

# Fixed Points and Contraction Factor Functions

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**Running title:** Fixed points and contraction factor functions

**Abstract:** In a complete metric space  $(X, d)$ , we define  $w$ -distance functions  $p : X \times X \rightarrow [0, \infty)$ , of which the metric  $d$  is a special case, and contraction factor functions  $r : X \times X \rightarrow [0, \infty)$  such that if

$$p(Tx, Ty) \leq r(x, y)p(x, y)$$

for all  $x, y \in X$ , then  $T : X \rightarrow X$  has a (unique) fixed point.

**Keywords:** contraction,  $w$ -distance, fixed point.

# 1 Introduction

Banach's Contraction Principle states that if  $(X, d)$  is a complete metric space and  $T : X \rightarrow X$  is a contraction, i.e., there exists a number  $0 \leq r < 1$  such that for every two points  $x, y \in X$ :

$$d(Tx, Ty) \leq rd(x, y), \tag{1}$$

then  $T$  has a unique fixed point. There exist numerous extensions of this result; Rakotch [1], for instance, considers the problem of defining contraction factor functions such that the Banach Contraction Principle remains valid when the constant  $r$  in (1) is replaced by a function  $r(x, y)$ . This allows  $\sup_{(x,y)} r(x, y) = 1$ , in which case  $T$  is no longer a contraction. The purpose of this note is to define:

- functions  $p : X \times X \rightarrow [0, \infty)$ , of which the distance function  $d$  is a special case, and
- contraction factor functions  $r : X \times X \rightarrow [0, \infty)$ , including those of Rakotch [1],

such that if

$$p(Tx, Ty) \leq r(x, y)p(x, y)$$

for all  $x, y \in X$ , then  $T : X \rightarrow X$  has a (unique) fixed point. The functions  $p$  are so-called  $w$ -distances, introduced and studied in a recent sequence of papers by Kada, Suzuki, and Takahashi [2], Suzuki and Takahashi [3], and Suzuki [4].

## 2 Preliminaries

Denote by  $\mathbb{N}$  the set of positive integers and by  $\mathbb{R}$  the set of real numbers. Let  $X$  be a metric space with metric  $d$ . Following [2, p. 381], we call a function  $p : X \times X \rightarrow [0, \infty)$  a  $w$ -distance on  $X$  if the following conditions hold:

- $p$  satisfies the triangle inequality, i.e.,  $\forall x, y, z \in X : p(x, z) \leq p(x, y) + p(y, z)$ ;
- $p(x, \cdot) : X \rightarrow [0, \infty)$  is lower semicontinuous for every  $x \in X$ , i.e., if a sequence  $(y_m)$  in  $X$  converges to  $y \in X$ , then  $p(x, y) \leq \liminf_{m \rightarrow \infty} p(x, y_m)$ ;
- for every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for each  $x, y, z \in X$ : if  $p(z, x) \leq \delta$  and  $p(z, y) \leq \delta$ , then  $d(x, y) \leq \varepsilon$ .

The metric  $d$  is a  $w$ -distance. Examples of many other  $w$ -distances are found in [2] and [3, Lemma 1]. Kada *et al.* [2, Lemma 1] prove:

**Lemma 2.1** Let  $(X, d)$  be a metric space and let  $p$  be a  $w$ -distance on  $X$ . Consider points  $x, y, z \in X$ , a sequence  $(x_n)$  in  $X$ , and sequences  $(\alpha_n)$  and  $(\beta_n)$  in  $[0, \infty)$  converging to zero. The following claims hold:

- (a) If  $p(x_n, x_m) \leq \alpha_n$  for all  $m, n \in \mathbb{N}$  with  $m > n$ , then  $(x_n)$  is a Cauchy sequence in  $(X, d)$ .
- (b) If  $p(x_n, y) \leq \alpha_n$  and  $p(x_n, z) \leq \beta_n$  for all  $n \in \mathbb{N}$ , then  $y = z$ . In particular, if  $p(x, y) = p(x, z) = 0$ , then  $y = z$ .

Generalizing Rakotch [1], we define a family of functions that take over the role of the contraction factors in the original statement of Banach's Contraction Principle and its variants.

**Definition 2.2** Let  $(X, d)$  be a metric space and let  $p$  be a  $w$ -distance on  $X$ . A function  $r : X \times X \rightarrow [0, \infty)$  is a *contraction factor function* if there exists a function  $f : (0, \infty) \rightarrow [0, 1)$  such that  $p(x, y) \geq \varepsilon$  for some  $x, y \in X$  and  $\varepsilon > 0$  implies  $r(x, y) \leq f(\varepsilon)$ . The set of all contraction factor functions is denoted  $F(p)$ .

Definition 2.2 implies that  $r(x, y) \in [0, 1)$  for all  $x, y \in X$  with  $p(x, y) > 0$ . Rakotch [1, Def. 2] takes  $p = d$ , and considers only functions  $\beta$  on  $X \times X$  for which there exists a function  $\alpha : [0, \infty) \rightarrow \mathbb{R}$  such that

- (a)  $\forall x, y \in X : \beta(x, y) = \alpha(d(x, y))$ , i.e.,  $\beta(x, y)$  only depends on the distance between  $x$  and  $y$ ;
- (b)  $\alpha(\varepsilon) \in [0, 1)$  for all  $\varepsilon > 0$ ;
- (c)  $\alpha$  is a decreasing function, i.e., if  $\varepsilon_1 \geq \varepsilon_2 \geq 0$ , then  $\alpha(\varepsilon_1) \leq \alpha(\varepsilon_2)$ .

Notice that  $\beta(x, y) = \alpha(d(x, y)) \in [0, 1)$  if  $d(x, y) > 0$ . If  $d(x, y) = 0$ , then for arbitrary  $\varepsilon > 0$ , decreasingness of  $\alpha$  implies that  $\beta(x, y) = \alpha(d(x, y)) \geq \alpha(\varepsilon) \geq 0$ . Hence  $\beta$  is a function into  $[0, \infty)$ . Defining  $f(\varepsilon) = \alpha(\varepsilon)$  for all  $\varepsilon > 0$ , it follows that  $d(x, y) \geq \varepsilon$  for some  $x, y \in X$  and  $\varepsilon > 0$  implies  $\beta(x, y) = \alpha(d(x, y)) \leq \alpha(\varepsilon) = f(\varepsilon)$ . Consequently, the functions  $\beta$  considered by Rakotch are contained in  $F(d)$ , the class of contraction factor functions for the  $w$ -distance  $p = d$ .

Apart from the fact that Definition 2.2 allows contraction factor functions to be defined for an arbitrary  $w$ -distance, it extends the functions of Rakotch in two directions: the values do not just depend on the distance between two points, and the monotonicity assumption is omitted.

### 3 The contraction theorem

After proving auxiliary results in Proposition 3.1, a contraction result is provided in Theorem 3.2.

**Proposition 3.1** *Let  $(X, d)$  be a metric space, let  $p$  be a  $w$ -distance on  $X$ , and  $T : X \rightarrow X$  a function from  $X$  into itself. Assume there exists a function  $r \in F(p)$  such that*

$$\forall x, y \in X : p(Tx, Ty) \leq r(x, y)p(x, y). \quad (2)$$

*Let  $x_0 \in X$  and  $x_n = T^n x_0$  for all  $n \in \mathbb{N}$ . The following claims hold:*

- (a) *The sequence  $(p(x_0, x_n))$  in  $[0, \infty)$  is bounded.*
- (b) *There exists a sequence  $(\alpha_n)$  in  $[0, \infty)$  converging to zero, such that*

$$p(x_n, x_m) \leq \alpha_n \quad (3)$$

*for all  $m, n \in \mathbb{N}$  with  $m > n$ .*

**Proof.** Fix a function  $f$  as in Definition 2.2. Then  $r(x, y) \in [0, 1)$  for all  $x, y \in X$  with  $p(x, y) > 0$ . Hence, repeated application of (2) yields that

$$\forall n \in \mathbb{N} : p(x_{n+1}, x_n) \leq p(x_1, x_0), \quad (4)$$

and

$$\forall k, n, p \in \mathbb{N} \cup \{0\} : \text{if } n > k, \text{ then } p(x_n, x_{n+p}) \leq p(x_k, x_{k+p}). \quad (5)$$

**Proof of (a):** Let  $\varepsilon > 0$ . Then  $f(\varepsilon) \in [0, 1)$  by Definition 2.2, so  $R := \max\{\varepsilon, \frac{p(x_0, x_1) + p(x_1, x_0)}{1 - f(\varepsilon)}\}$  is well-defined. Let  $n \in \mathbb{N}$ . We prove that  $p(x_0, x_n) \leq R$ . This is clear if  $p(x_0, x_n) < \varepsilon$ , so assume that  $p(x_0, x_n) \geq \varepsilon$ . Then  $0 \leq r(x_0, x_n) \leq f(\varepsilon) < 1$  by Definition 2.2. Using this inequality, the triangle inequality, (2) and (4), it follows that

$$\begin{aligned} p(x_0, x_n) &\leq p(x_0, x_1) + p(x_1, x_{n+1}) + p(x_{n+1}, x_n) \\ &\leq p(x_0, x_1) + r(x_0, x_n)p(x_0, x_n) + p(x_1, x_0) \\ &\leq p(x_0, x_1) + f(\varepsilon)p(x_0, x_n) + p(x_1, x_0), \end{aligned}$$

which implies

$$p(x_0, x_n) \leq \frac{p(x_0, x_1) + p(x_1, x_0)}{1 - f(\varepsilon)} \leq R.$$

**Proof of (b):** We prove:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ such that } n \geq N, p \in \mathbb{N} \text{ implies } p(x_n, x_{n+p}) < \varepsilon. \quad (6)$$

This implies (3): for each  $k \in \mathbb{N}$ , (6) implies the existence of  $N(k) \in \mathbb{N}$  such that

$$\forall n \geq N(k), \forall p \in \mathbb{N} : p(x_n, x_{n+p}) < \frac{1}{k}. \quad (7)$$

Without loss of generality, one can take  $N(k) < N(k+1)$  for each  $k \in \mathbb{N}$ . Take  $R$  such that  $p(x_0, x_n) \leq R$  for all  $n \in \mathbb{N}$ . For each  $n \in \mathbb{N}$ , define

$$\alpha_n = \begin{cases} R & \text{if } n < N(1), \\ \frac{1}{k} & \text{if } N(k) \leq n < N(k+1) \text{ for some } k \in \mathbb{N}. \end{cases}$$

Then  $(\alpha_n)$  is a sequence in  $[0, \infty)$  converging to zero. To see that  $p(x_n, x_{n+p}) \leq \alpha_n$  for all  $n, p \in \mathbb{N}$ , notice: firstly, if  $n < N(1)$ , then  $p(x_n, x_{n+p}) \leq p(x_0, x_p) \leq R = \alpha_n$  by (5), and secondly, if  $N(k) \leq n < N(k+1)$  for some  $k \in \mathbb{N}$ , then  $p(x_n, x_{n+p}) \leq \frac{1}{k} = \alpha_n$  by (7).

Remains to prove (6). Let  $\varepsilon > 0$ . Then  $f(\varepsilon) \in [0, 1)$  by Definition 2.2. Choose  $N \in \mathbb{N}$  such that

$$Rf(\varepsilon)^N < \varepsilon. \quad (8)$$

Let  $n \geq N, p \in \mathbb{N}$ . We prove that  $p(x_n, x_{n+p}) < \varepsilon$ . Repeated application of (2) yields

$$p(x_n, x_{n+p}) \leq p(x_0, x_p) \prod_{k=0}^{n-1} r(x_k, x_{k+p}).$$

Combining this with the assumption that  $R$  is a bound for the sequence  $(p(x_0, x_n))$ , we find

$$p(x_n, x_{n+p}) \leq R \prod_{k=0}^{n-1} r(x_k, x_{k+p}). \quad (9)$$

Discern two cases.

**Case 1:** If  $p(x_k, x_{k+p}) < \varepsilon$  for some  $k \in \{0, \dots, n-1\}$ , then (5) yields that  $p(x_n, x_{n+p}) \leq p(x_k, x_{k+p}) < \varepsilon$ .

**Case 2:** If  $p(x_k, x_{k+p}) \geq \varepsilon$  for every  $k \in \{0, \dots, n-1\}$ , then Definition 2.2 implies that  $r(x_k, x_{k+p}) \leq f(\varepsilon)$  for every  $k \in \{0, \dots, n-1\}$ . Using (9) and (8) yields

$$p(x_n, x_{n+p}) \leq R \prod_{k=0}^{n-1} r(x_k, x_{k+p}) \leq Rf(\varepsilon)^n \leq Rf(\varepsilon)^N < \varepsilon.$$

This finishes the proof. □

The preliminary work in Proposition 3.1 paves the road for our contraction result.

**Theorem 3.2** *Let  $(X, d)$  be a complete metric space, let  $p$  be a  $w$ -distance on  $X$ , and  $T : X \rightarrow X$  a function from  $X$  into itself. If there exists a function  $r \in F(p)$  such that*

$$\forall x, y \in X : p(Tx, Ty) \leq r(x, y)p(x, y), \quad (10)$$

*then  $T$  has a unique fixed point  $x \in X$ . This fixed point satisfies  $p(x, x) = 0$ .*

**Proof.** Let  $x_0 \in X$  and  $x_n = T^n x_0$  for all  $n \in \mathbb{N}$ . Proposition 3.1(b) and Lemma 2.1(a) imply that  $(x_n)$  is a Cauchy sequence. Since  $(X, d)$  is complete,  $(x_n)$  has a limit  $x \in X$ . We show that  $Tx = x$ . Consider a sequence  $(\alpha_n)$  as in Proposition 3.1(b). Since  $p(x_n, \cdot)$  is lower semicontinuous and  $x_m \rightarrow x$ , it follows from (3) that

$$\forall n \in \mathbb{N} : p(x_n, x) \leq \liminf_{m \rightarrow \infty} p(x_n, x_m) \leq \alpha_n, \quad (11)$$

and, using (10) and (11), that

$$\forall n \in \mathbb{N} : p(x_n, Tx) = p(Tx_{n-1}, Tx) \leq p(x_{n-1}, x) \leq \alpha_{n-1}. \quad (12)$$

From (11), (12), and part (b) of Lemma 2.1, it follows that  $Tx = x$ , i.e., that  $x$  is a fixed point of  $T$ . To see that  $p(x, x) = 0$ , suppose — to the contrary — that  $p(x, x) > 0$ . Then  $r(x, x) \in [0, 1)$  by Definition 2.2; by (10) and the fact that  $x$  is a fixed point, it follows that:

$$p(x, x) = p(Tx, Tx) \leq r(x, x)p(x, x) < p(x, x),$$

a contradiction. Finally, to prove that  $x$  is the *unique* fixed point of  $T$ , suppose that  $y \in X$  satisfies  $Ty = y$ . Analogous to the proof that  $p(x, x) = 0$ , it follows that  $p(x, y) = 0$ , so part (b) of Lemma 2.1 implies that  $x = y$ . □

## 4 Concluding remarks

Theorem 3.2, replacing the metric  $d$  of a metric space  $(X, d)$  with  $w$ -distances recently introduced in Kada *et al.* [2], and the contraction factors by contraction factor functions as in Definition 2.2, provides generalizations of the classical Banach Contraction Principle, the contraction theorem of Rakotch [1, p. 463], and more recent results by Suzuki and Takahashi [3, Thm. 2].

If  $T$  itself does not satisfy (10), but some power  $T^n$  ( $n \in \mathbb{N}$ ) of  $T$  does, the conclusion of Theorem 3.2 still holds: according to Theorem 3.2,  $T^n$  has a unique fixed point  $x$ , but

$$Tx = T(T^n x) = T^n(Tx)$$

indicates that  $Tx$  is also a fixed point of  $T^n$ . Hence  $Tx = x$ , i.e.,  $x$  is a fixed point of  $T$ . The fact that  $x$  is the unique fixed point of  $T$  and  $p(x, x) = 0$  follows in the same way as before.

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