LETTER TO THE EDITOR

On the Bose-Einstein condensation of a perfect gas

An article by Fowler and Jones[1]) on the Bose-Einstein condensation of a perfect gas has prompted a further investigation of this problem.

As is well known, a perfect gas, consisting of a very great number N of Bose-Einstein particles, enclosed in a box of volume V, has a transition temperature T_0, defined by:

$$\frac{N}{V}\left(\frac{h^2}{2\pi mkT_0}\right)^{3/2} = \sum_1^\infty j^{-3/2} = 2.612 \tag{1}$$

at which there is a discontinuity in the second derivative, with respect to the temperature, of the total energy E of the gas, and below which the pressure of the gas is independent of the volume, as with a real gas in the coexistence region. This phenomenon is therefore called Bose-Einstein condensation [2]) [3]).

Now the question presents itself, whether this phenomenon will also occur with another number w of dimensions, or with a more general form of the field. Therefore (while taking the usual well-known formulae of Bose statistics to be correct) we have investigated w-dimensional potential fields of such a shape as to give sequences of eigen-values of the form:

$$\varepsilon_i = \varepsilon_{s_1,\ldots,s_w} = \text{const.}\,\frac{h^2}{m}\left[\frac{s_1^a - 1}{a_1^2} + \ldots. + \frac{s_w^a - 1}{a_w^2}\right] \tag{2}$$

where a is a number between 1 and 2; $s_1, \ldots., s_w$ are the w necessary quantum numbers, and $a_1, \ldots., a_w$ are certain "characteristic lengths" of the field. (The lowest level, with $s_1, \ldots., s_w = 1$, has been taken as the zero point of the energy scale). For $a = 2$ the potential field is that of the w-dimensional rectangular box, with side lengths $a_1, \ldots., a_w$; for $a = 1$ we obtain the w-dimensional harmonic oscillator field (with appropriate choice of the constant, the $a_1, \ldots., a_w$ can be considered as the half-axes of the (w-dimensional ellipsoidal) classical "livingspace" of the particles at the lowest level).

In an article, still to be published [4]), it will be shown, that *the occurrence and character of the transition temperature T_0 depend on the value of the number $q = w/a$ (quotient of the number of dimensions and the exponent of the quantum number in the energy eigenvalue). For $q \leq 1$ there is no such point T_0; for $q > 1$ a transition point exists,* and is defined by an equation

$$\text{const.}\,\nu\,.\,T_0^{-q} = \sum_1^\infty j^{-q} \equiv \zeta(q) \tag{3}$$

in which ν is the quantity

$$\nu = \frac{N}{\prod_{v=1}^{w} a_v^{2/a}} \tag{4}$$

The character of the transition point, is as follows: For $1 < q < {}^3/_2$,

E, dE/dT and d^2E/dT^2 are continuous at $T = T_0$; discontinuities occur in higher derivatives. For $q = {}^3/_2$ (just the case of the 3-dimensional box, $w = 3$, $\alpha = 2$), d^2E/dT^2 shows a finite discontinuity, for ${}^3/_2 < q \leq 2$ d^2E/dT^2 shows an infinite discontinuity at $T = T_0$. For $q > 2$, the specific heat dE/dT shows a finite discontinuity at $T = T_0$, so that we have a λ-point.

A well-defined transition point T_0 appears only with a very great number N of particles (theoretically only for infinite N); the transition temperature T_0 is finite only [in the limiting case $N = \infty$] if ν is finite.

For the box, $\alpha = 2$, ν is equal to the (mean) density of the gas (the number of particles per unit volume, $n = N/\prod_{v=1}^{w} a_v$). If, with $N \to \infty$, ν tends to zero or infinity, then T_0 also tends to zero or infinity respectively, i.e., the transition point, while becoming sharper, at the same time tends to zero or infinity. The first case occurs e.g. when, with $\alpha < 2$, one keeps the "density" $n = N/\prod_{v=1}^{w} a_v$ a constant while $N \to \infty$; the second case when for example one introduces $N = \infty$ particles in a field with finite characteristical lengths $a_1, \ldots, a_w$ (e.g. a box of finite dimensions).

With $N \to \infty$ and finite ν, in the case $q > 1$ (transition point) and at temperatures below T_0, the number N_0 of particles in the lowest state also tends to infinity, in such a way as to remain a finite fraction of the total number N and the formula

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_0}\right)^q \tag{5}$$

holds. London's [3]) well-known formula for N_0/N is the special case for $q = {}^3/_2$ (3-dimensional box).

Like Fowler and Jones [1]), but unlike London [3]) we carried out the mathematical proofs and calculations without approximating the (for finite N) discrete energy spectrum (2) by a continuous one. Our method can only be used with potential fields of special form (e.g. with a rectangular box, but not with a box of arbitrary form). Fields of still more general form can probably only be tackled with a "continuous spectrum" approximation; in that case London's "mixed continuous discrete spectrum" method is to be preferred, as it avoids some of the mathematical difficulties which are encountered when a completely "continuous spectrum" method is used.

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REFERENCES

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