Analytical Solutions and Conservation Laws of Models Describing Heat Transfer Through Extended Surfaces

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Declaration

I declare that this project is my own, unaided work. It is being submitted as partial fulfilment of the Degree of Master of Science at the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination in any other University.

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Abstract

The search for solutions to the important differential equations arising in extended surface heat transfer continues unabated. Extended surfaces, in the form of longitudinal fins are considered. First we consider the steady state problem and then the transient heat transfer models. Here, thermal conductivity and heat transfer coefficient are assumed to be functions of temperature. Thermal conductivity is considered to be given by the power law in one case and by the linear function of temperature in the other; whereas heat transfer coefficient is only given by the power law. Explicit analytical expressions for the temperature profile, fin efficiency and heat flux for steady state problems are derived using the one-dimensional Differential Transform Method (1D DTM). The obtained results from 1D DTM are compared with the exact solutions to verify the accuracy of the proposed method. The results reveal that the 1D DTM can achieve suitable results in predicting the solutions of these problems. The effects of some physical parameters such as the thermo-geometric fin parameter and thermal conductivity gradient, on temperature distribution are illustrated and explained. Also, we apply the two-dimensional Differential Transform Method (2D DTM) to models describing transient heat transfer in longitudinal fins. Furthermore, conservation laws for transient heat conduction equations are derived using the direct method and the multiplier method, and finally we find Lie point symmetries associated with the conserved vectors.

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Chapter 1

Introduction

1.1 Literature review

In the study of heat transfer, an extended surface (also known as a fin) is a solid that extends from a hot body to increase the rate of heat transfer to the surrounding fluid. A growing number of engineering applications are concerned with energy transport by requiring the rapid movement of heat form a hot body. The heat transfer mechanism of a fin is to conduct heat from a heat source to its surface by its thermal conduction, and then dissipate heat to the ambient fluid by the effect of thermal convection. In particular fins are used extensively in various industrial applications such as the cooling of computer processors, air conditioning, oil carrying pipe lines and so on. A well documented review of heat transfer in extended surfaces is presented by Kraus *et al.* [1].

The problems on heat transfer particularly in fins continue to be of scientific interest. These problems are modeled by highly nonlinear differential equations which are difficult to solve exactly. For the cases where the heat transfer coefficient is constant, exact solutions of the temperature profile and the rate

of heat transfer can be obtained easily [2]. Recently, the study of fins in boiling liquids has been increasing enormously and it has been found that the heat transfer coefficient is not constant but depends on the temperature difference between a surface and the adjacent fluid [3]. Problems arising in engineering have this type of dependence[1] and our study is devoted to the analysis of fin performance under such circumstances. Thus, the resulting equations becomes strictly nonlinear even in the simplest one-dimensional analysis [4].

Moitsheki *et al* [5, 6] have attempted to construct exact solutions for the steady state problems arising in heat flow through fins. A number of techniques, for example Lie symmetry analysis [5], He's variational iteration method [7], Adomain decomposition methods [8], homotopy perturbation methods [9], homotopy analysis methods [10], methods of successive approximations [11] and other approximation methods [12] have been used to determine solutions of the nonlinear differential equations describing heat transfer in fins. Recently, the solutions of the nonlinear ordinary differential equations (ODEs) arising in extended surface heat transfer have been constructed using the 1D DTM [13, 14], (see also [15, 16]). The DTM is an analytical method based on the Taylor series expansion and was first introduced by Zhou [17] in 1986. Most of the designs of systems employing extended surfaces are based on steady state analysis and much research has been focused on solving such problems. This is adequate for most applications [1], but there are some extended surfaces which require transient response analysis, for example, fins in heat exchangers and solar energy systems. The 2D DTM has been applied by Jarafi *et al* [18], to solve nonlinear Gas Dynamic and Klein-Gordon equations. Ndlovu and Moitsheki [19] applied the 2D DTM to transient heat conduction problem for

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[1]Thermal conductivity and heat transfer coefficients depending on temperature and given by linear or power law.

heat transfer in longitudinal rectangular fins (this has been applied, as far we know, for the first time in these problems). To illustrate the capability and reliability of the 2D DTM, Bhadauria *et al* [20] solved linear and non-linear diffusion equations for various cases, (see also [21]). Merdan and Gökdoğan [22] found exact and approximate solutions of the heat equation in the cast-mould heterogeneous domain using 2D DTM. The results indicated that the solutions obtained by this method are reliable, useful and that it is an effective method for decoupling partial differential equations (PDEs). Exact solutions can also be obtained from the known forms of the series solutions. Symmetry analysis of the transient one-dimensional fin problem was performed by Vaneeva *et al* [23], Pakdemirli and Sahin [24], Moitsheki and Harley [25], Bokhari *et al* [26] and many more. However, imposed boundary conditions were not invariant under any Lie point symmetry. As such, approximate solutions may be sought.

As far as we know, conservation laws of fin equations have not yet been constructed. Different approaches of constructing conservation laws has been studied and compared for some differential equation in fluid mechanics by Naz *et al* [27]. We choose some of these methods which are appropriate for our problems, for example we are working with heat transfer equations which do not have a Lagrangian, thus we cannot apply Noether approach to our problem. As such one may apply the method of partial Lagrangian and partial Noether as formulated by Kara and Mahomed [28]. In this work, we construct conservation laws using (i) the direct method [28] and (ii) the method of multipliers [29, 30]. We also find Lie point symmetries associated with the conserved vectors. From the paper by Moitsheki and Harley [25], it was found that the boundary conditions of the transient problems were not invariant under any Lie point symmetry. As such, invariant solutions were not found. Therefore, we provide analytical solutions to transient equations using the 2D DTM in

Chapter 5. We adopt terminology analytical solutions to be series solutions obtained using DTM and exact solutions as expressions given in terms of fundamental trigonometric functions, logarithmic functions and so on.

1.2 Aims and objectives of the dissertation

The main objective of this dissertation is to contribute to the fundamental understanding of heat transfer through extended surfaces. Fins of different profiles will be investigated and their performance will be compared using fin efficiency, heat flux and other thermal properties. First we aim to apply the 1D DTM to the nonlinear steady state fin problems. Analytical solutions will also be derived for transient state problems using the 2D DTM. Furthermore we construct conservation laws and determine associated Lie point symmetries for transient heat transfer problems.

1.3 Outline of the dissertation

The outline of this dissertation is as follows

- In Chapter 2, the definitions and basic operations of the proposed methods of solutions are presented.

- Mathematical models are derived in Chapter 3.

- Chapter 4 deals with analysis of heat transfer problems in fins of different profiles. The results presented in Chapter 4 are from a research article by Ndlovu and Moitsheki which is under review through the journal Mathematical Problems in Engineering, see [13].

- In Chapter 5, analytical solutions are derived using 2D DTM for transient heat transfer in rectangular straight fins and convex parabolic fins. Some of the results presented in Chapter 5 are from a research article by Ndlovu and Moitsheki which is in press at Communications in Nonlinear Science and Numerical Simulations see [19].

- Two methods of constructing conservation laws are employed in Chapter 6. We further find Lie point symmetries associated with the conserved vectors. The results presented in Chapter 6 are from a research article by Ndlovu and Moitsheki which is under review through the Central European Journal of Physics, see [56].

- In Chapter 7 we will summarize our findings and provide conclusions.

Chapter 2

Formulae and theory

2.1 Introduction

In this chapter we will present a brief theory of the differential transformation and Lie group analysis which will be used to solve the models provided in Chapter 3. Furthermore, we briefly introduce the two methods of constructing conservation laws.

2.2 Differential Transform Methods

2.2.1 1D-Differential Transform Method

Let $\phi(t)$ be an analytic function in a domain $\mathcal{D}$. The Taylor series expansion function of $\phi(t)$ with the center located at $t = t_j$ is given by [17]

$$\phi(t) = \sum_{\kappa=0}^{\infty} \frac{(t - t_j)^{\kappa}}{\kappa!} \left[\frac{d^{\kappa}\phi(t)}{dt^{\kappa}} \right]_{t=t_j}, \ \forall \ t \in \mathcal{D}. \tag{2.1}$$

The particular case of Eqn (2.1) when $t_j = 0$ is referred to as the Maclaurin

series expansion of $\phi(t)$ and is expressed as,

$$\phi(t) = \sum_{\kappa=0}^{\infty} \frac{t^{\kappa}}{\kappa!} \left[\frac{d^{\kappa}\phi(t)}{dt^{\kappa}} \right]_{t=0}, \quad \forall \, t \in \mathcal{D}. \tag{2.2}$$

The differential transform of $\phi(t)$ is defined as follows;

$$\Phi(\kappa) = \sum_{\kappa=0}^{\infty} \frac{\mathcal{H}^{\kappa}}{\kappa!} \left[\frac{d^{\kappa}\phi(t)}{dt^{\kappa}} \right]_{t=0}, \tag{2.3}$$

where $\phi(t)$ is the original analytic function and $\Phi(\kappa)$ is the transformed function. The differential spectrum of $\phi(t)$ is confined within the interval $t \in [0, \mathcal{H}]$, where $\mathcal{H}$ is a constant. From Eqns (2.2) and (2.3), the differential inverse transform of $\phi(t)$ is defined as follows,

$$\phi(t) = \sum_{\kappa=0}^{\infty} \left(\frac{t}{\mathcal{H}} \right)^{\kappa} \Phi(\kappa), \tag{2.4}$$

and if $\phi(t)$ is approximated by a finite series, then

$$\phi(t) = \sum_{\kappa=0}^{r} \left(\frac{t}{\mathcal{H}} \right)^{\kappa} \Phi(\kappa). \tag{2.5}$$

It is clear that the concept of differential transformation is based upon the Taylor series expansion. The values of the function $\Phi(\kappa)$ are referred to as discrete, i.e., $\Phi(0)$ is known as the zero discrete, $\Phi(1)$ as the first discrete, etc. With more discrete available, it is possible to restore the unknown function more precisely. The function $\phi(t)$ consists of the T-function $\Phi(\kappa)$, and its value is given by the sum of the T-function with $(t/\mathcal{H})^{\kappa}$ as its coefficient. In real applications, at the right choice of the constant $\mathcal{H}$, the discrete values of the spectrum reduce rapidly with larger values of argument κ [31].

Some of the useful mathematical operations performed by the differential transform method are given by the following theorems.

Theorem 2.1. *If* $\phi(t) = x(t) \pm z(t)$, *then* $\Phi(\kappa) = X(\kappa) \pm Z(\kappa)$.

Theorem 2.2. *If* $\phi(t) = \alpha x(t)$, *then* $\Phi(\kappa) = \alpha X(\kappa)$.

Theorem 2.3. *If* $\phi(t) = \frac{d\phi(t)}{dt}$, *then* $\Phi(\kappa) = (\kappa + 1)\Phi(\kappa + 1)$.

Theorem 2.4. *If* $\phi(t) = \frac{d^2\phi(t)}{dt^2}$, *then* $\Phi(\kappa) = (\kappa + 1)(\kappa + 2)\Phi(\kappa + 2)$.

Theorem 2.5. *If* $\phi(t) = \frac{d^s\phi(t)}{dt^s}$, *then* $\Phi(\kappa) = (\kappa+1)(\kappa+2)\cdots(\kappa+s)\Phi(\kappa+s)$.

Theorem 2.6. *If* $\phi(t) = x(t)z(t)$, *then* $\Phi(\kappa) = \sum_{i=0}^{\kappa} X(i)Z(\kappa - i)$.

Theorem 2.7. *If* $\phi(t) = t^s$, *then* $\Phi(\kappa) = \delta(\kappa - s)$.

Theorem 2.8. *If* $\phi(t) = \exp(\lambda t)$, *then* $\Phi(\kappa) = \frac{\lambda^\kappa}{\kappa!}$.

Theorem 2.9. *If* $\phi(t) = (1+t)^s$, *then* $\Phi(\kappa) = \frac{s(s-1)\cdots(s-\kappa+1)}{\kappa!}$.

Theorem 2.10. *If* $\phi(t) = \sin(\omega t + \alpha)$, *then* $\Phi(\kappa) = \frac{\omega^\kappa}{\kappa!}\sin(\frac{\pi\kappa}{2!} + \alpha)$.

Theorem 2.11. *If* $\phi(t) = \cos(\omega t + \alpha)$, *then* $\Phi(\kappa) = \frac{\omega^\kappa}{\kappa!}\cos(\frac{\pi\kappa}{2!} + \alpha)$.

The Kronecker delta function $\delta(\kappa - s)$ is given by

$$
\delta(\kappa - s) = \begin{cases} 1 & \text{if } \kappa = s \\ 0 & \text{if } \kappa \neq s. \end{cases}
$$

2.2.2 2D-Differential Transform Method

Based on the one-dimensional differential transform method, the basic fundamental operations of the two dimensional differential are defined as follows

$$
\Phi(\kappa, s) = \sum_{\kappa=0}^{\infty}\sum_{s=0}^{\infty} \frac{1}{\kappa! s!} \left[\frac{\partial^{\kappa+s}\phi(t, x)}{\partial t^\kappa \partial x^s} \right]_{(0,0)}. \tag{2.6}
$$

The differential inverse transform of $\Phi(\kappa, s)$ is defined as

$$
\phi(t, x) = \sum_{\kappa=0}^{\infty}\sum_{s=0}^{\infty} \Phi(\kappa, s) t^\kappa x^s, \tag{2.7}
$$

and from Eqs (2.6) and (2.7) it can be concluded that

$$
\phi(t, x) = \sum_{\kappa=0}^{\infty}\sum_{s=0}^{\infty} \frac{1}{\kappa! s!} \left[\frac{\partial^{\kappa+s}\phi(t, x)}{\partial t^\kappa \partial x^s} \right]_{(0,0)} t^\kappa x^s. \tag{2.8}
$$

In real applications, the function $\phi(t, x)$ is approximated by a finite series, and Eqn (2.7) can be written as:

$$\phi(t, x) = \sum_{\kappa=0}^{m} \sum_{s=0}^{n} \Phi(\kappa, s) t^{\kappa} x^{s}, \tag{2.9}$$

Eqn (2.9) implies that

$$\phi(t, x) = \sum_{\kappa=m+1}^{\infty} \sum_{s=n+1}^{\infty} \Phi(\kappa, s) t^{\kappa} x^{s}, \tag{2.10}$$

is negligibly small.

Some of the useful mathematical operations performed by the differential transform method are given by the following theorems.

Theorem 2.12. *If* $\phi(t, x) = x(\tau, x) \pm z(t, x), \quad then \quad \Phi(\kappa, s) = X(\kappa, s) \pm Z(\kappa, s).$

Theorem 2.13. *If* $\phi(t, x) = \alpha x(t, x), \quad then \quad \Phi(\kappa, s) = \alpha X(\kappa, s).$

Theorem 2.14. *If* $\phi(t, x) = \frac{\partial \phi(t, x)}{dt}, \quad then \quad \Phi(\kappa, s) = (\kappa + 1) \Phi(\kappa + 1, s).$

Theorem 2.15. *If* $\phi(t, x) = \frac{\partial \phi(t, x)}{dx}, \quad then \quad \Phi(\kappa, s) = (s + 1) \Phi(\kappa, s + 1).$

Theorem 2.16. *If* $\phi(t, x) = \frac{\partial^{r+q} \phi(t, x)}{\partial t^{r} \partial x^{q}}, \quad then$

$\Phi(\kappa, s) = (\kappa + 1)(\kappa + 2) \ldots (\kappa + r)(s + 1)(s + 2) \ldots (s + q) \Phi(\kappa + r, s + q).$

Theorem 2.17. *If* $\phi(t, x) = x(t, x) z(t, x), \quad then$

$\Phi(\kappa, s) = \sum_{i=0}^{\kappa} \sum_{j=0}^{s} X(i, s - j) Z(\kappa - i, j).$

Theorem 2.18. *If* $\phi(t, x) = t^{m} x^{n}, \quad then \quad \Phi(\kappa, s) = \delta(\kappa - m) \delta(s - n).$

Theorem 2.19. *If* $\phi(t, x) = \frac{\partial x(t, x)}{\partial t} \frac{\partial z(t, x)}{\partial x}, \quad then$

$\Phi(\kappa, s) = \sum_{i=0}^{\kappa} \sum_{j=0}^{s} (i + 1)(\kappa - i + 1) X(i + 1, s - j) Z(k - i + 1, j).$

Theorem 2.20. *If* $\phi(t, x) = x^{m} \exp(at), \quad then \quad \Phi(\kappa, s) = \frac{a^{s}}{s!} \delta(\kappa - m).$

Theorem 2.21. *If* $\phi(t, x) = x^{m} \sin(at + b), \quad then$

$\Phi(\kappa, s) = \frac{a^{s}}{s!} \delta(\kappa - m) \sin(\frac{s\pi}{2} + b).$

Theorem 2.22. *If* $\phi(t, x) = x^{m} \cos(at + b), \quad then$

$\Phi(\kappa, s) = \frac{a^{s}}{s!} \delta(\kappa - m) \cos(\frac{s\pi}{2} + b).$

2.3 Lie point symmetries

A complete theory and application of Lie symmetry methods maybe found in text such as those of [32, 33, 34, 35, 36, 37]. Here we provide a brief outline. We consider a second order partial differential equation of the form

$$F(\tau, x, \theta, \theta_\tau, \theta_x, \theta_{\tau x}, \theta_{\tau\tau}, \theta_{xx}) = 0, \tag{2.11}$$

where τ and x are two independent variables and θ is the dependent variable. The subscripts denote partial differentiation. Basically we seek transformations of the form

$$\bar{x} = x + \epsilon\xi^1(\tau, x, \theta) + \mathcal{O}(\epsilon^2)$$
$$\bar{\tau} = \tau + \epsilon\xi^2(\tau, x, \theta) + \mathcal{O}(\epsilon^2) \tag{2.12}$$
$$\bar{\theta} = \theta + \epsilon\eta(\tau, x, \theta) + \mathcal{O}(\epsilon^2),$$

generated by

$$\mathcal{Y} = \xi^1(\tau, x, \theta)\frac{\partial}{\partial t} + \xi^2(\tau, x, \theta)\frac{\partial}{\partial x} + \eta(\tau, x, \theta)\frac{\partial}{\partial\theta}. \tag{2.13}$$

The transformations (2.12) are equivalent to the $r-$parameter group of transformations which leave Eqn (2.11) invariant. Calculation of symmetries involves determining the functions ξ^1, ξ^2 and η. Given second order Eqn (2.11), the invariance criterion is given by

$$\mathcal{Y}^{[2]}F(\tau, x, \theta, \theta_\tau, \theta_x, \theta_{\tau x}, \theta_{\tau\tau}, \theta_{xx})|_{F=0} = 0. \tag{2.14}$$

Here $\mathcal{Y}^{[2]}$ is the second prolongation of $\mathcal{Y}$ and is given by

$$\mathcal{Y}^{[2]} = \mathcal{Y} + \zeta_1\frac{\partial}{\partial\theta_\tau} + \zeta_2\frac{\partial}{\partial\theta_x} + \zeta_{11}\frac{\partial}{\partial\theta_{\tau\tau}} + \zeta_{12}\frac{\partial}{\partial\theta_{\tau x}} + \zeta_{22}\frac{\partial}{\partial\theta_{xx}} \tag{2.15}$$

where

$$\zeta_i = D_i(\eta) - \theta_l D_i(\xi^l), \quad i = 1, 2 \tag{2.16}$$

$$\zeta_{ij} = D_j(\zeta_i) - \theta_{il} D_j(\xi^l), \quad i, j = 1, 2 \tag{2.17}$$

with k summed from 1 to 2. The total derivatives with respect to τ and x are given by,

$$D_1 = D_\tau = \frac{\partial}{\partial \tau} + \theta_\tau \frac{\partial}{\partial \theta} + \theta_{\tau\tau} \frac{\partial}{\partial \theta_\tau} + \theta_{x\tau} \frac{\partial}{\partial \theta_x} + \cdots, \tag{2.18}$$

$$D_2 = D_x = \frac{\partial}{\partial x} + \theta_x \frac{\partial}{\partial \theta} + \theta_{\tau x} \frac{\partial}{\partial \theta_\tau} + \theta_{xx} \frac{\partial}{\partial \theta_x} + \cdots. \tag{2.19}$$

The invariance criterion (2.14) results in an overdetermined system of partial equations, which can be solved. Any linear combination of symmetries obtained may reduce the number of variables of a PDE by one. Furthermore, one may reduce the order of the ordinary differential equation (ODE) by one or completely integrate.

The group invariant solution, $\theta = \mathcal{W}(\tau, x)$ of the nonlinear partial differential equation is obtained by solving

$$\mathcal{Y}(\theta - \mathcal{W}(\tau, x)) \Big|_{\theta = \mathcal{W}(\tau, x)} = 0. \tag{2.20}$$

2.4 Conservation laws

A very useful application of symmetry is to the construction of conservation laws. A conservation law for a system of partial differential equations is a divergence expression which vanishes on solutions of a system of PDEs [38]. Conservation laws play a pivotal role in the study of partial differential equations (DEs) and in many applications. The mathematical idea of conservation laws comes from the formulation of physical laws such as for mass, energy and momentum. Furthermore, conservation laws have applications in the study of PDEs such as in showing existence and uniqueness of solutions for hyperbolic systems of PDEs [39].

Consider a lth order PDE,

$$F(\mathbf{x}, u, u_{(1)}, u_{(2)}, \ldots, u_{(l)}) = 0, \tag{2.21}$$

where $\mathbf{x}$ denotes n independent variables, u denote the dependent variable and $u_{(i)}$ denotes all the partial derivatives of order i. For an arbitrary PDE we write

$$D_i(T^i) = 0, \tag{2.22}$$

where T^i are differential functions of finite order. We define (2.22) as a conservation law for Eqn (2.21) if it satisfies the following equation.

$$D_i\big[T^i(\mathbf{x}, u, u_{(1)}, u_{(2)}, \ldots, u_{(l)})\big] = 0. \tag{2.23}$$

This can also be written as

$$D_i T^i\big|_{F=0} = 0. \tag{2.24}$$

The vector $T = (T^1, T^1, \ldots, T^n)$ is called a conserved vector.

A Lie point symmetry generator

$$\mathcal{Y} = \xi^i(x, u)\frac{\partial}{\partial x^i} + \eta(x, u)\frac{\partial}{\partial u} \tag{2.25}$$

is said to be associated with the conserved vector $T^i = (T^1, \ldots, T^n)$ for Eqn (2.21) if [40]

$$\mathcal{Y}(T^i) + T^i D_l(\xi^l) - T^l D_l(\xi^i) = 0, \ \ i = 1, \ldots, n. \tag{2.26}$$

In (2.26), $\mathcal{Y}$ is prolonged appropriately. The conserved vectors can be determined from (2.26) [28]. We determine conservation laws using two techniques outlined below.

2.4.1   Direct method

This approach was first used by Laplace [41], and gives all local conservation laws. Equation (2.22) is a conservation law. The direct method uses (2.22)

subject to the (2.21) being satisfied as the determining equation for the conserved vectors. The components $T^1, \ldots, T^n$ are obtained by separating the resulting equation according to powers and products of the derivatives of u.

2.4.2   Multiplier method

This approach involves the variational derivative

$$D_i T^i = \Lambda^\alpha F, \qquad (2.27)$$

where Λ^α are the characteristics. The characteristics are the multipliers which make the equation exact. This method is sometimes referred to this as the characteristic method.

2.5   Concluding remarks

In this Chapter, we have provided brief accounts of the methods to be used in analyzing the problems considered in this dissertation. Full accounts of these methods maybe be found in acknowledged references.

Chapter 3

Mathematical models describing heat transfer in longitudinal fins

3.1 Introduction

In this Chapter we derive the models describing heat conduction in longitudinal fins of various profiles. The imposed boundary conditions for heat transfer are the adiabatic conditions at the fin tip and the constant base temperature.

3.2 Mathematical models

A physical system consisting of a longitudinal one dimensional fin of cross-sectional area A_c is shown in Figure 3.1. The perimeter of the fin is denoted by P and its length by L. The fin is attached to a fixed prime surface of temperature T_b and extends to an ambient fluid of temperature T_a. The fin thickness at the prime surface is given by δ_b and its profile is given by $F(X)$. We assume that the fin is initially at ambient temperature. At time $t = 0$, the temperature at the base of the fin is suddenly changed from T_a to T_b and the

problem is to establish the temperature distribution in the fin for all $t \geq 0$. Based on the one dimensional heat conduction, the energy balance equation is then given by (see e.g. [1])

$$\rho c \frac{\partial T}{\partial t} = \frac{\delta_b}{2} \frac{\partial}{\partial X}\left(F(X)K(T)\frac{\partial T}{\partial X}\right) - \frac{P}{A_c}H(T)(T - T_a), \ 0 \leq X \leq L. \quad (3.1)$$

where K and H are the non-uniform thermal conductivity and heat transfer coefficients depending on the temperature (see e.g. [1]), ρ is the density, c is the specific heat capacity, T is the temperature distribution, t is the time and X is the space variable. Assuming that the fin tip is adiabatic (insulated) and the base temperature is kept constant, then the boundary conditions are given by [1],
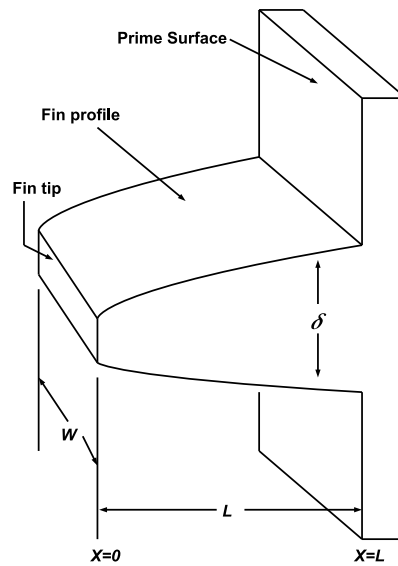
Figure 3.1: Schematic representation of a longitudinal fin of an unspecified profile.

$$T(t, L) = T_b \ \text{ and } \ \left.\frac{\partial T}{\partial X}\right|_{X=0} = 0, \quad (3.2)$$

and initially the fin is kept at the ambient temperature,

$$T(0, X) = T_a. \tag{3.3}$$

Introducing the following dimensionless variables,

$$x = \frac{X}{L}, \; \tau = \frac{k_a t}{\rho c_v L^2}, \; \theta = \frac{T - T_a}{T_b - T_a}, \; h = \frac{H}{h_b},$$
$$k = \frac{K}{k_a}, \; M^2 = \frac{P h_b L^2}{A_c k_a} \; \text{and} \; f(x) = \frac{\delta_b}{2} F(X), \tag{3.4}$$

reduces Eqn (3.1) to

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left(f(x)k(\theta)\frac{\partial \theta}{\partial x}\right) - M^2 h(\theta)\theta, \; 0 \le x \le 1 \tag{3.5}$$

and Eqn (3.5) admits the following boundary conditions,

$$\theta(0, x) = 0, \; 0 \le x \le 1; \; \theta(\tau, 1) = 1; \tag{3.6}$$

$$\left.\frac{\partial \theta}{\partial x}\right|_{x=0} = 0. \tag{3.7}$$

The dimensionless variable M is the thermo-geometric fin parameter, θ is the dimensionless temperature, x is the dimensionless space variable, k is the dimensionless thermal conductivity, k_a is the thermal conductivity of the fin at ambient temperature, h_b is the heat transfer coefficient at the fin base. For most industrial applications, the heat transfer coefficient maybe given as a power law [2, 42],

$$H(T) = h_b\left(\frac{T - T_a}{T_b - T_a}\right)^n, \tag{3.8}$$

where n and h_b are constants. The constant n may vary between -6.6 and 5. However, in most practical applications it lies between -3 and 3 [42]. The exponent n represents laminar film boiling or condensation when $n = -1/4$, laminar natural convection when $n = 1/4$, turbulent natural convection when $n = 1/3$, nucleate boiling when $n = 2$, radiation when $n = 3$ and $n = 0$ implies a constant heat transfer coefficient. Exact solutions may be constructed

for the steady-state one-dimensional differential ODE describing temperature distribution in a straight fin when the thermal conductivity is a constant and $n = -1, 0, 1$ and 2 [42].

In dimensionless variables we have $h(\theta) = \theta^n$. We consider the two distinct cases of the thermal conductivity as follows;

(a) the power law

$$K(T) = k_a \left(\frac{T - T_a}{T_b - T_a} \right)^m,$$
(3.9)

with m being a constant and

(b) the linear function [43]

$$K(T) = k_a[1 + \gamma(T - T_a)].$$
(3.10)

The dimensionless thermal conductivity given by the power law and the linear function of temperature are $k(\theta) = \theta^m$ and $k(\theta) = 1 + \beta\theta$, respectively. Here the thermal conductivity gradient is $\beta = \gamma(T_b - T_a)$.

The dimensionless steady problem is given by

$$\frac{d}{dx}\left[f(x)k(\theta)\frac{d\theta}{dx} \right] - M^2 h(\theta)\theta = 0, \ 0 \leq x \leq 1,$$
(3.11)

and the boundary conditions are

$$\theta(1) = 1, \ \text{at the prime surface}$$
(3.12)

and

$$\frac{d\theta}{dx}\bigg|_{x=0} = 0, \ \text{at the fin tip.}$$
(3.13)

For steady heat transfer, we consider various fin profiles including the longitudinal rectangular $f(x) = 1$, the longitudinal convex parabolic $f(x) = \sqrt{x}$ and the exponential profile $f(x) = e^{ax}$ with a being the constant (see also [44]). We only consider the rectangular and the convex parabolic fin profiles for transient heat conduction.

3.3 Concluding remarks

In this chapter, we have provided the models for heat transfer in longitudinal fins of various profiles. We note that the heat transfer coefficient is given as a power law with the exponent n, and thermal conductivity is given both as power law with exponent m on one hand, and it is given as a linear function of temperature on the other. The steady state exact solutions exist only when both the exponent of the thermal conductivity and heat transfer coefficient are the same. In this dissertation we construct steady state analytical solutions for distinct power laws and transient state solutions. Exact solutions for transient state processes, with temperature dependent thermal properties, do not exist.

Chapter 4

Analytical solutions for steady heat transfer in fins of various profiles

4.1 Introduction

In this Chapter we consider steady state models describing temperature distribution in longitudinal fins of different profiles. It is well known that exact solutions for ODEs such as Eqn (3.11) exist only when thermal conductivity and the term containing the heat transfer coefficient are connected by differentiation (or simply if the ODE such as Eqn (3.11) is linearizable) [45]. We determine the analytical solutions for the non-linearizable Eqn (3.11), firstly when thermal conductivity is given by the power law and secondly as a linear function of temperature. In both cases and throughout this dissertation, the heat transfer coefficient is assumed to be a power law function of temperature. These assumptions of the thermal properties are physically realistic. We have noticed that DTM runs into difficulty when the exponent of the power law of

the thermal conductivity is given by fractional values and also when the function $f(x)$ is given in terms of fractional powers. One may follow Moradi and Ahmadikia [44] by introducing a new variable to deal with fractional powers of $f(x)$, and on the other hand it is possible to remove the fractional exponent of the heat transfer coefficient by fundamental laws of exponent and binomial expansion.

Remark. A nonlinear ODE such as Eqn (3.11) may admit the DTM solution if $f(x)$ is a constant or exponential function. However, if $f(x)$ is a power law then such an equation admits a DTM solution if the product

$$f(x) \cdot f'(x) = \alpha,$$

holds, where α is a real constant.

Proof. Introducing the new variable $y = f(x)$, it follows from chain rule that Eqn (3.11) becomes

$$\alpha \frac{dy}{dx} \frac{d}{dy} \left[k(\theta) \frac{d\theta}{dy} \right] - M^2 \theta^{n+1} = 0. \tag{4.1}$$

$\square$

Example 1. If $f(x) = x^\sigma$, then $y = x^\sigma$ transforms Eqn (3.11) into

$$\frac{d}{dy} \left[k(\theta) \frac{d\theta}{dy} \right] - 4M^2 y \theta^{n+1} = 0, \tag{4.2}$$

only if $\sigma = \frac{1}{2}$. We note that the condition $f(x)f'(x) = \alpha$ can be solved for $f(x)$ to give

$$f(x) = (2\alpha x + \vartheta)^{\frac{1}{2}} \tag{4.3}$$

where ϑ is a constant.

This example implies that the DTM may only be applicable to problems describing heat transfer in fins with convex parabolic profile. In the next subsections we present analytical solutions for Eqn (3.11) with various functions $f(x)$ and $k(\theta)$.

4.2 Power law thermal conductivity

The temperature dependent thermal conductivity of some materials such as Gallium Nitride (GaN) and Aluminum Nitride (AlN) can be modeled by the power law [46]. Experimental data indicates that the exponent of the power law of these materials is positive for lower temperatures and negative for higher temperatures [47].

4.2.1 Comparison of exact and analytical (DTM) solutions given $m = n$

In this section we consider a model describing temperature distribution in a longitudinal rectangular fin with both thermal conductivity and heat transfer coefficient being functions of temperature given by the power law (see e.g. [5]). The exact solution of Eqn (3.11) when both the power laws are given by the same exponent is given by [5]

$$\theta(x) = \left[\frac{\cosh(M\sqrt{n+1}x)}{\cosh(M\sqrt{n+1})} \right]^{\frac{1}{n+1}}. \tag{4.4}$$

We use this exact solution as a benchmark or validation of the DTM. The effectiveness of the DTM is determined by comparing the exact and the analytical solutions. We compare the results for the cases $n = 1$ and $n = 2$ with fixed values of M.

Case $n = 1$

Applying the DTM to Eqn (3.11) with the power law thermal conductivity, $f(x) = 1$ (rectangular profile) and given $\mathcal{H} = 1$, one obtains the following

recurrence relation

$$\sum_{i=0}^{\kappa} \Big[\Theta(i)(\kappa - i + 1)(\kappa - i + 2)\Theta(\kappa - i + 2)$$

$$+ (i + 1)\Theta(i + 1)(\kappa - i + 1)\Theta(\kappa - i + 1) - M^2\Theta(i)\Theta(\kappa - i) \Big] = 0. \quad (4.5)$$

Exerting the transformation to the boundary condition (3.13) at a point $x = 0$,

$$\Theta(1) = 0. \tag{4.6}$$

The other boundary conditions are considered as follows;

$$\Theta(0) = a \tag{4.7}$$

where a is a constant. Eqn (4.5) is an iterative formula of constructing the power series solution as follows:

$$\Theta(2) = \frac{M^2 a}{2} \tag{4.8}$$

$$\Theta(3) = 0 \tag{4.9}$$

$$\Theta(4) = -\frac{M^4 a}{24} \tag{4.10}$$

$$\Theta(5) = 0 \tag{4.11}$$

$$\Theta(6) = \frac{19 M^6 a}{720} \tag{4.12}$$

$$\Theta(7) = 0 \tag{4.13}$$

$$\Theta(8) = -\frac{559 M^8 a}{40320} \tag{4.14}$$

$$\Theta(9) = 0 \tag{4.15}$$

$$\Theta(10) = \frac{29161 M^{10} a}{3628800} \tag{4.16}$$

$$\Theta(11) = 0 \tag{4.17}$$

$$\Theta(12) = -\frac{2368081 M^{12} a}{479001600} \tag{4.18}$$

$$\Theta(13) = 0 \tag{4.19}$$

$$\Theta(14) = \frac{276580459 M^{14} a}{87178291200} \tag{4.20}$$

$$\Theta(15) = 0 \tag{4.21}$$

$$\vdots$$

These terms may be taken as far as desired. Substituting Eqns (4.6) to (4.20) into (2.4), we obtain the following series solutions

$$\theta(x) = a + \frac{M^2 a}{2} x^2 - \frac{M^4 a}{24} x^4 + \frac{19 M^6 a}{720} x^6 - \frac{559 M^8 a}{40320} x^8$$
$$+ \frac{29161 M^{10} a}{3628800} x^{10} - \frac{2368081 M^{12} a}{479001600} x^{12} + \frac{276580459 M^{14} a}{87178291200} x^{14} + \dots \tag{4.22}$$

To obtain the value of a, we substitute the boundary condition (3.12) into (4.22) at the point $x = 1$. Thus, we have,

$$\theta(1) = a + \frac{M^2 a}{2} - \frac{M^4 a}{24} + \frac{19 M^6 a}{720} - \frac{559 M^8 a}{40320}$$
$$+ \frac{29161 M^{10} a}{3628800} - \frac{2368081 M^{12} a}{479001600} + \frac{276580459 M^{14} a}{87178291200} + \dots = 1. \tag{4.23}$$

We then obtain the expression for $\theta(x)$ upon substituting the obtained value of a into Eqn (4.22).

Case $n = 2$

Applying the DTM to Eqn (3.11) for $n = 2$, one obtains the following recurrence relation

$$\sum_{i=0}^{\kappa} \sum_{l=0}^{\kappa-i} \Bigg[\Theta(l)\Theta(i)(\kappa - i - l + 1)(\kappa - i - l + 2)\Theta(\kappa - i - l + 2)$$
$$+ 2\Theta(l)(i + 1)\Theta(i + 1)(\kappa - i - l + 1)\Theta(\kappa - i - l + 1)$$
$$- M^2 \Theta(l)\Theta(i)\Theta(\kappa - i - l) \Bigg] = 0. \tag{4.24}$$

Substituting the transformed boundary conditions into (4.24), we obtain the following series solution

$$
\begin{aligned}
\theta(x) = a &+ \frac{M^2 a}{2}x^2 - \frac{M^4 a}{8}x^4 + \frac{23M^6 a}{240}x^6 - \frac{1069M^8 a}{13440}x^8 \\
&+ \frac{9643M^{10} a}{134400}x^{10} - \frac{1211729M^{12} a}{17740800}x^{12} + \frac{217994167M^{14} a}{3228825600}x^{14} + \ldots . \quad (4.25)
\end{aligned}
$$

The constant a may be obtained using the boundary condition at the fin base. The comparison of the exact solutions and the DTM solutions are reflected in Tables 4.1 and 4.2 for different values of n. We observe a very small absolute error on the results. Furthermore, the comparison of the exact and the analytical solutions are depicted in Figures 4.1 and 4.2.

Table 4.1: Results of the DTM and Exact Solutions for $n = 1$, $M = 0.7$

x	DTM	Exact	Error
0	0.808093014	0.80809644	0.000003426
0.1	0.810072036	0.81007547	0.000003434
0.2	0.815999549	0.81600301	0.000003459
0.3	0.825847770	0.82585127	0.000003500
0.4	0.839573173	0.83957673	0.000003559
0.5	0.857120287	0.85712392	0.000003633
0.6	0.878426299	0.87843002	0.000003722
0.7	0.903426003	0.90342982	0.000003814
0.8	0.932056648	0.93206047	0.000003820
0.9	0.964262558	0.96426582	0.000003263
1.0	1.000000000	1.00000000	0.000000000

Table 4.2: Results of the DTM and Exact Solutions for $n = 2,\ M = 0.5$

x	DTM	Exact	Error
0	0.894109126	0.894109793	0.000000665
0.1	0.895226066	0.895226732	0.000000666
0.2	0.898568581	0.898569249	0.000000668
0.3	0.904112232	0.904112905	0.000000672
0.4	0.911817796	0.911818474	0.000000678
0.5	0.921633352	0.921634038	0.000000685
0.6	0.933496900	0.933497594	0.000000694
0.7	0.947339244	0.947339946	0.000000702
0.8	0.963086933	0.963087627	0.000000694
0.9	0.980665073	0.980665659	0.000000586
1.0	1.000000000	1.000000000	0.000000000
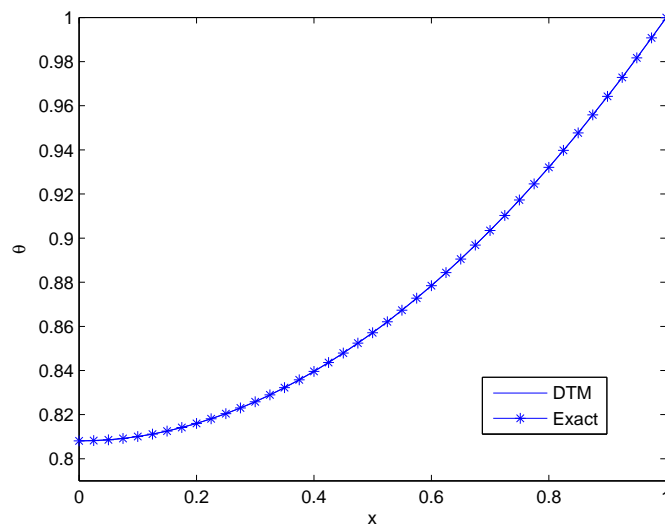
Figure 4.1: Comparison of analytical and exact solutions, $n = 1,\ M = 0.7$.
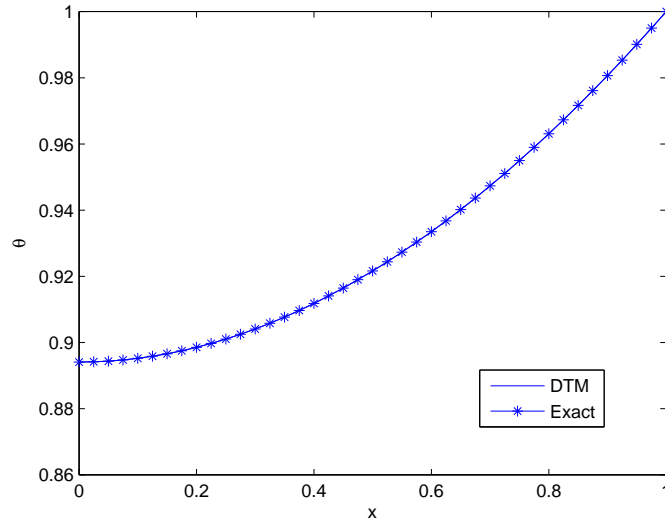
Figure 4.2: Comparison of analytical and exact solutions, $n = 2$, $M = 0.5$.

4.2.2 Distinct exponents, $m \neq n$

(i) The exponential profile and power law thermal conductivity

In this section we present solutions for equation describing heat transfer in a fin with exponential profile and power law thermal conductivity and heat transfer coefficient. That is, given Eqn (3.11) with $f(x) = e^{\sigma x}$, and both heat transfer coefficient and thermal conductivity are being given as power law functions of temperature, we construct analytical solutions. In our analysis we consider $n = 2$ and 3 indicating the fin subject to nucleate boiling, and radiation into free space at zero absolute temperature, respectively.

Firstly given $n = 3$ and $m = 2$ and applying the DTM one obtains the following recurrence revelation;

$$\sum_{i=0}^{\kappa}\sum_{l=0}^{\kappa-i}\sum_{p=0}^{\kappa-i-l}\left[\frac{\sigma^p}{p!}\Theta(l)\Theta(i)(\kappa-i-l-p+1)(\kappa-i-l-p+2)\times\right.$$

$$\Theta(\kappa-i-l-p+2)+2\frac{\sigma^p}{p!}\Theta(l)(i+1)\Theta(i+1)(\kappa-i-l-p+1)\times$$

$$\Theta(\kappa-i-l-p+1)+2\sigma\frac{\sigma^p}{p!}\Theta(l)\Theta(i)(\kappa-i-l-p+1)\times$$

$$\left.\Theta(\kappa-i-l-p+1)-M^2\Theta(p)\Theta(l)\Theta(i)\Theta(\kappa-i-l-p)\right]=0. \quad (4.26)$$

We recall the transformed prescribed boundary conditions (4.6) and (4.7). Eqn (4.26) is an iterative formula of constructing the power series solution as follows:

$$\Theta(2)=\frac{M^2a^2}{2} \quad (4.27)$$

$$\Theta(3)=-\frac{\sigma M^2a^2}{3} \quad (4.28)$$

$$\Theta(4)=\frac{M^2a^2(3\sigma^2-2M^2a)}{24} \quad (4.29)$$

$$\Theta(5)=-\frac{M^2a^2(\sigma^3-4\sigma M^2a)}{30} \quad (4.30)$$

$$\Theta(6)=\frac{M^2a^2(5\sigma^4-78\sigma^2M^2a+58M^4a^2)}{720} \quad (4.31)$$

$$\Theta(7)=\frac{M^2a^2\sigma(\sigma^4-50\sigma^2M^2a+146M^4a^2)}{840} \quad (4.32)$$

$$\vdots$$

The above process is continuous and one may consider as many terms as desired (but bearing in mind that DTM converges quite fast). Substituting Eqns (4.6), (4.7) and (4.27) to (4.32) into (2.4), we obtain the following series solution

$$\theta(x)=a+\frac{M^2a^2}{2}x^2-\frac{\sigma M^2a^2}{3}x^3+\frac{M^2a^2(3\sigma^2-2M^2a)}{24}x^4$$

$$-\frac{M^2a^2(\sigma^3-4\sigma M^2a)}{30}x^5+\frac{M^2a^2(5\sigma^4-78\sigma^2M^2a+58M^4a^2)}{720}x^6$$

$$-\frac{M^2a^2\sigma(\sigma^4-50\sigma^2M^2a+146M^4a^2)}{840}x^7+\dots \quad (4.33)$$

To obtain the value of a, we substitute the boundary condition (3.12) into (4.33) at the point $x = 1$. That is,

$$\theta(1) = a + \frac{M^2 a^2}{2} - \frac{\sigma M^2 a^2}{3} + \frac{M^2 a^2 (3\sigma^2 - 2M^2 a)}{24}$$
$$- \frac{M^2 a^2 (\sigma^3 - 4\sigma M^2 a)}{30} + \frac{M^2 a^2 (5\sigma^4 - 78\sigma^2 M^2 a + 58M^4 a^2)}{720}$$
$$- \frac{M^2 a^2 \sigma (\sigma^4 - 50\sigma^2 M^2 a + 146M^4 a^2)}{840} + \ldots = 1. \qquad (4.34)$$

Substituting this value of a into Eqn (4.33) one finds the expression for $\theta(x)$. On the other hand, given $(n, m) = (2, 3)$ one obtains the solution

$$\theta(x) = a + \frac{M^2}{2} x^2 - \frac{\sigma M^2}{3} x^3 + \frac{M^2 (\sigma^2 a - 2M^2)}{8a} x^4$$
$$- \frac{\sigma M^2 (2\sigma^2 a - 21M^2)}{60a} x^5 + \frac{M^2 (180M^4 - 186\sigma^2 M^2 a + 5\sigma^4 a^2)}{720a^2} x^6$$
$$- \frac{\sigma M^2 (440M^4 - 111\sigma^2 M^2 a + \sigma^4 a^2)}{840a^2} x^7 + \ldots \qquad (4.35)$$

Here, the constant a maybe obtained by evaluating the boundary condition $\theta(1) = 1$. The solutions (4.33) and (4.35) are depicted in Figures 4.3 and 4.4, respectively.

Figure 4.3: Temperature distribution in an exponential fin, $(n, m) = (3, 2)$.

Figure 4.4: Temperature distribution in an exponential fin, $(n, m) = (2, 3)$.

(ii) **The rectangular profile and power law thermal conductivity**

In this section we provide a detailed construction of analytical solutions for the heat transfer in a longitudinal rectangular fin with a power law thermal conductivity, that is we consider Eqn (3.11) with $f(x) = 1$ and $k(\theta) = \theta^m$. Taking the differential transform of Eqn (3.11) with $f(x) = 1$ and $k(\theta) = \theta^m$ for $(n, m) = (2, 3)$ we get

$$\sum_{i=0}^{\kappa} \sum_{l=0}^{\kappa-i} \sum_{p=0}^{\kappa-i-l} \left[\Theta(p)\Theta(l)\Theta(i)(\kappa - i - l - p + 1)(\kappa - i - l - p + 2) \times \right.$$

$$\Theta(\kappa - i - l - p + 2) + 3\Theta(p)\Theta(l)(i + 1)\Theta(i + 1) \times$$

$$\left. (\kappa - i - l - p + 1)\Theta(\kappa - i - l - p + 1) \right]$$

$$-M^2 \sum_{i=0}^{\kappa} \sum_{l=0}^{\kappa-i} \Theta(l)\Theta(i)\Theta(\kappa - i - l) = 0. \tag{4.36}$$

Equation (4.36) is a recurrence relation of constructing an analytical solution. Recalling the boundary conditions and following a similar approach for $(n, m) = (3, 2)$, we obtain the analytical expressions of the solutions as follows;

Case $(n, m) = (2, 3)$

$$\theta(x) = a + \frac{M^2}{2}x^2 - \frac{M^4}{4a}x^4 + \frac{M^6}{4a^2}x^6 - \frac{33M^8}{122a^3}x^8 + \frac{127M^{10}}{336a^4}x^{10}$$
$$- \frac{1889M^{12}}{3696a^5}x^{12} + \frac{160649M^{14}}{224224a^6}x^{12} - \frac{7225M^{16}}{7007a^7}x^{16} + \dots \qquad (4.37)$$

Case $(n, m) = (3, 2)$

$$\theta(x) = a + \frac{a^2 M^2}{2}x^2 - \frac{a^3 M^4}{12}x^4 + \frac{29a^4 M^6}{360}x^6 - \frac{307a^5 M^8}{5040}x^8$$
$$+ \frac{23483a^6 M^{10}}{453600}x^{10} - \frac{125893a^7 M^{12}}{2721600}x^{12} + \frac{1635899a^8 M^{14}}{38102400}x^{14}$$
$$- \frac{23417113a^9 M^{16}}{571536000}x^{16} + \dots \qquad (4.38)$$

The constant a is obtained by solving the appropriate $\theta(x)$ at the fin base boundary condition. The analytical solutions in Eqns (4.37) and (4.38) are depicted in Figures 4.5 and 4.6, respectively.
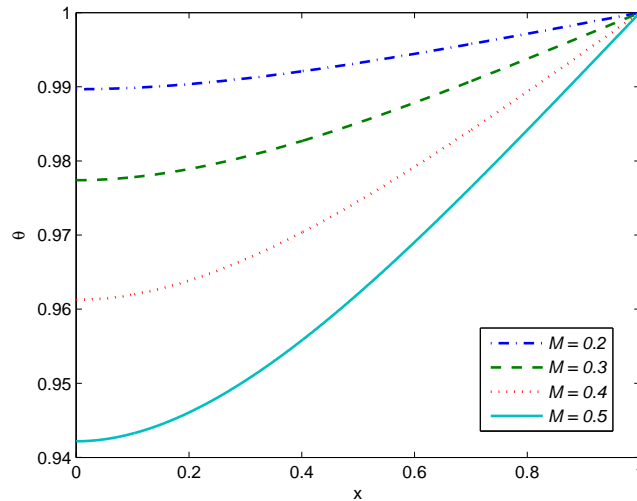

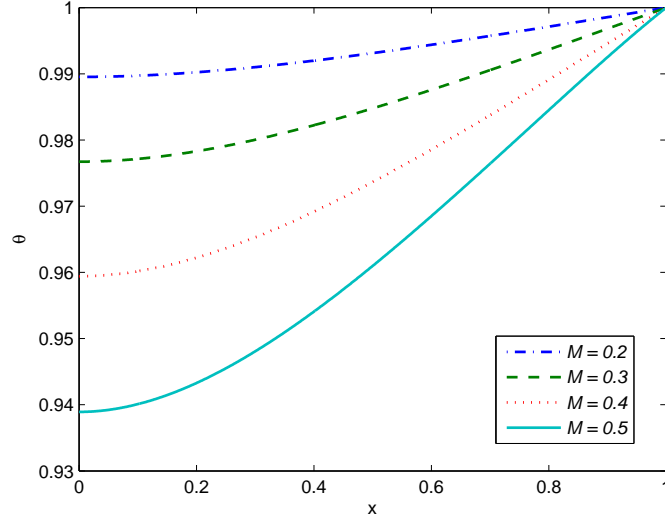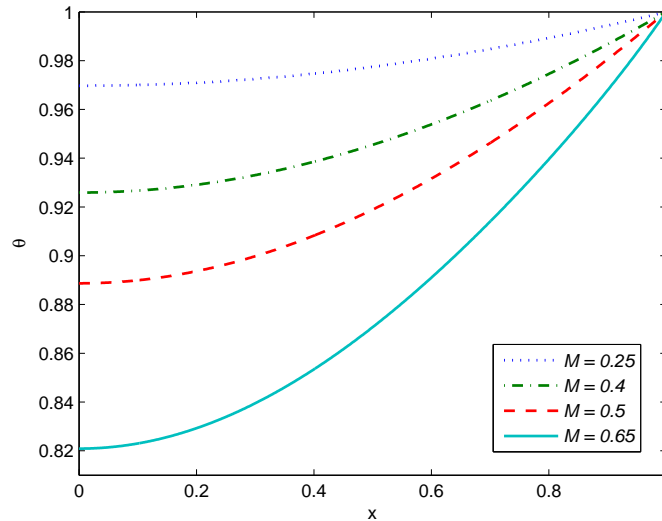Figure 4.5: Temperature distribution in a rectangular fin, $(n, m) = (2, 3)$.
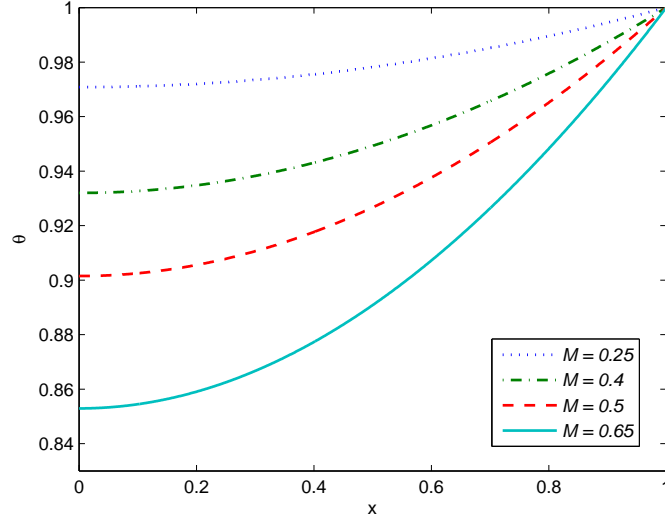
Figure 4.6: Temperature distribution in a rectangular fin, $(n, m) = (3, 2)$.

(iii) The convex profile and power law thermal conductivity

In this section we present solutions for the equation describing the heat transfer in a fin with convex parabolic profile and power law thermal conductivity. We consider Eqn (4.2), with the definition $y = x^{1/2}$. Here we consider the values $\{(n, m) = (2, 3); (3, 2)\}$. With $(n, m) = (2, 3)$, Eqn (4.2) reduces to

$$\frac{d}{dy}\left[\theta^3 \frac{d\theta}{dy}\right] - 4M^2 y \theta^3 = 0. \tag{4.39}$$

Taking the differential transform of Eqn (4.39) we obtain the following relation

$$\sum_{i=0}^{\kappa} \sum_{l=0}^{\kappa-i} \sum_{p=0}^{\kappa-i-l} \Bigg[\Theta(p)\Theta(l)\Theta(i)(\kappa - i - l - p + 1)(\kappa - i - l - p + 2) \times$$

$$\Theta(\kappa - i - l - p + 2) + 3\Theta(p)\Theta(l)(i + 1)\Theta(i + 1) \times$$

$$(\kappa - i - l - p + 1)\Theta(\kappa - i - l - p + 1)$$

$$- 4M^2 \delta(p - 1)\Theta(l)\Theta(i)\Theta(\kappa - i - l - p) \Bigg] = 0, \tag{4.40}$$

and substituting the transformed boundary conditions, we obtain the following

series solution,

$$\theta(x) = a + \frac{2M^2}{3}x^{3/2} - \frac{2M^4}{5a}x^3 + \frac{23M^6}{45a^2}x^{9/2} - \frac{1909M^8}{2475a^3}x^6$$
$$+ \frac{329222M^{10}}{259875a^4}x^{15/2} - \frac{174361597M^{12}}{79521750a^5}x^9 + \dots \quad (4.41)$$

Similarly for $(n, m) = (3, 2)$ we obtain the following solution

$$\theta(x) = a + \frac{2a^2M^2}{3}x^{3/2} - \frac{4a^3M^4}{45}x^3 + \frac{4a^4M^6}{27}x^{9/2} - \frac{992a^5M^8}{7425}x^6$$
$$+ \frac{30064a^6M^{10}}{212625}x^{15/2} - \frac{295216a^7M^{12}}{1893375}x^9 + \dots \quad (4.42)$$

The constant a is obtained by evaluating the appropriate $\theta(x)$ at the fin base boundary condition. The solutions in Eqns (4.41) and (4.42) are depicted in Figure 4.7 and 4.8, respectively.
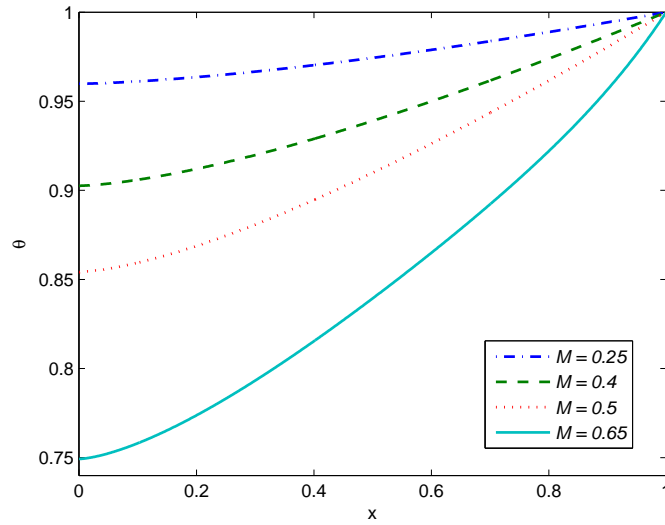
Figure 4.7: Temperature distribution in a convex fin, $(n, m) = (2, 3)$.
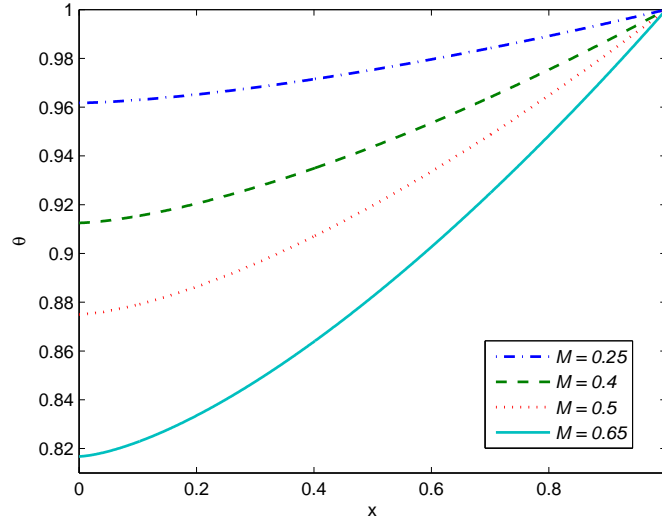
Figure 4.8: Temperature distribution in a convex fin, $(n, m) = (3, 2)$.

4.3 Linear thermal conductivity

In many engineering applications and many materials, thermal conductivity depends linearly on temperature (See e.g [48])

4.3.1 The rectangular profile, $f(x) = 1$

In this section we present solutions for the equation representing the heat transfer in a fin with rectangular profile and thermal conductivity depending linearly on temperature. That is, we consider Eqn (3.11) with $f(x) = 1$ and $k(\theta) = 1 + \beta\theta$. With these properties, taking the differential transform of Eqn (3.11) for $n = 0$ we get

$$
\begin{aligned}
&(\kappa + 1)(\kappa + 2)\Theta(\kappa + 2) \\
&+ \beta \sum_{i=0}^{\kappa} \big[\Theta(i)(\kappa - i + 1)(\kappa - i + 2)\Theta(\kappa - i + 2) \\
&+ \beta \sum_{i=0}^{\kappa} (i + 1)\Theta(i + 1)(\kappa - i + 1)\Theta(\kappa - i + 1)\big] - M^2\Theta(\kappa) = 0.
\end{aligned} \tag{4.43}
$$

Following a similar approach described above for other values of n we obtain the following analytical solutions upon substitution of the boundary conditions,

Case $n = 0$

$$\theta(x) = a + \frac{M^2 a}{2(1 + \beta a)} x^2 - \frac{M^4 a(-1 + 2\beta a)}{24(1 + \beta a)^3} x^4 + \frac{M^6 a(1 - 16\beta a + 28\beta^2 a^2)}{720(1 + \beta a)^5} x^6$$

$$- \frac{M^8 a(-1 + 78\beta a - 600\beta^2 a^2 + 896\beta^3 a^3)}{40320(1 + \beta a)^7} x^8$$

$$+ \frac{M^{10} a(1 - 332\beta a + 7812\beta^2 a^2 - 39896\beta^3 a^3 + 51184\beta^4 a^4)}{3628800(1 + \beta a)^9} x^{10}$$

$$+ \ldots. \tag{4.44}$$

Case $n = 1$

$$\theta(x) = a + \frac{M^2 a^2}{2(1 + \beta a)} x^2 - \frac{M^4 a^3(-1 + 2\beta a)}{24(1 + \beta a)^3} x^4$$

$$+ \frac{M^6 a^4(10 - 16\beta a + 19\beta^2 a^2)}{720(1 + \beta a)^5} x^6$$

$$- \frac{M^8 a^5(-80 + 342\beta a - 594\beta^2 a^2 + 559\beta^3 a^3)}{40320(1 + \beta a)^7} x^8$$

$$+ \frac{M^{10} a^6(1000 - 7820\beta a + 24336\beta^2 a^2 - 36908\beta^3 a^3 + 29161\beta^4 a^4)}{3628800(1 + \beta a)^9} x^{10}$$

$$+ \ldots. \tag{4.45}$$

Case $n = 2$

$$\theta(x) = a + \frac{M^2 a^3}{2(1 + \beta a)}x^2 + \frac{M^4 a^5}{8(1 + \beta a)^3}x^4 + \frac{M^6 a^7(3 + 2\beta^2 a^2)}{80(1 + \beta a)^5}x^6$$
$$+ \frac{M^8 a^9(49 - 20\beta a + 66\beta^2 a^2 - 40\beta^3 a^3)}{4480(1 + \beta a)^7}x^8$$
$$+ \frac{M^{10} a^{11}(427 - 440\beta a + 1116\beta^2 a^2 - 1020\beta^3 a^3 + 672\beta^4 a^4)}{134400(1 + \beta a)^9}x^{10}$$
$$+ \ldots. \tag{4.46}$$

Case $n = 3$

$$\theta(x) = a + \frac{M^2 a^4}{2(1 + \beta a)}x^2 + \frac{M^4 a^7(4 + \beta a)}{24(1 + \beta a)^3}x^4 + \frac{M^6 a^{10}(52 + 32\beta a + 25\beta^2 a^2)}{720(1 + \beta a)^5}x^6$$
$$+ \frac{M^8 a^{13}(1288 + 1020\beta a + 1212\beta^2 a^2 - 95\beta^3 a^3)}{40320(1 + \beta a)^7}x^8$$
$$+ \frac{M^{10} a^{16}(52024 + 45688\beta a + 77184\beta^2 a^2 - 680\beta^3 a^3 + 15025\beta^4 a^4)}{3628800(1 + \beta a)^9}x^{10}$$
$$+ \ldots. \tag{4.47}$$

The constant a may be obtained from the boundary condition on the appropriate solution. The solutions in Eqns (4.44), (4.45), (4.46) and (4.47) are depicted in Figure 4.9.

4.3.2 The convex parabolic profile, $f(x) = x^{1/2}$

In this section we present solutions for the equation describing the heat transfer in a fin with convex parabolic profile and the thermal conductivity depending linearly on temperature. That is, we consider Eqn (4.2) with $k(\theta) = 1 + \beta\theta$ and $f(x) = x^{1/2}$,

$$\frac{d}{dx}\left[x^{1/2}(1 + \beta\theta)\frac{d\theta}{dx}\right] - M^2\theta^{n+1} = 0. \tag{4.48}$$

Figure 4.9: Temperature profile in a longitudinal rectangular fin with linear thermal conductivity and varying values of n. Here $\beta = 0.5$ and $M = 1.5$ are fixed.

Substituting $y = x^{1/2}$ Eqn (4.48) reduces to

$$(1 + \beta\theta)\frac{d^2\theta}{dy^2} + \beta\left(\frac{d\theta}{dy}\right)^2 - 4M^2y\theta^{n+1} = 0. \tag{4.49}$$

Taking the differential transform of Eqn (4.49) for $n = 0$ we get

$$(\kappa + 1)(\kappa + 2)\Theta(\kappa + 2)$$
$$+ \beta\sum_{i=0}^{\kappa}\Theta(i)(\kappa - i + 1)(\kappa - i + 2)\Theta(\kappa - i + 2)$$
$$+ \beta\sum_{i=0}^{\kappa}(i + 1)\Theta(i + 1)(\kappa - i + 1)\Theta(\kappa - i + 1)$$
$$- 4M^2\sum_{i=0}^{\kappa}\delta(i - 1)\Theta(\kappa - i) = 0. \tag{4.50}$$

Following a similar approach described above for other values of n we obtain the following analytical solutions upon substitution of the boundary conditions,

Case $n = 0$

$$\theta(x) = a + \frac{2M^2c}{3(1 + \beta a)}x^{3/2} - \frac{2M^4a(-2 + 3\beta a)}{45(1 + \beta a)^3}x^3$$
$$+ \frac{M^6a(2 - 25\beta a + 33\beta^2a^2)}{405(1 + \beta a)^5}x^{9/2}$$
$$- \frac{M^8a(-10 + 599\beta a - 3582\beta^2a^2 + 4059\beta^3a^3)}{66825(1 + \beta a)^7}x^6 + \dots \quad (4.51)$$

Case $n = 1$

$$\theta(x) = a + \frac{2M^2a^2}{3(1 + \beta a)}x^{3/2} - \frac{2M^4a^3(-4 + \beta a)}{45(1 + \beta a)^3}x^3$$
$$+ \frac{2M^6a^4(9 - 11\beta a + 10\beta^2a^2)}{405(1 + \beta a)^5}x^{9/2}$$
$$- \frac{2M^8a^5(-330 + 1118\beta a - 1584\beta^2a^2 + 1093\beta^3a^3)}{66825(1 + \beta a)^7}x^6 + \dots \quad (4.52)$$

Case $n = 2$

$$\theta(x) = a + \frac{2M^2a^3}{3(1 + \beta a)}x^{3/2} + \frac{2M^4a^5(6 + \beta a)}{45(1 + \beta a)^3}x^3$$
$$+ \frac{M^6a^7(16 + 3\beta a + 7\beta^2a^2)}{135(1 + \beta a)^5}x^{9/2}$$
$$+ \frac{M^8a^9(1160 - 107\beta a + 1116\beta^2a^2 - 367\beta^3a^3)}{22275(1 + \beta a)^7}x^6 + \dots \quad (4.53)$$

Case $n = 3$

$$\theta(x) = a + \frac{2M^2 a^4}{3(1 + \beta a)} x^{3/2} + \frac{2M^4 a^7 (8 + 3\beta a)}{45(1 + \beta a)^3} x^3$$
$$+ \frac{4M^6 a^{10}(23 + 17\beta a + 9\beta^2 a^2)}{405(1 + \beta a)^5} x^{9/2}$$
$$+ \frac{2M^8 a^{13}(5000 + 4868\beta a + 4356\beta^2 a^2 + 363\beta^3 a^3)}{66825(1 + \beta a)^7} x^6 + \dots . \qquad (4.54)$$

The constant a may be obtained from the boundary condition on the appropriate $\theta(x)$. The solutions in Eqns (4.51), (4.52), (4.53) and (4.54) are depicted in Figure 4.10.
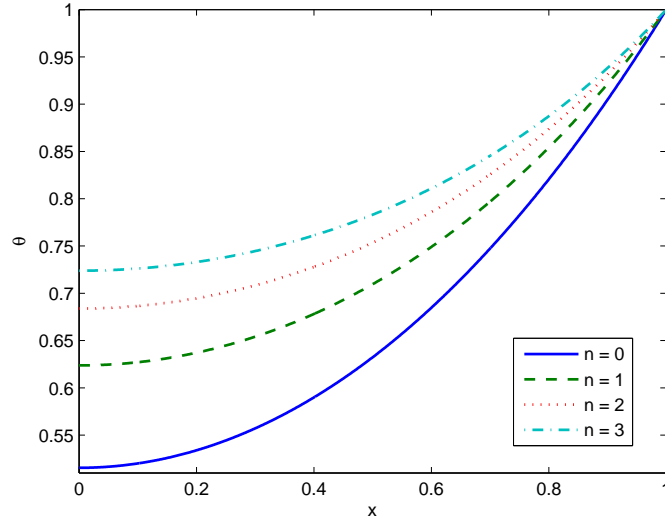

Figure 4.10: Temperature profile in a longitudinal convex parabolic fin with linear thermal conductivity and varying values of n. Here $\beta = 0.5$ and $M = 1.5$ are fixed.

4.3.3   The exponential profile, $f(x) = \mathrm{e}^{\sigma x}$

In this section we present solutions for the heat transfer equation in a fin with exponential profile and thermal conductivity depending linearly on temperature. That is, we consider Eqn (3.11) with $k(\theta) = 1 + \beta\theta$ and $f(x) = \mathrm{e}^{\sigma x}$, where σ is a constant,

$$\frac{d}{dx}\left[\mathrm{e}^{\sigma x}(1 + \beta\theta)\frac{d\theta}{dx}\right] - M^2\theta^{n+1} = 0. \tag{4.55}$$

Taking the differential transform of Eqn (4.55) for $n = 0$ we obtain the following recurrence relation;

$$\sum_{i=0}^{\kappa}\frac{\sigma^i}{i!}(\kappa - i + 1)(\kappa - i + 2)\Theta(\kappa - i + 2)$$

$$+ \beta\sum_{i=0}^{\kappa}\sum_{t=0}^{\kappa-i}\frac{\sigma^t}{t!}(\kappa - i - t + 1)(\kappa - i - t + 2)\Theta(\kappa - i - t + 2)$$

$$+ \sigma\sum_{i=0}^{\kappa}\frac{\sigma^i}{i!}(\kappa - i + 1)\Theta(\kappa - i + 1)$$

$$+ \sigma\beta\sum_{i=0}^{\kappa}\sum_{t=0}^{\kappa-i}\frac{\sigma^i}{i!}(\kappa - i - t + 1)\Theta(i)\Theta(\kappa - i - t + 1)$$

$$+ \beta\sum_{i=0}^{\kappa}\sum_{t=0}^{\kappa-i}\frac{\sigma^i}{i!}(i + 1)(\kappa - i - t + 1)\Theta(i + 1)\Theta(\kappa - i - t + 1)$$

$$- M^2\Theta(\kappa) = 0. \tag{4.56}$$

Following a similar approach described above for other values of n we obtain the following analytical solutions upon substitution of the boundary conditions,

Case $n = 0$

$$\theta(x) = a + \frac{M^2 a}{2(1 + \beta a)} x^2 - \frac{\sigma M^2 a(2 + \beta a + \beta^2 a)}{6(1 + \beta a)^2} x^3$$
$$+ \frac{M^2 a(M^2(1 - 2\beta^2 a) + \sigma^2(3 + 3\beta a + \beta^3 a^2 + \beta^4 a^2 + \beta^2 a(3 + a)))}{24(1 + \beta a)^3} x^4$$
$$+ \dots . \tag{4.57}$$

Case $n = 1$

$$\theta(x) = a + \frac{M^2 a^2}{2(1 + \beta a)} x^2 - \frac{\sigma M^2 a^2(2 + \beta a + \beta^2 a)}{6(1 + \beta a)^2} x^3$$
$$+ \frac{M^2 a^2(M^2 a(2 + \beta a - 2\beta^2 a) + \sigma^2(3 + 3\beta a + \beta^3 a^2 + \beta^4 a^2 + \beta^2 a(3 + a)))}{24(1 + \beta a)^3} x^4$$
$$+ \dots . \tag{4.58}$$

Case $n = 2$

$$\theta(x) = a + \frac{M^2 a^3}{2(1 + \beta a)} x^2 - \frac{\sigma M^2 a^3(2 + \beta a + \beta^2 a)}{6(1 + \beta a)^2} x^3$$
$$+ \frac{M^2 a^3(M^2 a^2(3 + 2\beta a - 2\beta^2 a) + \sigma^2(3 + \beta a(3 + \beta^2 a + \beta^3 a + \beta(3 + a))))}{24(1 + \beta a)^3} x^4$$
$$+ \dots . \tag{4.59}$$

Case $n = 3$

$$\theta(x) = a + \frac{M^2 a^4}{2(1 + \beta a)} x^2 - \frac{\sigma M^2 a^4 (2 + \beta a + \beta^2 a))}{6(1 + \beta a)^2} x^3$$

$$+ \frac{M^2 a^4 (M^2 a^3 (4 + 3\beta a - 2\beta^2 a) + \sigma^2 (3 + \beta a(3 + \beta^2 a + \beta^3 a + \beta(3 + a))))}{24(1 + \beta a)^3} x^4$$

$$+ \dots . \tag{4.60}$$

The constant a may be obtained from the boundary condition on the appropriate $\theta(x)$. The solutions in Eqns (4.57), (4.58), (4.59) and (4.60) are depicted in Figure 4.11.
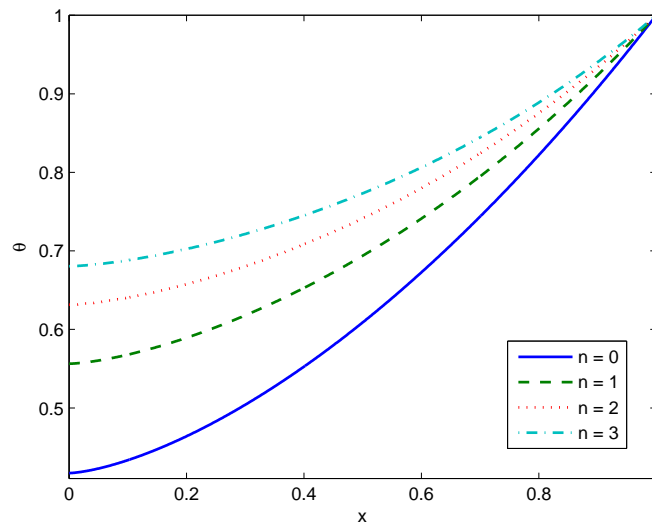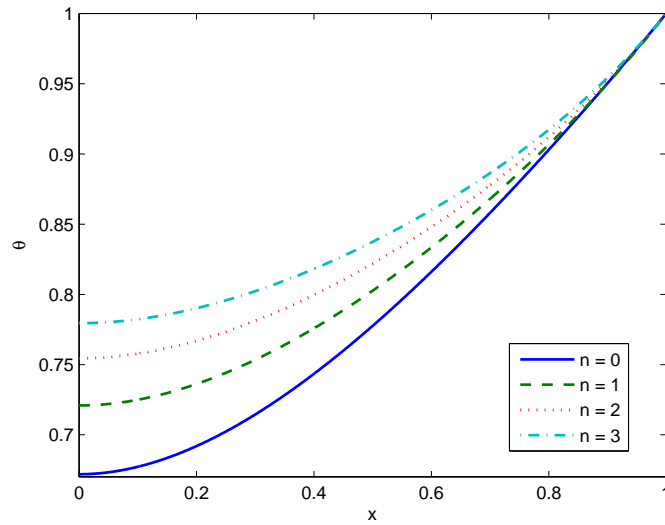

Figure 4.11: Temperature profile in a longitudinal fin of exponential profile with linear thermal conductivity and varying values of n. Here $\beta = 0.5$ and $M = 1.5$ are fixed.

4.4 Fin efficiency and heat flux

4.4.1 Fin efficiency

The heat transfer rate from a fin is given by Newton's second law of cooling,

$$Q = \int_0^L PH(T)(T - T_a)dx. \tag{4.61}$$

Fin efficiency is defined as the ratio of the fin heat transfer rate to the rate that would be if the entire fin were at the base temperature and is given by (see e.g. [1])

$$\eta = \frac{Q}{Q_{ideal}} = \frac{\int_0^L PH(T)(T - T_a)dX}{Ph_bL(T_b - T_a)}. \tag{4.62}$$

In dimensionless variables we have

$$\eta = \int_0^1 \theta^{n+1}dx. \tag{4.63}$$

We consider the solutions (4.44), (4.45), (4.46) and (4.47) and depict the fin efficiency (4.63) in Figure 4.12.
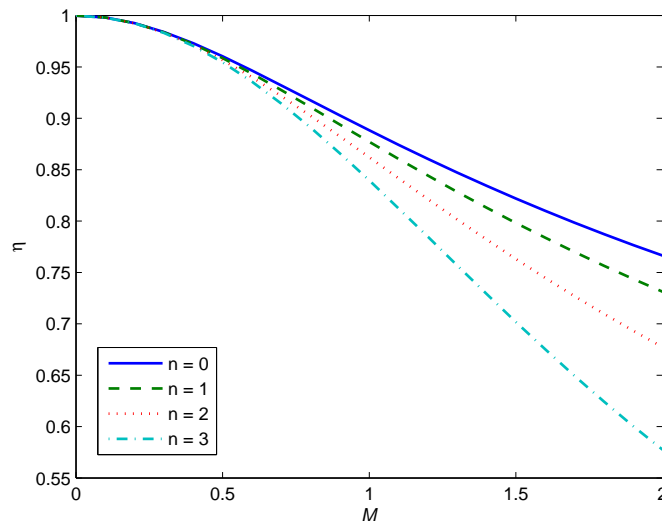
Figure 4.12: Fin efficiency of a longitudinal rectangular fin. Here $\beta = 0.75..$

4.4.2 Heat flux

The fin base heat flux is given by the Fourier's law

$$q_b = A_c K(T) \frac{dT}{dx}. \tag{4.64}$$

The total heat flux of the fin is given by [1]

$$q = \frac{q_b}{A_c H(T)(T_b - T_a)}. \tag{4.65}$$

Introducing the dimensionless variable as described in Chapter 3, implies

$$q = \frac{1}{Bi} \frac{k(\theta)}{h(\theta)} \frac{d\theta}{dx}, \tag{4.66}$$

where the dimensionless parameter $Bi = \dfrac{h_b L}{k_a}$ is the Biot number. We consider a number of cases for the thermal conductivity and the heat transfer coefficient.

With linear thermal conductivity and power law heat transfer coefficient Eqn (4.66) becomes

$$q = \frac{1}{Bi}(1 + \beta\theta)\theta^{-n}\frac{d\theta}{dx}. \tag{4.67}$$

The heat flux in Eqn (4.67) at the base of the fin is plotted in Figure 4.13.

For power law thermal conductivity and heat transfer coefficient Eqn (4.66) becomes

$$q = \frac{1}{Bi}\theta^{m-n}\frac{d\theta}{dx}. \tag{4.68}$$

Not surprisingly, heat flux in one dimensional fins is higher given values $Bi \ll 1$. The heat flux in Eqn (4.68) is plotted in Figures 4.14, 4.15 and 4.16.

4.5 Discussion of results

The DTM has resulted in some interesting observations and study. We have observed in Figures 4.1 and 4.2, an excellent agreement between the analytical
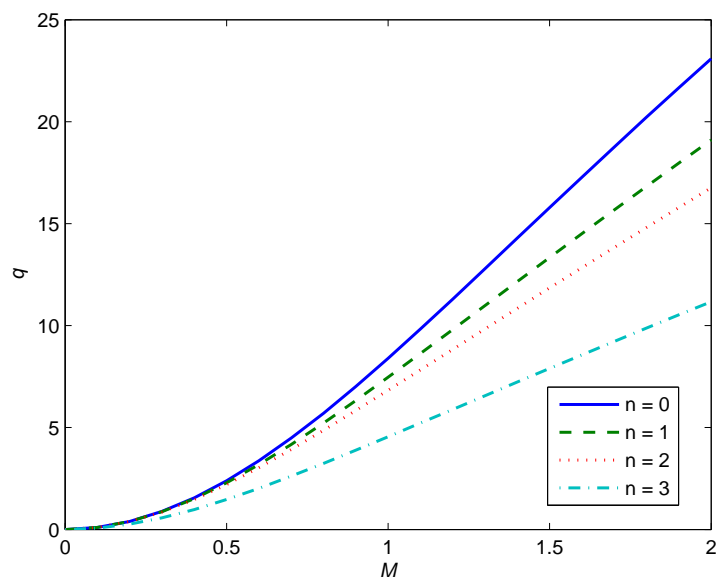
Figure 4.13: Base heat flux in a longitudinal rectangular fin with linear thermal conductivity. Here $\beta = 0.1.$.
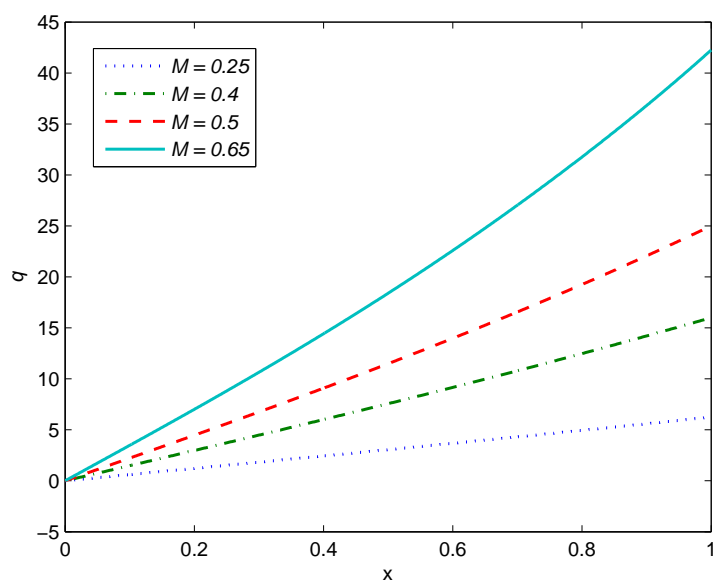
Figure 4.14: Heat flux across a longitudinal rectangular fin with power law thermal conductivity. Here $n - m > 0$, $Bi = 0.01.$.
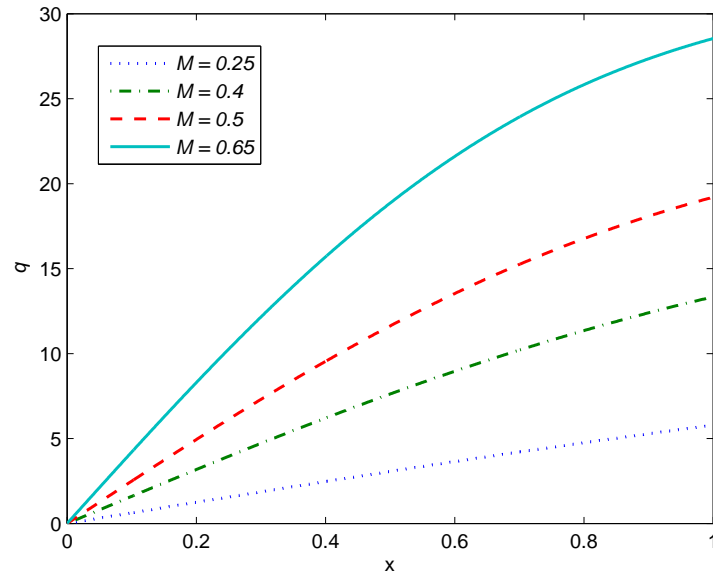
Figure 4.15: Heat flux across a longitudinal rectangular fin with power law thermal conductivity. Here $n - m < 0$, $Bi = 0.01$..
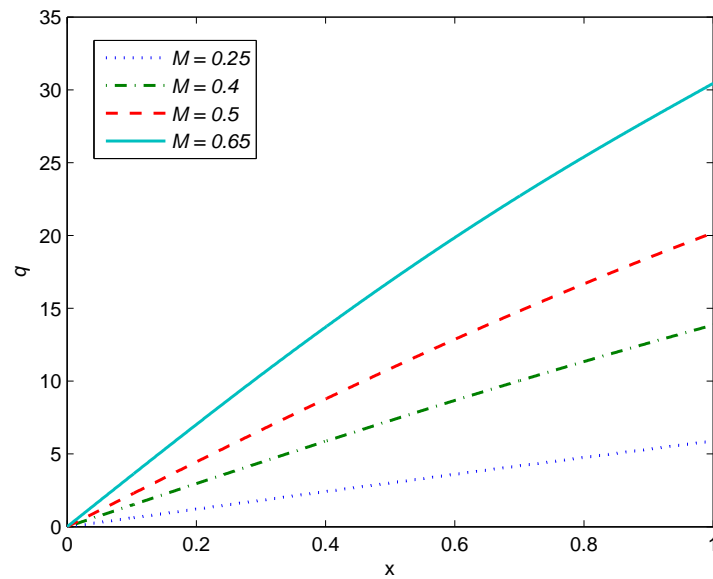
Figure 4.16: Base heat flux in a longitudinal rectangular fin with power law thermal conductivity. Here $n - m = 0$, $Bi = 0.01$..

solutions generated by DTM and the exact solution obtained in [5]. In particular, we considered a fin problem in which both thermal conductivity and heat transfer coefficient are both given by the same power law. Furthermore, we notice from Table 4.1 that an absolute error of approximately $3.5e - 005$ is produced by DTM of order $O(15)$. In Table 4.2 an absolute error of approximately $6.5e - 006$ is produced for the same order. This confirms that the DTM converges faster and can provide accurate results with minimum computational effort. As such, a tremendous confidence in the DTM in terms of the accuracy and effectiveness was built and thus we used this method to solve other problems for which exact solutions are harder to construct.

In the Figures 4.3, 4.4, 4.5, 4.6, 4.7 and 4.8, we observe that the fin temperature increases with the decreasing values of the thermo-geometric fin parameter. Here, the values of the exponents are fixed. Also, we observe that fin temperature is higher when $n - m > 0$, that is, when heat transfer coefficient is higher than the thermal conductivity. We observe in Figures 4.9, 4.10 and 4.11, that the fin temperature increases with the increasing values of n. Furthermore, it appears that the fin with exponential profile performs least in transferring the heat from the base, since the temperature in such a fin is much higher than that of the rectangular and the convex parabolic profiles. In other words, heat dissipation to the fluid surrounding the extended surface is much faster in longitudinal fins of rectangular and convex parabolic profiles. In Figure 4.12, fin efficiency decreases with increasing thermo-geometric fin parameter. Also, fin efficiency increases with increasing values of n. It is easy to show that the thermo-geometric fin parameter is directly proportional to the aspect ratio (extension factor) with square root of the Biot number being the proportionality constant. As such shorter fins are more efficient than longer ones. The increased Biot number results in less efficient fin whenever

the space is confined, that is where the length of the fin cannot be increased. Figure 4.13, depicts the heat flux at the fin base. The amount of heat energy dissipated from the fin base is of immense interest in engineering [49]. We observe in Figure 4.13 that the base heat flux increases with the thermo-geometric fin parameter for considered values of the exponent n (see also [49]). Figures 4.14, 4.15 and 4.16 display the heat flux across the fin length. We note that the heat flux across the fin length increases with increasing values of the thermo-geometric fin parameter.

The performance of fins is sometimes expressed in terms of the fin effectiveness ε defined as [50],

$$\varepsilon = \frac{q}{A_c H(T)(T_b - T_a)}, \tag{4.69}$$

where A_c is cross sectional area at the fin base. q represents the rate of heat transfer from this area if no fins are attached to the surface. Fin effectiveness is then the ratio of the fin heat transfer rate to the heat transfer rate of the object if it had no fin. The physical significance of effectiveness of fin can be summarized below

○ An effectiveness of $\varepsilon = 1$ indicates that the addition of a fin to the surface does not affect heat transfer at all. That is, heat conducted to the fin through the base area A_c is equal to the heat transferred from the same area A_c to the surrounding medium.

○ An effectiveness of $\varepsilon < 1$ indicates that the fin actually acts as an insulation, slowing down the heat transfer from the surface. This situation can occur when fins made of low thermal conductivity materials are used.

○ An effectiveness of $\varepsilon > 1$ indicates that the fins are enhancing heat transfer from the surface, as they should. However, the use of fins cannot

be justified unless ε is sufficiently larger than 1. Finned surfaces are designed on the basis of maximizing effectiveness of a specified cost or minimizing cost for a desired effectiveness.

4.6 Concluding remarks

In this chapter we have successfully applied the 1D DTM to highly nonlinear problems arising in heat transfer through longitudinal fins of various profiles. Both thermal conductivity and heat transfer coefficient are given as functions of temperature. The DTM agreed well with exact solutions when the thermal conductivity and heat transfer coefficient are given by the same power law. A rapid convergence to the exact solution was observed. Following the confidence in DTM built by the results mentioned, we then solved various problems. Obtained results have been shown in tables and figures listed in this chapter.

The results obtained in this chapter are significant improvements on the known results. In particular, both the heat transfer coefficient and thermal conductivity are allowed to be given by the power law functions of temperature, and also we considered three fin profiles. We note that exact solutions are difficult if not impossible to construct when the exponents of the power laws for the heat transfer coefficient and thermal conductivity are distinct.

Chapter 5

Application of the 2D DTM to transient heat transfer problems

5.1   Introduction

In this Chapter we extend the 1D DTM in order to solve partial differential equations. The resulting method is referred to as the 2D DTM. We saw from the previous chapter that the differential transform method offers great advantages of straightforward applicability, computational efficiency and high accuracy. In the following sections we attempt to solve transient heat conduction equation where the thermal conductivity is given by a linear function of temperature. We consider heat transfer in fins of convex parabolic profile and the rectangular profile. Using 2D DTM to solve PDEs consists of three main steps which are; transforming the PDE into algebraic equations, solving the equations, and inverting the solution of algebraic equations to obtain a series solution or an approximate solution.

When thermal conductivity depends linearly on temperature, Eqn (3.5) is

then given by the following nonlinear equation,

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial}{\partial x}\left(f(x)(1+\beta\theta)\frac{\partial\theta}{\partial x}\right) - M^2\theta^{n+1} \tag{5.1}$$

The initial condition is,

$$\theta(0,x) = 0,\ 0 \le x \le 1, \tag{5.2}$$

the boundary conditions are,

$$\theta(\tau,1) = 1, \tag{5.3}$$

and

$$\left.\frac{\partial\theta}{\partial x}\right|_{x=0} = 0. \tag{5.4}$$

5.2 Rectangular profile

Here we consider a fin of a rectangular profile $f(x) = 1$ for transient heat conduction through fins.

5.2.1 Solution for $n = 1$

Taking the two-dimensional differential transform of eqn (5.1) with $f(x) = 1$ and $n = 1$, we obtain the following recurrence relation

$$(k+1)\Theta(k+1,h) = (h+1)(h+2)\Theta(k,h+2)$$

$$+ \beta\sum_{i=0}^{\kappa}\sum_{j=0}^{h}\Theta(\kappa-i,j)(h+1-j)(h+2-j)\Theta(i,h+2-j)$$

$$+ \beta\sum_{i=0}^{\kappa}\sum_{j=0}^{h}(j+1)\Theta(\kappa-i,j+1)(h+1-j)\Theta(i,h+1-j)$$

$$- M^2\sum_{i=0}^{\kappa}\sum_{j=0}^{h}\Theta(i,h-j)\Theta(\kappa-i,j), \tag{5.5}$$

where $\Theta(\kappa, h)$ is the differential transform of $\theta(\tau, x)$.

Taking the two-dimensional differential transform of the initial condition (5.2) and boundary condition (5.4) we obtain the following transformations respectively,

$$\Theta(0, h) = 0, \ h = 0, 1, 2, \ldots \tag{5.6}$$

$$\Theta(\kappa, 1) = 0, \ \kappa = 0, 1, 2, \ldots \tag{5.7}$$

We consider the other boundary condition as follows,

$$\Theta(\kappa, 0) = a, \ a \in \mathbb{R}, \ \kappa = 1, 2, 3, \ldots \tag{5.8}$$

where the constant a can be determined from the boundary condition (5.7) at each time step after obtaining the series solution.

Substituting Eqns (5.6)-(5.8) into (5.5) we obtain the following,

$$\Theta(1, 2) = a \tag{5.9}$$

$$\Theta(2, 2) = \frac{1}{2}(3a - 2\beta a^2 + a^2 M^2) \tag{5.10}$$

$$\Theta(3, 2) = \frac{1}{2}(4a - 5\beta a^2 + 2\beta^2 a^3 + 2a^2 M^2 - \beta a^3 M^2) \tag{5.11}$$

$$\Theta(1, 4) = \frac{1}{12}(3a - 2\beta a^2 + a^2 M^2) \tag{5.12}$$

$$\Theta(2, 4) = \frac{1}{24}(12a - 33\beta a^2 + 10\beta^2 a^3 + 10a^2 M^2 - 5\beta a^3 M^2) \tag{5.13}$$

$$\vdots$$

Substituting Eqns (5.6)-(5.13) into (2.9) we obtain the following series solution,

$$\begin{aligned}
\theta(\tau, x) = {} & a\tau + a\tau^2 + a\tau x^2 + \frac{1}{2}(3a - 2\beta a^2 + a^2 M^2)\tau^2 x^2 \\
& + a\tau^3 + \frac{1}{2}(4a - 5\beta a^2 + 2\beta^2 a^3 + 2a^2 M^2 - \beta a^3 M^2)\tau^3 x^2 \\
& + a\tau^4 + \frac{1}{12}(3a - 2\beta a^2 + a^2 M^2)\tau x^4 \\
& + \frac{1}{24}(12a - 33\beta a^2 + 10\beta^2 a^3 + 10a^2 M^2 - 5\beta a^3 M^2)\tau^2 x^4 + \ldots \tag{5.14}
\end{aligned}$$

The constant a can be determined from the boundary condition (5.2) at each time step. To obtain the value of a, we substitute the boundary condition (5.2) into (5.14) at the point $x = 1$. Thus, we have,

$$\theta(\tau, 1) = a\tau + a\tau^2 + a\tau + \frac{1}{2}(3a - 2\beta a^2 + a^2 M^2)\tau^2$$
$$+ a\tau^3 + \frac{1}{2}(4a - 5\beta a^2 + 2\beta^2 a^3 + 2a^2 M^2 - \beta a^3 M^2)\tau^3$$
$$+ a\tau^4 + \frac{1}{12}(3a - 2\beta a^2 + a^2 M^2)\tau$$
$$+ \frac{1}{24}(12a - 33\beta a^2 + 10\beta^2 a^3 + 10a^2 M^2 - 5\beta a^3 M^2)\tau^2$$
$$+ \ldots = 1 \tag{5.15}$$

We then obtain the expression for $\theta(\tau, x)$ upon substituting the obtained value of a into equation (5.14). Using the first 40 terms of the power series solution we plot the solution (5.14) for various parameters as shown in Figures 5.1, 5.2 and 5.3.

5.2.2 Solution for $n = 0$

$$\theta(\tau, x) = a\tau + a\tau^2 + \frac{1}{2}(2a + aM^2)\tau x^2 + \frac{1}{2}(3a - 2a^2\beta + aM^2 - a^2\beta M^2)\tau^2 x^2$$
$$+ a\tau^3 + \frac{1}{2}(4a - 5a^2\beta + 2a^3\beta^2 + aM^2 - 2a^2\beta M^2 + a^3\beta^2 M^2)\tau^3 x^2$$
$$+ a\tau^4 + \frac{1}{24}(6a - 4a^2\beta + 4aM^2 - 2a^2\beta M^2 + aM^4)\tau x^4 + \ldots \tag{5.16}$$

5.2.3 Solution for $n = 2$

$$\theta(\tau, x) = a\tau + a\tau^2 + a\tau x^2 + \frac{1}{2}(3a - 2a^2\beta)\tau^2 x^2 + a\tau^3$$
$$+ \frac{1}{2}(4a - 5a^2\beta + 2a^3\beta^2 + a^3 M^2)\tau^3 x^2 + a\tau^4$$
$$+ \frac{1}{12}(3a - 2a^2\beta)\tau x^4 + \frac{1}{24}(12a - 33a^2\beta + 10a^3\beta^2 + 3a^3 M^2)\tau^2 x^4 + \ldots.$$

$$(5.17)$$

5.2.4 Solution for $n = 3$

$$\theta(\tau, x) = a\tau + a\tau^2 + a\tau x^2 + \frac{1}{2}(3a - 2a^2\beta)\tau^2 x^2 + a\tau^3$$
$$+ \frac{1}{2}(4a - 5a^2\beta + 2a^3\beta^2)\tau^3 x^2 + a\tau^4 + \frac{1}{12}(3a - 2a^2\beta)\tau x^4$$
$$+ \frac{1}{24}(12a - 33a^2\beta + 10a^3\beta^2)\tau^2 x^4 + \ldots \qquad (5.18)$$

Plots of solutions (5.16), (5.17) and (5.18) for various parameters are shown in Figures 5.4-5.8.

5.3 Convex parabolic profile

Here we consider a fin of a convex profile $f(x) = x^{1/2}$ for transient heat conduction through fins. We make a substitution $y = x^{1/2}$ to eliminate fractional powers. Eqn (5.1) then simplifies to

$$4y\frac{\partial\theta}{\partial\tau} = (1 + \beta\theta)\frac{\partial^2\theta}{\partial y^2} + \beta\left(\frac{\partial\theta}{\partial y}\right)^2 - 4M^2 y\theta^{n+1} \qquad (5.19)$$

5.3.1 Solution for $n = 1$

Taking the two-dimensional differential transform of eqn (5.19) for $n = 1$, we obtain

$$4 \sum_{i=0}^{\kappa} \sum_{j=0}^{h} (i+1)\Theta(i+1, h-j)\delta(j-1)\delta(k-i) = (h+1)(h+2)\Theta(k, h+2)$$

$$+ \beta \sum_{i=0}^{\kappa} \sum_{j=0}^{h} \Theta(\kappa - i, j)(h+1-j)(h+2-j)\Theta(i, h+2-j)$$

$$+ \beta \sum_{i=0}^{\kappa} \sum_{j=0}^{h} (j+1)\Theta(\kappa - i, j+1)(h+1-j)\Theta(i, h+1-j)$$

$$- 4M^2 \sum_{i=0}^{\kappa} \sum_{t=0}^{k-i} \sum_{j=0}^{h} \sum_{p=0}^{h-j} \Theta(i, h-j-p)\Theta(t, j)\delta(\kappa - i - t)\delta(p-1) \quad (5.20)$$

Substituting Eqns (5.6)-(5.8) into Eqn (5.20) we obtain the following,

$$\Theta(1, 3) = \frac{4a}{3} \quad (5.21)$$

$$\Theta(2, 3) = -\frac{2}{3}(-3a + 2a^2\beta - a^2 M^2) \quad (5.22)$$

$$\Theta(3, 3) = \frac{2}{3}(4a - 5a^2\beta + 2a^3\beta^2 + 2a^2 M^2 - a^3\beta M^2) \quad (5.23)$$

$$\Theta(4, 3) = -\frac{2}{3}(-5a + 9a^2\beta - 7a^3\beta^2 + 2a^4\beta^3 - 3a^2 M^2 + 3a^3\beta M^2 - a^4\beta^2 M^2) \quad (5.24)$$

$$\vdots$$

Substituting Eqns (5.6)-(5.8) and Eqns (5.21)-(5.24) into Eqn (2.9) we obtain the following series solution,

$$\theta(\tau, x) = a\tau + a\tau^2 + \frac{4a}{3}\tau x^{3/2} - \frac{2a}{3}(-3 + 2a\beta - aM^2)\tau^2 x^{3/2} + a\tau^3$$

$$+ \frac{2a}{3}(4 - 5a\beta + 2a^2\beta^2 + 2aM^2 - a^2\beta M^2)\tau^3 x^{3/2} + a\tau^4 + \dots \quad (5.25)$$

5.3.2 Solution for $n = 0$

$$\theta(\tau, x) = a\tau + a\tau^2 + \frac{2a}{3}(2 + M^2)\tau x^{3/2} - \frac{2a}{3}(-3 + 2a\beta - M^2 + a\beta M^2)\tau^2 x^{3/2}$$
$$+ \frac{2a}{3}(4 - 5a\beta + 2a^2\beta^2 + M^2 - 2a\beta M^2 + a^2\beta^2 M^2)\tau^3 x^{3/2} + a\tau^3 + \ldots$$

$$(5.26)$$

5.3.3 Solution for $n = 2$

$$\theta(\tau, x) = a\tau + a\tau^2 + \frac{4a}{3}\tau x^{3/2} - \frac{2a}{3}(-3 + 2a\beta - aM^2)\tau^2 x^{3/2} + a\tau^3$$
$$+ \frac{2a}{3}(4 - 5a\beta + 2a^2\beta^2 + 2aM^2 - a^2\beta M^2)\tau^3 x^{3/2} + a\tau^4 + \ldots \quad (5.27)$$

Plots of solutions (5.25) and (5.27) are shown in Figures 5.9 and 5.10 respectively.

5.4 Discussion of results

Some interesting results were obtained using the 2D DTM. We observe in Figure 5.1 that the fin temperature increases with time. This transient temperature profile approaches the steady state solution as time evolves. We observe in Figures 5.2 and 5.9 that the temperature decreases with increasing values of the thermo-geometric fin parameter. We note that the thermo-geometric fin parameter $M = (Bi)^{1/2}E$, where $Bi = h_b\delta/k_a$ is the Biot number and $E = L/\delta$ is the aspect ratio or the extension factor. Evidently, small values of M corresponds to the relatively short and thick fins of high conductivity and high values of M correspond to longer and thin fins of poor conductivity [48]. A fin is an excellent dissipater at small values of M. As M increases the convective
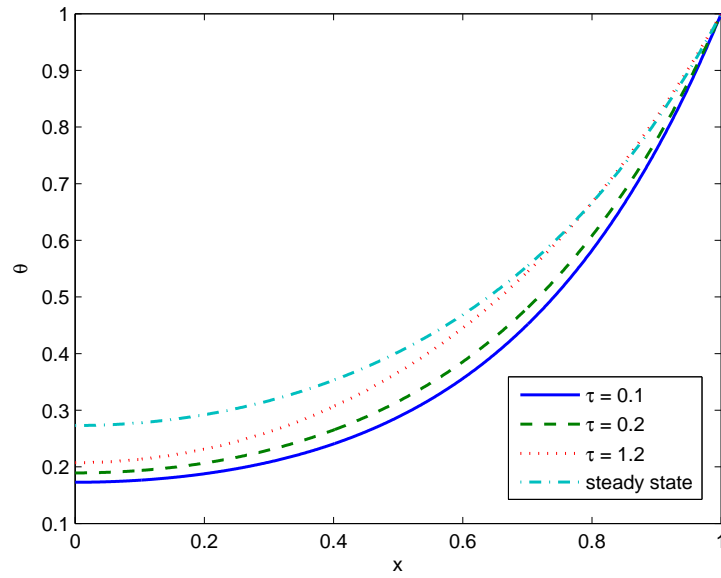
Figure 5.1: Transient temperature distribution in a rectangular fin with linear thermal conductivity, $n = 1, \beta = 1, M = 6$.
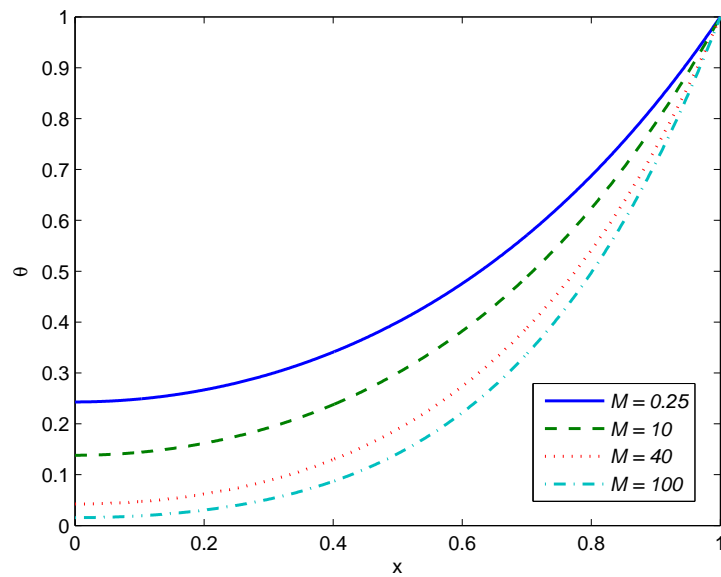
Figure 5.2: Transient temperature distribution in a rectangular fin with linear thermal conductivity, $n = 1, \beta = 1, \tau = 1.2$.
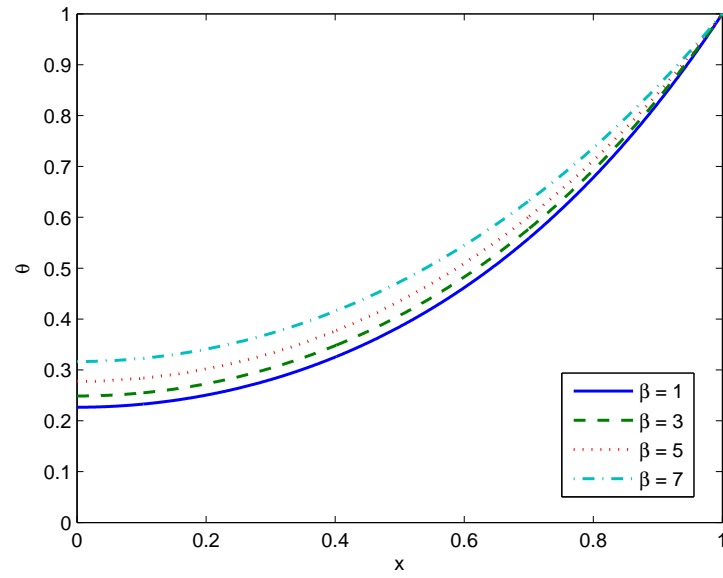
Figure 5.3: Transient temperature distribution in a rectangular fin with linear thermal conductivity, $n = 1, M = 2.5, \tau = 1.2$.
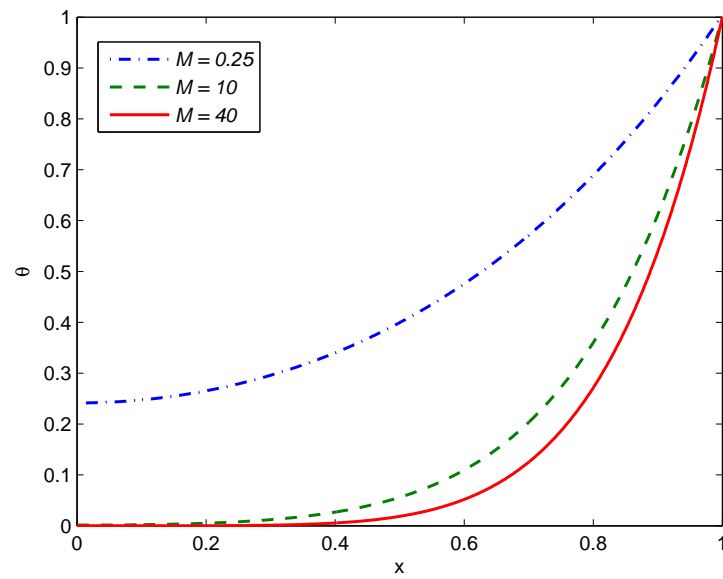
Figure 5.4: Transient temperature distribution in a rectangular fin with linear thermal conductivity, $n = 0, \beta = 1, \tau = 1.2$.
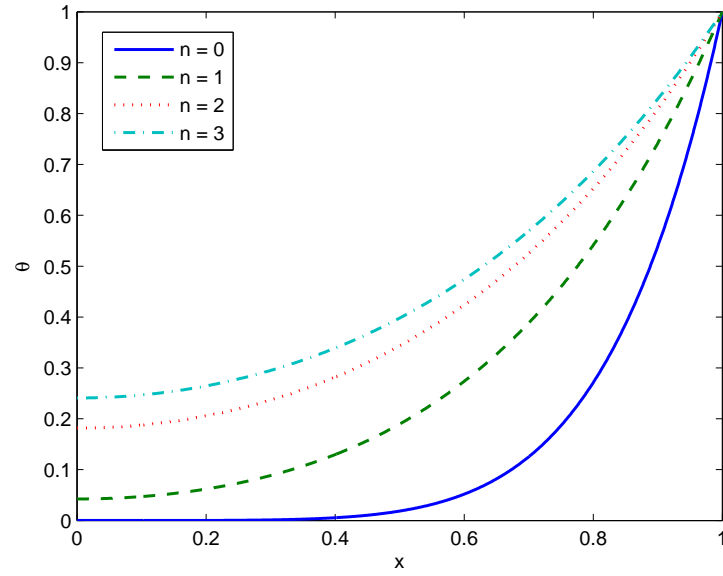
Figure 5.5: Transient temperature distribution in a rectangular fin with linear thermal conductivity, $M = 40, \beta = 1, \tau = 1.2$.
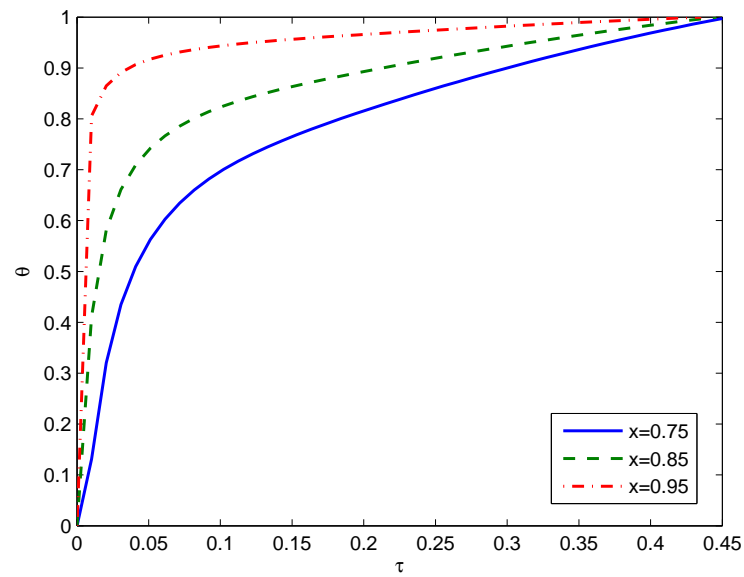
Figure 5.6: Transient temperature distribution in a rectangular fin with linear thermal conductivity, $M = \beta = n = 1$.
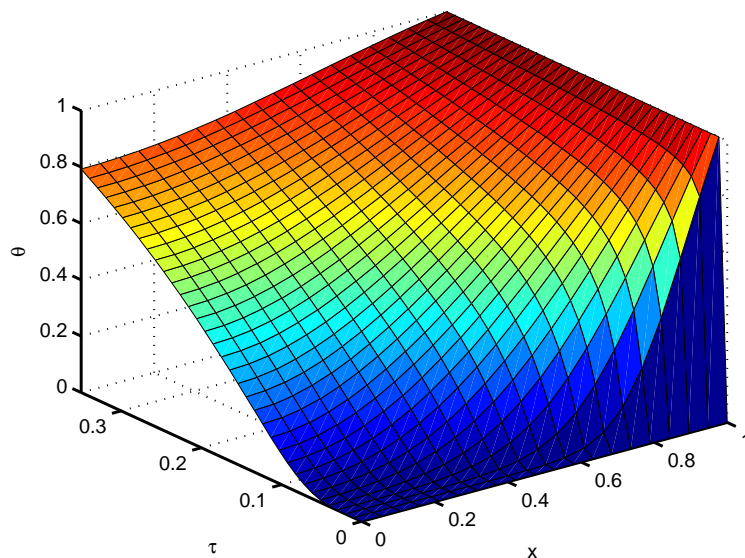
Figure 5.7: Transient temperature distribution in a rectangular fin with linear thermal conductivity, $M = \beta = n = 1$.
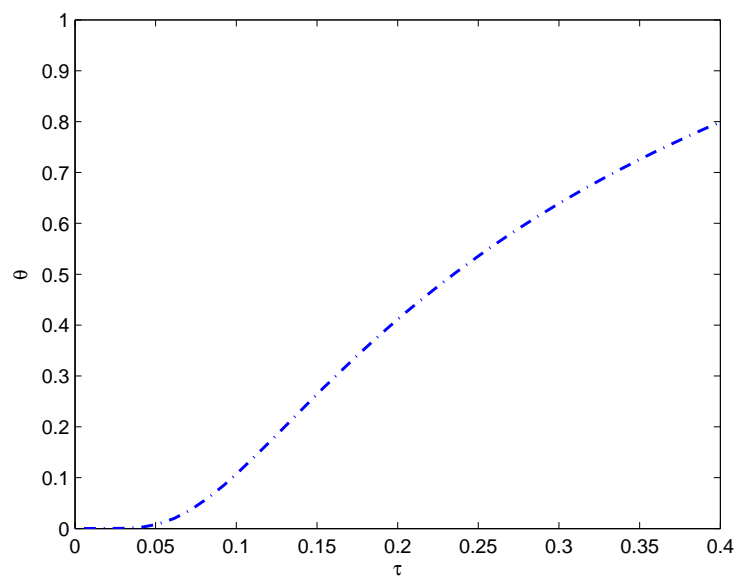
Figure 5.8: Transient temperature distribution in a rectangular fin with linear thermal conductivity, $M = 0.75, \beta = 1, n = 1$.
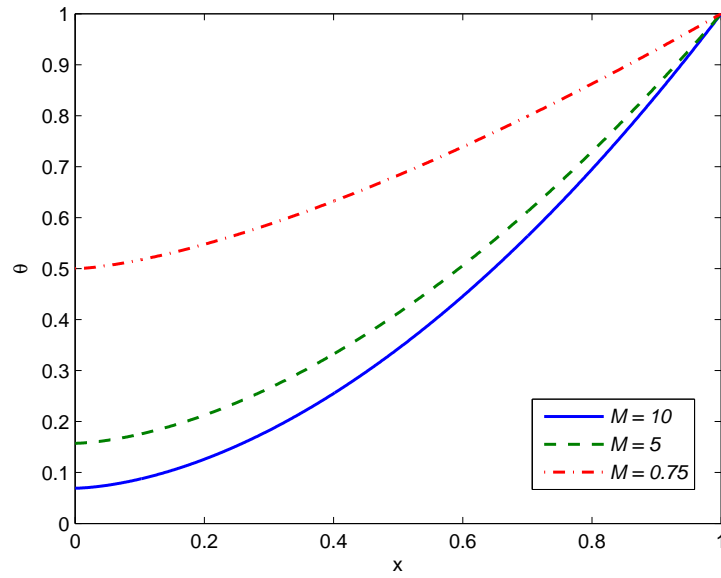
Figure 5.9: Transient temperature distribution in a convex parabolic fin with linear thermal conductivity, $\beta = 1, \tau = 0.2$.
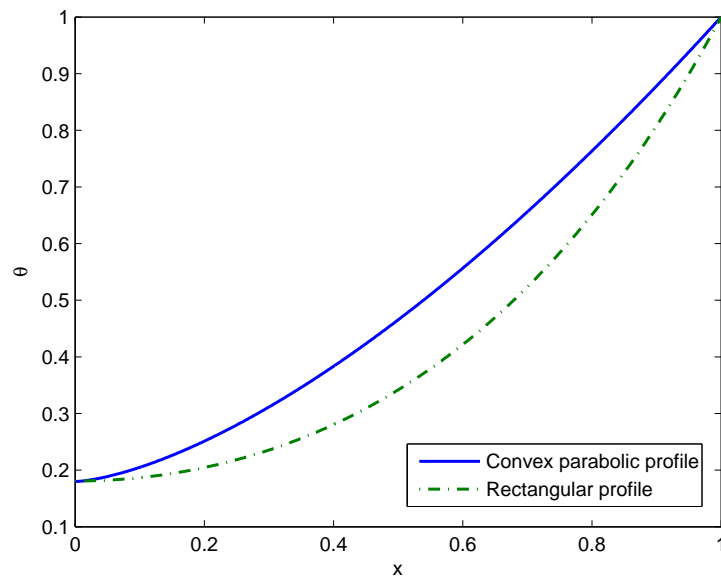
Figure 5.10: Transient temperature distribution in a rectangular and convex parabolic fin with linear thermal conductivity, $n = 2, M = 6, \beta = 1, \tau = 1.2$.

heat loss increases and the temperature profile becomes steeper reflecting high base heat flow rates (see e.g. [51]).

We observe in Figure 5.6, that the closer one gets to the base of the fin, the initial temperature gets to unity, even at small time scale and this correspond to the imposed initial condition. In Figure 5.3, we observe that the temperature in the fin increases with the increasing values of the thermal conductivity gradient. We note that $\beta = 0$ implies that the thermal conductivity is a constant given by k_a. The curves with $\beta > 0$, such as those in Figure 5.3, corresponds to the fin material whose thermal conductivity increases as temperature increases from T_a to T_b [51]. The converse of the curves with $\beta < 0$ holds according to [51]. In Figure 5.7, we plot the three dimensional distribution of temperature in a rectangular fin. The fin temperature at radiation is much higher than at the nucleate boiling and constant heat transfer coefficient as observed in Figure 5.5. As such, the temperature distribution in the fin increase with increasing values of n (this is due to fact that heat transfer coefficient over the fin with $n = 3$ is smaller than that for $n = 0$). The fin tip temperature gradually increases as time evolves as observed in Figure 5.8. For a fixed value of n we compared the temperature distribution in a rectangular and convex parabolic fin profiles as depicted in Figure 5.10. We note that the temperature distribution in a fin is always greater in a convex profile than in a rectangular profile. Thus the rectangular fin is efficient in dissipating heat to the ambient fluid. However, the profile for the temperature in convex parabolic is much steeper than that of the rectangular profile. This indicates high flow rates in convex parabolic fins.

5.5 Concluding remarks

We have successfully applied the 2D DTM to transient heat conduction problems for heat transfer in longitudinal rectangular and convex parabolic fins. Some interesting results are obtained and heat transfer processes are compared for the two profiles, namely the rectangular and the convex parabolic. We have studied the effects of the parameters appearing in the models on the temperature distribution. On the other hand, we obtained solutions which approach the steady state as time evolves.

Chapter 6

Conservation laws and associated Lie point symmetries for equations describing transient heat transfer in fins

6.1 Introduction

In this Chapter we will derive conservation laws for the two PDEs describing transient heat transfer through extended surfaces given two cases of the thermal conductivity term. An elegant way of constructing conservation laws is by use of the Noether's theorem [52]. This theorem states that for Euler-Lagrange differential equations, to each Noether symmetry associated with a Lagrangian there corresponds a conservation law which can be determined explicitly by means of a formula. The application of Noether's theorem depends on the knowledge of a Lagrangian. The considered fin equations in this dissertation do not have a Lagrangian, thus we stick to the approaches which were briefly

introduced in chapter 2. One may follow the approach of partial Lagrangian
and partial Noether as formulated by Kara and Mahomed [28]. The equations
to be considered are;

the nonlinear PDE with power law thermal conductivity

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left(\theta^m \frac{\partial \theta}{\partial x}\right) - M^2 \theta^{n+1}, \tag{6.1}$$

the nonlinear PDE with linear thermal conductivity

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial x}\left((1 + \beta\theta)\frac{\partial \theta}{\partial x}\right) - M^2 \theta^{n+1}. \tag{6.2}$$

Here m and n are arbitrary constants. Equations (6.1) and (6.2) describe
heat transfer in longitudinal fins of rectangular profile.

6.2 Nonlinear equation with power law thermal conductivity

6.2.1 Direct method

A conservation law for (6.1) satisfies

$$D_1 T^1 + D_2 T^2 \bigg|_{(6.1)} = 0, \tag{6.3}$$

where D_1 and D_2 are the total derivatives given by (2.18) and (2.19) respectively.

For simplicity and convenience we seek the conserved vectors of the form

$$T^1 = (\tau, x, \theta, \theta_x), \quad T^2 = (\tau, x, \theta, \theta_x). \tag{6.4}$$

Conserved vectors which depend on θ_τ and θ_{xx} may also considered but the
computations become laborious. Substituting (6.4) into (6.3) we obtain the

following partial differential equation,

$$\left(\frac{\partial T^1}{\partial \tau} + \theta_\tau \frac{\partial T^1}{\partial \theta} + \theta_{x\tau} \frac{\partial T^1}{\partial \theta_x} + \frac{\partial T^2}{\partial x} + \theta_x \frac{\partial T^2}{\partial \theta} + \theta_{xx} \frac{\partial T^2}{\partial \theta_x} \right) \Bigg|_{(6.1)} = 0. \qquad (6.5)$$

Expanding Eqn (6.5) we obtain,

$$\frac{\partial T^1}{\partial \tau} + \left[m\theta^{m-1} \left(\frac{\partial \theta}{\partial x} \right)^2 + \theta^m \frac{\partial^2 \theta}{\partial x^2} - M^2 \theta^{n+1} \right] \frac{\partial T^1}{\partial \theta} +$$

$$\theta_{x\tau} \frac{\partial T^1}{\partial \theta_x} + \frac{\partial T^2}{\partial x} + \theta_x \frac{\partial T^2}{\partial \theta} + \theta_{xx} \frac{\partial T^2}{\partial \theta_x} = 0. \qquad (6.6)$$

Since T^1 and T^2 are independent of the second derivatives of θ, we can separate
(6.6) by second derivatives of θ:

$$\theta_{xx} \quad : \quad \theta^m \frac{\partial T^1}{\partial \theta} + \frac{\partial T^2}{\partial \theta_x} = 0, \qquad (6.7)$$

$$\theta_{x\tau} \quad : \quad \frac{\partial T^1}{\partial \theta_x} = 0, \qquad (6.8)$$

$$\text{rem} \quad : \quad \frac{\partial T^1}{\partial \tau} + m\theta^{m-1}\theta_x^2 \frac{\partial T^1}{\partial \theta} + \frac{\partial T^2}{\partial x}$$

$$+ \theta_x \frac{\partial T^2}{\partial \theta} - M^2 \theta^{n+1} \frac{\partial T^1}{\partial \theta} = 0. \qquad (6.9)$$

Eqn (6.8) implies that $T^1 = T^1(\tau, x, \theta)$. Thus, we can integrate (6.7) with
respect to θ_x to obtain an explicit expression for T^2,

$$T^2 = -\theta^m \frac{\partial T^1}{\partial \theta} \theta_x + A(\tau, x, \theta). \qquad (6.10)$$

Substituting T^2 into (6.9) and separating by θ_x:

$$\theta_x^2 \quad : \quad -\theta^m \frac{\partial^2 T^1}{\partial \theta^2} = 0, \qquad (6.11)$$

$$\theta_x \quad : \quad -\theta^m \frac{\partial^2 T^1}{\partial \theta \partial x} + \frac{\partial A}{\partial \theta} = 0 \qquad (6.12)$$

$$\text{rem} \quad : \quad \frac{\partial T^1}{\partial \tau} + \frac{\partial A}{\partial x} - M^2 \theta^{n+1} \frac{\partial T^1}{\partial \theta} = 0. \qquad (6.13)$$

Integrating (6.11) we obtain an explicit expression for T^1,

$$T^1 = B(\tau, x)\theta + C(\tau, x). \qquad (6.14)$$

Thus, Eqn (6.12) can be written as,

$$-\theta^m \frac{\partial B}{\partial x} + \frac{\partial A}{\partial \theta} = 0. \tag{6.15}$$

Integrating (6.15) we obtain an explicit expression for A,

$$A = \frac{\theta^{m+1}}{m+1} \frac{\partial B}{\partial x} + D(\tau, x), \qquad m \neq -1. \tag{6.16}$$

Substituting A and T^1 into (6.13), we obtain,

$$\theta \frac{\partial B}{\partial \tau} + \frac{\partial C}{\partial \tau} + \frac{\theta^{m+1}}{m+1} \frac{\partial^2 B}{\partial x^2} + \frac{\partial D}{\partial x} - M^2 \theta^{n+1} B = 0 \tag{6.17}$$

We now consider two cases to separate (6.17) by powers of θ:

Case $m \neq n$ $(m \neq 0, n \neq 0, n \neq -1)$

$$\theta \quad : \quad \frac{\partial B}{\partial \tau} = 0, \tag{6.18}$$

$$\theta^{m+1} \quad : \quad \frac{1}{m+1} \frac{\partial^2 B}{\partial x^2} = 0, \tag{6.19}$$

$$\theta^{n+1} \quad : \quad -M^2 B = 0, \tag{6.20}$$

$$\text{rem} \quad : \quad \frac{\partial C}{\partial \tau} + \frac{\partial D}{\partial x} = 0. \tag{6.21}$$

Eqn (6.18) implies that $B = B(x)$, Eqn (6.19) implies that $B = ax + b$, where $a, b \in \mathbb{R}$, (6.20) implies that $M = 0$ or if $M \neq 0$ then $B = 0$ and we get trivial conservation laws. Eqn (6.21) is directly satisfied, which implies that $D(\tau, x) = 0$, $C(\tau, x) = 0$. For non trivial conservation laws we get,

$$A = \frac{\theta^{m+1}}{m+1} a. \tag{6.22}$$

When $\theta(\tau, x)$ is a solution of (6.1)

$$D_\tau \big[(ax+b)\theta \big] + D_x \left[-\theta^m (ax+b)\theta_x + a \frac{\theta^{m+1}}{m+1} \right] = 0. \tag{6.23}$$

A conserved vector of the partial differential equation (6.1) with $M = 0$ is
therefore a linear combination of the two conserved vectors

$$T^1 = \theta, \quad T^2 = -\theta^m \theta_x, \tag{6.24}$$

$$T^1 = x\theta, \quad T^2 = -x\theta^m \theta_x + \frac{\theta^{m+1}}{m+1}. \tag{6.25}$$

The above conserved vectors are for the special case $M = 0$ otherwise when
$M \neq 0$ then we get trivial conservation laws. One may assume

$$T^i = T(\tau, x, \theta, \theta_x, \theta_\tau, \theta_{\tau x}, \theta_{\tau\tau}, \theta_{xx}) \tag{6.26}$$

which is computationally expensive. We consider the second case below.

Case $m = n$

$$\theta^{m+1} \quad : \quad \frac{1}{m+1} \frac{\partial^2 B}{\partial x^2} - M^2 B = 0, \tag{6.27}$$

$$\theta \quad : \quad \frac{\partial B}{\partial \tau} = 0, \tag{6.28}$$

$$\text{rem} \quad : \quad \frac{\partial C}{\partial \tau} + \frac{\partial D}{\partial x} = 0. \tag{6.29}$$

We note that this case applies for $m = n \neq -1$ and $m = n \neq 0$. The general
solution of Eqn (6.27) is given by,

$$B(x) = c_1 e^{\sqrt{1+mM}x} + c_2 e^{-\sqrt{1+mM}x}, \tag{6.30}$$

and thus,

$$A = \frac{\theta^{m+1}}{m+1}(c_1 \omega e^{\omega x} - c_2 \omega e^{-\omega x}), \tag{6.31}$$

where, $\omega = \sqrt{1+mM}$.

$$D_\tau\left[(c_1 e^{\omega x} + c_2 e^{-\omega x})\theta\right] +$$

$$D_x\left[-\theta^m(c_1 e^{\omega x} + c_2 e^{-\omega x})\theta_x + \frac{\theta^{m+1}}{m+1}(c_1 \omega e^{\omega x} + c_2 \omega e^{-\omega x})\right] = 0. \tag{6.32}$$

The non trivial conserved vectors are,

$$T^1 = \theta e^{\omega x}, \quad T^2 = e^{\omega x}\left(\frac{\theta^{m+1}}{m+1}\omega - \theta_x\theta^m\right), \tag{6.33}$$

$$T^1 = \theta e^{-\omega x}, \quad T^2 = e^{-\omega x}\left(\frac{\theta^{m+1}}{m+1}\omega - \theta_x\theta^m\right). \tag{6.34}$$

Equation (6.16) applies only for $m \neq -1$. A special case $m = -1$, results in extra conserved quantities as follows;

$$A = \ln\theta\frac{\partial B}{\partial x} + D(\tau, x). \tag{6.35}$$

A conserved vector of the PDE (6.1) with $m = -1$ is a linear combination of the two conserved vectors,

$$T^1 = \theta, \quad T^2 = -\theta^{-1}\theta_x, \tag{6.36}$$

$$T^1 = x\theta, \quad T^2 = -x\theta^{-1}\theta_x + \ln\theta. \tag{6.37}$$

We also observe that the separation of Eqn (6.17) requires $m \neq 0$, $n \neq 0$ and $n \neq -1$. Given $n = -1$ we obtain the following conserved vectors,

$$T^1 = \theta, \quad T^2 = -\theta^m\theta_x + M^2x, \tag{6.38}$$

$$T^1 = x\theta, \quad T^2 = x\left(\frac{M^2x}{2} - \theta^m\theta_x\right) + \frac{\theta^{m+1}}{m+1}. \tag{6.39}$$

6.2.2   Multiplier method

A multiplier $\Lambda(t, x, u)$ of Eqn (6.1) has the property that

$$\Lambda\left(\frac{\partial\theta}{\partial\tau} - \frac{\partial}{\partial x}\left(\theta^m\frac{\partial\theta}{\partial x}\right) + M^2\theta^{n+1}\right) = D_1T^1 + D_2T^2, \tag{6.40}$$

for all functions $\theta(\tau, x)$. We are going to consider multipliers of the form $\Lambda = \Lambda(\tau, x, \theta)$. Multipliers which depend on the first order and higher derivatives can also be considered but the calculations become more complicated [53]. The right hand side of Eqn (6.40) is a divergence expression, thus the determining equation for the multiplier Λ is

$$E_\theta \left[\Lambda \left(\frac{\partial \theta}{\partial \tau} - \frac{\partial}{\partial x} \left(\theta^m \frac{\partial \theta}{\partial x} \right) + M^2 \theta^{n+1} \right) \right] = 0 \tag{6.41}$$

where E_θ is the standard Euler operator given by

$$E_\theta = \frac{\partial}{\partial \theta} - D_x \frac{\partial}{\partial \theta_x} - D_\tau \frac{\partial}{\partial \theta_\tau} + D_x^2 \frac{\partial}{\partial \theta_{xx}} + D_x D_\tau \frac{\partial}{\partial \theta_{x\tau}} + D_\tau^2 \frac{\partial}{\partial \theta_{\tau\tau}} - \dots . \tag{6.42}$$

The expansion of Eqn (6.41) yields

$$\theta_\tau \frac{\partial \Lambda}{\partial \theta} - \theta^m \theta_{xx} \frac{\partial \Lambda}{\partial \theta} + (n+1)M^2 \Lambda \theta^n + M^2 \theta^{n+1} \frac{\partial \Lambda}{\partial \theta}$$
$$- \frac{\partial \Lambda}{\partial \tau} - \theta_\tau \frac{\partial \Lambda}{\partial \theta} - \theta^m \frac{\partial^2 \Lambda}{\partial x^2} - \theta^m \theta_x \frac{\partial^2 \Lambda}{\partial \theta \partial x} - 2\theta_x \theta^m \frac{\partial^2 \Lambda}{\partial \theta \partial x}$$
$$- \theta_x^2 m \theta^{m-1} \frac{\partial \Lambda}{\partial \theta} - \theta_x^2 \theta^m \frac{\partial^2 \Lambda}{\partial \theta^2} - \theta^m \frac{\partial \Lambda}{\partial \theta} = 0 \tag{6.43}$$

Since Eqn (6.43) is satisfied for all functions $\theta(\tau, x)$, it can be separated by equating the coefficients of the partial derivatives of $\theta(\tau, x)$. The coefficients of θ_{xx} give

$$\frac{\partial \Lambda}{\partial \theta} = 0 \tag{6.44}$$

and therefore $\Lambda = \Lambda(\tau, x)$. Equation (6.43) reduces to

$$(n+1)M^2 \theta^n \Lambda - \frac{\partial \Lambda}{\partial \tau} - \theta^m \frac{\partial^2 \Lambda}{\partial x^2} = 0. \tag{6.45}$$

We consider two cases when separating Eqn (6.45) by powers of θ:

Case $m \neq n$

$$\theta^n \quad : \quad (n+1)M^2\Lambda = 0, \tag{6.46}$$

$$\theta^m \quad : \quad \frac{\partial^2\Lambda}{\partial x^2} = 0, \tag{6.47}$$

$$\theta^0 \quad : \quad \frac{\partial\Lambda}{\partial\tau} = 0. \tag{6.48}$$

We also note that the case requires $m \neq 0$ and $n \neq 0$. Thus $n = -1$, $\Lambda = \Lambda(x)$

where

$$\Lambda(x) = c_1 + c_2 x \tag{6.49}$$

and c_1 and c_2 are constants. From Eqns (6.40) and (6.49) and by performing

elementary manipulations,

$$(c_1 + c_2 x)\left(\frac{\partial\theta}{\partial\tau} - \frac{\partial}{\partial x}\left(\theta^m \frac{\partial\theta}{\partial x}\right) + M^2\right) = D_\tau\big[(c_1 + c_2 x)\theta\big]$$

$$+ D_x\left[-\theta^m(c_1 + c_2 x)\theta_x + c_2\frac{\theta^{m+1}}{m+1} + M^2 x\left(\frac{c_2 x}{2} + c_1\right)\right] \tag{6.50}$$

for all functions $\theta(\tau, x)$.

Thus when $\theta(\tau, x)$ is a solution of Eqn (6.1), then

$$D_\tau\big[(c_1 + c_2 x)\theta\big] + D_x\left[-\theta^m(c_1 + c_2 x)\theta_x + c_2\frac{\theta^{m+1}}{m+1} + M^2 x\left(\frac{c_2 x}{2} + c_1\right)\right]. \tag{6.51}$$

A conserved vector of the PDE (6.1) with a multiplier of the form $\Lambda(\tau, x, u)$ is

therefore a linear combination of the two conserved vectors

$$T^1 = \theta, \quad T^2 = -\theta^m\theta_x + M^2 x, \tag{6.52}$$

$$T^1 = x\theta, \quad T^2 = x\left(\frac{M^2 x}{2} - \theta^m\theta_x\right) + \frac{\theta^{m+1}}{m+1}. \tag{6.53}$$

We observe in Eqn (6.53) that $m \neq -1$ because $m \neq n$ and $n = -1$. The

conserved vectors (6.52) and (6.53) therefore form a basis of conserved vectors

of the PDE (6.1) with multipliers of the form $\Lambda(\tau, x, \theta)$ for the case where $m \neq n$ with $n = -1$. We note that these conservation laws are the same as those obtained using the direct method for $m \neq -1$. We consider the second case below.

Case $m = n$,

$$\theta^m \quad : \quad \frac{\partial^2 \Lambda}{\partial x^2} - (m+1)M^2\Lambda = 0, \tag{6.54}$$

$$\theta^0 \quad : \quad \frac{\partial \Lambda}{\partial \tau} = 0. \tag{6.55}$$

Thus $\Lambda = \Lambda(x)$ where

$$\Lambda(x) = c_1 e^{\omega x} + c_2 e^{-\omega x} \tag{6.56}$$

and c_1 and c_2 are constants. Substituting Eqn (6.56) into Eqn (6.40) and by performing elementary manipulations, we obtain the conserved vectors (6.33) and (6.34).

6.3 Nonlinear equation with linear thermal conductivity

6.3.1 Direct method

A conservation law for (6.2) satisfies

$$D_1 T^1 + D_2 T^2 \bigg|_{(6.2)} = 0, \tag{6.57}$$

We seek the conserved vectors of the form

$$T^1 = (\tau, x, \theta, \theta_x), \quad T^2 = (\tau, x, \theta, \theta_x). \tag{6.58}$$

Substituting Eqn (6.58) into Eqn (6.57) we obtain the following PDE,

$$\frac{\partial T^1}{\partial \tau} + \theta_{xx}\left[(1+\beta\theta)\frac{\partial T^1}{\partial \theta} + \frac{\partial T^2}{\partial \theta_x}\right] + \beta\theta_x^2\frac{\partial T^1}{\partial \theta}$$
$$- M^2\theta^{n+1}\frac{\partial T^1}{\partial \theta} + \theta_{x\tau}\frac{\partial T^1}{\partial \theta_x} + \frac{\partial T^2}{\partial x} + \theta_x\frac{\partial T^2}{\partial \theta} = 0. \tag{6.59}$$

Since T^1 and T^2 are independent of the second derivatives of θ, we can separate
Eqn (6.59) by second derivatives of θ,

$$\theta_{xx} \quad : \quad (1+\beta\theta)\frac{\partial T^1}{\partial \theta} + \frac{\partial T^2}{\partial \theta_x} = 0, \tag{6.60}$$

$$\theta_{x\tau} \quad : \quad \frac{\partial T^1}{\partial \theta_x} = 0, \tag{6.61}$$

$$\text{rem} \quad : \quad \frac{\partial T^1}{\partial \tau} + \beta\theta_x^2\frac{\partial T^1}{\partial \theta} - M^2\theta^{n+1}\frac{\partial T^1}{\partial \theta} + \frac{\partial T^2}{\partial x} + \theta_x\frac{\partial T^2}{\partial \theta} = 0. \tag{6.62}$$

Eqn (6.61) implies that $T^1 = T^1(\tau, x, \theta)$ and we can thus integrate Eqn (6.60)
with respect to θ_x to obtain an explicit expression for T^2,

$$T^2 = -(1+\beta\theta)\frac{\partial T^1}{\partial \theta}\theta_x + A(\tau, x, \theta) \tag{6.63}$$

where A is an arbitrary function. Substituting T^2 into Eqn (6.62) results in

$$\frac{\partial T^1}{\partial \tau} - M^2\theta^{n+1}\frac{\partial T^1}{\partial \theta} + \frac{\partial A}{\partial x} - (1+\beta\theta)\theta_x\frac{\partial^2 T^1}{\partial \theta \partial x}$$
$$- \theta_x\frac{\partial A}{\partial \theta} - (1+\beta\theta)\theta_x^2\frac{\partial^2 T^1}{\partial \theta^2} = 0. \tag{6.64}$$

We can then separate (6.64) by first derivatives of θ,

$$\theta_x^2 \quad : \quad (1+\beta\theta)\frac{\partial^2 T^1}{\partial \theta^2} = 0, \tag{6.65}$$

$$\theta_x \quad : \quad (1+\beta\theta)\frac{\partial^2 T^1}{\partial \theta \partial x} - \frac{\partial A}{\partial \theta} = 0, \tag{6.66}$$

$$\text{rem} \quad : \quad \frac{\partial T^1}{\partial \tau} + \frac{\partial A}{\partial x} - M^2\theta^{n+1}\frac{\partial T^1}{\partial \theta} = 0. \tag{6.67}$$

Equation (6.65) implies that

$$\frac{\partial^2 T^1}{\partial \theta^2} = 0 \tag{6.68}$$

and integrating gives the following expression for T^1,

$$T^1 = B(\tau, x)\theta + C(\tau, x) \tag{6.69}$$

where B and C are arbitrary functions. Substituting T^1 into Eqn (6.66) and integrating we get the following equation for A,

$$A = \left(\theta + \frac{\beta}{2}\theta^2\right)\frac{\partial B}{\partial x} + D(\tau, x) \tag{6.70}$$

where D is an arbitrary function.

Eqn (6.67) simplifies to

$$\theta\frac{\partial B}{\partial \tau} + \frac{\partial C}{\partial \tau} + \left(\theta + \frac{\beta}{2}\theta^2\right)\frac{\partial^2 B}{\partial x^2} + \frac{\partial B}{\partial x} - M^2\theta^{n+1}B = 0 \tag{6.71}$$

and Eqn (6.71) can be separated by powers of θ:

$$\theta \quad : \quad \frac{\partial B}{\partial \tau} + \frac{\partial^2 B}{\partial x^2} = 0, \tag{6.72}$$

$$\theta^2 \quad : \quad \frac{\beta}{2}\frac{\partial^2 B}{\partial x^2} = 0, \tag{6.73}$$

$$\theta^{n+1} \quad : \quad -M^2 B = 0, \tag{6.74}$$

$$\text{rem} \quad : \quad \frac{\partial C}{\partial \tau} + \frac{\partial D}{\partial x} = 0. \tag{6.75}$$

To obtain Eqns (6.72) to (6.75) the separation requires $n \neq 1$, $n \neq 0$ and $n \neq -1$. Following a similar approach of constructing conservation laws, we also investigate these special cases and state the results. Equations (6.72) and (6.73) imply that $B = B(x)$ and $B = c_1 x + c_2$, where c_1 and c_2. From Eqn (6.74), $B \neq 0$ implies that $M = 0$. Equation (6.75) is directly satisfied, C and D are both equal to zero. A is then given by

$$A = c_1\left(\theta + \beta\frac{\theta^2}{2}\right). \tag{6.76}$$

A conserved vector of the partial differential equation (6.2) is therefore a linear combination of the two conserved vectors

$$T^1 = \theta, \quad T^2 = -\theta_x(1 + \beta\theta), \tag{6.77}$$

$$T^1 = x\theta, \quad T^2 = -x\theta_x(1 + \beta\theta) + \theta + \beta\frac{\theta^2}{2}. \tag{6.78}$$

We now investigate the special cases $n = -1$, $n = 0$ and $n = 1$. For $n = -1$,

separating Eqn (6.71) by powers of θ gives:

$$\theta \quad : \quad \frac{\partial B}{\partial \tau} + \frac{\partial^2 B}{\partial x^2} = 0, \tag{6.79}$$

$$\theta^2 \quad : \quad \frac{\beta}{2}\frac{\partial^2 B}{\partial x^2} = 0, \tag{6.80}$$

$$\text{rem} \quad : \quad \frac{\partial C}{\partial \tau} + \frac{\partial D}{\partial x} = M^2 B. \tag{6.81}$$

Equations (6.79) and (6.80) imply that $B(x) = ax + b$ where a and b are

constants. Substituting B into (6.81) and setting C to zero, we obtain the

following conserved vectors

$$T^1 = \theta, \quad T^2 = -(1 + \beta\theta)\theta_x + M^2 x, \tag{6.82}$$

$$T^1 = x\theta, \quad T^2 = x\left(\frac{M^2 x}{2} - \theta_x(1 + \beta\theta)\right) + \theta + \beta\frac{\theta^2}{2}. \tag{6.83}$$

The other cases gives conserved vectors that were obtained originally for the

general cases.

6.3.2   Multiplier method

A multiplier Λ for Eqn (6.2) has the property that

$$\Lambda\left(\frac{\partial\theta}{\partial\tau} - \frac{\partial}{\partial x}\left((1 + \beta\theta)\frac{\partial\theta}{\partial x}\right) + M^2\theta^{n+1}\right) = D_1 T^1 + D_2 T^2 \tag{6.84}$$

for all functions $\theta(\tau, x)$ and the determining equation is

$$E_\theta\left[\Lambda\left(\frac{\partial\theta}{\partial\tau} - \frac{\partial}{\partial x}\left((1 + \beta\theta)\frac{\partial\theta}{\partial x}\right) + M^2\theta^{n+1}\right)\right] = 0, \tag{6.85}$$

where E_θ is the standard Euler operator given by Eqn (6.42). The expansion
of Eqn (6.85) yields

$$
\begin{aligned}
&\theta_\tau \frac{\partial \Lambda}{\partial \theta} - \theta_{xx} \frac{\partial \Lambda}{\partial \theta} - \beta \theta_{xx} \Lambda - \beta \theta \theta_{xx} \frac{\partial \Lambda}{\partial \theta} - \beta \theta_x^2 \frac{\partial \Lambda}{\partial \theta} + M^2 (n+1) \theta^n \Lambda \\
&+ M^2 \theta^{n+1} \frac{\partial \Lambda}{\partial \theta} + 2\beta \theta_x \frac{\partial \Lambda}{\partial x} + 2\beta \theta_x^2 \frac{\partial \Lambda}{\partial \theta} + 2\beta \theta_{xx} \Lambda - \frac{\partial \Lambda}{\partial \tau} - \theta_\tau \frac{\partial \Lambda}{\partial \theta} \\
&- \frac{\partial^2 \Lambda}{\partial x^2} - \theta_x \frac{\partial^2 \Lambda}{\partial \theta \partial x} - \theta_x \frac{\partial^2 \Lambda}{\partial \theta \partial x} - \theta_x^2 \frac{\partial^2 \Lambda}{\partial \theta^2} - \theta_{xx} \frac{\partial \Lambda}{\partial \theta} - \beta \theta \frac{\partial^2 \Lambda}{\partial x^2} - \beta \theta_x \frac{\partial \Lambda}{\partial x} \\
&- \beta \theta_x \frac{\partial^2 \Lambda}{\partial \theta \partial x} - \beta \theta_x \frac{\partial \Lambda}{\partial x} - \beta \theta_x^2 \frac{\partial \Lambda}{\partial \theta} - \beta \theta_{xx} \Lambda - \beta \theta \theta_x \frac{\partial^2 \Lambda}{\partial \theta \partial x} - \beta \theta_x^2 \frac{\partial \Lambda}{\partial \theta} \\
&- \beta \theta \theta_x^2 \frac{\partial^2 \Lambda}{\partial \theta^2} - \beta \theta \theta_{xx} \frac{\partial \Lambda}{\partial \theta} = 0.
\end{aligned}
\tag{6.86}
$$

Since Eqn (6.86) is satisfied for all functions $\theta(\tau, x)$ it can be separated by
equating the coefficients of the partial derivatives of $\theta(\tau, x)$. The coefficients
of θ_{xx} give

$$
(1 + \beta \theta) \frac{\partial \Lambda}{\partial \theta} = 0
\tag{6.87}
$$

and $(1 + \beta \theta) \neq 0$, thus $\Lambda = \Lambda(\tau, x)$. Eqn (6.86) reduces to

$$
M^2 (n+1) \theta^n \Lambda - \frac{\partial \Lambda}{\partial \tau} - \frac{\partial^2 \Lambda}{\partial x^2} = 0.
\tag{6.88}
$$

Separating Eqn (6.88) by powers of θ gives

$$
\theta^n \quad : \quad (n+1) M^2 \Lambda = 0,
\tag{6.89}
$$

$$
\theta^0 \quad : \quad \frac{\partial \Lambda}{\partial \tau} + \frac{\partial^2 \Lambda}{\partial x^2} = 0.
\tag{6.90}
$$

We note that the separation requires $n \neq 0$. Equation (6.89) implies that
$n = -1$ or $\Lambda = 0$. From Eqn (6.90), setting each term to zero we observe that
$\Lambda = \Lambda(x)$ and

$$
\Lambda(x) = c_1 x + c_2
\tag{6.91}
$$

where c_1 and c_2 are constants. From Eqns (6.84) and (6.91) and by performing

elementary manipulations,

$$(c_1 + c_2 x)\left(\frac{\partial \theta}{\partial \tau} - \frac{\partial}{\partial x}\left((1 + \beta\theta)\frac{\partial \theta}{\partial x}\right) + M^2\right) = D_\tau[(c_1 x + c_2)\theta]+$$

$$D_x\left[-(1 + \beta\theta)(c_1 x + c_2)\theta_x + \left(\theta + \beta\frac{\theta^2}{2}\right)c_1 + M^2 x\left(\frac{c_1 x}{2} + c_2\right)\right] \quad (6.92)$$

for all functions $\theta(\tau, x)$. Thus when $\theta(\tau, x)$ is a solution of (6.2)

$$D_\tau\left[(c_1 x + c_2)\theta\right]$$

$$+ D_x\left[-(1 + \beta\theta)(c_1 x + c_2)\theta_x + \left(\theta + \beta\frac{\theta^2}{2}\right)c_1 + M^2 x\left(\frac{c_1 x}{2} + c_2\right)\right] = 0.$$

$$(6.93)$$

A conserved vector of the PDE (6.2) with a multiplier of the form $\Lambda(t, x, u)$

is therefore a linear combination of the two conserved vectors

$$T^1 = \theta, \quad T^2 = -(1 + \beta\theta)\theta_x + M^2 x, \quad (6.94)$$

$$T^1 = x\theta, \quad T^2 = x\left(\frac{M^2 x}{2} - \theta_x(1 + \beta\theta)\right) + \theta + \beta\frac{\theta^2}{2}. \quad (6.95)$$

The conserved vectors (6.94) and (6.95) therefore form a basis of conserved

vectors of the PDE (6.2) with multipliers of the form $\Lambda(\tau, x, \theta)$. We note that

the direct method and the multiplier method gives the same results.

6.4   Conserved vectors and associated point symmetries

We have derived conserved vectors for the two transient heat conduction equa-

tions. We notice that the conserved vectors obtained using the direct method

for the transient equation with linear thermal conductivity were independent

of the thermo-geometric fin parameter M. Also, for the power law thermal

conductivity we obtained conserved vectors that are independent of M for the general case where $m \neq n$ using the direct method, while the method of multipliers gave physically sound conservation laws.

We find Lie point symmetries associated with the conserved vectors for the transient equation with power law thermal conductivity. From Section (2.4) the determining equation for the Lie point symmetries $\mathcal{Y}$ associated with the conserved vector $T = (T^1, T^2)$ is,

$$\mathcal{Y}(T^i) + T^i D_l(\xi^l) - T^l D_l(\xi^i) = 0. \tag{6.96}$$

Eqn (6.96) consists of two components

$$\mathcal{Y}(T^1) + T^1 D_2(\xi^2) - T^2 D_2(\xi^1) = 0, \tag{6.97}$$

$$\mathcal{Y}(T^2) + T^2 D_1(\xi^1) - T^1 D_1(\xi^2) = 0. \tag{6.98}$$

6.4.1 Case $m = n$ $(m \neq -1,\ m \neq 0)$

Substituting the elementary conserved vector (6.33) into (6.97) and (6.98) gives

$$\xi^2 \theta \omega e^{\omega x} + \eta e^{\omega x} + \theta e^{\omega x} \frac{\partial \xi^2}{\partial x} + \theta \theta_x e^{\omega x} \frac{\partial \xi^2}{\partial \theta} - e^{\omega x} \theta^{m+1} \frac{\omega}{m+1} \frac{\partial \xi^1}{\partial x}$$

$$- \frac{\omega}{m+1} \theta_x \theta^{m+1} e^{\omega x} \frac{\partial \xi^1}{\partial \theta} + \theta_x \theta^m e^{\omega x} \frac{\partial \xi^1}{\partial x} + \theta_x^2 \theta^m e^{\omega x} \frac{\partial \xi^1}{\partial \theta} = 0. \tag{6.99}$$

and

$$\xi^2 \frac{\omega^2}{m+1} e^{\omega x} \theta^{m+1} - \xi^2 \theta_x \theta^m \omega e^{\omega x} + \eta \omega e^{\omega x} \theta^m - \eta \theta_x m \theta^{m-1} e^{\omega x}$$

$$- \theta^m e^{\omega x} \frac{\partial \eta}{\partial x} - \theta^m e^{\omega x} \theta_x \frac{\partial \eta}{\partial \theta} + \theta^m e^{\omega x} \theta_\tau \frac{\partial \xi^1}{\partial x} + \theta^m e^{\omega x} \theta_\tau \theta_x \frac{\partial \xi^1}{\partial \theta}$$

$$+ \theta_x \theta^m e^{\omega x} \frac{\partial \xi^2}{\partial x} + \theta^m e^{\omega x} \theta_x^2 \frac{\partial \xi^2}{\partial \theta} + \frac{\omega}{m+1} e^{\omega x} \theta^{m+1} \frac{\partial \xi^1}{\partial \tau}$$

$$+ \frac{\omega}{m+1} e^{\omega x} \theta^{m+1} \theta_\tau \frac{\partial \xi^1}{\partial \theta} - \theta_x \theta^m e^{\omega x} \frac{\partial \xi^1}{\partial \tau} - \theta_x \theta^m e^{\omega x} \theta_\tau \frac{\partial \xi^1}{\partial \theta}$$

$$- \theta e^{\omega x} \frac{\partial \xi^2}{\partial \theta} - \theta e^{\omega x} \theta_\tau \frac{\partial \xi^2}{\partial \theta} = 0. \tag{6.100}$$

Separating Eqn (6.99) by derivatives of θ we find

$$\theta_x^2 \quad : \quad \theta^m e^{\omega x} \frac{\partial \xi^1}{\partial \theta} = 0, \tag{6.101}$$

$$\theta_x \quad : \quad \theta e^{\omega x} \frac{\partial \xi^2}{\partial \theta} - \frac{\omega}{m+1} \theta^{m+1} e^{\omega x} \frac{\partial \xi^1}{\partial \theta} + \theta^m e^{\omega x} \frac{\partial \xi^1}{\partial x} = 0, \tag{6.102}$$

$$\text{rem} \quad : \quad \xi^2 \theta \omega e^{\omega x} + \eta e^{\omega x} + \theta e^{\omega x} \frac{\partial \xi^2}{\partial x} - e^{\omega x} \theta^{m+1} \frac{\omega}{m+1} \frac{\partial \xi^1}{\partial x} = 0. \tag{6.103}$$

Separating Eqn (6.100) by derivatives of θ we find

$$\theta_x^2 \quad : \quad \theta^m e^{\omega x} \frac{\partial \xi^2}{\partial \theta} = 0, \tag{6.104}$$

$$\theta_x \quad : \quad -\xi^2 \theta^m \omega e^{\omega x} - \eta m \theta^{m-1} e^{\omega x} - \theta^m e^{\omega x} \frac{\partial \eta}{\partial \theta}$$

$$+ \theta^m e^{\omega x} \frac{\partial \xi^2}{\partial x} - \theta^m e^{\omega x} \frac{\partial \xi^1}{\partial \tau} = 0, \tag{6.105}$$

$$\theta_\tau \quad : \quad \theta^m e^{\omega x} \frac{\partial \xi^1}{\partial x} + \frac{\omega}{m+1} e^{\omega x} \theta^{m+1} \frac{\partial \xi^1}{\partial \theta} - \theta e^{\omega x} \frac{\partial \xi^2}{\partial \theta} = 0, \tag{6.106}$$

$$\text{rem} \quad : \quad \xi^2 \frac{\omega^2}{m+1} e^{\omega x} \theta^{m+1} + \eta \omega e^{\omega x} \theta^m - \theta^m e^{\omega x} \frac{\partial \eta}{\partial x}$$

$$+ \frac{\omega}{m+1} e^{\omega x} \theta^{m+1} \frac{\partial \xi^1}{\partial \tau} - \theta e^{\omega x} \frac{\partial \xi^2}{\partial \tau} = 0. \tag{6.107}$$

From Eqns (6.101), (6.102) and (6.104)

$$\xi^1 = \xi^1(\tau), \tag{6.108}$$

and

$$\xi^2 = \xi^2(\tau, x). \tag{6.109}$$

Equation (6.103) reduces to

$$\xi^2 \theta \omega + \eta + \theta \frac{\partial \xi^2}{\partial x} = 0, \tag{6.110}$$

thus

$$\eta = -\xi^2 \theta \omega - \theta \frac{\partial \xi^2}{\partial x}, \tag{6.111}$$

and

$$\frac{\partial \eta}{\partial x} = -\omega \theta \frac{\partial \xi^2}{\partial x} - \theta \frac{\partial^2 \xi^2}{\partial x^2}. \tag{6.112}$$

Substituting Eqns (6.111) and (6.112) into Eqn (6.107) gives

$$\xi^2 \frac{\omega^2}{m+1}\theta^{m+1} - \omega^2\theta^{m+1}\xi^2 - \omega\theta^{m+1}\frac{\partial\xi^2}{\partial x} + \omega\theta^{m+1}\frac{\partial\xi^2}{\partial x}$$
$$+ \theta^{m+1}\frac{\partial^2\xi^2}{\partial x^2} + \frac{\omega}{m+1}\theta^{m+1}\frac{\partial\xi^1}{\partial\tau} - \theta\frac{\partial\xi^2}{\partial\tau} = 0. \tag{6.113}$$

Separating Eqn (6.113) by powers of θ gives

$$\theta \quad : \quad \frac{\partial\xi^2}{\partial\tau} = 0, \tag{6.114}$$

$$\theta^{m+1} \quad : \quad \xi^2\frac{\omega^2}{m+1} - \omega^2\xi^2 + \frac{\partial^2\xi^2}{\partial x^2} + \frac{\omega}{m+1}\frac{\partial\xi^1}{\partial\tau} = 0. \tag{6.115}$$

Differentiating Eqn (6.115) with respect to τ and solving ξ^1 gives

$$\xi^1(\tau) = a_1\tau + a_2 \tag{6.116}$$

where a_1 and a_2 are constants.

Substituting ξ^1 into Eqn (6.105) and solving for ξ^2 we obtain

$$\xi^2(x) = \frac{a_1}{m\omega} + a_2 e^{-\frac{\omega mx}{2+m}}, \tag{6.117}$$

where $m \neq -2$. With the explicit expressions for ξ^1 and ξ^2, substituting into
Eqn (6.111) gives

$$\eta = -\omega\theta\left(\frac{a_1}{m\omega} + a_2 e^{-\frac{\omega mx}{2+m}}\right) + \theta\left(\frac{a_2}{m\omega}e^{-\frac{\omega mx}{2+m}}\right). \tag{6.118}$$

The Lie point symmetries are given by

$$\mathcal{Y}_1 = \frac{\partial}{\partial\tau} + e^{-\frac{\omega mx}{2+m}}\frac{\partial}{\partial x} + \left\{\left(\frac{1}{m\omega} - \omega\right)\theta\exp\left(-\frac{m\omega x}{2+m}\right)\right\}\frac{\partial}{\partial\theta}, \tag{6.119}$$

$$\mathcal{Y}_2 = \tau\frac{\partial}{\partial\tau} + \frac{1}{m\omega}\frac{\partial}{\partial x} - \frac{\theta}{m}\frac{\partial}{\partial\theta}. \tag{6.120}$$

6.4.2 Case $m \neq n$ ($n = -1$, $m \neq -1$, $m \neq 0$)

The method of multipliers gave conserved vectors that depend on the M.
Substituting the elementary conserved vector (6.52) into (6.97) and (6.98)

gives

$$\eta + \theta \frac{\partial \xi^2}{\partial x} + \theta \theta_x \frac{\partial \xi^2}{\partial \theta} + \theta^m \theta_x \frac{\partial \xi^1}{\partial x} + \theta^m \theta_x^2 \frac{\partial \xi^1}{\partial \theta} - xM^2 \frac{\partial \xi^1}{\partial x} - xM^2 \frac{\partial \xi^1}{\partial \theta} = 0.$$

(6.121)

and

$$\xi^2 M^2 - m\eta \theta_x \theta^{m-1} - \theta^m \frac{\partial \eta}{\partial x} - \theta^m \theta_x \frac{\partial \eta}{\partial \theta} + \theta_\tau \theta^m \frac{\partial \xi^1}{\partial x} - \theta_\tau \theta_x \theta^m \frac{\partial \xi^1}{\partial \theta}$$
$$+ \theta_x \theta^m \frac{\partial \xi^2}{\partial x} + \theta^m \theta_x^2 \frac{\partial \xi^2}{\partial \theta} - \theta^m \theta_x \frac{\partial \xi^1}{\partial \tau} - \theta_\tau \theta_x \theta^m \frac{\partial \xi^1}{\partial \theta} + xM^2 \frac{\partial \xi^1}{\partial \tau}$$
$$+ xM^2 \theta_\tau \frac{\partial \xi^1}{\partial \theta} - \theta \frac{\partial \xi^2}{\partial \tau} - \theta \theta_\tau \frac{\partial \xi^2}{\partial \theta} = 0.$$

(6.122)

Separating Eqn (6.121) by derivatives of θ we find

$$\theta_x^2 \quad : \quad \theta^m \frac{\partial \xi^1}{\partial \theta} = 0,$$

(6.123)

$$\theta_x \quad : \quad \theta \frac{\partial \xi^2}{\partial \theta} + \theta^m \frac{\partial \xi^1}{\partial x} - xM^2 \frac{\partial \xi^1}{\partial \theta} = 0,$$

(6.124)

$$\text{rem} \quad : \quad \eta + \theta \frac{\partial \xi^2}{\partial x} - xM^2 \frac{\partial \xi^1}{\partial x} = 0.$$

(6.125)

Separating Eqn (6.122) by derivatives of θ we find

$$\theta_x^2 \quad : \quad \theta^m \frac{\partial \xi^2}{\partial \theta} = 0,$$

(6.126)

$$\theta_x \quad : \quad -m\eta \theta^{m-1} - \theta^m \frac{\partial \eta}{\partial \theta} + \theta^m \frac{\partial \xi^2}{\partial x} - \theta^m \frac{\partial \xi^1}{\partial \tau} = 0,$$

(6.127)

$$\theta_\tau \quad : \quad \theta^m \frac{\partial \xi^1}{\partial x} + xM^2 \frac{\partial \xi^1}{\partial \theta} - \theta \frac{\partial \xi^2}{\partial \theta} = 0,$$

(6.128)

$$\text{rem} \quad : \quad \xi^2 M^2 - \theta^m \frac{\partial \eta}{\partial x} + xM^2 \frac{\partial \xi^1}{\partial \tau} - \theta \frac{\partial \xi^2}{\partial \tau} = 0.$$

(6.129)

From Eqns (6.123), (6.126) and (6.124)

$$\xi^1 = \xi^1(\tau),$$

(6.130)

and

$$\xi^2 = \xi^2(\tau, x).$$

(6.131)

Equation (6.125) reduces to

$$\eta + \theta \frac{\partial \xi^2}{\partial x} = 0, \tag{6.132}$$

thus

$$\eta = -\theta \frac{\partial \xi^2}{\partial x}, \tag{6.133}$$

and

$$\frac{\partial \eta}{\partial x} = -\theta \frac{\partial^2 \xi^2}{\partial x^2}. \tag{6.134}$$

Substituting Eqns (6.133) and (6.134) into Eqn (6.129) gives

$$\xi^2 M^2 + \theta^{m+1} \frac{\partial \xi^2}{\partial x^2} + x M^2 \frac{\partial \xi^1}{\partial \tau} - \theta \frac{\partial \xi^2}{\partial \tau} = 0. \tag{6.135}$$

Separating Eqn (6.135) by powers of θ gives

$$\theta \quad : \quad \frac{\partial \xi^2}{\partial \tau} = 0, \tag{6.136}$$

$$\theta^{m+1} \quad : \quad \frac{\partial^2 \xi^2}{\partial x^2} = 0, \tag{6.137}$$

$$\theta^0 \quad : \quad \xi^2 M^2 + x M^2 \frac{\partial \xi^1}{\partial \tau} = 0. \tag{6.138}$$

From Eqns (6.136) and (6.137) we get that

$$\xi^2(x) = ax + b \tag{6.139}$$

where a and b are constants.

Substituting ξ^2 into Eqn (6.127) we obtain

$$(m+2)a - \frac{\partial \xi^1}{\partial \tau} = 0. \tag{6.140}$$

Equation (6.140) implies that

$$\xi^1(\tau) = a(m+2)\tau + b. \tag{6.141}$$

Substituting the expression of ξ^2 into Eqn (6.133) we get the following expression for η

$$\eta = -a\theta. \tag{6.142}$$

The Lie point symmetries are given by

$$\mathcal{Y}_1 = (m+2)\tau\frac{\partial}{\partial\tau} + x\frac{\partial}{\partial x} - \theta\frac{\partial}{\partial\theta}, \tag{6.143}$$

$$\mathcal{Y}_2 = \frac{\partial}{\partial\tau} + \frac{\partial}{\partial x}. \tag{6.144}$$

6.5 Concluding remarks

The two methods considered, namely, the direct method and the method of multipliers yielded the same results for conservation laws for transient heat conduction through fins. Two conservation laws were derived for nonlinear heat equation with power law thermal conductivity, one of which was the elementary conservation law. Two cases were considered, (i) exponent of the thermal conductivity being the same as the exponent of the heat transfer coefficient, (ii) distinct exponents. Conserved vectors which depended on at most first order partial derivatives were considered [56].

Higher order conservation laws could be investigated by considering conserved vectors which depend on higher order spatial derivatives. The analysis will be computationally laborious and may best be done with the aid of computer programs. Furthermore, non trivial conservation laws may be constructed if the thermal conductivity is the same as the heat transfer coefficient. This is a very ideal/special case and may not always be true for heat transfer in fins.

Chapter 7

Conclusions

In this dissertation, the DTM was used to construct analytical-series solutions for nonlinear heat conduction equations for heat transfer in longitudinal fins. The 1D DTM was used to solve the steady state ordinary differential equations. A comparison was made between the 1D DTM series solutions of 15 terms and the exact solutions. It is found that the solutions agreed well with the exact solution and very small absolute errors were noted. Analytical solutions for the boundary value problems were obtained for special cases of the exponent n that signify the types of movement of heat through the fin.

Perhaps the notable advantage of the DTM is the generalization of the Taylor method to problems involving unusual derivative procedures such as fractional, fuzzy or q-derivative [54]. Some generalizations have been made by Obidat *et al.* [55] and they referred to their new method as the Generalized Differential Transform Method (GDTM). This showed great improvement compared to the Fractional Differential Transform Method (FDTM) introduced by Arikoglu and Ozkol [57].

We note that given differential equations such as Eqns (3.11) and (3.5) with a power law heat transfer coefficient of a fractional exponent, then one

can easily remove the fraction by basic exponential rules and employment of the binomial expansion. However, using the DTM, one runs into difficulty if the power law thermal conductivity in the same equation is given by the fractional exponent. We do not know whether these observations call for the 'modified' DTM to solve problems arising in heat flow through fins with other profiles, such as longitudinal triangular, concave parabolic and so on, and also with fractional power law thermal conductivity.

For the transient state problems, we employed the 2D DTM methods and the effects of thermal parameters appearing in the equation on temperature have been studied. Finally we sort the conservation laws and associated Lie point symmetries. We note that the direct method and the multiplier method gave similar results for some special cases of exponents of power laws. Mhlongo *et al.*[58], have shown that exact solutions maybe constructed for these cases.

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