Branching Lévy Processes with inhomogeneous breeding potentials

submitted by

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for the degree of Doctor of Philosophy

of the

University of Bath

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July 2012

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Summary

The object of study in this thesis is a number of different models of branching Lévy processes in inhomogeneous breeding potential. We employ some widely-used spine techniques to investigate various features of these models for their subsequent comparison. The thesis is divided into 5 chapters.

In the first chapter we introduce the general framework for branching Markov processes within which we are going to present all our results.

In the second chapter we consider a branching Brownian motion in the potential $\beta|\cdot|^p$, $\beta > 0$, $p \geq 0$. We give a new proof of the result about the critical value of p for the explosion time of the population. The main advantage of the new proof is that it can be easily generalised to other models.

The third chapter is devoted to continuous-time branching random walks in the potential $\beta|\cdot|^p$, $\beta > 0$, $p \geq 0$. We give results about the explosion time and the rightmost particle behaviour comparing them with the known results for the branching Brownian motion.

In the fourth chapter we look at general branching Lévy processes in the potential $\beta|\cdot|^p$, $\beta > 0$, $p \geq 0$. Subject to certain assumptions we prove some results about the explosion time and the rightmost particle. We exhibit how the corresponding results for the branching Brownian motion and and the branching random walk fit into the general structure.

The last chapter considers a branching Brownian motion with branching taking place at the origin on the local time scale. We present some results about the population dynamics and the rightmost particle behaviour. We also prove the Strong Law of Large Numbers for this model.

Acknowledgements

Firstly, I would like to thank my supervisor Simon Harris, who introduced me to the branching processes and was keeping me interested in the subject all these years.

I would also like to thank all the members of the Probability Laboratory and, more generally, all the members of the Mathematical Department for maintaining a great academic atmosphere. A separate word of thanks goes to Prof. Andreas Kyprianou, who has shown an interest in my research and provided me with some useful comments.

I would certainly like to thank my parents for their support.

I would like to thank my office mates: Susan, Tom, Jensen, Parkpoom, Marion, Chris, Kuntalee and all the new people for being nice office mates.

I would like to thank Ray, Matt, Geoff, Fynn, Euan, Melina, Kristian, Vaios, Curdin, Elvijs, James C., James L., Chris H. and about a hundred others for all the good time we've had together.

My final word of thanks goes to the nameless staff members at the Parade Bar for the high-quality service.

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Chapter 1

Introduction

This thesis is devoted to the study of branching Lévy processes. They are a natural generalisation of Branching Brownian Motion (BBM), a model extensively studied over the last few decades. Let us mention the paper of H.P. McKean [26] from 1975 as one of the earliest works in this subject.

In this chapter we give some general definitions and state a few fundamental results valid within an even bigger class of branching Markov processes. The major reference for this chapter is the work of Hardy and Harris [19], where all the proofs and further references can be found. We shall use the BBM model whenever we need an example to illustrate some general idea.

1.1 Some definitions and notation

Let us begin with the description of a general branching Markov process, which is a sufficiently large class of processes for us to consider.

Initially we have one particle at position x. It moves in space according to a certain Markov process. If it has position X_t at time t then it splits at instantaneous rate $\beta(X_t)$ at time t and we assume that $\int_0^t \beta(X_s)\mathrm{d}s$ is well-defined for all $t > 0$. The function $\beta(\cdot)$ is called the branching rate (or the potential).

By splitting we mean that the original particle dies, and at the position where it died it is replaced with a number of new particles (children). If the death occured at location y then the number of children is $1 + A(y)$, where $A(y)$ is a random variable with the probability distribution given by

$$\mathbb{P}(A(y) = k) = p_k(y), \text{ where } k \in \{0, 1, 2, ...\}. \tag{1.1}$$

Each new particle then, independently of the others and of its parent, stochastically repeats the behaviour of the initial particle.

So if the initial particle moved like a Brownian motion in $\mathbb{R}$ we would roughly see a picture as in Figure 1-1 below.

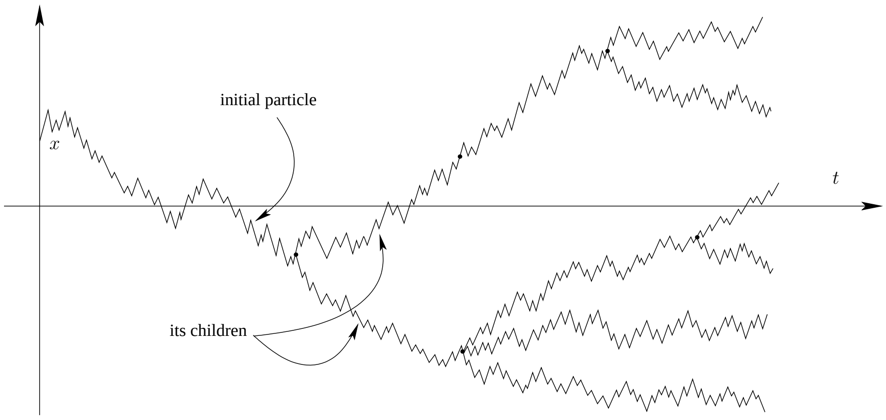


Figure 1-1: *Branching process*

Remark 1.1. *Let us note that the number of children of any particle is ≥ 1. Thus the process is guaranteed to survive and we don't need to worry about its extinction.*

We reserve the letter m for the mean of $A(\cdot)$:

$$m(y) := \mathbb{E}A(y) = \sum_{k\geq 0} kp_k(y). \tag{1.2}$$

In our later applications we shall always take $A(\cdot)$ to be spatially independent. That is, $A(y) = A$ for all y and consequently $m(y) = m$ for all y.

Example 1.2. *It is possible to simplify the model even further by taking $A \equiv 1$. In such model each particle when it dies produces exactly two children. The corresponding branching process is then called binary or dyadic branching process. In this case $m = 1$.*

Example 1.3. *One should also keep in mind the degenerate case when $A \equiv 0$ or, equivalently, $m = 0$. In this instance a particle when it dies has always only one descendant, so the branching process reduces to a single-particle Markov process $(X_t)_{t\geq 0}$. The same is true if the branching rate satisfies $\beta(\cdot) \equiv 0$.*

Remark 1.4. *We said that $\beta(\cdot)$ is the instantaneous branching rate. It means that conditional on its path the (initial) particle will not split by time t with probability $e^{-\int_0^t \beta(X_s)\mathrm{d}s}$. Or, if we take away the conditioning, the actual probability of this event is $\mathbb{E}\Big(e^{-\int_0^t \beta(X_s)\mathrm{d}s}\Big)$.*

Alternatively, given that the particle is at position X_t at time t, the probability that it splits in the time interval $[t, t+h)$ is $\beta(X_t)h + o(h)$.

Sometimes it makes more sense to talk about the cumulative branching rate which equals to $\int_0^t \beta(X_s)\mathrm{d}s$.

In the simplest models $\beta(\cdot)$ is a constant function. That is, $\beta(\cdot) \equiv \beta$.

In all our applications the underlying Markov process will be an $\mathbb{R}$-valued Lévy process.

We label particles according to the Ulam-Harris convention. That is, we call the original particle $\varnothing$. Its children are then labelled 1, 2, 3, ... and children of particle u ($\neq \varnothing$) are labelled $u1$, $u2$, $u3$, ... So e.g. a particle with label 132 would be the second child of the third child of the first child of the initial ancestor $\varnothing$.

For two labels v and u we shall write $v < u$ to indicate that v is an ancestor of u (but not u itself). We shall say $v \leq u$ if $v < u$ or $v = u$. We shall also write $|u|$ for the generation of u. E.g. $|\varnothing| = 0$, $|132| = 3$.

Definition 1.5. *N_t is the set of (labels of) particles alive at time t.*

Below we give an illustration of the last couple of paragraphs.

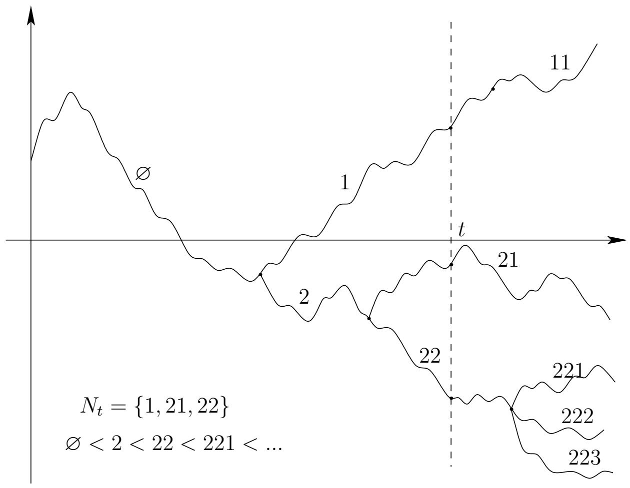


Figure 1-2: *Particle labelling*

Let us introduce some more notation that we are going to need.

Definition 1.6.

- X^u_t *is the position of a particle u ($\in N_t$) at time t*
- S_u *is the fission time (or death time) of particle u*
- σ_u *is the lifetime of particle u, so $S_u = \sum_{v \leq u} \sigma_v$*
- $A_u := (\#$ *of children of particle u*$) - 1$
- $\left(X^u_s\right)_{0 \leq s \leq t}$ *is the path of particle $u \in N_t$. That is, for $0 \leq s \leq t$ we take X^u_s to be the position of the unique ancestor of u alive at time s*
- $\mathbb{X}_t := \{(u, X^u_t) : u \in N_t\}, \ t \geq 0$

To the branching process $\mathbb{X}$ we associate the probability measure P^x (where x is the starting position of the first particle). We also let $\big(\mathcal{F}_t\big)_{t\geq 0}$ denote the natural filtration of our branching process, so that $\mathcal{F}_t$ contains the information about the paths of all the particles as well as their genealogy up to time t. Formally, we define

$$\mathcal{F}_t := \sigma\Big(\big(u, X^u, \sigma_u\big) : S_u \leq t; \big(u, X^u_s : s \in [S_u - \sigma_u, t)\big) : t \in [S_u - \sigma_u, S_u)\Big).$$

As always we write $\mathcal{F}_\infty$ for $\sigma\big(\cup_{t\geq 0} \mathcal{F}_t\big)$.

1.2 Spines

Description of the process given in the previous section is sufficient for understanding the questions studied in this thesis. However all these questions will be answered using different spine techniques. In this section we shall introduce spines.

Definition 1.7. *A set $\xi = \{\emptyset,\ u_1,\ u_2,\ u_3,\ ...\ \}$ is a spine if u_1 is a child of $\emptyset$, u_2 is a child of u_1, u_3 is a child of u_2, etc. In other words, a spine is a distinguished infinite line of descent.*

Definition 1.8. *For a given spine ξ we define the process*

$$\xi_t := X^u_t \quad \text{if } u \in N_t \cap \xi \quad , \quad t \geq 0.$$

That is, $\big(\xi_t\big)_{t\geq 0}$ is the path corresponding to spine ξ. (Note that $N_t \cap \xi$ always has exactly one element in it.)

$\xi = \{\emptyset,\ 2,\ 22,\ 221,\ ...\ \}$ as shown in Figure 1-3 below is a spine. Its path $(\xi_t)_{t\geq 0}$ is drawn with the bold line.

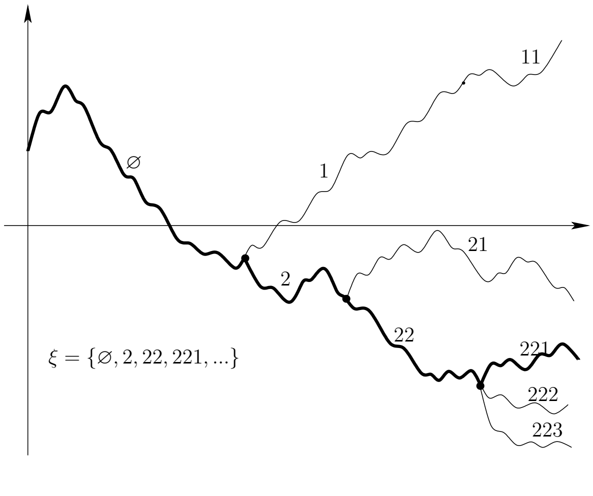


Figure 1-3: *An example of a spine*

We can extend the original branching process by identifying a spine. Our sample space is therefore the space of all possible realisations of $\mathbb{X}$ with a distinguished spine ξ. Let us mention a couple more quantities, which we associate to a spine.

Definition 1.9. *For a given spine ξ we shall write $node_t(\xi)$ for the unique particle u in $N_t \cap \xi$. So that $node_t(\xi)$ is the particle in the spine that is alive at time t.*

We also define n_t to be the number of fissions that have occured along the path of the spine by time t, so $n_t = |node_t(\xi)|$.

The spine process that we shall always assume can be described as follows. We start with the initial particle $\varnothing$. Whenever the current particle of the spine splits, we choose one of its children uniformly at random to continue the spine. One important observation is that for a particle $u \in N_t$

$$P^x(u \in \xi \big| \mathcal{F}_t) = \prod_{v<u} \frac{1}{1 + A_v}. \tag{1.3}$$

In the special case of binary branching

$$P^x(u \in \xi \big| \mathcal{F}_t) = 2^{-|u|}. \tag{1.4}$$

In the Figure 1-4 below we show in brackets the probability of particles belonging to the spine.

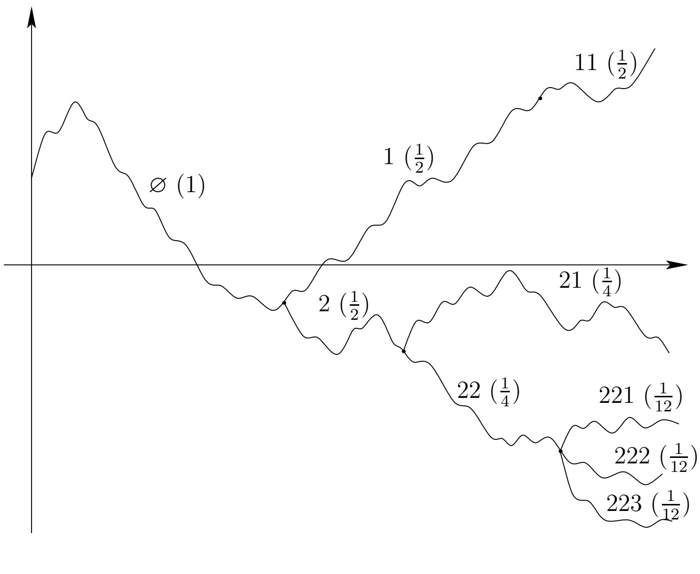


Figure 1-4: *Particle weights*

Definition 1.10. *$\tilde{P}^x$ is the extension of the probability measure P^x, under which the spine is chosen uniformly as described above.*

Hence $P^x = \tilde{P}^x|_{\mathcal{F}_\infty}$. We shall write $\tilde{E}^x$ for the expectation with respect to $\tilde{P}^x$ and E^x for the expectation with respect to P^x.

For more details about the spine construction one should see [19].

Remark 1.11. *Under the probability measure $\tilde{P}^x$ the spine process $\big(\xi_t\big)_{t\geq 0}$ has the same distribution as the Markov process $\big(X_t\big)_{t\geq 0}$ (corresponding to the motion of a single particle in the branching system).*

The next important step is to define a number of filtrations of our sample space, which contain different information about the process.

Definition 1.12 (Filtrations)**.**

- *$\mathcal{F}_t$ was defined earlier. It is the filtration which knows everything about the particles' motion and their genealogy, but it knows nothing about the spine.*
- *We also define $\tilde{\mathcal{F}}_t := \sigma\big(\mathcal{F}_t, node_t(\xi)\big)$. Thus $\tilde{\mathcal{F}}$ has all the information about the process including all the information about the spine. This will be the largest filtration.*
- *$\mathcal{G}_t := \sigma\big(\xi_s : \ 0 \leq s \leq t\big)$. This filtration only has information about the path of the spine process, but it can't tell which particle $u \in N_t$ is the spine particle at time t.*
- *$\tilde{\mathcal{G}}_t := \sigma\big(\mathcal{G}_t, \ (node_s(\xi) : 0 \leq s \leq t), \ (A_u : u < \xi_t)\big)$. This filtration knows everything about the spine including which particles make up the spine and how many children they have, but it doesn't know what is happening off the spine.*

We shall use these filtrations extensively for taking various conditional expectations. Let us note that $\mathcal{G}_t \subset \tilde{\mathcal{G}}_t \subset \tilde{\mathcal{F}}_t$ and $\mathcal{F}_t \subset \tilde{\mathcal{F}}_t$.

We finish this section with a couple of important observations.

Proposition 1.13. *Under $\tilde{P}^x$, conditional on the path of the spine, $(n_t)_{t\geq 0}$ is an inhomogeneous Poisson process with instantaneous jump rate $\beta(\xi_t)$. So conditional on $\mathcal{G}_t$, k splits take place along the spine by time t with probability*

$$\tilde{P}^x(n_t = k|\mathcal{G}_t) = \frac{(\int_0^t \beta(\xi_s)\mathrm{d}s)^k}{k!} e^{-\int_0^t \beta(\xi_s)\mathrm{d}s},$$

or, taking away the conditioning,

$$\tilde{P}^x(n_t = k) = \tilde{E}^x\Big(\frac{(\int_0^t \beta(\xi_s)\mathrm{d}s)^k}{k!} e^{-\int_0^t \beta(\xi_s)\mathrm{d}s}\Big).$$

Proposition 1.14. *Under $\tilde{P}^x$ the entire branching process (with the spine) can be described in the following way.*

- *the initial particle (the spine) moves like some given Markov process.*
- *At instantaneous rate $\beta(\cdot)$ it splits into a random number of particles.*
- *The number of particles has the distribution of a random variable $A(\cdot)$*
- *One of these particles (chosen uniformly at random) continues the spine. That is, it continues moving as the given Markov process and branching at rate $\beta(\cdot)$.*
- *The other particles initiate new independent P-branching processes from the position of the split*

1.3 Many-to-One theorem

The first very useful tool that we mention is the Many-to-One theorem. Let us state it in its general form as it was stated in [19].

Theorem 1.15 (Many-to-One)**.** *Let $f(t) \in m\mathcal{G}_t$. In other words, $f(t)$ is $\mathcal{G}_t$-measurable. Suppose it has the representation*

$$f(t) = \sum_{u \in N_t} f_u(t) \mathbf{1}_{\{node_t(\xi)=u\}},$$

where $f_u(t) \in m\mathcal{F}_t$, then

$$E^x\Big(\sum_{u \in N_t} f_u(t)\Big) = \tilde{E}^x\Big(f(t) e^{\int_0^t m(\xi_s)\beta(\xi_s)\mathrm{d}s}\Big).$$

Remark 1.16. *It was shown in the recent PhD thesis of M. Roberts [29] that any $f(t) \in m\mathcal{G}_t$ has the required representation*

$$f(t) = \sum_{u \in N_t} f_u(t) \mathbf{1}_{\{node_t(\xi)=u\}},$$

where $f_u(t) \in m\mathcal{F}_t$.

Here f is some functional $\hat{f}$ of the spine's path $\big(\xi_s\big)_{0 \le s \le t}$. That is, $f(t) = \hat{f}((\xi_s)_{s \in [0,t]})$ and f_u is the same functional of the path $\big(X_s^u\big)_{0 \le s \le t}$ of a particle $u \in N_t$. That is, $f_u(t) = \hat{f}((X_s^u)_{s \in [0,t]})$. Therefore the theorem reduces the expectation of a sum over particles $u \in N_t$ of functionals of paths of those particles to the expectation of a functional of only one particle.

Let us give a couple of examples to make things more clear.

Example 1.17.

- *Take* $f(t) = e^{\int_0^t \alpha(\xi_s)\mathrm{d}s}$ *for some function* α*. Then*

$$E^x\Big(\sum_{u\in N_t} e^{\int_0^t \alpha(X_s^u)\mathrm{d}s}\Big) = \tilde{E}^x\Big(e^{\int_0^t \alpha(\xi_s)+m(\xi_s)\beta(\xi_s)\mathrm{d}s}\Big).$$

- *Take* $f(t) = \mathbf{1}_{\{\sup_{s\in[0,t]}|\xi_s|\ \le\ c\}}$ *for some number* c*. Then*

$$E^x\Big(\sum_{u\in N_t} \mathbf{1}_{\{\sup_{s\in[0,t]}|X_s^u|\ \le\ c\}}\Big) = \tilde{E}^x\Big(\big(\mathbf{1}_{\{\sup_{s\in[0,t]}|\xi_s|\ \le\ c\}}\big)e^{\int_0^t m(\xi_s)\beta(\xi_s)\mathrm{d}s}\Big).$$

In the special case when the functional $f_u(t)$ only depends on the position of a particle u at time t (i.e. the endpoint of the path $(X_s^u)_{0\le s\le t}$), the Many-to-One theorem takes the following form:

Lemma 1.18 (special case of Many-to-One)**.** *Let* g *be some measurable function, then*

$$E^x\Big(\sum_{u\in N_t} g(X_t^u)\Big) = \tilde{E}^x\Big(g(\xi_t)e^{\int_0^t m(\xi_s)\beta(\xi_s)\mathrm{d}s}\Big).$$

Often we take g to be an indicator function of some event.

1.4 Additive martingales and changes of measure

In this section we give a construction of additive martingales, another very useful tool in the study of branching processes. One of the first mentions of these objects can be found in the paper of McKean [26]. They have been used vastly since then (see for example [27], [15] or [21]).

A typical additive martingale has the form

$$M_t = \sum_{u\in N_t} e^{-\int_0^t m(X_s^u)\beta(X_s^u)\mathrm{d}s} M_t^u\ , \qquad t\ge 0,$$

where M^u's are single-particle martingales w.r.t $(X_s^u)_{0\le s\le t}$. In the rest of this section we give a detailed sketch of the construction of $(M_t)_{t\ge 0}$.

From Proposition 1.13 we know that under $\tilde{P}^x$ the process $(\xi_t)_{t\ge 0}$ moves as some Markov process, and, conditional on the path of this process, $(n_t)_{t\ge 0}$ is a Poisson process with cumulative jump rate $\int_0^t \beta(\xi_s)\mathrm{d}s$. The following proposition as well as the whole subsequent construction in greater detail can be found in [19].

Proposition 1.19 (Scaling the birth rate along the spine)**.**

$$\tilde{M}_t^{(1)} := \Big(\prod_{v<node_t(\xi)} \big(1+m(\xi_{S_v})\big)\Big)e^{-\int_0^t m(\xi_s)\beta(\xi_s)\mathrm{d}s}\ , \qquad t\ge 0$$

is a martingale with respect to probability measure $\tilde{P}^x$ and filtration $(\tilde{\mathcal{G}}_t)_{t\geq 0}$. (Here $m(\cdot)$ is the mean of $A(\cdot)$ as in (1.2).)

If we define the new measure $\tilde{Q}_1^x$ via the Radon-Nikodym derivative

$$\frac{\mathrm{d}\tilde{Q}_1^x}{\mathrm{d}\tilde{P}^x}\bigg|_{\tilde{\mathcal{F}}_t} = \tilde{M}_t^{(1)} \ , \qquad t \geq 0$$

then under $\tilde{Q}_1^x$ the process $(n_t)_{t\geq 0}$, conditional on $\mathcal{G}_\infty$, becomes a Poisson process with cumulative jump rate $\int_0^t (m(\xi_s)+1)\beta(\xi_s)\mathrm{d}s$. That is,

$$\tilde{Q}_1^x(n_t = k|\mathcal{G}_t) = \frac{\left(\int_0^t (m(\xi_s)+1)\beta(\xi_s)\mathrm{d}s\right)^k}{k!} e^{-\int_0^t (m(\xi_s)+1)\beta(\xi_s)\mathrm{d}s}.$$

Example 1.20 (Binary branching)**.** *In the case of binary branching we have $m \equiv 1$. Therefore*

$$\tilde{M}_t^{(1)} = 2^{n_t} e^{-\int_0^t \beta(\xi_s)\mathrm{d}s} \ , \qquad t \geq 0,$$

$$\tilde{Q}_1^x(n_t = k|\mathcal{G}_t) = \frac{\left(\int_0^t 2\beta(\xi_s)\mathrm{d}s\right)^k}{k!} e^{-\int_0^t 2\beta(\xi_s)\mathrm{d}s},$$

so $\tilde{Q}_1^x$ simply doubles the jump rate of $(n_t)_{t\geq 0}$.

Proposition 1.21 (Biasing family sizes along the spine)**.**

$$\tilde{M}_t^{(2)} := \prod_{v < node_t(\xi)} \frac{1+A_v}{1+m(\xi_{S_v})} \ , \qquad t \geq 0$$

is also a $\tilde{P}^x$-martingale. (Here $1 + A_v$ is the number of children of particle v.) If we define the new measure $\tilde{Q}_2^x$ as

$$\frac{\mathrm{d}\tilde{Q}_2^x}{\mathrm{d}\tilde{P}^x}\bigg|_{\tilde{\mathcal{F}}_t} = \tilde{M}_t^{(2)} \ , \qquad t \geq 0$$

then under $\tilde{Q}_2^x$ the random variables A_v change their distribution in the following way:

$$\textit{if } v < \xi_t \textit{ then } Prob(A_v = k) = \frac{(1+k)p_k(\xi_{S_v})}{1+m(\xi_{S_v})} \ , \ k \in \{0, 1, 2, ...\}.$$

(Formally by $Prob(\ \cdot\ )$ we mean $\tilde{Q}_2^x\big(\ \cdot \ \mid \sigma(\mathcal{G}_t, node_s(\xi) : 0 \leq s \leq t)\big)$)

Example 1.22 (Binary branching)**.** *In the case of binary branching $A(\cdot) \equiv 1$, $m(\cdot) \equiv 1$ and therefore $\tilde{M}_t^{(2)} \equiv 1$, so no changes take place.*

Also suppose that we are given some mean-one positive $\tilde{P}^x$-martingale $(\tilde{M}_t^{(3)})_{t\geq 0}$ with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$, the natural filtration of $(\xi_t)_{t\geq 0}$. We use it to define the new measure $\tilde{Q}_3^x$ via the Radon-Nikodym derivative:

$$\frac{\mathrm{d}\tilde{Q}_3^x}{\mathrm{d}\tilde{P}^x}\bigg|_{\mathcal{F}_t} = \tilde{M}_t^{(3)}\ , \qquad t \geq 0, \tag{1.5}$$

Suppose that under $\tilde{Q}_3^x$ the spine moves like some new Markov process. Let us illustrate this with a classical example.

Example 1.23 (BBM)**.** *If the spine process $(\xi_t)_{t\geq 0}$ is a (standard) Brownian motion started from 0 then we can take $\tilde{M}^{(3)}$ to be a Girsanov martingale. Namely for some path $(\gamma(t))_{t\geq 0}$ such that $\int_0^t \gamma(s)^2 \mathrm{d}s < \infty\ \forall t \geq 0$ we can take*

$$\tilde{M}_t^{(3)} = e^{\int_0^t \gamma(s)\mathrm{d}\xi_s - \frac{1}{2}\int_0^t \gamma(s)^2 \mathrm{d}s}\ , \qquad t \geq 0.$$

Then under $\tilde{Q}_3^0$ the spine process $(\xi_s)_{0\leq s\leq t}$ moves like a (standard) Brownian motion with drift $\int_0^t \gamma(s)\mathrm{d}s$.

Given such martingales $\tilde{M}^{(1)}$, $\tilde{M}^{(2)}$ and $\tilde{M}^{(3)}$ we have the following result.

Proposition 1.24.

$$\tilde{M}_t := \tilde{M}_t^{(1)}\tilde{M}_t^{(2)}\tilde{M}_t^{(3)}\ , \qquad t \geq 0$$

is a martingale w.r.t the probability measure $\tilde{P}^x$ and filtration $(\tilde{\mathcal{G}}_t)_{t\geq 0}$. Moreover, probability measure $\tilde{Q}^x$ defined as

$$\frac{\mathrm{d}\tilde{Q}^x}{\mathrm{d}\tilde{P}^x}\bigg|_{\tilde{\mathcal{F}}_t} = \tilde{M}_t = \prod_{v<node_t(\xi)} (1+A_v)e^{-\int_0^t m(\xi_s)\beta(\xi_s)\mathrm{d}s}\tilde{M}_t^{(3)}\ , \qquad t \geq 0 \tag{1.6}$$

has the effect of changing the motion of the spine in space (according to the martingale $\tilde{M}^{(3)}$) as well as scaling the birth rate along the spine (according to the martingale $\tilde{M}^{(1)}$) and size-biasing the families along the spine (according to the martingale $\tilde{M}^{(2)}$).

Under $\tilde{Q}^x$ the behaviour of the whole branching process (with the spine) can be described in the following way.

Proposition 1.25 (Branching process under $\tilde{Q}^x$)**.**

- *The initial particle (the spine) moves like the measure-changed Markov process.*
- *At instantaneous rate $\big(1+m(\cdot)\big)\beta(\cdot)$ it splits into a random number of particles.*
- *The number of particles follows the distribution*

$$\Big(\frac{(1+k)p_k(\cdot)}{1+m(\cdot)}\ :\ k = 0, 1, 2, ...\Big).$$

- *One of these particles (chosen uniformly at random) continues the spine. I.e. it continues moving as the measure-changed Markov process and branching at rate* $\big(1+m(\cdot)\big)\beta(\cdot)$ *producing a biased number of children.*

- *The other particles initiate unbiased branching processes from the position of the split*

Remark 1.26. *Note that although* (1.6) *only defines* $\tilde{Q}^x$ *on events in* $\cup_{t\geq 0}\tilde{\mathcal{F}}_t$*, Carathéodory's extension theorem tells that* $\tilde{Q}^x$ *has a unique extension on* $\tilde{\mathcal{F}}_\infty := \sigma(\cup_{t\geq 0}\tilde{\mathcal{F}}_t)$ *and thus* (1.6) *implicitly defines* $\tilde{Q}^x$ *on* $\tilde{\mathcal{F}}_\infty$*.*

Example 1.27 (Binary BBM)**.** *Let us consider a simple model of BBM with binary branching and homogeneous branching rate* $\beta(\cdot) \equiv \beta$ *with the initial particle started from* 0*. For some* $\gamma > 0$ *let* $\tilde{M}^{(3)}_t = e^{\gamma\xi_t - \frac{1}{2}\gamma^2 t}$*,* $t \geq 0$ *and*

$$\tilde{M}_t = 2^{n_t} e^{-\beta t} e^{\gamma\xi_t - \frac{1}{2}\gamma^2 t}\ , \qquad t \geq 0.$$

Then under $\tilde{Q}^0$ *the spine process moves as a Brownian motion with (instantaneous) linear drift* γ*. Births occur along the spine at rate* 2β *and each time two children are born of which one continues the spine and the other starts an unbiased branching process. An illustration is given in Figure 1-5 below.*

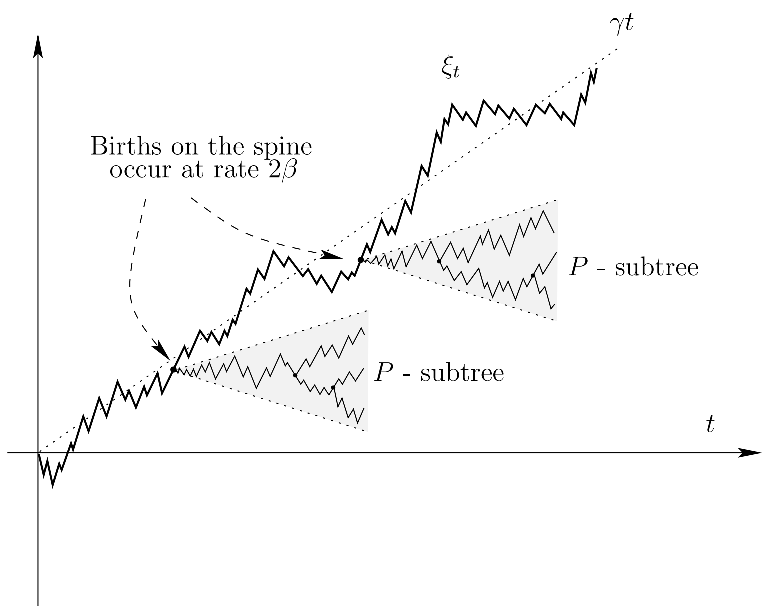


Figure 1-5: *Branching process under* $\tilde{Q}^0$

Note that $\tilde{M}^{(3)}$ must be some function of $(\xi_s)_{0\leq s\leq t}$. For each particle $u \in N_t$ let us denote by $(M^u_s)_{0\leq s\leq t}$ the same function of $(X^u_s)_{0\leq s\leq t}$.

Thus $\tilde{M}_t$ (recall equation (1.6)) has the following representation

$$\tilde{M}_t = \sum_{u\in N_t}\prod_{v<u}(1+A_v)e^{-\int_0^t m(X_s^u)\beta(X_s^u)\mathrm{d}s}M_t^u \mathbf{1}_{\{\xi_t=u\}}.$$

Then, by projecting $\tilde{M}$ onto the filtration $\mathcal{F}$ and recalling (1.3) we get the following martingale w.r.t $\mathcal{F}$ and P^x.

$$\begin{aligned} M_t &:= \tilde{E}^x\left(\tilde{M}_t|\mathcal{F}_t\right) \\ &= \sum_{u\in N_t} M_t^u e^{-\int_0^t m(X_s^u)\beta(X_s^u)\mathrm{d}s} \times \prod_{v<u}(1+A_v)\times \tilde{E}^x\Big(\mathbf{1}_{\{\xi_t=u\}}|\mathcal{F}_t\Big) \\ &= \sum_{u\in N_t} M_t^u e^{-\int_0^t m(X_s^u)\beta(X_s^u)\mathrm{d}s}. \end{aligned} \tag{1.7}$$

Martingales of the form (1.7) will be referred to as additive martingales. Note that by the Many-to-One Theorem (Theorem 1.15) $E(M_t) = 1$.

Finally let us note that if we define $Q^x := \tilde{Q}^x|_{\mathcal{F}_\infty}$, where $\mathcal{F}_\infty = \sigma(\cup_{t\geq 0}\mathcal{F}_t)$, then

$$\left.\frac{\mathrm{d}Q^x}{\mathrm{d}P^x}\right|_{\mathcal{F}_t} = M_t\ , \qquad t\geq 0. \tag{1.8}$$

We finish this section with an example.

Example 1.28 (Binary BBM)**.** *Consider the model from Example 1.27. We've had*

$$\tilde{M}_t = 2^{n_t}e^{\gamma\xi_t-(\beta+\frac{1}{2}\gamma^2)t}\ , \qquad t\geq 0.$$

The corresponding additive martingale is

$$M_t = \sum_{u\in N_t} e^{\gamma X_t^u-(\beta+\frac{1}{2}\gamma^2)t}\ , \qquad t\geq 0.$$

1.5 Spine decomposition

Here the basic idea, already seen in Proposition 1.14, is that the tree made from the paths of all the particles can be decomposed into the spine's path and the subtrees initiated from it. Each of those subtrees has the same law as the original branching process started at time S_u from the position ξ_{S_u} for $u\in\xi$. On the illustration below we have the spine process drawn with a bold black line and different subtrees on the spine drawn in different colours.
The proof of the following theorem as well as some further discussion can be found in [19].

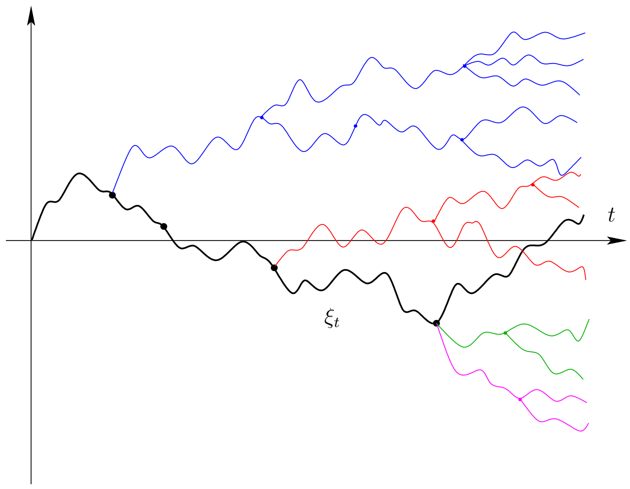


Figure 1-6: *Spine decomposition*

Theorem 1.29 (Spine decomposition)**.** *As a consequence of Proposition 1.14 and the martingale property of M we have the following decomposition:*

$$E^{\tilde{Q}^x}\left(M_t|\tilde{\mathcal{G}}_\infty\right) = \tilde{M}_t^{(3)} e^{-\int_0^t m(\xi_s)\beta(\xi_s)\mathrm{d}s} + \sum_{u<node_t(\xi)} A_u \tilde{M}_{S_u}^{(3)} e^{-\int_0^{S_u} m(\xi_s)\beta(\xi_s)\mathrm{d}s}, \qquad t \geq 0$$

Recall that S_u's for $u < node_t(\xi)$ are just the birth times along the spine before time t. We shall refer to the first term of this decomposition as the spine term or $spine(t)$ and to the second one as the sum term or $sum(t)$.

This theorem is very helpful in analysing the asymptotic behaviour of $(M_t)_{t\geq 0}$. For example, as we shall see later, it is useful in deciding whether M is uniformly integrable or not. In [19] Hardy and Harris used it to investigate L^p convergence of a family of additive martingales.

Remark 1.30. *We shall often assume (without loss of generality) that the branching process starts from* 0 *and in such cases we shall write P in place of P^0 and similarly for $\tilde{P}$, Q and $\tilde{Q}$.*

Chapter 2

Branching Brownian Motion in a supercritical potential

In this chapter we consider binary branching Brownian motion with branching rate $\beta(x) = \beta|x|^p$, where $\beta > 0$, $p \geq 0$. That is, single particles in the system move as standard Brownian motions splitting into two new particles at instantaneous rate $\beta|\cdot|^p$.

This model has been a subject of study before. J. Harris and S. Harris in [22] investigated the asymptotic growth of the rightmost particle in the case $p \in [0, 2]$. The asymptotic properties of the population growth in the case $p \in [0, 2)$ are studied in the paper of J. Berestycki, É. Brunet, J. Harris, S. Harris and M. Roberts [3].

However, one fundamental question one needs to answer before studying various aspects of the model is whether the population size stays finite or explodes in finite time. Itô and McKean proved in their book [23] that if $p \in [0, 2]$ then almost surely the number of particles stays finite at any time, whereas if $p > 2$ the number of particles almost surely explodes in finite time.

The proof of Itô and McKean was in the spirit of Borel-Cantelli lemma and relied on knowing the distribution of passage time of a Brownian motion to a given level. The latter made it difficult to adapt the proof to processes other than Brownian motion.

We wish to give an alternative proof using spine techniques discussed in the first chapter. The work of J. Harris and S. Harris [22] shows how in the case $p \in [0, 2]$ to prove that with positive probability in the branching process there is a path, which asymptotically grows like some given deterministic function. We shall use a slightly modified argument to show that in the supercritical case ($p > 2$) with positive probability there is a path in the branching process which drifts to infinity in finite time. That will be sufficient to deduce the almost sure population explosion in finite time as we shall see later. In Chapter 3 we shall adapt our proof to a branching Random walk.

2.1 Introduction

Let us take a binary branching Brownian motion with the branching rate $\beta(x) = \beta|x|^p$, where $\beta > 0$, $p \geq 0$. We shall denote by P^x the law of such process if we want to emphasize that it starts from x. Otherwise we shall assume that it starts from 0 and denote its law by P.

Definition 2.1. *We define the explosion time as*

$$T_{explo} := \sup\{t : |N_t| < \infty\}.$$

In this section we give an overview of the properties of T_{explo}. Most of the things we say can be found in [23] in one form or another, so we shall not go into too much detail.

We start with the following observation:

Proposition 2.2.

$$P^x\big(T_{explo} = \infty\big) = P^y\big(T_{explo} = \infty\big) \qquad \forall x, y \in \mathbb{R}.$$

Proof. Take any x and $y \in \mathbb{R}$ and start a branching Brownian motion from x. Let T_y be the first passage time of the process to level y. That is,

$$T_y := \inf\{t \;:\; \exists u \in N_t \text{ s.t. } X^u_t = y\}.$$

$T_y < \infty$ because a Brownian motion started from any level x will hit any level y. Then by the strong Markov property of the branching process the subtree initiated from y at time T_y has the same law as a BBM started from y (see Figure 2-1 for an illustration).

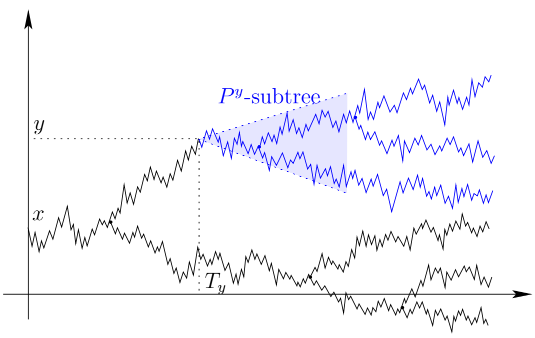


Figure 2-1: *P^y-subtree*

Consequently, if the explosion does not happen in the big tree started from x, it cannot happen in its subtree started from y. Thus

$$P^x\big(T_{explo} = \infty\big) \le P^y\big(T_{explo} = \infty\big).$$

Since x and y were arbitrary it follows that

$$P^x\big(T_{explo} = \infty\big) = P^y\big(T_{explo} = \infty\big) \qquad \forall x, y \in \mathbb{R}.$$

□

One important corollary of the previous result is the following 0-1 law.

Corollary 2.3.

$$P\big(T_{explo} = \infty\big) \in \{0, 1\}.$$

Proof. If X_1 is the position of the first split then from the branching property we have

$$P\big(T_{explo} = \infty\big) = E\Big(\big(P^{X_1}(T_{explo} = \infty)\big)^2\Big) = \big(P\big(T_{explo} = \infty\big)\big)^2.$$

Thus $P(T_{explo} = \infty) \in \{0, 1\}$. □

Remark 2.4. *This argument (Proposition 2.2 + Corollary 2.3) can be used to derive zero-one laws for various other events. We shall see in later chapters an alternative way to present this argument.*

It is also worth mentioning that it was crucial in the proof that a Brownian motion hits any point on the real line. Without this property the proof would not work.

Let us state another useful fact.

Proposition 2.5. *Take some deterministic time $t > 0$.*

$$\textit{If } P\big(T_{explo} < t\big) = 0 \textit{ then } P^x\big(T_{explo} < t\big) = 0\ \forall x.$$

Proof. Take any $\epsilon \in (0, t)$. Let T_x be the hitting time of level x as in Proposition 2.2. Then there is a positive probability that the process will hit level x before time ϵ. Then

$$\underbrace{P\big(T_{explo} < t\big)}_{=0} \ge P\big(T_{explo} < t, T_x < \epsilon\big) \ge P\big(T^x_{explo} < t - \epsilon, T_x < \epsilon\big),$$

where T^x_{explo} is the explosion time of the subtree started from x (drawn in blue in Figure 2-2 below)

$$= E\Big(P\big(T^x_{explo} < t - \epsilon, T_x < \epsilon | T_x\big)\Big) = \underbrace{P\big(T_x < \epsilon\big)}_{>0} P^x\big(T_{explo} < t - \epsilon\big).$$

Thus

$$P^x\big(T_{explo} < t - \epsilon\big) = 0 \quad \forall \epsilon > 0.$$

Letting $\epsilon \downarrow 0$ we get the result. $\square$

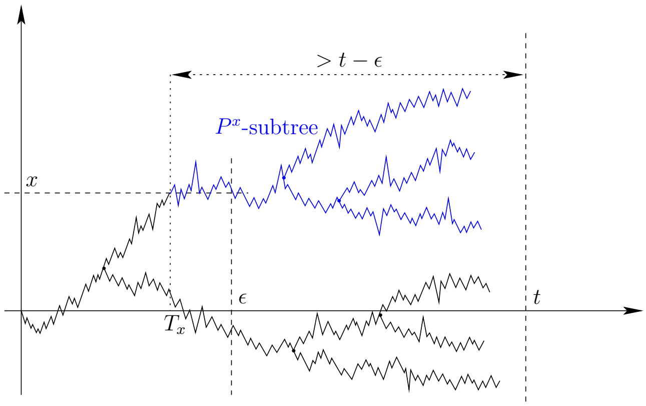


Figure 2-2: *Illustration of Proposition 2.5*

As a consequence of Proposition 2.5 we get the following corollary.

Corollary 2.6. *Let $t > 0$ be any deterministic time.*

$$\textit{if } P\big(T_{explo} \geq t\big) = 1 \textit{ then } P\big(T_{explo} = \infty\big) = 1.$$

The result follows by induction since if the original tree almost surely does not explode by time t then none of its subtrees initiated at time t will explode by time $2t$ and one can repeat this argument any number of times.

Proof. If $P(T_{explo} < t) = 0$, then by Proposition 2.5 $P^x(T_{explo} < t) = 0$ $\forall x \in \mathbb{R}$. Let $t_n := t(1 - \frac{1}{2^n})$ for $n \in \mathbb{N}$. Then

$$P^x(T_{explo} \leq t_n) = 0 \ \forall x \in \mathbb{R}, \ \forall n \in \mathbb{N}. \tag{2.1}$$

Let $T_n := \sum_{i=1}^n t_i$. Then

$$P(T_{explo} \leq T_1) = 0.$$

Suppose $P(T_{explo} \leq T_n) = 0$ for some $n \geq 1$. Then

$$\begin{aligned} P(T_{explo} \leq T_{n+1}) =& P(T_n < T_{explo} \leq T_{n+1}) \\ =& P(T_n < T_{explo} \leq T_n + t_{n+1}) \\ \leq& E\Big(\sum_{u \in N_{T_n}} P^{X^u_{T_n}}\big(T_{explo} \leq t_n\big)\Big) = 0 \text{ by (2.1)}. \end{aligned}$$

Thus $P(T_{explo} \leq T_n) = 0$ for all $n \geq 1$ and since $T_n \to \infty$ it follows that $P(T_{explo} < \infty) = 0$. □

The main result of this chapter is the following dichotomy

Theorem 2.7.
a) If $p \leq 2$ then $T_{explo} = \infty$ P-a.s.
b) If $p > 2$ then $T_{explo} < \infty$ P-a.s.

Case a) is easy and we give its proof now. Case b) is more involved and we devote the next section to its proof.

Proof of Theorem 2.7 a). We use the fact

$$E\big(|N_t|\big) < \infty \;\Rightarrow\; |N_t| < \infty \; P\text{-a.s.} \;\Rightarrow\; T_{explo} > t \; P\text{-a.s.}$$

A simple application of the Many-to-One lemma (see Lemma 1.18) gives us

$$E\Big(|N_t|\Big) = E\Big(\sum_{u \in N_t} 1\Big) = \tilde{E}\Big(e^{\int_0^t \beta(\xi_s)\mathrm{d}s}\Big) = \tilde{E}\Big(e^{\int_0^t \beta|\xi_s|^p \mathrm{d}s}\Big),$$

where under $\tilde{P}$ the spine process $(\xi_t)_{t \geq 0}$ is a standard Brownian motion.

It then follows that

$$\begin{aligned}
\tilde{E}\Big(e^{\int_0^t \beta|\xi_s|^p \mathrm{d}s}\Big) \leq & \tilde{E}\Big(e^{t\beta \sup_{0\leq s\leq t}|\xi_s|^p}\Big) = \tilde{E}\Big(e^{t\beta (\sup_{0\leq s\leq t}|\xi_s|)^p}\Big) \\
\leq & \tilde{E}\Big(e^{t\beta (\sup_{0\leq s\leq t}\xi_s)^p} + e^{t\beta (\sup_{0\leq s\leq t}-\xi_s)^p}\Big) \\
= & 2\tilde{E}\Big(e^{t\beta (\sup_{0\leq s\leq t}\xi_s)^p}\Big) \\
= & 2\int_0^\infty e^{t\beta x^p} \frac{2}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \mathrm{d}x
\end{aligned}$$

using the well-known fact that $\sup_{0 \leq s \leq t} \xi_s \overset{d}{=} |\xi_t| \overset{d}{=} |N(0,t)|$.

Thus we see that if $p < 2$ then $E(|N_t|) < \infty$ $\forall t > 0$, and if $p = 2$ then $E(|N_t|) < \infty$ for $t < \frac{1}{\sqrt{2\beta}}$. In either case we have that $E(|N_t|) < \infty$ for some $t > 0$, and hence $T_{explo} > t$ P-a.s. Then by Corollary 2.6 we deduce that $T_{explo} = \infty$ P-a.s. □

To end this section let us mention that the distribution of T_{explo} is known to be the solution of a generalised version of the FKPP equation.

Proposition 2.8 (Itô and McKean)**.** *Let $u(t,x) := P^x(T_{explo} < t)$. Then $u(t,x)$ solves the following partial differential equation:*

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u(1-u)\beta|x|^p \\ u(0,x) = 0 \\ 0 \leq u \leq 1 \end{cases} \tag{2.2}$$

Note that $u \equiv 0$ is always a solution of this equation. Moreover, if $p \le 2$ it can be shown using analytic methods that this is the only solution (see [23]).

2.2 Population explosion in the case $p > 2$

Consider binary BBM started from 0 with branching rate $\beta|x|^p$, where $\beta > 0$, $p > 2$. In this section we prove that for such model $T_{explo} < \infty$ P-a.s.

Proof of Theorem 2.7 b). We shall prove that for any deterministic $T > 0$

$$P\Big(T_{explo} \le T\Big) > 0. \tag{2.3}$$

This would tell us that $T_{explo} < \infty$ P-a.s. by Corollary 2.3 and would also give a non-trivial solution of the differential equation (2.2).

Let us suppose for the rest of this section that (2.3) is false. That is, $\exists\, T > 0$ s.t.

$$P\Big(T_{explo} \le T\Big) = 0. \tag{2.4}$$

In other words, $|N_T| < \infty$ P-a.s. Fix this T for the rest of the proof.

Under the assumption (2.4) that there is no explosion before time T we can perform the usual spine construction on $[0, T)$. That is, if the original process restricted to $[0, T)$ is defined under the probability measure P with $\{\mathcal{F}_t\}_{t\in[0,T)}$ its natural filtration, then we can define the BBM process with the spine process $(\xi_t)_{t\in[0,T)}$ on the filtration $\{\tilde{\mathcal{F}}_t\}_{t\in[0,T)}$ under probability measure $\tilde{P}$ in the usual way. Then $P = \tilde{P}\big|_{\mathcal{F}_T}$, where $\mathcal{F}_T = \sigma\big(\cup_{t\in[0,T)} \mathcal{F}_t\big)$. Similarly we define $\mathcal{G}_T$, $\tilde{\mathcal{G}}_T$ and $\tilde{\mathcal{F}}_T$.

Then we can consider a $\tilde{P}$-martingale of the form (1.6)

$$\tilde{M}(t) := e^{-\int_0^t \beta|\xi_s|^p \mathrm{d}s} 2^{n_t} \times e^{\int_0^t g'(s)\mathrm{d}\xi_s - \int_0^t \frac{1}{2} g'(s)^2 \mathrm{d}s}\ , \qquad t \in [0, T),$$

where $g : [0, T) \to \mathbb{R}$ is a function in $C^1\big([0, T)\big)$ satisfying $\int_0^t g'(s)^2 \mathrm{d}s < \infty$ $\forall t \in [0, T)$. Here we have used the classical Girsanov martingale in the place of martingale (1.5).

And via the Radon-Nikodym derivative we define a new measure $\tilde{Q}$

$$\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}\tilde{P}}\bigg|_{\tilde{\mathcal{F}}_t} = \tilde{M}(t), \qquad t \in [0, T).$$

Under this measure the spine process diffuses as $\xi_t = \tilde{B}_t + g(t)$, where $\tilde{B}$ is a $\tilde{Q}$-Brownian motion. P-subtrees are born along the spine at instantaneous rate $2\beta|\xi_t|^p$. These subtrees don't explode (up to time T) by Proposition 2.5. Then we define the measure $Q := \tilde{Q}\big|_{\mathcal{F}_T}$ so that

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}P}\right|_{\mathcal{F}_t} = M(t), \qquad t \in [0, T), \tag{2.5}$$

where

$$M(t) := \sum_{u \in N_t} \exp\Big(\int_0^t g'(s)\mathrm{d}X_s^u - \int_0^t \big(\frac{1}{2}g'(s)^2 + \beta|X_s^u|^p\big)\mathrm{d}s\Big) \tag{2.6}$$

is an additive P-martingale (recall (1.7)).

We'll be interested in paths g which explode at time T. In particular we consider paths of the form $g(s) = c(T-s)^{-d} - cT^{-d}$ for c, d some positive constants so that $g(s) \to \infty$ as $s \to T$ and $g(0) = 0$. There is the 'critical' path $g_*(s) = c_*(T-s)^{-d_*} - c_*T^{-d_*}$, where

$$c_* = \Big(\frac{2}{\sqrt{2\beta}(p-2)}\Big)^{\frac{2}{p-2}}, \ d_* = \frac{2}{p-2}, \tag{2.7}$$

which solves the equation

$$\frac{1}{2}g'(s)^2 = \beta g(s)^p \tag{2.8}$$

(ignoring the normalising constant cT^{-d}). The meaning of this equation will become apparent later. Let us mention that equation (2.8) comes up quite often in the BBM model. For example in the model with subcritical branching rate ($p \leq 2$) the solution of this equation describes the asymptotic growth of the rightmost particle.

For our martingale $(M_t)_{t\in[0,T)}$ we need to take a path which increases faster than $g_*(s)$. So we let

$$g(s) = c(T-s)^{-d_*} - cT^{-d_*} \tag{2.9}$$

for some $c > c_*$ (e.g. $c = c_* + 1$).

In Figure 2-3 we see how the branching process would typically look like under the probability measure $\tilde{Q}$.

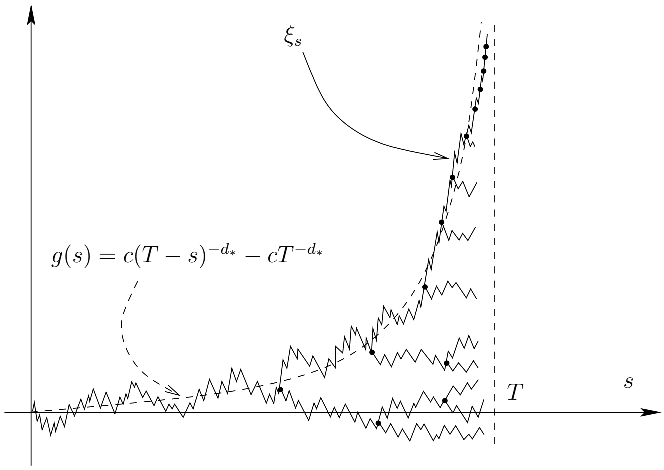


Figure 2-3: *BBM under* $\tilde{Q}$

We now recall the following measure-theoretic result taken from the book of R. Durrett [13].

Theorem 2.9 (Durrett)**.** *Let μ be a finite measure and ν a probability measure on $(\Omega, \mathcal{F})$. Let $\mathcal{F}_n \uparrow \mathcal{F}$ be σ-fields (i.e., $\sigma(\cup\mathcal{F}_n) = \mathcal{F}$). Let μ_n and ν_n be the restrictions of μ and ν to $\mathcal{F}_n$.*

Suppose $\mu_n \ll \nu_n$ for all n. Let $X_n = \frac{\mathrm{d}\mu_n}{\mathrm{d}\nu_n}$ and let $X = \limsup X_n$. Then

$$\mu(A) = \int_A X \mathrm{d}\nu + \mu(A \cap \{X = \infty\}).$$

This theorem gives Lebesgue's decomposition of measure μ into absolutely continuous and singular parts. In our case Theorem 2.9 takes form of the following lemma.

Lemma 2.10. *Let M be as in* (2.6) *above with function g as in* (2.9) *and let measure Q be as in* (2.5)*. Then for events $A \in \mathcal{F}_T$*

$$Q\big(A\big) = \int_A \limsup_{t \to T} M(t) \mathrm{d}P + Q\big(A \cap \{\limsup_{t \to T} M(t) = \infty\}\big). \tag{2.10}$$

Our aim is to show that $\limsup_{t \to T} M(t) < \infty$ Q-a.s. This will enable us to deduce that for $A \in \mathcal{F}_T$ $P(A) > 0$ whenever $Q(A) > 0$. In particular, knowing that under Q there is a particle that drifts to infinity, we can deduce that this also happens with positive P-probability.

Let us consider the spine decomposition (recall Theorem 1.29)

$$E^{\tilde{Q}}\Big(M(t)\big|\tilde{\mathcal{G}}_T\Big) = sum(t) + spine(t), \qquad t \in [0, T), \tag{2.11}$$

$$\text{where} \quad spine(t) = \exp\Big(\int_0^t g'(s)\mathrm{d}\xi_s - \int_0^t \big(\frac{1}{2}g'(s)^2 + \beta|\xi_s|^p\big)\mathrm{d}s\Big) \tag{2.12}$$

$$\begin{aligned} \text{and} \quad sum(t) &= \sum_{u < node_t(\xi)} \exp\Big(\int_0^{S_u} g'(s)\mathrm{d}\xi_s - \int_0^{S_u} \big(\frac{1}{2}g'(s)^2 + \beta|\xi_s|^p\big)\mathrm{d}s\Big) \\ &= \sum_{u < node_t(\xi)} spine(S_u). \end{aligned} \tag{2.13}$$

We want to show that $\limsup_{t \to T} E^{\tilde{Q}}(M(t)\big|\tilde{\mathcal{G}}_T) < \infty$ $\tilde{Q}$-a.s. We start by proving the following assertion.

Proposition 2.11. *$\exists$ some $\tilde{Q}$-a.s. finite positive random variables C', C'' and a random time $T' \in [0, T)$ such that $\forall t > T'$*

$$spine(t) \leq C' \exp\Big(-C'' \int_0^t (T - s)^{-pd^*}\mathrm{d}s\Big).$$

Proof of Proposition 2.11. Under $\tilde{Q}$, $\mathrm{d}\xi_s = \mathrm{d}\tilde{B}_s + g'(s)\mathrm{d}s$ (where $\tilde{B}$ is a standard Brownian motion). So,

$$spine(t) = \exp\Big(\int_0^t g'(s)\mathrm{d}\tilde{B}_s + \int_0^t \big(\frac{1}{2}g'(s)^2 - \beta|\tilde{B}_s + g(s)|^p\big)\mathrm{d}s\Big),$$

where $g(s) = c\big(T-s\big)^{-d_*} - cT^{-d_*}$. Then

$$\begin{aligned}\int_0^t g'(s)^2\mathrm{d}s =& \int_0^t c^2 d_*^2\big(T-s\big)^{-2(d_*+1)}\mathrm{d}s \\ =& C_1\big(T-t\big)^{-2d_*-1} - C_2 \ \to \infty \text{ as } t \to T,\end{aligned}$$

where C_1, C_2 are some positive constants. Then

$$\frac{\int_0^t g'(s)\mathrm{d}\tilde{B}_s}{\int_0^t g'(s)^2\mathrm{d}s} \to 0 \text{ as } t \to T \quad \tilde{Q}\text{-a.s.} \tag{2.14}$$

since by the Dubins-Schwarz Theorem

$$\Big(\int_0^t g'(s)\mathrm{d}\tilde{B}_s\Big)_{t\in[0,T)} \overset{d}{=} \Big(\tilde{B}_{\int_0^t g'(s)^2\mathrm{d}s}\Big)_{t\in[0,T)},$$

and $\frac{\tilde{B}_t}{t} \to 0$ as $t \to \infty$ $\tilde{Q}$-a.s. Also, $g(s) \to \infty$ as $s \to T$, whereas $\sup_{s\in[0,T)}|\tilde{B}_s| < \infty$ $\tilde{Q}$-a.s., so

$$\frac{|\tilde{B}_s + g(s)|}{g(s)} \to 1 \text{ as } s \to T \quad \tilde{Q}\text{-a.s.} \tag{2.15}$$

Therefore for any ϵ, $\delta > 0$ we can find random times T_δ, $T_\epsilon \in [0,T)$ s.t.

$$\begin{aligned}-(1-\epsilon)g(s)^p \ &\le\ |\tilde{B}_s + g(s)|^p \ \le\ (1+\epsilon)g(s)^p \quad \forall s > T_\epsilon \quad \text{by (2.15)} \\ -\delta\int_0^t g'(s)^2\mathrm{d}s \ &\le\ \int_0^t g'(s)\mathrm{d}\tilde{B}_s \ \le\ \delta\int_0^t g'(s)^2\mathrm{d}s \quad \forall t > T_\delta \quad \text{by (2.14)}\end{aligned}$$

So $\forall t > T_\delta \vee T_\epsilon$ we have

$$\begin{aligned}spine(t) \le& C_3\exp\Big(\int_{T_\delta\vee T_\epsilon}^t \big((1+2\delta)\frac{1}{2}g'(s)^2 - \beta(1-\epsilon)g(s)^p\big)\mathrm{d}s\Big) \\ =& C_4\exp\Big(\int_0^t \big((1+2\delta)\frac{1}{2}g'(s)^2 - \beta(1-\epsilon)g(s)^p\big)\mathrm{d}s\Big),\end{aligned}$$

where C_3, C_4 are some $\tilde{Q}$-a.s. finite positive random variables, which don't depend on t.

Substituting $g(s) = c\big(T-s\big)^{-d_*} - cT^{-d_*}$ into this we get that for all $t > T_\delta \vee T_\epsilon$

$$spine(t) \leq C_5 \exp\Big(\underbrace{\big((1+2\delta)\frac{1}{2}c^2 d_*^2 - \beta(1-\epsilon)c^p\big)}_{:=C_{\delta,\epsilon}} \underbrace{\int_0^t \big(T-s\big)^{-pd_*} \mathrm{d}s}_{:=h(t)} \Big). \tag{2.16}$$

Let us note that:

$$\begin{aligned}
1) \quad h(t) &= \int_0^t \big(T-s\big)^{-pd_*} \mathrm{d}s = \Big[\frac{1}{pd_* - 1}\big(T-s\big)^{1-pd_*}\Big]_0^t \\
&= \frac{p-2}{p+2}\Big(\big(T-t\big)^{-\frac{p+2}{p-2}} - T^{-\frac{p+2}{p-2}}\Big) \to \infty \text{ as } t \to T \\
2) \quad C_{\delta,\epsilon} &= (1+2\delta)\frac{1}{2}c^2 d_*^2 - \beta(1-\epsilon)c^p \\
&= \delta\big(c^2 d_*^2\big) + \epsilon\big(\beta c^p\big) + \ c^2 \underbrace{\big(\frac{1}{2}d_*^2 - \beta c^{p-2}\big)}_{<\frac{1}{2}d_*^2 - \beta c_*^{p-2} = 0} < 0 \text{ for } \epsilon, \delta \text{ chosen small enough}
\end{aligned}$$

1) and 2) together show that $spine(t) \to 0$ $\tilde{Q}$-a.s. and this occurs 'rapidly'. To finish the proof of Proposition 2.11 let $C' = C_5$, $C'' = C_{\delta,\epsilon}$ and $T' = T_\delta \vee T_\epsilon$. $\square$

Next we look at the *sum* term.

$$\begin{aligned}
sum(t) &= \sum_{u < node_t(\xi)} spine(S_u) \\
&= \Big(\sum_{u<node_t(\xi),\ S_u \leq T'} spine(S_u)\Big) + \Big(\sum_{u<node_t(\xi),\ S_u > T'} spine(S_u)\Big) \\
&\leq \sum_{u<node_t(\xi),\ S_u \leq T'} spine(S_u) \\
&\quad + \sum_{u<node_t(\xi),\ S_u > T'} C' \exp\Big(-C'' \int_0^{S_u} (T-s)^{-pd^*} \mathrm{d}s\Big)
\end{aligned}$$

using Proposition 2.11. The first sum is $\tilde{Q}$-a.s. bounded since it only counts births up to time T'. Call an upper bound on the first sum C_6. Then we have

$$sum(t) \leq \ C_6 + C' \sum_{n=1}^{\infty} \exp\Big(-C'' \int_0^{S_n} \big(T-s\big)^{-pd_*} \mathrm{d}s\Big), \tag{2.17}$$

where S_n is the time of the n^{th} birth on the spine.

The birth process along the spine $(n_t)_{t \in [0,T)}$ conditional on the path of the spine is time-inhomogeneous Poisson process (or Cox process) with birth rate $2\beta|\xi_t|^p$ at time t

(recall Proposition 1.13). Thus

$$\frac{n_t}{\int_0^t 2\beta|\xi_s|^p \mathrm{d}s} \to 1 \quad \tilde{Q}\text{-a.s. as } t \to T.$$

Also from (2.15)

$$\frac{\int_0^t 2\beta|\xi_s|^p \mathrm{d}s}{\int_0^t 2\beta g(s)^p \mathrm{d}s} \to 1 \quad \tilde{Q}\text{-a.s. as } t \to T.$$

Hence

$$\frac{n_t}{\int_0^t 2\beta g(s)^p \mathrm{d}s} \to 1 \quad \tilde{Q}\text{-a.s. as } t \to T \tag{2.18}$$

and also

$$\frac{n}{\int_0^{S_n} 2\beta g(s)^p \mathrm{d}s} \to 1 \quad \tilde{Q}\text{-a.s. as } n \to \infty.$$

So for some $\tilde{Q}$-a.s. finite positive random variable C_7 we have

$$\int_0^{S_n} (T-s)^{-pd_*} \mathrm{d}s = \int_0^{S_n} g(s)^p + T^{-pd_*} \mathrm{d}s \geq C_7 n \quad \forall n.$$

Substituting this into (2.17) we get that

$$sum(t) \leq C_6 + C' \sum_{n=1}^{\infty} \exp\Big(- C'' C_7 n \Big).$$

Thus $sum(t)$ is $\tilde{Q}$-a.s. bounded by some finite random variable. We deduce that

$$\limsup_{t\to T} E^{\tilde{Q}}\Big(M(t) \big| \tilde{\mathcal{G}}_T \Big) = \limsup_{t\to T} \big(spine(t) + sum(t) \big) < \infty \qquad \tilde{Q}\text{-a.s.}$$

By Fatou's lemma

$$E^{\tilde{Q}}\Big(\liminf_{t\to T} M(t) \big| \tilde{\mathcal{G}}_T \Big) \leq \liminf_{t\to T} E^{\tilde{Q}}\Big(M(t) \big| \tilde{\mathcal{G}}_T \Big) \leq \limsup_{t\to T} E^{\tilde{Q}}\Big(M(t) \big| \tilde{\mathcal{G}}_T \Big) < \infty \quad \tilde{Q}\text{-a.s.}$$

Then $\liminf_{t\to T} M(t) < \infty$ $\tilde{Q}$-a.s. and hence also Q-a.s. Since $\frac{1}{M(t)}$ is a positive Q-supermartingale on $[0, T)$, it must converge Q-a.s., hence

$$\limsup_{t\to T} M(t) = \liminf_{t\to T} M(t) < \infty \qquad Q\text{-a.s.} \tag{2.19}$$

This is our sought result. That is, we have shown that

$$\limsup_{t\to T} M(t) < \infty \qquad Q\text{-a.s.,}$$

where

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}P}\right|_{\mathcal{F}_t} = M(t), \qquad t \in [0,T).$$

Lemma 2.10 now tells us that for events $A \in \mathcal{F}_T$

$$Q(A) = \int_A \limsup_{t\to T} M_t \mathrm{d}P.$$

Thus $Q(A) > 0 \Rightarrow P(A) > 0$. Let us consider the event

$$A := \Big\{|N_t| \to \infty \text{ as } t \to T\Big\} \in \mathcal{F}_T.$$

From (2.18) we recall that

$$\begin{aligned}
&\tilde{Q}\Big(\frac{n_t}{\int_0^t 2\beta g(s)^p \mathrm{d}s} \to 1 \text{ as } t \to T\Big) = 1 \\
\Rightarrow &\tilde{Q}\Big(n_t \to \infty \text{ as } t \to T\Big) = 1 \\
\Rightarrow &\tilde{Q}\Big(|N_t| \to \infty \text{ as } t \to T\Big) = 1 \\
\Rightarrow &Q\Big(|N_t| \to \infty \text{ as } t \to T\Big) = 1 \\
\Rightarrow &P\Big(|N_t| \to \infty \text{ as } t \to T\Big) > 0 \\
\Rightarrow &P\Big(T_{explo} \leq T\Big) > 0,
\end{aligned}$$

which contradicts (2.4). Therefore it must be the case that

$$P(T_{explo} \leq T) > 0 \ \forall T > 0$$

and consequently from Corollary 2.3

$$T_{explo} < \infty \ P\text{-a.s.}$$

□

Chapter 3

Branching random walks

This chapter is devoted to the study of continuous-time binary branching random walks with inhomogeneous branching rate $\beta(x) = \beta|x|^p$, $\beta > 0$, $p \geq 0$.

We prove that the population almost surely explodes in finite time if $p > 1$ and stays finite otherwise. For the proof we adapt the methods from Chapter 2.

In the case $p \leq 1$ we give the asymptotic growth of the rightmost particle. For that we use ideas from the paper of J. Harris and S. Harris [22] that considers BBM in a similar inhomogeneous potential.

3.1 Introduction

3.1.1 Description of the model

We consider a binary branching process started from 0, where branching occurs at instantaneous rate $\beta(\cdot) = \beta|\cdot|^p$ and single particles move according to a continuous-time random walk.

By continuous-time random walk we mean a $\mathbb{Z}$-valued process $(X_t)_{t\geq 0}$ under some probability measure $\mathbb{P}$, which starts from 0 and makes jumps up or down of size 1 at constant rate λ in each direction.

Thus $(X_t)_{t\geq 0}$ can be viewed as a compound Poisson process:

$$X_t = \sum_{i=1}^{P_t} W_i \ , \qquad t \geq 0,$$

where W_i's are i.i.d. random variables with $\mathbb{P}(W_1 = 1) = \mathbb{P}(W_1 = -1) = \frac{1}{2}$ and P_t is a Poisson process

$$(P_t)_{t\geq 0} \stackrel{d}{=} PP(2\lambda).$$

Alternatively, we can write

$$X_t = X_t^+ - X_t^- , \qquad t \geq 0,$$

where $(X_t^+)_{t\geq 0}$ and $(X_t^-)_{t\geq 0}$ are two independent Poisson processes of rate λ. A typical sample path of $(X_t)_{t\geq 0}$ can be seen in Figure 3-1 below.

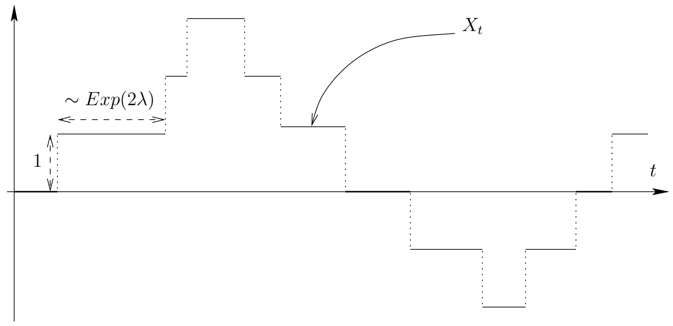


Figure 3-1: *Sample path of a random walk*

We are going to need the following basic fact about $(X_t)_{t\geq 0}$:

Proposition 3.1. *The process $(X_t)_{t\geq 0}$ is recurrent in the sense that $\forall n \in \mathbb{Z}$*

$$\limsup_{t\to\infty} \mathbf{1}_{\{X_t=n\}} = 1 \quad \mathbb{P}\text{-}a.s.$$

In other words the process $(X_t)_{t\geq 0}$ visits every state $n \in \mathbb{Z}$ infinitely often.

Let us note that the model studied in this chapter is very similar to the BBM model considered in Chapter 2 with the only difference that single particles move as a continuous-time random walk rather than a Brownian motion. Thus we'll be interested in comparing results for the two models.

3.1.2 Main results

Recall Definition 2.1 of the explosion time:

$$T_{explo} = \sup\{t : |N_t| < \infty\}.$$

We have the following dichotomy for T_{explo}.

Theorem 3.2. *Consider branching random walk in the potential $\beta(x) = \beta|x|^p$.*
a) If $p \leq 1$ then $T_{explo} = \infty$ P-a.s.
b) If $p > 1$ then $T_{explo} < \infty$ P-a.s.

Let us also define the process of the rightmost particle as

Definition 3.3.

$$R_t := \sup_{u \in N_t} X_t^u \ , \qquad t \geq 0.$$

When $p \in [0,1]$, we prove the following result about the asymptotic behaviour of R_t.

Theorem 3.4. *Consider branching random walk in the potential $\beta(x) = \beta|x|^p$.*

a) if $p = 0$ **then**

$$\lim_{t \to \infty} \frac{R_t}{t} = \hat{a} := \lambda(\hat{\theta} - \frac{1}{\hat{\theta}}) \qquad P - a.s., \tag{3.1}$$

where $\hat{\theta}$ is the unique solution of

$$g(\theta) = \frac{\beta}{\lambda} \quad on\ (1, \infty) \tag{3.2}$$

$$and\ g(\theta) = \big(\theta - \frac{1}{\theta}\big) \log \theta - \big(\theta + \frac{1}{\theta}\big) + 2$$

b) if $p \in (0,1)$ **then**

$$\lim_{t \to \infty} \Big(\frac{\log t}{t}\Big)^{\hat{b}} R_t = \hat{c} \qquad P - a.s., \tag{3.3}$$

where $\hat{b} = \frac{1}{1-p}$ and $\hat{c} = \Big(\frac{\beta(1-p)^2}{p}\Big)^{\hat{b}}$.

c) if $p = 1$ **then**

$$\lim_{t \to \infty} \frac{\log R_t}{\sqrt{t}} = \sqrt{2\beta} \qquad P - a.s. \tag{3.4}$$

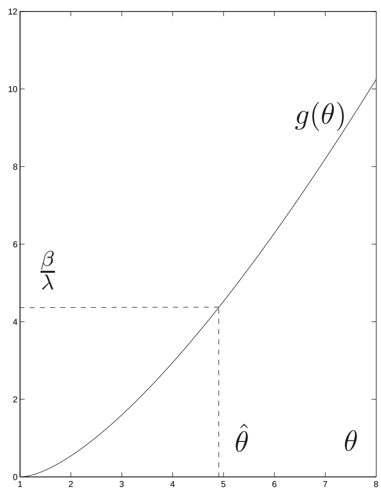


Figure 3-2: *Plot of $g(\theta)$ from Theorem 3.4 a)*

Part a) of Theorem 3.4 is a special case of a result proved by Biggins in [6] and [7].

3.1.3 Comparison with BBM

Theorem 3.2 must be compared with Theorem 2.7 in Chapter 2. We observe that in the BBM model $p = 2$ is the critical value for population explosion, whereas in the branching random walk model the critical value is $p = 1$.

Also Theorem 3.4 should be compared with the following result from [22]:

Theorem 3.5 (J. Harris and S. Harris)**.**
Consider binary BBM in the potential $\beta(x) = \beta|x|^p$, $p \in [0,2]$.

a) if $p \in [0,2)$ **then**

$$\lim_{t\to\infty} \frac{R_t}{t^{\hat{b}}} = \hat{a} \qquad P-a.s. \tag{3.5}$$

where $\hat{b} = \frac{2}{2-p}$ and $\hat{a} = \left(\frac{\beta}{2}(2-p)^2\right)^{\frac{1}{2-p}}$.

b) if $p = 2$ **then**

$$\lim_{t\to\infty} \frac{\log R_t}{t} = \sqrt{2\beta} \qquad P-a.s. \tag{3.6}$$

We see that even if we take spatially-independent branching rate $\beta(\cdot) \equiv \beta$, the two models will behave differently. Thus we conclude that the spatial motion of particles has a crucial effect on the behaviour of the model.

A heuristic way to recover results from Theorem 3.5 for the BBM model is to consider the expected number of particles at time t staying close to a curve f in the sense that is made precise in [3]. That is, we look at

$$E\Big(\sum_{u\in N_t} \mathbf{1}_{\{X^u_s \approx f(s)\ \forall s\in[0,t]\}}\Big).$$

Then the Many-to-One lemma (Lemma 1.18) reduces this to the expectation of a single Brownian motion $(\xi_t)_{t\geq 0}$:

$$E\Big(\sum_{u\in N_t} \mathbf{1}_{\{X^u_s \approx f(s)\ \forall s\in[0,t]\}}\Big) = \tilde{E}\Big(\mathbf{1}_{\{\xi_s \approx f(s)\ \forall s\in[0,t]\}} e^{\int_0^t \beta|\xi_s|^p \mathrm{d}s}\Big).$$

That can then be approximated by the Schilder's theorem:

$$\log \tilde{E}\Big(\mathbf{1}_{\{\xi_s \approx f(s)\ \forall s\in[0,t]\}} e^{\int_0^t \beta|\xi_s|^p \mathrm{d}s}\Big) \sim \int_0^t \beta f(s)^p - \frac{1}{2} f'(s)^2 \mathrm{d}s.$$

Hence the expected number of particles following the function f either grows exponentially or decays exponentially in t depending on the growth rate of f. The critical function f which solves the equation

$$\frac{1}{2} f'(s)^2 = \beta f(s)^p$$

in fact corresponds to the position of the rightmost particle. The rigorous proof requires showing that almost surely there exists a particle staying close to the critical curve f.

In principle a similar argument can be used for a branching random walk. Using heuristic methods which involve some large deviations theory we can get that

$$\log E\Big(\sum_{u\in N_t} \mathbf{1}_{\{X^u_s \approx f(s)\ \forall s\in[0,t]\}}\Big) \sim \int_0^t \beta f(s)^p - \Lambda\big(f'(s)\big)\mathrm{d}s,$$

where $\Lambda : [0,\infty) \to [0,\infty)$ is the rate function for the random walk and

$$\Lambda(x) = 2\lambda + x\log\big(\frac{\sqrt{x^2+4\lambda^2}+x}{2\lambda}\big) - \sqrt{x^2+4\lambda^2} \sim x\log x \text{ as } x\to\infty.$$

This heuristic argument actually gives the asymptotics of the rightmost particle from Theorem 3.4.

3.1.4 Outline of the chapter

In Section 3.2 we introduce a family of one-particle martingales. We also present some other relevant one-particle results, which will be used in later sections. Section 3.2 is self-contained and can be read out of the context of branching processes.

In Section 3.3 we prove Theorem 3.2 about the explosion time by adapting the proof of Theorem 2.7 from Chapter 2.

In Section 3.4 we give a proof of Theorem 3.4 about the rightmost particle using the ideas from [22].

3.2 One-particle results

In the analysis of the BBM model in [22] one crucial component was exponential martingales, also known as Girsanov martingales. They were used in place of martingale $\tilde{M}^{(3)}$ in (1.5) and conditioned the spine process to stay close to a given deterministic path.

In this section we introduce a family of martingales for continuous-time random walks, which will play the same role as the Girsanov martingales in the BBM model.

3.2.1 Changes of measure for Poisson processes

For this section let the time set for all the processes be $[0,T)$, where $T\in(0,\infty]$.

Suppose we are given a Poisson process $(Y_t)_{t\in[0,T)} \overset{d}{=} PP(\lambda)$ under a probability measure $\mathbb{P}$. Let us denote by J_i the time of the i^{th} jump of $(Y_t)_{t\in[0,T)}$. Then we have the following result.

Lemma 3.6. *Let $\theta : [0,T) \to [0,\infty)$ be a locally-integrable function. That is, $\int_0^t \theta(s)\mathrm{d}s < \infty$ $\forall t \in [0,T)$. Then the following process is a $\mathbb{P}$-martingale:*

$$M_t := e^{\int_0^t \log\theta(s)\mathrm{d}Y_s + \lambda\int_0^t (1-\theta(s))\mathrm{d}s} = \Big(\prod_{i:J_i\le t}\theta(J_i)\Big)e^{\lambda\int_0^t(1-\theta(s))\mathrm{d}s} \quad , \, t \in [0,T).$$

Here $\mathrm{d}Y$ puts a delta function at jump times of Y. That is, for any function f, $\int_0^t f(s)\mathrm{d}Y_s := \sum_{i:J_i\le t} f(J_i)$.

Example 3.7. *If we take $\theta(\cdot) \equiv \theta$ then*

$$M_t = \theta^{Y_t}e^{\lambda(1-\theta)t} \quad , \, t \in [0,T)$$

and it is well-known that this is a martingale. In fact we have already seen it in Chapter 1 (recall Example 1.20).

Our next result tells us what effect the martingale $(M_t)_{t\in[0,T)}$ has on the process $(Y_t)_{t\in[0,T)}$ when used as a Radon-Nikodym derivative.

Lemma 3.8. *Let $(\hat{\mathcal{F}}_t)_{t\in[0,T)}$ be the natural filtration of $(Y_t)_{t\in[0,T)}$. Define the new measure $\mathbb{Q}$ as*

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\hat{\mathcal{F}}_t} = M_t \quad , \, t \in [0,T).$$

Then under the new measure $\mathbb{Q}$

$$(Y_t)_{t\in[0,T)} \stackrel{d}{=} IPP\big(\lambda\theta(t)\big),$$

where $IPP\big(\lambda\theta(t)\big)$ stands for time-inhomogeneous Poisson process of instantaneous jump rate $\lambda\theta(t)$.

Example 3.9. *If we take $\theta(\cdot) \equiv \theta$ then under the new measure $\mathbb{Q}$*

$$(Y_t)_{t\in[0,T)} \stackrel{d}{=} PP\big(\lambda\theta\big).$$

Thus M has the effect of scaling the jump rate of Y by the factor of θ.

To prove Lemma 3.6 and Lemma 3.8 we shall first prove the following identity.

Proposition 3.10.

$$\mathbb{E}\Big(e^{\int_0^t \log\theta(s)\mathrm{d}Y_s}\mathbf{1}_{\{Y_t=k\}}\Big) = e^{-\lambda t}\frac{\lambda^k}{k!}\Big(\int_0^t \theta(s)\mathrm{d}s\Big)^k \quad \forall k \in \mathbb{N}, \tag{3.7}$$

where $\mathbb{E}$ is the expectation associated with $\mathbb{P}$ (and this will be the case throughout this section).

Proof of Proposition 3.10. For $k = 0$, $\mathbb{P}(Y_t = 0) = e^{-\lambda t}$, so equality in (3.7) holds trivially. Let us suppose that $k \geq 1$. Then

$$\mathbb{E}\Big(e^{\int_0^t \log\theta(s)\mathrm{d}Y_s}\mathbf{1}_{\{Y_t=k\}}\Big) = \mathbb{E}\Big(\prod_{i=1}^{k}\theta(J_i)\ \mathbf{1}_{\{Y_t=k\}}\Big) = \mathbb{E}\Big(\prod_{i=1}^{k}\theta(J_i)\ \mathbf{1}_{\{J_k\leq t<J_{k+1}\}}\Big),$$

where J_i's are the jump times of Y. Also $J_i = S_0 + ... + S_{i-1}$ where S_i's are the holding times, and it is known that $S_i \sim Exp(\lambda)\ \forall i$ and that S_i's are independent. Hence

$$\begin{aligned}
\mathbb{E}\Big(\prod_{i=1}^{k}\theta(J_i)\mathbf{1}_{\{J_k<t\leq J_{k+1}\}}\Big) &= \mathbb{E}\Big(\prod_{i=1}^{k}\theta\big(\sum_{j=0}^{i-1}S_j\big)\ \mathbf{1}_{\big\{\sum_{j=0}^{k-1}S_j\leq t<\sum_{j=0}^{k}S_j\big\}}\Big)\\
&= \mathbb{E}\Big(\prod_{i=1}^{k}\theta\big(\sum_{j=0}^{i-1}S_j\big)\ \mathbf{1}_{\big\{\sum_{j=0}^{k-1}S_j\leq t\big\}}\mathbf{1}_{\big\{S_k>t-\sum_{j=0}^{k-1}S_j\big\}}\Big)\\
&= \int_{\sum_{j=0}^{k-1}x_j\leq t,x_k>t-\sum_{j=0}^{k-1}x_j}\Big(\prod_{i=1}^{k}\theta\big(\sum_{j=0}^{i-1}x_j\big)\Big)\lambda^{k+1}e^{-\lambda\sum_{j=0}^{k}x_j}\mathrm{d}\mathbf{x}
\end{aligned}$$

$$\begin{aligned}
&= \lambda^k e^{-\lambda t}\int_{\sum_{j=0}^{k-1}x_j\leq t}\prod_{i=1}^{k}\theta\big(\sum_{j=0}^{i-1}x_j\big)\ \mathrm{d}x_{k-1}\cdots\mathrm{d}x_0 \quad \text{(after integrating out } x_k\text{)}\\
&= \lambda^k e^{-\lambda t}\int_0^t\int_0^{t-x_0}\cdots\int_0^{t-x_0-...-x_{k-2}}\theta(x_0)\times\cdots\times\theta(x_0+\cdots+x_{k-1})\mathrm{d}x_{k-1}\cdots\mathrm{d}x_0,
\end{aligned}$$

where for the $k = 1$ case we only have one integral going from 0 to t. Then, after making the natural change of variables $y_i = x_0 + \cdots + x_i,\ i = 0, \cdots, k-1$, we get

$$\begin{aligned}
&\lambda^k e^{-\lambda t}\int_0^t\int_{y_0}^t\cdots\int_{y_{k-2}}^t\theta(y_0)\theta(y_1)\times\cdots\times\theta(y_{k-1})\ \mathrm{d}y_{k-1}\ \cdots\ \mathrm{d}y_0\\
=&\lambda^k e^{-\lambda t}\int_0^t\cdots\int_0^t\mathbf{1}_{\{y_0<y_1<\cdots<y_{k-1}\}}\theta(y_0)\times\cdots\times\theta(y_{k-1})\ \mathrm{d}y_{k-1}\ \cdots\ \mathrm{d}y_0\\
=&\lambda^k e^{-\lambda t}\frac{1}{k!}\Big(\int_0^t\theta(y)\ \mathrm{d}y\Big)^k.
\end{aligned}$$

by the obvious symmetry. □

With identity (3.7) we can now prove lemmas 3.6 and 3.8.

Proof of Lemma 3.6. Firstly note that

$$\begin{aligned}
\mathbb{E}M_t &= \mathbb{E}e^{\int_0^t\log\theta(s)\mathrm{d}Y_s+\lambda\int_0^t(1-\theta(s))\mathrm{d}s}\\
&= \mathbb{E}\Big(\sum_{k=0}^{\infty}e^{\int_0^t\log\theta(s)\mathrm{d}Y_s}\mathbf{1}_{\{Y_t=k\}}\Big)e^{\lambda\int_0^t(1-\theta(s))\mathrm{d}s}
\end{aligned}$$

$$= \Big(\sum_{k=0}^{\infty} e^{-\lambda t}\frac{\lambda^k}{k!}\big(\int_0^t \theta(s)\mathrm{d}s\big)^k\Big)e^{\lambda\int_0^t(1-\theta(s))\mathrm{d}s}$$

$$= 1. \tag{3.8}$$

Secondly we note that

$$\begin{aligned}\mathbb{E}(M_t|\hat{\mathcal{F}}_s) &= M_s\mathbb{E}\Big(e^{\int_s^t \log\theta(u)\mathrm{d}Y_u+\lambda\int_s^t(1-\theta(u))\mathrm{d}u}|\hat{\mathcal{F}}_s\Big)\\ &= M_s\mathbb{E}\Big(e^{\int_0^{t-s} \log\phi(u)\mathrm{d}\tilde{Y}_u+\lambda\int_0^{t-s}(1-\phi(u))\mathrm{d}u}|\hat{\mathcal{F}}_s\Big)\\ &= M_s\end{aligned}$$

by (3.8), where $(\tilde{Y}_u)_{u\in[0,T-s)} = (Y_{s+u}-Y_s)_{u\in[0,T-s)}$ is a Poisson process independent of $\hat{\mathcal{F}}_s$, and $\phi(u) = \theta(s+u)$.

Therefore we see that $(M_t)_{t\geq 0}$ is a $\mathbb{P}$ - martingale. □

Let us now check that under probability measure $\mathbb{Q}$ we have
$(Y_t)_{t\in[0,T)} \overset{d}{=} IPP\big(\lambda\theta(t)\big)$.

Proof of Lemma 3.8.
It is sufficient to check that for $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_n \leq t_n < T$
and for $0 \leq k_1 \leq k_2 \leq \cdots \leq k_n$

$$\begin{aligned}&\mathbb{Q}(Y_{t_1}-Y_{s_1}=k_1,\ Y_{t_2}-Y_{s_2}=k_2,\ \cdots,\ Y_{t_n}-Y_{s_n}=k_n)\\ =&\mathbb{Q}(Y_{t_1}-Y_{s_1}=k_1)\times\mathbb{Q}(Y_{t_2}-Y_{s_2}=k_2)\times\cdots\times\mathbb{Q}(Y_{t_n}-Y_{s_n}=k_n)\\ =&\frac{1}{k_1!}\Big(\int_{s_1}^{t_1}\lambda\theta(u)\mathrm{d}u\Big)^{k_1}e^{-\int_{s_1}^{t_1}\lambda\theta(u)\mathrm{d}u}\times\cdots\times\frac{1}{k_n!}\Big(\int_{s_n}^{t_n}\lambda\theta(u)\mathrm{d}u\Big)^{k_n}e^{-\int_{s_n}^{t_n}\lambda\theta(u)\mathrm{d}u}\end{aligned} \tag{3.9}$$

Let us prove (3.9) by induction on n. Suppose $n = 1$. Then the distribution of a single increment is

$$\begin{aligned}\mathbb{Q}(Y_{t_1}-Y_{s_1}=k_1) &= \mathbb{E}\big(M_{t_1}\mathbf{1}_{\{Y_{t_1}-Y_{s_1}=k_1\}}\big)\\ &= \mathbb{E}\Big(\mathbb{E}\big(M_{t_1}\mathbf{1}_{\{Y_{t_1}-Y_{s_1}=k_1\}}|\hat{\mathcal{F}}_{s_1}\big)\Big)\\ &= \mathbb{E}\Big(\mathbb{E}\big(M_{s_1}e^{\int_{s_1}^{t_1}\log\theta(u)\mathrm{d}Y_u+\lambda\int_{s_1}^{t_1}(1-\theta(u))\mathrm{d}u}\mathbf{1}_{\{Y_{t_1}-Y_{s_1}=k_1\}}|\hat{\mathcal{F}}_{s_1}\big)\Big) \qquad (3.10)\\ &= \mathbb{E}\Big(M_{s_1}\mathbb{E}\big(e^{\int_0^{t_1-s_1}\log\phi(u)\mathrm{d}\tilde{Y}_u+\lambda\int_0^{t_1-s_1}(1-\phi(u))\mathrm{d}u}\mathbf{1}_{\{\tilde{Y}_{t_1-s_1}=k_1\}}|\hat{\mathcal{F}}_{s_1}\big)\Big)\\ &\overset{(3.7)}{=} e^{-\lambda(t_1-s_1)}\frac{\lambda^{k_1}}{k_1!}\Big(\int_0^{t_1-s_1}\phi(u)\mathrm{d}u\Big)^{k_1}e^{\lambda\int_0^{t_1-s_1}(1-\phi(u))\mathrm{d}u}\\ &= \frac{1}{k_1!}\Big(\int_{s_1}^{t_1}\lambda\theta(u)\mathrm{d}u\Big)^{k_1}e^{-\int_{s_1}^{t_1}\lambda\theta(u)\mathrm{d}u}\end{aligned}$$

Here, as in the proof of Lemma 3.6, $(\tilde{Y}_u)_{u\in[0,T-s_1)} = (Y_{s_1+u} - Y_{s_1})_{u\in[0,T-s_1)}$ is a Poisson process independent of $\hat{\mathcal{F}}_{s_1}$ and $\phi(u) = \theta(s_1 + u)$.

Suppose now that (3.9) holds for n increments. Then

$$
\begin{aligned}
&\mathbb{Q}(Y_{t_1} - Y_{s_1} = k_1,\ \cdots,\ Y_{t_n} - Y_{s_n} = k_n,\ Y_{t_{n+1}} - Y_{s_{n+1}} = k_{n+1})\\
=&\mathbb{E}\Big(M_{t_{n+1}}\mathbf{1}_{\{Y_{t_1}-Y_{s_1}=k_1,\ \cdots,\ Y_{t_n}-Y_{s_n}=k_n\}}\mathbf{1}_{\{Y_{t_{n+1}}-Y_{s_{n+1}}=k_{n+1}\}}\Big)\\
=&\mathbb{E}\Big(\mathbb{E}\big(M_{t_{n+1}}\mathbf{1}_{\{Y_{t_1}-Y_{s_1}=k_1,\ \cdots,\ Y_{t_n}-Y_{s_n}=k_n\}}\mathbf{1}_{\{Y_{t_{n+1}}-Y_{s_{n+1}}=k_{n+1}\}}|\hat{\mathcal{F}}_{s_{n+1}}\big)\Big)\\
=&\mathbb{E}\Big(M_{s_{n+1}}\mathbf{1}_{\{Y_{t_1}-Y_{s_1}=k_1,\ \cdots,\ Y_{t_n}-Y_{s_n}=k_n\}}\\
&\quad\mathbb{E}\big(e^{\int_{s_{n+1}}^{t_{n+1}}\log\theta(u)\mathrm{d}Y_u+\lambda\int_{s_{n+1}}^{t_{n+1}}(1-\theta(u))\mathrm{d}u}\mathbf{1}_{\{Y_{t_{n+1}}-Y_{s_{n+1}}=k_{n+1}\}}|\hat{\mathcal{F}}_{s_{n+1}}\big)\Big)\\
=&\mathbb{Q}(Y_{t_1} - Y_{s_1} = k_1,\ Y_{t_2} - Y_{s_2} = k_2,\ \cdots,\ Y_{t_n} - Y_{s_n} = k_n)\mathbb{Q}(Y_{t_{n+1}} - Y_{s_{n+1}} = k_{n+1})
\end{aligned}
$$

For the last line we used (3.10). Thus we see that (3.9) follows by induction. □

3.2.2 "Integration by parts" and applications

Proposition 3.11 (Integration by parts for time-inhomogeneous Poisson processes)**.** *For $T \in (0, \infty]$ let $f \in C^1\big([0, T)\big)$ and $(Y_t)_{t\in[0,T)} \stackrel{d}{=} IPP(r(t))$ defined on some probability space, where $r : [0, T) \to [0, \infty)$ is a locally-integrable function. Then*

$$\int_0^t f(s)\mathrm{d}Y_s = f(t)Y_t - \int_0^t f'(s)Y_s\mathrm{d}s,$$

where $\mathrm{d}Y$ counts the jumps of Y. That is, if $J_1, J_2, \cdots$ are the jump times of Y then

$$\int_0^t f(s)\mathrm{d}Y_s = \sum_{i=1}^{Y_t} f(J_i).$$

Proof. On the right hand side we have

$$
\begin{aligned}
f(t)Y_t - \int_0^t f'(s)Y_s\ \mathrm{d}s &= f(t)Y_t - \sum_{i=1}^{Y_t-1}\int_{J_i}^{J_{i+1}} f'(s)Y_s\mathrm{d}s - \int_{J_{Y_t}}^t f'(s)Y_s\mathrm{d}s\\
&= f(t)Y_t - \sum_{i=1}^{Y_t-1} i\int_{J_i}^{J_{i+1}} f'(s)\mathrm{d}s - Y_t\int_{J_{Y_t}}^t f'(s)\mathrm{d}s\\
&= f(t)Y_t - \sum_{i=1}^{Y_t-1} i\big(f(J_{i+1}) - f(J_i)\big) - Y_t\big(f(t) - f(J_{Y_t})\big)\\
&= \sum_{i=1}^{Y_t} f(J_i).
\end{aligned}
$$

□

Note that we didn't need to know the distribution of jump times in the proof, so the proof works for a class of processes larger than time-inhomogeneous Poisson processes.

As one application of the above result we get the asymptotic behaviour of $\int_0^t f(s)\mathrm{d}Y_s$ as $t \to T$. Before we present it let us mention a simple result about the asymptotic growth of Y_t as $t \to T$.

Proposition 3.12. *Let $(Y_t)_{t\in[0,T)} \stackrel{d}{=} IPP(r(t))$ as in the previous proposition. If $\lim_{t\to T} \int_0^t r(s)\mathrm{d}s = \infty$ then*

$$\frac{Y_t}{\int_0^t r(s)\mathrm{d}s} \to 1 \text{ a.s. as } t \to T.$$

Proof. Let $R(t) := \int_0^t r(s)ds$. It is a well-known fact that $(Y_t)_{t\in[0,T)} \stackrel{d}{=} (Z_{R(t)})_{t\in[0,T)}$, where $(Z_t)_{t\geq 0}$ is a $PP(1)$. It is also well-known that $\frac{Z_t}{t} \to 1$ a.s. as $t \to \infty$. Thus

$$\begin{aligned} &\frac{Z_{R(t)}}{R(t)} \to 1 \text{ a.s. as } t \to T \\ \Rightarrow &\frac{Y_t}{\int_0^t r(s)\mathrm{d}s} = \frac{Y_t}{R(t)} \to 1 \text{ a.s. as } t \to T. \end{aligned}$$

That finishes the proof of Proposition 3.12. For completeness let us also prove that $\frac{Z_t}{t} \to 1$ as $t \to \infty$:

$(Z_{i+1} - Z_i)_{i\geq 0}$ are independent $Po(1)$ random variables, so for $n \in \mathbb{N}$

$$\frac{Z_n}{n} = \frac{(Z_1 - Z_0) + \cdots + (Z_n - Z_{n-1})}{n} \to 1 \text{ a.s. as } n \to \infty$$

by the Strong Law of Large Numbers.

More generally, $Z_{\lfloor t\rfloor} \leq Z_t \leq Z_{\lceil t\rceil}$, so

$$\frac{Z_{\lfloor t\rfloor}}{t} \leq \frac{Z_t}{t} \leq \frac{Z_{\lceil t\rceil}}{t},$$

but $\frac{Z_{\lfloor t\rfloor}}{t} = \frac{Z_{\lfloor t\rfloor}}{\lfloor t\rfloor}\frac{\lfloor t\rfloor}{t} \to 1$ a.s. and similarly $\frac{Z_{\lceil t\rceil}}{t} \to 1$. Thus

$$\frac{Z_t}{t} \to 1 \text{ a.s. as } t \to \infty.$$

□

Now let us put together Propositions 3.11 and 3.12 to get the asymptotic behaviour of $\int_0^t f(s)\mathrm{d}Y_s$, which will be useful to us later in this chapter.

Proposition 3.13. *Let $(Y_t)_{t\in[0,T)} \stackrel{d}{=} IPP(r(t))$ as before. Let $f : [0,T) \to [0,\infty)$ be differentiable such that $f'(t) \geq 0$ for t large enough and let $r : [0,T) \to [0,\infty)$ be locally* *integrable. Suppose r and f satisfy the following two conditions:*

1. $\int_0^t r(s)\mathrm{d}s \to \infty$ *as* $t \to T$

2. $c := \limsup_{t\to T} \frac{f(t)\int_0^t r(s)\mathrm{d}s}{\int_0^t f(s)r(s)\mathrm{d}s} < \infty$

Then

$$\frac{\int_0^t f(s)\mathrm{d}Y_s}{\int_0^t f(s)r(s)\mathrm{d}s} \to 1 \text{ a.s. as } t \to T.$$

Note that the second condition is generally rather restrictive, but it is satisfied by the functions that we consider in this chapter. Also, since f is non-decreasing, $c \geq 1$.

Proof. Fix $\epsilon > 0$. By Proposition 3.12 there exists a random time $T_\epsilon < T$ such that $\forall t \geq T_\epsilon$:

$$1-\epsilon \leq \frac{Y_t}{\int_0^t r(s)\mathrm{d}s} \leq 1+\epsilon$$
$$\Rightarrow (1-\epsilon)\int_0^t r(s)\mathrm{d}s \leq Y_t \leq (1+\epsilon)\int_0^t r(s)\mathrm{d}s$$

Also we can assume that $f'(t) \geq 0$ for $t \geq T_\epsilon$. Hence $\forall t \geq T_\epsilon$ using Proposition 3.11 we have

$$\frac{\int_0^t f(s)\mathrm{d}Y_s}{\int_0^t f(s)r(s)\mathrm{d}s} = \frac{f(t)Y_t - \int_0^t f'(s)Y_s\mathrm{d}s}{\int_0^t f(s)r(s)\mathrm{d}s}$$

$$\leq \frac{(1+\epsilon)f(t)\int_0^t r(s)\mathrm{d}s - \int_0^{T_\epsilon} Y_s f'(s)\mathrm{d}s - (1-\epsilon)\int_{T_\epsilon}^t f'(s)\int_0^s r(v)\mathrm{d}v\mathrm{d}s}{\int_0^t f(s)r(s)\mathrm{d}s}$$
$$= \frac{(1+\epsilon)f(t)\int_0^t r(s)\mathrm{d}s - (1-\epsilon)\int_0^t f'(s)\int_0^s r(v)\mathrm{d}v\mathrm{d}s + A_\epsilon}{\int_0^t f(s)r(s)\mathrm{d}s},$$

where $A_\epsilon = -\int_0^{T_\epsilon} f'(s)Y_s\mathrm{d}s + (1-\epsilon)\int_0^{T_\epsilon} f'(s)\int_0^s r(v)\mathrm{d}v\mathrm{d}s$ is an a.s. finite r.v.,

$$= \frac{(1+\epsilon)f(t)\int_0^t r(s)\mathrm{d}s - (1-\epsilon)\left\{[f(s)\int_0^s r(v)\mathrm{d}v]_0^t - \int_0^t f(s)r(s)\mathrm{d}s\right\} + A_\epsilon}{\int_0^t f(s)r(s)\mathrm{d}s}$$
$$= 2\epsilon\frac{f(t)\int_0^t r(s)\mathrm{d}s}{\int_0^t f(s)r(s)\mathrm{d}s} + (1-\epsilon) + \frac{A_\epsilon}{\int_0^t f(s)r(s)\mathrm{d}s}.$$

Thus, by taking the lim sup as $t \to T$ and using condition 2. of the Proposition we get:

$$\limsup_{t\to T} \frac{\int_0^t f(s)\mathrm{d}Y_s}{\int_0^t f(s)r(s)\mathrm{d}s} \leq 1 + \epsilon(2c-1).$$

Similarly we have

$$\frac{\int_0^t f(s)\mathrm{d}Y_s}{\int_0^t f(s)r(s)\mathrm{d}s} \geq -2\epsilon\frac{f(t)\int_0^t r(s)\mathrm{d}s}{\int_0^t f(s)r(s)\mathrm{d}s} + (1+\epsilon) + \frac{B_\epsilon}{\int_0^t f(s)r(s)\mathrm{d}s},$$

where B_ϵ is some almost surely finite random variable, so

$$\liminf_{t\to T} \frac{\int_0^t f(s)\mathrm{d}Y_s}{\int_0^t f(s)r(s)\mathrm{d}s} \geq 1 - \epsilon(2c-1).$$

Since ϵ was arbitrary we deduce that

$$\frac{\int_0^t f(s)\mathrm{d}Y_s}{\int_0^t f(s)r(s)\mathrm{d}s} \to 1 \text{ as } t \to T.$$

$\square$

3.2.3 Changes of measure for continuous-time random walks

Let $T \in (0,\infty]$. Recall the continuous-time random walk $(X_t)_{t\in[0,T)}$ defined under some probability measure $\mathbb{P}$ from subsection 3.1.1. As it was already mentioned we can write

$$X_t = X_t^+ - X_t^- \ , \ t \in [0,T),$$

where $(X_t^+)_{t\to[0,T)}$, $(X_t^-)_{t\in[0,T)} \stackrel{d}{=} PP(\lambda)$ independently of each other.

From Lemmas 3.6 and 3.8 we get the following result.

Proposition 3.14. *Let θ^+, θ^- : $[0,T) \to [0,\infty)$ be two locally-integrable functions. Then the following process is a $\mathbb{P}$-martingale:*

$$M_t := e^{\int_0^t \log\theta^+(s)\mathrm{d}X_s^+ + \lambda\int_0^t (1-\theta^+(s))\mathrm{d}s \ + \ \int_0^t \log\theta^-(s)\mathrm{d}X_s^- + \lambda\int_0^t(1-\theta^-(s))\mathrm{d}s} \ , \ t \in [0,T). \quad (3.11)$$

Moreover, if we define the new measure $\mathbb{Q}$ as

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\hat{\mathcal{F}}_t} = M_t \quad , \ t \in [0,T),$$

where $(\hat{\mathcal{F}}_t)_{t\in[0,T)}$ is the natural filtration of $(X_t)_{t\in[0,T)}$, then under $\mathbb{Q}$

$$(X_t^+)_{t\in[0,T)} \stackrel{d}{=} IPP\big(\lambda\theta^+(t)\big),$$

$$(X_t^-)_{t\in[0,T)} \stackrel{d}{=} IPP\big(\lambda\theta^-(t)\big).$$

In other words the martingale M used as the Radon-Nikodym derivative has the effect of scaling the rate of upward jumps by the factor of $\theta^+(t)$ and the rate of downward jumps by the factor $\theta^-(t)$ at time t.

Furthermore from Propositions 3.12 and 3.13 we know that $\mathbb{Q}$-a.s.

$$\lim_{t\to T}\frac{X_t^+}{\int_0^t\lambda\theta^+(s)\mathrm{d}s}=1,$$
$$\lim_{t\to T}\frac{X_t^-}{\int_0^t\lambda\theta^-(s)\mathrm{d}s}=1,$$
$$\lim_{t\to T}\frac{\int_0^t f(s)\mathrm{d}X_s^+}{\int_0^t\lambda\theta^+(s)f(s)\mathrm{d}s}=1,$$
$$\lim_{t\to T}\frac{\int_0^t f(s)\mathrm{d}X_s^-}{\int_0^t\lambda\theta^-(s)f(s)\mathrm{d}s}=1$$

provided that θ^+, θ^- and f satisfy the conditions of Propositions 3.12 and 3.13.

3.3 Explosion

Consider a branching random walk in the potential $\beta(\cdot)=\beta|\cdot|^p$, where $\beta>0$, $p\geq 0$, as it was described in Section 1. In this section we give the proof of Theorem 3.2. We shall apply the same methods as we did for the BBM model in Chapter 2.

3.3.1 $p\leq 1$ case

Let us first prove part a) of Theorem 3.2 which is a lot simpler than part b).

Proof of Theorem 3.2 a). We wish to show that if $p\leq 1$ then $P(T_{explo}=\infty)=1$. As for the BBM case it is sufficient to show that $E(|N_t|)<\infty$ for some $t>0$.

By the Many-to-One lemma (Lemma 1.18)

$$E\Big(|N_t|\Big)=E\Big(\sum_{u\in N_t}1\Big)=\tilde{E}\Big(e^{\int_0^t\beta|\xi_s|^p\mathrm{d}s}\Big),$$

where $(\xi_t)_{t\geq 0}$ is a continouos-time random walk under $\tilde{P}$. Then $\xi_t=\xi_t^+-\xi_t^-$, where $(\xi_t^+)_{t\geq 0}$ and $(\xi_t^-)_{t\geq 0}$ are two independent Poisson processes with jump rate λ. Therefore

$$\begin{aligned}\tilde{E}\Big(e^{\int_0^t\beta|\xi_s|^p\mathrm{d}s}\Big)&\leq\tilde{E}\Big(e^{t\beta\sup_{0\leq s\leq t}|\xi_s|^p}\Big)\\&=\tilde{E}\Big(e^{t\beta\sup_{0\leq s\leq t}|\xi_s^+-\xi_s^-|^p}\Big)\leq\tilde{E}\Big(e^{t\beta\sup_{0\leq s\leq t}\big((\xi_s^+)^p\vee(\xi_s^-)^p\big)}\Big)\\&=\tilde{E}\Big(e^{t\beta\big((\xi_t^+)^p\vee(\xi_t^-)^p\big)}\Big)\leq\tilde{E}\Big(e^{t\beta\big((\xi_t^+)^p+(\xi_t^-)^p\big)}\Big)\\&=\Big[\tilde{E}\Big(e^{t\beta(\xi_t^+)^p}\Big)\Big]^2\leq\Big[\tilde{E}\Big(e^{t\beta\xi_t^+}\Big)\Big]^2\end{aligned}$$

because ξ^+ is supported on $\{0, 1, 2, ...\}$ whence $(\xi_t^+)^p \leq \xi_t^+$ for $p \in [0, 1]$. Then

$$\begin{aligned}\tilde{E}\left(e^{t\beta\xi_t^+}\right) &= \sum_{n=0}^{\infty} e^{\beta tn} \tilde{P}(\xi_t^+ = n) \\ &= \sum_{n=0}^{\infty} e^{\beta tn} \frac{(\lambda t)^n}{n!} e^{-\lambda t} = \exp\left\{e^{\beta t}\lambda t - \lambda t\right\} < \infty \quad \forall t \geq 0.\end{aligned}$$

Thus $E(|N_t|) < \infty$ for all $t > 0$ and this finishes the proof Theorem 3.2 a). □

3.3.2 $p > 1$ case

Let us now prove part b) of Theorem 3.2.

Proof of Theorem 3.2 b). We wish to show that if $p > 1$ then $P(T_{explo} < \infty) = 1$. As in Chapter 2 it would be sufficient to prove that $P(T_{explo} \leq T) > 0$ for any $T > 0$. Assume for contradiction that $\exists T > 0$ s.t.

$$P(T_{explo} \leq T) = 0. \tag{3.12}$$

Fix this T for the rest of this subsection. The key steps of the proof can then be summarised as follows:

1. We choose appropriate functions θ^+, $\theta^- : [0, T) \to [0, \infty)$ for the one-particle martingale of the form (3.11) from Proposition 3.14, such that under the new measure the process goes to ∞ at time T.

2. For this choice of θ^+ and θ^- we define additive martingale $(M(t))_{t \in [0,T)}$ and the corresponding probability measure Q.

3. We show that $\limsup_{t \to T} M(t) < \infty$ Q-a.s.

4. We deduce that $Q \ll P$ on $\mathcal{F}_T$, whence with positive P-probability one particle goes to ∞ at time T giving infinitely many births along its path.

5. We get a contradiction to (3.12).

To avoid unnecessary repetitions we shall omit some details, which can be found in Chapter 2.

We start by defining the new measure $\tilde{Q}$ via the spine martingale $\tilde{M}$ as in Proposition 1.24.

The spine process $(\xi_t)_{t \in [0,T)}$ can be written as the difference of processes $(\xi_t^+)_{t \in [0,T)}$ and $(\xi_t^-)_{t \in [0,T)}$, where under $\tilde{P}$, ξ^+ and ξ^- are two independent $PP(\lambda)$ processes.

Then recalling one-particle martingale (3.11) from Proposition 3.14 and letting $\theta^-(\cdot) \equiv 1$ there we take

$$\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}\tilde{P}}\Big|_{\tilde{\mathcal{F}}_t} = \tilde{M}(t),$$

where

$$\tilde{M}(t) = 2^{n_t} e^{-\beta \int_0^t |\xi_s|^p \mathrm{d}s} \exp\Big(\int_0^t \log\theta^+(s)\mathrm{d}\xi_s^+ + \int_0^t \lambda(1-\theta^+(s))\mathrm{d}s\Big), \quad t \in [0,T) \tag{3.13}$$

and $\theta^+ : [0,T) \to [0,\infty)$ is some function to be defined a little later. One important feature of θ^+ is that it explodes at time T.

This gives rise to additive martingale $(M(t))_{t\in[0,T)}$ and probability measure Q such that

$$\frac{\mathrm{d}Q}{\mathrm{d}P}\Big|_{\mathcal{F}_t} = M(t)\ ,\ t \in [0,T) \tag{3.14}$$

and

$$M(t) = \sum_{u\in N_t} \exp\Big(\int_0^t \log\theta^+(s)\mathrm{d}X_u^+(s) + \int_0^t \lambda\big(1-\theta^+(s)\big)\mathrm{d}s - \beta\int_0^t |X_u(s)|^p\mathrm{d}s\Big)\ ,\ t \in [0,T), \tag{3.15}$$

where for a particle $u \in N_t$, $(X_u^+(s))_{s\in[0,t]}$ is the process of its positive jumps.

There are lots of choices of θ^+ that will make the proof work. The natural form of θ^+ to look at is

$$\theta^+(s) = \frac{c_1}{\lambda(p-1)}\big(T-s\big)^{-c_2}\Big(\log\big(\frac{T}{T-s}\big)\Big)^{c_3}, \qquad s \in [0,T)$$

for $c_1, c_2, c_3 > 0$ (see Figure 3-3 below). Again, we are only interested in the asymptotic growth of $\theta^+(s)$ as $s \to T$, so it doesn't really matter what values it takes away from T. Just as in the BBM model there is the 'critical' path

$$\theta_*^+(s) = \frac{c_1^*}{\lambda(p-1)}\big(T-s\big)^{-c_2^*}\Big(\log\big(\frac{T}{T-s}\big)\Big)^{c_3^*},$$

where

$$c_1^* = \Big(\frac{p}{\beta(p-1)^2}\Big)^{\frac{1}{p-1}},\ c_2^* = \frac{p}{p-1},\ c_3^* = \frac{1}{p-1}. \tag{3.16}$$

So that if we pick a path which grows faster than θ_*^+ then $spine(t) \to 0$. Thus we take

$$\theta^+(s) = \frac{c_1}{\lambda(p-1)}\big(T-s\big)^{-c_2^*}\Big(\log\big(\frac{T}{T-s}\big)\Big)^{c_3^*} \tag{3.17}$$

for some $c_1 > c_1^*$ (e.g. $c_1 = c_1^* + 1$).

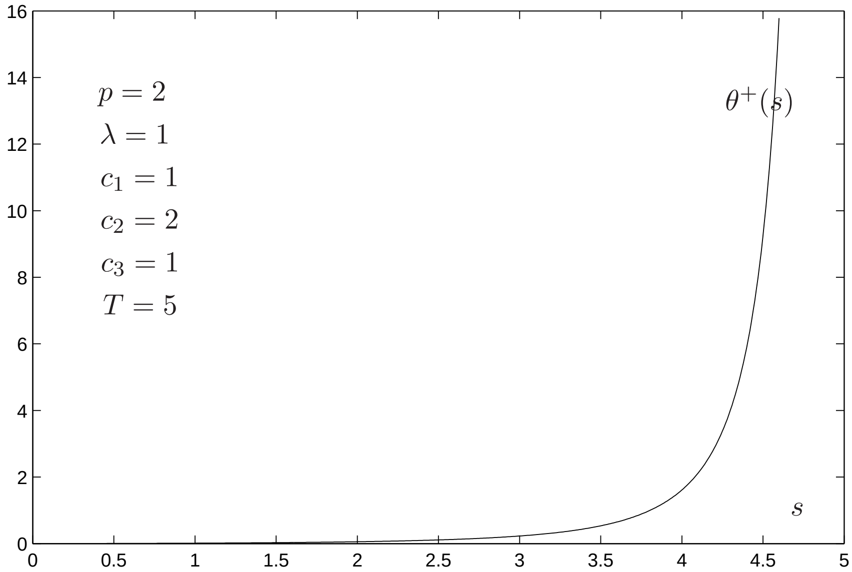


Figure 3-3: *Plot of $\theta^+(s)$ when $p > 1$*

As a special case of Theorem 2.9 we have the following lemma.

Lemma 3.15. *Let M be defined as in* (3.15) *with function θ^+ as in* (3.17)*. Let the probability measure Q be as in* (3.14)*. Then for events $A \in \mathcal{F}_T$*

$$Q(A) = \int_A \limsup_{t\to T} M(t)\mathrm{d}P + Q\big(A \cap \{\limsup_{t\to T} M(t) = \infty\}\big). \tag{3.18}$$

Our aim is again to show that $\limsup_{t\to T} M(t) < \infty$ $\tilde{Q}$-a.s.

The spine decomposition (recall Theorem 1.29) tells us that

$$E^{\tilde{Q}}\Big(M(t)\big|\tilde{\mathcal{G}}_T\Big) = sum(t) + spine(t),$$

where

$$spine(t) = \exp\Big(\int_0^t \log\theta^+(s)\mathrm{d}\xi_s^+ + \int_0^t \lambda\big(1-\theta^+(s)\big)\mathrm{d}s - \int_0^t \beta|\xi_s|^p\mathrm{d}s\Big)$$

and

$$sum(t) = \sum_{u<node_t(\xi)} e^{\int_0^{S_u}\log\theta^+(s)\mathrm{d}\xi_s^+ + \int_0^{S_u}\lambda\big(1-\theta^+(s)\big)\mathrm{d}s - \int_0^{S_u}\beta|\xi_s|^p\mathrm{d}s}.$$

If we can show that $\limsup_{t\to T} E^{\tilde{Q}}(M(t)|\tilde{\mathcal{G}}_T) < \infty$ $\tilde{Q}$-a.s. then it will follow that $\limsup_{t\to T} M(t) < \infty$ $\tilde{Q}$-a.s.

We start by proving the following assertion about the spine term.

Proposition 3.16. *There exist some $\tilde{Q}$-a.s. finite positive random variables C', C'' and a random time $T' \in [0,T)$ such that $\forall t > T'$*

$$spine(t) \le C' \exp\Big(- C'' \int_0^t \Big(\frac{1}{T-s} \log\big(\frac{T}{T-s}\big)\Big)^{\frac{p}{p-1}} \mathrm{d}s\Big).$$

Note that as $t \to T$

$$\int_0^t \Big(\frac{1}{T-s} \log\big(\frac{T}{T-s}\big)\Big)^{\frac{p}{p-1}} \mathrm{d}s \sim (p-1)\Big(\frac{1}{T-t}\Big)^{\frac{1}{p-1}} \Big(\log\big(\frac{T}{T-t}\big)\Big)^{\frac{p}{p-1}} \to \infty.$$

Proof of Proposition 3.16. Under $\tilde{Q}$ the process $(\xi_t^+)_{t\in[0,T)}$ is an inhomogeneous Poisson process of rate $\lambda\theta^+(t)$ as it follows from Proposition 3.14. Also $(\xi_t^-)_{t\in[0,T)}$ is a Poisson process of rate λ, which must be bounded on $[0,T)$.

Simple calculus tells us that for constants $k_1 > 1$, $k_2 > 0$

$$\int_0^t (T-s)^{-k_1} \Big(\log\big(\frac{T}{T-s}\big)\Big)^{k_2} \mathrm{d}s \sim \frac{1}{k_1 - 1}(T-t)^{-k_1+1}\Big(\log\big(\frac{T}{T-t}\big)\Big)^{k_2} \text{ as } t \to T.$$

Hence one can check that the following are true as $t \to T$ for θ^+ defined in (3.17):

- $\int_0^t \lambda\theta^+(s)\mathrm{d}s \;\sim\; c_1 (T-t)^{-\frac{1}{p-1}} \Big(\log\big(\frac{T}{T-t}\big)\Big)^{\frac{1}{p-1}} \to \infty$
- $\log\theta^+(t)\lambda\theta^+(t) \;\sim\; \dfrac{c_1 p}{(p-1)^2}(T-t)^{-\frac{p}{p-1}}\Big(\log\big(\frac{T}{T-t}\big)\Big)^{\frac{p}{p-1}}$
- $\limsup_{t\to T} \dfrac{\log\theta^+(t)\int_0^t \lambda\theta^+(s)\mathrm{d}s}{\int_0^t \log\theta^+(s)\;\lambda\theta^+(s)\mathrm{d}s} = 1 < \infty$

Hence from Proposition 3.12 and Proposition 3.13 we have that

- $\dfrac{\xi_t}{\int_0^t \lambda\theta^+(s)\mathrm{d}s} \to 1\; \tilde{Q}$ -a.s.
- $\dfrac{\int_0^t \log\theta^+(s)\mathrm{d}\xi_s^+}{\int_0^t \log\theta^+(s)\;\lambda\theta^+(s)\mathrm{d}s} \to 1\; \tilde{Q}$ -a.s.

Combining these observations we get that $\forall \epsilon > 0\; \exists\; \tilde{Q}$-a.s. finite time T_ϵ such that $\forall t > T_\epsilon$ the following inequalities are true:

$$\begin{aligned}
\int_0^t \log\theta^+(s)\mathrm{d}\xi_s^+ &< (1+\epsilon)\int_0^t \log\theta^+(s)\;\lambda\theta^+(s)\mathrm{d}s,\\
\log\theta^+(t)\lambda\theta^+(t) &< (1+\epsilon)\frac{c_1 p}{(p-1)^2}(T-t)^{-\frac{p}{p-1}}\Big(\log\frac{T}{T-t}\Big)^{\frac{p}{p-1}},\\
|\xi_t| &> (1-\epsilon)c_1(T-t)^{-\frac{1}{p-1}}\Big(\log\frac{T}{T-t}\Big)^{\frac{1}{p-1}},\\
\lambda\big(1-\theta^+(t)\big) &< 0.
\end{aligned}$$

Thus, for $t > T_\epsilon$ we have

$$\begin{aligned}
spine(t) &= \exp\Big(\int_0^t \log\theta^+(s)\mathrm{d}\xi_s^+ + \int_0^t \lambda\big(1-\theta^+(s)\big)\mathrm{d}s - \int_0^t \beta|\xi_s|^p\mathrm{d}s\Big)\\
&\le C_\epsilon \exp\Big\{\int_0^t (1+\epsilon)^2\frac{c_1 p}{(p-1)^2}(T-s)^{-\frac{p}{p-1}}\Big(\log\frac{T}{T-s}\Big)^{\frac{p}{p-1}}\\
&\qquad -\beta\Big((1-\epsilon)c_1(T-s)^{-\frac{1}{p-1}}\big(\log\frac{T}{T-s}\big)^{\frac{1}{p-1}}\Big)^p\mathrm{d}s\Big\}\\
&= C_\epsilon \exp\Big\{\Big((1+\epsilon)^2 c_1\frac{p}{(p-1)^2} - \beta(1-\epsilon)^p c_1^p\Big)\int_0^t\Big(\frac{1}{T-s}\log\frac{T}{T-s}\Big)^{\frac{p}{p-1}}\mathrm{d}s\Big\},
\end{aligned}$$

where C_ϵ is some a.s. finite random variable, which doesn't depend on t. Then

$$\begin{aligned}
(1+\epsilon)^2 c_1\frac{p}{(p-1)^2} - \beta(1-\epsilon)^p c_1^p &= c_1(1-\epsilon)^p\beta\Big(\frac{(1+\epsilon)^2}{(1-\epsilon)^p}\frac{p}{\beta(p-1)^2} - c_1^{p-1}\Big)\\
&= c_1(1-\epsilon)^p\beta\Big(\frac{(1+\epsilon)^2}{(1-\epsilon)^p}(c_1^*)^{p-1} - c_1^{p-1}\Big)\\
&< 0
\end{aligned}$$

for ϵ small enough. So letting $T' = T_\epsilon$, $C' = C_\epsilon$ and $C'' = (1+\epsilon)^2 c_1\frac{p}{(p-1)^2} - \beta(1-\epsilon)^p c_1^p$ we finish the proof of Proposition 3.16. $\square$

We now look at the *sum* term:

$$\begin{aligned}
sum(t) &= \sum_{u<node_t(\xi)} spine(S_u)\\
&= \Big(\sum_{u<node_t(\xi),\ S_u\le T'} spine(S_u)\Big) + \Big(\sum_{u<node_t(\xi),\ S_u>T'} spine(S_u)\Big)\\
&\le \sum_{u<node_t(\xi),\ S_u\le T'} spine(S_u)\\
&\qquad + \sum_{u<node_t(\xi),\ S_u>T'} C'\exp\Big(-C''\int_0^{S_u}\Big(\frac{1}{T-s}\log\big(\frac{T}{T-s}\big)\Big)^{\frac{p}{p-1}}\mathrm{d}s\Big)
\end{aligned}$$

using Proposition 3.16. The first sum is $\tilde{Q}$-a.s. bounded since it only counts births up to time T'. Call an upper bound on the first sum C_1. Then we have

$$sum(t) \le C_1 + C'\sum_{n=1}^{\infty}\exp\Big(-C''\int_0^{S_n}\Big(\frac{1}{T-s}\log\big(\frac{T}{T-s}\big)\Big)^{\frac{p}{p-1}}\mathrm{d}s\Big), \qquad (3.19)$$

where S_n is the time of the n^{th} birth on the spine.

The birth process along the spine $(n_t)_{t\in[0,T)}$ conditional on the path of the spine is time-inhomogeneous Poisson process (or Cox process) with birth rate $2\beta|\xi_t|^p$ at time t

(recall Proposition 1.13). Thus

$$\frac{n_t}{\int_0^t 2\beta|\xi_s|^p \mathrm{d}s} \to 1 \quad \tilde{Q}\text{-a.s. as } t \to T.$$

Also

$$\int_0^t |\xi_s|^p \mathrm{d}s \sim \int_0^t c_1^p (T-s)^{-\frac{p}{p-1}} \Big(\log \big(\frac{T}{T-s} \big) \Big)^{\frac{p}{p-1}} \mathrm{d}s \quad \tilde{Q}\text{-a.s. as } t \to T.$$

Hence

$$n_t \sim 2\beta c_1^p \int_0^t (T-s)^{-\frac{p}{p-1}} \Big(\log \big(\frac{T}{T-s} \big) \Big)^{\frac{p}{p-1}} \mathrm{d}s \quad \tilde{Q}\text{-a.s. as } t \to T. \tag{3.20}$$

So for some $\tilde{Q}$-a.s. finite positive random variable C_2 we have

$$\int_0^{S_n} (T-s)^{-\frac{p}{p-1}} \Big(\log \big(\frac{T}{T-s} \big) \Big)^{\frac{p}{p-1}} \mathrm{d}s \geq C_2 n \quad \forall n.$$

Then substituting this into (3.19) we get

$$sum(t) \leq C_1 + C' \sum_{n=1}^{\infty} e^{-C'' C_2 n},$$

which is bounded $\tilde{Q}$-a.s. We have thus shown that

$$\limsup_{t \to T} E^{\tilde{Q}} \Big(M(t) \big| \tilde{\mathcal{G}}_T \Big) < \infty \qquad \tilde{Q}\text{-a.s.}$$

Exactly the same argument as in the proof of Theorem 2.7 b) gives us that

$$\limsup_{t \to T} M(t) < \infty \qquad Q\text{-a.s.}$$

From Lemma 3.15 it now follows that for events $A \in \mathcal{F}_T$

$$Q(A) = \int_A \limsup_{t \to T} M(t) \mathrm{d}P.$$

Therefore $Q(A) > 0 \Rightarrow P(A) > 0$. Let us consider the event

$$A := \Big\{ |N_t| \to \infty \text{ as } t \to T \Big\} \in \mathcal{F}_T.$$

From (3.20) we have

$$
\begin{aligned}
&\tilde{Q}\Big(n_t \to \infty \text{ as } t \to T\Big) = 1 \\
\Rightarrow &Q\Big(|N_t| \to \infty \text{ as } t \to T\Big) = \tilde{Q}\Big(|N_t| \to \infty \text{ as } t \to T\Big) = 1 \\
\Rightarrow &P\Big(|N_t| \to \infty \text{ as } t \to T\Big) > 0 \\
\Rightarrow &P\Big(T_{explo} \leq T\Big) > 0,
\end{aligned}
$$

which contradicts the initial assumption (3.12). Therefore we must have

$$P(T_{explo} \leq T) > 0 \ \forall T > 0$$

and hence

$$T_{explo} < \infty \ P\text{-a.s.}$$

This completes the proof of Theorem 3.2 $\square$

Remark 3.17. *Recall Proposition 2.8. Replacing $\frac{\partial^2}{\partial x^2}$ in (2.2) with the infinitesimal generator of a continuous-time random walk, we get an equation solved by $u(t,x) := P^x(T_{explo} \leq t)$:*

$$
\begin{cases}
\dfrac{\partial u}{\partial t} = \lambda\big(u(t,x+1) + u(t,x-1) - 2u(t,x)\big) + u(t,x)(1-u(t,x))\beta|x|^p \\
\qquad\qquad\qquad\qquad u(0,x) = 0, \ 0 \leq u \leq 1
\end{cases}
\tag{3.21}
$$

3.4 The rightmost particle

This section is devoted to the proof of Theorem 3.4. The method of proof comes from [22] and is based on the analysis of a family of additive martingales defined below.

3.4.1 Additive martingales

Take the spine process $(\xi_t)_{t\geq 0}$, which under the probability measure $\tilde{P}$ is a continuous-time random walk. As it was noted earlier, we can write $\xi_t = \xi_t^+ - \xi_t^-$, where $(\xi_t^+)_{t\geq 0}$ is the process of positive jumps of ξ and $(\xi_t^-)_{t\geq 0}$ the porcess of its negative jumps. Then ξ^+ and ξ^- are independent processes and $(\xi_t^+)_{t\geq 0}$, $(\xi_t^-)_{t\geq 0} \overset{d}{=} PP(\lambda)$.

Let $\theta = (\theta^+, \theta^-)$, where θ^+, θ^- : $[0,\infty) \to [0,\infty)$ are two locally-integrable functions. From Lemma 3.6 we have that for a given θ the following is a martingale with respect to $\tilde{P}$:

$$e^{\int_0^t \log\theta^+(s)\mathrm{d}\xi_s^+ + \int_0^t \log\theta^-(s)\mathrm{d}\xi_s^- + \lambda\int_0^t 2-\theta^+(s)-\theta^-(s)\mathrm{d}s}, \qquad t \geq 0. \tag{3.22}$$

In the general setting decribed in Chapter 1 this would correspond to the martingale $\tilde{M}^{(3)}$ in (1.5). Now, recalling (1.6), we define a $\tilde{P}$-martingale w.r.t filtration $(\tilde{\mathcal{G}}_t)_{t\geq 0}$:

$$\tilde{M}_\theta(t) := e^{-\beta\int_0^t |\xi_s|^p \mathrm{d}s} 2^{n_t} \times \exp\Big(\int_0^t \log\theta^+(s)\mathrm{d}\xi_s^+ + \int_0^t \lambda(1-\theta^+(s))\mathrm{d}s \\ + \int_0^t \log\theta^-(s)\mathrm{d}\xi_s^- + \int_0^t \lambda(1-\theta^-(s))\mathrm{d}s\Big) \tag{3.23}$$

and the corresponding probability measure $\tilde{Q}_\theta$:

$$\frac{\mathrm{d}\tilde{Q}_\theta}{\mathrm{d}\tilde{P}}\bigg|_{\tilde{\mathcal{F}}_t} = \tilde{M}_\theta(t)\ , \qquad t\geq 0. \tag{3.24}$$

Under $\tilde{Q}_\theta$ the branching process has the following description:

- The initial particle (the spine) moves like a biased random walk. That is, at time t it jumps up at instantaneous rate $\lambda\theta^+(t)$ and jumps down at instantaneous rate $\lambda\theta^-(t)$.
- When it is at position x it splits into two new particles at instantaneous rate $2\beta(x)$.
- One of these particles (chosen uniformly at random) continues the spine. I.e. it continues moving as a biased random walk and branching at rate $2\beta(\cdot)$.
- The other particle initiates an unbiased branching process (as under P) from the position of the split.

Further, if we recall (1.7) and (1.8), we can define the probability measure $Q_\theta := \tilde{Q}_\theta|_{\mathcal{F}_\infty}$ so that

$$\frac{\mathrm{d}Q_\theta}{\mathrm{d}P}\bigg|_{\mathcal{F}_t} = M_\theta(t)\ , \qquad t\geq 0, \tag{3.25}$$

where $M_\theta(t)$ is the additive martingale

$$M_\theta(t) = \sum_{u\in N_t} \exp\Big(\int_0^t \log\theta^+(s)\mathrm{d}X_u^+(s) + \int_0^t \log\theta^-(s)\mathrm{d}X_u^-(s) \\ + \int_0^t \lambda\big(2-\theta^+(s)-\theta^-(s)\big)\mathrm{d}s - \beta\int_0^t |X_u(s)|^p\mathrm{d}s\Big) \tag{3.26}$$

and $(X_u^+(s))_{0\leq s\leq t}$ is the process of positive jumps of particle u, $(X_u^-(s))_{0\leq s\leq t}$ is the process of its negative jumps.

Having defined this family of martingales we can control the behaviour of the spine process via the choice of parameter θ.

In the BBM model in [22] this was achived with the use of exponential (Girsanov) martingales.

3.4.2 Convergence properties of M_θ (under Q_θ)

Before we proceed with the proof, let us emphasise that each M_θ is a positive P-martingale and so it converges almost surely to a finite limit $M_\theta(\infty)$ under P.

The following result will be crucial for us.

Theorem 3.18. *Consider the branching random walk in the potential $\beta(x) = \beta|x|^p$. Let M_θ be the additive martingale as defined in (3.26). Then for different values of p we have the following.*

Case A *($p = 0$), homogeneous case:*

Recall the function $g(\cdot)$ from (3.2). Let $\hat{\theta} \in (1, \infty)$ be the unique solution of $g(\theta) = \frac{\beta}{\lambda}$.

Take a constant $\theta_0 > 1$ and consider $\theta = (\theta^+, \theta^-)$, where $\theta^+(\cdot) \equiv \theta_0$ and $\theta^-(\cdot) \equiv \frac{1}{\theta_0}$. Then

i) $\theta_0 < \hat{\theta} \;\Rightarrow\; M_\theta$ is UI and $M_\theta(\infty) > 0$ a.s. (under P).

ii) $\theta_0 > \hat{\theta} \;\Rightarrow\; M_\theta(\infty) = 0$ P-a.s.

Case B *($p \in (0, 1)$), inhomogeneous non-explosive case:*

$$\text{Let } \hat{b} = \frac{1}{1-p}, \quad \hat{c} = \Big(\frac{\beta(1-p)^2}{p}\Big)^{\hat{b}} \text{ as in (3.3).}$$

Consider $\theta = (\theta^+, \theta^-)$, where $\theta^-(\cdot) \equiv 1$ and for given $b > 1$, $c > 0$

$$\theta^+(s) := \frac{c}{\lambda(1-p)} \frac{s^{b-1}}{(\log(s+2))^b} \;,\; s \geq 0 \quad (\textit{see Figure 3-4 below}).$$

Then

i) $b = \hat{b}$ and $c < \hat{c} \;\Rightarrow\; M_\theta$ is UI and $M_\theta(\infty) > 0$ P-a.s. (the same is true if $b < \hat{b}$)

ii) $b = \hat{b}$ and $c > \hat{c} \;\Rightarrow\; M_\theta(\infty) = 0$ P-a.s. (the same is true if $b > \hat{b}$)

Case C *($p = 1$), inhomogeneous near explosive case:*

Again, consider $\theta = (\theta^+, \theta^-)$, where $\theta^-(\cdot) \equiv 1$ and for given $\alpha > 0$

$$\theta^+(s) := e^{\alpha\sqrt{s}} \;,\; s \geq 0 \quad (\textit{see Figure 3-4 below})$$

Then

i) $\alpha < \sqrt{2\beta} \;\Rightarrow\; M_\theta$ is UI and $M_\theta(\infty) > 0$ P-a.s.

ii) $\alpha > \sqrt{2\beta} \;\Rightarrow\; M_\theta(\infty) = 0$ P-a.s.

The importance of this Theorem comes from the fact that if the martingale M_θ is P-uniformly integrable and $M_\theta(\infty) > 0$ P-a.s. then, as we shall see later, the measures P and Q_θ are equivalent on $\mathcal{F}_\infty$.

Since under $\tilde{Q}_\theta$ the spine process satisfies

$$\frac{\xi_t}{\int_0^t \lambda(\theta^+(s) - \theta^-(s))\mathrm{d}s} \to 1 \text{ a.s. as } t \to \infty$$

it would then follow that under P there is a particle with such asymptotic behaviour too. That would give the lower bound on the rightmost particle:

$$\liminf_{t\to\infty} \frac{R_t}{\int_0^t \lambda(\theta^+(s)-\theta^-(s))\mathrm{d}s} \geq 1,$$

which we can then optimise over θ^+ and θ^-.

The upper bound on the rightmost particle needs a slightly different approach, which we present in the last subsection.

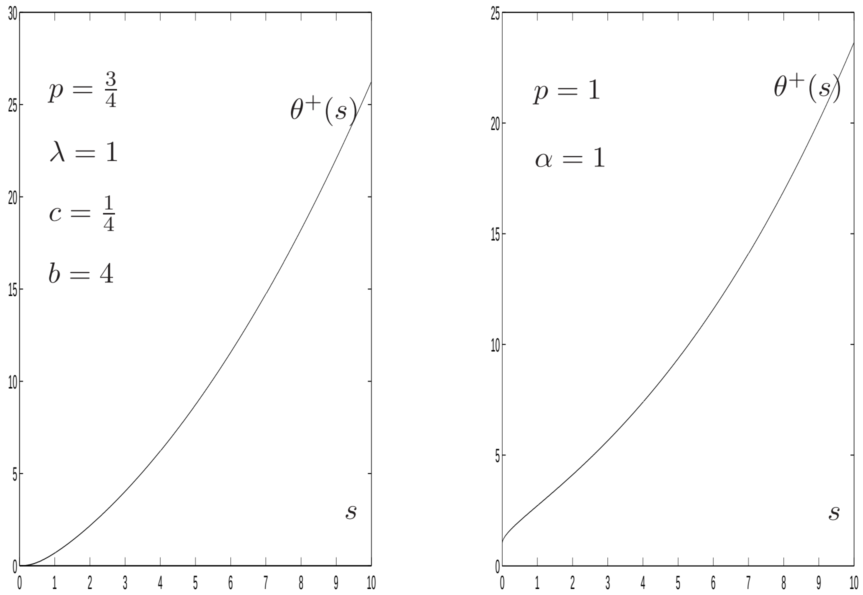


Figure 3-4: *Plots of $\theta^+(s)$ when $p\in(0,1]$*

Remark 3.19. *Let us note that the only important feature of $\theta^+(\cdot)$ in cases **B** and **C** is its asymptotic growth. By this we mean that we have freedom in defining $\theta(\cdot)$ as long as we keep*

$$\theta^+(t) \sim \frac{c}{\lambda(1-p)}\frac{t^{b-1}}{(\log t)^b} \text{ as } t\to\infty \text{ in Case } \mathbf{A}$$

and

$$\log\theta^+(t) \sim \alpha\sqrt{t} \text{ as } t\to\infty \text{ in Case } \mathbf{B}.$$

Remark 3.20. *Parts A ii), B ii) and C ii) of Theorem 3.18 will not be used in the proof of our main result, Theorem 3.4. We included them to better illustrate the behaviour of martingales M_θ.*

Recall Theorem 2.9. It gives the following decomposition of the probability measure Q_θ.

Lemma 3.21. *Let M_θ be a martingale of the form* (3.26) *and let Q_θ be the corresponding probability measure defined via* (3.25)*. Then for events $A \in \mathcal{F}_\infty$*

$$Q_\theta(A) = \int_A \limsup_{t\to\infty} M_\theta(t)\mathrm{d}P + Q_\theta\big(A \cap \{\limsup_{t\to\infty} M_\theta(t) = \infty\}\big) \tag{3.27}$$

By taking $A = \Omega$ we see that

$$1 = E(\limsup_{t\to\infty} M_\theta(t)) + Q_\theta(\limsup_{t\to\infty} M_\theta(t) = \infty)$$

and so immediate consequences of this lemma are:

1) $Q_\theta(\limsup_{t\to\infty} M_\theta(t) = \infty) = 1 \;\Leftrightarrow\; \limsup_{t\to\infty} M_\theta(t) = 0$ P-a.s. So to prove parts A ii), B ii) and C ii) of Theorem 3.18 we need to show that $\limsup_{t\to\infty} M_\theta(t) = \infty$ Q_θ-a.s.

2) $Q_\theta(\limsup_{t\to\infty} M_\theta(t) < \infty) = 1 \Rightarrow EM_\theta(\infty) = 1$ and so in this case $P(M_\theta(\infty) > 0) > 0$ and also M_θ is L^1-convergent w.r.t P as it follows from Scheffe's Lemma. Thus M_θ is P-uniformly integrable. So to prove the uniform integrability in parts A i), B i) and C i) of Theorem 3.18 we need to show that $\limsup_{t\to\infty} M_\theta(t) < \infty$ Q_θ-a.s.

The fact that $M_\theta(\infty) > 0$ P-a.s. (in parts A i), B i) and C i)) requires a separate proof, which we shall give at the end of this subsection.

Proof of Theorem 3.18: uniform integrability in A i), B i), C i). We start with proving that for the given values of θ in A i), B i) and C i) M_θ is UI. As we just said above, it is sufficient to prove that

$$\limsup_{t\to\infty} M_\theta(t) < \infty \ Q_\theta\text{-a.s.} \tag{3.28}$$

for the given paths θ. And we have already seen how to do this using the spine decomposition.

Recall that

$$E^{\tilde{Q}_\theta}(M_\theta(t)|\tilde{\mathcal{G}}_\infty) = spine(t) + sum(t), \tag{3.29}$$

where

$$\begin{aligned} spine(t) = \exp\Big(&\int_0^t \log\theta^+(s)\mathrm{d}\xi_s^+ + \int_0^t \log\theta^-(s)\mathrm{d}\xi_s^- \\ &+ \lambda\int_0^t (2 - \theta^+(s) - \theta^-(s))\mathrm{d}s - \beta\int_0^t |\xi_s|^p\mathrm{d}s\Big) \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} sum(t) &= \sum_{u \in node_t(\xi)} spine(S_u) \\ &= \sum_{u < node_t(\xi)} \exp\Big(\int_0^{S_u} \log\theta^+(s)\mathrm{d}\xi_s^+ + \int_0^{S_u} \log\theta^-(s)\mathrm{d}\xi_s^- \\ &\quad + \lambda \int_0^{S_u} (2-\theta^+(s)-\theta^-(s))\mathrm{d}s - \beta\int_0^{S_u} |\xi_s|^p \mathrm{d}s \Big), \end{aligned} \tag{3.31}$$

where $\{S_u : u \in \xi\}$ is the set of fission times along the spine.

We shall prove the following fact.

Proposition 3.22.

$$\limsup_{t\to\infty} E^{\tilde{Q}_\theta}(M_\theta(t)|\tilde{\mathcal{G}}_\infty) < \infty \ \tilde{Q}_\theta\text{-}a.s.$$

Then (3.28) follows from Proposition 3.22 just as we have seen it in Chapter 2:

$$\begin{aligned} E^{\tilde{Q}_\theta}(\liminf_{t\to\infty} M_\theta(t)|\tilde{\mathcal{G}}_\infty) &\le \liminf_{t\to\infty} E^{\tilde{Q}_\theta}(M_\theta(t)|\tilde{\mathcal{G}}_\infty) \\ &\le \limsup_{t\to\infty} E^{\tilde{Q}_\theta}(M_\theta(t)|\tilde{\mathcal{G}}_\infty) < \ +\infty \quad \tilde{Q}_\theta\text{-a.s.}, \end{aligned}$$

by conditional Fatou's lemma. Hence

$$\liminf_{t\to\infty} M_\theta(t) < \infty \ \tilde{Q}_\theta\text{-a.s.}$$

and thus also Q_θ-a.s. Since $\Big(\dfrac{1}{M_\theta(t)}\Big)_{t\ge 0}$ is a positive Q_θ-supermartingale (as it follows from the definition of Q_θ) it must converge Q_θ-a.s. So $M_\theta(t)$ also converges Q_θ-a.s.

Thus

$$\limsup_{t\to\infty} M_\theta(t) = \liminf_{t\to\infty} M_\theta(t) < \infty \ Q_\theta\text{-a.s.}$$

It remains to prove Proposition 3.22. The cases $p = 0$, $p \in (0,1)$ and $p = 1$ need slightly different approach and so will be dealt with separately.

Proof of Proposition 3.22: Case A $(p = 0)$. We start by looking at the spine term (3.30). The following proposition gives us a useful bound on $spine(t)$.

Proposition 3.23. *There exist some positive constant C'' and a $\tilde{Q}_\theta$-a.s. finite time T' such that $\forall t > T'$*

$$spine(t) \le e^{-C''t}.$$

Proof of Proposition 3.23. We are given parameter $\theta = (\theta^+, \theta^-)$, where $\theta^+(\cdot) \equiv \theta_0$ and $\theta^-(\cdot) \equiv \frac{1}{\theta_0}$. Under $\tilde{Q}_\theta$, $(\xi_t^+)_{t\ge0} \overset{d}{=} PP(\lambda\theta_0)$ and $(\xi_t^-)_{t\ge0} \overset{d}{=} PP(\frac{\lambda}{\theta_0})$.

From Proposition 3.12 we know that

$$\frac{\int_0^t \log\theta_0 \mathrm{d}\xi_s^+}{\lambda\theta_0\log\theta_0 t} = \frac{\xi_t^+}{\lambda\theta_0 t} \to 1 \quad \tilde{Q}_\theta\text{-a.s.}$$

Hence there exists a $\tilde{Q}_\theta$-a.s. finite time T_ϵ^+ such that

$$(1-\epsilon)\lambda\theta_0 t \le \xi_t^+ \le (1+\epsilon)\lambda\theta_0 t \quad \forall t > T_\epsilon^+$$

Similarly there exists a $\tilde{Q}_\theta$-a.s. finite time T_ϵ^- such that

$$(1-\epsilon)\frac{\lambda}{\theta_0}t \le \xi_t^- \le (1+\epsilon)\frac{\lambda}{\theta_0}t \quad \forall t > T_\epsilon^-$$

Letting $T_\epsilon = T_\epsilon^- \vee T_\epsilon^+$ we get

$$\begin{aligned}
spine(t) &\le \exp\Big(\lambda(1+\epsilon)\theta_0\log\theta_0 t + \lambda(1-\epsilon)\frac{1}{\theta_0}\log\big(\frac{1}{\theta_0}\big)t \\
&\qquad + \lambda\big(2-\theta_0-\frac{1}{\theta_0}\big)t - \beta t\Big) \\
&= \exp\Big(\big(\lambda\big[g(\theta_0)+\epsilon\big(\theta_0+\frac{1}{\theta_0}\big)\log\theta_0\big]-\beta\big)t\Big) \qquad \forall t \ge T_\epsilon.
\end{aligned}$$

Since $\theta_0 < \hat{\theta}$ and $g(\cdot)$ is increasing (see Figure 3-2) we have

$$g(\theta_0) < g(\hat{\theta}) = \frac{\beta}{\lambda}.$$

Hence for ϵ small enough

$$\lambda\Big(g(\theta_0)+\epsilon\big(\theta_0+\frac{1}{\theta_0}\big)\log\theta_0\Big)-\beta < 0.$$

We thus let $T' = T_\epsilon$ and $C'' = -\lambda\Big(g(\theta_0)+\epsilon\big(\theta_0+\frac{1}{\theta_0}\big)\log\theta_0\Big)-\beta$ to finish the proof of Proposition 3.23. $\square$

Now, for $t > T'$ the sum term is

$$\begin{aligned}
sum(t) &= \sum_{u<node_t(\xi)} spine(S_u) \\
&= \Big(\sum_{u<node_t(\xi),\ S_u\le T'} spine(S_u)\Big) + \Big(\sum_{u<node_t(\xi),\ S_u>T'} spine(S_u)\Big) \\
&\le \Big(\sum_{u<node_t(\xi),\ S_u\le T'} spine(S_u)\Big) + \Big(\sum_{u<node_t(\xi),\ S_u>T'} e^{-C''S_u}\Big)
\end{aligned}$$

using Proposition 3.23 for the inequality. The first sum is $\tilde{Q}_\theta$-a.s. bounded since it only counts births up to time T'. Call an upper bound on the first sum C_1.

Then we have

$$sum(t) \le \; C_1 + \sum_{n=1}^{\infty} e^{-C'' S_n}, \tag{3.32}$$

where S_n is the time of the n^{th} birth on the spine.

The birth process along the spine $(n_t)_{t\in[0,\infty)}$ is a Poisson process with rate 2β (Recall Proposition 1.13). Thus

$$\frac{n_t}{t} \to 2\beta \quad \tilde{Q}_\theta\text{-a.s. as } t \to \infty.$$

Hence

$$\frac{S_n}{n} \to \frac{1}{2\beta} \quad \tilde{Q}_\theta\text{-a.s. as } t \to \infty. \tag{3.33}$$

So for some $\tilde{Q}_\theta$-a.s. finite positive random variable C_2 we have

$$S_n \ge C_2 n \quad \forall n.$$

Then substituting this into (3.32) we get

$$sum(t) \le C_1 + \sum_{n=1}^{\infty} e^{-C'' C_2 n},$$

which is bounded $\tilde{Q}_\theta$-a.s. We have thus shown that

$$\limsup_{t\to\infty} E^{\tilde{Q}_\theta}\Big(M_\theta(t) \big| \tilde{\mathcal{G}}_\infty \Big) < \infty \qquad \tilde{Q}_\theta\text{-a.s.}$$

$\square$

Proof of Proposition 3.22: Case B ($p \in (0,1)$). We are given parameter $\theta = (\theta^+, \theta^-)$, where $\theta^-(\cdot) \equiv 1$, $\theta^+(s) = \dfrac{c}{\lambda(1-p)} \dfrac{s^{b-1}}{(\log(s+2))^b}$, $s \ge 0$. Again, we start by giving an upper bound on the spine term (3.30).

Proposition 3.24. *There exist some $\tilde{Q}_\theta$-a.s. finite positive random variables C', C'' and a random time $T' < \infty$ such that $\forall t > T'$*

$$spine(t) \le C' \exp\Big(- C'' \int_0^t \frac{s^{bp}}{(\log(s+2))^{bp}} \mathrm{d}s \Big).$$

Proof of Proposition 3.24. Simple calculus tells us that for constants $k_1 > 0$ and $k_2 \in \mathbb{R}$

$$\int_0^t s^{k_1} \big(\log(s+2)\big)^{k_2} \mathrm{d}s \sim \frac{1}{k_1+1} t^{k_1+1} \big(\log(t+2)\big)^{k_2} \text{ as } t \to \infty. \tag{3.34}$$

Hence one can check that the following are true as $t \to \infty$ for θ^+:

- $\int_0^t \lambda\theta^+(s)\mathrm{d}s \;\sim\; \dfrac{c}{b(1-p)}\dfrac{t^b}{(\log(t+2))^b} \to \infty$
- $\log\theta^+(t)\lambda\theta^+(t) \;\sim\; \dfrac{c(b-1)}{1-p}\dfrac{t^{b-1}}{(\log(t+2))^{b-1}}$
- $\limsup_{t\to\infty} \dfrac{\log\theta^+(t)\int_0^t \lambda\theta^+(s)\mathrm{d}s}{\int_0^t \log\theta^+(s)\ \lambda\theta^+(s)\mathrm{d}s} = 1 < \infty$

Under $\tilde{Q}_\theta$ the process $(\xi_t^+)_{t\in[0,\infty)}$ is an inhomogeneous Poisson process with jump rate $\lambda\theta^+(t)$ as it follows from Proposition 3.14 and $(\xi_t^-)_{t\in[0,\infty)}$ is a Poisson process of rate λ.

Hence from Proposition 3.12 and Proposition3.13 we have that

- $\dfrac{\xi_t^+}{\int_0^t \lambda\theta^+(s)\mathrm{d}s} \to 1\ \tilde{Q}_\theta$ -a.s.
- $\dfrac{\xi_t^-}{\lambda t} \to 1\ \tilde{Q}_\theta$-a.s.
- $\dfrac{\int_0^t \log\theta^+(s)\mathrm{d}\xi_s^+}{\int_0^t \log\theta^+(s)\ \lambda\theta^+(s)\mathrm{d}s} \to 1\ \tilde{Q}_\theta$ -a.s.

Since $\int_0^t \lambda\theta^+(s)\mathrm{d}s \gg \lambda t$ the first two equations give

$$\frac{\xi_t}{\int_0^t \lambda\theta^+(s)\mathrm{d}s} = \frac{\xi_t^+ - \xi_t^-}{\int_0^t \lambda\theta^+(s)\mathrm{d}s} \to 1\ \tilde{Q}_\theta\text{-a.s.}$$

Combining the previous observations we get that $\forall\epsilon > 0\ \exists\ \tilde{Q}_\theta$-a.s. finite time T_ϵ such that $\forall t > T_\epsilon$ the following inequalities are true:

- $(1-\epsilon)\int_0^t \log\theta^+(s)\ \lambda\theta^+(s)\mathrm{d}s \;<\; \int_0^t \log\theta^+(s)\mathrm{d}\xi_s^+ \;<\; (1+\epsilon)\int_0^t \log\theta^+(s)\ \lambda\theta^+(s)\mathrm{d}s$
- $(1-\epsilon)\dfrac{c(b-1)}{1-p}\dfrac{t^{b-1}}{(\log(t+2))^{b-1}} < \log\theta^+(t)\ \lambda\theta^+(t) < (1+\epsilon)\dfrac{c(b-1)}{1-p}\dfrac{t^{b-1}}{(\log(t+2))^{b-1}}$
- $(1-\epsilon)\dfrac{c}{b(1-p)}\dfrac{t^b}{(\log(t+2))^b} \;<\; \xi_t \;<\; (1+\epsilon)\dfrac{c}{b(1-p)}\dfrac{t^b}{(\log(t+2))^b}$
- $-\epsilon\dfrac{t^{b-1}}{(\log(t+2))^{b-1}} \;<\; \lambda(1-\theta^+(t)) \;<\; 0$

Thus, for $t > T_\epsilon$ we have

$$\begin{aligned} spine(t) &= \exp\Big\{\int_0^t \log\theta^+(s)\mathrm{d}\xi_s^+ + \lambda\int_0^t (1-\theta^+(s))\mathrm{d}s - \beta\int_0^t |\xi_s|^p\mathrm{d}s\Big\} \\ &< C_\epsilon \exp\Big\{\int_0^t (1+\epsilon)^2\frac{c(b-1)}{1-p}\frac{s^{b-1}}{(\log(s+2))^{b-1}} \\ &\qquad - \beta\Big((1-\epsilon)\frac{c}{b(1-p)}\frac{s^b}{(\log(s+2))^b}\Big)^p\mathrm{d}s\Big\}, \end{aligned}$$

where C_ϵ is some a.s. finite random variable, which doesn't depend on t.

Then if $b < \hat{b}$ it is true that $bp > b-1$ and so $s^{b-1} \ll s^{bp}$. Hence the negative term in the exponential dominates the positive one and this proves Proposition 3.24.

If $b = \hat{b}$ (that is, if $bp = b-1$) but $c < \hat{c}$ then

$$\frac{s^{\hat{b}-1}}{(\log(s+2))^{\hat{b}-1}} = \Big(\frac{s^{\hat{b}}}{(\log(s+2))^{\hat{b}}}\Big)^p$$

but

$$\begin{aligned}
&(1+\epsilon)^2 \frac{c(\hat{b}-1)}{1-p} - \beta(1-\epsilon)^p \Big(\frac{c}{\hat{b}(1-p)}\Big)^p \\
=&(1+\epsilon)^2 c \frac{p}{(1-p)^2} - \beta(1-\epsilon)^p c^p \\
=&c^p(1-\epsilon)^p \frac{p}{(1-p)^2}\Big(\frac{(1+\epsilon)^2}{(1-\epsilon)^p} c^{1-p} - \hat{c}^{1-p}\Big) \\
<&0
\end{aligned}$$

for ϵ sufficiently small. So letting $T' = T_\epsilon$, $C' = C_\epsilon$ and $C'' = (1+\epsilon)^2 c\frac{p}{(1-p)^2} - \beta(1-\epsilon)^p c^p$ we prove Proposition 3.24. $\square$

For the sum term we have when $t > T'$

$$\begin{aligned}
sum(t) =& \sum_{u < node_t(\xi)} spine(S_u) \\
=& \Big(\sum_{u < node_t(\xi),\ S_u \le T'} spine(S_u)\Big) + \Big(\sum_{u < node_t(\xi),\ S_u > T'} spine(S_u)\Big) \\
\le& \sum_{u < node_t(\xi),\ S_u \le T'} spine(S_u) \\
&+ \sum_{u < node_t(\xi),\ S_u > T'} C' \exp\Big(-C'' \int_0^{S_u} \frac{s^{bp}}{(\log(s+2))^{bp}} \mathrm{d}s\Big)
\end{aligned}$$

using Proposition 3.24 for the inequality. The first sum is $\tilde{Q}_\theta$-a.s. bounded since it only counts births up to time T'. Call an upper bound on the first sum C_1. Then we have

$$sum(t) \le\ C_1 + C' \sum_{n=1}^{\infty} \exp\Big(-C'' \int_0^{S_n} \frac{s^{bp}}{(\log(s+2))^{bp}} \mathrm{d}s\Big), \tag{3.35}$$

where S_n is the time of the n^{th} birth on the spine.

The birth process along the spine $(n_t)_{t\in[0,\infty)}$ conditional on the path of the spine is time-inhomogeneous Poisson process (or Cox process) with jump rate $2\beta|\xi_t|^p$ at time t

(recall Proposition 1.13). Thus

$$\frac{n_t}{\int_0^t 2\beta|\xi_s|^p \mathrm{d}s} \to 1 \quad \tilde{Q}_\theta\text{-a.s. as } t \to \infty.$$

Also

$$\int_0^t |\xi_s|^p \mathrm{d}s \sim \Big(\frac{c}{b(1-p)}\Big)^p \int_0^t \frac{s^{bp}}{(\log(s+2))^{bp}} \mathrm{d}s \quad \tilde{Q}_\theta\text{-a.s. as } t \to \infty.$$

Hence

$$n_t \sim 2\beta\Big(\frac{c}{b(1-p)}\Big)^p \int_0^t \frac{s^{bp}}{(\log(s+2))^{bp}} \mathrm{d}s \quad \tilde{Q}_\theta\text{-a.s. as } t \to \infty. \tag{3.36}$$

So for some $\tilde{Q}_\theta$-a.s. finite positive random variable C_2 we have

$$\int_0^{S_n} \frac{s^{bp}}{(\log(s+2))^{bp}} \mathrm{d}s \geq C_2 n \quad \forall n.$$

Then substituting this into (3.35) we get

$$sum(t) \leq C_1 + C' \sum_{n=1}^{\infty} e^{-C'' C_2 n},$$

which is bounded $\tilde{Q}_\theta$-a.s. We have thus shown that

$$\limsup_{t\to\infty} E^{\tilde{Q}_\theta}\Big(M_\theta(t)\big|\tilde{\mathcal{G}}_\infty\Big) < \infty \qquad \tilde{Q}_\theta\text{-a.s.}$$

□

Proof of Proposition 3.22: Case C ($p = 1$). We are given $\theta = (\theta^+, \theta^-)$, where $\theta^-(\cdot) \equiv 1$, $\theta^+(s) = e^{\alpha\sqrt{s}}$. We prove the following upper bound on *spine*(t).

Proposition 3.25. *There exist some $\tilde{Q}_\theta$-a.s. finite positive random variables C', C'' and a random time $T' < \infty$ such that $\forall t > T'$*

$$spine(t) \leq C' \exp\Big(-C'' \int_0^t \sqrt{s} e^{\alpha\sqrt{s}} \mathrm{d}s\Big).$$

Proof of Proposition 3.25. Simple calculus tells us that for constant $k \geq 0$

$$\int_0^t s^k e^{\alpha\sqrt{s}} \mathrm{d}s \sim \frac{2}{\alpha} t^{k+\frac{1}{2}} e^{\alpha\sqrt{t}} \text{ as } t \to \infty. \tag{3.37}$$

Hence as $t \to \infty$

- $\int_0^t \lambda\theta^+(s)\mathrm{d}s \ \sim\ \lambda\frac{2}{\alpha}\sqrt{t}e^{\alpha\sqrt{t}} \to \infty$
- $\log\theta^+(t)\lambda\theta^+(t) \ \sim\ \lambda\alpha\sqrt{t}e^{\alpha\sqrt{t}}$

- $\limsup_{t\to\infty} \dfrac{\log\theta^+(t)\int_0^t \lambda\theta^+(s)\mathrm{d}s}{\int_0^t \log\theta^+(s)\ \lambda\theta^+(s)\mathrm{d}s} = 1 < \infty$

As in Case B it then follows that

- $\dfrac{\xi_t}{\int_0^t \lambda\theta^+(s)\mathrm{d}s} \to 1\ \tilde{Q}_\theta$ -a.s.

- $\dfrac{\int_0^t \log\theta^+(s)\mathrm{d}\xi_s^+}{\int_0^t \log\theta^+(s)\ \lambda\theta^+(s)\mathrm{d}s} \to 1\ \tilde{Q}_\theta$ -a.s.

Combining the previous observations we get that $\forall\epsilon > 0\ \exists\ \tilde{Q}_\theta$-a.s. finite time T_ϵ such that $\forall t > T_\epsilon$ the following inequalities are true:

- $(1-\epsilon)\int_0^t \log\theta^+(s)\ \lambda\theta^+(s)\mathrm{d}s\ <\ \int_0^t \log\theta^+(s)\mathrm{d}\xi_s^+\ <\ (1+\epsilon)\int_0^t \log\theta^+(s)\ \lambda\theta^+(s)\mathrm{d}s$
- $(1-\epsilon)\lambda\alpha\sqrt{t}e^{\alpha\sqrt{t}}\ <\ \log\theta^+(t)\ \lambda\theta^+(t)\ <\ (1+\epsilon)\lambda\alpha\sqrt{t}e^{\alpha\sqrt{t}}$
- $(1-\epsilon)\lambda\frac{2}{\alpha}\sqrt{t}e^{\alpha\sqrt{t}}\ <\ \xi_t\ <\ (1+\epsilon)\lambda\frac{2}{\alpha}\sqrt{t}e^{\alpha\sqrt{t}}$
- $-\epsilon\sqrt{t}e^{\alpha\sqrt{t}}\ <\ \lambda(1-\theta^+(t))\ <\ 0$

Thus for $t > T_\epsilon$

$$\begin{aligned} spine(t) &= \exp\Big\{\int_0^t \log\theta^+(s)\mathrm{d}\xi_s^+ + \lambda\int_0^t (1-\theta^+(s))\mathrm{d}s - \beta\int_0^t |\xi_s|\mathrm{d}s\Big\} \\ &< C_\epsilon \exp\Big\{\int_0^t (1+\epsilon)\alpha\lambda\sqrt{s}e^{\alpha\sqrt{s}} - \beta\big((1-\epsilon)\lambda\frac{2\sqrt{s}}{\alpha}e^{\alpha\sqrt{s}}\big)\mathrm{d}s\Big\} \end{aligned}$$

for some finite random variable C_ϵ. Then for $\alpha < \sqrt{2\beta}$ we have that

$$(1+\epsilon)\alpha - \beta(1-\epsilon)\frac{2}{\alpha} < 0$$

provided ϵ was chosen small enough and this proves Proposition 3.25. $\square$

We then deal with $sum(t)$ in the usual way:

$$\begin{aligned} sum(t) \le &\sum_{u<node_t(\xi),\ S_u\le T'} spine(S_u) \\ &\quad + \sum_{u<node_t(\xi),\ S_u>T'} C'\exp\Big(-C''\int_0^{S_u}\sqrt{s}e^{\alpha\sqrt{s}}\mathrm{d}s\Big) \\ &\le C_1 + C'\sum_{n=1}^{\infty}\exp\Big(-C''\int_0^{S_n}\sqrt{s}e^{\alpha\sqrt{s}}\mathrm{d}s\Big), \end{aligned} \tag{3.38}$$

where $C_1 < \infty$ and S_n is the time of the n^{th} birth on the spine.

The birth process along the spine $(n_t)_{t\in[0,\infty)}$ satisfies

$$\frac{n_t}{\int_0^t 2\beta|\xi_s|^p \mathrm{d}s} \to 1 \quad \tilde{Q}_\theta\text{-a.s. as } t \to \infty.$$

Hence

$$n_t \sim \frac{4\beta\lambda}{\alpha}\int_0^t \sqrt{s}e^{\alpha\sqrt{s}}\mathrm{d}s \quad \tilde{Q}_\theta\text{-a.s. as } t \to \infty. \tag{3.39}$$

So for some $\tilde{Q}_\theta$-a.s. finite positive random variable C_2 we have

$$\int_0^{S_n} \sqrt{s}e^{\alpha\sqrt{s}}\mathrm{d}s \geq C_2 n \quad \forall n.$$

Then substituting this into (3.38) we get

$$sum(t) \leq C_1 + C'\sum_{n=1}^{\infty} e^{-C''C_2 n},$$

which gives

$$\limsup_{t\to\infty} E^{\tilde{Q}_\theta}\Big(M_\theta(t)\big|\tilde{\mathcal{G}}_\infty\Big) < \infty \qquad \tilde{Q}_\theta\text{-a.s.}$$

□

This completes the proof of Proposition 3.22 and hence also the proof of uniform integrability and the fact that $P(M_\theta(\infty) > 0) > 0$ in Theorem 3.18. □

Proof of Theorem 3.18: parts A ii), B ii), C ii). Since one of the particles at time t is the spine, we have

$$\begin{aligned} M_\theta(t) \ \geq \ \exp\Big(&\int_0^t \log(\theta^+(s))\mathrm{d}\xi_s^+ \ + \ \int_0^t \log(\theta^-(s))\mathrm{d}\xi_s^- \\ &+ \lambda\int_0^t (2 - \theta^+(s) - \theta^-(s))\mathrm{d}s \ - \ \beta\int_0^t |\xi_s|^p \mathrm{d}s\Big) \ = \ spine(t). \end{aligned}$$

For the paths θ in parts ii) of Theorem 3.18 one can check (following the same analysis as in the proof of parts i) of the Theorem) that $spine(t) \to \infty$ $\tilde{Q}_\theta$-a.s. Thus

$$\limsup_{t\to\infty} M_\theta(t) = \infty \ \tilde{Q}_\theta\text{-a.s.}$$

and so also Q_θ-a.s. Recalling (3.27) we see that $M_\theta(\infty) = 0$ P-a.s. for the proposed choices of θ. □

It remains to show that in parts A i), B i) and C i) of Theorem 3.18
$P(M_\theta(\infty) > 0) = 1$. The following lemma will do the job.

Lemma 3.26. *Let $q : \mathbb{Z} \to [0,1]$ be such that $M_t := \prod_{u\in N_t} q(X_u(t))$ is a P-martingale. Then $q(x) \equiv q \in \{0,1\}$.*

Proof of Lemma 3.26. Since M_t is a martingale and one of the particles alive at time t is the spine we have

$$q(x) = E^x M_t = \tilde{E}^x M_t \leq \tilde{E}^x q(\xi_t).$$

So $q(\xi_t)$ is a positive $\tilde{P}$-submartingale. Since it is bounded it converges $\tilde{P}$-a.s. to some limit q_∞. We also know that under $\tilde{P}$, $(\xi_t)_{t\geq 0}$ is a continuous-time random walk, which is recurrent (recall Proposition 3.1). Recurrence of $(\xi_t)_{t\geq 0}$ implies that $q_\infty \equiv q(0)$ and that $q(x)$ is constant in x.

Now suppose for contradiction that $q(0) \in (0,1)$. Then

$$M_t = \prod_{u\in N_t} q(X_u(t)) = q(0)^{|N_t|} \to 0$$

because $|N_t| \to \infty$. Since M is bounded it is uniformly integrable, so $q(0) = EM_\infty = 0$, which is a contradiction. So $q(0) \notin (0,1)$ and thus $q(0) \in \{0,1\}$. □

Proof of Theorem 3.18: positivity of limits in A i), B i), C i). We apply Lemma 3.26 to $q(x) = P^x(M_\theta(\infty) = 0)$. By the tower propery of conditional expectations and the branching Markov property we have

$$q(x) = E^x\Big(P^x\big(M_\theta(\infty) = 0\big|\mathcal{F}_t\big)\Big) = E^x\Big(\prod_{u\in N_t} q\big(X_u(t)\big)\Big)$$

whence $\prod_{u\in N_t} q(X_u(t))$ is a P-martingale. Also $E(M_\theta(\infty)) = M_\theta(0) = 1 > 0$. Therefore $P(M_\theta(\infty) = 0) \neq 1$. So by Lemma 3.26 $P(M_\theta(\infty) = 0) = 0$. □

One should note that the above argument is very similar to the zero-one law we proved in Chapter 2 (see Proposition 2.2, Corollary 2.3 and Remark 2.4).

Let us summarise what we have shown in this subsection. Suppose parameter θ is chosen as in parts *A i)*, *B i)* or *C i)* of Theorem 3.18. We have proved that in those cases:

1. $\limsup_{t\to\infty} M_\theta(t) < \infty$ Q_θ-a.s.

2. M_θ is P-uniformly integrable

3. $M_\theta(\infty) > 0$ P-a.s.

Thus from Lemma 3.21 for events $A \in \mathcal{F}_\infty$

$$Q_\theta(A) = E\big(\mathbf{1}_A M_\theta(\infty)\big)$$

and also

$$Q_\theta(A) = 1 \Leftrightarrow P(A) = 1. \tag{3.40}$$

In other words Q_θ and P are equivalent.

3.4.3 Lower bound on the rightmost particle

Now we can apply (3.40) to get lower bounds for Theorem 3.4.

Proposition 3.27. *Let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be as defined in Theorem 3.4. Then for different values of p we have the following.*

Case A $(p = 0)$*:*

$$\liminf_{t\to\infty} \frac{R_t}{t} \geq \hat{a} \ P\text{-a.s.}$$

Case B $(p \in (0,1))$*:*

$$\liminf_{t\to\infty} \Big(\frac{\log t}{t}\Big)^{\hat{b}} R_t \geq \hat{c} \ P\text{-a.s.}$$

Case C $(p = 1)$*:*

$$\liminf_{t\to\infty} \frac{\log R_t}{\sqrt{t}} \geq \sqrt{2\beta} \ P\text{-a.s.}$$

Proof.

Case A $(p = 0)$:

We consider $\theta = (\theta^+, \theta^-)$, where $\theta^+(\cdot) \equiv \theta_0$, $\theta^-(\cdot) \equiv \frac{1}{\theta_0}$ and $\theta_0 < \hat{\theta}$. Let $a_0 := \lambda(\theta_0 - \frac{1}{\theta_0})$. Take the event

$$B_{a_0} := \Big\{ \exists \text{ infinite line of descent } u : \liminf_{t\to\infty} \frac{X_u(t)}{t} = a_0 \Big\} \in \mathcal{F}_\infty.$$

Then

$$\begin{aligned}
&\tilde{Q}_\theta\big(\lim_{t\to\infty} \frac{\xi_t}{t} = a_0\big) = 1 \\
\Rightarrow &\tilde{Q}_\theta(B_{a_0}) = 1 \\
\Rightarrow &Q_\theta(B_{a_0}) = 1 \\
\Rightarrow &P(B_{a_0}) = 1 \qquad \text{by (3.40)} \\
\Rightarrow &P\big(\liminf_{t\to\infty} \frac{R_t}{t} \geq a_0\big) = 1.
\end{aligned}$$

Taking the limit $\theta_0 \nearrow \hat{\theta}$ we get $a_0 \nearrow \hat{a}$ and thus

$$P\big(\liminf_{t\to\infty} \frac{R_t}{t} \geq \hat{a}\big) = 1.$$

Case B $(p \in (0,1))$:

Consider $\theta = (\theta^+, \theta^-)$, where $\theta^-(\cdot) \equiv 1$, $\theta^+(s) = \dfrac{c}{\lambda(1-p)} \dfrac{s^{\hat{b}-1}}{(\log(s+2))^{\hat{b}}}$ and $c < \hat{c}$. Take the event

$$B_c := \Big\{ \exists u : \liminf_{t\to\infty} \Big(\frac{\log t}{t}\Big)^{\hat{b}} X_u(t) = c \Big\}.$$

Same argument as above gives that

$$\begin{aligned} & P(B_c) = 1 \\ \Rightarrow & P\Big(\liminf_{t\to\infty} \Big(\frac{\log t}{t}\Big)^{\hat{b}} R_t \geq c \Big) = 1 \qquad \forall c < \hat{c} \\ \Rightarrow & P\Big(\liminf_{t\to\infty} \Big(\frac{\log t}{t}\Big)^{\hat{b}} R_t \geq \hat{c} \Big) = 1. \end{aligned}$$

<u>**Case C** $(p = 1)$</u>:

Consider $\theta = (\theta^+, \theta^-)$, where $\theta^-(\cdot) \equiv 1$, $\theta^+(s) = e^{\alpha\sqrt{s}}$ and $\alpha < \sqrt{2\beta}$. Take the event

$$B_\alpha := \Big\{ \exists u : \liminf_{t\to\infty} \frac{\log X_u(t)}{\sqrt{t}} = \sqrt{2\beta} \Big\}.$$

Same argument as above gives that

$$\begin{aligned} & P(B_\alpha) = 1 \\ \Rightarrow & P\Big(\liminf_{t\to\infty} \frac{\log R_t}{\sqrt{t}} \geq \alpha \Big) = 1 \qquad \forall \alpha < \sqrt{2\beta} \\ \Rightarrow & P\Big(\liminf_{t\to\infty} \frac{\log R_t}{\sqrt{t}} \geq \sqrt{2\beta} \Big) = 1. \end{aligned}$$

□

3.4.4 Upper bound on the rightmost particle

To complete the proof of Theorem 3.4 and hence the whole section we need to prove the following result.

Proposition 3.28. *Let $\hat{a}$, $\hat{b}$ and $\hat{c}$ be as defined in Theorem 3.4. Then for different values of p we have the following.*

<u>**Case A** $(p = 0)$</u>*:*

$$\limsup_{t\to\infty} \frac{R_t}{t} \leq \hat{a} \ P\text{-}a.s.$$

<u>**Case B** $(p \in (0,1))$</u>*:*

$$\limsup_{t\to\infty} \Big(\frac{\log t}{t}\Big)^{\hat{b}} R_t \leq \hat{c} \ P\text{-}a.s.$$

Case C $(p = 1)$*:*

$$\limsup_{t\to\infty} \frac{\log R_t}{\sqrt{t}} \le \sqrt{2\beta} \ P\text{-}a.s.$$

To prove Proposition 3.28 we shall assume for contradiction that it is false. Then we shall show that under such assumption certain additive P-martingales will diverge to ∞ contradicting the Martingale Convergence Theorem.

We start by proving the following 0-1 law.

Lemma 3.29. *For all a_0, b, c, α > 0*

In **Case A** $(p = 0)$*:*

$$P\Big(\limsup_{t\to\infty} \frac{R_t}{t} \le a_0\Big) \in \{0, 1\}.$$

In **Case B** $(p \in (0, 1))$*:*

$$P\Big(\limsup_{t\to\infty} \big(\frac{\log t}{t}\big)^b R_t \le c\Big) \in \{0, 1\}.$$

In **Case C** $(p = 1)$*:*

$$P\Big(\limsup_{t\to\infty} \frac{\log R_t}{\sqrt{t}} \le \alpha\Big) \in \{0, 1\}.$$

Proof. We consider

in **Case A** $(p = 0)$:

$$q_1(x) = P^x\Big(\limsup_{t\to\infty} \frac{R_t}{t} \le a_0\Big),$$

in **Case B** $(p \in (0, 1))$:

$$q_2(x) = P^x\Big(\limsup_{t\to\infty} \big(\frac{\log t}{t}\big)^b R_t \le c\Big),$$

in **Case C** $(p = 1)$:

$$q_3(x) = P^x\Big(\limsup_{t\to\infty} \frac{\log R_t}{\sqrt{t}} \le \alpha\Big).$$

Then in Case A

$$q_1(x) = E^x\Big(P^x\big(\limsup_{t\to\infty} \frac{R_t}{t} \le a_0 \big| \mathcal{F}_t\big)\Big) = E^x\Big(\prod_{u\in N_t} q_1\big(X_u(t)\big)\Big)$$

so that $\prod_{u\in N_t} q_1(X_u(t))$ is a martingale. Similarly $\prod_{u\in N_t} q_2(X_u(t))$ and $\prod_{u\in N_t} q_3(X_u(t))$ are martingales in cases B and C respectively. Applying Lemma 3.26 to $q_1(\cdot)$, $q_2(\cdot)$ and $q_3(\cdot)$ we obtain the required result. $\square$

Proof of Proposition 3.28. The first step of the proof is slightly different for cases A, B and C, so we do it for the 3 cases separately.

Case A ($p = 0$)

Let us suppose for contradiction that $\exists a_0 > \hat{a}$ such that

$$P\Big(\limsup_{t\to\infty} \frac{R_t}{t} > a_0\Big) = 1. \tag{3.41}$$

Choose any $a_1 \in (\hat{a}, a_0)$ and take $\theta = (\theta^+, \theta^-)$, where $\theta^+(\cdot) \equiv \theta_A$, $\theta^-(\cdot) = \dfrac{1}{\theta_A}$ and θ_A is the unique solution of

$$a_1 = \lambda\big(\theta_A - \frac{1}{\theta_A}\big).$$

Let

$$f_A(s) := a_1 s.$$

Case B ($p \in (0,1)$)

Let us suppose for contradiction that $\exists c_0 > \hat{c}$ such that

$$P\Big(\limsup_{t\to\infty} \big(\frac{\log t}{t}\big)^{\hat{b}} R_t > c_0\Big) = 1. \tag{3.42}$$

Choose any $c_1 \in (\hat{c}, c_0)$ and take $\theta = (\theta^+, \theta^-)$, where $\theta^+(s) = \theta_B(s)$, $\theta^-(s) = \dfrac{1}{\theta_B(s)}$ and

$$\theta_B(s) = \frac{c_1}{\lambda(1-p)} \frac{s^{\hat{b}-1}}{(\log(s+2))^{\hat{b}}}.$$

Let

$$f_B(s) := c_1\Big(\frac{s}{\log(s+2)}\Big)^{\hat{b}}.$$

Case C ($p = 1$)

Let us suppose for contradiction that $\exists \alpha_0 > \sqrt{2\beta}$ such that

$$P\Big(\limsup_{t\to\infty} \frac{\log R_t}{\sqrt{t}} > \alpha_0\Big) = 1. \tag{3.43}$$

Choose any $\alpha_1 \in (\sqrt{2\beta}, \alpha_0)$ and take $\theta = (\theta^+, \theta^-)$, where $\theta^+(s) = \theta_C(s)$, $\theta^-(s) = \dfrac{1}{\theta_C(s)}$ and

$$\theta_C(s) = \frac{1}{\sqrt{s+1}} e^{\alpha_1\sqrt{s}}.$$

Let

$$f_C(s) := e^{\alpha_1\sqrt{s}}.$$

The next step in the proof is the same for cases A, B and C.

Let us write f to denote f_A, f_B and f_C. We define $D(f)$ to be the space-time region bounded above by the curve $y = f(t)$ and below by the curve $y = -f(t)$.

Under P the spine process $(\xi_t)_{t\geq 0}$ is a continuous-time random walk and so $\dfrac{|\xi_t|}{t} \to 0$ P-a.s. as $t \to \infty$. Hence there exists an a.s. finite random time $T' < \infty$ such that $\xi_t \in D(f)$ for all $t > T'$.

Since $(\xi_t)_{t\geq 0}$ is recurrent it will spend an infinite amount of time at position $y = 1$. During this time it will be giving birth to offspring at rate β. This assures us of the existence of an infinite sequence $\{T_n\}_{n\in\mathbb{N}}$ of birth times along the path of the spine when it stays at $y = 1$ with $0 \leq T' \leq T_1 < T_2 < \ldots$ and $T_n \nearrow \infty$.

Denote by u_n the label of the particle born at time T_n, which does not continue the spine. Then each particle u_n gives rise to an independent copy of the Branching random walk under P started from ξ_{T_n} at time T_n. Almost surely, by assumptions (3.41), (3.42) and (3.43), each u_n has some descendant that leaves the space-time region $D(f)$.

Let $\{v_n\}_{n\in\mathbb{N}}$ be the subsequence of $\{u_n\}_{n\in\mathbb{N}}$ of those particles whose first descendent leaving $D(f)$ does this by crossing the upper boundary $y = f(t)$. Since the breeding potential is symmetric and the particles u_n are born in the upper half-plane, there is at least probability $\dfrac{1}{2}$ that the first descendant of u_n to leave $D(f)$ does this by crossing the positive boundary curve. Therefore P-a.s. the sequence $\{v_n\}_{n\in\mathbb{N}}$ is infinite.

Let w_n be the decsendent of v_n, which exits $D(f)$ first and let J_n be the time when this occurs. That is,

$$J_n = \inf\big\{t : X_{w_n}(t) \geq f(t)\big\}.$$

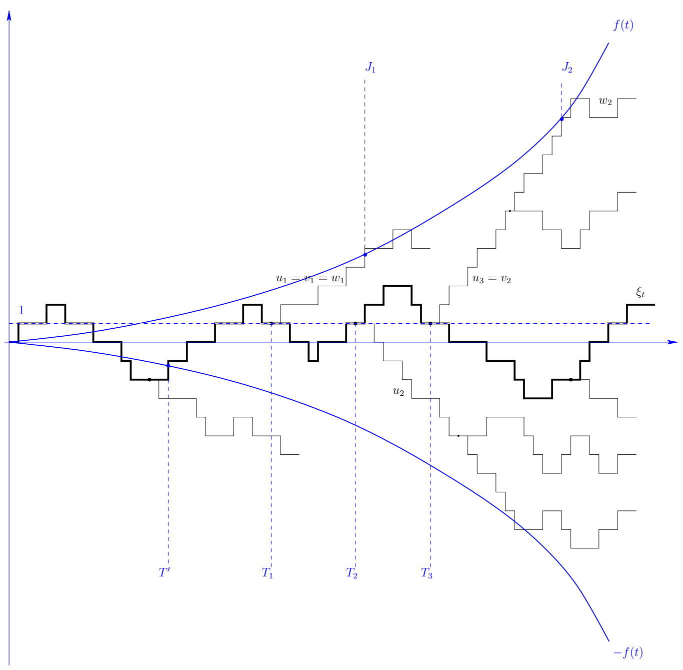


Figure 3-5: *Illustration to Proposition 3.28*

Note that the path of particle w_n satisfies

$$|X_{w_n}(s)| < f(s) \quad \forall s \in [T', J_n).$$

Clearly $J_n \to \infty$ as $n \to \infty$. To obtain a contradiction we shall show that the additive martingale M_θ fails to converge along the sequence of times $\{J_n\}_{n\geq 1}$, where θ was defined above differently for cases A, B and C. Thus for the last bit of the proof we have to look at cases A, B and C separately again.

<u>**Case A** $(p = 0)$</u>

$$\begin{aligned}
M_\theta(J_n) &= \sum_{u\in N_{J_n}} \exp\Big\{ \int_0^{J_n} \log\theta_A \mathrm{d}X_u^+(s) + \int_0^{J_n} \log\big(\frac{1}{\theta_A}\big)\mathrm{d}X_u^-(s) \\
&\quad + \lambda\int_0^{J_n} \big(2-\theta_A-\frac{1}{\theta_A}\big)\mathrm{d}s - \beta\int_0^{J_n} 1\mathrm{d}s\Big\} \\
&\geq \exp\Big\{ \int_0^{J_n} \log\theta_A \mathrm{d}X_{w_n}^+(s) + \int_0^{J_n} \log\big(\frac{1}{\theta_A}\big)\mathrm{d}X_{w_n}^-(s) \\
&\quad + \lambda\int_0^{J_n} \big(2-\theta_A-\frac{1}{\theta_A}\big)\mathrm{d}s - \beta\int_0^{J_n} 1\mathrm{d}s\Big\} \\
&= \exp\Big\{ \log\theta_A X_{w_n}^+(J_n) - \log\theta_A X_{w_n}^-(J_n) + \lambda\big(2-\theta_A-\frac{1}{\theta_A}\big)J_n - \beta J_n\Big\} \\
&= \exp\Big\{ \log\theta_A X_{w_n}(J_n) + \lambda\big(2-\theta_A-\frac{1}{\theta_A}\big)J_n - \beta J_n\Big\} \\
&\geq \exp\Big\{ a_1 J_n \log\theta_A + \lambda\big(2-\theta_A-\frac{1}{\theta_A}\big)J_n - \beta J_n\Big\} \\
&= \exp\Big\{ \Big(\lambda\big((\theta_A-\frac{1}{\theta_A}\big)\log\theta_A + \lambda\big(2-\theta_A-\frac{1}{\theta_A}\big) - \beta\Big)J_n\Big\} \\
&= \exp\Big\{ \Big(\lambda g(\theta_A) - \beta\Big)J_n\Big\}.
\end{aligned}$$

Then since $g(\cdot)$ is increasing, $\theta_A > \hat\theta$ and $g(\hat\theta) = \frac{\beta}{\lambda}$ it follows that

$$\lambda g(\theta_A) - \beta > 0$$

and thus $M_\theta(J_n) \to \infty$ as $n \to \infty$, which is a contradiction. Therefore assumption (3.41) is wrong and we must have that $\forall a_0 > \hat a$

$$P\Big(\limsup_{t\to\infty} \frac{R_t}{t} > a_0\Big) \neq 1.$$

It follows from Lemma 3.29 that $\forall a_0 > \hat{a}$

$$\begin{aligned}
&P\Big(\limsup_{t\to\infty}\frac{R_t}{t} > a_0\Big) = 0\\
\Rightarrow &P\Big(\limsup_{t\to\infty}\frac{R_t}{t} \le a_0\Big) = 1\\
\Rightarrow &P\Big(\limsup_{t\to\infty}\frac{R_t}{t} \le \hat{a}\Big) = 1
\end{aligned}$$

after taking the limit $a_0 \searrow \hat{a}$. This proves Proposition 3.28 in Case A.

Case B ($p \in (0,1)$)

$$\begin{aligned}
M_\theta(J_n) = &\sum_{u\in N_{J_n}} \exp\Big\{\int_0^{J_n}\log\theta_B(s)\mathrm{d}X_u^+(s) + \int_0^{J_n}\log\big(\frac{1}{\theta_B(s)}\big)\mathrm{d}X_u^-(s)\\
&+\lambda\int_0^{J_n}\big(2-\theta_B(s)-\frac{1}{\theta_B(s)}\big)\mathrm{d}s - \beta\int_0^{J_n}|X_u(s)|^p\mathrm{d}s\Big\}\\
\ge &\exp\Big\{\int_0^{J_n}\log\theta_B(s)\mathrm{d}X_{w_n}^+(s) + \int_0^{J_n}\log\big(\frac{1}{\theta_B(s)}\big)\mathrm{d}X_{w_n}^-(s)\\
&+\lambda\int_0^{J_n}\big(2-\theta_B(s)-\frac{1}{\theta_B(s)}\big)\mathrm{d}s - \beta\int_0^{J_n}|X_{w_n}(s)|^p\mathrm{d}s\Big\}.
\end{aligned}$$

Applying the integration by parts formula from Proposition 3.11 we get

$$\begin{aligned}
&\exp\Big\{\log\theta_B(J_n)X_{w_n}^+(J_n) - \int_0^{J_n}\frac{\theta_B'(s)}{\theta_B(s)}X_{w_n}^+(s)\mathrm{d}s\\
&\quad -\log\theta_B(J_n)X_{w_n}^-(J_n) + \int_0^{J_n}\frac{\theta_B'(s)}{\theta_B(s)}X_{w_n}^-(s)\mathrm{d}s\\
&\quad +\lambda\int_0^{J_n}\big(2-\theta_B(s)-\frac{1}{\theta_B(s)}\big)\mathrm{d}s - \beta\int_0^{J_n}|X_{w_n}(s)|^p\mathrm{d}s\Big\}\\
=&\exp\Big\{\log\theta_B(J_n)X_{w_n}(J_n) - \int_0^{J_n}\frac{\theta_B'(s)}{\theta_B(s)}X_{w_n}(s)\mathrm{d}s\\
&\quad +\lambda\int_0^{J_n}\big(2-\theta_B(s)-\frac{1}{\theta_B(s)}\big)\mathrm{d}s - \beta\int_0^{J_n}|X_{w_n}(s)|^p\mathrm{d}s\Big\}\\
\ge &C\exp\Big\{\log\theta_B(J_n)f_B(J_n) - \int_0^{J_n}\frac{\theta_B'(s)}{\theta_B(s)}f_B(s)\mathrm{d}s\\
&\quad +\lambda\int_0^{J_n}\big(2-\theta_B(s)-\frac{1}{\theta_B(s)}\big)\mathrm{d}s - \beta\int_0^{J_n}f_B(s)^p\mathrm{d}s\Big\}
\end{aligned}$$

using the facts that $X_{w_n}(J_n) \ge f_B(J_n)$ and $|X_{w_n}(s)| < f_B(s)$ for $s \in [T', J_n)$ and where C is some P-a.s positive random variable. Now asymptotic properties of $\theta_B(\cdot)$ and $f_B(\cdot)$ of the form 3.34 give us that for any $\epsilon > 0$ and n large enough the above expression is

$$\geq C_\epsilon \exp\Big\{(\hat{b}-1)c_1\frac{(J_n)^{\hat{b}}}{(\log J_n)^{\hat{b}-1}}(1-\epsilon) - \beta c_1^p \frac{1}{\hat{b}}\frac{(J_n)^{\hat{b}}}{(\log J_n)^{\hat{b}-1}}(1+\epsilon)\Big\}$$

for some P-a.s. positive random variable C_ϵ. Then since $c_1 > \hat{c}$

$$\begin{aligned}
&(\hat{b}-1)c_1(1-\epsilon) - \beta c_1^p \frac{1}{\hat{b}}(1+\epsilon)\\
=&c_1^p(\hat{b}-1)(1-\epsilon)\Big(c_1^{1-p} - \frac{1+\epsilon}{1-\epsilon}\beta\frac{1}{\hat{b}(\hat{b}-1)}\Big)\\
=&c_1^p(\hat{b}-1)(1-\epsilon)\Big(c_1^{1-p} - \hat{c}^{1-p}\frac{1+\epsilon}{1-\epsilon}\Big)\\
>&0
\end{aligned}$$

for ϵ small enough. Thus $M_\theta(J_n) \to \infty$ as $n \to \infty$, which is a contradiction. Therefore assumption (3.42) is wrong and we must have that $\forall c_0 > \hat{c}$

$$P\Big(\limsup_{t\to\infty}\,(\frac{\log t}{t})^{\hat{b}}R_t > c_0\Big) \neq 1.$$

It follows from Lemma 3.29 that $\forall c_0 > \hat{c}$

$$\begin{aligned}
&P\Big(\limsup_{t\to\infty}\,(\frac{\log t}{t})^{\hat{b}}R_t > c_0\Big) = 0\\
\Rightarrow &P\Big(\limsup_{t\to\infty}\,(\frac{\log t}{t})^{\hat{b}}R_t \leq c_0\Big) = 1\\
\Rightarrow &P\Big(\limsup_{t\to\infty}\,(\frac{\log t}{t})^{\hat{b}}R_t \leq \hat{c}\Big) = 1
\end{aligned}$$

after taking the limit $c_0 \searrow \hat{c}$. This proves Proposition 3.28 in Case B.

Case C ($p = 1$)

Essentially the same argument as in Case B gives that for any $\epsilon > 0$ and n large enough

$$M_\theta(J_n) \geq C_\epsilon \exp\Big\{(1-\epsilon)\alpha_1\sqrt{J_n}e^{\alpha_1\sqrt{J_n}} - (1+\epsilon)\frac{2\beta}{\alpha_1}\sqrt{J_n}e^{\alpha_1\sqrt{J_n}}\Big\}$$

for some $C_\epsilon > 0$ P-a.s. Then since $\alpha_1 > \sqrt{2\beta}$

$$(1-\epsilon)\alpha_1 - (1+\epsilon)\frac{2\beta}{\alpha_1} > 0$$

for ϵ chosen sufficiently small. Therefore $M_\theta(J_n) \to \infty$, which is a contradiction. Hence $\forall \alpha_0 > \sqrt{2\beta}$

$$P\Big(\limsup_{t\to\infty}\frac{\log R_t}{\sqrt{t}} \leq \alpha_0\Big) = 1$$

and therefore

$$P\Big(\limsup_{t\to\infty}\frac{\log R_t}{\sqrt{t}}\le\sqrt{2\beta}\Big)=1.$$

This finishes the proof of Proposition 3.28 and also Theorem 3.4 $\square$

Chapter 4

Branching Lévy processes

In this chapter we study branching processes in the potential $\beta(x) = \beta|x|^p$, $p \geq 0$, $\beta > 0$, where single particles in the system move according to a given Lévy process. Since Brownian motion and a continuous-time random walk are special instances of a Lévy process, we shall see how earlier results from [22] and Chapter 3 of this thesis fit into the general framework.

The class of all the Lévy processes is quite large, and earlier proofs cannot be generalised to all the members of this class. An important restriction one has to impose on the processes is that they must have finite exponential moments. This will assure us of the existence of exponential martingales, which are crucial in the analysis.

The case of homogeneous branching ($p = 0$) has been studied by J. Biggins in [6] and [7], where he gave the asymptotic growth of the rightmost particle. We shall give an alternative proof using spine techniques. We shall then extend this result subject to some further restrictions on the underlying Lévy processes to the case $p \in (0, 1)$, which we show to be non-explosive.

4.1 Lévy processes

In this preliminary section we give some general information about Lévy processes that we need to know in order to understand the rest of the chapter.

There are numerous books on the general theory of Lévy processes. Let us mention [1], [31], [4], [25]. Everything we shall claim about Lévy processes in this section can be found in one of these books.

Definition 4.1 (Lévy process)**.** *An $\mathbb{R}$-valued process $(X_t)_{t\geq 0}$ on some probability space is said to be a Lévy process under probability $\mathbb{P}$ if*

- $X_0 = 0$ $\mathbb{P}$*-a.s*
- *The paths of X are $\mathbb{P}$-a.s. càdlàg (that is, right continuous with left limits)*

- *For* $0 \leq s \leq t$, $X_t - X_s \stackrel{d}{=} X_{t-s}$
- *For* $0 \leq s \leq t$, $X_t - X_s$ *is independent of* $\sigma(X_u : u \leq s)$

4.1.1 Characterisation of Lévy processes

From Definition 4.1 it is easy to check that for $\gamma \in \mathbb{R}$

$$\mathbb{E}e^{i\gamma X_t} = e^{\Psi(\gamma)t} \qquad \forall t \geq 0 \tag{4.1}$$

for some function $\Psi : \mathbb{R} \to \mathbb{C}$, which is known as the characteristic exponent of the process $(X_t)_{t\geq 0}$. Lévy-Khintchine formula gives the following characterisation of Ψ.

Theorem 4.2 (Lévy-Khintchine)**.** *There exists a triplet* (a, σ, Π), *which we shall call a Lévy triplet, where* $a \in \mathbb{R}$, $\sigma \geq 0$ *and* Π *is a measure on* $\mathbb{R}\backslash\{0\}$ *satisfying* $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(\mathrm{d}x) < \infty$, *such that*

$$\Psi(\gamma) = ia\gamma - \frac{1}{2}\sigma^2\gamma^2 + \int_{\mathbb{R}\backslash\{0\}} \left(e^{i\gamma x} - i\gamma x \mathbf{1}_{|x|<1} - 1\right)\Pi(\mathrm{d}x). \tag{4.2}$$

The triplet (a, σ, Π) fully describes the distribution of a Lévy process $(X_t)_{t\geq 0}$. We shall refer to a, σ and Π as the drift term, the diffusion parameter and the jump measure respectively.

Note that Π might blow up at the origin, e.g. if $\Pi(\mathrm{d}x) = \frac{1}{|x|^{1+\alpha}}\mathrm{d}x$, $\alpha \in (0, 2)$.

Example 4.3. *Let us give a few examples.*

- *Standard Brownian motion is a Lévy process with* $a = 0$, $\sigma = 1$, $\Pi = 0$ *and* $\Psi(\gamma) = -\frac{1}{2}\gamma^2$.
- *Poisson process with jump rate* λ *is a Lévy process with* $a = 0$, $\sigma = 0$, $\Pi = \lambda\delta_1$ *and* $\Psi(\gamma) = \lambda(e^{i\gamma} - 1)$.
- *More generally, a Compound Poisson process with jump rate* λ *and jump distribution* $F(\mathrm{d}x)$ *is a Lévy process with* $a = \lambda\int_{0<|x|<1} xF(\mathrm{d}x)$, $\sigma = 0$, $\Pi(\mathrm{d}x) = \lambda F(\mathrm{d}x)$ *and* $\Psi(\gamma) = \lambda\int_{\mathbb{R}\backslash\{0\}}\left(e^{i\gamma x} - 1\right)F(\mathrm{d}x)$.
- *Stable process with exponent* $\alpha \in (0, 2)$ *and the property that* $X_t \stackrel{d}{=} t^{1/\alpha}X_1$ $\forall t > 0$ *is a Lévy process with* $a = 0$, $\sigma = 0$, $\Pi(\mathrm{d}x) = \frac{c}{|x|^{1+\alpha}}\mathrm{d}x$ *for some* $c > 0$. *It has characteristic exponent* $\Psi(\gamma) = -C|\gamma|^\alpha$ *for some* $C > 0$.

The following well-known theorem describes a general Lévy process as an independent sum of a Brownian motion with a drift, a Compound Poisson process and a certain square-integrable martingale.

Theorem 4.4 (Lévy-Itô decomposition)**.** *Given a Lévy triplet* (a, σ, Π) *there exist three independent Lévy processes* $(X_t^{(1)})_{t\geq 0}$, $(X_t^{(2)})_{t\geq 0}$ *and* $(X_t^{(3)})_{t\geq 0}$ *on some probability space such that:*

- $X^{(1)}$ *is a Brownian motion with diffusion parameter* σ *and linear drift* a*, so that it has the characteristic exponent*

$$\Psi^{(1)}(\gamma) = ai\gamma - \frac{1}{2}\sigma^2\gamma^2,$$

- $X^{(2)}$ *is a Compound Poisson process with jump rate* $\Pi(\mathbb{R}\backslash(-1,1))\mathbf{1}_{\{|x|\geq 1\}}$ *and jump distribution* $\frac{\Pi(\mathrm{d}x)}{\Pi(\mathbb{R}\backslash(-1,1))}$*, so that it has the characteristic exponent*

$$\Psi^{(2)}(\gamma) = \int_{|x|\geq 1} (e^{i\gamma x} - 1)\Pi(\mathrm{d}x),$$

- $X^{(3)}$ *is a square-integrable martingale with the characteristic exponent*

$$\Psi^{(3)}(\gamma) = \int_{0<|x|<1} (e^{i\gamma x} - i\gamma x - 1)\Pi(\mathrm{d}x).$$

Thus a general characteristic exponent from (4.2) can be decomposed as

$$\Psi(\gamma) = \Psi^{(1)}(\gamma) + \Psi^{(2)}(\gamma) + \Psi^{(3)}(\gamma),$$

where $\Psi^{(1)}(\gamma)$, $\Psi^{(2)}(\gamma)$ and $\Psi^{(3)}(\gamma)$ correspond to a Brownian motion with a drift, a compound Poisson process and some square-integrable martingale.

4.1.2 Recurrence and point-recurrence

Let us now define various notions of recurrence of a Lévy process, that we are going to need later.

Definition 4.5. *A Lévy process* $(X_t)_{t\geq 0}$ *is recurrent if*

$$\liminf_{t\to\infty} |X_t| = 0 \qquad \mathbb{P}\text{-}a.s.$$

In other words, X *returns to any open neighbourhood of* 0 *infinitely often.*

The following standard result can be found in [31] for example.

Proposition 4.6. *Suppose that a Lévy process* $(X_t)_{t\geq 0}$ *is integrable. That is,* $\mathbb{E}|X_1| < \infty$*. Then*

$$(X_t)_{t\geq 0} \textit{ is recurrent } \Leftrightarrow \mathbb{E}X_1 = 0.$$

Moreover, it is true for a non-degenerate recurrent Lévy process that $\limsup_{t\to\infty} X_t = \infty$ and $\liminf_{t\to\infty} X_t = -\infty$ $\mathbb{P}$-a.s.

A stronger notion is the notion of point-recurrence:

Definition 4.7. *A Lévy process $(X_t)_{t\geq 0}$ is point-recurrent if*

$$\limsup_{t\to\infty} \mathbf{1}_{\{X_t=0\}} = 1 \qquad \mathbb{P}\text{-}a.s.$$

In other words, X returns to $\{0\}$ infinitely often.

Example 4.8. *The following processes are point-recurrent and thus also recurrent:*

- *Brownian motion.*
- *Continuous-time random walk with jumps of size ± 1, which we have studied in Chapter 3.*
- *Recurrent processes which experience jumps in only one direction. Such processes are called spectrally-negative or spectrally-positive depending on the direction of jumps. Point-recurrence follows from the fact that the process can get to the upper/lower half-plane from the lower/upper half-plane only by continuously crossing the line $x = 0$.*
- *Symmetric alpha-stable processes with exponent $\alpha \in (1, 2)$. For the proof of point-recurrence see [31].*

Example 4.9. *For an example of a recurrent process which is not point-recurrent consider a compound Poisson process which makes jumps of magnitude 1 at rate 1 and jumps of magnitude $-\sqrt{2}$ at rate $\frac{1}{\sqrt{2}}$. Such process has 0 mean and so it is recurrent, but it will never return to 0 after it made its first jump.*

A lot more discussion about point-recurrence can be found in the book of Sato [31].

4.1.3 Laplace exponent and Legendre transform

As we already mentioned in the preface, we would have to impose the following restriction on the Lévy processes that we consider.

Assumption 4.10. *There exist $\gamma^-, \gamma^+ \in (0, \infty]$ such that*

$$\mathbb{E}e^{\gamma X_1} < \infty \qquad \forall \gamma \in (-\gamma^-, \gamma^+). \tag{4.3}$$

It is actually quite a strong restriction, which doesn't allow heavy-tailed jumps in either direction. Nevertheless, it still leaves us with a large class of Lévy processes to look at. Assumption 4.10 will be imposed on all the Lévy processes we consider throughout the rest of this chapter unless we specifically say that it isn't.

From Definition 4.1 subject to Assumption 4.10 it is easy to check that

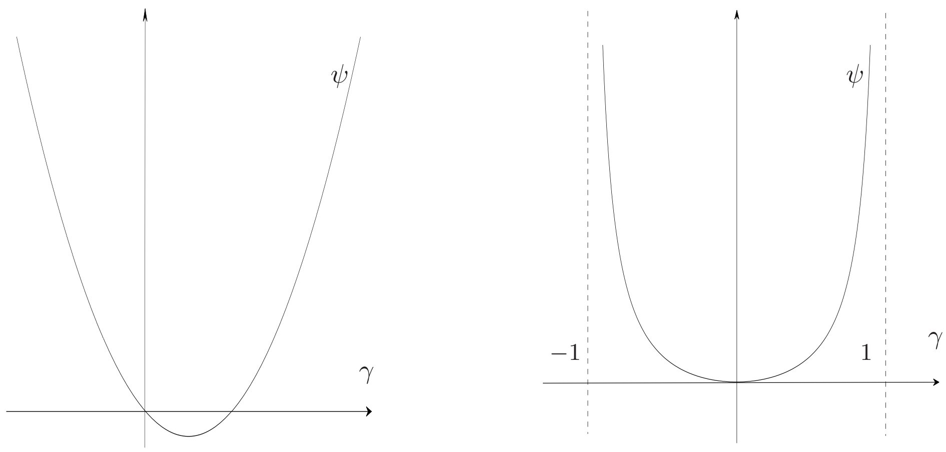
$$\mathbb{E}e^{\gamma X_t} = e^{\psi(\gamma)t} \qquad \forall t \geq 0 \tag{4.4}$$

for some function $\psi : (-\gamma^-, \gamma^+) \to \mathbb{R}$, which is known as the Laplace exponent of the process X. Analytic extension of the characteristic exponent Ψ gives us the following formula.

Proposition 4.11. *For* $\gamma \in (-\gamma^-, \gamma^+)$

$$\psi(\gamma) = a\gamma + \frac{1}{2}\sigma^2\gamma^2 + \int_{\mathbb{R}\setminus\{0\}} (e^{\gamma x} - \gamma x \mathbf{1}_{|x|<1} - 1)\Pi(\mathrm{d}x). \tag{4.5}$$

Note that (4.3) tells that Π must have tails which decay (at least) exponentially fast. On Figure 4-1 below one can see some examples of a Laplace exponent.



(a) $\psi(\gamma) = \frac{1}{2}\gamma^2 - \gamma$, $a = -1$, $\sigma = 1$, $\Pi = 0$ (b) $\psi(\gamma) = \frac{\gamma^2}{1-\gamma^2}$, $a = 0$, $\sigma = 0$, $\Pi(\mathrm{d}x) = \frac{1}{2}e^{-|x|}\mathrm{d}x$

Figure 4-1: *Plots of* $\psi(\gamma)$

The function ψ is infinitely differentiable (see [25]) and consequently has the following properties.

Proposition 4.12 (Properties of ψ)**.** *If we rule out the degenerate case* $(X_t)_{t\geq 0} \equiv 0$ *then:*

- $\psi(0) = 0$, $\psi'(0) = \mathbb{E}X_1$
- $\psi(\gamma) \nearrow \infty$ *as* $\gamma \nearrow \gamma^+$ *or* $\gamma \searrow -\gamma^-$

- *$\psi''(\gamma) > 0\ \forall \gamma \in (-\gamma^-, \gamma^+)$, i.e. ψ is strictly convex and $\psi'(\gamma)$ is strictly increasing on $(-\gamma^-, \gamma^+)$*

Since $\psi'(\gamma)$ is increasing it must converge to a limit as $\gamma \to \gamma^+$. Thus we have two possible behaviours of ψ (and this will be important later):

Case (**I**): $\lim_{\gamma\to\gamma^+} \psi'(\gamma) = \infty$. This is the most common case, happening if, for example, X makes positive jumps.

Case (**II**): $\lim_{\gamma\to\gamma^+} \psi'(\gamma) < \infty$. This is a somewhat degenerate case, which will be easy to handle. Note that we must necessarily have $\gamma^+ = \infty$ in this situation and so we can define $\psi'(\infty) := \lim_{\gamma\to\infty} \psi'(\gamma)$.

Since function ψ is convex, we can also define its Legendre transform Λ as follows:

Definition 4.13. *For $x \in \left[\psi'(0),\ \lim_{\gamma\to\gamma^+} \psi'(\gamma)\right)$*

$$\Lambda(x) := \sup_{p\geq 0}\{xp - \psi(p)\}.$$

Note the domain of Λ. Here as before

$$\lim_{\gamma\to\gamma^+} \psi'(\gamma) = \begin{cases} \infty & \textit{in Case (I)} \\ \psi'(\infty) & \textit{in Case (II)} \end{cases}$$

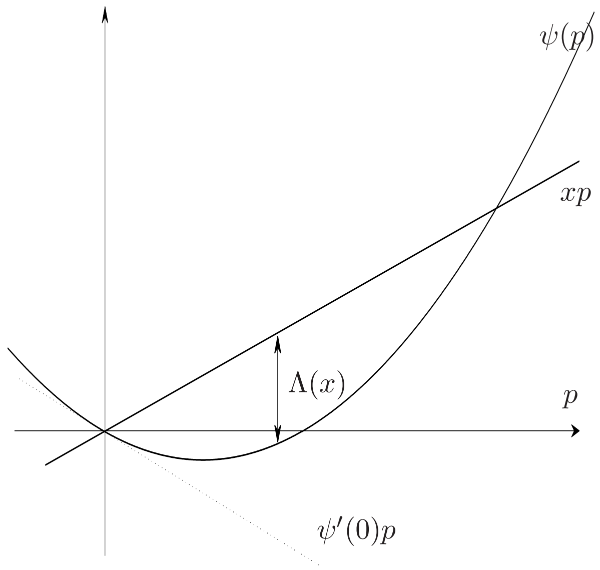


Figure 4-2: *Legendre transform*

We have the following identity for Λ that will often appear in this chapter:

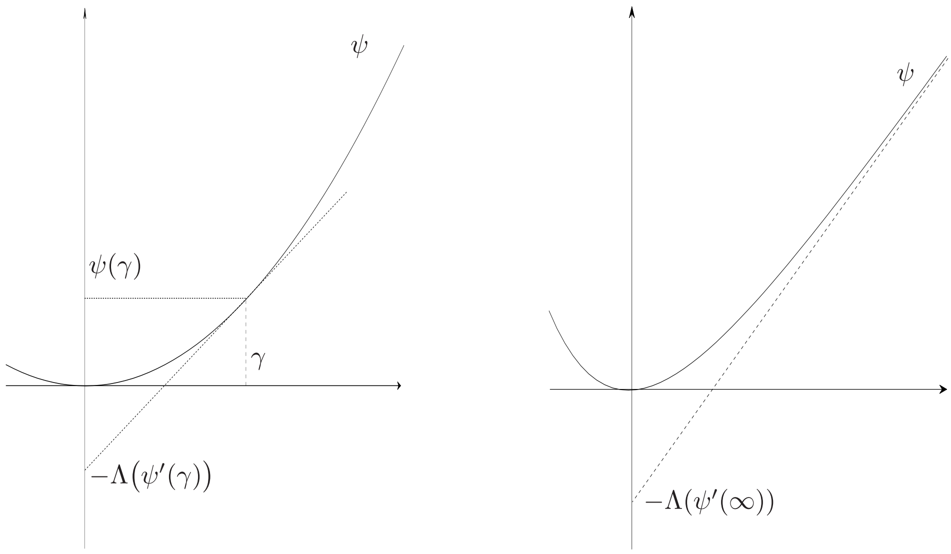
$$\Lambda\big(\psi'(\gamma)\big) = \gamma\psi'(\gamma) - \psi(\gamma) \qquad \forall\gamma \in [0, \gamma^{+}). \tag{4.6}$$

Thus Λ has the following geometrical interpetation: if we draw the tangent to ψ at a point γ then the intersection of this tangent with the y-axis would be $-\Lambda\big(\psi'(\gamma)\big)$ (see Figure 4-3(a)).

Note that in Case (II)

$$\begin{aligned} \Lambda(\psi'(\infty)) &:= \sup_{p}\{\psi'(\infty)p - \psi(p)\} \\ &= \lim_{\gamma\to\infty} \Lambda(\psi'(\gamma)) < \infty. \end{aligned}$$

See Figure 4-3(b) below for an illustartion.



(a) Case (**I**): $\psi(\gamma) = \frac{1}{2}\gamma^2$,$a = 0$, $\sigma = 1$, $\Pi = 0$ (b) Case (**II**): $\psi(\gamma) = \gamma + e^{-\gamma} - 1$, $a = 1$, $\sigma = 0$, $\Pi = \delta_{-1}$

Figure 4-3: *Illustartion for* Λ

Also Λ (in the domain $[\psi'(0), \lim_{\gamma\to\gamma^{+}\psi'(\gamma)})$) has the following useful properties.

Proposition 4.14 (Properties of Λ). *If we rule out the degenerate case $\psi(\cdot) \equiv 0$ then:*

- $\Lambda(\psi'(0)) = 0$,
- Λ *is strictly increasing, so* Λ^{-1} *is well-defined*
- Λ *is strictly convex*

Let us also observe that

- in Case (**I**) $\lim_{\gamma\to\gamma^+} \Lambda(\psi'(\gamma)) = \infty$
- in Case (**II**) $\lim_{\gamma\to\gamma^+} \Lambda(\psi'(\gamma)) = \Lambda(\psi'(\infty)) < \infty$

Example 4.15 (Case (**I**))**.**

- *Suppose $(X_t)_{t\geq 0}$ is a standard Brownian motion. Then $\psi(\gamma) = \frac{1}{2}\gamma^2$, the domain of Λ is $[0,\infty)$, $\Lambda(\psi'(\gamma)) = \frac{1}{2}\gamma^2$ and $\Lambda(x) = \frac{1}{2}x^2$.*
- *Suppose $(X_t)_{t\geq 0}$ is a continuous-time random walk from Chapter 3 that makes jumps of size ± 1 at rate λ. Then $\psi(\gamma) = \lambda(e^{\gamma} + e^{-\gamma} - 2)$, the domain of Λ is $[0,\infty)$, $\Lambda(\psi'(\gamma)) = \lambda(\gamma e^{\gamma} - \gamma e^{-\gamma} - e^{\gamma} - e^{-\gamma} + 2) \sim \lambda\gamma e^{\gamma}$ as $\gamma \to \infty$ and $\Lambda(x) = 2\lambda + x\log\left(\frac{\sqrt{x^2+4\lambda^2}+x}{2\lambda}\right) - \sqrt{x^2+4\lambda^2} \sim x\log x$ as $x\to\infty$ (recall discussion at the end of subsection 3.1.3).*

To finish the overview of Lévy processes we give an example of a process from the degenerate Case (**II**).

Example 4.16 (Case (**II**))**.** *Let $X_t^0 := t - P_t$, $t \geq 0$, where $(P_t)_{t\geq 0} \stackrel{d}{=} PP(1)$. So X^0 has constant linear upward drift and makes negative jumps of size 1 at rate 1.*

Then for such process $\psi(\gamma) = \gamma + e^{-\gamma} - 1$ (see Figure 4-3(b)), $\psi'(\gamma) = 1 - e^{-\gamma} \to 1$ as $\gamma\to\infty$, the domain of Λ is $[0,1)$ and $\Lambda(\psi'(\gamma)) = 1 - \gamma e^{-\gamma} - e^{-\gamma} \to 1$ as $\gamma\to\infty$.

Note that this process always stays below the line $x = t$ (see Figure 4-4).

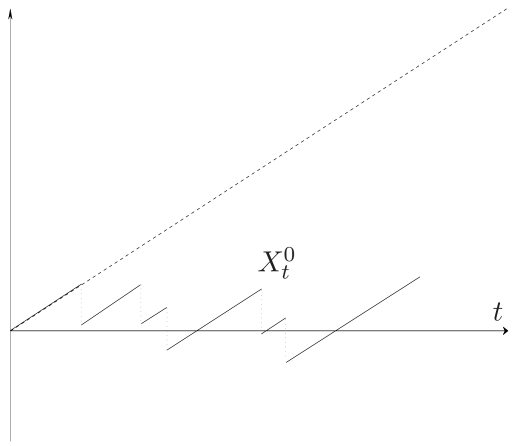


Figure 4-4: *Sample path of X_t^0 with $\psi(\gamma) = \gamma + e^{-\gamma} - 1$*

4.2 Branching model and main results

4.2.1 Description of the model

We are going to study a binary branching process started from 0, where branching occurs at instantaneous rate $\beta(\cdot) = \beta|\cdot|^p$, with $\beta > 0$ and $p \geq 0$ and single particles move according to a given Lévy process $(X_t)_{t\geq 0}$.

That is, we start with a single particle at the origin, which moves in $\mathbb{R}$ according to a certain Lévy process. At instantaneous rate $\beta|x|^p$, where x is the position of the particle, it splits into two new particles. The new particles then, independently of each other and of the past, stochastically repeat the behaviour of their parent starting from the position where it died.

4.2.2 Main Results

Recall Definition 3.3 of the rightmost particle:

$$R_t := \sup_{u \in N_t} X_t^u \ , \qquad t \geq 0.$$

We first state the following result in the simple case of homogeneous branching, which can be found in the works of Biggins (see [7], [6]).

Theorem 4.17 (Rightmost particle growth in the case $p = 0$)**.** *Consider a branching Lévy process in the homogeneous potential $\beta(\cdot) \equiv \beta$. Recall Assumption 4.10 on the domain of ψ, the Laplace exponent of X:*

$$\mathbb{E}e^{\gamma X_1} < \infty \qquad \forall \gamma \in (-\gamma^-, \gamma^+),$$

where γ^+, $\gamma^- \in (0, \infty]$. Under this assumption we have the following:

<u>*Case* (**I**) $\lim_{\gamma \to \gamma^+} \psi'(\gamma) = \infty$:</u>

$$\lim_{t \to \infty} \frac{R_t}{t} = \Lambda^{-1}(\beta) \ P\text{-}a.s.,$$

where Λ is the Legendre transform of ψ as given in Definition 4.13.

<u>*Case* (**IIa**) $\lim_{\gamma \to \gamma^+} \psi'(\gamma) = \psi'(\infty) < \infty$, $\beta < \Lambda(\psi'(\infty))$:</u>

$$\lim_{t \to \infty} \frac{R_t}{t} = \Lambda^{-1}(\beta) \ P\text{-}a.s.$$

as in the previous case.

<u>*Case* (**IIb**) $\lim_{\gamma \to \gamma^+} \psi'(\gamma) = \psi'(\infty) < \infty$, $\beta \geq \Lambda(\psi'(\infty))$:</u>

$$\lim_{t \to \infty} \frac{R_t}{t} = \psi'(\infty) \ P\text{-}a.s.$$

If we define the leftmost particle to be

$$L_t := \inf_{u \in N_t} X_t^u , \quad t \geq 0$$

then we can get the same results for L_t as for R_t by replacing $(X_t)_{t\geq 0}$ with $(-X_t)_{t\geq 0}$ in the theorem.

The proof of Theorem 4.17 heavily relies on Assumption 4.10. If we drop this assumption then in general we would expect the growth of R_t to be faster than linear. Some examples of this will be given in the next subsection.

Now take $p > 0$ and recall Definition 2.1 of the explosion time:

$$T_{explo} := \sup\{t : |N_t| < \infty\}.$$

For the next results we assume that $\gamma^+ = \gamma^- = \infty$ in Assumption 4.10.

Theorem 4.18 (Non-explosion)**.** *Consider a branching Lévy process in the potential $\beta(x) = \beta|x|^p$, $\beta > 0$, $p \in (0, 1]$, where single-particle process satisfies*

$$\mathbb{E}e^{\gamma X_1} < \infty \qquad \forall \gamma \in \mathbb{R}. \tag{4.7}$$

That is, $\gamma^+ = \gamma^- = \infty$. Then

$$T_{explo} = \infty \quad P\text{-a.s.}$$

Remark 4.19. *Assumption* (4.7) *for Theorem 4.18 in principle can be weakened, but we would then have to impose some additional assumption such as point-recurrence of the underlying Lévy process.*

Remark 4.20. *If $p = 0$ then the spatial component of the branching process has no effect on the population size $(|N_t|)_{t\geq 0}$. In fact in such setting, under no assumptions on the underlying Lévy process, $(|N_t|)_{t\geq 0}$ is a simple birth process and $|N_t| \overset{d}{=} Geom(e^{-\beta t})$ $\forall t > 0$, so $T_{explo} = \infty$ P-a.s.*

We shall discuss the case $p > 1$ in Section 4.3. Let us state it as a conjecture now.

Conjecture 4.21. *Consider a branching Lévy process in the potential $\beta(x) = \beta|x|^p$, $\beta > 0$, where one-particle motion satisfies condition* (4.7) *above. Then*

1. *if $(X_t)_{t\geq 0}$ is a Brownian motion with a linear drift*
   - $p \leq 2 \Rightarrow T_{explo} = \infty$ *P-a.s.*
   - $p > 2 \Rightarrow T_{explo} < \infty$ *P-a.s.*
2. *in all other cases*

- $p \leq 1 \Rightarrow T_{explo} = \infty$ P-*a.s.*
- $p > 1 \Rightarrow T_{explo} < \infty$ P-*a.s.*

In principle it seems possible to reduce the domain of ψ to $(-\gamma^-, \gamma^+)$, where γ^-, $\gamma^+ \in (0, \infty]$, in Theorem 4.18 and Conjecture 4.21 by imposing some other constraints, but for simplicity we shall adopt condition (4.7). However, if we drop Assumption 4.10 completely then in general we might expect the critical value of p to be smaller than 1, for certain branching Lévy processes we would even expect it to be 0. Some examples of this will be given in Section 4.3.

The next theorem gives the rightmost particle asymptotics in the case of inhomogeneous branching.

Theorem 4.22 (Rightmost particle growth in the case $p \in (0, 1)$)**.** *Consider a branching Lévy process in the potential $\beta(x) = \beta|x|^p$, $\beta > 0$, $p \in (0, 1)$, where the single-particle process fulfills the following conditions:*

1. $\psi(\gamma) = \log \mathbb{E} e^{\gamma X_1} < \infty \ \forall \gamma \in \mathbb{R}$,
2. *for all* $\delta > 0$ $\psi''(\gamma) < \psi'(\gamma)^{1+\delta}$ *for all* γ *large enough,*
3. $(X_t)_{t \geq 0}$ *is point-recurrent in the sense of Definition 4.7,*
4. $(X_t)_{t \geq 0}$ *is symmetric in the sense that* $(X_t)_{t \geq 0} \overset{d}{=} (-X_t)_{t \geq 0}$

Then the rightmost particle satisfies

$$\lim_{t \to \infty} \frac{R_t}{f(t)} = 1 \quad P\text{-}a.s., \tag{4.8}$$

where $f(t) = F^{-1}(t)$ *and*

$$F(t) := \int_0^t \frac{1}{\Lambda^{-1}(\beta(s))} \mathrm{d}s \ , \ t \geq 0 \tag{4.9}$$

is a strictly-increasing function. In particular, f is a nontrivial positive solution of the first-order autonomous differential equation

$$\Lambda(f'(s)) = \beta(f(s)) = \beta f(s)^p \ , \ s \geq 0 \ , \quad f(0) = 0. \tag{4.10}$$

Observe that we have forced $\mathbb{E} X_1 = \psi'(0) = 0$ and $\lim_{\gamma \to \infty} \psi'(\gamma) = \infty$. Thus the domain of Λ is $[0, \infty)$.

Also note that condition 1 guarantees non-explosion (recall Theorem 4.18).

Condition 2 adds some regularity to the Laplace exponent $\psi(\cdot)$ and is naturally satisfied by most of the Lévy processes that we consider. One simple way to ensure condition 2 is e.g. to take the jump measure to be supported on a set bounded above.

Condition 3 will be necessary for zero-one laws similar to Lemma 3.26.

Condition 4 on symmetry of the particles' motion is reasonable due to symmetry of the potential $\beta(\cdot)$.

To check that $f(t)$ is well-defined note that since $\Lambda(0) = 0$ and Λ is convex, there exist $t_0 > 0$ and $c > 0$ such that $\Lambda(t) \leq ct \ \forall t \in [0, t_0]$. Hence

$$F(t_0) = \int_0^{t_0} \frac{1}{\Lambda^{-1}(\beta s^p)} \mathrm{d}s \leq \int_0^{t_0} \frac{c}{\beta s^p} \mathrm{d}s < \infty.$$

Remark 4.23. *Note that in* (4.8) *it is sufficient to know only the first-order asymptotics of f defined via* (4.9).

If we drop condition 4 about symmetry of the underlying Lévy process then in general Theorem 4.22 may not hold for the reason explained in Section 4.6. However we can still prove the same lower bound on R_t assuming that the process $(X_t)_{t\geq 0}$ is making positive jumps or is a Brownian motion.

Theorem 4.24 (Lower bound on the rightmost particle in the case $p \in (0, 1)$ under weaker assumptions)**.** *Consider a branching Lévy process in the potential $\beta(x) = \beta|x|^p$, $\beta > 0$, $p \in (0, 1)$, where the single-particle process fulfills the following conditions:*

1. $\psi(\gamma) = \log \mathbb{E}e^{\gamma X_1} < \infty \ \forall \gamma \in \mathbb{R}$,
2. *for all $\delta > 0$ $\psi''(\gamma) < \psi'(\gamma)^{1+\delta}$ for all γ large enough,*
3. *$(X_t)_{t\geq 0}$ is point-recurrent in the sense of Definition 4.7,*
4. *$(X_t)_{t\geq 0}$ makes positive jumps (that is, $\Pi((0, \infty)) \neq 0$) or is a Brownian motion.*

Then the rightmost particle satisfies

$$\liminf_{t\to\infty} \frac{R_t}{f(t)} \geq 1 \quad P\text{-a.s.}, \tag{4.11}$$

where $f(t) = F^{-1}(t)$ and

$$F(t) := \int_0^t \frac{1}{\Lambda^{-1}(\beta(s))} \mathrm{d}s. \tag{4.12}$$

Note that we have again forced the process $(X_t)_{t\geq 0}$ to belong to Case (I).

4.2.3 Examples

Example 4.25 (Branching Brownian Motion)**.** *If $(X_t)_{t\geq 0}$ is a standard Brownian motion then its Laplace exponent is $\psi(\gamma) = \frac{1}{2}\gamma^2$ and the Legendre transform of ψ is $\Lambda(x) = \frac{1}{2}x^2$. Thus if $p = 0$ then from Theorem 4.17 we have*

$$\lim_{t\to\infty} \frac{R_t}{t} = \Lambda^{-1}(\beta) = \sqrt{2\beta} \ P\text{-a.s.}$$

and if $p > 0$ *then*

$$F(t) = \frac{1}{\sqrt{2\beta}} \frac{2}{2-p} t^{\frac{2-p}{2}} \ \ t \geq 0,$$

$$f(t) = \Big(\sqrt{2\beta}\frac{2-p}{2}t\Big)^{\frac{2}{2-p}},$$

which agrees with Theorem 3.5 Also equation 4.10 which gives the growth of the right-most particle becomes

$$\frac{1}{2}f'(s)^2 = \beta f(s)^p \ , \ s \geq 0 \ , \ f(0) = 0$$

and we have already seen it in Subsection 3.1.3.

Example 4.26 (Branching Random Walk)**.** *If* $(X_t)_{t\geq 0}$ *is a continuous-time random walk that makes jumps of size* ± 1 *at rate* λ *then its Laplace exponent is* $\psi(\gamma) = \lambda(e^{\gamma} + e^{-\gamma} - 2)$ *and the Legendre transform of* ψ *is*

$$\Lambda(x) = 2\lambda + x\log\Big(\frac{\sqrt{x^2+4\lambda^2}+x}{2\lambda}\Big) - \sqrt{x^2+4\lambda^2} \sim x\log x \ \text{ as } x \to \infty$$

If $p = 0$ *then from Theorem 3.4 a) we know that*

$$\lim_{t\to\infty}\frac{R_t}{t} = \lambda(\hat{\theta} - \frac{1}{\hat{\theta}}) \qquad P - a.s.,$$

where $\hat{\theta}$ *is the unique solution of* $g(\theta) = \frac{\beta}{\lambda}$ *and*

$$g(\theta) = (\theta - \frac{1}{\theta})\log(\theta) - (\theta + \frac{1}{\theta}) + 2$$

It is easy to check that

$$\Lambda\big(\lambda(\hat{\theta} - \frac{1}{\hat{\theta}})\big) = 2\lambda + \lambda(\hat{\theta} - \frac{1}{\hat{\theta}})\log\hat{\theta} - \lambda(\hat{\theta} + \frac{1}{\hat{\theta}}) = \lambda g(\hat{\theta}) = \beta.$$

Thus

$$\lim_{t\to\infty}\frac{R_t}{t} = \lambda(\hat{\theta} - \frac{1}{\hat{\theta}}) = \Lambda^{-1}(\beta) \qquad P - a.s.,$$

so Theorem 4.17 is consistent with Theorem 3.4 a) from Chapter 3.

If $p > 0$ *then since* $\Lambda(x) \sim x\log x$ *one can check that*

$$F(t) = \int_0^t \frac{1}{\Lambda^{-1}(\beta s^p)}\mathrm{d}s \sim \frac{p}{\beta(1-p)}t^{1-p}\log t \ \text{ as } t \to \infty$$

and

$$f(t) \sim \Big(\frac{p}{\beta(1-p)^2}\Big)^{\frac{1}{1-p}}\Big(\frac{t}{\log t}\Big)^{\frac{1}{1-p}},$$

so Theorem 4.22 is consistent with Theorem 3.4 b) from Chapter 3.

Example 4.27. *Recall the process $X_t^0 = t - P_t$, where $(P_t)_{t\geq 0} \stackrel{d}{=} PP(1)$, from Example 4.16. This process belongs to Case (II). Its sample path can be seen in Figure 4-4. The Laplace exponent of X^0 is $\psi(\gamma) = \gamma + e^{-\gamma} - 1$ and the Legendre transform of ψ defined on $[0,1)$ is*

$$\Lambda(x) = x + (1-x)\log(1-x)$$

Thus $\psi'(\infty) = 1$ and $\Lambda(\psi'(\infty)) = 1$. Consider the branching system with branching rate $\beta(\cdot) \equiv \beta$. Theorem 4.17 says that

- *if $\beta < 1$ then $\lim_{t\to\infty} \frac{R_t}{t} = \Lambda^{-1}(\beta)$ P-a.s.*
- *if $\beta \geq 1$ then $\lim_{t\to\infty} \frac{R_t}{t} = 1$ P-a.s.*

Example 4.28. *Consider a symmetric α-stable process $(X_t)_{t\geq 0}$, where $\alpha \in (0,2)$ and $X_t \stackrel{d}{=} t^{\frac{1}{\alpha}} X_1$ $\forall t > 0$. This process does not satisfy Assumption 4.10 due to heavy tails. Hence Theorem 4.17 can't be applied. It is known that $\mathbb{P}(X_1 > x) \sim \frac{c}{x^\alpha}$ as $x \to \infty$ for some constant c. So for a fast-increasing function $f(t)$ we have*

$$\mathbb{P}(X_t > f(t)) = \mathbb{P}(X_1 > f(t)t^{-\frac{1}{\alpha}}) \sim \frac{ct}{f(t)^\alpha}.$$

If we now consider a branching system with $p = 0$ then the Many-to-One Lemma says that the expected number of particles above the line $f(t)$ at time t is

$$e^{\beta t}\mathbb{P}(X_t > f(t)) \sim e^{\beta t}\frac{ct}{f(t)^\alpha}.$$

Thus if $f(t) = e^{\gamma t}$, where $\gamma \in (0, \frac{\beta}{\alpha})$ then the expected number of particles above the line $f(t)$ will be increasing rapidly suggesting exponential growth of the rightmost particle.

4.2.4 Outline of the chapter

In Section 4.3 we prove Theorem 4.18 about non-explosion and discuss Conjecture 4.21.

In Section 4.4 we introduce a family of one-particle martingales and prove some associated one-particle results that we are going to use in later sections.

Section 4.5 is devoted to the proof of Theorem 4.17 about the rightmost particle in the model with homogeneous branching.

In Section 4.6 we present proofs of Theorems 4.17 and 4.24 about the rightmost particle in the model with inhomogeneous branching.

4.3 Non-explosion

Let us prove Theorem 4.18. That is, subject to the condition on finite exponential moments we want to show that in the branching system with the potential $\beta(x) = \beta|x|^p$, where $\beta > 0$, $p \in (0,1]$ we have

$$T_{explo} = \infty \quad P\text{-a.s.}$$

The proof uses the same argument as we have used in Theorem 2.7 a) and Theorem 3.2 a).

Proof of Theorem 4.18. By the Many-to-One Lemma (Lemma 1.18) for any $t \geq 0$

$$E\big(|N_t|\big) = E\Big(\sum_{u \in N_t} 1\Big) = \tilde{E}\Big(e^{\int_0^t \beta |\xi_s|^p \mathrm{d}s}\Big),$$

where $(\xi_t)_{t \geq 0}$ moves as the given Lévy process under $\tilde{P}$. For $p \in (0,1]$ $|x|^p \leq |x| + 1$, so

$$\tilde{E}\Big(e^{\int_0^t \beta |\xi_s|^p \mathrm{d}s}\Big) \leq \tilde{E}\Big(e^{\int_0^t \beta (|\xi_s|+1) \mathrm{d}s}\Big) = e^{\beta t}\tilde{E}\Big(e^{\int_0^t \beta |\xi_s| \mathrm{d}s}\Big).$$

Next note that by Jensen's inequality for any locally-integrable function f

$$e^{\int_0^t f(s) \mathrm{d}s} \leq \frac{1}{t}\int_0^t e^{f(s)t} \mathrm{d}s. \tag{4.13}$$

To see this take $U \sim Uniform([0,t])$, $X := tf(U)$. Then

$$\frac{1}{t}\int_0^t e^{f(s)t} \mathrm{d}s = \mathbb{E}e^X \geq e^{\mathbb{E}X} = e^{\int_0^t f(s) \mathrm{d}s}.$$

Thus applying (4.13) we get

$$\tilde{E}\Big(e^{\int_0^t \beta |\xi_s| \mathrm{d}s}\Big) \leq \tilde{E}\Big(\frac{1}{t}\int_0^t e^{\beta |\xi_s| t} \mathrm{d}s\Big).$$

Then since

$$\begin{aligned}
\int_0^t \tilde{E}\Big(e^{\beta t |\xi_s|}\Big) \mathrm{d}s &\leq \int_0^t \tilde{E}\Big(e^{\beta t \xi_s} + e^{-\beta t \xi_s}\Big) \mathrm{d}s \\
&= \int_0^t e^{\psi(\beta t)s} + e^{\psi(-\beta t)s} \mathrm{d}s \\
&= \frac{1}{t}\Big(\frac{1}{\psi(\beta t)}\big(e^{\psi(\beta t)t} - 1\big) + \frac{1}{\psi(-\beta t)}\big(e^{\psi(-\beta t)t} - 1\big)\Big) < \infty
\end{aligned}$$

we have by Fubini's Theorem that

$$\tilde{E}\Big(\frac{1}{t}\int_0^t e^{\beta |\xi_s| t} \mathrm{d}s\Big) = \frac{1}{t}\int_0^t \tilde{E}\Big(e^{\beta t |\xi_s|}\Big) \mathrm{d}s < \infty$$

and hence

$$E\big(|N_t|\big) \leq e^{\beta t}\tilde{E}\Big(\frac{1}{t}\int_0^t e^{\beta |\xi_s| t} \mathrm{d}s\Big) < \infty \quad \forall t \geq 0.$$

Thus $T_{explo} = \infty$ P-a.s. $\square$

We can also verify that $E(|N_t|) = \infty$ if p is too large. Note that this is not sufficient to deduce that $T_{explo} = \infty$ P-a.s., but it gives us some evidence to believe that this might be the case.

For any $t > 0$ take any $t_0 \in (0, t)$ and x a large number. Then

$$\begin{aligned}
E\big(|N_t|\big) &= \tilde{E}\Big(e^{\int_0^t \beta|\xi_s|^p \mathrm{d}s}\Big) \\
&\geq \tilde{E}\Big(e^{\int_0^t \beta|\xi_s|^p \mathrm{d}s} \mathbf{1}_{\{|\xi_{t_0}|>x+1\}} \mathbf{1}_{\{\sup_{s\in[t_0,t]} |\xi_s - \xi_{t_0}|<1\}}\Big) \\
&\geq \tilde{E}\Big(e^{\int_{t_0}^t \beta|\xi_s|^p \mathrm{d}s} \mathbf{1}_{\{|\xi_{t_0}|>x+1\}} \mathbf{1}_{\{\sup_{s\in[t_0,t]} |\xi_s - \xi_{t_0}|<1\}}\Big) \\
&\geq e^{\beta(t-t_0)x^p} \tilde{P}\big(|\xi_{t_0}| > x+1\big) \underbrace{\tilde{P}\big(\sup_{s\in[t_0,t]} |\xi_s - \xi_{t_0}| < 1\big)}_{>0}
\end{aligned}$$

If we now let $x \to \infty$ we see that if $\tilde{P}\big(|\xi_{t_0}| > x+1\big)$ decays a lot slower than e^{-x^p} then $E(|N_t|) = \infty$.

As $(\xi_t)_{t\geq 0}$ is a Lévy process, ξ_{t_0} is an infinitely-divisible random variable, so let us quote the following result about its tail behaviour from [34] (Chapter IV, Corollary 9.9):

Proposition 4.29 (F.W. Steutel and K. Van Harn)**.** *A non-degenerate infinitely-divisible random variable X has a normal distribution iff it satisfies*

$$\limsup_{x\to\infty} \frac{-\log \mathbb{P}(|X| > x)}{x \log x} = \infty.$$

In other words, unless X has a normal distribution, $\mathbb{P}(|X| > x) \geq e^{-Cx\log x}$ for some $C > 0$ and x large enough. Thus if we take $p > 1$ then $x^p \gg x \log x$, so unless $(\xi_t)_{t\geq 0}$ is a Brownian motion with a linear drift we have

$$E\big(|N_t|\big) = \infty \quad \forall t \geq 0.$$

If $(\xi_t)_{t\geq 0}$ is a Brownian motion with a linear drift then

$$E\big(|N_t|\big) = \infty \text{ or } < \infty \quad \forall t \geq 0.$$

according to whether $p > 2$ or $p \leq 2$.

Also if we for example take $(\xi_t)_{t\geq 0}$ to be an α-stable process with $\alpha \in (0, 2)$, which no longer satisfies exponential moments assumption, then $\tilde{P}\big(|\xi_{t_0}| > x+1\big) \sim \frac{c}{x^\alpha}$ for some constant c and hence $E(|N_t|) = \infty$ for any $p > 0$.

4.4 One-particle results

In this self-contained section we introduce a family of exponential martingales for a rather general Lévy process and derive some useful results concerning the asymptotic growth of the Lévy process under the changed measure.

In later sections we are going to use these martingales in place of martingale $\tilde{M}^{(3)}$ from (1.5) to condition the spine process to stay close to a deterministic path of our choice.

4.4.1 Simple changes of measure for Lévy processes

Let $(X_t)_{t\geq 0}$ be a Lévy process under a probability measure $\mathbb{P}$. Suppose that X satisfies Assumption 4.10 That is, there exist γ^+, $\gamma^- \in (0, \infty]$ such that

$$\mathbb{E}e^{\gamma X_1} < \infty \qquad \forall \gamma \in (-\gamma^-, \gamma^+).$$

For $\gamma \in (-\gamma^-, \gamma^+)$ let $\psi(\gamma) = \log \mathbb{E}e^{\gamma X_1}$ be the Laplace exponent of X. Then we have the following well known results, which can be found for example in [25].

Theorem 4.30. *Take $\gamma \in (-\gamma^-, \gamma^+)$. Then the following process is a $\mathbb{P}$-martingale:*

$$M_t := e^{\gamma X_t - \psi(\gamma)t} \quad , \quad t \geq 0. \tag{4.14}$$

Proof. It is clear that $\mathbb{E}M_t = 1$, and if $(\hat{\mathcal{F}}_t)_{t\geq 0}$ is the natural filtration of $(X_t)_{t\geq 0}$ then for $s \leq t$

$$\begin{aligned}
\mathbb{E}(M_t|\hat{\mathcal{F}}_s) &= \mathbb{E}(e^{\gamma X_t - \psi(\gamma)t}|\hat{\mathcal{F}}_s) \\
&= e^{\gamma X_s - \psi(\gamma)s}\mathbb{E}(e^{\gamma(X_t - X_s) - \psi(\gamma)(t-s)}|\hat{\mathcal{F}}_s) \\
&= M_s\mathbb{E}(e^{\gamma X_{t-s} - \psi(\gamma)(t-s)}) = M_s.
\end{aligned}$$

□

Theorem 4.31. *Let the measure $\mathbb{Q}$ be defined as*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\bigg|_{\hat{\mathcal{F}}_t} = M_t \quad , t \in [0, \infty).$$

Then under $\mathbb{Q}$, $(X_t)_{t\geq 0}$ is a Lévy process with parameters $(\hat{a}, \sigma, \hat{\Pi})$, where

$$\hat{a} = a + \gamma\sigma^2 + \int_{|x|\in(0,1)} x(e^{\gamma x} - 1)\Pi(\mathrm{d}x)$$

and $\hat{\Pi}(\mathrm{d}x) = e^{\gamma x}\Pi(\mathrm{d}x)$.

To see this observe that $\forall \alpha \in (-\gamma^- - \gamma,\ \gamma^+ - \gamma)$

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\big(e^{\alpha X_1}\big) &= \mathbb{E}\Big(e^{\alpha X_1} e^{\gamma X_1 - \psi(\gamma)}\Big)\\ &= \mathbb{E}\Big(e^{(\alpha+\gamma)X_1 - \psi(\alpha+\gamma)}\Big) e^{\psi(\alpha+\gamma)-\psi(\gamma)}\\ &= e^{\psi(\alpha+\gamma)-\psi(\gamma)}.\end{aligned}$$

Thus the Laplace exponent of X with respect to $\mathbb{Q}$ is

$$\begin{aligned}\psi^{\mathbb{Q}}(\alpha) &= \psi(\alpha+\gamma) - \psi(\gamma)\\ &= a(\alpha+\gamma) + \frac{1}{2}\sigma^2(\alpha+\gamma)^2 + \int_{\mathbb{R}\setminus\{0\}} \big(e^{(\alpha+\gamma)x} - (\alpha+\gamma)x\mathbf{1}_{|x|<1} - 1\big)\Pi(\mathrm{d}x)\\ &\quad - a\gamma - \frac{1}{2}\sigma^2\gamma^2 - \int_{\mathbb{R}\setminus\{0\}} \big(e^{\gamma x} - \gamma x\mathbf{1}_{|x|<1} - 1\big)\Pi(\mathrm{d}x)\\ &= \alpha\Big(a + \gamma\sigma^2 + \int_{|x|\in(0,1)} x(e^{\gamma x} - 1)\Pi(\mathrm{d}x)\Big)\\ &\quad + \frac{1}{2}\sigma^2\alpha^2 + \int_{\mathbb{R}\setminus\{0\}} \big(e^{\alpha x} - \alpha x\mathbf{1}_{|x|<1} - 1\big)e^{\gamma x}\Pi(\mathrm{d}x).\end{aligned}$$

Note that the exponential moments of X under $\mathbb{Q}$ give us all the n^{th} moments of X. In particular we have:

$$\mathbb{E}^{\mathbb{Q}} X_t = \psi'(\gamma)t,$$

$$var^{\mathbb{Q}}(X_t) = \psi''(\gamma)t.$$

Example 4.32.

- *Take $(X_t)_{t\geq 0}$ to be a standard Brownian motion. Then $(a,\sigma,\Pi) = (0,1,0)$, $\psi(\gamma) = \frac{1}{2}\gamma^2$, so*

  $$M_t = e^{\gamma X_t - \frac{1}{2}\gamma^2 t} \quad , \qquad t \geq 0$$

  and under the new measure $\mathbb{Q}$, $(X_t)_{t\geq 0}$ is a Lévy process with parameters $(\hat{a},\sigma,\hat{\Pi}) = (\gamma,1,0)$, in other words, a Brownian motion with linear drift γ.

- *Take $(X_t)_{t\geq 0}$ to be a Poisson process with rate λ. Then $(a,\sigma,\Pi) = (0,0,\lambda\delta_1)$, $\psi(\gamma) = \lambda(e^\gamma - 1)$, so*

  $$M_t = e^{\gamma X_t - \lambda(e^\gamma - 1)t} = \theta^{X_t} e^{\lambda(1-\theta)t} \quad , \qquad t \geq 0,$$

  where $\theta = e^\gamma$ (Recall Example 3.7). Under the new measure $\mathbb{Q}$, $(X_t)_{t\geq 0}$ is a Lévy process with parameters

  $$(\hat{a},\sigma,\hat{\Pi}) = (0,0,\lambda e^\gamma\delta_1) = (0,0,\lambda\theta\delta_1),$$

  that is, $(X_t)_{t\geq 0}$ a Poisson process with jump rate $\lambda e^\gamma = \lambda\theta$.

Thus we see that martingale (4.14) is the natural generalisation of martingales seen in Example 1.27 and Example 3.7.

However martingale (4.14) does not include the martingales from Example 1.23 and Lemma 3.6. In the next subsection we deal with this issue.

4.4.2 Stochastic integrals and more advanced changes of measure

In this subsection we are going to use stochastic integrals with respect to $(X_t)_{t\geq 0}$. Construction of such objects as well as their properties can be found in the book of D. Applebaum [1] or his lecture notes on this topic [2].

In this thesis we are only going to consider very simple cases of stochastic integrals where the integrand is a 'nice' deterministic function. For such integrals the reader does not need to be familiar with the general theory of stochastic calculus for Lévy processes.

Suppose for this subsection that we are given a function $\gamma : [0,\infty) \to (-\gamma^-, \gamma^+)$, which is differentiable and satisfies $\int_0^t \psi(\gamma(s))\mathrm{d}s < \infty$ $\forall t \geq 0$. For such function we consider the integral $\int_0^t \gamma(s)\mathrm{d}X_s$. The following integration-by-parts formula, which can be found e.g. in [1] reduces it to a Lebesgue integral.

Proposition 4.33 (Integration by parts)**.**

$$\int_0^t \gamma(s)\mathrm{d}X_s = X_t\gamma(t) - \int_0^t X_s\gamma'(s)\mathrm{d}s.$$

The next result generalises Theorem 4.30 and can be found in [1].

Theorem 4.34. *The following process is a $\mathbb{P}$-martingale:*

$$M_t := e^{\int_0^t \gamma(s)\mathrm{d}X_s - \int_0^t \psi(\gamma(s))\mathrm{d}s}, \quad t \geq 0. \tag{4.15}$$

If we now define the measure $\mathbb{Q}$ as

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\hat{\mathcal{F}}_t} = M_t \quad, t \in [0,\infty)$$

then under $\mathbb{Q}$, $(X_t)_{t\geq 0}$ in general can not be characterised in a nice way. It is no longer a Lévy process nor a time-changed Lévy process. It can be thought of as a process with independent increments which has the instantaneous drift $\hat{a}_t = a + \gamma(t)\sigma^2 + \int_{|x|\leq 1} x(e^{\gamma(t)x} - 1)\Pi(\mathrm{d}x)$ at time t, the diffusion parameter σ and the instantaneous jump measure $\hat{\Pi}_t(\mathrm{d}x) = e^{\gamma(t)x}\Pi(\mathrm{d}x)$.

In the special cases of a Brownian motion and a Poisson process $(X_t)_{t\geq 0}$ has a nice characterisation under $\mathbb{Q}$.

Example 4.35.

- *Take $(X_t)_{t\geq 0}$ to be a standard Brownian motion. Then*

$$M_t = e^{\int_0^t \gamma(s)\mathrm{d}X_s - \frac{1}{2}\int_0^t \gamma(s)^2 \mathrm{d}s}, \qquad t \geq 0$$

  and under $\mathbb{Q}$, $(X_t)_{t\geq 0}$ is a standard Brownian motion with drift $\int_0^t \gamma(s)\mathrm{d}s$ (Recall Example 1.23).

- *Take $(X_t)_{t\geq 0}$ to be a Poisson process with jump rate λ. Then*

$$M_t = e^{\int_0^t \gamma(s)\mathrm{d}X_s - \lambda\int_0^t (e^{\gamma(s)}-1)\mathrm{d}s} = e^{\int_0^t \log\theta(s)\mathrm{d}X_s + \lambda\int_0^t (1-\theta(s))\mathrm{d}s}, \qquad t \geq 0,$$

  where $\theta(t) = e^{\gamma(t)}$. Under $\mathbb{Q}$, $(X_t)_{t\geq 0}$ is a time-inhomogeneous Poisson process with instantaneous jump rate $\lambda e^{\gamma(t)} = \lambda\theta(t)$. (Recall Lemma 3.6 and Lemma 3.8.)

We can still easily compute all the moments of X_t under $\mathbb{Q}$.

Proposition 4.36. *Suppose $\alpha \in \mathbb{R}$ is such that $\alpha + \gamma(t) \in (-\gamma^-, \gamma^+)$ and $\int_0^t \psi(\alpha+\gamma(s))\mathrm{d}s < \infty$ $\forall t > 0$ then*

$$\mathbb{E}^{\mathbb{Q}}\big(e^{\alpha X_t}\big) = e^{\int_0^t \psi(\alpha+\gamma(s)) - \psi(\gamma(s))\mathrm{d}s},$$

$$\mathbb{E}^{\mathbb{Q}}\big(X_t\big) = \int_0^t \psi'(\gamma(s))\mathrm{d}s,$$

$$var^{\mathbb{Q}}\big(X_t\big) = \int_0^t \psi''(\gamma(s))\mathrm{d}s.$$

Proof.

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\big(e^{\alpha X_t}\big) &= \mathbb{E}\Big(e^{\alpha X_t} e^{\int_0^t \gamma(s)\mathrm{d}X_s - \int_0^t \psi(\gamma(s))\mathrm{d}s}\Big)\\ &= \mathbb{E}\Big(e^{\int_0^t (\alpha+\gamma(s))\mathrm{d}X_s - \int_0^t \psi(\alpha+\gamma(s))\mathrm{d}s}\Big) e^{\int_0^t \psi(\alpha+\gamma(s)) - \psi(\gamma(s))\mathrm{d}s}\\ &= e^{\int_0^t \psi(\alpha+\gamma(s)) - \psi(\gamma(s))\mathrm{d}s}.\end{aligned}$$

Differentiating with respect to α n times and letting $\alpha = 0$ gives the n^{th} moment of X_t. In particular, we get $\mathbb{E}^{\mathbb{Q}}X_t$ and $var^{\mathbb{Q}}X_t$. □

The most important feature of $(X_t)_{t\geq 0}$ under $\mathbb{Q}$ to us will be its almost sure asymptotic behaviour. This issue is addressed in the next subsection.

4.4.3 Strong Laws of Large Numbers

Let us start with a well-known result, which can be found for example in [31] or [4].

Theorem 4.37 (SLLN)**.** *Suppose $(X_t)_{t\geq 0}$ is a Lévy process such that $\mathbb{E}|X_1| < \infty$. Then*

$$\frac{X_t}{t} \to \mathbb{E}X_1 \qquad \mathbb{P}\text{-}a.s.$$

Note that we don't even need to assume finite exponential moments for this theorem.

Corollary 4.38. *Take $(X_t)_{t\geq 0}$ to be a Lévy process under $\mathbb{P}$ and consider the martingale $M_t = e^{\gamma X_t - \psi(\gamma)t}$ and the corresponding measure $\mathbb{Q}$ from theorems 4.30 and 4.31. Then*

$$\frac{X_t}{t} \to \psi'(\gamma) \qquad \mathbb{Q}\text{-}a.s.$$

We now wish to prove that if we take the martingale $M_t = e^{\int_0^t \gamma(s)\mathrm{d}X_s - \int_0^t \psi(\gamma(s))\mathrm{d}s}$ from Theorem 4.34 and the corresponding measure $\mathbb{Q}$ then under some additional assumptions on $\psi(\cdot)$ and $\gamma(\cdot)$ we have

$$\frac{X_t}{\mathbb{E}^{\mathbb{Q}}X_t} = \frac{X_t}{\int_0^t \psi'(\gamma(s))\mathrm{d}s} \to 1 \qquad \mathbb{Q}\text{-a.s.}$$

Theorem 4.39. *Let $(X_t)_{t\geq 0}$ be a Lévy process under probability measure $\mathbb{P}$ that satisfies*

$$\mathbb{E}e^{\gamma X_1} < \infty \qquad \forall \gamma \in \mathbb{R}$$

and let $\psi(\gamma) = \log \mathbb{E}e^{\gamma X_1}$ as always. Note that we assumed that γ^-, $\gamma^+ = \infty$

Suppose we are given a function $\gamma : [0, \infty) \to \mathbb{R}$ which satisfies:

1. $\psi'(\gamma(t)) \geq 0$, $\int_0^t \psi(\gamma(s))\mathrm{d}s$, $\int_0^t \psi'(\gamma(s))\mathrm{d}s$, $\int_0^t \psi''(\gamma(s))\mathrm{d}s < \infty \quad \forall t \geq 0$
2. $\dfrac{\int_n^{n+1} \psi'(\gamma(s))\mathrm{d}s}{\int_0^n \psi'(\gamma(s))\mathrm{d}s} \to 0$ *as* $n \to \infty$
3. $\exists \delta > 0$ *such that for n large enough* $\dfrac{\int_0^n \psi''(\gamma(s))\mathrm{d}s}{\left(\int_0^n \psi'(\gamma(s))\mathrm{d}s\right)^2} \leq \dfrac{1}{n^{1+\delta}}$

For such a function γ define the martingale

$$M_t := e^{\int_0^t \gamma(s)\mathrm{d}X_s - \int_0^t \psi(\gamma(s))\mathrm{d}s}, \quad t \geq 0$$

and the corresponding measure $\mathbb{Q}$ as

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\hat{\mathcal{F}}_t} = M_t, \quad t \geq 0,$$

where $(\hat{\mathcal{F}}_t)_{t\geq 0}$ is the natural filtration of $(X_t)_{t\geq 0}$. Then

$$\frac{X_t}{\int_0^t \psi'(\gamma(s))\mathrm{d}s} \to 1 \text{ as } t \to \infty \qquad \mathbb{Q}\text{-}a.s.$$

Note that conditions 1 - 3 say that

1. $\psi'(\gamma(t)) \geq 0$, $\mathbb{E}^{\mathbb{Q}} X_t < \infty$, $var^{\mathbb{Q}} X_t < \infty \quad \forall t \geq 0$
2. $\dfrac{\mathbb{E}^{\mathbb{Q}}(X_{n+1} - X_n)}{\mathbb{E}^{\mathbb{Q}} X_n} \to 0$ as $n \to \infty$
3. $\exists \delta > 0$ such that for n large enough $\dfrac{var^{\mathbb{Q}} X_n}{\left(\mathbb{E}^{\mathbb{Q}} X_n\right)^2} \leq \dfrac{1}{n^{1+\delta}}$.

Although these conditions may appear restrictive, they will be naturally satisfied by the functions γ that we consider in later sections.

Proof. Take any $\epsilon > 0$. Then using Chebyshev's inequality and condition 3

$$\begin{aligned}
&\mathbb{Q}\Big(\big|X_n - \int_0^n \psi'(\gamma(s))\mathrm{d}s\big| > \epsilon \int_0^n \psi'(\gamma(s))\mathrm{d}s\Big) \\
\leq &\frac{var^{\mathbb{Q}} X_n}{\epsilon^2 (\mathbb{E}^{\mathbb{Q}} X_n)^2} \\
\leq &\frac{1}{\epsilon^2 n^{1+\delta}}
\end{aligned}$$

for n large enough from condition 3. Thus

$$\sum_{n \geq 1} \mathbb{Q}\Big(\big|X_n - \int_0^n \psi'(\gamma(s))\mathrm{d}s\big| > \epsilon \int_0^n \psi'(\gamma(s))\mathrm{d}s\Big) < \infty.$$

Hence by the Borel-Cantelli lemma

$$\mathbb{Q}\Big(\Big\{\big|X_n - \int_0^n \psi'(\gamma(s))\mathrm{d}s\big| > \epsilon \int_0^n \psi'(\gamma(s))\mathrm{d}s\Big\} \text{ for infinitely many } n \in \mathbb{N}\Big) = 0.$$

Therefore since ϵ was arbitrary it follows that for $n \in \mathbb{N}$ and $n \to \infty$, we have

$$\frac{|X_n - \int_0^n \psi'(\gamma(s))\mathrm{d}s|}{\int_0^n \psi'(\gamma(s))\mathrm{d}s} \to 0 \quad \mathbb{Q}\text{-a.s.} \tag{4.16}$$

In other words,

$$\frac{X_n}{\int_0^n \psi'(\gamma(s))\mathrm{d}s} \to 1 \qquad \mathbb{Q}\text{-a.s.}$$

We now wish to prove this convergence along the reals. Fix $n \in \mathbb{N}$. Then for each fixed $n \in \mathbb{N}$ the process

$$\Big(X_{n+t} - X_n - \int_n^{n+t} \psi'(\gamma(s))\mathrm{d}s\Big)_{t \in [0,1]}$$

is a $\mathbb{Q}$-martingale since $\forall t \in [0,1]$

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big|X_{n+t} - X_n - \int_n^{n+t} \psi'(\gamma(s))\mathrm{d}s\Big| \\
\leq &\mathbb{E}^{\mathbb{Q}}\Big|X_{n+t}\Big| + \mathbb{E}^{\mathbb{Q}}\Big|X_n\Big| + \int_n^{n+t} \psi'(\gamma(s))\mathrm{d}s < \infty
\end{aligned}$$

and for $0 \leq s \leq t \leq 1$, with $(\bar{\mathcal{F}}_t)_{t\geq 0}$ the natural filtration of $(X_{n+t} - X_n)_{t\in[0,1]}$

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Big(X_{n+t} - X_n - \int_n^{n+t} \psi'(\gamma(s))\mathrm{d}s\Big|\bar{\mathcal{F}}_s\Big) \\
=&X_{n+s} - X_n - \int_n^{n+s} \psi'(\gamma(s))\mathrm{d}s + \mathbb{E}^{\mathbb{Q}}\Big(X_{n+t} - X_{n+s} - \int_{n+s}^{n+t} \psi'(\gamma(s))\mathrm{d}s\Big) \\
=&X_{n+s} - X_n - \int_n^{n+s} \psi'(\gamma(s))\mathrm{d}s.
\end{aligned}$$

Therefore we also have that

$$\Big(\Big|X_{n+t} - X_n - \int_n^{n+t} \psi'(\gamma(s))\mathrm{d}s\Big|\Big)_{t\in[0,1]}$$

is a positive $\mathbb{Q}$-submartingale and we can apply Doob's martingale inequality to it.

Take $\epsilon > 0$, then

$$\begin{aligned}
&\mathbb{Q}\Big(\sup_{t\in[n,n+1]}\Big|X_t - X_n - \int_n^t \psi'(\gamma(s))\mathrm{d}s\Big| > \epsilon\int_0^{n+1} \psi'(\gamma(s))\mathrm{d}s\Big) \\
\leq&\frac{var^{\mathbb{Q}}(X_{n+1} - X_n)}{\epsilon^2(\int_0^{n+1} \psi'(\gamma(s))\mathrm{d}s)^2} \\
=&\frac{\int_n^{n+1} \psi''(\gamma(s))\mathrm{d}s}{\epsilon^2(\int_0^{n+1} \psi'(\gamma(s))\mathrm{d}s)^2} \\
\leq&\frac{\int_0^{n+1} \psi''(\gamma(s))\mathrm{d}s}{\epsilon^2(\int_0^{n+1} \psi'(\gamma(s))\mathrm{d}s)^2} \\
\leq&\frac{1}{\epsilon^2 n^{1+\delta}}
\end{aligned}$$

by condition 3. Thus by the Borel-Cantelli lemma

$$\begin{aligned}
\mathbb{Q}\Big(\Big\{\sup_{t\in[n,n+1]}\Big|X_t - X_n - \int_n^t \psi'(\gamma(s))\mathrm{d}s\Big| > \epsilon\int_0^{n+1} \psi'(\gamma(s))\mathrm{d}s\Big\} \\
\text{for infinitely many } n \in \mathbb{N}\Big) = 0.
\end{aligned}$$

Therefore

$$\frac{\sup_{t\in[n,n+1]}\left|X_t - X_n - \int_n^t \psi'(\gamma(s))\mathrm{d}s\right|}{\int_0^{n+1}\psi'(\gamma(s))\mathrm{d}s} \to 0 \text{ as } n\to\infty \quad \mathbb{Q}\text{-a.s.} \tag{4.17}$$

Combining (4.16) and (4.17) and using condition 2 which says that

$$\frac{\int_0^{\lfloor t\rfloor}\psi'(\gamma(s))\mathrm{d}s}{\int_0^t \psi'(\gamma(s))\mathrm{d}s} \to 1 \text{ as } t\to\infty$$

we get

$$\begin{aligned}
\frac{|X_t - \int_0^t \psi'(\gamma(s))\mathrm{d}s|}{\int_0^t \psi'(\gamma(s))\mathrm{d}s} &= \frac{|X_{\lfloor t\rfloor} - \int_0^{\lfloor t\rfloor}\psi'(\gamma(s))\mathrm{d}s + X_t - X_{\lfloor t\rfloor} - \int_{\lfloor t\rfloor}^t \psi'(\gamma(s))\mathrm{d}s|}{\int_0^t \psi'(\gamma(s))\mathrm{d}s} \\
&\le \frac{|X_{\lfloor t\rfloor} - \int_0^{\lfloor t\rfloor}\psi'(\gamma(s))\mathrm{d}s|}{\int_0^t \psi'(\gamma(s))\mathrm{d}s} + \frac{\left|X_t - X_{\lfloor t\rfloor} - \int_{\lfloor t\rfloor}^t \psi'(\gamma(s))\mathrm{d}s\right|}{\int_0^t \psi'(\gamma(s))\mathrm{d}s} \\
&\le \frac{|X_{\lfloor t\rfloor} - \int_0^{\lfloor t\rfloor}\psi'(\gamma(s))\mathrm{d}s|}{\int_0^t \psi'(\gamma(s))\mathrm{d}s} \\
&\quad + \frac{\sup_{r\in[\lfloor t\rfloor,\lfloor t\rfloor+1]}\left|X_r - X_{\lfloor t\rfloor} - \int_{\lfloor t\rfloor}^r \psi'(\gamma(s))\mathrm{d}s\right|}{\int_0^t \psi'(\gamma(s))\mathrm{d}s} \\
&\to 0 \text{ as } t\to\infty \quad \mathbb{Q}\text{-a.s.}
\end{aligned}$$

Hence

$$\frac{X_t}{\int_0^t \psi'(\gamma(s))\mathrm{d}s} \to 1 \qquad \mathbb{Q}\text{-a.s.}$$

□

From this we can now also derive a result about the asymptotic growth of the "stochastic" integral $\int_0^t \gamma(s)\mathrm{d}X_s$. Firstly let us compute the moments of $\int_0^t \gamma(s)\mathrm{d}X_s$ under $\mathbb{Q}$.

Proposition 4.40. *Let $\alpha\in\mathbb{R}$ be such that $\alpha+\gamma(t)\in(-\gamma^-,\gamma^+)$ and $\int_0^t \psi((\alpha+1)\gamma(s))\mathrm{d}s<\infty$ $\forall t>0$. Then*

$$\mathbb{E}^{\mathbb{Q}}\big(e^{\alpha\int_0^t \gamma(s)\mathrm{d}X_s}\big) = e^{\int_0^t \psi((\alpha+1)\gamma(s))-\psi(\gamma(s))\mathrm{d}s},$$

$$\mathbb{E}^{\mathbb{Q}}\big(X_t\big) = \int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s.$$

Proof.

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\big(e^{\alpha\int_0^{\gamma}(s)\mathrm{d}X_s}\big) &= \mathbb{E}\Big(e^{\alpha\int_0^t\gamma(s)\mathrm{d}X_s}e^{\int_0^t\gamma(s)\mathrm{d}X_s-\int_0^t\psi(\gamma(s))\mathrm{d}s}\Big)\\ &= \mathbb{E}\Big(e^{\int_0^t((\alpha+1)\gamma(s))\mathrm{d}X_s-\int_0^t\psi((\alpha+1)\gamma(s))\mathrm{d}s}\Big)e^{\int_0^t\psi((\alpha+1)\gamma(s))-\psi(\gamma(s))\mathrm{d}s}\\ &= e^{\int_0^t\psi((\alpha+1)\gamma(s))-\psi(\gamma(s))\mathrm{d}s}.\end{aligned}$$

Differentiating with respect to α and letting $\alpha = 0$ gives

$$\mathbb{E}^{\mathbb{Q}}\big(X_t\big) = \int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s.$$

□

Let us now prove the following result about $\mathbb{Q}$-a.s growth of $\int_0^t\gamma(s)\mathrm{d}X_s$.

Corollary 4.41 (Corollary to Theorem 4.39)**.** *Let a Lévy process $(X_t)_{t\geq 0}$ and a function $\gamma : [0,\infty) \to \mathbb{R}$ satisfy the assumptions of Theorem 4.39. Also assume that γ is differentiable with $\gamma'(t) \geq 0$ $\forall t$ and make two additional assumptions on $\gamma(\cdot)$:*

1. $\mathbb{E}^{\mathbb{Q}}\int_0^t\gamma(s)\mathrm{d}X_s = \int_0^t\gamma(s)\psi'(\gamma(s))\mathrm{d}s < \infty$ $\forall t \geq 0$
2. $\limsup_{t\to\infty}\dfrac{\gamma(t)\int_0^t\psi'(\gamma(s))\mathrm{d}s}{\int_0^t\gamma(s)\psi'(\gamma(s))\mathrm{d}s} < \infty$

Then

$$\frac{\int_0^t\gamma(s)\mathrm{d}X_s}{\mathbb{E}^{\mathbb{Q}}\int_0^t\gamma(s)\mathrm{d}X_s} = \frac{\int_0^t\gamma(s)\mathrm{d}X_s}{\int_0^t\gamma(s)\psi'(\gamma(s))\mathrm{d}s} \to 1 \quad \mathbb{Q}\text{-a.s.}$$

Proof. The proof is essentially the same as the proof of Proposition 3.13 for continuous-time random walks. We are going to put together Theorem 4.39 and Proposition 4.33.

Take $\delta > 0$. Then from Theorem 4.39 we know that $\exists$ $\mathbb{Q}$-a.s. finite random time T_δ such that

$$t \geq T_\delta \quad \Rightarrow \ 1-\delta \leq \frac{X_t}{\int_0^t\psi'(\gamma(s))\mathrm{d}s} \leq 1+\delta.$$

That is,

$$(1-\delta)\int_0^t\psi'\big(\gamma(s)\big)\mathrm{d}s \leq X_t \leq (1+\delta)\int_0^t\psi'\big(\gamma(s)\big)\mathrm{d}s.$$

So using the integration-by-parts formula from Proposition 4.33 we get for $t \geq T_\delta$

$$\begin{aligned}\int_0^t\gamma(s)\mathrm{d}X_s &= \gamma(t)X_t - \int_0^t\gamma'(s)X_s\mathrm{d}s\\ &\leq (1+\delta)\gamma(t)\int_0^t\psi'\big(\gamma(s)\big)\mathrm{d}s\\ &\qquad - \int_0^t\gamma'(s)(1-\delta)\int_0^s\psi'\big(\gamma(u)\big)\mathrm{d}u\mathrm{d}s + C_\delta,\end{aligned}$$

where

$$C_\delta = \int_0^{T_\delta} \gamma'(s)(1-\delta)\int_0^s \psi'\big(\gamma(u)\big)\mathrm{d}u\mathrm{d}s - \int_0^{T_\delta}\gamma'(s)X_s\mathrm{d}s$$

is some $\mathbb{Q}$-a.s. finite quantity which doesn't depend on t.

Then, using the deterministic integration-by-parts formula, we get

$$\int_0^t \gamma(s)\mathrm{d}X_s \leq (1+\delta)\int_0^t \gamma(s)\psi'\big(\gamma(s)\big)\mathrm{d}s + 2\delta\int_0^t \gamma'(s)\int_0^s \psi'\big(\gamma(u)\big)\mathrm{d}u\ \mathrm{d}s + C_\delta$$

Hence

$$\begin{aligned}\limsup_{t\to\infty}\frac{\int_0^t \gamma(s)\mathrm{d}X_s}{\int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s} &\leq 1+\delta+2\delta\limsup_{t\to\infty}\frac{\int_0^t \gamma'(s)\int_0^s \psi'\big(\gamma(u)\big)\mathrm{d}u\mathrm{d}s}{\int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s}\\ &= 1+\delta+2\delta\limsup_{t\to\infty}\Big(\frac{\gamma(t)\int_0^t \psi'\big(\gamma(s)\big)\mathrm{d}s}{\int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s}-1\Big)\\ &= 1+\delta c,\end{aligned}$$

where c is some finite constant. Thus after taking $\delta \to 0$ we get

$$\limsup_{t\to\infty}\frac{\int_0^t \gamma(s)\mathrm{d}X_s}{\int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s} \leq 1.$$

Similar argument shows that

$$\liminf_{t\to\infty}\frac{\int_0^t \gamma(s)\mathrm{d}X_s}{\int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s} \geq 1$$

completing the proof. □

4.5 The rightmost particle in the case of homogeneous branching ($p = 0$)

This section is dedicated to the proof of Theorem 4.17. The method of proof is going to be the same as the one we used for branching random walks in Section 3.4.

We are going to study a family of additive martingales derived from exponential martingales of the form (4.14). We shall see that the additive martingales either converge to a positive limit and are UI or converge to 0 depending on the value of the parameter γ. The critical value of the parameter will give us the first-order approximation of the rightmost particle.

4.5.1 Additive martingales

Take the spine process $(\xi_t)_{t\geq 0}$ which under the probability measure $\tilde{P}$ is a Lévy process with parameters (a, σ, Π) corresponding to the single-particle process. In particular for all $\gamma \in (-\gamma^-, \gamma^+)$ it satisfies:

$$\tilde{E}e^{\gamma\xi_1} < \infty.$$

From Theorem 4.30 we have that the following process is a $\tilde{P}$-martingale:

$$e^{\gamma\xi_t - \psi(\gamma)t} \qquad , t \geq 0. \tag{4.18}$$

We now substitute it for $\tilde{M}^{(3)}$ in equation (1.5) from the general setting described in Chapter 1. Hence, recalling (1.6), we define a $\tilde{P}$-martingale with respect to filtration $(\tilde{\mathcal{G}}_t)_{t\geq 0}$:

$$\tilde{M}_\gamma(t) := e^{-\beta t} 2^{n_t} \times e^{\gamma\xi_t - \psi(\gamma)t} , \qquad t \geq 0 \tag{4.19}$$

as well as the corresponding probability measure $\tilde{Q}_\gamma$:

$$\frac{\mathrm{d}\tilde{Q}_\gamma}{\mathrm{d}\tilde{P}}\bigg|_{\tilde{\mathcal{F}}_t} = \tilde{M}_\gamma(t) , \qquad t \geq 0. \tag{4.20}$$

Under $\tilde{Q}_\gamma$ the branching process has the following description:

- The initial particle (the spine) moves like a biased Lévy process with parameters $(\hat{a}, \sigma, \hat{\Pi})$ (recall Theorem 4.31).
- At rate 2β it splits into two new particles.
- One of these particles (chosen uniformly at random) continues the spine. I.e. it continues to move as a biased Lévy process and to branch at rate 2β.
- The other particle initiates an unbiased branching process where all the particles move as a Lévy process with parameters (a, σ, Π) and branch at rate β.

Projecting $\tilde{Q}_\gamma$ onto $\mathcal{F}_\infty$ in the usual way we get the probability measure $Q_\gamma := \tilde{Q}_\gamma|_{\mathcal{F}_\infty}$ and the corresponding additive martingale

$$M_\gamma(t) = \sum_{u\in N_t} \exp\Big(\gamma X^u_t - \psi(\gamma)t - \beta t\Big) , \qquad t \geq 0, \tag{4.21}$$

so that we have

$$\frac{\mathrm{d}Q_\gamma}{\mathrm{d}P}\bigg|_{\mathcal{F}_t} = M_\gamma(t) , \qquad t \geq 0. \tag{4.22}$$

Having defined this family of martingales we can control the behaviour of the spine process via the choice of parameter γ.

4.5.2 Convergence properties of M_γ (under Q_γ)

Just as before we want to show that either $M_\gamma(\infty) > 0$ a.s. and M_γ is UI under P or $M_\gamma(\infty) = 0$ P-a.s. depending on the value of γ.

Theorem 4.42. *Consider a branching Lévy process in the potential $\beta(\cdot) \equiv \beta$. Let M_γ be the additive martingale defined in* (4.21)*. Then we have the following for the three cases considered in Theorem 4.17.*

Case (**I**) *and Case* (**IIa**) $\underline{\beta < \lim_{\gamma\to\gamma^+} \Lambda(\psi'(\gamma))}$*:*

$$\textit{where } \lim_{\gamma\to\gamma^+} \Lambda(\psi'(\gamma)) = \begin{cases} \infty & \textit{in Case (I)} \\ \Lambda(\psi'(\infty)) < \infty & \textit{in Case (IIa)} \end{cases}$$

Let γ^* be the unique solution of $\Lambda(\psi'(\gamma)) = \beta$. So that

$$\Lambda\big(\psi'(\gamma^*)\big) = \gamma^* \psi'(\gamma^*) - \psi(\gamma^*) = \beta. \tag{4.23}$$

Then

i) if $\gamma \in [0, \gamma^*)$ then M_γ is U.I. and $M_\gamma(\infty) > 0$ a.s. under P

ii) if $\gamma \in (\gamma^*, \gamma^+)$ then $M_\gamma(\infty) = 0$ P-a.s.

Case (**IIb**) $\underline{\lim_{\gamma\to\gamma^+} \psi'(\gamma) = \psi'(\infty) < \infty,\ \beta \geq \Lambda(\psi'(\infty))}$*:*

$$\forall\gamma \geq 0,\ M_\gamma \textit{ is U.I. and } M_\gamma(\infty) > 0 \textit{ a.s. under } P.$$

The proof of this theorem will be essentially a modified version of the proof of Theorem 3.18.

If the martingale M_γ is P-uniformly integrable and $M_\gamma(\infty) > 0$ P-a.s. then we shall see that P and Q_γ are two equivalent measures on $\mathcal{F}_\infty$. Since under $\tilde{Q}_\gamma$ the spine process satisfies

$$\frac{\xi_t}{t} \to \psi'(\gamma) \text{ a.s.}$$

it will follow that P-a.s. there is a particle with such asymptotic behaviour. This will give us a lower bound on the rightmost particle.

Recalling Theorem 2.9 we have the following decomposition of the probability measure Q_γ (see also Lemma 3.21).

Lemma 4.43. *Let M_γ be a martingale of the form* (4.21) *and let Q_γ be the corresponding probability measure defined via* (4.22)*. Then for events $A \in \mathcal{F}_\infty$*

$$Q_\gamma(A) = \int_A \limsup_{t\to\infty} M_\gamma(t)\mathrm{d}P + Q_\gamma\big(A \cap \{\limsup_{t\to\infty} M_\gamma(t) = \infty\}\big). \tag{4.24}$$

To prove Theorem 4.42 we shall need the following simple corollary of this lemma.

Corollary 4.44.

- $M_\gamma(\infty) = 0$ *P-a.s.* $\Leftrightarrow \limsup_{t\to\infty} M_\gamma(t) = \infty$ *Q_γ-a.s.*
- $\limsup_{t\to\infty} M_\gamma(t) < \infty$ *Q_γ-a.s.* $\Rightarrow$ $EM_\gamma(\infty) = 1$, *M_γ is P-uniformly integrable and* $P(M_\gamma(\infty) > 0) > 0$

Also to show that $P(M_\gamma(\infty) > 0) > 0 \Rightarrow P(M_\gamma(\infty) > 0) = 1$ we need a zero-one law similar to Lemma 3.26. We shall present this result now before we proceed with the proof of Theorem 4.42.

Lemma 4.45. *Consider a branching Lévy process started from* 0 *in the potential* $\beta(\cdot) \equiv \beta$. *Let* $q \in [0,1]$ *be such that*

$$M_t := \prod_{u\in N_t} q = q^{|N_t|} \ , \ t \geq 0$$

is a P-martingale. Then

$$q \in \{0,1\}.$$

Proof of Lemma 4.45. If $q < 1$ then since $|N_t| \to \infty$ P-a.s.

$$M_t = q^{|N_t|} \to 0 \quad \text{as } t \to \infty \ P\text{-a.s.}$$

so by the monotone convergence theorem

$$q = E(M_0) = E(M_\infty) = 0$$

which is a contradiction unless $q = 0$. Thus $q \in \{0,1\}$. ◻

Corollary 4.46.

$$P(M_\gamma(\infty) = 0) \in \{0,1\} \tag{4.25}$$

Proof of Corollary 4.46. By taking $q(x) := P^x(M_\gamma(\infty) = 0)$ we see that $\forall x \in \mathbb{R}$

$$q(x) = E^x\Big(P^x\big(M_\gamma(\infty) = 0\big|\mathcal{F}_t\big)\Big) = E^x\Big(\prod_{u\in N_t} q(X^u_t)\Big).$$

Hence $\prod_{u\in N_t} q(X^u_t)$ is a P-martingale. Also

$$q(x) = P^x(M_\gamma(\infty) = 0) = P(e^{\gamma x} M_\gamma(\infty) = 0) = P(M_\gamma(\infty) = 0).$$

Thus $q(x) \equiv q$ and by Lemma 4.45 $q = P(M_\gamma(\infty) = 0) \in \{0,1\}$. ◻

Proof of Theorem 4.42: uniform integrability of M_γ and positivity of the limit. Take $\gamma \in [0,\gamma^*)$ in Case (I) and Case (IIa) or any $\gamma \geq 0$ in Case (IIb). To show that under

P M_γ is U.I. and $M_\gamma(\infty) > 0$ a.s. it is sufficient to prove that

$$\limsup_{t\to\infty} M_\gamma(t) < \infty \;\; Q_\gamma\text{-a.s.} \tag{4.26}$$

as it follows from Corollary 4.44 and Corollary 4.46.

Just as in the earlier chapters we are going to use the spine decomposition of $M_\gamma(t)$ to prove (4.26).

Proposition 4.47.

$$\limsup_{t\to\infty} E^{\tilde{Q}_\gamma}(M_\gamma(t)|\tilde{\mathcal{G}}_\infty) < \infty \;\; \tilde{Q}_\gamma\text{-}a.s.$$

Proof of Proposition 4.47. Recall that

$$E^{\tilde{Q}_\gamma}(M_\gamma(t)|\tilde{\mathcal{G}}_\infty) = spine(t) + sum(t), \tag{4.27}$$

where

$$spine(t) = \exp\Big(\gamma\xi_t - \psi(\gamma)t - \beta t\Big) \tag{4.28}$$

and

$$\begin{aligned} sum(t) &= \sum_{u\in node_t(\xi)} spine(S_u) \\ &= \sum_{u<node_t(\xi)} \exp\Big(\gamma\xi_{S_u} - \psi(\gamma)S_u - \beta S_u\Big), \end{aligned} \tag{4.29}$$

where $\{S_u : u \in \xi\}$ is the set of fission times along the spine.

We start by proving that the spine term (4.28) decays exponentially fast.

Proposition 4.48. *There exist some positive constant C'' and a $\tilde{Q}_\gamma$-a.s. finite time T' such that $\forall t > T'$*

$$spine(t) \le e^{-C''t}.$$

Proof of Proposition 4.48. Under $\tilde{Q}_\gamma$ the process $(\xi_t)_{t\ge0}$ is a Lévy process with mean $\psi'(\gamma)t$ (recall Theorem 4.31) so it satisfies

$$\frac{\xi_t}{\psi'(\gamma)t} \to 1 \quad \tilde{Q}_\gamma\text{-a.s.}$$

as it follows from Theorem 4.37. Hence for all $\epsilon > 0$ there exists a $\tilde{Q}_\gamma$-a.s. finite time T_ϵ such that

$$(1-\epsilon)\psi'(\gamma)t \le \xi_t \le (1+\epsilon)\psi'(\gamma)t \quad \forall t > T_\epsilon.$$

Thus

$$spine(t) \le \exp\Big((1+\epsilon)\gamma\psi'(\gamma)t - \psi(\gamma)t - \beta t\Big) = \exp\Big(\big(\Lambda\big(\psi'(\gamma)\big) + \epsilon\gamma\psi'(\gamma) - \beta\big)t\Big) \quad \forall t > T_\epsilon$$

using (4.6). Since the map $\Lambda(\psi'(\gamma)) = \gamma\psi'(\gamma) - \psi(\gamma)$ is increasing in γ (for $\gamma \geq 0$) it follows that

$$\Lambda(\psi'(\gamma)) < \begin{cases} \Lambda(\psi'(\gamma^*)) & \text{in Case (I) and Case (IIa)} \\ \Lambda(\psi'(\infty)) & \text{in Case (IIb)} \end{cases}$$

Thus in all the cases $\Lambda(\psi'(\gamma)) - \beta < 0$ and so for ϵ sufficiently small

$$\Lambda\big(\psi'(\gamma)\big) + \epsilon\gamma\psi'(\gamma) - \beta < 0.$$

Taking $T' = T_\epsilon$ for such an ϵ and $C'' = -(\Lambda(\psi'(\gamma)) + \epsilon\gamma\psi'(\gamma) - \beta)$ we complete the proof of Proposition 4.48 $\square$

Now, for $t > T'$ the sum term is

$$\begin{aligned} sum(t) &= \sum_{u<node_t(\xi)} spine(S_u) \\ &= \Big(\sum_{u<node_t(\xi),\ S_u \leq T'} spine(S_u) \Big) + \Big(\sum_{u<node_t(\xi),\ S_u > T'} spine(S_u) \Big) \\ &\leq \Big(\sum_{u<node_t(\xi),\ S_u \leq T'} spine(S_u) \Big) + \Big(\sum_{u<node_t(\xi),\ S_u > T'} e^{-C'' S_u} \Big) \end{aligned}$$

using Proposition 4.48. The first sum is $\tilde{Q}_\gamma$-a.s. bounded since it only counts births up to time T'. Call an upper bound on the first sum C_1. Then we have

$$sum(t) \leq \ C_1 + \sum_{n=1}^{\infty} e^{-C'' S_n}, \tag{4.30}$$

where S_n is the time of the n^{th} birth on the spine.

The birth process along the spine $(n_t)_{t\in[0,T)}$ is a Poisson process with rate 2β (Recall Proposition 1.13). Thus

$$\frac{n_t}{t} \to 2\beta \quad \tilde{Q}_\gamma\text{-a.s. as } t \to \infty.$$

Hence

$$\frac{S_n}{n} \to \frac{1}{2\beta} \quad \tilde{Q}_\gamma\text{-a.s. as } t \to \infty. \tag{4.31}$$

So for some $\tilde{Q}_\gamma$-a.s. finite positive random variable C_2 we have

$$S_n \geq C_2 n \quad \forall n.$$

Then substituting this into (4.30) we get

$$sum(t) \leq C_1 + \sum_{n=1}^{\infty} e^{-C'' C_2 n},$$

which is bounded $\tilde{Q}_\gamma$-a.s. We have thus shown that

$$\limsup_{t\to\infty} E^{\tilde{Q}_\gamma}\Big(M_\gamma(t)\big|\tilde{\mathcal{G}}_\infty\Big) < \infty \qquad \tilde{Q}_\gamma\text{-a.s.}$$

proving Proposition 4.47 □

Now from Proposition 4.47 we get the sought result (4.26) by the usual argument:

$$\begin{aligned} E^{\tilde{Q}_\gamma}(\liminf_{t\to\infty} M_\gamma(t)|\tilde{\mathcal{G}}_\infty) &\leq \liminf_{t\to\infty} E^{\tilde{Q}_\gamma}(M_\gamma(t)|\tilde{\mathcal{G}}_\infty) \\ &\leq \limsup_{t\to\infty} E^{\tilde{Q}_\gamma}(M_\gamma(t)|\tilde{\mathcal{G}}_\infty) < \ +\infty \quad \tilde{Q}_\gamma\text{-a.s.} \end{aligned}$$

by conditional Fatou's lemma. Hence

$$\liminf_{t\to\infty} M_\gamma(t) < \infty \ \tilde{Q}_\gamma\text{-a.s.}$$

and thus also Q_γ-a.s. Since $\left(\dfrac{1}{M_\gamma(t)}\right)_{t\geq 0}$ is a positive Q_γ-supermartingale (as it follows from the definition of Q_γ) it must converge Q_γ-a.s. So $M_\gamma(t)$ also converges Q_γ-a.s.

Hence

$$\limsup_{t\to\infty} M_\gamma(t) = \liminf_{t\to\infty} M_\gamma(t) < \infty \ Q_\gamma\text{-a.s.}$$

We have thus proved that M_γ is uniformly integrable and has a strictly positive limit under P. □

Proof of Theorem 4.42: zero limits. In Case (I) and Case (IIa) let $\gamma \in (\gamma^*, \gamma^+)$. Then since one of the particles at time t is the spine we have

$$M_\gamma(t) \geq \exp\Big(\gamma\xi_t - \psi(\gamma)t - \beta t\Big) = spine(t).$$

It then can be checked using the same analysis as in the proof of part i) that $spine(t) \to \infty$ $\tilde{Q}_\gamma$-a.s. Recalling Corollary 4.44 we get that

$$M_\gamma(\infty) = 0 \ P\text{-a.s.}$$

□

4.5.3 Lower bound on the rightmost particle

Let $\gamma \in [0, \gamma^*)$ in Case (I) and Case (IIa) and $\gamma \in [0, \infty)$ in Case (IIb). We have shown in the previous subsection that for such values of γ:

1. $\limsup_{t\to\infty} M_\gamma(t) < \infty$
2. M_γ is P-uniformly integrable
3. $M_\gamma(\infty) > 0$ P-a.s.

Thus from Lemma 4.43 for events $A \in \mathcal{F}_\infty$

$$Q_\gamma(A) = E(\mathbf{1}_A M_\gamma(\infty))$$

and also

$$Q_\gamma(A) = 1 \Leftrightarrow P(A) = 1.$$

In other words Q_γ and P are equivalent on $\mathcal{F}_\infty$.

Let us exploit this fact to get a lower bound on the rightmost particle

Proposition 4.49.
Case (**I**) *and Case* (**IIa**)*:*

Let γ^* be the unique solution of $\Lambda(\psi'(\gamma)) = \beta$. Then

$$\liminf_{t\to\infty} \frac{R_t}{t} \geq \psi'(\gamma^*) = \Lambda^{-1}(\beta) \ P\text{-}a.s.$$

Case (**IIb**)*:*

$$\liminf_{t\to\infty} \frac{R_t}{t} \geq \psi'(\infty) \ P\text{-}a.s.$$

Proof. Consider the event

$$B_\gamma := \Big\{\exists \text{ infinite line of descent } u : \liminf_{t\to\infty} \frac{X^u_t}{t} = \psi'(\gamma)\Big\} \in \mathcal{F}_\infty.$$

Then

$$\begin{aligned}
&\tilde{Q}_\gamma(\lim_{t\to\infty} \frac{\xi_t}{t} = \psi'(\gamma)) = 1 \\
\Rightarrow &\tilde{Q}_\gamma(B_\gamma) = 1 \\
\Rightarrow &Q_\gamma(B_\gamma) = 1 \\
\Rightarrow &P(B_\gamma) = 1 \\
\Rightarrow &P\big(\liminf_{t\to\infty} \frac{R_t}{t} \geq \psi'(\gamma)\big) = 1.
\end{aligned}$$

Letting $\gamma \nearrow \gamma^*$ in Case (I) and Case (IIa) and $\gamma \nearrow \infty$ and Case (IIb) we obtain the required result. $\square$

4.5.4 Upper bound on the rightmost particle

Proposition 4.50.
Case (**I**) *and Case* (**IIa**)*:*

Let γ^* be the unique solution of $\Lambda(\psi'(\gamma)) = \beta$. Then

$$\limsup_{t\to\infty} \frac{R_t}{t} \le \psi'(\gamma^*) = \Lambda^{-1}(\beta) \ P\text{-}a.s.$$

Case (**IIb**)*:*

$$\limsup_{t\to\infty} \frac{R_t}{t} \le \psi'(\infty) \ P\text{-}a.s.$$

Proof.
Case (**I**) and Case (**IIa**)

Let us suppose for contradiction that there exists $\epsilon > 0$ such that

$$P\Big(\limsup_{t\to\infty} \frac{R_t}{t} > \psi'(\gamma^*) + \epsilon\Big) > 0.$$

From this assumption with positive probability there exists a sequence of times $(J_n)_{n\ge1}$, $J_n \to \infty$ and a sequence of particles $(w_n)_{n\ge1}$, $w_n \in N_{J_n}$, such that

$$X^{w_n}_{J_n} > \big(\psi'(\gamma^*) + \epsilon\big) J_n.$$

Thus with positive probability for the additive martingale M_{γ^*} we have:

$$\begin{aligned}
M_{\gamma^*}(J_n) &\ge \exp\Big\{\gamma^* X^{w_n}_{J_n} - \psi(\gamma^*) J_n - \beta J_n\Big\} \\
&> \exp\Big\{\Big(\gamma^*\big(\psi'(\gamma^*) + \epsilon\big) - \psi(\gamma^*) - \beta\Big) J_n\Big\} \\
&= \exp\Big\{\Big(\Lambda\big(\psi'(\gamma^*)\big) + \epsilon\gamma^* - \beta\Big) J_n\Big\} \\
&= \exp\Big\{\epsilon\gamma^* J_n\Big\} \to \infty \text{ as } n \to \infty.
\end{aligned}$$

That is, we have shown that

$$P(M_{\gamma^*}(\infty) = \infty) > 0,$$

which contradicts the Martingale Convergence Theorem. So it must be that

$$P\Big(\limsup_{t\to\infty} \frac{R_t}{t} > \psi'(\gamma^*) + \epsilon\Big) = 0$$

for all $\epsilon > 0$. Letting $\epsilon \searrow 0$ we get

$$P\Big(\limsup_{t\to\infty} \frac{R_t}{t} \le \psi'(\gamma^*)\Big) = 1.$$

Case (**IIb**)

Let us suppose for contradiction that there exists $\epsilon > 0$ such that

$$P\Big(\limsup_{t\to\infty} \frac{R_t}{t} > \psi'(\infty) + \epsilon\Big) > 0.$$

From this assumption with positive probability there exists a sequence of times $(J_n)_{n\geq 1}$, $J_n \to \infty$ and a sequence of particles $(w_n)_{n\geq 1}$, $w_n \in N_{J_n}$, such that

$$X^{w_n}_{J_n} > \big(\psi'(\infty) + \epsilon\big) J_n.$$

Take $\gamma > \frac{\beta+1}{\epsilon}$. Then with positive probability for martingale M_γ we have:

$$\begin{aligned}
M_\gamma(J_n) &\geq \exp\Big\{\gamma X^{w_n}_{J_n} - \psi(\gamma)J_n - \beta J_n\Big\} \\
&> \exp\Big\{\Big(\gamma\big(\psi'(\infty) + \epsilon\big) - \psi(\gamma) - \beta\Big) J_n\Big\} \\
&> \exp\Big\{\Big(\gamma\psi'(\gamma) - \psi(\gamma) + \gamma\epsilon - \beta\Big) J_n\Big\} \\
&> \exp\Big\{\Big(\Lambda\big(\psi'(\gamma)\big) + 1\Big) J_n\Big\} \\
&\to \infty \text{ as } n \to \infty.
\end{aligned}$$

Thus

$$P(M_\gamma(\infty) = \infty) > 0,$$

which is a contradiction. So we have

$$P\Big(\limsup_{t\to\infty} \frac{R_t}{t} > \psi'(\infty) + \epsilon\Big) = 0$$

for all $\epsilon > 0$ and so

$$P\Big(\limsup_{t\to\infty} \frac{R_t}{t} \leq \psi'(\infty)\Big) = 1.$$

$\square$

Propositions 4.49 and 4.50 taken together prove Theorem 4.17 completing this section.

4.6 The rightmost particle in the case of inhomogeneous branching ($p \in (0, 1)$)

In this section we prove Theorem 4.22. We shall follow the same steps as in the previous section. Analysis of additive martingales derived from exponential martingales of the form (4.15) will play the crucial role in the proof.

4.6.1 Additive martingales

Recall the assumptions of Theorem 4.22 on the single-particle motion:

1. $\psi(\gamma) = \log \mathbb{E} e^{\gamma X_1} < \infty \ \forall \gamma \in \mathbb{R}$,
2. for all $\delta > 0$ $\psi''(\gamma) < \psi'(\gamma)^{1+\delta}$ for all γ large enough,
3. $(X_t)_{t\geq 0}$ is point-recurrent in the sense of Definition 4.7,
4. $(X_t)_{t\geq 0}$ is symmetric in the sense that $(X_t)_{t\geq 0} \stackrel{d}{=} (-X_t)_{t\geq 0}$.

Let us leave assumption 4 until the last subsection, where we shall prove the upper bound on the rightmost particle and replace it with a milder assumption from Theorem 4.24:

4^*. $(X_t)_{t\geq 0}$ makes positive jumps (that is, $\Pi((0,\infty)) \neq 0$) or it is a Brownian motion.

Subject to these assumptions we construct the branching process with the spine under the probability measure $\tilde{P}$ in the usual way.

Under $\tilde{P}$ the spine process $(\xi_t)_{t\geq 0}$ is a Lévy process satisfying assumptions 1 - 3 and 4^* above. For a function $\gamma : [0,\infty) \to \mathbb{R}$ such that $\int_0^t \psi(\gamma(s))\mathrm{d}s < \infty \ \forall t \geq 0$ we have from Theorem 4.34 that the following process is a $\tilde{P}$-martingale:

$$e^{\int_0^t \gamma(s)\mathrm{d}\xi_s - \int_0^t \psi(\gamma(s)} \qquad , \ t \geq 0. \tag{4.32}$$

We now substitute it for $\tilde{M}^{(3)}$ in equation (1.5) from the general setting described in Chapter 1. Hence, recalling (1.6), we define a $\tilde{P}$-martingale with respect to the filtration $(\tilde{\mathcal{G}}_t)_{t\geq 0}$:

$$\tilde{M}_\gamma(t) := e^{-\beta \int_0^t |\xi_s|^p \mathrm{d}s} 2^{n_t} \times \exp\Big(\int_0^t \gamma(s)\mathrm{d}\xi_s - \int_0^t \psi(\gamma(s))\mathrm{d}s\Big) \ , \qquad t \geq 0 \tag{4.33}$$

as well as the corresponding probability measure $\tilde{Q}_\gamma$:

$$\frac{\mathrm{d}\tilde{Q}_\gamma}{\mathrm{d}\tilde{P}}\Big|_{\tilde{\mathcal{F}}_t} = \tilde{M}_\gamma(t) \ , \qquad t \geq 0. \tag{4.34}$$

Under $\tilde{Q}_\gamma$ the branching process has the following description:

- The initial particle (the spine) moves like a measure-changed Lévy process with time-dependent drift $\hat{a}_t$, the diffusion parameter σ and time-dependent jump measure $\hat{\Pi}_t$.
- At rate 2β it splits into two new particles.

- One of these particles (chosen uniformly at random) continues the spine. I.e. it continues to move as a measure-changed Lévy process and to branch at rate 2β.
- The other particle initiates an unbiased branching process where all the particles move as a Lévy process with parameters (a, σ, Π) and branch at rate β.

Projecting $\tilde{Q}_\gamma$ onto $\mathcal{F}_\infty$ in the usual way we get the probability measure $Q_\gamma := \tilde{Q}_\gamma|_{\mathcal{F}_\infty}$ and the corresponding additive martingale

$$M_\gamma(t) = \sum_{u \in N_t} \exp\Big(\int_0^t \gamma(s)\mathrm{d}X_s^u + \int_0^t \psi(\gamma(s))\mathrm{d}s - \int_0^t \beta(X_s^u)\Big) \tag{4.35}$$

so that we have

$$\frac{\mathrm{d}Q_\gamma}{\mathrm{d}P}\bigg|_{\mathcal{F}_t} = M_\gamma(t) \ , \qquad t \geq 0. \tag{4.36}$$

Having defined this family of martingales we can control the behaviour of the spine process via the choice of parameter γ.

4.6.2 Convergence properties of M_γ (under Q_γ)

Before we state the main result let us note the following:

- $\mathbb{E}e^{\gamma X_1} < \infty \ \forall \gamma \in \mathbb{R} \Rightarrow \gamma^-, \gamma^+ = \infty$, so the domain of ψ, ψ' and ψ'' is $\mathbb{R}$. Also the process $(X_t)_{t\geq 0}$ is in Case (I). That is, $\psi'(\infty) = \infty$.
- $(X_t)_{t\geq 0}$ is point-recurrent $\Rightarrow$ $(X_t)_{t\geq 0}$ is recurrent $\Rightarrow$ $\psi'(0) = \mathbb{E}X_1 = 0$. So the domain of Λ is $[0, \psi'(\infty))$.
- In the rest of this section paths $\gamma : [0, \infty) \to \mathbb{R}$ are going to be positive and increasing. Hence $\int_0^t \psi(\gamma(s))\mathrm{d}s$, $\int_0^t \psi'(\gamma(s))\mathrm{d}s$, $\int_0^t \psi''(\gamma(s))\mathrm{d}s$, $\int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s < \infty \ \forall t \geq 0$.

Theorem 4.51 ($p \in (0,1)$)**.** *Consider a branching Lévy process in the potential $\beta(x) = \beta|x|^p$, $\beta > 0$, $p \in (0,1)$, where single particles satisfy:*

1. *$\psi(\gamma) = \log \mathbb{E}e^{\gamma X_1} < \infty \ \forall \gamma \in \mathbb{R}$,*
2. *for all $\delta > 0$ $\psi''(\gamma) < \psi'(\gamma)^{1+\delta}$ for all γ large enough,*
3. *$(X_t)_{t\geq 0}$ is point-recurrent,*
4. *$(X_t)_{t\geq 0}$ makes positive jumps or is a Brownian motion.*

Let M_γ be the additive martingale defined in (4.35)*. Then we have the following behaviours of M_γ for various functions γ.*

Let $f(t) = F^{-1}(t)$ as in Theorem 4.22, where

$$F(t) = \int_0^t \frac{1}{\Lambda^{-1}(\beta s^p)} \mathrm{d}s \ , \qquad t \geq 0.$$

Define

$$\gamma^*(t) := (\psi')^{-1}\big(f'(t)\big)$$

so that

$$f(t) = \int_0^t \psi'(\gamma^*(s))\mathrm{d}s$$

and also

$$\Lambda\big(\psi'(\gamma^*(t))\big) = \Lambda(f'(t)) = \beta f(t)^p = \beta\Big(\int_0^t \psi'(\gamma^*(s))\mathrm{d}s\Big)^p. \tag{4.37}$$

Then we have the following.
i) For $\epsilon \in (0,1)$ let $\gamma(t) = \big(\psi'\big)^{-1}\big((1-\epsilon)f'(t)\big)$, $t \geq 0$, so that $\psi'\big(\gamma(t)\big) = (1-\epsilon)f'(t) = (1-\epsilon)\psi'\big(\gamma^*(t)\big)$. Then

$$M_\gamma \text{ is U.I. and } M_\gamma(\infty) > 0 \ P\text{-a.s.}$$

ii) For $\epsilon > 0$ let $\gamma(t) = \big(\psi'\big)^{-1}\big((1+\epsilon)f'(t)\big)$, $t \geq 0$, so that $\psi'\big(\gamma(t)\big) = (1+\epsilon)f'(t) = (1+\epsilon)\psi'\big(\gamma^*(t)\big)$. Then

$$M_\gamma(\infty) = 0 \ P\text{-a.s.}$$

As always we have the following decomposition of the probability measure Q_γ:

Lemma 4.52. *For events $A \in \mathcal{F}_\infty$*

$$Q_\gamma(A) = \int_A \limsup_{t\to\infty} M_\gamma(t)\mathrm{d}P + Q_\gamma\big(A \cap \{\limsup_{t\to\infty} M_\gamma(t) = \infty\}\big). \tag{4.38}$$

Corollary 4.53.

- $M_\gamma(\infty) = 0$ *P-a.s.* $\Leftrightarrow \limsup_{t\to\infty} M_\gamma(t) = \infty$ *Q_γ-a.s.*
- $\limsup_{t\to\infty} M_\gamma(t) < \infty$ *Q_γ-a.s.* $\Rightarrow$ *$EM_\gamma(\infty) = 1$, M_γ is P-uniformly integrable and $P(M_\gamma(\infty) > 0) > 0$*

We also need to know that $P(M_\gamma(\infty) > 0) \in \{0,1\}$ and the next lemma helps us to resolve this issue.

Lemma 4.54. *Consider a branching Lévy process started from 0 in the potential $\beta(x) = \beta|x|^p$, $p \in (0,1)$, where $p \in (0,1)$ and the underlying Lévy process is point-recurrent.*

Let $q : \mathbb{R} \to [0,1]$ be a function such that

$$M_t := \prod_{u \in N_t} q\big(X_u(t)\big)$$

is a P-martingale. Then

$$q(0) \in \{0,1\}.$$

Proof of Lemma 4.54.

Since

$$q(0) = EM_t = \tilde{E}M_t \le \tilde{E}q(\xi_t)$$

we have that $(q(\xi_t))_{t\ge 0}$ is a positive submartingale, so

$$q(\xi_t) \to q_\infty \ \tilde{P}\text{-a.s.}$$

Since $(\xi_t)_{t\ge 0}$ is point-recurrent as in Definition 4.7 it returns to 0 infinitely often and hence $q(0) = q_\infty$. That is,

$$q(\xi_t) \to q(0) \ \tilde{P}\text{-a.s.}$$

However, if we define another independent spine process $(\tilde{\xi}_t)_{t\ge 0}$ we would have

$$M_t = \prod_{u\in N_t} q\big(X_u(t)\big) \le q(\xi_t)q(\tilde{\xi}_t) \text{ for } t \text{ large enough}$$

Thus by the uniform integrability of M_t and $q(\xi_t)q(\tilde{\xi}_t)$ we get that

$$q(0) = EM_\infty \le q_\infty^2 = q(0)^2$$

and hence $q(0) \in \{0,1\}$. □

Corollary 4.55.

$$P(M_\gamma(\infty) = 0) \in \{0,1\}. \tag{4.39}$$

Proof. Let $q(x) := P^x(M_\gamma(\infty) = 0)$. Then $\forall x \in \mathbb{R}$

$$q(x) = E^x\Big(P^x\big(M_\gamma(\infty) = 0\big|\mathcal{F}_t\big)\Big) = E^x\Big(\prod_{u\in N_t} q(X_t^u)\Big).$$

Hence $\prod_{u\in N_t} q(X_t^u)$ is a P-martingale and

$$P(M_\gamma(\infty) = 0) = q(0) \in \{0,1\}.$$

□

Proof of Theorem 4.51 part i). In view of Corollary 4.53 and Corollary 4.55 it is sufficient to show that

$$\limsup_{t\to\infty} M_\gamma(t) < \infty \ Q_\gamma\text{-a.s.} \tag{4.40}$$

Then absolutely identical argument to the one used in the case of homogeneous branching in Subsection 4.5.2 shows that it is actually enough to prove that

$$\limsup_{t\to\infty} E^{\tilde{Q}_\gamma}(M_\gamma(t)|\tilde{\mathcal{G}}_\infty) < \infty \ \tilde{Q}_\gamma\text{-a.s.} \tag{4.41}$$

The spine decomposition gives

$$E^{\tilde{Q}_\gamma}(M_\gamma(t)|\tilde{\mathcal{G}}_\infty) = spine(t) + sum(t), \tag{4.42}$$

where

$$spine(t) = \exp\Big(\int_0^t \gamma(s)\mathrm{d}\xi_s - \int_0^t \psi(\gamma(s))\mathrm{d}s - \int_0^t \beta|\xi_s|^p\mathrm{d}s\Big) \tag{4.43}$$

and

$$\begin{aligned} sum(t) &= \sum_{u\in node_t(\xi)} spine(S_u) \qquad (4.44)\\ &= \sum_{u<node_t(\xi)} \exp\Big(\int_0^{S_u} \gamma(s)\mathrm{d}\xi_s - \int_0^{S_u} \psi(\gamma(s))\mathrm{d}s - \int_0^{S_u} \beta|\xi_s|^p\mathrm{d}s\Big), \end{aligned}$$

where $\{S_u : u \in \xi\}$ is the set of fission times along the spine.

In order to estimate the spine term we need to know the following about the asymptotic behaviour of $(\xi_t)_{t\geq 0}$ under $\tilde{Q}_\gamma$:

Proposition 4.56.

$$\frac{\xi_t}{\int_0^t \psi'(\gamma(s))\mathrm{d}s} \to 1 \quad \tilde{Q}_\gamma\text{-}a.s., \tag{4.45}$$

$$\frac{\int_0^t \gamma(s)\mathrm{d}\xi_s}{\int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s} \to 1 \quad \tilde{Q}_\gamma\text{-}a.s. \tag{4.46}$$

Proof of Proposition 4.56. The result follows from Theorem 4.39 and Corollary 4.41. We only need to check the following three conditions on $\gamma(\cdot)$ and $\psi(\cdot)$:

$$\frac{\int_n^{n+1} \psi'(\gamma(s))\mathrm{d}s}{\int_0^n \psi'(\gamma(s))\mathrm{d}s} \to 0 \text{ as } n \to \infty, \tag{4.47}$$

$$\exists\delta > 0 \text{ s.t. for } n \text{ large enough } \frac{\int_0^n \psi''(\gamma(s))\mathrm{d}s}{(\int_0^n \psi'(\gamma(s))\mathrm{d}s)^2} \leq \frac{1}{n^{1+\delta}}, \tag{4.48}$$

$$\limsup_{t\to\infty} \frac{\gamma(t)\int_0^t \psi'(\gamma(s))\mathrm{d}s}{\int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s} < \infty. \tag{4.49}$$

The Lévy process that we are looking at either makes positive jumps (that is, $\Pi((0,\infty)) \neq 0$) or is a Brownian motion.

The case of a Brownian motion was considered in [22] where (4.45) and (4.46) were proved under weaker assumptions, so there is no need to repeat it. Thus let us restrict our attention to the process with positive jumps.

Firstly let us show that $\psi(\cdot)$ grows at least exponentially fast as $\gamma \to \infty$.

Since the Lévy process makes positive jumps there exists an interval (x_0, x_1), $0 < x_0 < x_1$ such that $\Pi\big((x_0, x_1)\big) > 0$. We then observe that from (4.5)

$$\begin{aligned}
\psi(\gamma) &= a\gamma + \frac{1}{2}\sigma^2\gamma^2 + \int_{\mathbb{R}\setminus\{0\}} \big(e^{\gamma x} - \gamma x \mathbf{1}_{|x|<1} - 1\big)\Pi(\mathrm{d}x) \\
&= a\gamma + \frac{1}{2}\sigma^2\gamma^2 + \int_{(-\infty,0)} \big(e^{\gamma x} - \gamma x \mathbf{1}_{|x|<1} - 1\big)\Pi(\mathrm{d}x) \\
&\quad + \int_{(0,\infty)} \big(e^{\gamma x} - \gamma x \mathbf{1}_{|x|<1} - 1\big)\Pi(\mathrm{d}x) \\
&\geq a\gamma + \frac{1}{2}\sigma^2\gamma^2 - \Pi\big((-\infty,-1]\big) + \int_{(-1,0)} \underbrace{\big(e^{\gamma x} - \gamma x - 1\big)}_{\geq 0}\Pi(\mathrm{d}x) \\
&\quad + \int_{(0,\infty)} \big(e^{\gamma x} - \gamma x \mathbf{1}_{|x|<1} - 1\big)\Pi(\mathrm{d}x) \\
&\geq a\gamma + \frac{1}{2}\sigma^2\gamma^2 - \Pi\big((-\infty,-1]\big) + \int_{(-1,0)} \big(e^{\gamma x} - \gamma x - 1\big)\Pi(\mathrm{d}x) \\
&\quad + (e^{\gamma x_0} - \gamma x_0 - 1)\Pi\big((x_0, x_1)\big),
\end{aligned}$$

which grows exponentially fast due to $e^{\gamma x_0}$. Similarly one can show that $\psi'(\cdot)$ grows at least exponentially fast and so $(\psi')^{-1}(\cdot)$ grows at most logarithmically fast. Then

$$\begin{aligned}
&\Lambda\big(\psi'(\gamma)\big) = \gamma\psi'(\gamma) - \psi(\gamma) \leq \gamma\psi'(\gamma) \\
\Rightarrow &\Lambda(x) \leq x\big((\psi')^{-1}(x)\big) \leq x^{1+\eta}
\end{aligned}$$

for any $\eta > 0$ and x large enough. On the other hand $\Lambda(x) \geq cx$ for some $c > 0$ and x large enough since Λ is convex. Thus we have shown that $\log \Lambda(x) \sim \log x$. Then

$$F(t) = \int_0^t \frac{1}{\Lambda(\beta s^p)} \mathrm{d}s \approx t^{1-p}$$

in the sense that $\forall \eta > 0$ and t large enough

$$t^{1-p-\eta} \leq F(t) \leq t^{1-p+\eta}.$$

Then $f(t) = F^{-1}(t) \approx t^{\frac{1}{1-p}}$ in the same sense that $\forall \eta > 0$ and t large enough

$$t^{\frac{1}{1-p}-\eta} \leq f(t) \leq t^{\frac{1}{1-p}+\eta}.$$

(Compare this with a more accurate result in Theorem 3.4 in the case of a continuous-time random walk.)

Thus it is also true that $\psi'(\gamma^*(t)) = f'(t) = \Lambda^{-1}(\beta f(t)^p) \approx t^{\frac{p}{1-p}}$ in the usual sense that $\forall \eta > 0$ and t large enough

$$t^{\frac{p}{1-p}-\eta} \le \psi'(\gamma^*(t)) \le t^{\frac{p}{1-p}+\eta}.$$

Then since the functions γ that we consider satisfy $\psi(\gamma(s)) = \alpha\psi(\gamma^*(s))$ for some $\alpha > 0$ it follows that $\forall \eta > 0$ and n large enough

$$\frac{\int_n^{n+1} \psi'(\gamma(s))\mathrm{d}s}{\int_0^n \psi'(\gamma(s))\mathrm{d}s} \le \frac{1}{n^{1-\eta}} \to 0 \text{ as } n \to \infty$$

proving (4.47).

To prove (4.48) we note that from condition 2 in Theorem 4.51 $\forall \delta > 0$ and γ large enough

$$\psi'(\gamma)^{1-\delta} \le \psi''(\gamma) \le \psi'(\gamma)^{1+\delta},$$

Hence $\forall \eta > 0$ and n large enough

$$\frac{\int_0^n \psi''(\gamma(s))\mathrm{d}s}{(\int_0^n \psi'(\gamma(s))\mathrm{d}s)^2} \le \frac{1}{n^{\frac{1}{1-p}-\eta}} = \frac{1}{n^{1+\frac{p}{1-p}-\eta}}.$$

Choosing η small enough then proves (4.48).

To prove (4.49) we note that $\forall \eta > 0$ and γ large enough

$$\begin{aligned} &\psi(\gamma) \le \eta\gamma\psi'(\gamma) \\ \Rightarrow &\Lambda(\psi'(\gamma)) = \gamma\psi'(\gamma) - \psi(\gamma) \sim \gamma\psi'(\gamma). \end{aligned} \tag{4.50}$$

Also, by differentiating (4.37) with respect to t we can check that functions $\gamma(\cdot)$ that we consider are increasing but $\gamma'(\cdot)$'s are decreasing. It then follows that $\forall \eta > 0$ and t large enough

$$\begin{aligned} \eta\gamma(t)\int_0^t \psi'(\gamma(s))\mathrm{d}s &> \eta\int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s \\ &\overset{(4.50)}{\ge} \int_0^t \psi(\gamma(s))\mathrm{d}s \\ &= \int_0^t\int_0^s \gamma'(u)\psi'(\gamma(u))\mathrm{d}u\mathrm{d}s \\ &> \int_0^t \gamma'(s)\int_0^s \psi'(\gamma(u))\mathrm{d}u\mathrm{d}s. \end{aligned}$$

Hence

$$\int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s = \gamma(t)\int_0^t \psi'(\gamma(s))\mathrm{d}s - \int_0^t \gamma'(s)\int_0^s \psi'(\gamma(u))\mathrm{d}u\mathrm{d}s$$
$$\sim \gamma(t)\int_0^t \psi'(\gamma(s))\mathrm{d}s$$

and thus

$$\limsup_{t\to\infty} \frac{\gamma(t)\int_0^t \psi'(\gamma(s))\mathrm{d}s}{\int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s} = 1 < \infty.$$

So conditions of Theorem 4.39 and Corollary 4.41 are satisfied and this proves Proposition 4.56. □

Let us now prove the following bound on the spine term.

Proposition 4.57. *There exist some $\tilde{Q}_\gamma$-a.s. finite positive random variables C', C'' and a random time $T' < \infty$ such that $\forall t > T'$*

$$\mathit{spine}(t) \le C' \exp\Big(- C'' \int_0^t \big(\int_0^s \psi'(\gamma(u))\mathrm{d}u\big)^p \mathrm{d}s\Big).$$

Proof of Proposition 4.57. Let us begin by observing the following two inequalities:

$$\Lambda\big((1-\epsilon)x\big) \le (1-\epsilon)\Lambda(x) \qquad \forall x \ge 0. \tag{4.51}$$

$$\psi(\gamma) \le \Lambda\big(\psi'(\gamma)\big) \text{ for } \gamma \text{ large enough.} \tag{4.52}$$

(4.51) follows from the convexity of Λ and the fact that $\Lambda(0) = 0$.

For (4.52) note that in the case of Brownian motion $\psi(\gamma) = \Lambda(\psi'(\gamma)) = \frac{1}{2}\sigma^2\gamma^2$. Otherwise $\Lambda(\psi'(\gamma)) \gg \psi(\gamma)$ as we have already seen in (4.50).

Then from Proposition 4.56 we have that for all $\delta > 0$ there exists $\tilde{Q}_\gamma$-a.s. finite random time T_δ such that $\forall t > T_\delta$

$$(1-\delta)\int_0^t \psi'(\gamma(s))\mathrm{d}s \le \xi_t \le (1+\delta)\int_0^t \psi'(\gamma(s))\mathrm{d}s$$

and

$$(1-\delta)\int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s \le \int_0^t \gamma(s)\mathrm{d}\xi_s \le (1+\delta)\int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s.$$

Thus $\forall t \geq T_\delta$

$$\begin{aligned}
spine(t) &= \exp\Big(\int_0^t \gamma(s)\mathrm{d}\xi_s - \int_0^t \psi\big(\gamma(s)\big)\mathrm{d}s - \int_0^{T_\delta} \beta|\xi_s|^p\mathrm{d}s - \int_{T_\delta}^t \beta|\xi_s|^p\mathrm{d}s\Big)\\
&\leq C_\delta \exp\Big((1+\delta)\int_0^t \gamma(s)\psi'(\gamma(s))\mathrm{d}s - \int_0^t \psi(\gamma(s))\mathrm{d}s\\
&\qquad - (1-\delta)^p\beta\int_0^t \Big(\int_0^s \psi'(\gamma(u))\mathrm{d}u\Big)^p\mathrm{d}s\Big)\\
&= C_\delta \exp\Big((1+\delta)\int_0^t \Lambda\big(\psi'(\gamma(s))\big)\mathrm{d}s + \delta\int_0^t \psi(\gamma(s))\mathrm{d}s\\
&\qquad - (1-\delta)^p\beta\int_0^t \Big(\int_0^s \psi'(\gamma(u))\mathrm{d}u\Big)^p\mathrm{d}s\Big),
\end{aligned}$$

where

$$C_\delta = \exp\Big(-\int_0^{T_\delta} \beta|\xi_s|^p\mathrm{d}s + \int_0^{T_\delta} \beta\Big|\int_0^s \psi'(\gamma(u))\mathrm{d}u\Big|^p\mathrm{d}s\Big)$$

is a $\tilde{Q}_\gamma$-a.s. finite random variable. Then by (4.52)

$$\begin{aligned}
spine(t) &\leq C'_\delta \exp\Big((1+\delta)\int_0^t \Lambda\big(\psi'(\gamma(s))\big)\mathrm{d}s + \delta\int_0^t \Lambda\big(\psi'(\gamma(s))\big)\mathrm{d}s\\
&\qquad - (1-\delta)^p\beta\int_0^t \Big(\int_0^s \psi'(\gamma(u))\mathrm{d}u\Big)^p\mathrm{d}s\Big),
\end{aligned}$$

where

$$C'_\delta = C_\delta \times \exp\Big(\delta\int_0^\tau \psi\big(\gamma(s)\big)\mathrm{d}s - \delta\int_0^\tau \Lambda\big(\psi'(\gamma(s))\big)\mathrm{d}s\Big)$$

and $\tau > 0$ is such that

$$\psi\big(\gamma(s)\big) \leq \Lambda\big(\psi'(\gamma(s))\big)\ \forall s \geq \tau.$$

Thus using (4.51) we have

$$\begin{aligned}
spine(t) &\leq C'_\delta \exp\Big((1+2\delta)\int_0^t \Lambda\big(\psi'(\gamma(s))\big)\mathrm{d}s - (1-\delta)^p\beta\int_0^t \Big(\int_0^s \psi'(\gamma(u))\mathrm{d}u\Big)^p\mathrm{d}s\Big)\\
&\leq C'_\delta \exp\Big((1+2\delta)(1-\epsilon)\int_0^t \Lambda\big(\psi'(\gamma^*(s))\big)\mathrm{d}s\\
&\qquad - (1-\delta)^p\beta\int_0^t \Big(\int_0^s \psi'(\gamma(u))\mathrm{d}u\Big)^p\mathrm{d}s\Big)\\
&= C'_\delta \exp\Big((1+2\delta)(1-\epsilon)\beta\int_0^t \Big(\int_0^s \psi'(\gamma^*(u))\mathrm{d}u\Big)^p\mathrm{d}s\\
&\qquad - (1-\delta)^p\beta\int_0^t \Big(\int_0^s \psi'(\gamma(u))\mathrm{d}u\Big)^p\mathrm{d}s\Big)\\
&\leq C'_\delta \exp\Big(\Big[(1+2\delta)(1-\epsilon) - (1-\delta)^p\Big]\beta\int_0^t \Big(\int_0^s \psi'(\gamma(u))\mathrm{d}u\Big)^p\mathrm{d}s\Big)
\end{aligned}$$

and for a given $\epsilon > 0$ we can choose $\delta > 0$ s.t.

$$c_\epsilon^- := (1+2\delta)(1-\epsilon) - (1-\delta)^p < 0.$$

So, choosing such a δ and letting $T' = T_\delta$, $C' = C'_\delta$ and $C'' = ((1+2\delta)(1-\epsilon)-(1-\delta)^p)\beta$ we prove Proposition 4.57. $\square$

For the sum term we have when $t > T'$

$$\begin{aligned} sum(t) &= \sum_{u<node_t(\xi)} spine(S_u) \\ &= \Big(\sum_{u<node_t(\xi),\ S_u \le T'} spine(S_u) \Big) + \Big(\sum_{u<node_t(\xi),\ S_u > T'} spine(S_u) \Big) \\ &\le \sum_{u<node_t(\xi),\ S_u \le T'} spine(S_u) \\ &\quad + \sum_{u<node_t(\xi),\ S_u > T'} C' \exp\Big(-C'' \int_0^{S_u} \Big(\int_0^s \psi'(\gamma(u))\mathrm{d}u \Big)^p \mathrm{d}s \Big) \end{aligned}$$

using Proposition 4.57 for the inequality. The first sum is $\tilde{Q}_\gamma$-a.s. bounded since it only counts births up to time T'. Call an upper bound on the first sum C_1. Then we have

$$sum(t) \le\ C_1 + C' \sum_{n=1}^{\infty} \exp\Big(-C'' \int_0^{S_n} \Big(\int_0^s \psi'(\gamma(u))\mathrm{d}u \Big)^p \mathrm{d}s \Big), \tag{4.53}$$

where S_n is the time of the n^{th} birth on the spine.

The birth process along the spine $(n_t)_{t\ge 0}$ conditional on the path of the spine is time-inhomogeneous Poisson process (or Cox process) with jump rate $2\beta|\xi_t|^p$ at time t (See Proposition 1.13). Thus

$$\frac{n_t}{\int_0^t 2\beta|\xi_s|^p \mathrm{d}s} \to 1 \quad \tilde{Q}_\gamma\text{-a.s. as } t \to \infty.$$

Also

$$\int_0^t |\xi_s|^p \mathrm{d}s \sim \int_0^t \Big(\int_0^s \psi'(\gamma(u))\mathrm{d}u \Big)^p \mathrm{d}s \quad \tilde{Q}_\gamma\text{-a.s. as } t \to \infty.$$

Hence

$$n_t \sim 2\beta \int_0^t \Big(\int_0^s \psi'(\gamma(u))\mathrm{d}u \Big)^p \mathrm{d}s \quad \tilde{Q}_\gamma\text{-a.s. as } t \to \infty. \tag{4.54}$$

So for some $\tilde{Q}_\gamma$-a.s. finite positive random variable C_2 we have

$$\int_0^{S_n} \Big(\int_0^s \psi'(\gamma(u))\mathrm{d}u \Big)^p \mathrm{d}s \ge C_2 n \quad \forall n.$$

Then substituting this into (4.53) we get

$$sum(t) \le C_1 + C' \sum_{n=1}^{\infty} e^{-C''C_2 n},$$

which is bounded $\tilde{Q}_\gamma$-a.s. We have thus shown that

$$\limsup_{t\to\infty} E^{\tilde{Q}_\gamma}\Big(M_\gamma(t)\big|\tilde{\mathcal{G}}_\infty\Big) < \infty \qquad \tilde{Q}_\gamma\text{-a.s.}$$

and hence

$$\limsup_{t\to\infty} M_\gamma(t) < \infty \qquad \tilde{Q}_\gamma\text{-a.s.}$$

□

Proof of Theorem 4.51 part ii). Since one of the particles at time t is the spine particle, we have

$$M_\gamma(t) \ge \exp\Big(\int_0^t \gamma(s)\mathrm{d}\xi_s - \int_0^t \psi(\gamma(s))\mathrm{d}s - \int_0^t \beta|\xi_s|^p \mathrm{d}s\Big) = spine(t).$$

For $\gamma(\cdot)$ satisfying $\psi'(\gamma(t)) = (1+\epsilon)\psi'(\gamma^*(t))$ one can check following the same analysis as in the proof of i) above that $spine(t) \to \infty$ $\tilde{Q}_\gamma$-a.s. Thus

$$\limsup_{t\to\infty} M_\gamma(t) = \infty \qquad \tilde{Q}_\gamma\text{-a.s.}$$

and so also $\tilde{Q}_\gamma$-a.s. Recalling Corollary 4.53 we see that $M_\gamma(\infty) = 0$ P-a.s. □

4.6.3 Lower bound on the rightmost particle

We can now prove Theorem 4.24, which will also provide us with the lower bound for Theorem 4.22.

Let $\gamma(\cdot)$ satisfy $\psi'(\gamma(t)) = (1-\epsilon)\psi'(\gamma^*(t)) = (1-\epsilon)f'(t)$. In the previous subsection we proved that

1. $\limsup_{t\to\infty} M_\gamma(t) < \infty$
2. M_γ is P-U.I.
3. $M_\gamma(\infty) > 0$ P-a.s.

Thus from Lemma 4.52 for events $A \in \mathcal{F}_\infty$

$$Q_\gamma(A) = E(\mathbf{1}_A M_\gamma(\infty))$$

and also

$$Q_\gamma(A) = 1 \Leftrightarrow P(A) = 1.$$

In other words Q_γ and P are equivalent on $\mathcal{F}_\infty$.

Let us exploit this fact to get a lower bound on the rightmost particle

Proof of Theorem 4.24. For $\epsilon \in (0,1)$ take $\gamma(\cdot)$ such that $\psi'(\gamma(t)) = (1-\epsilon)\psi'(\gamma^*(t)) = (1-\epsilon)f'(t)$ and consider the event

$$B_\gamma := \Big\{\exists \text{ infinite line of descent } u : \liminf_{t\to\infty} \frac{X_t^u}{\int_0^t \psi'(\gamma(s))\mathrm{d}s} = 1\Big\} \in \mathcal{F}_\infty.$$

Then

$$\begin{aligned}
&\tilde{Q}_\gamma\big(\lim_{t\to\infty} \frac{\xi_t}{\int_0^t \psi'(\gamma(s))\mathrm{d}s} = 1\big) = 1 \\
\Rightarrow &\tilde{Q}_\gamma(B_\gamma) = 1 \\
\Rightarrow &Q_\gamma(B_\gamma) = 1 \\
\Rightarrow &P(B_\gamma) = 1 \\
\Rightarrow &P\big(\liminf_{t\to\infty} \frac{R_t}{\int_0^t \psi'(\gamma(s))\mathrm{d}s} \geq 1\big) = 1 \\
\Rightarrow &P\big(\liminf_{t\to\infty} \frac{R_t}{f(t)} \geq 1-\epsilon\big) = 1.
\end{aligned}$$

Letting $\epsilon \searrow 0$ we obtain the required result. $\square$

4.6.4 Upper bound on the rightmost particle

In this subsection we complete the proof of Theorem 4.22 by establishing the appropriate upper bound.

According with Theorem 4.22 we impose an additional condition that the single-particle motion is symmetric in the sense that $(X_t)_{t\geq 0} \overset{d}{=} (-X_t)_{t\geq 0}$, which we did not need in the proof of the lower bound.

Proposition 4.58. *Consider a Branching Lévy process with a one-particle motion satisfying:*

1. $\psi(\gamma) = \log \mathbb{E}e^{\gamma X_1} < \infty \ \forall \gamma \in \mathbb{R}$,
2. *for all* $\delta > 0$ $\psi''(\gamma) < \psi'(\gamma)^{1+\delta}$ *for all* γ *large enough,*
3. $(X_t)_{t\geq 0}$ *is point-recurrent in the sense of Definition 4.7,*
4. $(X_t)_{t\geq 0}$ *is symmetric in the sense that* $(X_t)_{t\geq 0} \overset{d}{=} (-X_t)_{t\geq 0}$

Then

$$\limsup_{t\to\infty} \frac{R_t}{f(t)} \leq 1 \ P\text{-a.s.},$$

where $f(t) = \int_0^t \psi'(\gamma^*(s))\mathrm{d}s$ *as before.*

To prove Proposition 4.58 we shall assume for contradiction that it is false. Then we shall show that an additive martingale M_γ for the right choice of $\gamma(\cdot)$ will diverge to ∞ contradicting the Martingale Convergence Theorem.

We start by proving the following 0-1 law.

Lemma 4.59. *For all $c > 0$*

$$P\Big(\limsup_{t\to\infty} \frac{R_t}{f(t)} \leq c\Big) \in \{0, 1\}.$$

Proof. Let us consider

$$q(x) = P^x\Big(\limsup_{t\to\infty} \frac{R_t}{f(t)} \leq c\Big).$$

Then

$$q(x) = E^x\Big(P^x\big(\limsup_{t\to\infty} \frac{R_t}{f(t)} \leq c\big|\mathcal{F}_t\big)\Big) = E^x\Big(\prod_{u\in N_t} q\big(X_u(t)\big)\Big)$$

so that $\prod_{u\in N_t} q(X_u(t))$ is a P-martingale. Applying Lemma 4.54 to $q(x)$ we deduce that

$$P\Big(\limsup_{t\to\infty} \frac{R_t}{f(t)} \leq c\Big) = q(0) \in \{0, 1\}.$$

$\square$

Proof of Proposition 4.58. Let us suppose for contradiction that $\exists\epsilon > 0$ such that

$$P\Big(\limsup_{t\to\infty} \frac{R_t}{f(t)} > 1 + \epsilon\Big) = 1. \tag{4.55}$$

Let

$$h(t) := \big(1 + \frac{\epsilon}{2}\big)f(t)\ , \quad t \geq 0$$

and

$$\gamma(t) := (\psi')^{-1}\big((1 + \frac{\epsilon}{2})f'(t)\big)$$

so that

$$\int_0^t \psi'(\gamma(s))\mathrm{d}s = (1 + \frac{\epsilon}{2})f(t) = h(t).$$

We define $D(h)$ to be the space-time region bounded above by the curve $y = h(t)$ and below by the curve $y = -h(t)$.

Under P the spine process $(\xi_t)_{t\geq 0}$ is a Lévy process with zero mean and so $\dfrac{|\xi_t|}{t} \to 0$ P-a.s. as $t \to \infty$. Hence there exists an a.s. finite random time $T' < \infty$ such that $\xi_t \in D(h)$ for all $t > T'$.

Since $(\xi_t)_{t\geq 0}$ is point-recurrent there exists an interval $[a, b]$, $0 < a < b$ such that $(\xi_t)_{t\geq 0}$ will spend an infinite amount of time in this interval giving birth to offspring at rate $\geq \beta a^p$. This assures us of the existence of an infinite sequence $\{T_n\}_{n\in\mathbb{N}}$ of birth times along the path of the spine when it stays in $[a, b]$ with $0 \le T' \le T_1 < T_2 < \ldots$ and $T_n \nearrow \infty$.

Denote by u_n the label of the particle born at time T_n, which does not continue the spine. Then each particle u_n gives rise to an independent copy of the Branching Lévy process under P started from ξ_{T_n} at time T_n. Almost surely, by assumption (4.55), each u_n has some descendant that leaves the space-time region $D(h)$.

Let $\{v_n\}_{n\in\mathbb{N}}$ be the subsequence of $\{u_n\}_{n\in\mathbb{N}}$ of those particles whose first descendent leaving $D(h)$ does this by crossing the upper boundary $y = h(t)$. Since the branching process is symmetric and the particles u_n are born in the upper half-plane, there is at least probability $\dfrac{1}{2}$ that the first descendant of u_n to leave $D(h)$ does this by crossing the positive boundary curve. Therefore P-a.s. the sequence $\{v_n\}_{n\in\mathbb{N}}$ is infinite.

Let w_n be the descendent of v_n, which exits $D(h)$ first and let J_n be the time when this occurs. That is,

$$J_n = \inf\big\{t : X_{w_n}(t) \ge h(t)\big\}.$$

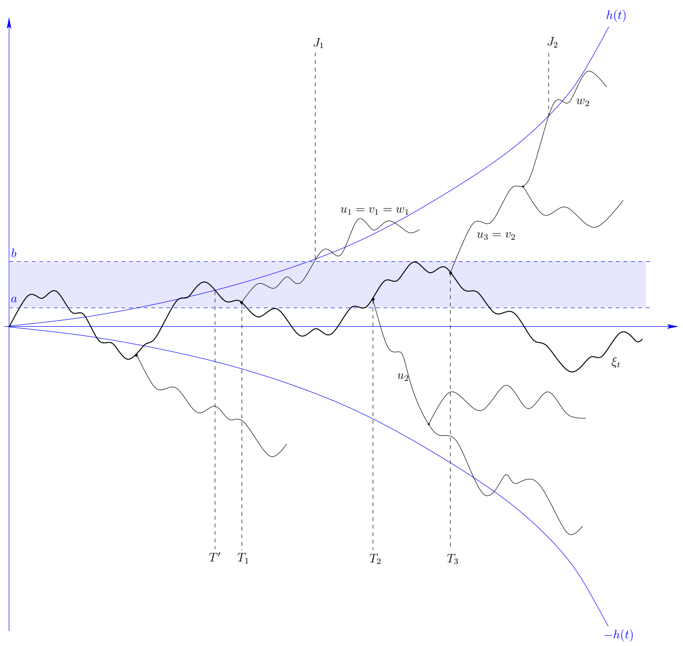


Figure 4-5: *Illustration to Proposition 4.58*

Note that the path of particle w_n satisfies

$$|X_{w_n}(s)| < h(s) \quad \forall s \in [T', J_n).$$

Clearly $J_n \to \infty$ as $n \to \infty$. To obtain a contradiction we shall show that the additive martingale M_γ fails to converge along the sequence of times $\{J_n\}_{n\geq 1}$.

$$\begin{aligned} M_\gamma(J_n) &= \sum_{u\in N_{J_n}} \exp\Big\{\int_0^{J_n}\gamma(s)\mathrm{d}X_s^u - \int_0^{J_n}\psi(\gamma(s))\mathrm{d}s - \int_0^{J_n}\beta|X_s^u|^p\mathrm{d}s\Big\} \\ &\geq \Big\{\int_0^{J_n}\gamma(s)\mathrm{d}X_s^{w_n} - \int_0^{J_n}\psi(\gamma(s))\mathrm{d}s - \int_0^{J_n}\beta|X_s^{w_n}|^p\mathrm{d}s\Big\}. \end{aligned}$$

Applying the integration-by-parts formula from Propostion 4.33 we get

$$\begin{aligned} &\exp\Big\{\gamma(J_n)X_{J_n}^{w_n} - \int_0^{J_n}\gamma'(s)X_s^{w_n}\mathrm{d}s - \int_0^{J_n}\psi(\gamma(s))\mathrm{d}s - \int_0^{J_n}\beta|X_s^{w_n}|^p\mathrm{d}s\Big\} \\ \geq& C\exp\Big\{\gamma(J_n)\int_0^{J_n}\psi'(\gamma(s))\mathrm{d}s - \int_0^{J_n}\gamma'(s)\int_0^s\psi'(\gamma(u))\mathrm{d}u\mathrm{d}s \\ &\quad - \int_0^{J_n}\psi(\gamma(s))\mathrm{d}s - \int_0^{J_n}\beta\Big(\int_0^s\psi'(\gamma(u))\mathrm{d}u\Big)^p\mathrm{d}s\Big\} \end{aligned}$$

using the facts that $X_{J_n}^{w_n} \geq h(J_n)$ and $|X_s^{w_n}| < h(s)$ for $s \in [T', J_n)$ and where C is some P-a.s positive random variable. Then applying the classical integration-by-parts formula we get

$$\begin{aligned} &C\exp\Big\{\int_0^{J_n}\gamma(s)\psi'(\gamma(s)) - \psi(\gamma(s))\mathrm{d}s - \int_0^{J_n}\beta\Big(\int_0^s\psi'\big(\gamma(u)\big)\mathrm{d}u\Big)^p\mathrm{d}s\Big\} \\ =&C\exp\Big\{\int_0^{J_n}\Lambda\Big(\psi'\big(\gamma(s)\big)\Big)\mathrm{d}s - \int_0^{J_n}\beta\Big(\int_0^s\psi'\big(\gamma(u)\big)\mathrm{d}u\Big)^p\mathrm{d}s\Big\} \\ =&C\exp\Big\{\int_0^{J_n}\Lambda\Big(\big(1+\frac{\epsilon}{2}\big)\psi'\big(\gamma^*(s)\big)\Big)\mathrm{d}s - \big(1+\frac{\epsilon}{2}\big)^p\int_0^{J_n}\beta\Big(\int_0^s\psi'\big(\gamma^*(u)\big)\mathrm{d}u\Big)^p\mathrm{d}s\Big\} \\ \geq&C\exp\Big\{\big(1+\frac{\epsilon}{2}\big)\int_0^{J_n}\Lambda\Big(\psi'\big(\gamma^*(s)\big)\Big)\mathrm{d}s - \big(1+\frac{\epsilon}{2}\big)^p\int_0^{J_n}\beta\Big(\int_0^s\psi'\big(\gamma^*(u)\big)\mathrm{d}u\Big)^p\mathrm{d}s\Big\} \\ =&C\exp\Big\{\Big[\big(1+\frac{\epsilon}{2}\big) - \big(1+\frac{\epsilon}{2}\big)^p\Big]\int_0^{J_n}\Lambda\Big(\psi'\big(\gamma^*(s)\big)\Big)\mathrm{d}s\Big\} \to \infty \end{aligned}$$

since $\Big[\big(1+\frac{\epsilon}{2}\big) - \big(1+\frac{\epsilon}{2}\big)^p\Big] > 0$.

Thus $M_\gamma(J_n) \to \infty$ as $n \to \infty$, which is a contradiction. Therefore assumption (4.55) is wrong and we must have that $\forall\epsilon > 0$

$$P\Big(\limsup_{t\to\infty}\frac{R_t}{f(t)} > 1+\epsilon\Big) = 0.$$

It follows after taking the limit $\epsilon \searrow 0$ that

$$P\Big(\limsup_{t\to\infty}\frac{R_t}{f(t)} \leq 1\Big) = 1.$$

□

Let us note that the above proof relied on the symmetry of the branching process. This symmetry guaranteed the existence of an infinite sequence $\{v_n\}_{n\in\mathbb{N}}$, which was crucial in the proof.

If we don't assume the symmetry of the particles' motion then we might see the following picture:

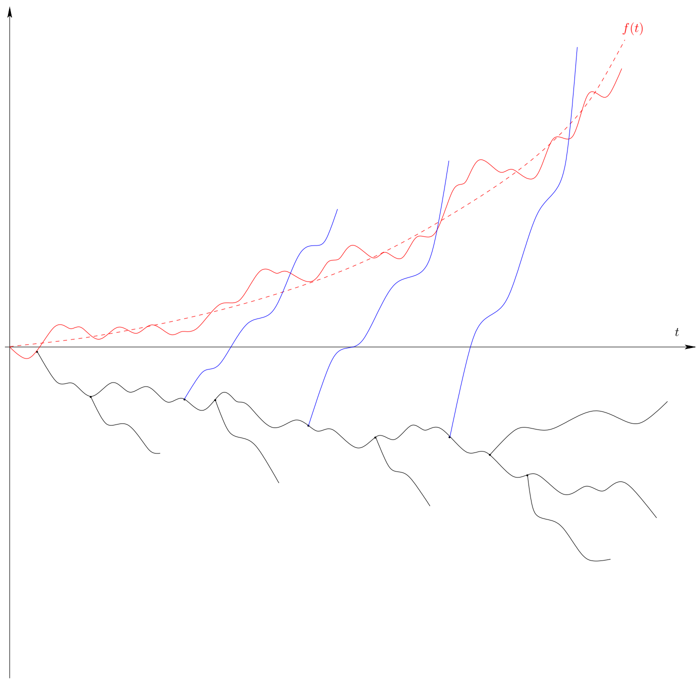


Figure 4-6: *Asymmetric branching process*

That is, we still have a lineage of particles staying near $f(t)$ (drawn in red in Figure 4-6 above), but due to asymmetry we might have that most of the particles' mass is concentrated in the lower half-plane. Then it is possible that those particles from the lower half-plane ocasionally go above $f(t)$ (such particles are drawn in blue in Figure 4-6). In this case $f(t)$ will underestimate the rightmost particle of the system.

Chapter 5

BBM with branching at the origin

In this chapter we study a branching Brownian motion in which binary fission takes place only at the origin at rate β on the local time scale. That is, the cumulative branching rate of each particle is βL_t, where L_t is its local time at 0 and β is a positive constant. Heuristically, we can think of the instantaneous branching rate as $\beta\delta_0(x)$ if we accept that $\int_0^t \delta_0(X_s)\mathrm{d}s = L_t$.

This model has been studied before in the context of superprocesses. See e.g. works of D. A. Dawson and K. Fleischmann [11], K. Fleischmann and J.F. Le Gall [17] or J. Engländer and D. Turaev [14]. In the discrete space similar models have been studied extensively as well. See e.g. some recent papers such as [9] or [12].

We shall prove results about the total number of particles in the system and the number of particles above the given line λt. In particular, we shall exhibit the asymptotic behaviour of the rightmost particle. We shall also prove the strong law of large numbers for the branching process, adapting the proof of J. Engländer, S.C. Harris and A.E. Kyprianou from [16].

5.1 Introduction

5.1.1 Local time of a Brownian motion

Basic information about local times and the excursion theory can be found in many textbooks on Brownian motion (see e.g. [28]). Also a good introduction is given in the paper of C. Rogers [30]. Let us give a very brief overview of this topic.

Suppose $(X_t)_{t\geq 0}$ is a standard Brownian motion on some probability space under probability measure $\mathbb{P}$. The following result due to Trotter is taken from [30].

Theorem 5.1 (Trotter)**.** *There exists a jointly continuous process* $\{L(t,x) : t \geq, x \in \mathbb{R}\}$

such that for all bounded measurable f, and all $t \geq 0$

$$\int_0^t f(X_s)\mathrm{d}s = \int_{-\infty}^{\infty} f(x)L(t,x)\mathrm{d}x.$$

In particular, for any Borel set A

$$\int_0^t \mathbf{1}_A(X_s)\mathrm{d}s = \int_A L(t,x)\mathrm{d}x,$$

so L is an occupation density.

Definition 5.2. *The process $(L(t,x))_{t\geq 0}$ is called the local time of $(X_t)_{t\geq 0}$ at x.*

The following corollary of Theorem 5.1 can be found e.g. in [28] and it is often usen as the definition of the local times.

Corollary 5.3. *Almost surely*

$$L(t,x) = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}_{\{X_s\in(x-\epsilon,x+\epsilon)\}}\mathrm{d}s$$

for every $x \in \mathbb{R}$ and $t \geq 0$.

For the rest of this chapter we shall only be concerned with the local time at 0, which we shall denote as L_t rather than $L(t,0)$. We recall a couple of well-known results.

Theorem 5.4 (Tanaka's formula)**.**

$$|X_t| = \int_0^t sgn(X_s)\mathrm{d}X_s + L_t,$$

where

$$sgn(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x \leq 0 \end{cases}$$

In a non-rigorous way this can be thought of as Itô's formula applied to $f(x) = |x|$, where $f'(x) = sgn(x)$, $f''(x) = 2\delta_0(x)$ (where δ_0 is the Dirac delta function). Then one can think of L_t as $\int_0^t \delta_0(X_s)\mathrm{d}s$.

Another useful result is the following theorem.

Theorem 5.5 (Lévy)**.** *Let $(S_t)_{t\geq 0}$ be the running supremum of X. That is, $S_t = \sup_{0\leq s\leq t} X_s$. Then*

$$(S_t, S_t - X_t)_{t\geq 0} \stackrel{d}{=} (L_t, |X_t|)_{t\geq 0}$$

and as a consequence $(|X_t| - L_t)_{t\geq 0}$ is a standard Brownian motion.

Corollary 5.6. $\forall t \geq 0$ *by the Reflection Principle*

$$L_t \stackrel{d}{=} S_t \stackrel{d}{=} |X_t| \stackrel{d}{=} |N(0,t)|.$$

5.1.2 Description of the model

Under probability measure P we construct the branching process in the following way. Initial particle starts moving from 0 according to a standard Brownian motion with the position at time t denoted by X_t.

If $(L_t)_{t\geq 0}$ is the local time of $(X_t)_{t\geq 0}$ at 0 then at cumulative rate βL_t (See Remark 1.4), where $\beta > 0$ is a given constant, the particle splits into two new ones. Note that since $(L_t)_{t\geq 0}$ only increases on the zero set of $(X_t)_{t\geq 0}$ the split can only occur at the position 0.

The new particles then independently of each other and of the past repeat the behaviour of their father.

5.1.3 Main results

In this subsection we list all our main theorems and propositions in the order that we are going to prove them.

Firstly we shall prove the following two lemmas about the expected population growth.

Lemma 5.7. *Recall that N_t is the set of particles alive at time t. Then*

$$E\Big(|N_t|\Big) \sim 2e^{\frac{\beta^2}{2}t} \text{ as } t \to \infty.$$

Lemma 5.8. *For $\lambda > 0$ let $N_t^{\lambda t} := \{u \in N_t : X_t^u > \lambda t\}$ be the set of particles at time t, which lie above λt. Then as $t \to \infty$*

$$\frac{1}{t}\log E\Big(|N_t^{\lambda t}|\Big) \to \Delta_\lambda := \begin{cases} \frac{1}{2}\beta^2 - \beta\lambda & \text{if } \lambda < \beta \\ -\frac{1}{2}\lambda^2 & \text{if } \lambda \geq \beta \end{cases}$$

Note that Δ_λ is < 0 or > 0 according to whether λ is $> \frac{\beta}{2}$ or $< \frac{\beta}{2}$ (see Figure 5-1 below). The next few results are concerned with the almost sure asymptotic behaviour of the population.

Theorem 5.9.

$$\lim_{t\to\infty} \frac{\log |N_t|}{t} = \frac{1}{2}\beta^2 \qquad P\text{-a.s.}$$

Theorem 5.10. *Take $\lambda > 0$. Then:*

1. *if $\lambda > \frac{\beta}{2}$ then $\lim_{t\to\infty} |N_t^{\lambda t}| = 0$ P- a.s.*

2. *if* $\lambda < \frac{\beta}{2}$ *then* $\lim_{t\to\infty} \dfrac{\log |N_t^{\lambda t}|}{t} = \Delta_\lambda = \frac{1}{2}\beta^2 - \beta\lambda$ *P-a.s.*

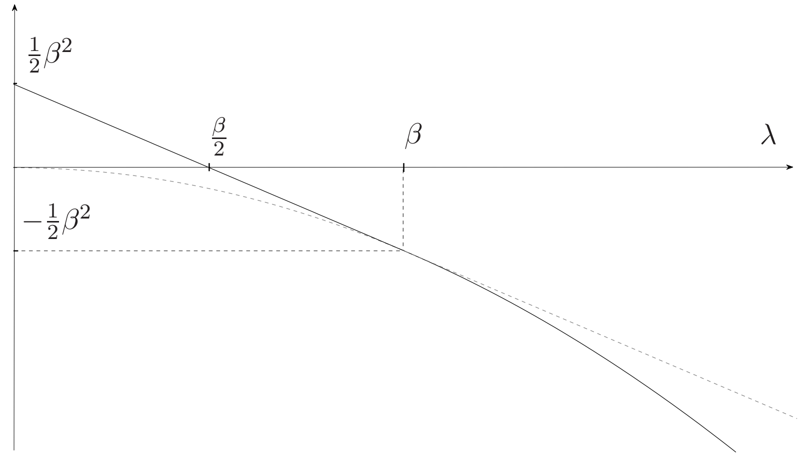


Figure 5-1: *Plot of* Δ_λ

From Theorem 5.10 we immediately get the growth of the rightmost particle.

Corollary 5.11. *Let* $(R_t)_{t\geq 0}$ *be the rightmost particle of the branching process. Then*

$$\lim_{t\to\infty} \frac{R_t}{t} = \frac{\beta}{2} \quad P\text{-a.s.}$$

We also give a bit more information about $|N_t^{\lambda t}|$ in the case $\lambda > \frac{\beta}{2}$.

Lemma 5.12. *For* $\lambda > \frac{\beta}{2}$

$$\lim_{t\to\infty} \frac{\log P(|N_t^{\lambda t}| \geq 1)}{t} = \Delta_\lambda = -\frac{1}{2}\lambda^2.$$

Our final result is the Strong Law of Large Numbers for the branching system.

Theorem 5.13 (SLLN)**.** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be some Borel-measurable bounded function. Then*

$$\lim_{t\to\infty} e^{-\frac{\beta^2}{2}t} \sum_{u\in N_t} f(X_t^u) = M_\infty \int f(x)\beta e^{-\beta|x|}\mathrm{d}x \quad P\text{-a.s.},$$

where M_∞ *is the almost sure limit of the* P*-uniformly integrable additive martingale*

$$M_t = \sum_{u\in N_t} \exp\big\{ -\beta|X_t^u| - \frac{1}{2}\beta^2 t\big\}.$$

One can observe that taking $f(\cdot) \equiv 1$ in Theorem 5.13 would give Lemma 5.7 and an even stronger result than in Theorem 5.9. However the proof of Theorem 5.13 relies on Theorem 5.9 and the proof of Theorem 5.9 relies on Lemma 5.7. Thus it is necessary that we prove our results in the presented order.

5.1.4 Outline of the chapter

In Section 5.2 we shall introduce a certain change of measure, which we shall then use to prove Lemma 5.7 and Lemma 5.8.

In Section 5.3 we shall present the additive martingale associated with the change of measure from Section 5.2 and discuss some of its properties. It will be the same martingale that features in Theorem 5.13.

We shall then prove Theorem 5.9 in Section 5.4 making some use of this additive martingale.

Section 5.5 is devoted to the proofs of Theorem 5.10, Corollary 5.11 and Lemma 5.12.

In Section 5.6 we prove Theorem 5.13.

5.2 Expected population growth

5.2.1 Brownian motion with drift towards the origin

In this self-contained subsection we present a family of single-particle martingales and the corresponding changes of measure. Let $(X_t)_{t\geq 0}$ be a standard Brownian motion under probability measure $\mathbb{P}$ and let $(L_t)_{t\geq 0}$ be its local time at the origin. Then Theorem 5.5 says that

$$(Z_t)_{t\geq 0} := (|X_t| - L_t)_{t\geq 0}$$

is also a standard Brownian Motion under $\mathbb{P}$. Hence for any $\gamma \in \mathbb{R}$

$$W_t = \exp\Big\{\gamma\big(|X_t| - L_t\big) - \frac{1}{2}\gamma^2 t\Big\} = \exp\Big\{\gamma Z_t - \frac{1}{2}\gamma^2 t\Big\}\ , \qquad t \geq 0$$

is a martingale (namely, a Girsanov martingale for Z). And more generally, for $\gamma(\cdot)$ a smooth path

$$\begin{aligned} W_t =& \exp\Big\{\int_0^t \gamma(s)\mathrm{d}Z_s - \frac{1}{2}\int_0^t \gamma^2(s)\mathrm{d}s\Big\} \\ \overset{\text{Tanaka}}{=}& \exp\Big\{\int_0^t \gamma(s)sgn(X_s)\mathrm{d}X_s - \frac{1}{2}\int_0^t \gamma^2(s)\mathrm{d}s\Big\} \end{aligned} \tag{5.1}$$

is a $\mathbb{P}$-martingale. Used as the Radon-Nikodym derivative it puts the instantaneous drift $sgn(X_t)\gamma(t)$ on the process $(X_t)_{t\geq 0}$. Let us restrict ourselves to the case $\gamma(\cdot) \equiv -\gamma < 0$ so that W puts the constant drift γ towards the origin on $(X_t)_{t\geq 0}$. The following result can be found in [8].

Proposition 5.14. *Let $\mathbb{Q}$ be the probability measure defined as*

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\hat{\mathcal{F}}_t} = \exp\Big\{ -\gamma\big(|X_t| - L_t\big) - \frac{1}{2}\gamma^2 t\Big\}\ , \qquad t \geq 0,$$

where $(\hat{\mathcal{F}}_t)_{t\geq 0}$ is the natural filtration of $(X_t)_{t\geq 0}$. Then under $\mathbb{Q}$, $(X_t)_{t\geq 0}$ has the transition density

$$p(t;x,y) = \frac{1}{2\sqrt{2\pi t}} \exp\Big(\gamma(|x|+|y|) - \frac{\gamma^2}{2}t - \frac{(x-y)^2}{2t}\Big) + \frac{\gamma}{4} Erfc\Big(\frac{|x|+|y|-\gamma t}{\sqrt{2t}}\Big)$$

with respect to the speed measure

$$m(\mathrm{d}y) = 2e^{-2\gamma|y|}\mathrm{d}y,$$

so that

$$\mathbb{Q}^x\big(X_t \in A\big) = \int_A p(t;x,y)m(\mathrm{d}y). \tag{5.2}$$

Here $Erfc(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\mathrm{d}u \sim \frac{1}{x\sqrt{\pi}}e^{-x^2}$ as $x \to \infty$.

It also has the stationary probability measure

$$\pi(\mathrm{d}x) = \gamma e^{-2\gamma|x|}\mathrm{d}x. \tag{5.3}$$

5.2.2 Expected asymptotic growth of $|N_t|$

Let us now consider the branching process as described in subsection 5.1.2 with the spine process $(\xi_t)_{t\geq 0}$ defined in the usual way under probability measure $\tilde{P}$. Let $(\tilde{L}_t)_{t\geq 0}$ be the local time of $(\xi_t)_{t\geq 0}$ at 0.

Recall the Many-to-one theorem (Theorem 1.15), which will take the following form in this chapter.

Theorem 5.15 (Many-to-One)**.** *Suppose $f(t) \in m\mathcal{G}_t$ has the representation $f(t) = \sum_{u\in N_t} f_u(t)\mathbf{1}_{\{node_t(\xi)=u\}}$, where $f_u(t) \in m\mathcal{F}_t$, then*

$$E\Big(\sum_{u\in N_t} f_u(t)\Big) = \tilde{E}\Big(f(t)e^{\beta\tilde{L}_t}\Big).$$

Similarly Lemma 1.18 will take the following form.

Lemma 5.16. *Let g be some measurable function, then*

$$E\Big(\sum_{u\in N_t} g(X_t^u)\Big) = \tilde{E}\Big(g(\xi_t)e^{\beta\tilde{L}_t}\Big).$$

To evaluate the expectation on the right hand side of the this lemma one might use the joint density of ξ_t and $\tilde{L}_t$ (see e.g. [24]):

$$\tilde{P}\big(\xi_t \in \mathrm{d}x, \tilde{L}_t \in \mathrm{d}y\big) = \frac{|x|+y}{\sqrt{2\pi t^3}} \exp\Big\{-\frac{(|x|+y)^2}{2t}\Big\}\mathrm{d}x\mathrm{d}y\ , \qquad x \in \mathbb{R},\ y > 0. \tag{5.4}$$

However we shall be instead using the change of measure introduced in Proposition 5.14.

Since $(\xi_t)_{t\geq 0}$ is a standard Brownian motion we can define the following $\tilde{P}$-martingale with respect to the filtration $(\mathcal{G}_t)_{t\geq 0}$, the natural filtration of $(\xi_t)_{t\geq 0}$:

$$\tilde{M}_t^\beta := e^{-\beta|\xi_t| + \beta\tilde{L}_t - \frac{1}{2}\beta^2 t}\ , \qquad t \geq 0. \tag{5.5}$$

We also define the corresponding probability measure $\tilde{Q}_\beta$ as

$$\frac{\mathrm{d}\tilde{Q}_\beta}{\mathrm{d}\tilde{P}}\Big|_{\mathcal{G}_t} = \tilde{M}_t^\beta\ , \qquad t \geq 0. \tag{5.6}$$

Then under $\tilde{Q}_\beta$, $(\xi_t)_{t\geq 0}$ has drift β towards the origin and from Proposition 5.14 we know its exact transition density as well as its stationary distribution.

Let us now exploit this change of measure to prove Lemma 5.7 and Lemma 5.8.

Proof of Lemma 5.7. From Lemma 5.16 we have

$$\begin{aligned} E\Big(|N_t|\Big) = E\Big(\sum_{u\in N_t} 1\Big) &= \tilde{E}\Big(e^{\beta\tilde{L}_t}\Big) = \tilde{E}\Big(e^{\beta\tilde{L}_t - \beta|\xi_t| - \frac{1}{2}\beta^2 t} e^{\beta|\xi_t| + \frac{1}{2}\beta^2 t}\Big) \\ &= \tilde{E}\Big(\tilde{M}_t^\beta e^{\beta|\xi_t| + \frac{1}{2}\beta^2 t}\Big) = E^{\tilde{Q}_\beta}\Big(e^{\beta|\xi_t|}\Big) e^{\frac{1}{2}\beta^2 t}. \end{aligned}$$

Then using the stationary measure from (5.3) we have

$$\begin{aligned} E^{\tilde{Q}_\beta}\Big(e^{\beta|\xi_t|}\Big) &\to \int_{-\infty}^{\infty} e^{\beta|x|}\pi(\mathrm{d}x) \\ &= \int_{-\infty}^{\infty} e^{\beta|x|}\beta e^{-2\beta|x|}\mathrm{d}x = \beta\int_{-\infty}^{\infty} e^{-\beta|x|}\mathrm{d}x = 2. \end{aligned}$$

Thus

$$E\Big(|N_t|\Big) \sim 2e^{\frac{\beta^2}{2}t}.$$

□

Alternatively we could have evaluated $\tilde{E}(e^{\beta\tilde{L}_t})$ explicitly using the fact that $\tilde{L}_t \overset{d}{=} |N(0,t)|$. We would find that

$$\tilde{E}(e^{\beta\tilde{L}_t}) = 2\Phi(\beta\sqrt{t})e^{\frac{\beta^2}{2}t},$$

where $\Phi(x) = \mathbb{P}(N(0,1) \leq x)$. However we don't need such precision as the main use of Lemma 5.7 will be in giving the upper bound for Theorem 5.9, which doesn't require such an accurate convergence result.

5.2.3 Expected asymptotic behaviour of $|N_t^{\lambda t}|$

Let us now prove that $\lim_{t\to\infty}\frac{1}{t}\log E(|N_t^{\lambda t}|)=\Delta_\lambda$, where $N_t^{\lambda t}=\{u\in N_t : X_t^u>\lambda t\}$ and

$$\Delta_\lambda=\begin{cases}\frac{1}{2}\beta^2-\beta\lambda & \text{if } \lambda<\beta\\ -\frac{1}{2}\lambda^2 & \text{if } \lambda\geq\beta\end{cases}$$

Proof of Lemma 5.8. Following the same steps as in Lemma 5.7 we get

$$\begin{aligned}
E\Big(|N_t^{\lambda t}|\Big) &= E\Big(\sum_{u\in N_t}\mathbf{1}_{\{X_t^u>\lambda t\}}\Big)=\tilde{E}\Big(e^{\beta\tilde{L}_t}\mathbf{1}_{\{\xi_t>\lambda t\}}\Big)\\
&= E^{\tilde{Q}_\beta}\Big(e^{\beta|\xi_t|}\mathbf{1}_{\{\xi_t>\lambda t\}}\Big)e^{\frac{1}{2}\beta^2 t}\\
&= E^{\tilde{Q}_\beta}\Big(e^{\beta\xi_t}\mathbf{1}_{\{\xi_t>\lambda t\}}\Big)e^{\frac{1}{2}\beta^2 t}\\
&= \int_{\lambda t}^\infty e^{\beta x}p(t;0,x)m(\mathrm{d}x)e^{\frac{1}{2}\beta^2 t}\\
&= \int_{\lambda t}^\infty e^{\beta x}\Big(\frac{1}{2\sqrt{2\pi t}}\exp\big(\beta x-\frac{\beta^2}{t}-\frac{x^2}{2t}\big)+\frac{\beta}{4}\mathrm{Erfc}\big(\frac{x-\beta t}{\sqrt{2t}}\big)\Big)2e^{-2\beta x}\mathrm{d}x\; e^{\frac{1}{2}\beta^2 t}\\
&= \underbrace{\Big(\int_{\lambda t}^\infty\frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}\mathrm{d}x\Big)}_{(1)}+\underbrace{\Big(\frac{\beta}{2}\int_{\lambda t}^\infty\mathrm{Erfc}\big(\frac{x-\beta t}{\sqrt{2t}}\big)e^{-\beta x}\mathrm{d}x\Big)e^{\frac{1}{2}\beta^2 t}}_{(2)}.
\end{aligned}$$

Then for some functions $\epsilon_i(t)$ satisfying $\log\epsilon_i(t)=o(t)$ we have the following:

$$\begin{aligned}
(1) &= \epsilon_1(t)e^{-\frac{\lambda^2}{2}t},\\
\text{if } \lambda\geq\beta \text{ then } (2) &= \epsilon_2(t)e^{-\frac{\lambda^2}{2}t},\\
\text{if } \lambda<\beta \text{ then } (2) &= \epsilon_3(t)\Big(e^{-\beta\lambda t}-e^{-\beta^2 t}\Big)e^{\frac{\beta^2}{2}t}+\epsilon_4(t)e^{\frac{-\beta^2}{2}t}\\
&= \epsilon_3(t)e^{-\beta\lambda t+\frac{\beta^2}{2}t}+\epsilon_5(t)e^{\frac{-\beta^2}{2}t}.
\end{aligned}$$

Here we have used that $\mathrm{Erfc}(x)\sim\frac{1}{x\sqrt{\pi}}e^{-x^2}$ as $x\to\infty$ and $\mathrm{Erfc}(x)\to 2$ as $x\to-\infty$.

Thus $E(|N_t^{\lambda t}|)=(1)+(2)$

$$=\begin{cases}\epsilon_6(t)e^{-\frac{\lambda^2}{2}t} & \text{if } \lambda\geq\beta\\ \epsilon_7(t)e^{-\beta\lambda t+\frac{\beta^2}{2}t} & \text{if } \lambda<\beta\end{cases}$$

which proves the result after taking the logarithm and dividing by t. □

Remark 5.17. *We see that $\lambda_c:=\frac{\beta}{2}$ is critical in the sense that for $\lambda>\lambda_c$ the expected number of particles above λt is decaying to 0 exponentially fast whereas for $\lambda<\lambda_c$ the* *expected number of particles above λt is growing exponentially. So we might guess that the rightmost particle satisfies $R_t \approx \frac{\beta}{2}t$. This will be proved in Section 5.6.*

Lemmas 5.7 and 5.8 were proved by S. Harris in his PhD thesis [20] using the excursion theory. The proofs that we presented here (using the change of measure) suggest the use of the certain additive martingale in the study of our model. The next section is devoted to this additive martingale.

5.3 The additive martingale

Let us substitute the martingale $(\tilde{M}_t^{\beta})_{t\geq 0}$ for the martingale $(\tilde{M}_t^{(3)})_{t\geq 0}$ in Proposition 1.24 to define the new probability measure $\tilde{Q}$:

$$\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}\tilde{P}}\bigg|_{\tilde{\mathcal{F}}_t} = 2^{n_t} e^{-\beta \tilde{L}_t} \tilde{M}_t^{\beta} \qquad , t \geq 0,$$

which has the effect of changing the spine's motion by adding the drift β towards the origin and doubling the branching rate along the spine. We then define $Q := \tilde{Q}|_{\mathcal{F}_\infty}$ so that

$$\begin{aligned}\frac{\mathrm{d}Q}{\mathrm{d}P}\bigg|_{\mathcal{F}_t} = M_t &:= \sum_{u \in N_t} \exp\Big\{\big(-\beta|X_t^u| + L_t^u - \frac{1}{2}\beta^2 t\big) - \beta L_t^u\Big\} \\ &= \sum_{u \in N_t} \exp\Big\{-\beta|X_t^u| - \frac{1}{2}\beta^2 t\Big\}, \qquad t \geq 0,\end{aligned} \tag{5.7}$$

where L_t^u is the local time at 0 of X_t^u. This additive martingale will help us estimate the almost sure growth of $|N_t|$ and will also be used in the section about the SLLN. The next theorem is a standard result for additive martingales, which we have already seen many times.

Theorem 5.18. *$(M_t)_{t\geq 0}$ is P-uniformly integrable and $M_\infty > 0$ P-almost surely.*

Proof. Note that $P(M_\infty > 0) \in \{0, 1\}$ as it follows from Lemma 4.54 since the proof of the lemma didn't depend on the branching rate of the process. Another proof can be found in [22].

Then as usual, for an event $A \in \mathcal{F}_\infty$ we have

$$Q(A) = \int_A \limsup_{t\to\infty} M_t \mathrm{d}P + Q\Big(A \cap \big\{\limsup_{t\to\infty} M_t = \infty\big\}\Big).$$

Hence to prove the theorem it is sufficient to show that

$$\limsup_{t\to\infty} M_t < \infty \quad Q\text{-a.s.} \tag{5.8}$$

Let us consider the spine decomposition of M_t

$$E^{\tilde{Q}}\Big(M_t\big|\tilde{\mathcal{G}}_\infty\Big) = spine(t) + sum(t),$$

where

$$spine(t) = \exp\Big\{-\beta|\xi_t| - \frac{1}{2}\beta^2 t\Big\}$$

and

$$sum(t) = \sum_{u<node_t(\xi)} \exp\Big\{-\beta|\xi_{S_u}| - \frac{1}{2}\beta^2 S_u\Big\}.$$

Note that under $\tilde{Q}$, $(\xi_t)_{t\geq 0}$ is a Brownian Motion with drift β towards the origin. Thus $\dfrac{\xi_t}{t} \to 0$ and $\dfrac{\tilde{L}_t}{t} \to \beta$ $\tilde{Q}$-a.s. and so

$$spine(t) \sim \exp\Big\{-\frac{1}{2}\beta^2 t\Big\} \quad \tilde{Q}\text{-a.s.}$$

In particular, there exists some random time T' and a constant $C > 0$ such that $\forall t > T'$

$$spine(t) \leq e^{-Ct}.$$

Then also

$$sum(t) \leq \sum_{u<node_t(\xi)} e^{-CS_u} \leq \sum_{n=1}^{\infty} e^{-CS_n}, \tag{5.9}$$

where S_n is the n^{th} birth on the spine. Let $(n_t)_{t\geq 0}$ be the number of births along the spine. Then given the path of the spine process $(n_t)_{t\geq 0}$ a time-inhomogeneous Poisson process (Cox process) with cumulative jump rate $2\beta\tilde{L}_t$. Hence

$$\begin{aligned}
&\frac{n_t}{2\beta\tilde{L}_t} \to 1 \quad \tilde{Q}\text{-a.s. as } t\to\infty\\
\Rightarrow&\frac{n_t}{2\beta^2 t} \to 1 \quad \tilde{Q}\text{-a.s. as } t\to\infty\\
\Rightarrow&\frac{n}{2\beta^2 S_n} \to 1 \quad \tilde{Q}\text{-a.s. as } n\to\infty.
\end{aligned}$$

Thus for some random variable $C_1 > 0$

$$S_n \geq C_1 n \qquad \forall n.$$

Substituting this into (5.9) we get

$$sum(t) \leq \sum_{n=1}^{\infty} e^{-CC_1 n}.$$

Thus $sum(t)$ is bounded by some $\tilde{Q}$-a.s. finite random variable. We deduce that

$$\limsup_{t\to\infty} E^{\tilde{Q}}\Big(M_t|\tilde{\mathcal{G}}_\infty\Big) = \limsup_{t\to\infty}\Big(spine(t) + sum(t)\Big) < \infty \quad \tilde{Q}\text{-a.s.}$$

Hence by the usual argument

$$\limsup_{t\to\infty} M_t < \infty \quad Q\text{-a.s.}$$

completing the proof of the theorem. □

The next theorem will be helpful in proving the Strong Law of Large Numbers in the last section.

Theorem 5.19. *For $p \in (1,2)$ $(M_t)_{t\geq 0}$ is L^p-convergent.*

Proof. We use similar proof as found in [19]. It is sufficient to show that $E\Big(M_t^p\Big)$ is bounded in t.

$$\begin{aligned} E\Big(M_t^p\Big) &= E\Big(M_t^{p-1}M_t\Big) = E^Q\Big(M_t^{p-1}\Big) = E^{\tilde{Q}}\Big(M_t^{p-1}\Big) \\ &= E^{\tilde{Q}}\Big(E^{\tilde{Q}}\Big(M_t^{p-1}|\tilde{\mathcal{G}}_\infty\Big)\Big). \end{aligned}$$

Then using the fact that for $a, b \geq 0$, $q \in (0,1)$ $(a+b)^q \leq a^q + b^q$ we see that

$$M_t^{p-1} \leq e^{\frac{\beta^2}{2}(p-1)t} \sum_{u\in N_t} e^{-\beta(p-1)|X_t^u|}.$$

And hence

$$\begin{aligned} E^{\tilde{Q}}\Big(M_t^{p-1}|\tilde{\mathcal{G}}_\infty\Big) &\leq \Big(E^{\tilde{Q}}\big(M_t|\tilde{\mathcal{G}}_\infty\big)\Big)^{p-1} \\ &\leq e^{-\frac{\beta^2}{2}(p-1)t-\beta(p-1)|\xi_t|} + \sum_{u<node_t(\xi)} e^{-\frac{\beta^2}{2}(p-1)S_u-\beta(p-1)|\xi_{S_u}|} \end{aligned}$$

using the spine decomposition. Then the same argument as in Theorem 5.18 completes the proof. □

5.4 Almost sure asymptotic growth of $|N_t|$

In this section we prove Theorem 5.9 saying that

$$\lim_{t\to\infty}\frac{\log|N_t|}{t} = \frac{1}{2}\beta^2 \quad P\text{-a.s.}$$

Proof of Theorem 5.9. Let us first prove the lower bound:

$$\liminf_{t\to\infty} \frac{\log|N_t|}{t} \ge \frac{1}{2}\beta^2 \quad P\text{-a.s.} \tag{5.10}$$

We observe that

$$M_t = \sum_{u\in N_t} \exp\Big\{-\beta|X_t^u| - \frac{1}{2}\beta^2 t\Big\} \le |N_t|e^{-\frac{1}{2}\beta^2 t}.$$

Hence

$$\begin{aligned} &\log M_t \le \log|N_t| - \frac{1}{2}\beta^2 t \\ \Rightarrow &\frac{\log|N_t|}{t} \ge \frac{1}{2}\beta^2 + \frac{\log M_t}{t} \\ \Rightarrow &\liminf_{t\to\infty} \frac{\log|N_t|}{t} \ge \frac{1}{2}\beta^2 \end{aligned}$$

using the fact that $\lim_{t\to\infty} M_t > 0$ P-a.s. proved in Theorem 5.18.

Let us now establish the upper bound:

$$\limsup_{t\to\infty} \frac{\log|N_t|}{t} \le \frac{1}{2}\beta^2 \quad P\text{-a.s.} \tag{5.11}$$

We first prove (5.11) on integer (or other lattice) times. Take $\epsilon > 0$. Then

$$P\big(|N_t|e^{-(\frac{1}{2}\beta^2+\epsilon)t} > \epsilon\big) \le \frac{E|N_t|e^{-(\frac{1}{2}\beta^2+\epsilon)t}}{\epsilon} \sim \frac{2}{\epsilon}e^{-\epsilon t}$$

using the Markov inequality and Theorem 5.7. So

$$\sum_{n=1}^{\infty} P\big(|N_n|e^{-(\frac{1}{2}\beta^2+\epsilon)n} > \epsilon\big) < \infty.$$

Thus by the Borel-Cantelli lemma

$$|N_n|e^{-(\frac{1}{2}\beta^2+\epsilon)n} \to 0 \ P\text{-a.s. as } n\to\infty.$$

Taking the logarithm we get

$$\big(-\frac{1}{2}\beta^2 - \epsilon\big)n + \log|N_n| \to -\infty.$$

Hence

$$\limsup_{n\to\infty} \frac{\log|N_n|}{n} \le \frac{1}{2}\beta^2 + \epsilon.$$

Taking the limit $\epsilon \to 0$ we get the desired result. To get the convergence over any real-valued sequence we note that $|N_t|$ is an increasing process and so

$$\begin{aligned}
&|N_t| \leq |N_{\lceil t \rceil}| \\
\Rightarrow &\frac{\log |N_t|}{t} \leq \frac{\lceil t \rceil}{t} \frac{\log |N_{\lceil t \rceil}|}{\lceil t \rceil} \\
\Rightarrow &\limsup_{t\to\infty} \frac{\log |N_t|}{t} \leq \limsup_{t\to\infty} \frac{\log |N_{\lceil t \rceil}|}{\lceil t \rceil} \leq \frac{1}{2}\beta^2.
\end{aligned}$$

(5.11) and (5.10) taken together prove Theorem 5.9. □

5.5 Almost sure asymptotic behaviour of $|N_t^{\lambda t}|$

In this section we prove Theorem 5.10. Namely, that

$$\frac{\log |N_t^{\lambda t}|}{t} \to \Delta_\lambda \ P\text{-a.s. if } \lambda < \frac{\beta}{2}$$

and

$$|N_t^{\lambda t}| \to 0 \ P\text{-a.s. if } \lambda > \frac{\beta}{2}.$$

Since the proof is quite long we break it down into two parts. In Subsection 5.5.1 we prove the upper bound and in Subsection 5.5.2 the lower bound. We also present the proofs of Lemma 5.12 and Corollary 5.11 in Subsections 5.5.3 and 5.5.4.

5.5.1 Upper bound

Lemma 5.20.

$$\limsup_{t\to\infty} \frac{\log |N_t^{\lambda t}|}{t} \leq \Delta_\lambda \quad P\text{-}a.s.$$

We start with the upper bound because it can be proved in a similar way that we proved the upper bound on $|N_t|$ (recall 5.11). The main difference comes from the fact that $(|N_t^{\lambda t}|)_{t\geq 0}$ is not an increasing process and so getting convergence along any real time sequence requires some extra work.

Proof. Take $\epsilon > 0$ and consider events

$$A_n = \Big\{ \sum_{u \in N_{n+1}} \mathbf{1}_{\{\sup_{s\in[n,n+1]} X_s^u \,\geq\, \lambda n\}} > e^{(\Delta_\lambda + \epsilon)n} \Big\}.$$

If we can show that $P(A_n)$ decays to 0 exponentially fast then by the Borel-Cantelli Lemma we would have $P(A_n \ i.o.) = 0$ and that would be sufficient to get the result.

By the Markov inequality and the Many-to-one theorem (Theorem 5.15) we have

$$
\begin{aligned}
P(A_n) &\le E\Big(\sum_{u\in N_{n+1}} \mathbf{1}_{\{\sup_{s\in[n,n+1]} X^u_s \,\ge\, \lambda n\}}\Big)e^{-(\Delta_\lambda+\epsilon)n}\\
&= \tilde{E}\Big(e^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{\sup_{s\in[n,n+1]}\xi_s \,\ge\, \lambda n\}}\Big)e^{-(\Delta_\lambda+\epsilon)n}\\
&= \tilde{E}\Big(e^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{\xi_{n+1}+\bar{\xi}_n \,\ge\, \lambda n\}}\Big)e^{-(\Delta_\lambda+\epsilon)n},
\end{aligned}
$$

where $\bar{\xi}_n := \sup_{s\in[n,n+1]}(\xi_s - \xi_{n+1})$ is a sequence of i.i.d. random variables $\overset{d}{=} \sup_{s\in[0,1]}\xi_s$ and $(\xi_t)_{t\ge 0}$ is a standard Brownian motion under $\tilde{P}$.

To give an upper bound on the expectation we split it according to whether $|\xi_{n+1}| < (\lambda-\delta)(n+1)$ or $\ge (\lambda-\delta)(n+1)$ for some small $\delta > 0$ to be chosen later.

$$
\begin{aligned}
&\tilde{E}\Big(e^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{\xi_{n+1}+\bar{\xi}_n \,\ge\, \lambda n\}}\Big)e^{-(\Delta_\lambda+\epsilon)n}\\
=&\underbrace{\tilde{E}\Big(e^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{\xi_{n+1}+\bar{\xi}_n \,\ge\, \lambda n\}}\mathbf{1}_{\{|\xi_{n+1}| \,>\, (\lambda-\delta)(n+1)\}}\Big)e^{-(\Delta_\lambda+\epsilon)n}}_{(1)}\\
+&\underbrace{\tilde{E}\Big(e^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{\xi_{n+1}+\bar{\xi}_n \,\ge\, \lambda n\}}\mathbf{1}_{\{|\xi_{n+1}| \,\le\, (\lambda-\delta)(n+1)\}}\Big)e^{-(\Delta_\lambda+\epsilon)n}}_{(2)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
(1) &\le \tilde{E}\Big(e^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{|\xi_{n+1}| \,>\, (\lambda-\delta)(n+1)\}}\Big)e^{-(\Delta_\lambda+\epsilon)n}\\
&= 2\tilde{E}\Big(e^{\beta\tilde{L}_{n+1}}\mathbf{1}_{\{\xi_{n+1} \,>\, (\lambda-\delta)(n+1)\}}\Big)e^{-(\Delta_\lambda+\epsilon)n}\\
&\approx e^{\Delta_{\lambda-\delta}n}e^{-(\Delta_\lambda+\epsilon)n},
\end{aligned}
$$

where we used Theorem 5.8 to estimate the expectation. This quantity decays exponentially fast for δ chosen small enough since Δ_λ is continuous in λ.

$$
\begin{aligned}
(2) &= E^{\tilde{Q}_\beta}\Big(e^{\beta|\xi_{n+1}|+\frac{1}{2}\beta^2(n+1)}\mathbf{1}_{\{\xi_{n+1}+\bar{\xi}_n \,\ge\, \lambda n\}}\mathbf{1}_{\{|\xi_{n+1}| \,\le\, (\lambda-\delta)(n+1)\}}\Big)e^{-(\Delta_\lambda+\epsilon)n}\\
&\lesssim E^{\tilde{Q}_\beta}\Big(\mathbf{1}_{\{\xi_{n+1}+\bar{\xi}_n \,\ge\, \lambda n\}}\mathbf{1}_{\{|\xi_{n+1}| \,\le\, (\lambda-\delta)(n+1)\}}\Big)e^{Kn}\\
&\le E^{\tilde{Q}_\beta}\Big(\mathbf{1}_{\{\bar{\xi}_n \,\ge\, \delta n+(\delta-\lambda)\}}\Big)e^{Kn}\\
&= \tilde{Q}_\beta\big(\bar{\xi}_1 \;\ge\; \delta n + (\delta-\lambda)\big)e^{Kn},
\end{aligned}
$$

where $K = \frac{1}{2}\beta^2 + \beta(\lambda-\delta) - (\Delta_\lambda+\epsilon)$. However $\tilde{Q}_\beta\big(\bar{\xi}_1 \;\ge\; \delta n + (\delta-\lambda)\big)$ decays faster than exponentially because for any $\theta > 0$, which we take to be large

$$
\tilde{Q}_\beta\big(\bar{\xi}_1 \;\ge\; \delta n\big) \le E^{\tilde{Q}_\beta}\Big(e^{\theta\bar{\xi}_1}\Big)e^{-\theta\delta n},
$$

but

$$E^{\tilde{Q}_\beta}\Big(e^{\theta\bar{\xi}_1}\Big) = \tilde{E}\Big(e^{\theta\bar{\xi}_1}e^{-\beta|\xi_1|+\beta\tilde{L}_1-\frac{1}{2}\beta^2}\Big) < \infty$$

by Cauchy-Schwarz inequality for example.

Thus we have shown that $P(A_n) = (1) + (2)$ decays exponentially fast. So by the Borel-Cantelli lemma $P(A_n \ i.o.) = 0$ and $P(A_n^c \ ev.) = 1$. That is,

$$\sum_{u\in N_{n+1}} \mathbf{1}_{\{\sup_{s\in[n,n+1]} X_s^u \ \geq \ \lambda n\}} \leq e^{(\Delta_\lambda+\epsilon)n} \text{ eventually.}$$

So there exists a P-almost surely finite time T_ϵ such that $\forall n > T_\epsilon$

$$\sum_{u\in N_{n+1}} \mathbf{1}_{\{\sup_{s\in[n,n+1]} X_s^u \ \geq \ \lambda n\}} \leq e^{(\Delta_\lambda+\epsilon)n}.$$

Then

$$\begin{aligned}
|N_t^{\lambda t}| &\leq \sum_{u\in N_{\lfloor t\rfloor+1}} \mathbf{1}_{\{\sup_{s\in[\lfloor t\rfloor,\ \lfloor t\rfloor+1]} X_s^u \ \geq \ \lambda\lfloor t\rfloor\}} \\
\Rightarrow |N_t^{\lambda t}| &\leq e^{(\Delta_\lambda+\epsilon)\lfloor t\rfloor} \qquad \text{for } t > T_\epsilon + 1,
\end{aligned}$$

which proves that

$$\limsup_{t\to\infty} \frac{\log|N_t^{\lambda t}|}{t} \leq \Delta_\lambda \quad P\text{-a.s.}$$

□

Remark 5.21. *Since $|N_t^{\lambda t}|$ takes only integer values we see that for $\lambda > \frac{\beta}{2}$ the inequality*

$$\limsup_{t\to\infty} \frac{\log|N_t^{\lambda t}|}{t} \leq \Delta_\lambda < 0$$

actually implies that $|N_t^{\lambda t}| \to 0$ P-a.s.

5.5.2 Lower bound

Before we present the proof of the lower bound of Theorem 5.10 let us give a heuristic argument, which this proof will be based upon.

Take $\lambda > 0$. Suppose we are given some large time t and we want to estimate the number of particles $u \in N_t$ such that $|X_t^u| > \lambda t$.

Let $p \in [0,1]$. At time pt the number of particles in the system is $|N_{pt}| \approx e^{\frac{1}{2}\beta^2 pt}$ by Theorem 5.9. If we ignore any branching that takes place in the time interval $(pt, t]$ then each of these particles will end up in the region $(-\infty, -\lambda t] \cup [\lambda t, \infty)$ at time t with probability $\gtrsim e^{-\frac{\lambda^2}{2(1-p)}t}$ using the standard estimate of the tail distribution of a normal random variable.

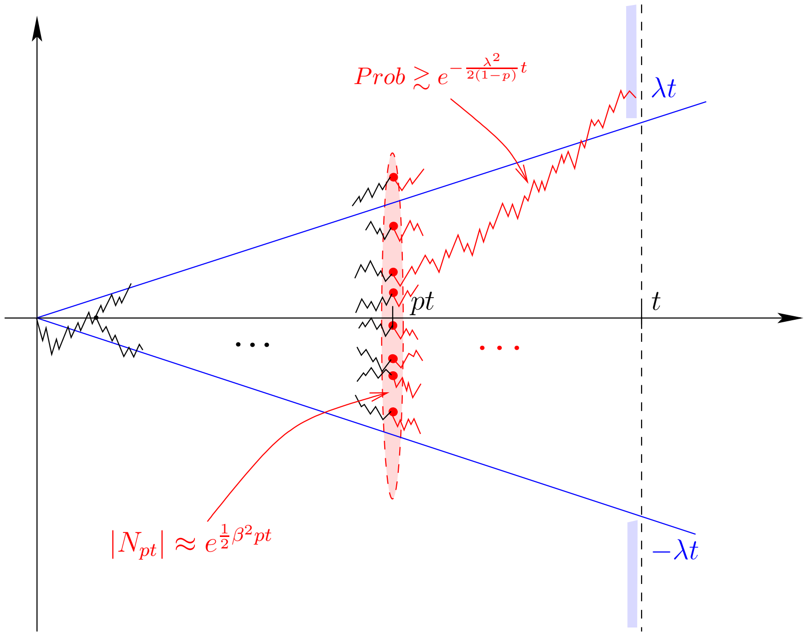


Figure 5-2: *Illustration for the lower bound of Theorem 5.10*

Thus a crude estimate gives us that the number of particles at time t in the region $(-\infty, -\lambda t] \cup [\lambda t, \infty)$ is

$$\gtrsim e^{-\frac{\lambda^2}{2(1-p)}t} \times |N_{pt}| \approx e^{-\frac{\lambda^2}{2(1-p)}t} \times e^{\frac{1}{2}\beta^2 pt}.$$

The value of p which maximises this expression is

$$p^* = \begin{cases} 0 & \text{if } \lambda \geq \beta \\ 1 - \frac{\lambda}{\beta} & \text{if } \lambda < \beta \end{cases}$$

and then

$$\frac{\log\left(e^{-\frac{\lambda^2}{2(1-p)}t} \times |N_{pt}|\right)}{t}\Bigg|_{p=p^*} \approx \begin{cases} -\frac{1}{2}\lambda^2 & \text{if } \lambda \geq \beta \\ \frac{1}{2}\beta^2 - \beta\lambda & \text{if } \lambda < \beta \end{cases} = \Delta_\lambda.$$

Let us now use this idea to give a formal proof of the following lemma.

Lemma 5.22. *Take $\lambda < \frac{\beta}{2}$. Then*

$$\liminf_{t\to\infty} \frac{\log |N_t^{\lambda t}|}{t} \geq \Delta_\lambda = \frac{1}{2}\beta^2 - \beta\lambda \quad P\text{-}a.s.$$

Proof. Take $p := 1 - \frac{\lambda}{\beta} \in (\frac{1}{2}, 1)$. For integer times n we shall consider particles alive at time pn (that is, particles in the set N_{pn}).

For each particle $u \in N_{pn}$ we can choose one descendant alive at time $n+1$. Let $\hat{N}_{n+1}$ be a set of such descendants (so that $|\hat{N}_{n+1}| = |N_{pn}|$).
Then, for $u \in \hat{N}_{n+1}$, paths $\big(X_t^u\big)_{t\in[pn,\ n+1]}$ correspond to independent Brownian motions (started at some unknown positions at time pn). Note that, wherever particle u is at time pn,

$$P\Big(|X_s^u| > \lambda s \ \forall s \in [n, n+1]\Big) \gtrsim e^{-\frac{\lambda^2}{2(1-p)}n} = e^{-\frac{1}{2}\beta\lambda n} =: q_n(\lambda)$$

using the tail estimate of the normal distribution. Take any small $\delta > 0$ to be specified later. Then by Theorem 5.9

$$|\hat{N}_{n+1}| = |N_{pn}| \geq e^{(\frac{1}{2}\beta^2 p - \delta)n} \text{ eventually.}$$

To prove Lemma 5.22 we take $\epsilon > 0$ and consider the events

$$B_n := \Big\{ \sum_{u\in\hat{N}_{n+1}} \mathbf{1}_{\{|X_s^u|>\lambda s \ \forall s\in[n,n+1]\}} < e^{(\Delta_\lambda - \epsilon)n} \Big\}.$$

We wish to show that $P(B_n \ i.o.) = 0$. Now,

$$\begin{aligned}
&P\Big(B_n \cap \big\{|\hat{N}_{n+1}| > e^{(\frac{1}{2}\beta^2 p-\delta)n}\big\}\Big) \\
=&P\Big(\Big\{ \sum_{u\in\hat{N}_{n+1}} \mathbf{1}_{\{|X_s^u|>\lambda s \ \forall s\in[n,n+1]\}} < e^{(\Delta_\lambda - \epsilon)n} \Big\} \cap \big\{|\hat{N}_{n+1}| > e^{(\frac{1}{2}\beta^2 p-\delta)n}\big\}\Big) \\
\leq&P\Bigg(\sum_{i=1}^{e^{(\frac{1}{2}\beta^2 p-\delta)n}} \mathbf{1}_{A_i} < e^{(\Delta_\lambda - \epsilon)n} \Bigg),
\end{aligned}$$

where A_i's are independent events with $P(A_i) \geq q_n(\lambda) \ \forall i$. Then

$$\begin{aligned}
P\Bigg(\sum_{i=1}^{e^{(\frac{1}{2}\beta^2 p-\delta)n}} \mathbf{1}_{A_i} < e^{(\Delta_\lambda - \epsilon)n} \Bigg) &= P\Big(e^{-\sum \mathbf{1}_{A_i}} > e^{-e^{(\Delta_\lambda-\epsilon)n}} \Big) \\
&\leq e^{e^{(\Delta_\lambda-\epsilon)n}} E\Big(e^{-\sum \mathbf{1}_{A_i}}\Big) \\
&= e^{e^{(\Delta_\lambda-\epsilon)n}} \prod_{i=1}^{e^{(\frac{1}{2}\beta^2 p-\delta)n}} E\Big(e^{-\mathbf{1}_{A_i}}\Big) \\
&\leq e^{e^{(\Delta_\lambda-\epsilon)n}} \prod \Big(1 - P(A_i)(1-e^{-1})\Big) \\
&\leq e^{e^{(\Delta_\lambda-\epsilon)n}} \prod \Big(1 - q_n(\lambda)(1-e^{-1})\Big)
\end{aligned}$$

$$\le e^{e^{(\Delta_\lambda-\epsilon)n}} \prod_{i=1}^{e^{(\frac{1}{2}\beta^2 p-\delta)n}} e^{-q_n(\lambda)(1-e^{-1})}$$

$$= \exp\Big\{e^{(\Delta_\lambda-\epsilon)n} - (1-e^{-1})q_n(\lambda)e^{(\frac{1}{2}\beta^2 p-\delta)n}\Big\}$$

$$= \exp\Big\{e^{(\Delta_\lambda-\epsilon)n} - (1-e^{-1})e^{(\Delta_\lambda-\delta)n}\Big\}.$$

This expression decays fast enough if we take $\delta < \epsilon$. Thus

$$P\Big(B_n \cap \big\{|\hat{N}_{n+1}| > e^{(\frac{1}{2}\beta^2 p-\delta)n}\big\} \text{ i.o.}\Big) = 0.$$

And since $P\Big(\big\{|\hat{N}_{n+1}| > e^{(\frac{1}{2}\beta^2 p-\delta)n}\big\} \text{ ev. }\Big) = 1$, we get that $P\Big(B_n \text{ i.o. }\Big) = 0$. That is,

$$\sum_{u\in\hat{N}_{n+1}} \mathbf{1}_{\{|X_s^u|>\lambda s\ \forall s\in[n,n+1]\}} \ge e^{(\Delta_\lambda-\epsilon)n} \text{ for } n \text{ large enough}$$

Now, since the process is symmetric, the probability that a particle $u \in \hat{N}_{n+1}$ such that $|X_s^u| > \lambda s\ \forall s \in [n, n+1]$ actually satisfies $X_s^u > \lambda s\ \forall s \in [n, n+1]$ is $\frac{1}{2}$. So applying the usual Borel-Cantelli argument once again we can for example prove that for some constant $C > 0$

$$\sum_{u\in\hat{N}_{n+1}} \mathbf{1}_{\{X_s^u>\lambda s\ \forall s\in[n,n+1]\}} \ge Ce^{(\Delta_\lambda-\epsilon)n} \text{ eventually}$$

Then

$$\sum_{u\in N_t} \mathbf{1}_{\{X_t^u>\lambda t\}} \ge \sum_{u\in\hat{N}_{\lfloor t\rfloor+1}} \mathbf{1}_{\{X_s^u>\lambda s\ \forall s\in[\lfloor t\rfloor,\lfloor t\rfloor+1]\}} \ge Ce^{(\Delta_\lambda-\epsilon)\lfloor t\rfloor}.$$

So for t large enough and some other constant C'

$$|N_t^{\lambda t}| \ge C'e^{(\Delta_\lambda-\epsilon)t}$$

and hence

$$\liminf_{t\to\infty} \frac{\log|N_t^{\lambda t}|}{t} \ge \Delta_\lambda.$$

□

Lemmas 5.20 and 5.22 together prove Theorem 5.10.

5.5.3 Decay of $P(|N_t^{\lambda t}| \ge 1)$ in the case $\lambda > \frac{\beta}{2}$

Theorem 5.10 told us that if $\lambda > \frac{\beta}{2}$ then $|N_t^{\lambda t}| \to 0$. Let us also prove that

$$\frac{\log P(|N_t^{\lambda t}| \ge 1)}{t} \to \Delta_\lambda = -\frac{1}{2}\lambda^2$$

in this case.

Proof of Lemma 5.12. Trivially

$$P(|N_t^{\lambda t}| \geq 1) \leq E|N_t^{\lambda t}|$$
$$\Rightarrow \limsup_{t\to\infty} \frac{\log P(|N_t^{\lambda t}| \geq 1)}{t} \leq \Delta_\lambda$$

by Theorem 5.8.

For the lower bound we use the same idea as in Lemma 5.22. Let us take

$$p = \begin{cases} 0 & \text{if } \lambda > \beta \\ 1 - \frac{\lambda}{\beta} & \text{if } \lambda \leq \beta \end{cases}$$

We define a set $\hat{N}_t$ as in Subsection 5.5.2 That is, for each particle $u \in N_{pt}$ we choose one descendent alive at time t (so that $\hat{N}_t \subset N_t$, $|\hat{N}_t| = |N_{pt}|$). Then for each $u \in \hat{N}_t$ wherever it is at time pt we have

$$P(|X_t^u| > \lambda t) \gtrsim e^{-\frac{\lambda^2}{2(1-p)}t} =: p_t(\lambda).$$

Then

$$P\Big(|N_t^{\lambda t}| \geq 1\Big) \geq \frac{1}{2} P\Big(|N_t^{\pm\lambda t}| \geq 1\Big),$$

where $N_t^{\pm\lambda t} := \{u \in N_t : |X_t^u| > \lambda t\}$. And thus for some $\delta > 0$ to be chosen later we have, ignoring any multiplicative constants of P,

$$\begin{aligned} P\Big(|N_t^{\lambda t}| \geq 1\Big) &\geq \frac{1}{2} P\Big(|N_t^{\pm\lambda t}| \geq 1,\ |N_{pt}| > \underbrace{e^{(\frac{1}{2}\beta^2 p - \delta)t}}_{:= n_t(\delta)}\Big) \\ &\geq P\Big(\bigcup_{u \in \hat{N}_t} \{|X_t^u| > \lambda t\},\ |\hat{N}_t| > n_t(\delta)\Big) \\ &\geq \Big(1 - \big(1 - p_t(\lambda)\big)^{n_t(\delta)}\Big) P\big(|\hat{N}_t| > n_t(\delta)\big). \end{aligned}$$

$P\big(|\hat{N}_t| > n_t(\delta)\big) \to 1$, so we can just ignore it. And

$$\begin{aligned} &\Big(1 - \big(1 - p_t(\lambda)\big)^{n_t(\delta)}\Big) \\ =& n_t(\delta) p_t(\lambda) - \binom{n_t(\delta)}{2} p_t(\lambda)^2 + \binom{n_t(\delta)}{3} p_t(\lambda)^3 - \cdots \\ \geq& n_t(\delta) p_t(\lambda) - n_t(\delta)^2 p_t(\lambda)^2 \Big(1 + n_t(\delta) p_t(\lambda) + n_t(\delta)^2 p_t(\lambda)^2 + \cdots\Big). \end{aligned}$$

Note that for δ small enough

$$n_t(\delta) p_t(\lambda) = e^{(\frac{1}{2}\beta^2 p - \delta)t} e^{-\frac{\lambda^2}{2(1-p)}t} = e^{(\Delta_\lambda - \delta)t} \ll 1.$$

Hence

$$P(|N_t^{\lambda t}| \geq 1) \geq e^{(\Delta_\lambda - \delta)t} + o\big(e^{(\Delta_\lambda - \delta)t}\big).$$

Therefore

$$\liminf_{t\to\infty} \frac{\log P(|N_t^{\lambda t}| \leq 1)}{t} \geq \Delta_\lambda.$$

This completes the proof of Lemma 5.12. □

5.5.4 The rightmost particle

As it was observed earlier (in Remark 5.17), the number of particles above the line λt grows exponentially if $\lambda < \frac{\beta}{2}$ and decays exponentially if $\lambda > \frac{\beta}{2}$. As a corollary of Theorem 5.10 we get that

$$\frac{R_t}{t} \to \frac{\beta}{2} \quad P\text{-a.s.},$$

where $(R_t)_{t\geq 0}$ is the rightmost particle of the branching process.

Proof of Corollary 5.11. Take $\lambda < \frac{\beta}{2}$. By Theorem 5.10 $|N_t^{\lambda t}| \geq 1$ $\forall t$ large enough, so $R_t \geq \lambda t$ for t large enough. Thus

$$\liminf_{t\to\infty} \frac{R_t}{t} \geq \lambda \quad P\text{-a.s.}$$

Letting $\lambda \nearrow \frac{\beta}{2}$ we get

$$\liminf_{t\to\infty} \frac{R_t}{t} \geq \frac{\beta}{2} \quad P\text{-a.s.}$$

Similarly, if we take $\lambda > \frac{\beta}{2}$ then by Theorem 5.10 $|N_t^{\lambda t}| = 0$ $\forall t$ large enough and so $R_t \leq \lambda t$ for t large enough. Hence

$$\limsup_{t\to\infty} \frac{R_t}{t} \leq \lambda \quad P\text{-a.s.}$$

So, letting $\lambda \searrow \frac{\beta}{2}$ we get

$$\limsup_{t\to\infty} \frac{R_t}{t} \leq \frac{\beta}{2} \quad P\text{-a.s.}$$

and this proves Corollary 5.11. □

Note that the rightmost particle (i.e. the extremal particle) in our model behaves very differently from the rightmost particle in the model with homogeneous branching. Figure 5-3 below illustrates the difference.

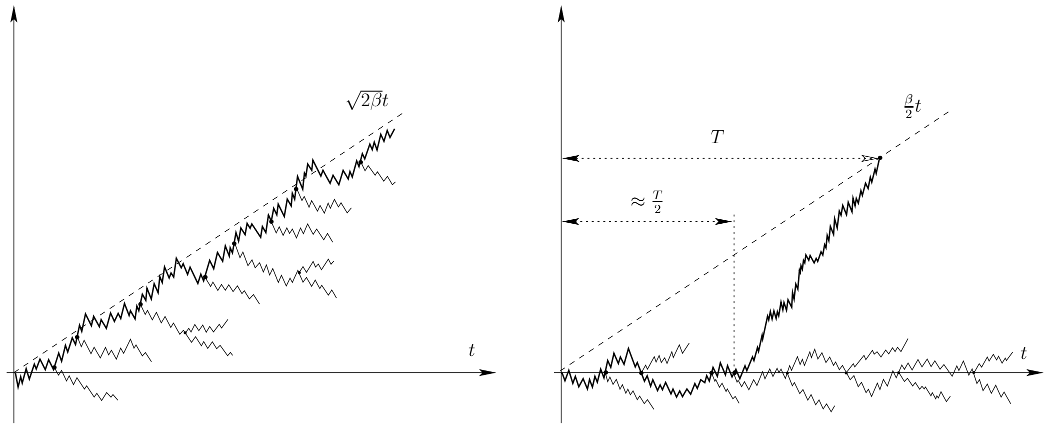


Figure 5-3: *Rightmost particle in models with homogeneous branching and branching at the origin*

On the left we see a branching Brownian motion with constant branching rate β. In such model with probability 1 there is a particle staying near the critical line $\sqrt{2\beta}t$ all the time. (Here the word particle is a bit ambiguous since we are really talking about an infinite line of descent, but this is a common description.)

On the right we see a BBM with branching rate $\beta\delta_0(x)$. Note that since branching is only allowed at the origin, no particle can stay close to a straight line λt, $\lambda > 0$ all the time. The optimal way for some particle to reach the critical line $\frac{\beta}{2}t$ at time T is to wait near the origin until the time $\frac{T}{2}$ in order to give birth to as many particles as possible, and then at time $\frac{T}{2}$ one of $\approx e^{\frac{\beta^2}{4}T}$ particles will have a good chance of reaching $\frac{\beta}{2}T$ at time T.

5.6 Strong law of large numbers

Recall the additive martingale $M_t = e^{-\frac{\beta^2}{2}t}\sum_{u\in N_t} e^{-\beta|X_t^u|}$, $t \geq 0$ from Section 5.3 and the measure $\pi(\mathrm{d}x) = \beta e^{-2\beta|x|}\mathrm{d}x$ from Proposition 5.14.

In this section we shall prove Theorem 5.13 which says that for a measurable bounded function $f(\cdot)$

$$\begin{aligned}\lim_{t\to\infty} e^{-\frac{\beta^2}{2}t}\sum_{u\in N_t} f(X_t^u) &= M_\infty \int_{-\infty}^{\infty} f(x)\beta e^{-\beta|x|}\mathrm{d}x \\ &= M_\infty \int_{-\infty}^{\infty} f(x)e^{\beta|x|}\pi(\mathrm{d}x) \ P\text{-a.s.} \qquad (5.12)\end{aligned}$$

Remark 5.23. *Observe that*

$$\begin{aligned}E\Big(e^{-\frac{\beta^2}{2}t}\sum_{u\in N_t} f(X_t^u)\Big) &= \tilde{E}\Big(e^{-\frac{\beta^2}{2}t} f(\xi_t)e^{\beta\tilde{L}_t}\Big)\\ &= \tilde{E}\Big(f(\xi_t)e^{\beta|\xi_t|}\big(e^{-\beta|\xi_t|+\beta\tilde{L}_t-\frac{\beta^2}{2}t}\big)\Big)\\ &= E^{\tilde{Q}_\beta}\Big(f(\xi_t)e^{\beta|\xi_t|}\Big) \to \int f(x)e^{\beta|x|}\pi(\mathrm{d}x)\end{aligned}$$

and that $(M_{t\geq 0})$ *is UI with* $EM_\infty = 1$.

Corollary 5.24. *Taking* $f(\cdot)\equiv 1$ *we get*

$$|N_t|e^{-\frac{1}{2}\beta^2 t} \to 2M_\infty \quad P\text{-a.s.} \tag{5.13}$$

This should be compared with results in Lemma 5.7 and Theorem 5.9.

Corollary 5.25.

$$\frac{\sum_{u\in N_t} f(X_t^u)}{|N_t|} \to \frac{1}{2}\int f(x)e^{\beta|x|}\pi(\mathrm{d}x) = \frac{\beta}{2}\int f(x)e^{-\beta|x|}\mathrm{d}x \quad P\text{-a.s.}$$

Proof. Dividing (5.12) by (5.13) gives the required result. □

The Strong Law of Large Numbers was proved in [16] for a large class of general diffusion processes and branching rates $\beta(x)$. In our case the branching rate is a generalised function $\beta\delta_0(x)$, which doesn't satisfy the conditions of [16]. Nevertheless we can adapt the proof to our model if we take the generalised principal eigenvalue $\lambda_c = \frac{\beta^2}{2}$ and eigenfunctions $\phi(x) = e^{-\beta|x|}$, $\tilde{\phi}(x) = \beta e^{-\beta|x|}$ in [16].

In the rest of this section we present the proof of Theorem 5.13. We only need to consider functions of the form $f(x) = e^{-\beta|x|}\mathbf{1}_{\{x\in B\}}$ for measurable sets B. After we prove the result for such functions we can derive the general result by approximating a general function with linear combinations of functions of the above form.

Proof of Theorem 5.13. Take B a bounded measurable set and for this set B let

$$U_t := e^{-\frac{\beta^2}{2}t}\sum_{u\in N_t} e^{-\beta|X_t^u|}\mathbf{1}_{\{X_t^u\in B\}} = e^{-\frac{\beta^2}{2}t}\sum_{u\in N_t} f(X_t^u)$$

So for example if $B=\mathbb{R}$ then $U_t = M_t$ and generally $U_t \leq M_t$. We wish to show that

$$U_t \to \pi(B)M_\infty \ \Big(= \int f(x)e^{\beta|x|}\pi(\mathrm{d}x)M_\infty\Big) \text{ as } t\to\infty.$$

The proof can be split into three parts.

Part I:

Let us take $K > 0$. At this stage it doesn't matter what K is, but in Part II of the proof we shall choose an appropriate value for it. Let $m_n := Kn$ (using the same notation as in [16]). Also fix $\delta > 0$. We first want to prove that

$$\lim_{n\to\infty} \Big| U_{(n+m_n)\delta} - E\Big(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta}\Big)\Big| = 0 \quad P\text{-a.s.} \tag{5.14}$$

We begin with the observation that

$$\forall s,t \geq 0 \qquad U_{s+t} = \sum_{u\in N_t} e^{-\frac{\beta^2}{2}t} U_s^{(u)}, \tag{5.15}$$

where conditional on $\mathcal{F}_t$, $U_s^{(u)}$ are independent copies of U_s started from X_t^u.

To prove (5.14) using the Borel-Cantelli lemma we need to show that for all $\epsilon > 0$

$$\sum_{n=1}^{\infty} P\Big(\Big|U_{(n+m_n)\delta} - E\Big(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta}\Big)\Big| > \epsilon\Big) < \infty. \tag{5.16}$$

Let us take any $p \in (1,2)$.Then

$$\begin{aligned}
&P\Big(\Big|U_{(n+m_n)\delta} - E\Big(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta}\Big)\Big| > \epsilon\Big) \\
\leq &\frac{1}{\epsilon^p} E\Big(\Big|U_{(n+m_n)\delta} - E\Big(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta}\Big)\Big|^p\Big).
\end{aligned}$$

Next we shall apply the following inequality, which was used in the proof of the SLLN in [16] and can also be found in [5]: if $p \in (1,2)$ and X_i are independent random variables with $\mathbb{E}X_i = 0$ (or they are martingale differences), then

$$\mathbb{E}\Big|\sum_{i=1}^n X_i\Big|^p \leq 2^p \sum_{i=1}^n \mathbb{E}|X_i|^p. \tag{5.17}$$

Then by (5.15)

$$U_{s+t} - E\Big(U_{s+t}|\mathcal{F}_t\Big) = \sum_{u\in N_t} e^{-\frac{\beta^2}{2}t}\Big(U_s^{(u)} - E\big(U_s^{(u)}|\mathcal{F}_t\big)\Big),$$

where conditional on $\mathcal{F}_t$, $U_s^{(u)} - E\big(U_s^{(u)}|\mathcal{F}_t\big)$ are independent with 0 mean. Thus applying (5.17) and Jensen's inequality we get

$$\begin{aligned}
&E\Big(\Big|U_{s+t} - E\Big(U_{s+t}|\mathcal{F}_t\Big)\Big|^p|\mathcal{F}_t\Big) \\
\leq &2^p e^{-p\frac{\beta^2}{2}t} \sum_{u\in N_t} E\Big(\Big|U_s^{(u)} - E(U_s^{(u)}|\mathcal{F}_t)\Big|^p|\mathcal{F}_t\Big) \\
\leq &2^p e^{-p\frac{\beta^2}{2}t} \sum_{u\in N_t} E\Big(2^{p-1}\Big(\big|U_s^{(u)}\big|^p + \big|E(U_s^{(u)}|\mathcal{F}_t)\big|^p\Big)|\mathcal{F}_t\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq 2^p e^{-p\frac{\beta^2}{2}t} \sum_{u\in N_t} E\Big(2^{p-1}\Big(\big|U_s^{(u)}\big|^p + E\big(\big|U_s^{(u)}\big|^p|\mathcal{F}_t\big)\Big)|\mathcal{F}_t\Big) \\
&= 2^{2p} e^{-p\frac{\beta^2}{2}t} \sum_{u\in N_t} E\Big(\big|U_s^{(u)}\big|^p|\mathcal{F}_t\Big).
\end{aligned} \tag{5.18}
$$

Hence by (5.18)

$$
\begin{aligned}
&\sum_{n=1}^{\infty} E\Big(\Big|U_{(n+m_n)\delta} - E\Big(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta}\Big)\Big|^p\Big) \\
\leq& 2^{2p} \sum_{n=1}^{\infty} e^{-p\frac{\beta^2}{2}\delta n} E\Big(\sum_{u\in N_{\delta n}} E^{X^u_{\delta n}}\big(U_{m_n\delta}\big)^p\Big) \\
\leq& 2^{2p} \sum_{n=1}^{\infty} e^{-p\frac{\beta^2}{2}\delta n} E\Big(\sum_{u\in N_{\delta n}} E^{X^u_{\delta n}}\big(M_{m_n\delta}\big)^p\Big) \\
=& 2^{2p} \sum_{n=1}^{\infty} e^{-p\frac{\beta^2}{2}\delta n} E\Big(\sum_{u\in N_{\delta n}} e^{-\beta p|X^u_{\delta n}|} E_0\big(M_{m_n\delta}\big)^p\Big) \\
\leq& \sum_{n=1}^{\infty} e^{-p\frac{\beta^2}{2}\delta n} e^{\frac{\beta^2}{2}\delta n} \times C,
\end{aligned}
$$

where C is some positive constant and we have used the Many-to-One Lemma (Lemma 5.16) and and Theorem 5.19 in the last inequality. Since $p > 1$ the sum is $< \infty$. This finishes the proof of (5.16) and hence (5.14).

<u>Part II</u>:

Let us now prove that

$$
\lim_{n\to\infty} \Big|E\Big(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta}\Big) - \pi(B)M_\infty\Big| = 0 \quad P\text{-a.s.} \tag{5.19}
$$

Together with (5.14) this will complete the proof of Theorem 5.13 along lattice times for functions $f(x)$ of the form $e^{-\beta|x|}\mathbf{1}_{\{x\in B\}}$.

We begin by noting that

$$
\begin{aligned}
E\Big(U_{s+t}|\mathcal{F}_t\Big) &= E\Big(\sum_{u\in N_t} e^{-\frac{\beta^2}{2}t} U_s^{(u)}|\mathcal{F}_t\Big) \\
&= \sum_{u\in N_t} e^{-\frac{\beta^2}{2}t} E^{X^u_t} U_s \\
&= \sum_{u\in N_t} e^{-\frac{\beta^2}{2}t} E^{X^u_t}\Big(\sum_{u\in N_s} e^{-\frac{\beta^2}{2}s-\beta|X^u_s|}\mathbf{1}_{\{X^u_s\in B\}}\Big) \\
&= \sum_{u\in N_t} e^{-\frac{\beta^2}{2}t} \tilde{E}^{X^u_t}\Big(e^{-\frac{\beta^2}{2}s-\beta|\xi_s|}\mathbf{1}_{\{\xi_s\in B\}} e^{\beta\tilde{L}_s}\Big)
\end{aligned}
$$

$$= \sum_{u\in N_t} e^{-\frac{\beta^2}{2}t-\beta|X_t^u|} E^{\tilde{Q}_\beta^{X_t^u}}\Big(\mathbf{1}_{\{\xi_s\in B\}}\Big)$$
$$= \sum_{u\in N_t} e^{-\frac{\beta^2}{2}t-\beta|X_t^u|} \int_B p(s, X_t^u, y)m(\mathrm{d}y),$$

where $\tilde{Q}_\beta$ and $p(\cdot)$ were defined in (5.6) and Proposition 5.14. Thus

$$E\Big(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta}\Big) = \sum_{u\in N_{n\delta}} e^{-\frac{\beta^2}{2}n\delta-\beta|X_{n\delta}^u|} \int_B p(m_n\delta, X_{n\delta}^u, y)m(\mathrm{d}y). \tag{5.20}$$

Recalling that $m_n = Kn$ where $K > 0$ we have

$$E\Big(U_{(n+m_n)\delta}|\mathcal{F}_{n\delta}\Big) = \sum_{u\in N_{n\delta}} e^{-\frac{\beta^2}{2}n\delta-\beta|X_{n\delta}^u|} \int_B p(Kn\delta, X_{n\delta}^u, y)m(\mathrm{d}y).$$

Now choose $M > \frac{\beta}{2}$ and consider events

$$C_n := \Big\{|X_{n\delta}^u| < Mn\delta \quad \forall u \in N_{n\delta}\Big\}.$$

Then

$$\sum_{u\in N_{n\delta}} e^{-\frac{\beta^2}{2}n\delta-\beta|X_{n\delta}^u|} \int_B p(Kn\delta, X_{n\delta}^u, y)m(\mathrm{d}y)$$
$$= \sum_{u\in N_{n\delta}} e^{-\frac{\beta^2}{2}n\delta-\beta|X_{n\delta}^u|} \int_B p(Kn\delta, X_{n\delta}^u, y)m(\mathrm{d}y)\ \mathbf{1}_{C_n^c}$$
$$+ \sum_{u\in N_{n\delta}} e^{-\frac{\beta^2}{2}n\delta-\beta|X_{n\delta}^u|} \int_B p(Kn\delta, X_{n\delta}^u, y)m(\mathrm{d}y)\ \mathbf{1}_{C_n}.$$

The first sum is 0 for n large enough by Corollary 5.11 (or even earlier by Theorem 5.10). To deal with the second sum we substitute the known transition density $p(\cdot)$:

$$\int_B p(Kn\delta, X_{n\delta}^u, y)m(\mathrm{d}y)\ \mathbf{1}_{C_n}$$
$$= \int_B \frac{1}{\sqrt{2\pi Kn\delta}} \exp\Big\{\beta\Big(|X_{n\delta}^u| - |y|\Big) - \frac{\beta^2}{2}Kn\delta - \frac{(X_{n\delta}^u - y)^2}{2Kn\delta}\Big\}$$
$$+ \frac{\beta}{2}Erfc\Big(\frac{|X_{n\delta}^u| + |y| - \beta Kn\delta}{\sqrt{2Kn\delta}}\Big)e^{-2\beta|y|}\ \mathrm{d}y\ \mathbf{1}_{C_n}.$$

Then for any given $M > \frac{\beta}{2}$ we can choose $K > \frac{2M}{\beta}$ and hence

$$\int_B \frac{1}{\sqrt{2\pi Kn\delta}} \exp\Big\{\beta\Big(|X_{n\delta}^u| - |y|\Big) - \frac{\beta^2}{2}Kn\delta - \frac{(X_{n\delta}^u - y)^2}{2Kn\delta}\Big\}\ \mathrm{d}y\ \mathbf{1}_{C_n}$$
$$\leq \exp\Big\{\Big(\beta M - \frac{\beta^2}{2}K\Big)n\delta\Big\} \times C' \ \to 0 \text{ as } n \to \infty,$$

where C' is some positive constant and

$$\int_B \frac{\beta}{2} Erfc\Big(\frac{|X^u_{n\delta}| + |y| - \beta K n\delta}{\sqrt{2Kn\delta}}\Big) e^{-2\beta|y|} \, \mathrm{d}y \, \mathbf{1}_{C_n} \to \int_B \beta e^{-2\beta|y|} \mathrm{d}y = \pi(B) \quad \text{as } n \to \infty$$

since $Erfc(x) \to 2$ as $x \to -\infty$ and $\mathbf{1}_{C_n} \to 1$ as $n \to \infty$. Then going back to (5.20) and we see that

$$\lim_{n\to\infty} \Big| E\Big(U_{(n+m_n)\delta} | \mathcal{F}_{n\delta}\Big) - \pi(B) M_{n\delta} \Big| = 0 \quad P\text{-a.s.}$$

and so also

$$\lim_{n\to\infty} \Big| E\Big(U_{(n+m_n)\delta} | \mathcal{F}_{n\delta}\Big) - \pi(B) M_{\infty} \Big| = 0 \quad P\text{-a.s.}$$

As it was mentioned earlier parts I and II together complete the proof of Theorem 5.13 along lattice times for functions of the form $f(x) = e^{-\beta|x|}\mathbf{1}_B(x)$. To see this put together (5.14) and (5.19) to get that

$$\lim_{n\to\infty} \Big| U_{(n+m_n)\delta} - \pi(B) M_{\infty} \Big| = 0 \quad P\text{-a.s.}$$

That is,

$$\lim_{n\to\infty} \Big| U_{n(K+1)\delta} - \pi(B) M_{\infty} \Big| = 0 \quad P\text{-a.s.}$$

Then $K+1$ can be absorbed into δ which stayed arbitrary throughout the proof. Also as it was mentioned earlier we can easily replace functions of the form $e^{-\beta|x|}\mathbf{1}_B(x)$ with any measurable functions. To see this we observe that given any meausurable set A and $\epsilon_1 > 0$ we can find constants $\underline{c}_1, \dots, \underline{c}_n, \bar{c}_1, \dots, \bar{c}_n$ and measurable sets $A_1, \dots, A_n$ such that

$$\Big(\sum_{i=1}^n \bar{c}_i \mathbf{1}_{A_i}(x) e^{-\beta|x|}\Big) - \epsilon_1 \le \mathbf{1}_A(x) \le \sum_{i=1}^n \bar{c}_i \mathbf{1}_{A_i}(x) e^{-\beta|x|}$$

and

$$\sum_{i=1}^n \underline{c}_i \mathbf{1}_{A_i}(x) e^{-\beta|x|} \le \mathbf{1}_A(x) \le \Big(\sum_{i=1}^n \underline{c}_i \mathbf{1}_{A_i}(x) e^{-\beta|x|}\Big) + \epsilon_1.$$

Similarly given any positive bounded measurable function f and $\epsilon_2 > 0$ we can find simple functions $\underline{f}$ and $\bar{f}$ such that

$$\bar{f}(x) - \epsilon_2 \le f(x) \le \bar{f}(x)$$

and

$$\underline{f}(x) \le f(x) \le \underline{f}(x) + \epsilon_2.$$

Thus given any positive bounded measurable function f and $\epsilon > 0$ we can find functions $\bar{f}^\epsilon(x)$ and $\underline{f}^\epsilon(x)$, which are linear combinations of functions of the form $e^{-\beta|x|}\mathbf{1}_A(x)$ such that

$$\bar{f}^\epsilon(x) - \epsilon \leq f(x) \leq \bar{f}^\epsilon(x)$$

and

$$\underline{f}^\epsilon(x) \leq f(x) \leq \underline{f}^\epsilon(x) + \epsilon.$$

Then

$$\begin{aligned} &\bar{f}^\epsilon(x)\beta e^{-\beta|x|} \leq (f(x)+\epsilon)\beta e^{-\beta|x|} \\ \Rightarrow &\int_{-\infty}^{\infty} \bar{f}^\epsilon(x)\beta e^{-\beta|x|}\mathrm{d}x \leq \int_{-\infty}^{\infty} f(x)\beta e^{-\beta|x|}\mathrm{d}x + 2\epsilon \end{aligned}$$

and hence P-almost surely we have

$$\begin{aligned} \limsup_{n\to\infty} e^{-\frac{\beta^2}{2}n\delta} \sum_{u\in N_{n\delta}} f(X^u_{n\delta}) &\leq \limsup_{n\to\infty} e^{-\frac{\beta^2}{2}n\delta} \sum_{u\in N_{n\delta}} \bar{f}^\epsilon(X^u_{n\delta}) \\ &= M_\infty \int_{-\infty}^{\infty} \bar{f}^\epsilon(x)\beta e^{-\beta|x|}\mathrm{d}x \\ &\leq M_\infty \Big(\int_{-\infty}^{\infty} f(x)\beta e^{-\beta|x|}\mathrm{d}x + 2\epsilon \Big). \end{aligned}$$

Since ϵ is arbitrary we get

$$\limsup_{n\to\infty} e^{-\frac{\beta^2}{2}n\delta} \sum_{u\in N_{n\delta}} f(X^u_{n\delta}) \leq M_\infty \int_{-\infty}^{\infty} f(x)\beta e^{-\beta|x|}\mathrm{d}x.$$

Similarly

$$\liminf_{n\to\infty} e^{-\frac{\beta^2}{2}n\delta} \sum_{u\in N_{n\delta}} f(X^u_{n\delta}) \geq M_\infty \int_{-\infty}^{\infty} f(x)\beta e^{-\beta|x|}\mathrm{d}x.$$

Also any bounded measurable function f can be written as a difference of two positive bounded measurable functions. This completes the proof of Theorem 5.13 with the limit taken along lattice times. Now let us finish the proof of the theorem by extending it to the continuous-time limit.

Part III:

As in the previous parts of the proof it is sufficient to consider functions of the form $f(x) = e^{-\beta|x|}\mathbf{1}_B(x)$ for measurable sets B.

Let us now take $\epsilon > 0$ and define the following interval

$$B^\epsilon(x) := B \cap \Big(-|x| - \frac{1}{\beta}\log(1+\epsilon),\ |x| + \frac{1}{\beta}\log(1+\epsilon)\Big).$$

Note that $y \in B^\epsilon(x)$ iff $y \in B$ and $e^{-\beta|y|} > \frac{e^{-\beta|x|}}{1+\epsilon}$. Furthermore, for $\delta, \epsilon > 0$ let

$$\Xi^{\delta,\epsilon}_B(x) := \mathbf{1}_{\{X^u_s \in B^\epsilon(x)\ \forall s\in[0,\delta]\ \forall u\in N_\delta\}}$$

and

$$\xi_B^{\delta,\epsilon}(x) := E^x\Big(\Xi_B^{\delta,\epsilon}(x)\Big).$$

Then for $t \in\ [n\delta,\ (n+1)\delta]$

$$\begin{aligned}U_t &= e^{-\frac{\beta^2}{2}t}\sum_{u\in N_t} e^{-\beta|X_t^u|}\mathbf{1}_{\{X_t^u\in B\}}\\ &= \sum_{u\in N_{n\delta}} e^{-\frac{\beta^2}{2}n\delta}U_{t-n\delta}^{(u)} \geq e^{-\frac{\beta^2}{2}n\delta}\sum_{u\in N_{n\delta}} U_{t-n\delta}^{(u)}\ \Xi_B^{\delta,\epsilon}(X_{n\delta}^u)\\ &\geq e^{-\frac{\beta^2}{2}n\delta}\sum_{u\in N_{n\delta}} e^{-\frac{\beta^2}{2}\delta}\ \frac{e^{-\beta|X_{n\delta}^u|}}{1+\epsilon}\ \Xi_B^{\delta,\epsilon}(X_{n\delta}^u) \qquad (5.21)\end{aligned}$$

because at time t there is at least one descendent of each particle alive at time $n\delta$. Let us consider the sum

$$e^{-\frac{\beta^2}{2}n\delta}\sum_{u\in N_{n\delta}} e^{-\beta|X_{n\delta}^u|}\Xi_B^{\delta,\epsilon}(X_{n\delta}^u).$$

Note that

$$\Xi_B^{\delta,\epsilon}(X_{n\delta}^u) \text{ are independent conditional on } \mathcal{F}_{n\delta}, \qquad (5.22)$$

$$E\Big(e^{-\frac{\beta^2}{2}n\delta}\sum_{u\in N_{n\delta}} e^{-\beta|X_{n\delta}^u|}\Xi_B^{\delta,\epsilon}(X_{n\delta}^u)\Big|\mathcal{F}_{n\delta}\Big) = e^{-\frac{\beta^2}{2}n\delta}\sum_{u\in N_{n\delta}} e^{-\beta|X_{n\delta}^u|}\xi_B^{\delta,\epsilon}(X_{n\delta}^u), \qquad (5.23)$$

and

$$\lim_{n\to\infty} e^{-\frac{\beta^2}{2}n\delta}\sum_{u\in N_{n\delta}} e^{-\beta|X_{n\delta}^u|}\xi_B^{\delta,\epsilon}(X_{n\delta}^u) = \int \xi_B^{\delta,\epsilon}(x)\pi(\mathrm{d}x)M_\infty. \qquad (5.24)$$

The last equation follows from the SLLN along lattice times which we already proved. Also we should point out that if we further let $\delta\to 0$, $\xi_B^{\delta,\epsilon}(x)$ will converge to $\mathbf{1}_B(x)$ and (5.24) will converge to $\pi(B)M_\infty$. Our next step then is to show that

$$\lim_{n\to\infty}\Big|e^{-\frac{\beta^2}{2}n\delta}\sum_{u\in N_{n\delta}} e^{-\beta|X_{n\delta}^u|}\Xi_B^{\delta,\epsilon}(X_{n\delta}^u) - e^{-\frac{\beta^2}{2}n\delta}\sum_{u\in N_{n\delta}} e^{-\beta|X_{n\delta}^u|}\xi_B^{\delta,\epsilon}(X_{n\delta}^u)\Big| = 0. \qquad (5.25)$$

In view of (5.22) and (5.23) we prove this using the method of Part I. That is, we exploit the Borel-Cantelli Lemma and in order to do that we need to show that for some $p\in(1,2)$

$$\begin{aligned}\sum_{n=1}^{\infty} E\Bigg(\Big|&e^{-\frac{\beta^2}{2}n\delta}\sum_{u\in N_{n\delta}} e^{-\beta|X_{n\delta}^u|}\Xi_B^{\delta,\epsilon}(X_{n\delta}^u)\\ &- E\Big(e^{-\frac{\beta^2}{2}n\delta}\sum_{u\in N_{n\delta}} e^{-\beta|X_{n\delta}^u|}\Xi_B^{\delta,\epsilon}(X_{n\delta}^u)\Big|\mathcal{F}_{n\delta}\Big)\Big|^p\Bigg) < \infty.\end{aligned}$$

A similar argument to the one used in Part I (see (5.18) gives us that

$$\sum_{n=1}^{\infty} E\bigg(\Big|e^{-\frac{\beta^2}{2}n\delta} \sum_{u\in N_{n\delta}} e^{-\beta|X^u_{n\delta}|}\Xi^{\delta,\epsilon}_B(X^u_{n\delta}) \\ - E\Big(e^{-\frac{\beta^2}{2}n\delta} \sum_{u\in N_{n\delta}} e^{-\beta|X^u_{n\delta}|}\Xi^{\delta,\epsilon}_B(X^u_{n\delta})\Big|\mathcal{F}_{n\delta}\Big)\Big|^p\bigg) \\ \leq \sum_{n=1}^{\infty} 2^{2p} e^{-p\frac{\beta^2}{2}n\delta} E\Big(\sum_{u\in N_{n\delta}} e^{-\beta p|X^u_{n\delta}|}\xi^{\delta,\epsilon}_B(X^u_{n\delta})\Big),$$

where $\Xi^{\delta,\epsilon}_B(X^u_{n\delta})$ is an indicator function and therefore raising it to the power p leaves it unchanged. Using once again the Many-to-One Lemma and the usual change of measure we have

$$\sum_{n=1}^{\infty} 2^{2p} e^{-p\frac{\beta^2}{2}n\delta} E\Big(\sum_{u\in N_{n\delta}} e^{-\beta p|X^u_{n\delta}|}\xi^{\delta,\epsilon}_B(X^u_{n\delta})\Big) \\ \leq \sum_{n=1}^{\infty} 2^{2p} e^{-p\frac{\beta^2}{2}n\delta} E\Big(\sum_{u\in N_{n\delta}} e^{-\beta p|X^u_{n\delta}|}\Big) \\ = \sum_{n=1}^{\infty} 2^{2p} e^{-(p-1)\frac{\beta^2}{2}n\delta} E^{\tilde{Q}_\beta}\Big(\sum_{u\in N_{n\delta}} e^{-\beta(p-1)|X^u_{n\delta}|}\Big).$$

Thus we have proved (5.25), which together with (5.24) implies that

$$\liminf_{n\to\infty} e^{-\frac{\beta^2}{2}n\delta} \sum_{u\in N_{n\delta}} e^{-\beta|X^u_{n\delta}|}\Xi^{\delta,\epsilon}_B(X^u_{n\delta}) = \liminf_{n\to\infty} e^{-\frac{\beta^2}{2}n\delta} \sum_{u\in N_{n\delta}} e^{-\beta|X^u_{n\delta}|}\xi^{\delta,\epsilon}_B(X^u_{n\delta}) \\ = \int \xi^{\delta,\epsilon}_B(x)\pi(\mathrm{d}x)M_\infty.$$

Putting this into (5.21) and letting $n = \lfloor \frac{t}{\delta} \rfloor$ gives us

$$\liminf_{t\to\infty} U_t \geq \frac{e^{-\frac{\beta^2}{2}\delta}}{1+\epsilon} \int \xi^{\delta,\epsilon}_B(x)\pi(\mathrm{d}x)M_\infty.$$

Letting $\delta, \epsilon \searrow 0$ we get

$$\liminf_{t\to\infty} U_t \geq \pi(B)M_\infty.$$

Since the same result also holds for B^c we can easily see that

$$\limsup_{t\to\infty} U_t \leq \pi(B)M_\infty.$$

Thus

$$\lim_{t\to\infty} U_t = \pi(B)M_\infty.$$

Then the same argument as at the end of Part II of the proof extends the result for functions of the form $\mathbf{1}_B(x)e^{-\beta|x|}$ to all bounded measurable functions. $\square$

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