rangle$. These are the localisations of the functor category fun-R at the Serre subcategory of functors annihilating M, for each M. They correspond to the identification of sorts in $\mathbb{L}_R^{eq+}$ where they have isomorphic evaluations on the module M. Note that the only pp-pairs that annihilate a model P as above are ϕ/ψ where the pp-formulas ϕ and ψ are equivalent modulo Th(Mod-R), i.e. the trivial pp-pairs that are closed on every R-module.

Every right module $M_R \in$ Mod-R gives rise to a functor M^{eq+} from $\mathbb{L}_R^{eq+}$ to Ab. The image of the full subcategory of sorts with $pdim \leq 1$ under the restriction of this functor is the category of pp-definable groups in M and pp-definable maps between them. Hence there are surjections from the set of objects in $\mathbb{L}_R^{eq+}$ to the set of generators for K_0(Mod-R), and to the set of generators for $K_0(M)$, for each R-module M.

Definition 6.4.1. *Every module $M \in$ Mod-R generates a definable subcategory*

$\langle M\rangle \subseteq$ Mod-R, and we write fun$\langle M\rangle$ to denote the category $(\langle M\rangle, Ab)^{\to\Pi}$, the category of functors from $\langle M\rangle$ to Ab that commute with direct products and direct limits.

The category fun$\langle M\rangle$ is equivalent to the category with objects the quotient groups $\phi(M)/\psi(M)$ for every pp-pair of $\mathcal{L}_R$, and arrows the pp-definable maps between them (by [30] 12.3.20).

Proposition 6.4.2. *([30] 3.2.15 in terms of imaginaries, [30] 12.3.19 in terms of fun-R) Let M be a right R-module. Then the category fun$\langle M\rangle$ is a localisation of the category fun-R by the Serre subcategory of functors that are zero when evaluated at M.*

For any functor $F \in (\text{mod-}R, Ab)$, let $\overrightarrow{F} \in (\text{Mod-}R, Ab)$ denote the extension of F to a functor on the whole module category Mod-R which commutes with direct limits. Every $M \in$ Mod-R gives rise to a Serre subcategory $\mathcal{S}_M := \{F \in \text{fun-}R : \overrightarrow{F}M = 0\}$ of fun-R. This category $\mathcal{S}_M$ has as objects all the functors in fun-R that annihilate the module M, i.e. the functors $F_{\phi/\psi}$ in fun-R with $\phi(M) = \psi(M)$. Recall, for any pp-pair $\psi \to \phi$, the functor $F_{\phi/\psi}$ is given by $F_{\phi/\psi} : A \mapsto \phi(A)/\psi(A)$, $\forall A \in$ mod-R. There is an equivalence of categories fun$\langle M\rangle \simeq$ fun-$R/\mathcal{S}_M$.

If R is a ring such that the category fun-R has finite type, then the definable subcategory $\mathcal{S}_M$ is generated by its indecomposable elements. Given the category of all pp-imaginaries for R-modules, $\mathbb{L}_R^{eq+}$, it is not immediately evident how to identify up to isomorphism the pp-sorts that are subsorts of the basic sorts. For some rings of finite representation type, the area of Auslander-Reiten theory provides a method for these calculations, as seen in Section 6.7.

In the sequel we investigate the isomorphisms of pp-sorts in $\mathcal{L}_{\mathbb{Z}_4}$ and $\mathcal{L}_{k[\varepsilon]}$. We calculate the generators for the rings $K_0(\text{Mod-}\mathbb{Z}_4)$ and $K_0(\text{Mod-}k[\varepsilon])$ and demonstrate, by means of examples, that there can be extra identifications among the generators in the Grothendieck rings of particular modules.

6.5 Rings of finite representation type

For certain classes of ring R, there are particular strategies for calculating the sorts that correspond to a generating set of the Grothendieck ring of R-modules. One such class is the rings of finite representation type.

Definition 6.5.1. *A ring is said to have* finite representation type *(or FRT) if every right module is a direct sum of indecomposable modules and there are only finitely many indecomposable modules up to isomorphism.*

When the ring R has finite representation type, there are only finitely many non-isomorphic indecomposable modules, so we may form their direct sum N and $\mathrm{End}_R(N)$ is a ring with unity. Let S denote the ring $\mathrm{End}_R(N)$. Then the category fun-$R \simeq \mathbb{L}_R^{eq+}$ is also equivalent to the category S-mod of finitely presented left S-modules, by Proposition 6.7.1 (or see [6], 4.9.4). The category of (finitely presented) S-modules is sometimes easier to calculate than the equivalent category fun-R. Every indecomposable S-module will be a direct summand of the evaluation $F_{\phi/\psi}(N)$ of some functor $F_{\phi/\psi} \in \mathbb{L}_R^{\mathrm{eq}+}$ on the module N.

From the construction, we know that the Grothendieck ring $K_0(\text{Mod-}R)$ will be generated by the representatives of the equivalence classes of pp-formulas in the basic sorts, i.e. the powers of the home sort. We can determine which pp-sorts are isomorphic to ones in the basic sorts. If we calculate a projective resolution for a functor $F_{\phi/\psi}$ in fun-R, this will determine whether the functor is isomorphic to one of the form F_ϕ, since we may check if $pdim(F) \leq 1$.

6.6 An extended example, Mod-$\mathbb{Z}_4$

We consider the example $R = \mathbb{Z}_4$, the ring of integers modulo 4. The ring R has finite representation type and its indecomposable (right) modules up to isomorphism are $\mathbb{Z}_4$ and $\mathbb{Z}_2$. In this section we explore the method this property admits for computing the generators of $K_0(\text{Mod-}R)$ and $K_0(M)$ for general $M \in \text{Mod-}R$.

We first identify the generators of $K_0(\text{Mod-}R)$, and then consider the localisation $\text{fun}\langle \mathbb{Z}_2 \rangle$ of the category fun-R, for the module $\mathbb{Z}_2 \in \text{Mod-}R$. Let i denote the inclusion $\mathbb{Z}_2 \hookrightarrow \mathbb{Z}_4$ given by $1_2 \mapsto 2_4$ and π denote the epimorphism $\mathbb{Z}_4 \twoheadrightarrow \mathbb{Z}_2$ given by $1_4 \mapsto 1_2$. Observe that $\pi i = 0 \in (\mathbb{Z}_2, \mathbb{Z}_2)$ and $i\pi = (-)2 \in (\mathbb{Z}_4, \mathbb{Z}_4)$. The Auslander-Reiten quiver of R is:

$$\Gamma(R) = \begin{tikzcd} \mathbb{Z}_2 \arrow[r, bend left, "i"] & \mathbb{Z}_4 \arrow[l, "\pi"] \end{tikzcd}$$

Let N be the module $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, the direct sum of one copy of each indecomposable module (up to isomorphism). We let $M^* = \mathbb{Z}_2^{(\aleph_0)} \oplus \mathbb{Z}_4^{(\aleph_0)}$. Then $T^* := Th(M^*)$ is a largest theory of R-modules and following the approach above, $K_0(\text{Mod-}\mathbb{Z}_4) = K_0(M^*)$. We define S to be the ring of endomorphisms of N:

$$S = \text{End}(N) = \begin{pmatrix} \mathbb{Z}_4 1_{\mathbb{Z}_4} & \mathbb{Z}_2 i \\ \mathbb{Z}_2 \pi & \mathbb{Z}_2 1_{\mathbb{Z}_2} \end{pmatrix}$$

Then N is a left S-module. The indecomposable S-modules are N , $0_2 \oplus (\mathbb{Z}_4 \cdot 2)$, $\mathbb{Z}_2 \oplus (\mathbb{Z}_4 \cdot 2)$, $\mathbb{Z}_2 \oplus \mathbb{Z}_4/(\mathbb{Z}_4 \cdot 2)$ and $\mathbb{Z}_2 \oplus 0_4$. These modules correspond to the pp-sorts $v = v$, $2|v$, $v2 = 0$, $(v = v/2|v)$ and $(v2 = 0/2|v)$ respectively in $\mathbb{L}_R^{eq+}$. Note that any pp-formula may be regarded as a pp-pair by taking it over the pp-formula $v = 0$ or $\overline{v} = \overline{0}$. The pp-pairs in the basic sorts among these five are $v = v$, $2|v$ and $v2 = 0$ corresponding to the indecomposable S-modules N, $0_2 \oplus (\mathbb{Z}_4 \cdot 2)$ and $\mathbb{Z}_2 \oplus (\mathbb{Z}_4 \cdot 2)$. Hence there are three non-isomorphic pp-subsorts of the home sort and it is the images of these three that are a generating set $K_0(\text{Mod-}\mathbb{Z}_4)$. Note also that there are no additional indecomposable pp-subsorts of higher powers of the home sort.

Localisation at the theory of a particular module. The R-module $M = \mathbb{Z}_2$ generates a definable subcategory $\langle \mathbb{Z}_2 \rangle \subset \text{Mod-}\mathbb{Z}_4$. The finitely presented functors from this category to Ab, form a category $\text{fun}\langle M \rangle$ equivalent to the quotient of fun-R by the subcategory of functors annihilating M. Of the indecomposables of fun-R; the pp-pairs $v = v, 2|v, v2 = 0, (v = v/2|v)$ and $(v2 = 0/2|v)$, the only one to annihilate $\mathbb{Z}_2$ is $2|v$. We evaluate $2|v$ on N, and obtain $0_2 \oplus (\mathbb{Z}_4 \cdot 2)$. Therefore the category

$\text{fun}\langle M\rangle$ is the quotient category $\text{fun-}R/\langle 2|v\rangle \simeq S\text{-mod}/\langle 0_2 \oplus (\mathbb{Z}_4 \cdot 2)\rangle$.

We may ask which of the indecomposable pp-sorts in fun-R become isomorphic in this quotient category. We take the quotient of S-mod by the subcategory generated by $0_2 \oplus (\mathbb{Z}_4 \cdot 2)$. The indecomposable S-module $0_2 \oplus (\mathbb{Z}_4 \cdot 2)$ is clearly annihilated in the quotient. The other indecomposables N, $\mathbb{Z}_2 \oplus \mathbb{Z}_4/(\mathbb{Z}_4 \cdot 2)$, $\mathbb{Z}_2 \oplus (\mathbb{Z}_4 \cdot 2)$ and $\mathbb{Z}_2 \oplus 0_4$ become definably isomorphic in the quotient category. Therefore there is one indecomposable, up to isomorphism, in the quotient category. Since S is of finite type, the functors corresponding to indecomposable pp-sorts generate the functor category and its quotients. This indecomposable module in the quotient category is the evaluation of the pp-sort $v = v$, i.e. the home sort.

Proposition 6.6.1. *In every $\mathbb{Z}_4$-module, every pp-formula defines a set that is isomorphic to a product of unary pp-sets. This implies that the Grothendieck ring of any $\mathbb{Z}_4$-module is generated by the representatives of these formulas.*

Proof. The proposition follows from the characterisation of all the indecomposable subsorts of powers of the home sort in $\mathbb{L}_R^{eq+}$ as being all subsorts of the first power, shown above. Since the functor category fun-$R \simeq \mathbb{L}_R^{eq+}$ has finite representation type, every functor (equivalently every pp-pair) is a direct sum of the indecomposable functors. As shown earlier in this section, the indecomposable functors in fun-$\mathbb{Z}_4$ are all subfunctors of the first power of the forgetful functor. In terms of $\mathbb{L}_R^{eq+}$, this is equivalent to every indecomposable sort being a unary pp-formula. □

We also give an explicit description of how the pp-definable sets are isomorphic to products of unary pp-sets. This is an example of the combinatorics of definable sets that is the foundation of constructing Grothendieck rings. We first consider $\mathbb{Z}_4$-linear equations, then conjunctions of such and finally the case when some of the variables are bound by existential quantifiers. For the remainder of this section, let $\phi_1(v)$ denote the formula $\exists w\, w2 = v$ and let $\phi_2(v)$ denote the formula $v2 = 0$.

A quantifier free positive primitive formula over $\mathbb{Z}_4$ will be a finite system of linear equations. A general $\mathbb{Z}_4$-linear equation with parameters is $\sum_{i=1}^{n} v_i \cdot r_i = c$ after grouping the parameter terms in the formula into one constant symbol. Without loss of generality this is possible and the resulting formula is logically equivalent to the original. The solution set is always definably isomorphic to (i) M^{n-1} or (ii) $\phi_2(M) \times M^{n-2}$ for example $(v_1 + v_2) \cdot 2 = 0$ or (iii) empty for example $v_1 \cdot 2 = c$ where $c \notin M \cdot 2$.

The conjunction of m such equations with variables among $v_1, \ldots, v_n$ will have solution set isomorphic to (i) $\phi_2(M)^k$, (ii) M^l, (iii) $\phi_2(M)^k \times M^l$ or (iv) empty. The most general pp-formulas in $\mathcal{L}_{\mathbb{Z}_4}$ are of the form $\exists \overline{w} \bigwedge_{j=1}^{m} \left(\sum v_i \cdot r_i + \sum w_k \cdot r_k = c_j\right)$. This allows us to define sets isomorphic to $\phi_1(M)$ too, and products with all the aforementioned sets.

Thus we can see explicitly how the solution set of a pp-formula in a module, $\psi(M) \subseteq M^n$ will be definably isomorphic to a product of pp-subsets of M^1. We know that the representatives of these unary pp-sets are generators for the whole ring $K_0(M)$.

The complete theories of $\mathbb{Z}_4$-modules.

In $\mathcal{L}_{\mathbb{Z}_4}$, the lattice of pp-conditions (up to logical equivalence) in one free variable is:

$$
\begin{array}{cl}
v = v & \\
\Big| & \\
v2 = 0 & \qquad \text{denoted } \phi_2(v) \\
\Big| & \\
\exists w \; w2 = v & \qquad \text{denoted } \phi_1(v) \\
\Big| & \\
v = 0 &
\end{array}
$$

Over the ring $\mathbb{Z}_4$, every module is isomorphic to a direct sum $\mathbb{Z}_2^{(\lambda)} \oplus \mathbb{Z}_4^{(\kappa)}$ for some λ, κ. Hence every $\mathbb{Z}_4$-module is elementarily equivalent to a direct summand of a module $M^* := \mathbb{Z}_2^{(\aleph_0)} \oplus \mathbb{Z}_4^{(\aleph_0)}$.

The index or invariant of each pp-pair in the lattice depends entirely on λ and κ. We have $Inv(M, \phi_1(v)/v = 0) = 2^\kappa, Inv(M, \phi_2(v)/\phi_1(v)) = 2^\lambda$ and $Inv(M, v = v/\phi_2(v)) = 2^\kappa$. The complete theories of $\mathbb{Z}_4$-modules are characterised by the values of these two invariants. Two modules $M_1 = \mathbb{Z}_2^{(\lambda)} \oplus \mathbb{Z}_4^{(\kappa)}$ and $M_2 = \mathbb{Z}_2^{(\mu)} \oplus \mathbb{Z}_4^{(\nu)}$ will have the same theory iff $\lambda \doteq \mu$ and $\kappa \doteq \nu$. Here the dotted equality means 'equal or both infinite'. For the Grothendieck ring of a $\mathbb{Z}_4$-module, the finite sets are represented by parameter sets and the only feature of the theory of a module that impacts its Grothendieck ring is whether λ and κ are infinite or not.

For the remainder of this section the notation ϕ_1 and ϕ_2 will always refer to these specific pp-formulas. Each of the simple intervals of the diagram is a trivial interval of the lattice, i.e. there are no intermediate pp-formulas. Therefore the pp-rank of a formula is simply the height in the lattice. In M^* the pp-rank of any $\psi \in pp^1(\mathbb{Z}_4)$ will equal the Morley rank of the set $\psi(M^*)$. For any $M \in \text{Mod-}\mathbb{Z}_4$, $pp^n(M)$ is generated by $pp^1(M)$ in the sense that the sets in $pp^n(M)$ for each n are definably isomorphic to products of sets in $pp^1(M)$. This is a consequence of the fact, shown above, that all the indecomposable pp-subsorts of powers of the home sort are subsorts of the first power of the home sort $(x = x)$.

The notions of Morley rank and Morley degree of definable sets are illuminating with this pp-lattice for Mod-$\mathbb{Z}_4$. For the module M^*, or any direct sum $\mathbb{Z}_2^{(\lambda)} \oplus \mathbb{Z}_4^{(\kappa)}$ with λ, κ both infinite, the solution sets of the pp-formulas in the lattice increase in Morley rank at every step, and they all have Morley degree 1. The set $\phi_1(M)$ of Morley rank 1 since it is infinite and minimal. The set $\phi_2(M)$ is an infinite union of cosets of $\phi_1(M)$ and hence must be of higher Morley rank.

If the exponent κ is a finite cardinal and λ is infinite, then the set $\phi_1(M)$ is finite, $\phi_2(M)$ is of Morley rank and degree 1, and M is of Morley rank 1 and Morley degree 2^κ since it is the union of 2^κ cosets of $\phi_2(M)$.

If κ is infinite and λ is finite, then $\phi_1(M)$ is a minimal infinite set with Morley rank and degree 1. The set $\phi_2(M)$ is a union of 2^λ cosets of $\phi_1(M)$ and hence has Morley rank 1 and degree 2^λ. In this case, M is of Morley rank 2 and Morley degree 1, since it contains infinitely many (2^κ) cosets of the Morley rank 1 set $\phi_1(M)$ but there is no definable, countable partition into sets of Morley rank 2.

Every parameter free pp-condition in one free variable is logically equivalent to one of these four shown in the lattice above. These pp-formulas define a chain of subgroups in any given module $M = \mathbb{Z}_2^\lambda \oplus \mathbb{Z}_4^\kappa$ and the invariants in said module are

$Inv(M, v = v \,/\, v2 = 0) = 2^{\kappa}$, $Inv(M, v2 = 0 \,/\, 2|v) = 2^{\lambda}$, $Inv(M, 2|v \,/\, v = 0) = 2^{\kappa}$.

6.7 Auslander-Reiten theory

For certain rings, the Auslander-Reiten quiver, defined below, is useful in calculating the indecomposable pp-sorts. In this section we include some of the relevant background material for its construction. In the following section we demonstrate the technique for the ring $k[\varepsilon]$.

We find the isomorphism classes of indecomposable right R-modules and compute the Auslander-Reiten quiver of R. If R has finite representation type, these classes will be finite in number. Then we set N equal to the direct sum of one representative from each isomorphism class of indecomposables. The module $M^* = N^{(\aleph_0)}$ is a model of the largest theory of R-modules. We then define S to be the ring $\mathrm{End}_R(N)$ and compute the Auslander-Reiten quiver of S.

Proposition 6.7.1. *([6], 4.9.4) The category S-Mod is equivalent to Fun-R := (mod-R, Ab). The evaluation of each functor $F \in$ (mod-R, Ab) on N gives an S-module, $F(N)$. This evaluation yields an equivalence of categories, (mod-R, Ab) $\simeq$ S-mod.*

Therefore the respective full subcategories of finitely presented objects will be equivalent, by Lemma 3.6.2, that is S-mod $\simeq$ (mod-R, $Ab)^{fp}$ =: fun-$R \simeq \mathbb{L}_R^{eq+}$.

From the equivalent category S-mod, we can identify the objects of fun-R that are subsorts of powers of the home sort. These are the pp-pairs whose isomorphism classes generate the Grothendieck ring K_0(Mod-R).

Definition 6.7.2. *A ring R is said to be an* artin algebra *if the centre of R is artinian, i.e. satisfies the descending chain condition on ideals, and R is a finitely generated module over its centre.*

In this section we introduce the definitions and material necessary to define the Auslander-Reiten quiver of an artin algebra. In the remainder of the chapter, we go on to calculate the Auslander-Reiten quiver for specific examples, and demonstrate the rôle of Auslander-Reiten theory in calculating Grothendieck rings, under certain conditions, described in the sequel.

Definition 6.7.3. *Let $g : B \to C$ be a morphism of right R-modules that is not a split epimorphism. If every morphism $h : M \to C$ that is not a split epimorphism, factors through g then we say that g is* right almost split *in Mod-R. We may omit the module category when it is clear from the context.*

$$\begin{tikzcd} B \arrow[rr, "g"] & & C \\ & M \arrow[ul, dashed, "\exists"] \arrow[ur, "h"'] & \end{tikzcd}$$

Definition 6.7.4. *The dual notion is a* left almost split *morphism.*

Definition 6.7.5. *The morphism $g : B \to C$ is called* right minimal *if every morphism $h : B \to B$ such that $g \circ h = g$ is an automorphism. If g is both right minimal and almost split, we say it is* minimal right almost split*.*

Definition 6.7.6. *An* almost split sequence *(also known as an Auslander-Reiten sequence) in Mod-R is an exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ satisfying the following equivalent conditions:*

- *f is left almost split and g is right almost split*
- *$End_R C$ is a local ring and f is left almost split*
- *$End_R A$ is a local ring and g is right almost split*
- *f is a minimal left almost split morphism*
- *g is a minimal right almost split morphism*

Remark. It follows from the definition, that if a sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is almost split, then A and C are indecomposable modules.

By ([4],1.16), the almost split sequences of any module category Mod-R are uniquely determined up to *isomorphism* by their initial module A (equivalently by their terminal module C). An isomorphism of almost split sequences consists of module isomorphisms that commute with the minimal almost split maps of the sequence, i.e. two almost split sequences are isomorphic if there exists module isomorphisms for the vertical maps such that the diagram below commutes.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\cong"'] & B \arrow[r, "g"] \arrow[d, "\cong"'] & C \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0 \end{tikzcd}$$

Therefore the map τ taking C to A if there exists an almost split sequence $0 \to A \to B \to C \to 0$ is well defined and also invertible, $C = \tau^{-1}A$. We call this map τ the *Auslander-Reiten translate*. M. Auslander and I. Reiten prove in [2],[3], that for R of finite representation type and for R an artin algebra, there exist in Mod-R, for every finitely generated, indecomposable non-projective module C, an Auslander-Reiten sequence of finitely generated modules ending in C. Also for every finitely generated, indecomposable non-injective module A there exists an Auslander-Reiten sequence of finitely generated modules beginning with A.

Definition 6.7.7. *A* quiver *is a labeled directed multigraph. A* representation of a quiver *Q in a module category is an assignment of a module M_i to each vertex i of Q and a module morphism $M_i \xrightarrow{M_\alpha} M_j$ to each arrow $\alpha : i \to j$ in Q.*

Definition 6.7.8. *The* Auslander-Reiten quiver *of R, denoted $\Gamma(R)$, has a vertex for every isomorphism class of finitely generated indecomposable R-modules.*

Let M, N be finitely generated R-modules. Then $rad(M,N)/rad^2(M,N)$ is an $(End(N)/JEnd(N), End(M)/JEnd(M))$-bimodule and suppose it has dimension b on the left dimension a on the right. Then there is an arrow in $\Gamma(R)$ from $[M]$ to $[N]$ labeled (a,b) where a and b are the dimensions of the bimodule $rad(M,N)/rad^2(M,N)$ (but we omit all the labels of the form $(1,1)$). All of the arrows of $\Gamma(R)$ arise in this way.

Auslander-Reiten quivers of artin algebras were introduced and developed by M. Auslander and I. Reiten in [2] and [3]. In the examples in chapter, we show how Auslander-Reiten quivers can be used to find the indecomposable pp-sorts corresponding to subfunctors of powers of the forgetful functor in fun-R.

6.8 The ring $k[\varepsilon]$

Overview. Let $R = k[\varepsilon]$ be the k-algebra over the quiver with one vertex and one non-identity arrow ε with $\varepsilon^2 = 0$. Then R may also be regarded as $R = k[\varepsilon] := k[X]/\langle X^2\rangle$. We wish ultimately to find the subsorts of powers of the home sort in $\mathbb{L}_R^{eq+}$. There is an approach for this calculation, making use of Auslander-Reiten theory, as follows:

- Find all the indecomposable right R-modules up to isomorphism.
- Compute the Auslander-Reiten quiver $\Gamma(R)$.
- Let N be the direct sum of one copy of each indecomposable R-modules.
- Compute its endomorphism ring $S = \mathrm{End}_R(N)$.
- Compute the Auslander-Reiten quiver $\Gamma(S)$.
- Use the equivalence S-mod $\simeq \mathbb{L}_R^{eq+}$ to identify the indecomposable pp-sorts that are subsorts of powers of the home sort.

These indecomposable basic sorts are the generators of the Grothendieck ring of any theory of R-modules. The relations between the generators will depend on the theory in question, and in some cases some of the generators may be identified if they are isomorphic over particular modules. This approach works well for $R = k[\varepsilon]$ because the ring S is of finite representation type, and so it is possible to compute $\Gamma(S)$ and calculate indecomposable pp-sorts from it. In general the techniques of this section may be brought to bear on any ring satisfying this condition.

The indecomposable R-modules. The ring R is an indecomposable projective right module over itself, and setting $S_1 := \mathrm{rad}(R)$, it is isomorphic to $\mathrm{top}(R) := R/\mathrm{rad}(R)$. As R is an artin algebra, mod-$R \simeq (R\text{-mod})^{op}$ and the projectives of mod-R are dual to the injectives of R-mod (by 3.1 of [2], II). The category mod-R has only one indecomposable projective, R_R, which is of length 2.

Hence there is only one indecomposable injective in R-mod, and it is of length 2, as module lengths are preserved by the duality mod-$R \simeq R$-mod. Now the ring as a left module over itself, ${}_RR$, is indecomposable and has length 2. It remains to show that R is injective. Now ${}_RR$ has simple socle, namely εR, so the injective hull of ${}_RR$ must be indecomposable and hence can only be ${}_RR$ itself. Thus R is an injective left R-module and also an injective right R-module since R is commutative.

Therefore R_R is the only indecomposable injective, and hence any $M \in$ Mod-R has a decomposition $M = R^{(\kappa)} \oplus M'$ where M' has no injective submodule.

Claim. Let $a \in M'$. Then length$(aR) \leq 1$. Assume for a contradiction that length$(aR) > 1$. Then $R \to aR$ is an isomorphism, but $aR \subseteq M'$ and this is a contradiction. Thus for each $a \in M'$, length$(aR) \leq 1$ and M' is a direct sum of copies of S_1, the radical rad(R). Thus the arbitrary right R-module M has the form $M = R^{(\kappa)} \oplus S_1^{(\lambda)}$. Therefore R and S_1 are the only indecomposable modules.

Recall that the vertices of the Auslander-Reiten quiver are the isomorphism classes of indecomposable f.g. R-modules. The **Auslander-Reiten quiver of** R is given below with the maps between indecomposables satisfying $\pi i = 0$, $i\pi = \varepsilon$.

$$\Gamma(R) = \begin{tikzcd} S_1 \arrow[r, bend left, "i"] & R \arrow[l, "\pi"] \end{tikzcd}$$

The module N and its endomorphism ring S. Let N be the direct sum of one copy of each indecomposable right R-module up to isomorphism, that is $N := R \oplus S_1$. Then $N \in$ Mod-R is a left module over its own endomorphism ring $S := \mathrm{End}_R(N)$, as is any module. We wish to find all the indecomposable S-modules. Considering the left action of S on $R \oplus S_1$, we may write

$$S = \begin{pmatrix} (R,R) & (S_1,R) \\ (R,S_1) & (S_1,S_1) \end{pmatrix} = \begin{pmatrix} k1_R \oplus k\varepsilon & ki \\ k\pi & k1_{S_1} \end{pmatrix}$$

The bracket notation (S_1, R) here means the R-module morphisms from S_1 to R. Observe that $R = k1_R \oplus k\varepsilon$ is a 2-dimensional k-vector space, and S is a 5-dimensional k-vector space as seen above in the right-hand matrix presentation, where the dimensions are $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. The decomposition of S into a direct sum of indecomposable projective left S-modules is given by:

$$S = \begin{pmatrix} R & 0 \\ k\pi & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & ki \\ 0 & k1_{S_1} \end{pmatrix} =: Q_1 \oplus Q_2$$

The indecomposable projective Q_2 defined above has radical $\begin{pmatrix} 0 & ki \\ 0 & 0 \end{pmatrix}$ and the indecomposable projective Q_1 defined above has radical $\begin{pmatrix} k\varepsilon & 0 \\ k\pi & 0 \end{pmatrix}$ since ${1_R}^2 = 1_R$, $\varepsilon^2 = 0$, ${1_S}^2 = 1_S$. Therefore T_1 below is simple:

$$T_1 := \mathrm{Top}(Q_1) = Q_1/\mathrm{rad}(Q_1) = \begin{pmatrix} R & 0 \\ k\pi & 0 \end{pmatrix} / \begin{pmatrix} k\varepsilon & 0 \\ k\pi & 0 \end{pmatrix} \cong \begin{pmatrix} k1_R & 0 \\ 0 & 0 \end{pmatrix}$$

The other indecomposable projective, Q_2, is isomorphic to the radical of Q_1, via right multiplication by $\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} \in S$. Recall $i\pi = \varepsilon$, and so

$$\begin{pmatrix} 0 & ki \\ 0 & k1_{S_1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} = \begin{pmatrix} k\varepsilon & 0 \\ k\pi & 0 \end{pmatrix}$$

Observe that since $\varepsilon^2 = 0$, but obviously ${1_R}^2 = 1_R$ and ${1_{S_1}}^2 = 1_{S_1}$ and these are a basis for the diagonal entries, we have

$$\mathrm{rad}(S) = \mathrm{Nil}(S) = \begin{pmatrix} k\varepsilon & ki \\ k\pi & 0 \end{pmatrix}$$

Hence, for $T_i := \mathrm{Top}(Q_i)$ we have:

$$S/\mathrm{rad}(S) = Q_1/\mathrm{rad}(Q_1) \oplus Q_2/\mathrm{rad}(Q_2) = T_1 \oplus T_2$$

Since $T_1 \cong \begin{pmatrix} k1_R & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathrm{rad}(Q_2) = \begin{pmatrix} 0 & ki \\ 0 & 0 \end{pmatrix}$ we see that

$$\begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \Rightarrow T_1 \simeq \mathrm{rad}(Q_2)$$

Therefore S is a 5-dimensional k-vector space and to give a decomposition into simple S-modules, we have (up to isomorphism):

$$S \quad = \quad Q_1 \oplus Q_2 \quad = \quad \begin{matrix} T_1 \\ T_2 \\ T_1 \end{matrix} \bigoplus \begin{matrix} T_2 \\ T_1 \end{matrix}$$

Then S may be regarded as a quiver algebra, shown below, where the arrows π and i in the quiver represent right multiplication by $\begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}$ respectively. So the quiver algebra has k-basis $\{e_1, e_2, \pi, i, \varepsilon = i\pi\}$.

$$S = k\left(\begin{tikzcd} 1 \arrow[r, bend left, "i"] & 2 \arrow[l, bend left, "\pi"] \end{tikzcd} \right) \quad \text{where } \pi i = 0$$

We calculate the **Auslander-Reiten quiver for** S in order to learn about the category of pp-sorts over R, since fun-$S \simeq \mathbb{L}_R^{eq+}$. First, we identify the indecomposable S-modules. The simple S-modules are T_1 and T_2 given above. We have the indecomposable projectives Q_1, Q_2 and it remains to find the indecomposable injective left S-modules. The indecomposable projective S-modules are given by the decomposition

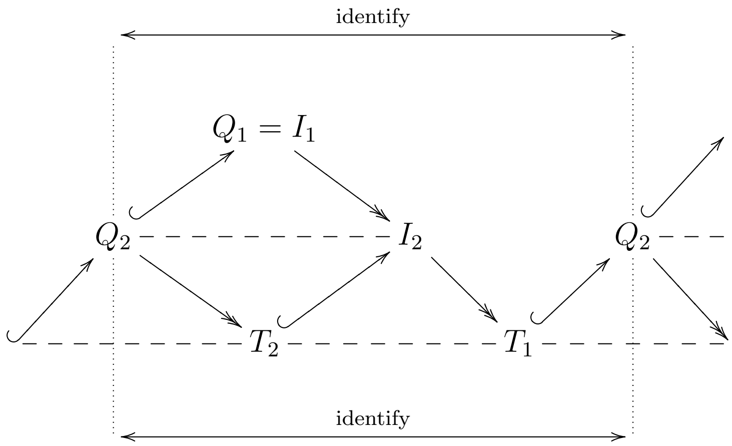
$$S_S = \begin{pmatrix} R & ki \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ k\pi & k1_{S_1} \end{pmatrix} =: P_1 \oplus P_2$$

Note that P_1, P_2 are k-vector spaces of dimension 3 and 2 respectively. The indecomposable injective left S-modules are the duals $P_i^* = \mathrm{Hom}_k(P_i, k)$ for $i = 1, 2$. By the duality of the projectives of mod-S and the injectives of S-mod, there must be exactly one indecomposable left S-module of dimension 3 and one of dimension 2.

Definition 6.8.1. *Given R-modules $A \subseteq B$, we say that B is an* essential extension *of A if whenever C is a nonzero submodule of B, $A \cap C \neq 0$.*

Q_1 is an essential extension of T_1 and is 3-dimensional, so Q_1 must be the unique indecomposable injective of dimension 3, say $Q_1 = I_1$. And $Q_1/\mathrm{soc}(Q_1)$ is 2-dimensional and an indecomposable extension of T_2, hence an essential extension. Therefore $I_2 := Q_1/\mathrm{soc}(Q_1)$ is the injective hull of T_2, and the second indecomposable injective.

Now we have computed all of the indecomposable injective and projective S-modules and we may compute the Auslander-Reiten quiver $\Gamma(S)$.



The dashed lines indicate almost split sequences. The Auslander-Reiten translate τ acts from right to left along each dashed line.

For this ring $R = k[\varepsilon]$, the ring $S = \mathrm{End}(N)$ is of finite representation type. There are five indecomposable left S-modules. Since S-mod $\simeq$ fun-R, these correspond to five indecomposable pp-sorts over R. The indecomposable S-modules are $Q_1 = I_1$, Q_2, T_1, $I_2 = Q_1/T_1$ and $T_2 = Q_2/T_1$. We will find the corresponding pp-sorts and deduce which of them are subsorts of basic sorts, i.e. powers of the home sort $x = x$.

The home sort is given by the pp-pair $(x = x/x = 0)$ or simply $(x = x)$, corresponding to the forgetful functor $(R_R, -)$. To present this as an S-module we evaluate this home sort on $N = R_R \oplus S_1$, which yields ${}_SN$. Now ${}_SN$ is 3-dimensional over k and generated by $\begin{pmatrix} 1_R \\ 0 \end{pmatrix}$, since

$$S \begin{pmatrix} 1_R \\ 0 \end{pmatrix} = \begin{pmatrix} k1_R \oplus k\varepsilon & ki \\ k\pi & k1_{S_1} \end{pmatrix} \begin{pmatrix} 1_R \\ 0 \end{pmatrix} = \begin{pmatrix} R \\ k\pi R \end{pmatrix} = \begin{pmatrix} R \\ S_1 \end{pmatrix} = N$$

We wish to find $N/\mathrm{rad}(N)$ or, more specifically, its dimension over k.

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1_R \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} \begin{pmatrix} 1_R \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1_{S_1} \end{pmatrix}$$

Since $\begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \pi & 0 \end{pmatrix} \in \mathrm{rad}(S)$ and $\begin{pmatrix} 1_R \\ 0 \end{pmatrix}$ generates ${}_SN$, we see that $\begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly independent elements of $\mathrm{rad}(N)$. Hence $N/\mathrm{rad}(N)$ is 1-dimensional and thus ${}_SN$ is indecomposable and has length 3. Therefore it is isomorphic to $Q_1 = I_1$.

Now we require pp-pairs in $\mathbb{L}_R^{eq+} \simeq$ fun-R that take the values of the other indecomposable S-modules. These were found by trial and error, a method that works fine when there are only a few indecomposable S-modules. Every pp-pair that gives an indecomposable S-module when evaluated on N must be indecomposable itself in fun-R. Evaluating pp-pairs on the module N_R, and considering the dimensions of the resulting modules and whether they are projective, injective and which are direct sums or quotients of the others, we find pp-pairs corresponding to the 5 indecomposable S-modules. The information is contained in the table below:

pp-pair	indecomposable	evaluation on N
$x = x$	Q_1	$(R \oplus S_1)$
$x\varepsilon = 0$	Q_2	$(\varepsilon k \oplus S_1)$
$\varepsilon \| x$	T_1	$(\varepsilon k \oplus 0)$
$\frac{x=x}{\varepsilon \mid x}$	Q_1/T_1	$(R/\varepsilon k \oplus S_1)$
$\frac{x\varepsilon=0}{\varepsilon \mid x}$	Q_2/T_1	$(0 \oplus S_1)$

A complete list of indecomposable sorts, up to isomorphism, is; $(x = x)$, $(x\varepsilon = 0)$, $(\varepsilon|x)$, $x = x/\varepsilon|x$ and $x\varepsilon = 0/\varepsilon|x$. The indecomposable subsorts of the home sort are $(x = x)$, $(x\varepsilon = 0)$ and $(\varepsilon|x)$ corresponding to the S-modules; N, $\varepsilon k \oplus S_1$ and $\varepsilon k \oplus 0$ respectively. These are the only indecomposable subsorts of any power of the home sort. Hence it is the images under χ_0 of these 3 pp-sorts, or more precisely the images under χ_0 of their solution sets in Def(P), that as indeterminates, along with $\mathbb{Z}$, generate the ring $K_0(\text{Mod-}k[\varepsilon])$.

Localisation at the theory of a particular module. On the module $N = R \oplus S_1$, all of the non-isomorphic pp-pairs have non-isomorphic evaluations . For an example of the localisation at the theory of a particular module where one or more of the pp-pairs may collapse, we consider the module $R = k[\varepsilon]$ as a right R-module over itself. Let M denote R_R.

By Proposition 6.4.2, there is a Serre subcategory of the functor category fun-R, defined by $\mathcal{S}_M := \{F \in \text{fun-}R : F(M) = 0\}$. Note that $F_{\phi/\psi}(M) = 0$ iff $\phi(M) = \psi(M)$. We wish to find fun$\langle M\rangle \simeq$ fun-$R/\mathcal{S}_M$. The category $\mathcal{S}_M$ will be generated by the indecomposable functors it contains. Therefore we simply check which of the sorts $(x = x)$, $(x\varepsilon = 0)$, $(\varepsilon|x)$, $(x = x)/(\varepsilon|x)$ and $(x\varepsilon = 0)/(\varepsilon|x)$ annihilates M. These were the indecomposable functors for Th(Mod-R.

The only indecomposable pp-sort to annihilate M is $\frac{x\varepsilon=0}{\varepsilon|x}$, which yields $R\varepsilon/R\varepsilon$ when evaluated on R_R. Hence we have

$$\mathrm{fun}\langle M\rangle \;=\; \text{fun-}R/\langle(x\varepsilon = 0)/(\varepsilon|x)\rangle \;\simeq\; S\text{-mod}/\langle 0 \oplus S_1\rangle$$

The indecomposable S-module generating the quotient category is $T_2 = 0 \oplus S_1$. Observe that T_1, Q_2 and Q_1/T_1 all become isomorphic in the quotient category S-mod$/\langle T_2\rangle$. Hence there are only two indecomposable sorts in fun$\langle M\rangle$ up to isomorphism, namely $x = x$ and $x\varepsilon = 0$. These are both subsorts of the home sort and hence $K_0(M)$ is generated by the two indeterminates and $\mathbb{Z}$.

Chapter 7

Modules over semisimple rings

7.1 Background and definitions

Definition 7.1.1. *A ring is said to be a right(left)* semisimple ring *if it is semisimple as a right(left) module over itself, i.e. a direct sum of simple modules. A ring is right semisimple iff it is left semisimple, so it is just said to be semisimple. A ring is semisimple iff it is artinian and has zero radical.*

We will see that the Grothendieck rings of modules over semisimple rings have a particularly neat characterisation.

By definition of the multiplication operation in the Grothendieck ring of a first order structure, the rings are necessarily commutative. Given $x, y \in K_0(M)$, there are sets $A, B \in \mathrm{Def}(M)$ such that $x = [A]$ and $y = [B]$. We have $xy = [A \times B], yx = [B \times A]$ and the two cartesian products are clearly in definable bijection. Therefore their representatives in the Grothendieck ring are equal.

Every commutative ring is isomorphic to one of the form $\mathbb{Z}[X_i : i \in I]/J$ where I is an arbitrary index set for the indeterminates X_i and J is an ideal with generators among the polynomials in $\{X_i : i \in I\}$. Hence the Grothendieck ring of any first order structure can be given this presentation.

We wish to characterise the rings R and modules $M \in \text{Mod-}R$ for which $K_0(M) \cong \mathbb{Z}[X_1, \ldots, X_n]$ for some natural number n. We show in this chapter that if R is a semisimple artinian ring, then any right R-module M has such a Grothendieck ring and so does the module category Mod-R. This condition is sufficient but not necessary. For example $K_0(\text{Mod-}\mathbb{Z}_4)$ and also the Grothendieck rings of any individual infinite $\mathbb{Z}_4$-module, considered in Section 6.6, are generated as rings over the integers by finitely many indeterminates as shown in Proposition 6.6.1.

For any ring R, we have $K_0(\text{Mod-}R)$ is of the form $\mathbb{Z}[X_i : i \in I]/J$. It would be interesting to characterise the rings R with I a finite set, and further to characterise when there is no nontrivial quotient, i.e. the ideal $J = 0$.

Definition 7.1.2. *A pp-sort $F = \phi/\psi \in \mathbb{L}_R^{eq+}$ is called an* indecomposable pp-sort*, if for all pp-sorts $G, H \in \mathbb{L}_R^{eq+}$ with $F = G \oplus H$, either G or H is isomorphic to the trivial sort $v = v/v = v$.*

Definition 7.1.3. *The category fun-R is said to have* finite representation type in the basic sorts *if there are only finitely many non-isomorphic indecomposable subfunctors of powers of the forgetful functor in fun-R $= (\text{mod-}R, Ab)^{fp}$, and every subfunctor of powers of the forgetful functor is a direct sum of indecomposables.*

This characterisation is equivalent to the category $\mathbb{L}_R^{eq+}$ having finitely many isomorphism classes of indecomposable pp-sorts, each defined by a pp-pair of the form $\phi(\overline{v})/\overline{v} = \overline{0}$, and every basic sort in $\mathbb{L}_R^{eq+}$ being a direct sum of these indecomposables.

Remark. If fun-R has finite representation type, then it has finite representation type in the basic sorts. The converse does not necessarily hold.

Conjecture. The rings R for which every R-module M will have its Grothendieck ring $K_0(M)$ generated over the integers $\mathbb{Z}$ by a finite set of indeterminates $X_1, \ldots, X_n$, are exactly those rings R where the category fun-R has *finite representation type in the basic sorts.*

Partial answer. As a partial answer to the conjecture, for any ring R with fun-R of finite representation type in the basic sorts, the ring

$$K_0(\text{Mod-}R) \cong \mathbb{Z}[X_i : i \in I]/J$$

has finitely many indeterminates $\{X_i : i \in I\} = \{X_1, \ldots, X_n\}$. in Chapter 6 it is shown that for any particular module M over such a ring, $K_0(M)$ has generators among those of $K_0(\text{Mod-}R)$, meaning the representatives of the same pp-formulas. Hence $K_0(M)$ is also known to have finitely many indeterminates, at most the same $\{X_1, \ldots, X_n\}$ as in $K_0(\text{Mod-}R)$. The other direction of the conjecture remains open. For example, there might exist a module M with infinitely many non-isomorphic indecomposable pp-sets, but enough relations between their representatives in $K_0(M)$ that a cofinite subset of them may be expressed as polynomials in terms of the others. Thus the possible relations between indecomposable pp-sorts is of interest.

Theorem 7.1.4. (Krull-Schmidt Theorem) *Let N be a finite length module over a ring S. If N can be expressed in two ways as a direct sum of indecomposable S-modules; $N = \bigoplus_{i=1}^{n} P_i$ and $N = \bigoplus_{j=1}^{m} Q_j$, then $n = m$ and the $P_1, \ldots, P_n$ are a permutation of the $Q_1, \ldots, Q_n$, up to isomorphisms.*

Let $M \in \text{Mod-}R$. Let $F_1, \ldots, F_n, G_1, \ldots, G_m$ be indecomposable pp-sorts in $\mathbb{L}_R^{eq+}$. Now the $F_i(M)$ and the $G_j(M)$ are all indecomposable modules over $S = \text{End}_R(M)$. Suppose the products $\Pi_i F_i$ and $\Pi_j G_j$ are equal or isomorphic in $\mathbb{L}_R^{eq+}$. From the assumption $\Pi_i F_i \cong \Pi_j G_j$ in the category of pp-imaginaries, we have that $\bigoplus_{i=1}^{n} F_i(M) \cong \bigoplus_{j=1}^{m} G_j(M)$ is an isomorphism of S-modules. Hence by the Krull-Schmidt Theorem, $n = m$ and the list $\{F_1(M), \ldots, F_n(M)\}$ is a permutation of the list $\{G_1(M), \ldots, G_n(M)\}$ up to isomorphisms.

Thus if we consider the relations between isomorphism classes of indecomposable pp-sorts, there are no nontrivial relations of the form $\Pi_i F_i \cong \Pi_j G_j$, i.e. equalities between monomials. However, we have $K_0(M)$ equal to a quotient ring of a polynomial ring for every module M, so there may be equalities between more complicated polynomials.

For any module M with $Th(M) = Th(M)^{(\aleph_0)}$, all of the invariants $Inv(M, \phi, \psi)$ are infinite or equal to 1. For example if P is the direct sum of one model of each complete theory of right R-modules, then $T^* := Th(P) = (T^*)^{(\aleph_0)}$.

Let $F_1', \ldots, F_n', G_1', \ldots, G_m'$ be products of indecomposable pp-pairs and suppose that $\bigsqcup_{i=1}^{n} F_i'(M) = \bigsqcup_{j=1}^{m} G_j'(M)$, using parameters if necessary to ensure all unions are disjoint. Then for each $1 \leq i \leq n$, we have $F_i'(M) \subseteq \bigsqcup_{j=1}^{m} G_j'(M)$, and for each

$1 \leq j \leq m$, we have $G'_j(M) \subseteq \bigsqcup_{i=1}^{n} F'_i(M)$. When the invariants for each pp-pair are infinite or equal to 1, the index between pp-pairs can only be infinite or equal to 1, and recall that a product of pp-pairs is given by a pp-pair. Therefore Neumann's Lemma implies that $F'_i(M) = G'_j(M)$ for some $1 \leq j \leq m$. This follows from the proof of Lemma 8.2.1. Then we have an equality between products of indecomposable pp-pairs and the Krull-Schmidt Theorem implies that that the indecomposable pp-pairs themselves are equal as shown above.

The argument immediately above shows that two distinct polynomials in the indecomposable pp-sorts, $\bigsqcup_{i=1}^{n} F'_i$ and $\bigsqcup_{j=1}^{m} G'_j$, will not become identified by having isomorphic images in $\widetilde{\mathrm{Def}}(M)$ when we evaluate them at the module M. However there is a further identification in $K_0(M)$, namely that $\bigsqcup_{i=1}^{n} F'_i(M) \sim \bigsqcup_{j=1}^{m} G'_j(M)$ if we may add some set to both terms (i.e. take the disjoint union with said set) and the results are definably isomorphic. Note though that this additional equivalence under $\sim$ will cause no further identification, because the definable set that is added to both sides is represented in $K_0(M)$ by some polynomial in the original indeterminates.

Therefore for any module M with $Th(M) = Th(M)^{(\aleph_0)}$, there are no nontrivial equalities in $K_0(M)$ between polynomials from $\mathbb{Z}[X_i : i \in I]$ where I is the set of non-isomorphic indecomposable pp-pairs. In particular, this holds for the monster model P introduced in Chapter 6 with $K_0(P) =: K_0(\text{Mod-}R)$.

7.2 Modules over semisimple rings

Let R be a semisimple ring. Then there exist $t, n_1, \ldots, n_t \in \mathbb{N}$ and division rings $R_1, \ldots, R_t$ such that R is isomorphic as a ring to $M_{n_1}(R_1) \times \ldots \times M_{n_t}(R_t)$, a direct sum of matrix rings over division rings. Note that every semisimple ring is of finite representation type.

Theorem 7.2.1. *Let N be a right R module regarded as an $\mathcal{L}_R$-structure, where R is a semisimple ring. Then for some $t > 0$, $R = R_1 \times \ldots \times R_t$ and $N = N_1 \oplus \ldots \oplus N_t$, where $N_i \in$ Mod-R_i and the Grothendieck ring of N will be:*

(a) $K_0(N) = K_0(N_1) \otimes \ldots \otimes K_0(N_t) = \mathbb{Z}[X_1] \otimes \ldots \otimes \mathbb{Z}[X_t] = \mathbb{Z}[X_1, \ldots, X_t]$*, if every summand* N_i *is infinite, and*

(b) $K_0(N) = K_0(N_1) \otimes \ldots \otimes K_0(N_t) \cong \mathbb{Z}[X_1] \otimes \ldots \otimes \mathbb{Z}[X_s] \otimes \mathbb{Z} \otimes \ldots \otimes \mathbb{Z} = \mathbb{Z}[X_1, \ldots, X_s]$*, where* $s < t$ *is the number of infinite summands* N_i *of* N*, if* N *has one or more finite summand.*

Proof of Theorem 7.2.1. The theorem follows from other results of this thesis. The ring R is semisimple so it is isomorphic to a finite direct sum of matrix rings over division rings, $R = M_{n_1}(R_1) \oplus \ldots \oplus M_{n_t}(R_t)$ as a decomposition of R-modules. This t is the t in the statement of the theorem. Therefore as a product of rings $R = M_{n_1}(R_1) \times \ldots \times M_{n_t}(R_t)$. Hence every module N in Mod-R is equal to a direct sum $N = N_1 \oplus \ldots \oplus N_t$ where each N_t is a module over the ring $M_{n_i}(R_i)$. Thus Theorem 3.5.1 yields that $K_0(N) = K_0(N_1) \otimes \ldots \otimes K_0(N_t)$ where the tensor product is over the ring of integers. The theorem has two cases to consider:

(a) Suppose each of the summands $N_i | N$ is infinite. For each $i = 1, \ldots, t$, the ring of $n_i \times n_i$ matrices over R_i is Morita equivalent to R_i itself. Hence the Grothendieck rings of the respective module categories are equal by Theorem 6.3.5 proved in the sequel, $K_0(\text{Mod-}R_i) = K_0(\text{Mod-}M_{n_i}(R_i))$.

Every module over a division ring, $M \in \text{Mod-}D$, has Grothendieck ring $K_0(M) \cong \mathbb{Z}[X]$ by Theorem 4.3.1. It follows from Section 6.1, that $K_0(\text{Mod-}R_i) = \mathbb{Z}[X]$ for every $i = 1, \ldots, t$. The ring of $n_i \times n_i$ matrices over R_i is Morita equivalent to the division ring R_i itself, for each i. Hence there is an equivalence of categories Mod-$R_i \equiv$ Mod-$M_{n_i}(R_i)$, and the categories fun-R_i and fun-$M_{n_i}(R_i)$ are also equivalent by the results of Section 3.6. Therefore by Theorem 6.3.5, the Grothendieck ring of any infinite $M_{n_i}(R_i)$-module is isomorphic to $\mathbb{Z}[X]$. Hence for each $1 \leq i \leq t$, the module N_i has Grothendieck ring $K_0(N_i)$ isomorphic to $\mathbb{Z}[X]$. The tensor product over $\mathbb{Z}$ of t such polynomial rings is the polynomial ring $\mathbb{Z}[X_1, \ldots, X_t]$ as required.

(b) Suppose that one or more of the summands $N_i | N$ is finite. Let $s < t$ be the number of infinite direct summands, $N_i \in \text{Mod-}M_{n_i}(R_i)$. We have $K_0(N) = K_0(N_1) \otimes \ldots \otimes K_0(N_t)$ as above. For those i with N_i infinite, we have $K_0(N_i) \cong \mathbb{Z}[X]$

as in case *(a)*, but for those i with N_i finite, we have $K_0(N_i) \cong \mathbb{Z}$ by Lemma 2.2.5. Then the tensor product over $\mathbb{Z}$ of s copies of $\mathbb{Z}[X]$, each with a distinct indeterminate, and $t - s$ copies of $\mathbb{Z}$ is isomorphic to $\mathbb{Z}[X_1, \ldots, X_s]$ as required. $\square$

Chapter 8

Definable bijections in theories of modules

8.1 Questions of interest

One of the most important notions in the study of combinatorics with definable sets is that of definable bijections. Questions of interest include; when does a module M satisfy the condition PHP, when does it satisfy the condition *onto*PHP, and is it possible for a finite set in $\mathrm{Def}(M)$ to be sent to $0 \in K_0(M)$ by the Euler characteristic χ_0 for any modules M? It is shown in Proposition 8.1.1, that an infinite module over an infinite division ring will satisfy PHP. The author conjectures that arbitrary modules will satisfy *onto*PHP. The conjecture remains open, but a weaker result is proved in Proposition 8.2.3.

Proposition 8.1.1. *Let M be an infinite module over an infinite division ring. Then M satisfies PHP.*

Proof. Every such module has Grothendieck ring isomorphic to $\mathbb{Z}[X]$ as shown in Theorem 4.3.1. For a module over a division ring, every set in $\mathrm{Def}(M)$ is definably isomorphic to one of the *representative sets* of the polynomials in $\mathbb{Z}[X]$. Assume for a contradiction that $M \not\models PHP$. Then there exist $A, B \in \mathrm{Def}(M)$ with $A \subsetneq B$ and a definable bijection $b : A \to B$. Now there exist $f(X), g(X) \in \mathbb{Z}[X]$ such that

$A \simeq S_{f(X)}$ and $B \simeq S_{g(X)}$. Therefore $[S_f] = [S_g]$, and by Lemma 4.7.4 $f(X) = g(X)$. But $A \subsetneq B \Rightarrow A \setminus B$ is nonempty. There exists a polynomial $h(X)$ such that $A \setminus B \simeq S_{h(X)}$. Hence we have

$$A = (A \setminus B) \sqcup B \Rightarrow [A] = [A \setminus B] + [B] \Rightarrow [S_{f(X)}] = [S_{h(X)}] + [S_{f(X)}]$$

$$\Rightarrow f(X) = f(X) + h(X) \Rightarrow h(X) = 0$$

But $S_0 = \emptyset$ and $A \setminus B$ is nonempty so they cannot be in (definable) bijection. This is the desired contradiction.

□

Conjecture A: Let M be a right R-module, regarded as an $\mathcal{L}_R$-structure. Then $M \models ontoPHP$.

Conjecture B: Let M be a right R-module with $M \equiv M^{(\aleph_0)}$, regarded as an $\mathcal{L}_R$-structure. Then $M \models ontoPHP$.

This thesis contains examples of rings over which every infinite module has non-trivial Grothendieck ring, including fields and all semisimple rings. The author has not completed the calculation of any Grothendieck rings of modules and found them to be trivial, although there are cases where this possibility has not been ruled out. It is known that $K_0(M) = 0$ if and only if $M \models ontoPHP$. **Conjecture A** is that no such module exists. The author believes that the character of all definable sets and functions, which must locally have an additive group-like structure prohibits the necessary cancellation from occurring. For any definable bijection b, there is a partition of the domain of b into finitely many definable sets, on each of which the restriction of b acts as the restriction of a pp-definable map.

This conjectured result would be in stark contrast to rings and fields regarded as $\mathcal{L}_{\text{rings}}$-structures. There many such examples known to have trivial Grothendieck ring including; the p-adic integers, the p-adic numbers [12], certain $\mathbb{Z}$-valued fields and fields of formal Laurent series [11]. These are all proved by constructing a definable bijection that yields $1 = 0$ in the relevant Grothendieck rings.

8.2 Theories with $T = T^{(\aleph_0)}$

Let M be a right R-module and suppose that $\mathrm{Th}(M) = T = T^{(\aleph_0)}$. Note that the direct sum of one model of each complete theory of R-modules, denoted $P \in \text{Mod-}R$, is always such a model. But we will work with an arbitrary module with this property.

Lemma 8.2.1. *Suppose M is a right R-module such that the theory of M as an $\mathcal{L}_R$-structure satisfies $Th(M) = T = T^{(\aleph_0)}$. Let $\alpha(\overline{v}), \beta_1(\overline{v}), \ldots, \beta_m(\overline{v})$ be pp-formulas. If $M \models \forall \overline{v}(\alpha(\overline{v}) \to \bigvee_{j=1}^m \beta_j(\overline{v}))$, then for some $j \in \{1, \ldots, m\}$, $M \models \forall \overline{v}(\alpha(\overline{v}) \to \beta_j(\overline{v}))$.*

Proof. Clearly $\alpha(\overline{v}) \leftrightarrow \bigvee_{j=1}^m (\alpha \wedge \beta_j(\overline{v}))$. Let $A = \alpha(M)$ and $B_j = \beta_j(M)$. Then $A = \bigcup_j A \cap B_j$ and by Neumann's Lemma, we may discard from the union all the sets $A \cap B_j$ having infinite index in A. Since $T = T^{(\aleph_0)}$, pp-pairs must have index 1 or infinite index. Therefore the remaining $A \cap B_j$ are of index 1. This means that the additive groups (defined by parameter free formulas) are equal, so the cosets are equal or disjoint. Therefore $A \subseteq \bigcup_{j=1}^m B_j$ implies that for some $1 \leq j_0 \leq m$, $A \subseteq B_{j_0}$. Hence $T = Th(M) \models \forall \overline{v}(\alpha(\overline{v}) \to \beta_{j_0}(\overline{v}))$. □

Suppose we have an $\mathcal{L}_R$-formula ρ and the theory $T = T^{(\aleph_0)}$ models the $\mathcal{L}_R$-sentence "ρ defines the graph of a bijection". The property of defining the graph of a bijection is clearly an *elementary* one, meaning expressible in the language. Let this bijection be denoted f. If it was proven that the image of f in the module M cannot be equal to the domain less one point, then this would establish the non-triviality of the ring $K_0(M)$, by ([24], 3.2). Proposition 8.2.3 below establishes that a class of 'simple' formulas can never define such a bijection f.

Lemma 8.2.2. *Let f be a definable bijection over M and let $\rho(M, \overline{c})$ be the graph of the bijection f. If there exist pp-formulas $\phi, \psi_1, \ldots, \psi_n$ such that*

$$M \models \forall \overline{v} \left(\rho(\overline{v}, \overline{c}) \leftrightarrow (\phi \wedge \bigwedge_{i=1}^{n} \neg \psi_i)(\overline{v}, \overline{c'}) \right)$$

from some parameters $\overline{c}$ and $\overline{c'}$ then the set $\phi(M, \overline{c'})$ will be the graph of a bijection.

Proof. Fix an element $\overline{y}$ in the image of f. Then there is a unique tuple, $\overline{m}$ say, such that $M \models \rho(\overline{m}, \overline{y}, \overline{c})$. We have $\phi(M, \overline{y}, \overline{c'}) = \rho(M, \overline{y}, \overline{c}) \cup \bigcup_i \psi_i(M, \overline{y}, \overline{c'})$ by definition of

ρ and since $M \models \forall \overline{v}\,(\psi_i(\overline{v}, \overline{c'}) \to \phi(\overline{v}, \overline{c'}))$. Hence $\phi(M, \overline{y}, \overline{c'}) = \{\overline{m}\} \cup \bigcup_i \psi_i(M, \overline{y}, \overline{c'})$. Suppose the LHS of the equation is infinite. Then by Neumann's Lemma we have $\phi(M, \overline{y}, overline c') = \bigcup_i \psi_i(M, \overline{y}, \overline{c'})$, but this is a contradiction since we know $\overline{m} \in (\phi \wedge \bigwedge_{i=1}^n \neg\psi_i)(M, \overline{y}, \overline{c'})$. Therefore $\phi(M, \overline{y}, \overline{c'})$ must be finite and since $T = T^{(\aleph_0)}$, it must be the singleton $\{\overline{m}\}$. Therefore $\phi(M, \overline{y}, \overline{c'})$ is a singleton for each $\overline{y}$ in the image of f.

It follows that $\phi(M, \overline{z}, \overline{c'})$ must be a singleton for every element $\overline{z}$ in $\pi_2(\phi(M), \overline{c'})$. Assume for a contradiction that $(\overline{x}_1, \overline{z}), (\overline{x}_2, \overline{z}) \in \phi(M, \overline{c'})$ with $\overline{x}_1 \neq \overline{x}_2$. The pp-set $\phi(M, \overline{c'})$ is a coset of an additive group, which is defined by the parameter free version of ϕ. The difference of two elements in the same coset of a group, $(\overline{x}_1, \overline{z}) - (\overline{x}_1, \overline{z}) = (\overline{x}_1 - \overline{x}_2, \overline{0})$, is an element of the additive group. Adding this group element to an element $(\overline{m}, \overline{y})$ of the coset $\phi(M, \overline{c'})$ will produce another element of the same coset. Hence

$$M \models \phi(\overline{x}_1, \overline{z}, \overline{c'}) \wedge \phi(\overline{x}_1, \overline{z}, \overline{c'}) \quad \Rightarrow \quad M \models \phi(\overline{m} + \overline{x}_1 - \overline{x}_2, \overline{y}, \overline{c'})$$

This contradicts the fact that $\phi(M, \overline{y}, \overline{c'})$ is a singleton. Therefore $\phi(M, \overline{z}, \overline{c'})$ is singleton for every $\overline{z} \in \pi_2(\phi(M, \overline{c'}))$. The symmetrical argument in the other projection shows that for every $\overline{x}$ in the first projection of $\phi(M, \overline{c'})$ there is a unique $\overline{z}$ such that $M \models \phi(\overline{x}, \overline{z}, \overline{c'})$. This establishes the claim.

□

Proposition 8.2.3. *Let f be a bijection whose graph is the solution set of a formula $\rho \in \mathcal{L}_R(M)$ of the form*

$$(\phi \wedge \bigwedge_{i=1}^{n} \neg\psi_i)(\overline{v}, \overline{c})$$

Then the image of f cannot be equal to the domain less one point.

Proof. We may assume wlog that each ψ_i is $\psi_i \wedge \phi$. The solution set of ρ is unchanged under this assumption, and it simplifies the calculation to have all the subset inclusions $\psi_i(M) \subset \phi(M)$. Let π_1, π_2 denote the natural projections onto the domain and image respectively. Note that the projections of a definable set are definable and that the projections of a coset of a pp-definable group are cosets of a pp-definable group.

Lemma 8.2.2 gives that solution set of the formula $\phi(\overline{v})$ is the graph of a bijection between two cosets of pp-subgroups. The function f is a restriction of this bijection to the domain $\pi_1(\rho(M))$.

We proceed by assuming for a contradiction that there is a unique element, s say, in the domain of f and not the image. Let

$$\pi_1\rho(M) = \pi_2\rho(M) \sqcup \{s\}$$

$$\pi_1(\phi \setminus \bigcup_{i=1}^{n} \psi_i(M)) = \pi_2(\phi \setminus \bigcup_{i=1}^{n} \psi_i(M)) \cup \{s\}$$

Since $\psi_i \to \phi$ for $1 \leq i \leq n$ and the set $\phi(M)$ is the graph of a one-to-one function, this can be rearranged to yield

$$\pi_1(\phi) \setminus \left(\bigcup_i \pi_1(\psi_i)\right) = \{s\} \cup \pi_2(\phi) \setminus \left(\bigcup_i \pi_2(\psi_i)\right)$$

Some of the sets ψ_i may be singletons, but otherwise they are infinite. Since each is a subset of ϕ, the projections $\pi_1\psi_i(M)$ and $\pi_2\psi_i(M)$ must be in bijection. If we denote the singleton sets among the $\psi_i(M)$ separately by their actual elements (a_i, b_i) and denote the infinite sets as before, we obtain:

$$\pi_1(\phi) \cup \bigcup_{i=k+1}^{n} \pi_2(\psi_i) \cup \{b_1, \ldots, b_k\} = \{s, a_1, \ldots, a_k\} \cup \pi_2(\phi) \cup \bigcup_{i=k+1}^{n} \pi_1(\psi_i)$$

after reordering.

Now $\pi_1(\phi) \subseteq \{s, a_1, \ldots, a_k\} \cup \pi_2(\phi) \cup \bigcup_{i=k+1}^{n} \pi_1(\psi_i)$. Observe that

$$\pi_1(\phi(M) \setminus \bigcup_{i=1}^{n} \psi_i(M)) \neq \emptyset \quad \Rightarrow \quad \pi_1\psi_i(M) \subsetneq \pi_1\phi(M)$$

and similarly for π_2. Thus the condition $M \equiv M^{(\aleph_0)}$ implies that $Inv(M, \pi_1\phi, \pi_1\psi_i)$ is infinite for every i, and Neumann's Lemma implies

$$\pi_1(\phi) \subseteq \{s, a_1, \ldots, a_k\} \cup \pi_2(\phi) \cup \bigcup_{i=k+1}^{n} \pi_1(\psi_i) \quad \Rightarrow \quad \pi_1\phi = \pi_2\phi$$

Hence we have:

$$\pi_1(\phi) \setminus \left(\bigcup_i \pi_1(\psi_i)\right) = \{s\} \cup \pi_2(\phi) \setminus \left(\bigcup_i \pi_2(\psi_i)\right)$$

$\Rightarrow$

$$\pi_1(\phi) \setminus \left(\bigcup_i \pi_1(\psi_i) \right) = \{s\} \cup \pi_1(\phi) \setminus \left(\bigcup_i \pi_2(\psi_i) \right)$$

$\Rightarrow$

$$\bigcup_i \pi_2(\psi_i) = \{s\} \cup \bigcup_i \pi_1(\psi_i)$$

$\Rightarrow$

$$\bigcup_{i=k+1}^{n} \pi_2(\psi_i) \cup \{b_1, \dots, b_k\} = \{s, a_1, \dots, a_k\} \cup \bigcup_{i=k+1}^{n} \pi_1(\psi_i)$$

For each $k < i \leq n$ the coset $\pi_2(\psi_i(M))$ is infinite and we have $\pi_2(\psi_i(M)) \subseteq \{s, a_1, \dots, a_k\} \cup \bigcup_{i=k+1}^{n} \pi_1(\psi_i(M))$. Hence by Lemma 8.2.1, $\pi_2(\psi_i(M)) = \pi_1(\psi_j(M))$ for some $k < j \leq n$. Also for every $k < i \leq n$ there exists $k < j \leq n$ such that $\pi_1(\psi_i(M)) = \pi_2(\psi_j(M))$. Therefore $\bigcup_{i=k+1}^{n} \pi_2(\psi_i)(M) = \bigcup_{i=k+1}^{n} \pi_1(\psi_i)(M)$, and we have

$$\bigcup_{i=k+1}^{n} \pi_1(\psi_i(M)) \cup \{b_1, \dots, b_k\} = \{s, a_1, \dots, a_k\} \cup \bigcup_{i=k+1}^{n} \pi_1(\psi_i(M))$$

Recall $s \in \pi_1(\phi(M)) \setminus (\bigcup_i \pi_1(\psi_i(M)))$ and therefore $s \notin \bigcup_{i=k+1}^{n} \pi_1(\psi_i(M))$.

Claim. For each $1 \leq j \leq k$, $a_j \in \bigcup_{i=k+1}^{n} \pi_1(\psi_i)$ iff $b_j \in \bigcup_{i=k+1}^{n} \pi_1(\psi_i)$.

Proof. If $a_j \in \pi_1(\psi_h)$ for some $h > k$, then $a_j \in \pi_1(\psi_h)(M) \cap \pi_1(\psi_j)(M)$. Hence there exist $x, y \in M$ such that $M \models \psi_h(a_j, x) \wedge \psi_j(a_j, y)$. This in turn implies $M \models \phi(a_j, x) \wedge \phi(a_j, y)$. Now $\phi(M)$ is the graph of a bijection, so $x = y$. Since $j \leq k$, $\psi_j(M) = \{(a_j, b_j)\}$ so $x = y = b_j$ and $\psi_j(M) \subset \psi_h(M)$. Thus

$$b_j \in \pi_2 \psi_h(M) \subseteq \bigcup_{i=k+1}^{n} \pi_2(\psi_i)(M) = \bigcup_{i=k+1}^{n} \pi_1(\psi_i)(M)$$

The converse is proved by a symmetrical argument, and hence the claim holds.

$\square$

This claim implies that we may reorder our indexing again and obtain

$$\bigcup_{i=k'+1}^{n} \pi_2(\psi_i) \sqcup \{b_1, \dots, b_{k'}\} = \{s, a_1, \dots, a_{k'}\} \sqcup \bigcup_{i=k'+1}^{n} \pi_1(\psi_i)$$

But this implies

$$\{b_1, \dots, b_{k'}\} = \{s, a_1, \dots, a_{k'}\}$$

Recall that $a_i = a_{i'}$ iff $b_i = b_{i'}$ since $M \models \psi_i(a_i, b_i) \Rightarrow M \models \phi(a_i, b_i)$ and ϕ defines a bijection. But $s \neq a_i$ for every $i \leq k$ because s is in the domain of the bijection f (it is the unique point in the domain but not the image) and $a_i \in \pi_1\psi_i(M)$ is not. In fact $\{a_i\} = \pi_1\psi_i(M)$. Thus we have the desired contradiction since the finite sets are of different sizes. Thus we have proved Proposition 8.2.3. □

Remarks. Over a module M, every definable bijection f will have some formula ρ, of the form $\bigvee_{j=1}^{m}(\phi_j \wedge \bigwedge_{i=1}^{n_j} \neg\psi_{ji})$, with solution set $\rho(M)$ equal to the graph of f. A proof has not been found, in this full generality, that f cannot map a set onto itself minus a point. Proposition 8.2.3 gives the desired result for $j = 1$ for modules with $Th(M) =: T = T^{(\aleph_0)}$.

A positive answer to **Conjecture A** would imply that for every module M over every ring R, the Grothendieck ring $K_0(M)$ is nontrivial. A positive answer to the weakened version, **Conjecture B**, would imply that for every ring R, the Grothendieck ring of the category of right R-modules $K_0(\text{Mod-}R)$ is nontrivial, and moreover that certain other modules have nontrivial Grothendieck rings. Note that this follows immediately from a positive answer to **Conjecture A**. Furthermore a proof of either conjecture might, depending on the nature of the proof, extend to a proof that not only is $1 \neq 0$ in the Grothendieck ring, but $1 + \ldots + 1 \neq 0$ in the Grothendieck ring, for every finite sum. This would imply that not only is the Grothendieck ring nontrivial but it includes $\mathbb{Z}$ as a subring.

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