Hedging Credit Risk Using Equity Derivatives

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Declaration

I declare that this is my own, unaided work. It is being submitted for the Degree of Master of Science to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other University.

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Abstract

Equity and credit markets are often treated as independent markets. In this dissertation our objective is to hedge a position in a credit default swap with either shares or share options. Structural models enable us to link credit risk to equity risk via the firm's asset value. With an extended version of the seminal Merton (1974) structural model, we value credit default swaps, shares and share options using arbitrage pricing theory. Since we are interested in hedging the change in value of a credit default swap dynamically, we use a jump-diffusion model for the firm's asset value in order to model the short term credit risk dynamics more accurately. Our mathematical model does not admit an explicit solutions for credit default swaps, shares and share options, thus we use a Brownian Bridge Monte Carlo procedure to value these financial products and to compute the *delta hedge ratios*. These delta hedge ratios measure the sensitivity of the value of a credit default swap with respect to either share or European share option prices. We apply these delta hedge ratios to simulated and market data, to test our hedging objective. The hedge performs well for the simulated data for both cases where the hedging instrument is either shares or share options. The hedging results with market data suggests that we are able to hedge the value of a credit default swap with shares, however it is more difficult with share options.

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Contents

Chapter 1

Introduction

1.1 Background to the Credit Derivatives Market

The rise in credit related events (i.e. bankruptcy of firms, default and deterioration of the credit quality of corporate and government bonds) throughout the financial world over the past decade has been met with a commensurate increase in interest in derivatives which depend in value solely on credit risk[1]. Until recently there did not exist risk management products for credit risk. The market for credit derivatives[2] began slowly in the early 1990's, sparked by the need to keep pace with the increasing demand for other over-the-counter derivative products. The sheer volume of business swamping the market meant that financial institutions could no longer rely on taking collateral or making loss provisions to manage their credit risk. They strived for a way to lay off illiquid credit exposures and to protect themselves against defaulting clients.

Traditionally, credit was sourced in the new issue market and placed with the end investors, where it remained irrevocably linked to the asset with which it was originally associated. With the innovation of credit derivatives it is now possible to separate credit risk from any financial obligation (i.e. loans, bonds, swaps) and manage it just like any other asset. The credit derivative tool kit has revolutionized credit risk management and fundamentally modernised the credit market. Before the advent of credit derivatives, credit risk management meant a strategy of portfolio diversification backed by credit line limits, with occasional sale of positions in the secondary market. These strategies are inefficient, largely because they do not separate the management of risk from the asset with which the credit risk is associated. Credit derivatives are significant as they allow the risk manager to isolate credit risk and manage it discreetly without interfering with customer relationships. They also form the first mechanism via which short sales of credit instruments can be performed with reasonable liquidity. For example, it is impossible to short-sell a bank loan, however this can be achieved synthetically by buying a credit derivative offering credit protection. Credit derivatives allow investors to be synthetically exposed to credit assets that were unavailable before, either because of regulations or because these assets were publicly restricted (e.g. bank loan).

The credit derivative market has experienced an exponential growth since its inception in the early 1990's. The total market notional for 2006 is estimated at $20,207 billion (US dollars) and is predicted to increase to $33,210 billion by 2008, according to surveys by the British Banker's Association (BBA) (see Barret & Ewan (2006)). Figure 1.1 shows estimates, from Barret & Ewan (2006), of the growth of the credit derivative market. It is not just the size of the market that has continued to grow, but also the diversity of credit derivative instruments. To highlight, a few recent products are: index trades, tranched index trades and equity-linked products. Single name credit default swaps (CDS) represent a substantial section of the market, according to the BBA 2006 survey, single name CDS represents 32.9% of the credit derivative market. The single name CDS is considered to be the fundamental instrument in the credit derivatives market, and is the focus of this dissertation. Since 1992, the International Swaps and Derivatives Association (ISDA) has standardised CDS contracts and other credit derivatives, allowing the involved parties to specify the terms of the transaction from a number of defined alternatives. Recently the ISDA has

[1] Credit risk is the risk of default on an obligatory payment. We give a thorough description of credit risk in Chapter 3.

[2] We give a detailed definition of credit derivatives in Chapter 3, for now it can be thought of as a financial instrument whose value depends on credit risk.

published the 2003 *ISDA Credit Derivatives Definitions*, which updates the 1999 version, and offers the basic framework for the documentation of privately negotiated credit derivative transactions. This standardisation has been a major development in the credit derivative market, since it has reduced legal ambiguities that hampered the market's growth in its early stages. Growth in this market has also been spurred by banks' recent endeavours to develop internal credit risk models to quantify regulatory capital requirements, which has been encouraged by the 1997 revised Basel Capital Accord and the new 2004 Basel II Capital Accord[3]. Banks have dominated the market as the biggest traders of credit derivatives. Other major market participants are securities firms, insurance companies and hedge funds.

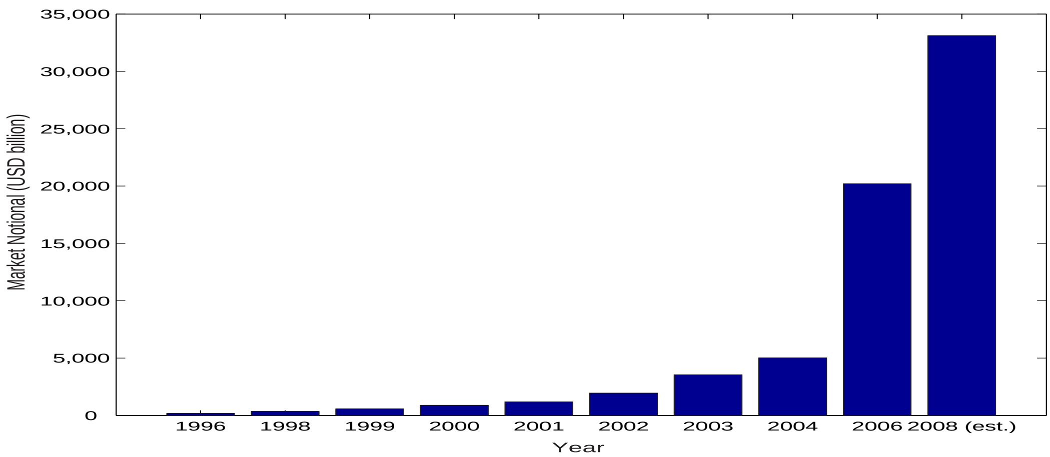

Figure 1.1: Growth of the credit derivative market according to the British Banker's Association 2006 survey.

1.2 Credit Risk and Equity

There have been several studies on the relationship between credit risk and equity, and it is widely agreed that there is a negative relationship between them. Vassalou & Xing (2004) and Charitou et al. (2004) concluded that there exists a statistically significant negative relationship between credit risk (as measured by the risk-neutral probability of default[4] derived from the Merton (1974) model) and stock returns, and this relationship is more significant for firms with a high default probability. Elizalde (2005) assumed a *global market credit risk factor* inherent in all bond prices. Using linear regression, he found that this global market credit risk is negatively correlated with equity indices (S&P 500 and Dow Jones). This negative relationship between credit risk and equity prices is intuitive. As a firm's credit risk increases, the firms financial health decreases (its propensity to default increases) and equity investors will become sceptical of realising a return on their investment, resulting in share prices decreasing. By construction, credit default swaps are the financial instruments with the most accurate reflection of a firm's credit risk[5]. By regressing CDS premiums on several equity factors, Zhang et al. (2005) found that equity returns are statistically significant in explaining the change in CDS premiums. By calculating the daily lead-lag cross-correlation coefficients, Acharya & Johnson (2005) and Byström (2005) found that share price changes lead CDS premium changes, that there is an inverse relationship between the two, and that this inverse relationship is strongest when the lag is zero.

The main objective of this dissertation is to quantify this relationship between CDS premium changes and stock returns in order to hedge[6] a position in a CDS with an equity position. To my knowledge the closest related studies to this objective are by Yu (2005) and Schefer & Strebulaev

[3]The set of regulatory banking capital adequacy requirements, compiled by the Bank for International Settlements (BIS) Basel Committee on Banking Supervision.

[4]This measure of credit risk will be discussed in detail in Chapter 4.

[5]In Chapter 4 we discuss credit default swaps in detail. There it is shown why it is value is considered to be the most accurate reflection of credit risk.

[6]A hedge is a trading strategy designed to reduce risk, the variability of the value of the position being hedged. Hedging will be discussed in Chapter 6.

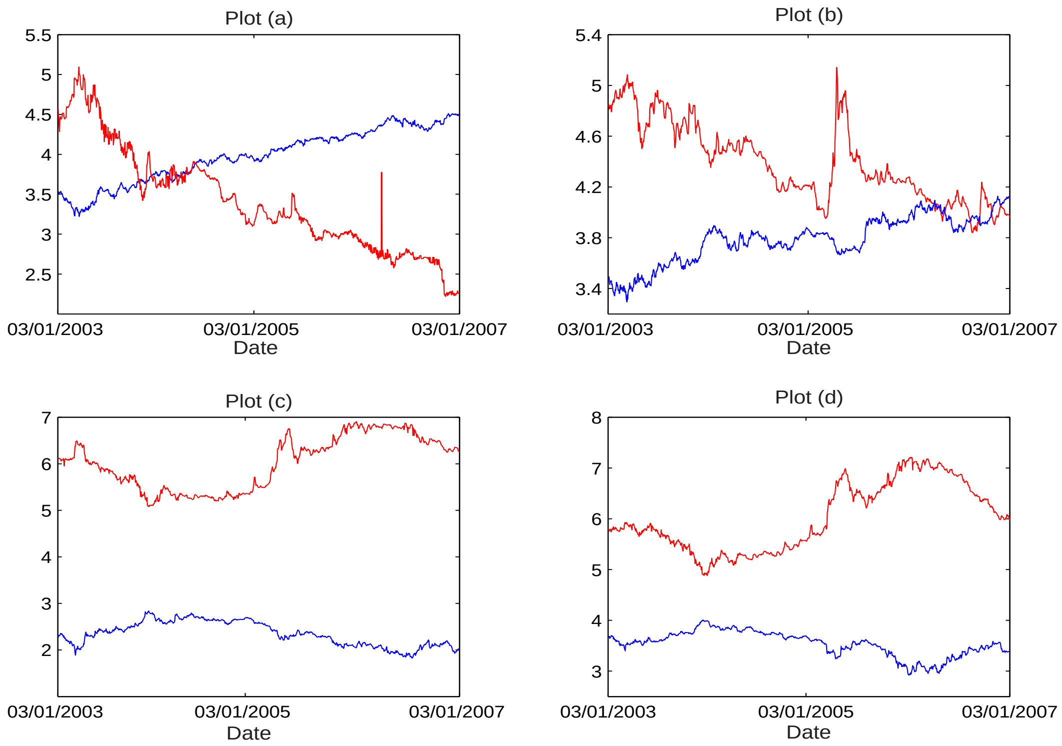

Figure 1.2: Times series plots of daily 5 years CDS premiums and daily share prices (the natural logarithm is taken of both time series for visual comparison) for 4 different NYSE listed companies: Boeing (Plot (a)), Daimler Chrysler (Plot(b)), Ford (Plot(c)) and General Motors (Plot (d)). The red line represents 5-yr CDS premiums and the blue line represents share prices.

(2004). Yu (2005) investigates how profitable their capital structure arbitrage trading strategy is. Capital structure arbitrage attempts to profit from temporary relative mispricings between market CDS spreads and theoretically implied equity CDS spreads. In this strategy they hedge their trading strategy with an equity position. However, the focus of this paper is the realised trading profit and not on the ability of their model to hedge their CDS position. Schefer & Strebulaev (2004) calculated hedge ratios, that measured the sensitivity of bond prices with equity, from the Merton model, and found that they successfully predict market sensitivities. To further illustrate the relationship between the CDS market and the equity market, four NYSE (New York Stock Exchange) listed companies' daily share prices are plotted in figure 1.2, with their corresponding daily 5 year CDS premiums. From these plots it can be seen there is a definite negative relationship between the two. Each of the firm's cross-correlation[7] (between the natural logarithm[8] of daily 5yr CDS premiums and the natural logarithm of daily share price) coefficients ρ also reveal a negative relationship: Boeing $\rho = -0.9662$, Daimler $\rho = -0.8582$, Ford $\rho = -0.9005$ and General Motors $\rho = -0.9392$.

1.3 Credit Risk Models

There exists two classes of credit risk models: structural and reduced form. Structural models originated with Black & Scholes (1973), and Merton (1974), and reduced form models with Jarrow & Turnbull (1995). Structural models are built on the premise that there is a fundamental process

[7] Pearson's product-moment cross-correlation with zero lag. This correlation coefficient $\rho \in [-1, 1]$, measures the linear dependency between two random variables. The correlation coefficient is 1 in the case of an increasing linear relationship, -1 in the case of a decreasing linear relationship, and some value in between in all other cases, indicating the degree of linear dependence between the variables. The closer the correlation coefficient is to either -1 or 1, the stronger the correlation between the random variables.

[8] In general, when calculating the correlation between two financial assets, we look at the natural logarithm of the assets' price, because a common assumption is that financial asset prices follow a log-normal distribution. Thus the natural logarithm of financial asset prices follow a normal distribution, which is the natural distribution of the random variables when calculating the Pearson's product-moment cross-correlation coefficient.

V_t, interpreted as the total value of the assets of the firm at time t. The firm's asset value V_t is the driving force behind the dynamics of the prices of all the securities issued by the firm (equity and debt), and all claims on the firm's value are modelled as derivative securities with the firm's value V_t as the underlying. In the structural framework, default is triggered when the asset value V_t is insufficient to pay back the outstanding debt. This is modelled as V_t crossing some default threshold barrier b_t. From the structural model one can model both credit risk and equity, the fundamental link being the firms' asset value V_t. This is the reason why we will focus on structural models in this dissertation. In a reduced form framework, default is modeled by a default process. The default process is usually defined as an exogenous one-jump process which can jump from no-default to default. Reduced form models use market prices of the firms defaultable instruments (such as bonds or credit default swaps) to extract their default intensity (the jump intensity) which can be used to calculate default probabilities and prices of other defaultable instruments. The value of the firm's assets and its capital structure are not modelled at all. They rely on the market as the only source of information regarding the firms credit structure (without considering any credit related information included in balance sheets or equity prices).

1.4 Research Objectives

There are three objectives that will be considered in this dissertation. Firstly, to establish a theoretical relationship between CDS values and equity prices, via the structural model framework and calculate the sensitivity of CDS values to equity prices (hedge ratios). The second objective will be to quantify these sensitivities with market data. The third objective will be to test whether these sensitivities can be used to hedge exposure to a CDS position, with an equity position, either using an equity option or the underlying equity.

Since we are using the structural model approach, the theoretical values of credit default swaps, ϕ_t, stocks, S_t, call stock options[9], φ_t, and put stock options ,$\overline{\varphi}_t$, will be dependent on the firm's asset value V_t. The sensitivity of credit default swap values, ϕ_t, to stock prices, S_t, call stock option prices, φ_t, and put stock option prices, $\overline{\varphi}_t$, will be evaluated by using the following partial derivatives[10]:

$$\frac{\partial \phi_t}{\partial S_t} = \frac{\partial \phi_t}{\partial V_t}\frac{\partial V_t}{\partial S_t}, \tag{1.1}$$

$$\frac{\partial \phi_t}{\partial \varphi_t} = \frac{\partial \phi_t}{\partial V_t}\frac{\partial V_t}{\partial \varphi_t}, \tag{1.2}$$

$$\frac{\partial \phi_t}{\partial \overline{\varphi}_t} = \frac{\partial \phi_t}{\partial V_t}\frac{\partial V_t}{\partial \overline{\varphi}_t}. \tag{1.3}$$

The above partial derivatives give the rate of change of CDS values with respect to stock, call stock option and put stock option prices, with all the other independent variables in the CDS pricing formula remaining fixed. These partial derivatives will determine the number of shares, and options on shares, that are needed to hedge the change in value of a CDS. We will name these partial derivatives *delta hedge ratios.* We will apply a simple hedging strategy according to these delta hedge ratios to determine if it is possible to hedge exposure to a CDS position with an equity position.

Intuitively, a credit instrument (eg. a corporate bond) should be used to hedge a CDS[11]. A less obvious approach would be to use a position in equity, since credit and equity markets are often considered to be distinct. By means of a structural credit risk model we are able to theoretically link these two markets. In Chapter 3, we will see that hedging a CDS with a credit instrument can be problematic or even impossible as there may not be any publicly available credit instruments. We analyse a novel alternative approach to hedging credit default swaps, by taking a position in the equity market. From this study we are also able to infer theoretical prices for equity (shares and share options) and credit default swaps from the credit and equity market, respectively. From these theoretical prices, it is possible to identify relative mispricings between theoretical (which has

[9] In this dissertation we only work with European options.

[10] We show in Chapter 6 that there is one-to-one relationship between CDS values, stock prices, stock option prices and the firm's asset value V, under our structural model framework. Since there is this one-to-one relationship, equations (1.1), (1.2) and (1.3) hold.

[11] This is intuitive, since a CDS can be seen as insurance policy on a credit instrument, thus a natural hedging instrument would be the credit instrument. We examine hedging a CDS with credit instruments in Chapter 3.

been inferred by a different market) and market prices, which is required to perform the capital structure arbitrage trading strategy.

1.5 Structure of the Dissertation

The three specific aims of this study are: to construct a mathematical model that will enable us to relate the credit and equity market dynamically, to examine methods of pricing of equity and credit products under our mathematical model, and finally calculate the delta hedge ratios and implement them with market data to asses if it is possible to hedge the value of a credit default swap with stock or stock options.

In the following, we briefly go over the content of each chapter. Chapter 2 outlines the mathematical underpinnings of credit risk models. This framework is based on the asset valuation framework developed by Harrison & Pliska (1981). It is extended to account for assets that have defaultable payments. Chapter 3 gives a brief overview of credit derivatives and a thorough description of credit default swaps and its intricacies. An approximate hedge-based pricing (replication) strategy (Schönbucher (2003)) of credit default swaps will be illustrated. A bond based pricing method (Hull & White (2000)), a popular method to price CDS, will also be explained. A generic CDS pricing formula will be presented, that is applicable to all credit risk models. Chapter 4 examines structural models. We review the seminal structural model, the Merton (1974) model, and comment on its strengths and weaknesses. We research extensions to this model, and analyse the significance of the additional features. We present our model that will be used to price and calculate the delta hedge ratios. We present several techniques available to estimate the parameters of a structural model. Chapter 5 we give the valuation formulae for CDS, stock and European stock options. Since our mathematical model does not admit explicit solutions for theses valuation formulae, we present Monte Carlo methods to calculate the valuation formulae. We investigate their computational accuracy and efficiency. We outline the calibration method which we used to estimate our model's parameters. In Chapter 6 we examine hedging under different market assumptions. We then present the method to calculate our delta hedge ratios and outline the method to implement them to hedge a position in a CDS with stock or stock options. We then test this hedging scheme with simulated and market data. We conclude with an examination of the results from these tests.

Chapter 2

Mathematical Framework

2.1 Introduction

The purpose of this chapter is to lay down the mathematical framework and describe the *arbitrage-free pricing* technique for *defaultable* securities. This is not a thorough description of arbitrage-free pricing theory. For a technical account the interest reader is referred to Protter (2005). The following section highlights some of the main ideas concerning the arbitrage-free pricing technique for *non-defaultable* contingent claims. We will extend this arbitrage-free pricing technique to price defaultable contingent claims. From the general arbitrage-free framework for defaultable claims, we will characterise two major credit risk models: the structural model and the reduced form model.

2.2 Arbitrage Pricing Theory

An arbitrage opportunity exists if it is possible to make a risk-free profit. The absence of arbitrage opportunities in the financial market model is the fundamental economic assumption for asset pricing. By assuming the financial market model is arbitrage-free, portfolios having identical cash flows must have the same price. By constructing an appropriate portfolio which yields a riskless return over an infinitesimally small period of time, Black & Scholes (1973) concluded that to avoid arbitrage opportunities the portfolio's instantaneous return must equal the prevailing risk-free rate. This observation led to their celebrated partial differential equation which can be explicitly solved for European vanilla options. The first mathematically rigorous framework for arbitrage-free pricing was developed by Harrison & Kreps (1979) and Harrison & Pliska (1981). We now review the main results of Harrison and Pliska, in order to extend these tools to price defaultable securities. For now, we assume all securities are non-defaultable and later in the chapter we include defaultable securities into the mathematical framework. All of the following definitions, propositions and theorems are taken from Šelić (2006) and Brigo & Mercurio (2006).

2.2.1 Financial Market Model

Consider a financial market with a fixed trading horizon $[0, T]$. On this time interval the uncertainty of the financial market is modelled by a filtered probability space $(\Omega, \mathscr{F}_T, \mathbb{F}, \mathbb{P})$, $\mathbb{P} \in \mathscr{P}$, where Ω is the sample space set containing all possible outcomes, $\mathbb{F}$ is right continuous filtration, $\mathbb{F} = \{\mathscr{F}_t : 0 \leq t \leq T\}$ (a dynamically evolving information structure), $\mathbb{P}$ is the real world (or statistical) probability measure assigned to event sets $A \in \mathbb{F}$ and $\mathscr{P}$ is a class of equivalent probability measures on $(\Omega, \mathscr{F}_T)$. The financial interpretation of modelling the uncertainty in a financial market with a *class* of equivalent probability measures, is that investors agree on which outcomes are possible, but their assignment of probability to these outcomes differ (see Musiela & Rutkowski (1997) for more details).

In the financial market there are $K + 1$ primary traded securities. These primary securities do not pay dividends. The price processes of these securities are denoted[1] by $\boldsymbol{V} = \{\boldsymbol{V}_t,\, 0 \leq t \leq T\}$, where $\boldsymbol{V}_t = \left(V_t^0,\, V_t^1, \ldots, V_t^K\right)'$. We model these processes with strictly positive semimartingales

[1] In the usual Harrison & Pliska (1981) framework, the primary securities are stock, however under the structural model the primary securities are the firm's total assets.

(see Protter (2005) for the mathematical definition of a semimartingale stochastic process). Semimartingales are conventionally used to model price processes as they are the most general stochastic process for which a stochastic integral can be reasonably defined. It can be shown (see Delbaen & Schachermayer (1998), Theorem 7.2) if the price processes are modelled by semimartingales and the model admits equivalent martingale measures, then our model is arbitrage-free[2]. The security indexed by 0 is the bank account process, and its price process evolves according to

$$dV_t^0 = r_t V_t^0 \, dt, \tag{2.1}$$

where $V_0^0 = 1$ and r_t is the instantaneous risk-free short rate at time t. The solution of (2.1) is

$$V_t^0 = \exp\left(\int_0^t r_s \, \mathrm{d}s\right).$$

Let $B(0,\, t) := V_t^0$ denote the bank account and if $s \le t$ then $B(s,\, t) = B(0,\, t)/B(0,\, s)$.

In addition to the primary securities, we have contingent claims[3] in the financial market. Let us denote the price process of the contingent claim by ξ_t and denote the cash flow process[4] generated from this contingent claim by U_t. The value of the contingent claim is dependent on the underlying primary securities. The concept of a replicating portfolio is used to price and hedge contingent claims. Suppose we create a trading strategy, with an initial inflow of cash and thereafter no additional injection of cash into the strategy, that can match all of the cashflows of the contingent claim for all $\omega \in \Omega$. By ensuring that the financial market is arbitrage-free, the contingent claim and the replicating portfolio have equal values at inception.

Definition 2.2.1. *A trading strategy is a $(K+1)-$dimensional process $\boldsymbol{\psi} = \{\boldsymbol{\psi}_t,\, 0 \le t \le T\}$, where $\boldsymbol{\psi}_t = \left(\psi_t^0,\, \psi_t^1, \ldots, \psi_t^K\right)$ and $\boldsymbol{\psi}_t$ is locally bounded and predictable.*

The k-th component ψ_t^k of trading strategy $\boldsymbol{\psi}_t$ at time t, represents the amount of the k-th primary security held at time t. The financial interpretation of the predictability condition on $\boldsymbol{\psi}_t$, is that an investor knows immediately before time t the number of units held in each security, and rebalances his portfolio after observing the prices of the securities at time t. The locally bounded condition is a technical stochastic integrability condition.

Associated with a trading strategy $\boldsymbol{\psi}$ is the *value process* $\vartheta(\boldsymbol{\psi}) = \{\vartheta_t(\boldsymbol{\psi}),\, 0 \le t \le T\}$, defined by

$$\vartheta_t(\boldsymbol{\psi}) = \boldsymbol{\psi}_t \,.\, \boldsymbol{V}_t = \sum_{k=0}^{K} \psi_t^k V_t^k,$$

and the *gains process* $\boldsymbol{G}(\boldsymbol{\psi}) = \{\boldsymbol{G}_t(\boldsymbol{\psi}),\, 0 \le t \le T\}$, defined by

$$G_t(\boldsymbol{\psi}) = \int_0^t \boldsymbol{\psi}_t \,.\, \mathrm{d}\boldsymbol{V}_t = \sum_{k=0}^{K} \int_0^t \psi_t^k \mathrm{d}V_t^k.$$

The value process $\vartheta(\boldsymbol{\psi})$ represents the market value of a portfolio following trading strategy $\boldsymbol{\psi}$, and the gains process $G(\boldsymbol{\psi})$ represents the cumulative capital gains when applying trading strategy $\boldsymbol{\psi}$. A trading strategy $\boldsymbol{\psi}$ is termed *self-financing* if its value process $\vartheta(\boldsymbol{\psi})$ changes only due to changes in the values of the primary securities (i.e. a trading strategy that requires no additional financing after the portfolio has been set up).

Definition 2.2.2. *A trading strategy $\boldsymbol{\psi}$ is termed self-financing if the associated value process satisfies*

$$\vartheta_t(\boldsymbol{\psi}) = \vartheta_0(\boldsymbol{\psi}) + G_t(\boldsymbol{\psi}). \tag{2.2}$$

The above relation (2.2) holds when the values of the primary securities are expressed in terms of a numèraire[5] N_t. A popular choice for a numèraire is the bank account process[6] $B(0, t)$. Let us denote the numèraire based value of the primary securities by $\hat{\boldsymbol{V}}_t = \boldsymbol{V}_t/N_t$, and the numèraire based value process, associated with trading strategy $\boldsymbol{\psi}$, by $\hat{\vartheta}_t(\boldsymbol{\psi}) = \boldsymbol{\psi}_t \,.\, (\boldsymbol{V}_t/N_t)$.

[2]The terms *equivalent martingale measure* and *arbitrage-free* will be explained later in the chapter.

[3]Contingent claims are financial securities whose prices depend on the values of other assets.

[4]For a European stock option the cash flow process will be a single payment at maturity if the option matures in-the-money. However, for a swap contract (e.g. an interest rate swap) the cash flow process will have many periodic payments till expiration of the contract.

[5]A numèraire is any positive non-dividend paying security.

[6]It will soon be seen why we choose the bank account process for our numèraire.

Proposition 2.2.1. *A trading strategy* $\boldsymbol{\psi}$ *is self-financing if and only if*

$$\hat{\vartheta}_t(\boldsymbol{\psi}) = \vartheta_0(\boldsymbol{\psi})/N_0 + \int_0^t \boldsymbol{\psi}_t \,.\, \mathrm{d}\hat{\boldsymbol{V}}_t.$$

Proof. See Geman et al. (1995) [§2, Prop. 1, p. 445-446]. □

Lets reiterate our contingent claim pricing problem in terms of our introduced notation. To determine ξ_t, the price of a contingent claim at time t, we need to create a self-financing trading strategy[7] $\boldsymbol{\psi}$ such that the value process of our strategy replicates all future cash flows generated by the contingent claim[8]. If we can find such a $\boldsymbol{\psi}$, then by no-arbitrage arguments, $\xi_t = \vartheta_t(\boldsymbol{\psi})$.

2.2.2 Equivalent Martingale Measure

At the center of the arbitrage-free pricing concept is the relationship between arbitrage-free market models and the existence of equivalent martingale measures (EMM).

Definition 2.2.3. *A probability measure* $\mathbb{Q}$ *is absolutely continuous with respect to another measure* $\mathbb{P}$ *(denoted* $\mathbb{Q} \ll \mathbb{P}$*) if for all* $A \in \mathscr{F}$ *where* $\mathbb{P}(A) = 0$ *we also have* $\mathbb{Q}(A) = 0$*. If* $\mathbb{Q} \ll \mathbb{P}$ *and* $\mathbb{P} \ll \mathbb{Q}$ *then the two measures are equivalent (denoted by* $\mathbb{Q} \sim \mathbb{P}$*).*

Let us fix $\mathbb{P}$ as the *real world* measure.

Definition 2.2.4. *A probability measure* $\mathbb{Q}^N \sim \mathbb{P}$ *is an equivalent martingale measure with respect to a chosen numèraire* N_t *if all the numèraire based price processes of the primary assets in the financial market satisfy*[9]

$$\frac{V_t}{N_t} = \mathbb{E}^{\mathbb{Q}^N}\left(\frac{V_T}{N_T}\middle|\mathscr{F}_t\right),$$

for all $0 \le t \le T$*; i.e. all the numèraire based asset price processes* $\hat{V}_t$ *are* $\mathscr{F}_t$*-martingales under measure* $\mathbb{Q}^N$*.*

Definition 2.2.5. *There exists an arbitrage opportunity in the financial market if there exists a self-financing trading strategy* $\boldsymbol{\psi}$ *such that the value process satisfies the following set of conditions:*

$$\vartheta_t(\boldsymbol{\psi}) = 0, \quad \mathbb{P}(\vartheta_t(\boldsymbol{\psi}) \ge 0) = 1, \quad \mathbb{P}(\vartheta_t(\boldsymbol{\psi}) > 0) > 0, \quad \text{for some } t > 0$$

Before we link the existence of an equivalent martingale measure to an arbitrage-free market, we need to place restrictions on the choice of self-financing strategies. These restrictions will remove doubling and suicide strategies which are able to generate arbitrage opportunities.

Definition 2.2.6. *Given our financial market admits an equivalent martingale measure* $\mathbb{Q}^N$*, a self-financing trading strategy* $\boldsymbol{\psi}$ *will be defined as being* $\mathbb{Q}^N$*-admissable if its associated numèraire based portfolio price process* $\hat{\vartheta}_t(\boldsymbol{\psi}) \ge 0\, \forall t \in [0, T]$ *and has the* $\mathscr{F}_t$*-martingale property under* $\mathbb{Q}^N$*; i.e.*

$$\mathbb{E}^{\mathbb{Q}^N}\Big(\hat{\vartheta}_T(\boldsymbol{\psi})\big|\mathscr{F}_t\Big) = \hat{\vartheta}_t(\boldsymbol{\psi}).$$

Note that a number of different definitions of admissibility appear in the literature, we will use the above definition of admissibility. To exclude doubling and suicide strategies we restrict self-financing strategies to be $\mathbb{Q}^N$-admissable.

The connection between an arbitrage-free market and the existence of a martingale measure is known as the *Fundamental Theorem of Asset Pricing.* This theorem states that for a financial market with primary securities $\boldsymbol{V}$, the existence of an equivalent martingale measure is *essentially* equivalent to an arbitrage-free market (no arbitrage opportunities exist). Delbaen & Schachermayer (1998) prove, for certain conditions on the numèraire based asset processes $\hat{\boldsymbol{V}}$, an absence of arbitrage implies the existence of an equivalent martingale measure. We examine the reverse implication of the existence of an equivalent martingale measure $\mathbb{Q}^N$ implying the absence of arbitrage opportunities.

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[7]This replication argument does not hold for all self-financing strategies: we must exclude *suicide strategies* and *doubling strategies*. Self-financing strategies must be *admissable*, see Harrison & Pliska (1981) for more details.

[8]The replicating portfolio ϑ_t represents the value of all the future cash flow at time t. By investing in this replication portfolio it will generate all future cashflows $U_s,\ t < s \le T$ almost surely.

[9]The expression $\mathbb{E}^{\mathbb{Q}^N}(.)$ represents the expectation under the measure $\mathbb{Q}^N$.

Theorem 2.2.1. *The existence of an equivalent martingale measure $\mathbb{Q}^N$ is sufficient to ensure that the financial model is arbitrage free*

Proof. We prove the theorem by contradiction. Let an $\mathbb{Q}^N$-admissable self-financing strategy $\boldsymbol{\psi}$ give rise to an arbitrage opportunity. Assuming without loss of generality that

$$\vartheta_0(\boldsymbol{\psi}) = 0 \Rightarrow \hat{\vartheta}_0(\boldsymbol{\psi}) = 0, \tag{2.3}$$

then from Definition 2.2.5 $\mathbb{P}(\vartheta_T(\boldsymbol{\psi}) \geq 0) = 1$ and $\mathbb{P}(\vartheta_T(\boldsymbol{\psi}) > 0) > 0$, which in turn implies that $\mathbb{P}(\hat{\vartheta}_T(\boldsymbol{\psi}) \geq 0) = 1$ and $\mathbb{P}(\hat{\vartheta}_T(\boldsymbol{\psi}) > 0) > 0$. Since $\mathbb{Q}^N \sim \mathbb{P}$, we will also have $\mathbb{Q}^N(\hat{\vartheta}_T(\boldsymbol{\psi}) > 0) > 0$ and $\mathbb{Q}^N(\hat{\vartheta}_T(\boldsymbol{\psi}) \geq 0) = 1$, which implies that

$$\mathbb{E}^{\mathbb{Q}^N}\Big(\hat{\vartheta}_T(\boldsymbol{\psi})\Big) > 0. \tag{2.4}$$

In contrast note that because $\boldsymbol{\psi}$, is a $\mathbb{Q}^N$-admissable strategy, its associated value process will have the martingale property under $\mathbb{Q}^N$; i.e.

$$\mathbb{E}^{\mathbb{Q}^N}\Big(\hat{\vartheta}_T(\boldsymbol{\psi})\Big) = \hat{\vartheta}_0(\boldsymbol{\psi}). \tag{2.5}$$

From (2.3), this implies that

$$\mathbb{E}^{\mathbb{Q}^N}\Big(\hat{\vartheta}_T(\boldsymbol{\psi})\Big) = 0. \tag{2.6}$$

We therefore have a contradiction between the existence of an arbitrage opportunity which implies that (2.4) is true and the existence of an equivalent martingale measure $\mathbb{Q}^N$ that implies that (2.5) is true. □

Formation of an Equivalent Martingale Measure

In this section we present the Radon-Nikodým derivative which characterizes an equivalent martingale measure. When two measures are equivalent, it is possible to express the first in terms of the second through the Radon-Nikodým derivative.

Definition 2.2.7. *A martingale ρ_t on $(\Omega, \mathscr{F}_T, \mathbb{F}, \mathbb{P})$ which has the following properties:*

$$\mathbb{Q}(A) = \int_A \rho_t(\omega)\,\mathrm{d}\mathbb{P}(\omega), \;\; A \in \mathscr{F}_t,$$

and

$$\mathbb{E}^{\mathbb{P}}(\rho_T) = 1.$$

is called the Radon-Nikodým derivative[10] *of $\mathbb{Q}$ with respect to $\mathbb{P}$ restricted to $\mathscr{F}_t$. The Radon-Nikodým derivative can be written in a more concise form as*

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = \rho_t.$$

Also $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}$ refers to $\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_T}$. The Radon-Nikodým derivative is often used to find the expected value of a random variable X, with respected to a equivalent measure. It can be easily seen that the following holds:

$$\mathbb{E}^{\mathbb{Q}}(X) = \int_\Omega X(\omega)\,\mathrm{d}\mathbb{Q}(\omega) = \int_\Omega X(\omega)\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\,\mathrm{d}\mathbb{P}(\omega) = \mathbb{E}^{\mathbb{P}}\left(X\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right).$$

Definition 2.2.8. *The Doléans-Dade exponential $\mathscr{E}(D)$ is the unique solution of the stochastic differential equation*

$$\mathrm{d}\mathscr{E}(D)_t = \mathscr{E}(D)_{t^-}\,\mathrm{d}D_t, \;\; \mathscr{E}(D)_0 = 1. \tag{2.7}$$

The explicit solution to (2.7) is given by

$$\mathscr{E}(D)_t = \exp\left(D_t - \frac{1}{2}[D,D]^c_t\right)\prod_{0\leq u\leq t}(1+\Delta D_u)e^{-\Delta D_u}, \tag{2.8}$$

[10] Sometimes ρ_t is called the Radon-Nikodým density process.

where D_{t^-} is the left continuous version of D_t, $\Delta D_t = D_t - D_{t^-}$ and $[D, D]^c$ is the path-by-path continuous part of the quadratic variation process[11] $[D, D]_t$

$$[D, D]^c_t = [D, D]_t - \sum_{0 \le u \le t} (\Delta D_u)^2.$$

Theorem 2.2.2. *Consider $\mathbb{Q} \sim \mathbb{P}$ and the Radon-Nikodým derivative process ρ_t. Suppose that there exists a $\mathbb{P}$-local martingale D with $D_0 = D_{0^-} = 0$ satisfying*

$$\Delta D_t > -1, \;\; 0 \le t \le T \tag{2.9}$$

$$\mathbb{E}^{\mathbb{P}}[\mathscr{E}(D)_t] = 1. \tag{2.10}$$

Then there exists a one-to-one correspondence between ρ and D, given by

$$\rho_t = \mathscr{E}(D)_t, \;\; 0 \le t \le T$$

Proof. See Musiela & Rutkowski (1997) [§10.1.4, p. 245-246]. □

From Theorem 2.2.2 it can be seen that the Radon-Nikodým derivative process is the Doléans-Dade exponential of a local martingale, when conditions (2.9) and (2.10) are satisfied.

The next theorem, will provide us with the semi-martingale decomposition under an equivalent probability measure $\mathbb{Q}$, for a semi-martingale that was initially defined in probability space $(\Omega, \mathscr{F}_T, \mathbb{F}, \mathbb{P})$. It will also provide us with an equation which can be used to find the explicit form of the Radon-Nikodým derivative which characterizes an equivalent martingale measure.

Theorem 2.2.3. ***Girsanov's Theorem.*** *Let X be a continuous semi-martingale under probability space $(\Omega, \mathscr{F}_T, \mathbb{F}, \mathbb{P})$ with decomposition*

$$X_t = X_0 + M_t + A_t, \quad 0 \le t \le T$$

where M is a continuous local martingale and A is a continuous finite variation process. Let $\mathbb{Q} \sim \mathbb{P}$ and let the Radon-Nikodým derivative $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \mathscr{E}(D)_T$ be defined by the Doléans-Dade exponential of a local martingale D, satisfying conditions from Theorem 2.2.2. Then X is a continuous semi-martingale under $\mathbb{Q}$ with decomposition

$$X_t = X_0 + L_t + C_t, \quad 0 \le t \le T, \tag{2.11}$$

where L is a $\mathbb{Q}$-local martingale[12]

$$L_t = M_t - \langle M, D\rangle_t, \quad 0 \le t \le T,$$

and C is a $\mathbb{Q}$ finite variation process

$$C_t = A_t + \langle M, D\rangle_t, \quad 0 \le t \le T.$$

In particular, X is a local martingale under $\mathbb{Q}$ if and only if

$$A_t + \langle M, D\rangle_t = 0, \quad 0 \le t \le T. \tag{2.12}$$

Proof. See Protter (2005) [Thm. 39, p. 135-136]. □

Relating the above theory to our financial model, X_t represents a single numèraire based price processes $\hat{V}_t$. From equation (2.12) we can solve for D, after one has specified a stochastic process for the evolution of the primary asset price process V_t. With the explicit form of D we can determine the Radon-Nikodým derivative $\frac{\mathrm{d}\mathbb{Q}^N}{\mathrm{d}\mathbb{P}}$ from Theorem 2.2.3, which characterizes an *equivalent martingale measure* for our financial model. From equation (2.11) we can determine the form of the numèraire based price processes under measure $\mathbb{Q}^N$. For an application of the Girsanov's Theorem under a Black & Scholes framework, refer to Šelić (2006). See Øksendal & Sulem (2005) for examples of Girsanov's Theorem for different specifications of X_t.

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[11] See Protter (2005) for the definition of a quadratic variation process.

[12] The expression $\langle X, Y\rangle$ denotes the *conditional quadratic covariation* of processes X and Y. See Protter (2005) for the definition of a conditional quadratic covariation process.

2.2.3 Arbitrage-Free Pricing of Contingent Claims

The purpose of this section is to calculate an arbitrage-free price at time t, of a contingent claim ξ, which has a maturity at T and generates stochastic cashflow according to the process U_t, $0 \le t \le T$. To facilitate the pricing technique we accumulate all future cashflows to time T. Let

$$Y_{t,\,T} := \int_t^T B(s,\,T)\,\mathrm{d}U_s.$$

The above introduced term $Y_{t,\,T}$ represents the value at maturity T of all future cash flows, after time t, generated by contingent claim ξ. Note $Y_{t,\,T}$ is a stochastic variable, at time t.

Definition 2.2.9. *A contingent claim ξ initiated at $t = 0$, which generates cashflows U_s, $0 \le s \le T$ and matures at T, is defined as attainable if there exists a $\mathbb{Q}^N$-admissable self-financing strategy $\boldsymbol{\psi}$ that replicates the cashflow $Y_{t,\,T}$ at maturity T, i.e.*

$$\vartheta_T(\boldsymbol{\psi}) = Y_{t,\,T}.$$

The following proposition, proved by Harrison & Pliska (1981), provides the mathematical characterization of the arbitrage-free price associated with any attainable contingent claim.

Proposition 2.2.2. *Assume there exists an equivalent martingale measure $\mathbb{Q}^N$ and let ξ be an attainable contingent claim with cash flow process U_t. Then, an arbitrage-free price process $\{\xi_t,\, 0 \le t \le T\}$ at time t, for the contingent claim ξ is given by the following martingale based pricing formula*

$$\frac{\xi_t}{N_t} = \mathbb{E}^{\mathbb{Q}^N}\!\left(\frac{Y_{t,\,T}}{N_T}\,\middle|\,\mathscr{F}_t\right)$$

Proof.

$$\begin{aligned} \xi_t &= \vartheta_t(\boldsymbol{\psi}) \\ &= N_t\hat{\vartheta}_t(\boldsymbol{\psi}) \\ &= N_t\mathbb{E}^{\mathbb{Q}^N}\!\left(\hat{\vartheta}_T(\boldsymbol{\psi})\,\middle|\,\mathscr{F}_t\right) \\ \Rightarrow \frac{\xi_t}{N_t} &= \mathbb{E}^{\mathbb{Q}^N}\!\left(\frac{Y_{t,\,T}}{N_T}\,\middle|\,\mathscr{F}_t\right) \end{aligned}$$

□

The uniqueness of the arbitrage-free price (2.13) is determined by the uniqueness of the equivalent martingale measure.

Definition 2.2.10. *A financial market is complete if and only if every contingent claim is attainable.*

Harrison & Pliska (1983) proved the following fundamental result linking market completeness and a unique arbitrage-free price.

Theorem 2.2.4. *A financial market model is complete if and only if $\mathbb{Q}^N$ the equivalent martingale measure associated with numèraire N is unique.*

Proof. Harrison & Pliska (1983). □

We will often use the bank account process $B(0,\,t)$ as the numèraire, since this replaces the problem of estimating the instantaneous rate of return of the primary traded securities[13] with estimating the riskless rate which is observable in the market. When we use the bank account process $B(0,\,t))$ as the numèraire, we will denote the corresponding equivalent martingale measure by $\mathbb{Q} := \mathbb{Q}^{B(0,\,t)}$. Throughout the dissertation we will use this equivalent martingale measure for pricing. This measure $\mathbb{Q}$ is named the *risk-neutral measure.*

[13]The rate of return on the traded securities differs for each investor's risk preference depends (see Rosenberg & Engle (2002)).

2.3 Financial Securities Dependent on the Default Event

The following section provides a general credit risk model, which can be used to price credit risk sensitive instruments. This general model will form a basis for both the structural and reduced-form credit risk models. The following concepts are adapted from Bielecki & Rutkowski (2002).

2.3.1 General Credit Risk Model

We fix a finite horizon date $T > 0$. The financial market is modelled by a filtered probability space $(\Omega, \mathscr{G}_T, \mathbb{G}, \mathbb{P})$, $\mathbb{P} \in \mathscr{P}$, where $\mathbb{G}$ is a right continuous filtration, $\mathbb{G} = \{\mathscr{G}_t : 0 \leq t \leq T\}$. The financial model consists of $K + 1$ primary traded securities, denoted by $\boldsymbol{V} = \{\boldsymbol{V}_t,\, 0 \leq t \leq T\}$, where $\boldsymbol{V}_t = \left(V_t^0,\, V_t^1, \ldots, V_t^K\right)'$. Let the security $V_t^0 \equiv B(0,t)$ be the bank account process. The bank account process evolves according to (2.1).

In addition to the primary securities, we have a defaultable claim in the financial market[14]. Let us denote the price process of the defaultable contingent claim by $\tilde{\xi}_t$. The defaultable contingent claim has a maturity of $t_n \leq T$. Let us denote the cash flow process generated from this defaultable claim by $\tilde{U}_t$, where $t \in [0,\, t_n]$. Suppose that the filtration $\mathbb{G} = \{\mathscr{G}_t : 0 \leq t \leq T\}$ is sufficiently rich to support the following objects:

- the primary traded securities $\boldsymbol{V}$,
- the instantaneous risk-free interest rate r_t,
- the default time τ^*,
- the promised payment X_{t_n} of contingent claim $\tilde{\xi}$. The promised payment X_{t_n} is to be paid at time $t_n \leq T$, if the default event has not occurred, i.e. $\tau^* > t_n$,
- the promised dividend process I_t of contingent claim $\tilde{\xi}$, i.e. the stream of promised payments that occur until default or until maturity.
- the recovery claim $\tilde{X}_{t_n}$, which represent the recovery payoff of contingent claim $\tilde{\xi}$ paid at time t_n if $\tau^* \leq t_n$.
- the recovery process $\tilde{Z}_t$, which specifies the recovery payment of contingent claim $\tilde{\xi}$ paid at the time of default τ, if $\tau^* \leq t_n$.

The cash flow process is defined by the quintuple $\tilde{U} = \left(X_{t_n},\, I,\, \tilde{X}_{t_n},\, \tilde{Z},\, \tau^*\right)$. We specify the different constituents of $\tilde{U}_t$, since the payments have different recovery payoff schemes. In case of a defaultable coupon bond, it is often postulated that in case of default the future coupons, are lost (zero recovery scheme). However a strictly positive fraction of the bond's face value, X_{t_n}, is usually received by the bondholder.

There exists a filtration $\mathbb{F} = \{\mathscr{F}_t : 0 \leq t \leq T\}$ and $\mathbb{F} \subseteq \mathbb{G}$. Another filtration is defined to differentiate between the two major credit risk models: structural models and reduced-form models. We postulate that the processes $\boldsymbol{V}_t$, r_t, I_t and $\tilde{Z}_t$ are progressively measurable with respect to the filtration $\mathbb{F}$, and that X_{t_n} and $\tilde{X}_{t_n}$ are $\mathscr{F}_{t_n}$-measurable. We assume without mentioning that all random objects introduced above satisfy suitable integrability conditions needed to find the explicit solution to Equation (2.13), the price of the defaultable contingent claim.

Definition 2.3.1. *A random time τ is a $\mathbb{F}$-stopping time if and only if*

$$(\tau \leq t) \in \mathscr{F}_t.$$

In the general credit risk model, default time τ^* is a $\mathbb{F}$-stopping time. The distinguishing factor between the structural model and the reduced-form model is the approach to modelling the default time. In general, within the structural framework, τ^* is a $\mathbb{F}$-stopping time and within the reduced-form framework τ^* is a $\mathbb{G}$-stopping time. We will later discuss how τ^* is modelled in these two different credit risk frameworks.

The payment after default is known as the *recovery payment.* It is often represented as a percentage of the *promised payment* X_{t_n}, which is known as the *recovery rate* R. If no default

[14]For notational simplicity we consider one defaultable claim. The framework can be extended to more than one defaultable claim.

occurs before the end of the maturity of the defaultable contingent claim ($\tau^* > t_n$), all the promised payments X_{t_n} and I_t, $\forall t \in [0,\, t_n]$ will be made at the respective payment dates. If default occurs before maturity t_n, depending on the recovery payoff scheme adopted, either the amount Z_{τ^*} is paid at default time τ^*, or the amount $\tilde{X}_{t_n}$ is paid at maturity t_n. In a general setting we will consider both recovery schemes, and the cashflow will be described by the quintuple $\tilde{U} = \left(X_{t_n},\, I,\, \tilde{X}_{t_n},\, \tilde{Z},\, \tau^*\right)$. In practical situations $\tilde{X}_{t_n} = 0$ or $\tilde{Z} = 0$ depending if recovery is at *default* or at *maturity*, respectively.

2.3.2 Arbitrage-Free Valuation Formula

Suppose the market is arbitrage-free, in the sense that there exists an equivalent martingale measure. Let the bank account process be our choice of numèraire: $N_t = B(0,\, t)$. Let's introduce the *hazard process* $H_t := \mathbb{1}_{\{\tau^* \leq t\}}$, which equals one if default occurs before or at time t, and equals zero if default occurs after time t. From the introduced constituents of the cashflow process of the defaultable contingent claim ξ, we can mathematically define $\tilde{U}_t$.

Definition 2.3.2. *The cashflow process $\tilde{U}_t$ of defaultable contingent claim $\tilde{\xi}$ which matures at time t_n, equals*

$$\tilde{U}_t = \mathbb{1}_{\{t \geq t_n\}} \left(X_{t_n} \mathbb{1}_{\{\tau^* > t_n\}} + \tilde{X}_{t_n} \mathbb{1}_{\{\tau^* \leq t_n\}}\right) + \int_0^t (1 - H_s)\, \mathrm{d}I_s + \int_0^t \tilde{Z}_s\, \mathrm{d}H_s, \quad \text{for} 0 \leq t \leq t_n$$

If default occurs at some time t, the promised dividend payment at time t, $I_t - I_{t^-}$, is not passed on to the holder of the contingent claim, thus

$$\int_0^t (1 - H_s)\, \mathrm{d}I_s = \int_0^t \mathbb{1}_{\{\tau^* > t\}}\, \mathrm{d}I_s = I_{\tau^{*-}} \mathbb{1}_{\{\tau^* \leq t\}} + I_t \mathbb{1}_{\{\tau^* > t\}}.$$

Furthermore, we have

$$\int_0^t \tilde{Z}_s\, \mathrm{d}H_s = \tilde{Z}_{\tau^* \wedge t} \mathbb{1}_{\{\tau^* \leq t\}} = \tilde{Z}_{\tau^*} \mathbb{1}_{\{\tau^* \leq t\}},$$

where $\tau^* \wedge t = \min(\tau^*,\, t)$.

Let $\tilde{Y}_{t,\, t_n}$ represent all the accumulated cashflows of the defaultable contingent claim $\tilde{\xi}$ after time t, to its maturity t_n

$$\begin{aligned} \tilde{Y}_{t,\, t_n} &:= \int_t^{t_n} B(s,\, t_n)\, \mathrm{d}\tilde{U}_s \\ &= X_{t_n} \mathbb{1}_{\{\tau^* > t_n\}} + \tilde{X}_{t_n} \mathbb{1}_{\{\tau^* \leq t_n\}} + B(\tau^*,\, t_n) I_{\tau^{*-}} \mathbb{1}_{\{t \leq \tau^* \leq t_n\}} \\ &\quad + I_{t_n} \mathbb{1}_{\{\tau^* > t_n\}} + B(\tau^*,\, t_n) \tilde{Z}_{\tau^*} \mathbb{1}_{\{\tau^* \leq t_n\}}. \end{aligned}$$

The price process of the defaultable contingent claim $\tilde{\xi}_t$, represents the time-t value of all future cash flows after time t, associated with the defaultable contingent claim. Thus the value of $\tilde{\xi}$ at its maturity is 0, since there are no more future cashflows. Using our arbitrage-free pricing theory we are able to price the defaultable contingent claim at time t.

Definition 2.3.3. *The price process of the defaultable contingent claim $\tilde{\xi}$, which settles at time t_n, is given by*

$$\tilde{\xi}_t = \mathbb{E}^{\mathbb{Q}} \left(\frac{\tilde{Y}_{t,\, t_n}}{B(t,\, t_n)} \middle| \mathscr{G}_t \right) \qquad \forall\, t \in [0,\, t_n]. \tag{2.13}$$

Remember that $\mathbb{Q}$ represents the *risk-neutral measure*. We will often refer to the risk-neutral default probability which refers to the probability of default occuring before some maturity T, under measure $\mathbb{Q}$, i.e. $\mathbb{Q}(\tau^* \leq T)$. The price process for the defaultable contingent claim in the structural and reduced-form model differs only by the filtration, which the expectation is conditioned on.

2.3.3 Reduced-Form Model

In the reduced-form framework, default is modelled as an arbitrary jump-process that jumps from a *non-default* state to a *default* state. Thus the hazard process H is determined by this arbitrary default jump-process. Let $\mathbb{H}$ be the filtration generated by the process H i.e., $\mathscr{H}_t = \sigma(H_s \,:\, s \leq t) =$

$\sigma(\{\tau^* \leq s\}, : s \leq t)$. Under the reduced form-framework the enlarged filtration includes market observable information $\mathbb{F}$ and unobservable default related information $\mathbb{H}$, i.e. $\mathbb{G} = \mathbb{H} \bigvee \mathbb{F}$. Thus the default time τ^* is not measurable with respect to market observable information $\mathbb{F}$, i.e. τ^* is not a stopping-time with respect to the filtration $\mathbb{F}$. A reduced-form model does not give any economic fundamental reason for the arrival of default. A poisson process (jump-process), with intensity parameter λ_t is usually used to model default[15]. The intensity parameter can either be deterministic or stochastic. Mathematically, we can define the default time in terms of the intensity parameter by

$$\tau^* = \inf\{t \geq 0 : H(\lambda_t, t) \geq 1\}.$$

Encoded in the market prices of defaultable instruments is the market's assessment of the default risk of the obligor. In the reduced-form framework we estimate the intensity parameter in the jump-process by calibrating the market prices of standard credit instruments (eg. corporate bonds) to the theoretical prices calculated from the model. Once we have estimated the jump-process parameters, we can price exotic credit instruments (eg. CDS). See Bielecki & Rutkowski (2002) and Schönbucher (2003) for more details on the different specifications of the reduced-form model and for technical details about attainability and hedging of defaultable contingent claims under the reduced-form framework.

2.3.4 Structural Model

In the structural model framework, default is modelled as the firm's asset level reaching a critical default threshold b, which represents the value of the firms liabilities. The primary securities $\boldsymbol{V}$ of the financial market model will represent the asset value of different firms. Mathematically, we can define the default time as

$$\tau^* = \inf\{t \geq 0 : V_t \leq b_t\}.$$

The default threshold is assumed to be $\mathbb{F}$-measurable, and the default time τ^* is a $\mathbb{F}$-stopping time. Thus with observable[16] market information $\mathbb{F}$ we can determine if a company has defaulted.

Under the structural model, the value of all securities (shares and bonds) issued by a firm depends on the value of the firm's assets V. This is the reason why we focus our attention on structural models, since it provides us with a link between equity and debt instruments. The structural model provides economic fundamental reasoning to model default and a more ambitious aim of providing a link between equity and credit. See Chapter 4 for a more detailed review on structural models.

2.4 Discussion

In this chapter we have introduced the topic of arbitrage-free pricing for non-defaultable contingent claims. We introduced a general credit risk model, upon which we utilised arbitrage-free pricing theory, to price defaultable contingent claims. From the general credit risk model, we have shown how to distinguish between the two major credit risk models: structural models and reduced-form models. For a more rigorous technical outlook on general arbitrage-free pricing theory, the interest reader is referred to Musiela & Rutkowski (1997) and Protter (2005), and for a more specialized scope on the arbitrage-free pricing theory for defaultable claims see Bielecki & Rutkowski (2002).

[15] Often the reduced-form models are termed *intensity-based* models.

[16] There is an issue of observability, since the asset value of a firm is not a publicly traded security. If the primary securities V are not observable then we can not replicate contingent claims on these primary securities. Arbitrage-free pricing theory hinges on the fact that we are able to replicate the value of the contingent claims. However, Ericsson & Reneby (1999) argue that if just one of the firm's issued securities is publicly traded then it is possible to replicate the value of the firm's asset value, with that publicly traded security and a non-defaultable bond. This is sufficient to replicate the value contingent claims on the firm's asset value. Many academics just assume the firm's asset value is publicly traded.

Chapter 3

Credit Default Swaps

3.1 Introduction

In this chapter we define exactly the term *credit risk*, briefly discuss credit ratings, and introduce credit derivatives, particularly credit default swaps and how credit default swaps operate in the market. We will explain CDS replicating strategies, where the replication instruments are credit securities. Besides being a good exercise to become familiar with the common financial instruments and their payoff peculiarities, the replicating strategies enable us to understand better the link between the CDS market and the underlying cash (bond) market. The bond based pricing method (Hull & White (2000)) will be described. It is considered to be the market standard to value a CDS. Finally, a generic arbitrage-free pricing formula for a CDS will be constructed. The following description of credit risk and credit default swaps are derived from Schönbucher (2003).

3.2 Credit Risk

The traditional definition of credit risk is the risk that an obligor does not honour his payment obligations. More generally, credit risk encompasses any kind of credit-linked events, such as: changes in the credit quality (including downgrades or upgrades in credit ratings), variations of credit spreads[1], and the default event. Credit risk is intrinsically linked to the payment obligations of an obligor. The obligor is contractually bound to honour all his obligations as long as he is able to. If not, a workout procedure is entered, the obligor loses control of his assets and an independent agent attempts to pay off the creditors using the obligor's assets. The workout procedure involves significant losses to the obligor. These losses stem from the costs of employing the independent agent and from the decrease in the firm's business activity, due to the degradation of the firm's credibility. These losses provide an incentive for the obligor to ensure his solvency, and he only defaults if he really cannot pay his obligations. Thus default almost invariably entails a loss to the creditors.

The major components of credit risk (according to Schönbucher (2003)) are *arrival risk*, *timing risk* and *recovery risk*.

- *Arrival risk* refers to the uncertainty whether a default will occur within a given time horizon. The measure of arrival risk is the probability of default. The probability of default describes the distribution of the indicator function $\mathbb{1}_{\{\tau^*<T\}}$ (default before the time horizon). Where τ^* is the default time and T is the specified time horizon.

- *Timing risk* refers to the uncertainty about the precise time of default. The underlying unknown quantity of timing risk is the time of default τ^*. Knowledge about timing risk includes knowledge about arrival risk for all T. Timing risk is described by the probability distribution function (pdf) of the random variable τ^*.

- *Recovery risk* refers to the uncertainty of the losses if a default has happened. In recovery risk the uncertain quantity is the payoff that a creditor receives after a default. Market

[1] A credit spread measures the excess in return on a defaultable asset over an equivalent non-defaultable asset.

convention is to express this payoff as a proportion of the notional value of the claim, this proportion is known as the recovery rate R. Recovery risk is described by the pdf of the recovery rate R. The pdf of R is a conditional distribution, conditional upon default.

3.3 Credit Ratings

In corporate finance, the credit rating assesses the credit worthiness of a corporation's debt issues. The credit rating is a financial indicator to potential investors of debt securities such as corporate bonds. Credit ratings are assigned by credit rating agencies and have letter designations to denote the credit rating. There are three main credit rating agencies Standard & Poor's, Fitch, and Moody's. Moody's assigns bond credit ratings of Aaa, Aa, A, Baa, Ba, B, Caa, Ca, C . Standard & Poor's and Fitch assign bond credit ratings of AAA, AA, A, BBB, BB, B, CCC, CC, C, D. These ratings are from good to poor credit worthiness. Bonds rated BBB (or Baa) and higher are called *investment grade* bonds. Bonds rated lower than investment grade are colloquially referred to as *junk* bonds.

3.4 Credit Derivatives

A credit derivative is an over-the-counter (OTC) derivative security whose value depends on the underlying firm's (or firms') creditworthiness. Credit derivatives are categorized into two classes, the first class consists of credit derivatives that are linked exclusively to the default event. This class encompasses credit default swaps, default put options and general credit insurance products. The second class of credit derivatives are instruments whose payoff is primarily determined by changes in the credit quality of the underlying firm, instead of a default event. This class encompasses credit spread derivatives and credit ratings based derivatives.

The common features amongst credit derivatives is their ability to transfer credit risk from one counterparty to another, and their payoff is materially affected by credit risk. The value of a credit derivative fluctuates according to the market's perception of the underlying credit risk. The primary objective of this research is to provide a theoretical relationship between these fluctuations in credit derivative values (resulting from changes in credit risk) and equity and equity derivative prices. Specifically, credit default swaps will be examined.

3.5 Credit Default Swap

A credit default swap is a contract between two parties, the protection seller and the protection buyer. The protection seller agrees to pay the protection buyer a *default payment* if a *credit event* occurs to a third-party reference credit. A reference credit/entity is an entity whose defaults trigger the credit event. The credit event is a precisely agreed default event, which is defined with respect to the reference credit and reference credit assets. The 2003 *ISDA Credit Derivatives Definitions* include the following as possible definitions for a credit event[2]:

- bankruptcy, filing for protection
- failure to pay
- obligation default
- obligation acceleration
- repudiation/moratorium
- restructuring

To trigger the last four items of the above list, a certain material threshold must be exceeded and a grace period must have lapsed, this is to ensure that only genuine defaults trigger the *default payment* of the CDS, and not technical errors or minor legal disagreements.

The *default payment* of a CDS is constructed to mirror the losses incurred by the reference entity's creditors. Default payments can be settled by either of the following two methods:

[2]See *http://www.credit-deriv.com/isdadefinitions.htm* (24/08/2006) for a description of each default event.

- Physical delivery (by the protection buyer) of one or several of the reference credit assets against repayment at par[3] (by the protection seller).
- A cash settlement equivalent to par minus post-default market value of the reference credit assets.

Reference assets are a set of assets issued by the reference credit. Reference assets have two possible purposes in a CDS contract:

- The determination of the credit event. For example, missed payments on the reference assets.
- The calculation of the value of the *default payment*:
  - In the case where physical settlement is agreed on, reference assets are the set of deliverable assets that are applicable for the default payment.
  - In the case where cash settlement is agreed on, reference assets determine a basis for the price determination mechanism for the default payment.

Any reference credit asset can be chosen, and the purpose of the reference credit asset can also be specified. Common examples of assets that are used for reference assets are loans and liquidly traded bonds. Depending on availability, traded credit assets are commonly used to determine the default payment, since price discovery is more transparent than for non-traded credit assets. Loans are usually earmarked for determining the credit event. Different sets of reference assets are used for different purposes. By adjusting the set of reference credit assets that determine the credit event, the counterparties can agree to focus on the credit risk of an individual credit asset issued by the reference credit; or they can widen the coverage to any of the reference entity's obligations, thus capturing the reference entity's credit risk completely. Usually, the set of deliverable reference credit assets is less comprehensive than the set of reference credit assets that determine the credit event. As liquidity in defaulted assets can be very low, the set of deliverable reference assets usually includes more than one bond issue, of the same seniority[4], by the reference credit. This delivery option enhances the CDS value for the protection buyer.

For this default protection, the protection buyer pays the protection seller a fixed fee (premium) at regular time intervals (quarterly or semi-annually) until default or maturity, whichever comes first. This premium is calculated by multiplying the CDS spread rate by the notional value of the CDS, adjusted for the day count convention. In the market the CDS spread rate is quoted in basis points[5]. The notional value is equal to the sum of the par values of the reference assets that are used for the default payment. For example, if the CDS notional value is $10 000 000, and the set of deliverable assets is bonds with $1 000 000 par value, the default payment is settled by delivering 10 of these bonds. If the default payment is settled by a cash settlement then the post-default market value is calculated using 10 of these bonds. The CDS spread is calculated so the value of the newly minted CDS contract is zero. This spread is usually called the *fair* or *par* CDS spread. The standard CDS market day count convention is actual/360. The first fee is usually payable at the end of the first time interval, and if a default occurs between two fee payment dates, the protection buyer pays at the time of default the fraction of the next fee payment that has accrued until the time of default.

With regard to cash settlement, since liquidity and manipulation of distressed assets is a real concern a robust procedure is needed to calculate the post-default market value of the reference assets. Therefore in a CDS contract, several dealers are agreed upon to provide quotes for the reference assets, and an average is taken after eliminating the highest and lowest quotes. This is repeated, sometimes several times, in order to eliminate the influence of temporary liquidity holes. Most CDS are settled by physical delivery, since cash settlement is so complex. Cash settlement is only chosen when there may not be any physical assets to deliver, for example if the reference credit may have not issued enough bonds, or the reference assets are not tradeable, such as loans. A physical delivery settlement is not entirely without problems either. If many investors speculated on the default of a particular firm by buying a CDS (buying default protection), they would have to buy the reference assets in the event of a default, in order to sell them to the CDS sellers at their par value. This demand for the reference assets may push their prices artificially high, corroding the value of the CDS for the buyer.

[3]The par value is the principal value of the reference credit asset, also sometimes called the notional value.

[4]Seniority refers to the order of repayment in the event of bankruptcy. Senior debt must be repaid before subordinated debt is repaid.

[5]A basis point is one hundredth of a percentage.

Let us introduce notation for a CDS. Suppose a CDS contract, with maturity at time $t_n = T$, is initiated at time $t_0 = 0$, and that the premiums are paid at time points $0 < t_1, t_2, \ldots, t_n = T$. Let us denote the notional value of the CDS by N and the par CDS spread is denoted by $\tilde{p}_{0,\,t_n}$. Note that the par CDS spread is dependent on the CDS initiation time t_0 and the CDS maturity t_n. Sometimes we will denote $\tilde{p}_{0,\,t_n}$ with the shorthand notation $\tilde{p}$, if it is obvious what the initiation at maturity times are. Mathematically, the CDS premium payments paid by the CDS buyer at some general time t_i, is represented by

$$N\tilde{p}_{0,\,t_n}(t_i - t_{i-1}).$$

Recall that τ^* represents the time of default. If default occurs before time t_i, the premium payment made by the CDS buyer at default is expressed as

$$N\tilde{p}_{0,\,t_n}(\tau^* - t_{i-1}).$$

Recall that R represents the recovery rate of a bond. Mathematically, we represent the default payment paid by the CDS seller, as

$$N(1 - R).$$

Let us denote the value of the CDS contract at some future time $t_f > t_0$ by

$$\phi(t_0,\, t_f,\, T).$$

By construction the value of a CDS contract at initiation is equal to zero, i.e $\phi(t_0,\, t_0,\, T) = 0$. However as the credit risk of the reference credit changes, the CDS contract will posses some value.

3.6 Replication-Based CDS Pricing

The principle of replication-based pricing, is that if two portfolios have equivalent future payoffs (occurring at the same time), then the current value of both of these portfolios must be equal, else an arbitrage (risk-free profit) opportunity exists. For a CDS there exist several simple approximate replication strategies (see Schönbucher (2003)). In this section we will highlight two approximate CDS replicating strategies. Replicating strategies are important and popular in practice since they provide estimates and bounds for pricing. They are useful to spot mispricings in the market because they only rely on payoff comparisons, and the results are robust because they are independent of any assumed pricing model. This section is based Section 2.7 of Schönbucher (2003).

3.6.1 Replication Instruments

The following two financial instruments will be used in the CDS replicating strategies. Note that all listed credit instruments below mature at time T and have a principal of 1:

1. *Default-free bonds*: Default-free[6] coupon bonds either have fixed or floating coupons. The fixed-coupon bond carries a coupon of c. The price of a default-free fixed-coupon bond at time $t < T$ is denoted by $D^c(t,\, T)$. If the last coupon was paid at time $t_{i-1} < T$, the floating-coupon bond carries a coupon of $c' = L(t_{i-1},\, t_i)$ at time $t_i \leq T$. The price of a default-free floating-coupon bond[7] at time $t < T$ is denoted by $D^{c'}(t,\, T)$. The notation $L(t_{i-1},\, t_i)$ denotes the LIBOR interest rate for time interval $[t_{i-1},\, t_i]$. LIBOR (London Interbank Offered Rate) is the interest rate offered by banks on deposits from other banks in Eurocurrency markets. LIBOR represents the interest rate at which banks lend money amongst each other. Technically it is not a default-free rate, but it is common practice to use it as a floating[8] default free interest rate, since the default risk amongst banks is negligible compared to companies in other sectors. We will use the LIBOR rate to represent a generic floating default-free interest rate. We also denote $D(t, T)$, as the time t value of a default-free zero-coupon bond. Note that $D(t,T) = \mathbb{E}^{\mathbb{Q}}\left[B(t,\, T)^{-1}\right]$, where $B(t,\, T)$ is the bank account process.

[6]Treasury bonds are often regarded as default-free bonds, since their default risk is negligible compared to corporate bonds. However, treasury bonds can and have defaulted (eg. Russia's, August 1998, default on its Ruble-denominated GKO short-term debt obligations.)

[7]The notation for the price of a fixed and floating coupon bond differs by the superscript, if it is c it is a fixed coupon and if it is c' it is a floating coupon bond.

[8]The interest rate is not necessarily constant.

2. *Defaultable bonds*: Defaultable coupon bonds issued by the reference credit could either have fixed or floating coupons. The fixed-coupon bond carries a coupon of $\tilde{c}$. The price of a defaultable fixed-coupon bond at time $t < T$ is denoted $\widetilde{D}^{\tilde{c}}(t,\, T)$. If the last coupon was paid at time $t_{i-1} < T$, the floating-coupon bond carries a coupon[9] of $\tilde{c}' = L(t_{i-1},\, t_i) + p^{par}$ at time $t_i \leq T$. The price of a defaultable floating-coupon bond at time $t < T$ is denoted by $\widetilde{D}^{\tilde{c}'}(t,\, T)$. We also denote $\widetilde{D}(t,\, T)$, as the time t value of a defaultable zero-coupon bond.

Note that some of these replication instruments might not be available. Most reference credits only issue fixed-coupon bonds, if they issue any bonds at all. In some cases these bonds will contain call provisions or an equity convertibility option, which makes them unsuitable as CDS replication instruments.

3.6.2 Short Positions in Bonds

A trading strategy that yields a positive payoff in the case of a default is critical for the hedging of credit risk. This can be done by taking a short position in a defaultable bond. In the literature, a short position is treated as a negative portfolio position without considering the complicated underlying trading strategies that have to be implemented in reality. We will outline two possible methods to create a short position in a defaultable bond.

Repo Transaction

Repo (Repurchase) transactions are commonly used for short sales and refinancing of Treasury (government) bonds and corporate (defaultable) bonds. We will focus on the short sale of a defaultable bond. For the following explanation we will use a defaultable fixed-coupon bond $\widetilde{D}^{\tilde{c}}$. A repo transaction is between two counterparties **A** and **B**. Counterparty **A** owns the defaultable bond. A repurchase transaction consists of a sales component and a repurchase component:

- *Sale*: At current time $t = 0$ **A** sells the defaultable bond to **B** for $\widetilde{D}^{\tilde{c}}(0, T)$.
- *Repurchase*: Also at time $t = 0$, **A** and **B** enter into a repurchase agreement. Counterparty **A** agrees to buy back the defaultable bond from **B**, at a specified time $t_1 > 0$ for the forward price K. This agreement is binding on both counterparties.

The forward price is the current (spot) price, adjusted for any accrued coupon payments, accumulated by the *repo rate* r^{repo}:

$$K = \widetilde{D}^{\tilde{c}}(0, T)\, [1 + r^{\text{repo}}]^{t_1}$$

where the *repo rate* is an annual rate. For **B** to create a short position on the defaultable bond, **B** must further:

- Sell the defaultable bond in the market at $t = 0$ for the spot price $\widetilde{D}^{\tilde{c}}(0, T)$.
- Buy back the defaultable bond at $t = t_1$ at the prevailing market price $\widetilde{D}^{\tilde{c}}(t_1, T)$, in order to deliver the bond back to **A** in exchange for payment K.

Counterparty **B**'s profit (or loss) is determined by how much $\widetilde{D}^{\tilde{c}}(t_1, T)$ falls below (or rises above) the forward price K. A repo transaction is an effective method to speculate on falling bond prices. From counterparty **B**'s perspective, the *repo* transaction has achieved creating a short position in the defaultable bond. If **B** is unable to deliver the bond to **A**, due to financial stresses, **A** will remain with the cash he received for the bond. If **A** is unable to repurchase the bond from **B**, due to financial stresses, **B** will remain with the bond. From counterparty **A**'s perspective, **A** has refinanced his position at the *repo rate*. The principal determinant of the price to implement this short position is the repo rate that **B** must pay **A**.

Forward Contracts

A short position could also be made by selling the defaultable bond in a forward transaction. This requires a counterparty that is willing to buy a defaultable bond. A forward contract is an agreement between two parties to buy or sell an asset for a pre-agreed price, at a specified future point in time. It is generally easier to find an owner of an defaultable bond that would like to refinance his position through a repo transaction than creating a new position in the bond.

[9] The spread above the LIBOR, p^{par}, will be explained later, for now consider it to be an arbitrary constant.

3.6.3 CDS Replicating Strategies

Before we demonstrate the CDS replicating strategies, we need to make some simplifying assumptions about the default payment of a CDS:

- We consider a defaultable bond issued by the reference credit to be the only deliverable asset of the CDS.
- We assume that the default payment takes place at the time of the credit event[10]. The time delay through grace periods, dealer polls, etc. is ignored.
- We assume that coupon payments of coupon bearing bonds and the CDS premium payments, occur on the same dates which are denoted by $t_1, t_2, \ldots, t_n = T$. We also abstract from the bond and CDS market day count conventions.

A CDS is constructed such that the credit risk of a defaultable bond issued by the reference credit is hedged. Thus a portfolio of a combined position in a defaultable bond and a CDS (written on the same reference credit that issued the defaultable bond), should trade close to the price of an equivalent default-free bond. This is the intuition behind the CDS replicating strategy.

Replicating Strategy with Fixed-Coupon Bonds

Consider the two portfolios that are constructed at $t = 0$ and unwound at $t = T$ or at time of the default[11] τ^* whichever comes first. Note that the principal values for the bond and the notional value of the CDS are assumed to be 1 for notational brevity.

Portfolio I

- A long position in one defaultable coupon bond, which pays a coupon of $\tilde{c}$ at increasing dates $0 < t_1, t_2, \ldots, t_n = T$. The principal is paid at maturity T.
- A long position in one CDS on this defaultable coupon bond. The CDS premium is $\tilde{p}$.
- If the reference entity defaults before T, the portfolio is unwound at time of default, $t = \tau^*$.

Portfolio II

- A long position in one default-free coupon bond with corresponding coupon and maturity payment dates as the defaultable coupon bond and with a coupon value of $c = \tilde{c} - \tilde{p}$.
- If the reference entity defaults before T, the default-free bond is sold at time of default, $t = \tau^*$.

The concept of *Portfolio I* is that it can be considered a synthetic default-free bond, because it is protected from default risk by the CDS. From Table 3.1, it can be seen that in the event of no default, the survival cash flows (cash flows at times $t_1, t_2, \ldots, t_n$) of *Portfolio I* and *Portfolio II* are equivalent. Assuming the payoffs at time of default ($t = \tau^*$) are also equivalent, the initial values of *Portfolio I* and *Portfolio II* will be the equal. Provided the payoffs at default are the same, we have

$$\widetilde{D}^{\tilde{c}}(0,T) = D^c(0,T) = B(0,T) + (\tilde{c} - \tilde{p}) \sum_{i=1}^{n} B(0,t_i). \tag{3.1}$$

From the above equation (3.1), the fair CDS premium $\tilde{p}$ for a newly minted CDS contract can be solved. To uniquely determine $\tilde{p}$ we need the term structure of default-free interest rates and the initial price price of the defaultable fixed coupon bond, $\widetilde{D}^c(0,T)$.

However, the payoffs of *Portfolio I* and *Portfolio II* at the time of default are not equivalent. In the event of default, the payoff of the CDS is equal to notional value less the recovery of the reference defaultable bond. The value of the reference defaultable bond at default is equal to its recovery value, thus the value of *Portfolio I* would be equal to the notional value of the defaultable bond. However, at the time of default, the value of *Portfolio II* will depend on the then prevailing term structure of default-free interest rates. The value of the default-free bond will almost certainly

[10] This assumption is applied throughout the dissertation.
[11] We use the term default and credit event interchangeably.

Time	Portfolio I Defaultable Bond	 CDS	Portfolio II Default-Free Bond
$t = 0$	$-\widetilde{D}^{\tilde{c}}(0, T)$	0	$-D^c(0, T)$
$t = t_1, \ldots, t_{n-1}$	$\tilde{c}$	$-\tilde{p}$	$\tilde{c} - \tilde{p}$
$t = T$	$1 + \tilde{c}$	$-\tilde{p}$	$1 + \tilde{c} - \tilde{p}$
$t = \tau^*$	Recovery	1-Recovery	$D^c(\tau^*, T)$

Table 3.1: Payoff of the two portfolios of the fixed-coupon bond replication strategy.

differ from the par (notional) value. At $t = \tau^*$ the difference in value between *Portfolio I* and *Portfolio II* will be $1 - D^c(\tau^*, T)$.

For the two portfolios to have equivalent payoffs at default, $D^c(\tau^*, T)$ must be equal to its notional value. There are several reasons why $D^c(\tau^*, T)$ will differ from 1. Firstly, the initial value of the default-free bond may have already been off par, i.e. $D^c(0, T) \neq 1$. Secondly, the term structure of interest rates are dynamic, even if the default-free bond was trading at par initially, there is no reason to believe it will trade near par at some random future time point, except at maturity T. Finally, there is the matter of accrued interest. At the coupon dates t_i, the price of the bond will drop by the coupon payment amount $\tilde{c} - \tilde{p}$, then it will tend to increase again until the next coupon payment date. All these reasons make this replication strategy an approximate not an exact arbitrage relationship. The unknown future value of the default-free coupon bond at a random future time point, is the reason why this is not an exact arbitrage relationship.

Replication Strategy with Floating-Coupon Bonds

We can eliminate at least two of these sources of default payoff uncertainty. This can be achieved by replacing the default-free bond from the previous replication strategy with a default-free floating-rate bond. A default-free floater pays a coupon of $L(t_{i-1}, t_i)$ at time t_i and pays its principal value 1 at maturity T. Since all the payoffs of a default-free floater can be replicated by investing 1 at the time of issue and then rolling over this investment at the default-free short-term interest (Libor) rate, each of the ex-coupon date[12], t_i^+, values of a default-free floating bond, $D^{c'}(t_i^+, T)$, will be equal to 1. In order to achieve matching payoffs in survival we must have a defaultable bond that pays a floating coupon. Furthermore, we would like the defaultable bond to trade at par (value equal to 1) from the outset, to match the initial value of the default-free floater. Defaultable bonds with this feature are called par floaters. Let $\widetilde{D}^{\tilde{c}'}(0, T)$ denote the value of a par floater at $t = 0$, with a maturity of T and a floating-rate coupon of $\tilde{c}'_i = L(t_{i-1}, t_i) + p^{par}$, where the par spread p^{par} is chosen such that the value of the par floater is par (equal to 1) at time of issue $t = 0$.

Portfolio I

- A long position in one defaultable par floater, which pays a coupon of $\tilde{c}'_i = L(t_{i-1}, t_i) + p^{par}$, at dates $0 < t_1, t_2, \ldots, t_n = T$. The principal is paid at maturity T.
- A long position in one CDS on this defaultable par floater. The CDS premium is $\tilde{p}$ and the notional value is 1.
- If the reference entity defaults before T, the portfolio is unwound at the time of default, $t = \tau^*$.

Portfolio II

- A long position in one default-free floating-coupon bond with corresponding maturity and payment dates as the defaultable par floater and with a coupon value of $c' = L(t_{i-1}, t_i)$.
- If the reference entity defaults before T, the default-free floating-coupon bond is sold at time of default, $t = \tau^*$.

[12]The notation t_i^+ denotes the time immediately after the coupon, due at time t_i, has been paid.

Unlike the previous replication strategy, the initial values of *Portfolio I* and *Portfolio II* are identical. However, at time of default the cash flows differ. This difference will only be the accrued interest on the default-free floater, if default occurs between coupon payment dates. If the time of default τ^* occurs in the interval $[t_{i-1}, t_i]$, the value of the default-free par floater including the accrued interest is

$$D^{c'}(\tau) = 1 + L(t_{i-1}, t_i)\,(\tau^* - t_{i-1})\,.$$

Portfolio I on the other hand pays out the notional 1 at default. The difference in default payoff is $L(t_{i-1}, t_i)\,(\tau^* - t_{i-1})$. Adjustments can be made to the notional value of the CDS to compensate for this[13]. The difference in the default payoff is small.

	Portfolio I		**Portfolio II**
Time	Par Floater	CDS	Default-Free Bond
$t = 0$	-1	0	-1
$t = t_1, \dots, t_n$	$L(t_{i-1}, t_i) + p^{par}$	$-\tilde{p}$	$L(t_{i-1}, t_i)$
$t = T$	$1 + L(t_{i-1}, t_i) + p^{par}$	$-\tilde{p}$	$1 + L(t_{i-1}, t_i)$
$t = \tau^*$	Recovery	1-Recovery	$D^{c'}(\tau^*, T)$

Table 3.2: Payoff of the two portfolios of the floating-coupon bond replication strategy.

To summarise: the initial cash flow has been exactly matched, while the cash flow at default has been approximately matched. The survival payoffs differ by the difference between the credit default spread $\tilde{p}$ and the par spread p^{par}. However, these payoffs must coincide, in order to avoid any arbitrage opportunities[14]. Thus the par spread must equal the CDS premium

$$p^{par} = \tilde{p}.$$

We can rearrange the portfolios of the par floater replication strategy, to replicate a short and a long position in a CDS. To create a synthetic long position in a CDS, one needs to short sell the defaultable par floater and buy the default-free floater. To create a synthetic short position in a CDS, one needs to short sell the default-free floater and buy the defaultable floater. Thus a long position in the CDS can be roughly hedged by short selling the default-free floater and buying the defaultable floater.

Problems with the Replicating Strategy

There are several problems with the CDS replicating strategy[15]. It is difficult to implement a short position on a defaultable bond, since there may be a shortage of supply of these bonds. Repo rates change with the supply and demand of corporate bonds, thus the replication strategy is exposed to changing repo rates. At default a par floater will differ from par, making the replication strategy an approximation. The only available default-free and defaultable bonds may not have the same payment structure as the CDS. Finally, it is impossible to apply this method if the reference entity does not issue any defaultable bonds.

3.7 Bond Price-Based CDS Pricing

From the market prices of defaultable securities, one can extract the market's assessment of the issuer's (issuer of the defaultable securities) default risk. By comparing the prices of an obligor's defaultable asset and a similar default-free asset, one can infer measurements (for example probability of default) of the obligor's credit risk. Using this comparison method a simple credit risk model can be constructed, to price many vanilla credit derivatives, without any further modelling effort. This comparison method is also the basis for reduced form modelling. We will now overview the Hull & White (2000) comparison method, since it is viewed as the market standard to price credit default swaps. We will use similar notation as used in Hull & White (2000).

[13] See Schönbucher (2003) for a rule of thumb for this adjustment.
[14] We assume the payoff at default is negligibly different.
[15] We focussed on two CDS replication instruments, defaultable bonds and default-free bonds; there are other CDS replication instruments that can be used and the interested reader is referred to Schönbucher (2003). However, note that none of these replicating strategies are exact.

3.7.1 Bond Price-Based Framework

To price a CDS we need the risk-neutral default probability[16] of the reference entity, for different time intervals. This method infers the risk-neutral default probability from the prices of bonds issued by the reference entity. If we assume that the only reason a defaultable bond sells for less than a similar default-free bond is the possibility of default[17], it follows that:

$$\text{Value of Default-Free Bond} \; - \text{Value of Defaultable Bond} = \text{Value of Cost of Default.}$$

We can infer the risk-neutral probability of the reference entity defaulting at different future times, by calculating the value of the cost of defaults on a range of different maturing bonds issued by the reference entity.

Here is an example given in Hull & White (2000), to clarify how to estimate the risk-neutral default probability: Suppose that a 5-year zero-coupon default-free bond with a face (notional) value of 100, and a similar 5-year zero-coupon bond issued by a firm yields 5% and 5.5% respectively (rates are expressed with continuous compounding). The value of the default-free bond is

$$100B(0,\,5) = 100e^{-0.05\times 5} = 77.8801,$$

and the value of the corporate bond is

$$100\widetilde{B}(0,\,5) = 100e^{-0.055\times 5} = 75.9572.$$

Thus, the value of the cost of default is the difference between the two

$$100B(0,\,5) - 100\widetilde{B}(0,\,5) = 77.8801 - 75.9572 = 1.9229.$$

Let $q(0,\,5)$ denote the risk-neutral probability of default during the 5-year life of the defaultable bond, i.e. $\mathbb{Q}(\tau^* \leq 5)$. An assumption on the recovery rate must also be made to determine the risk-neutral default probability. For this example we assume the recovery rate is 0, this means the holder of the bond receives nothing after default. The value of the loss in the event of a default, in the risk-neutral world is

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{100}{B(0,\,5)}\mathbb{1}_{\{\tau^*\leq 5\}}\right) = 100q(0,\,5)e^{-0.05\times 5}.$$

By comparing the bond values we have calculated the value of the loss of default to be 1.9229. Thus it follows that

$$100q(0,\,5)e^{-0.05\times 5} = 1.9229,$$

so the risk-neutral probability of the bond defaulting before maturity is $q(0,\,5) = 0.0247$. In practice, extracting risk-neutral default probabilities from bond prices is more complicated than this, because recovery rates are non-zero and random, and most corporate bonds yield coupons.

In their paper Hull and White assume that there exists a set of n fixed coupon bonds that are issued by the reference entity. It is also assumed that default can only happen on any of the bond maturity dates, interest rates are deterministic, and recovery rates are known. It is possible to generalise the analysis to allow defaults to occur on any date, and to allow stochastic interest and recovery rates. Suppose the maturity of the jth bond is t_j with $0 < t_1 < t_2 < t_3 < \ldots < t_n$. Define:

- $\widetilde{D}^c(0,\,t_j)$: Value of the jth (defaultable) bond at present time.
- $D^c(0,\,t_j)$: Value of the jth default-free bond today, promising the same cash flows as the jth defaultable bond.
- $F_j(0,\,t)$: Forward price of the jth default-free bond for a forward contract maturing at time t, where $t < t_j$.

[16] Remember that this is the default probability under measure $\mathbb{Q}$.

[17] However there are studies that have noticed that market credit spreads are influenced by non-default factors, such as tax differentials, liquidity, and other market risk factors (see Delianedis & Geske (2001)).

- $v(0, t)$: Present value of 1$ received at time t (present time being $t_0 = 0$).
- $C_j(t)$: Claim made by the holders of the jth defaultable bond if there is a default at time t, $t \leq t_j$.
- $R_j(t)$: Recovery rate for holders of the jth defaultable bond in the event of a default at time t, $t \leq t_j$.
- α_{ij}: Present value of the loss, relative to a default-free bond, from a default on the jth bond at time t_i.
- q_i: The risk-neutral default probability at t_i.

Hull & White assume the value of the reference asset (bond) just after default is equal to the recovery rate times the sum of its face value and the accrued interest up to the default time (this sum is known as the claim amount)

$$RN[1 + A(\tau^*)],$$

where $A(\tau^*)$ is the accrued interest on the reference asset at the time of default τ^*, as a percent of its face value. Thus the default payoff of the CDS is

$$N - RN[1 + A(\tau^*)] = N[1 - R - A(\tau^*)].$$

Since interest rates are deterministic, the no-default value at time t of the jth bond is $F_j(0, t)$. If there is a default at time $\tau^* = t$, the bondholder makes a recovery at $R_j(t)$ on a claim $C_j(t)$. It follows that

$$\alpha_{ij} = v(0, t_i)[F_j(0, t_i) - R_j(t_i)C_j(t_i)]$$

The risk-neutral default probability of the loss α_{ij} being incurred is q_i. The total present value of the losses on the jth bond is therefore given by:

$$D^c(0, t_j) - \widetilde{D}^c(0, t_j) = \sum_{i=1}^{j} q_i \alpha_{ij}$$

This equation allows the risk-neutral probabilities of default to be determined inductively:

$$q_j = \frac{D^c(0, t_j) - \widetilde{D}^c(0, t_j) - \sum_{i=1}^{j-1} q_i \alpha_{ij}}{\alpha_{jj}}.$$

3.7.2 Extensions to Situation where Defaults can happen at any Time

In the event of default by the reference entity, we will assume, for ease of exposition, that all the bonds have the same seniority[18] and the recovery rate is independent of time. We will denote this constant recovery rate by R. The previous analysis to derive the risk-neutral default probability, assumed that default can only occur on bond maturity dates. We can extend this analysis to allow the default event to occur any time. We will refer to $q(t)$ as the risk-neutral default probability density function. We assume that $q(t)$ is equal to a constant $q(t_{i-1}, t_i)$ for $t_{i-1} < t < t_i$. Thus $q(t)$ is a step function. The notation $q(t_{i-1}, t_i)$ represents the risk-neutral default probability density function in the time interval $[t_{i-1}, t_i]$. Setting

$$\beta_{ij} = \int_{t_{i-1}}^{t_i} v(0, t)[F_j(0, t) - RC_j(t)]\,\mathrm{d}t,$$

where β_{ij} is the present value of the loss, relative to a default-free bond, from a default on the jth bond in the time interval $[t_{i-1}, t_i]$. A similar analysis used in the discrete case[19], gives:

$$q(t_{j-1}, t_j) = \frac{D^c(0, t_j) - \widetilde{D}^c(0, t_j) - \sum_{i=1}^{j-1} q(t_{i-1}, t_i)\beta_{ij}}{\beta_{jj}}.$$

[18] If all bonds have the same seniority, they all default at the same time and the recovery rates are the same for all the bonds.

[19] See Hull & White (2000) for more details.

3.7.3 CDS Pricing

We can now consider the valuation of a credit default swap. We assume that default events, risk-free rates, and recovery rates are mutually independent. The notional principal of the CDS equals N, and the CDS premiums are paid annually. Define:

- T: Life of the credit default swap.
- $q(t)$: The risk-neutral default probability density function at time t.
- $u(t)$: Present value of payments at the rate of \$1 per year on payment dates between time zero and time t.
- $e(t)$: Present value of an accrual payment at time t equal to $t - t^*$ where t^* is the CDS premium payment date immediately preceding time t (this is needed to value the CDS if a default occurs between payment dates).
- w: Total payments paid per year, by the CDS buyer.
- $\tilde{p}_{0,T}$: Value of w that causes the CDS to have a value of zero at $t = 0$.
- π: The risk-neutral probability of no credit event occurring during the life of the swap.
- $A(t)$: Accrued interest on the reference asset at time t as a percent of face value.

The value of π is one minus the risk-neutral probability that a credit event will occur by time T. It can be calculated from the probability density function $q(t)$:

$$\pi = 1 - \int_0^T q(s)\,\mathrm{d}s.$$

The CDS buyer's premium payments last until a credit event (default) occurs or until maturity of the CDS T, whichever comes first. The risk-neutral expected present value of the CDS buyer's payments is

$$\int_0^T w[u(s) + e(s)]q(s)\,\mathrm{d}s + \pi w u(T). \tag{3.2}$$

The first term of expression (3.2), represents the expected present value of the premium payments if a default occurs during the life of the swap. The second term of expression (3.2), represents the expected present value of the premium payments if a default does not occur during the life of the swap. Given the assumption of the claim amount, the risk-neutral expected present value of the default payoff of the CDS is

$$\int_0^T [1 - R(1 + A(s))]q(s)v(0,\, s)\,\mathrm{d}s.$$

The value of the CDS to the buyer is equal to the expected present value of the default payoff minus the expected payoff of the CDS payments made by the buyer

$$\int_0^T [1 - R(1 + A(s))]q(s)v(0,\, s)\,\mathrm{d}s - w\int_0^T [u(s) + e(s)]q(s)\,\mathrm{d}s + \pi w u(T)$$

The CDS spread $\tilde{p}_{0,T}$ is the value of w that makes the above expression equal to zero:

$$\tilde{p}_{0,T} = \frac{\int_0^T [1 - R(1 + A(s))]q(s)v(0,\, s)\,\mathrm{d}s}{\int_0^T [u(s) + e(s)]q(s)\,\mathrm{d}s + \pi w u(T)}$$

This CDS spread $\tilde{p}_{0,T}$ is the total of the payments per year, as a percent of the notional principal, for a newly minted CDS contract.

3.8 Generic CDS Pricing Model

The CDS is initiated at time $t > 0$. The notional value of the credit default swap is N. The CDS buyer makes periodic premium payments until default or maturity T. The premium dates are $t_1, t_2, \ldots, t_n = T$, where $t < t_1 < t_2 < \ldots < t_n$. Provided that a credit event has not occurred until time t_i, the i^{th} premium the CDS buyer pays at time t_i is equal to $\tilde{p}_{t,T}(t_i - t_{i-1})N$. This provides protection for the period $[t_{i-1}, t_i]$. The value $\tilde{p}_{t,T}$ is known as the annualised credit default swap spread. If a credit event occurs before time t_i, the CDS buyer does not pay the i^{th} premium payment or any of the following scheduled premium payments. The payment made by the CDS buyer at time t_i can be written mathematically as

$$\tilde{p}_{t,T}(t_i - t_{i-1})N\mathbb{1}_{\{\tau^* > t_i\}},$$

where τ^* is the default time. An additional accrued premium payment must be paid at time of default, to pay for the protection from the last premium payment date till time of default. Thus if default occurs between payment dates $[t_{i-1}, t_i]$, then the accrued premium payment paid at τ^* is equal to $\tilde{p}_{t,T}(\tau^* - t_{i-1})N$. The time–$t$ value of the total payment made by the CDS buyer is therefore given by

$$N\tilde{p}_{t,T}\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n}\mathbb{1}_{\{t_i \geq t\}}\left(e^{-r(t_i - t)}(t_i - t_{i-1})\mathbb{1}_{\{\tau^* > t_i\}} + e^{-r(\tau^* - t)}(\tau^* - t_{i-1})\mathbb{1}_{\{t_{i-1} < \tau^* < t_i\}}\right)\right]. \tag{3.3}$$

The default payoff of a credit default swap depends upon the level of the recovery rate of the reference assets upon default. For this general case we consider the recovery rate to be stochastic. If a credit event occurs before maturity T, the seller of the CDS makes a payment of $N(1 - R_{\tau^*})$, where R_{τ^*} is the recovery rate at the time of default. The time–t value of the default payoff of the CDS is

$$N\mathbb{E}^{\mathbb{Q}}\left[e^{-r(\tau^* - t)}(1 - R_{\tau^*})\mathbb{1}_{\{\tau^* \leq T\}}\right]. \tag{3.4}$$

The time–t value of the CDS for the CDS buyer is given by the value of the default payoff (3.4) of the CDS minus the value of the total payments (3.3) made by the CDS buyer

$$\begin{aligned} &N\mathbb{E}^{\mathbb{Q}}\left[e^{-r(\tau^* - t)}(1 - R_{\tau^*})\mathbb{1}_{\{\tau^* \leq T\}}\right] - N\tilde{p}_{t,T}\mathbb{E}^{\mathbb{Q}}\Bigg[\sum_{i=1}^{n}\mathbb{1}_{\{t_i \geq t\}}\Bigg(e^{-r(t_i - t)}(t_i - t_{i-1})\mathbb{1}_{\{\tau^* > t_i\}} \\ &+ e^{-r(\tau^* - t)}(\tau^* - t_{i-1})\mathbb{1}_{\{t_{i-1} < \tau^* < t_i\}}\Bigg)\Bigg] \end{aligned} \tag{3.5}$$

The CDS spread $\tilde{p}_{t,T}$ is chosen such that the value of a T–maturity CDS contract at time t, when it was initiated, is equal to 0. The CDS spread is also referred to as the fair or par spread, since this premium makes the value of the payments from the CDS buyer and seller equal (fair). Thus $\tilde{p}_{t,T}$ denote the CDS premium that makes the value of a CDS, with maturity T, equal to zero at time t. We can calculate $\tilde{p}_{t,T}$ by equating (3.5) to 0 and solving for $\tilde{p}_{t,T}$:

$$\tilde{p}_{t,T} = \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-r(\tau^* - t)}(1 - R_{\tau^*})\mathbb{1}_{\{\tau^* \leq T\}}\right]}{\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n}\mathbb{1}_{\{t_i \geq t\}}\left(e^{-r(t_i - t)}(t_i - t_{i-1})\mathbb{1}_{\{\tau^* > t_i\}} + e^{-r(\tau^* - t)}(\tau^* - t_{i-1})\mathbb{1}_{\{t_{i-1} < \tau^* < t_i\}}\right)\right]}. \tag{3.6}$$

Let $\phi(t, t_f, T)$ denote the time t_f–value of a CDS contract, which was issued at t and has a maturity of T. At time of issue t the CDS spread is chosen such that the value of the CDS contract is equal to 0. At a future time point $t_f > t$, the credit risk of the reference entity may have changed. Thus the CDS spread for providing protection from the future time point t_f until the same maturity T, will change if the market's perception of the reference entity's credit risk has also changed. If the spread for time interval $[t_f, T]$ is higher than the spread for the time interval $[t, T]$, i.e. $\tilde{p}_{t,T} < \tilde{p}_{t_f,T}$, then the original contract issued at t, has a positive value for the CDS buyer and a negative value for the CDS seller, at time t_f. This is because the spread that the CDS buyer is paying for default protection is cheaper than the spread being offered at time t_f. The CDS seller has a negative value since he is receiving a spread lower than the time t_f market spread. The value of CDS contract, with maturity T, at some time $t_f \geq t$, after it was initially issued at time t, is equal to

$$\begin{aligned}\phi(t,t_f,T) =& N\left(\tilde{p}_{t_f,T}-\tilde{p}_{t,T}\right)\mathbb{E}^{\mathbb{Q}}\Bigg[\sum_{i=1}^{n}\mathbb{1}_{\{t_i\geq t_f\}}\Big(e^{-r(t_i-t_f)}(t_i-t_{i-1})\mathbb{1}_{\{\tau^*>t_i\}}\\ &+e^{-r(\tau^*-t_f)}(\tau^*-t_{i-1})\mathbb{1}_{\{t_{i-1}<\tau^*<t_i\}}\Big)\Bigg].\end{aligned} \tag{3.7}$$

We have not specified a model for the default time, so the above general valuation formulae (3.7) is applicable to both structural and reduced form models.

3.9 Discussion

In this chapter we have defined credit risk, and credit derivatives. In particular we discussed credit default swaps in detail. We attempted to replicate the cashflows of a CDS with credit instruments (defaultable and default-free bonds). However, even with simplifying assumptions on the credit default swaps and the replication instrument's cashflows, we found that we can not replicate the CDS cashflows exactly. However, from this replicating strategy we are able to obtain approximations for the market CDS spread and observe the risk which a CDS is insuring. We reviewed the Hull & White (2000) method to determine the CDS spread. This reduced-form type method is considered to be the market standard to calculate CDS spreads. Finally, we presented a generic CDS pricing formula, which is applicable to any credit risk model. We will apply this pricing formula in our assumed structural model.

Chapter 4

Structural Models

4.1 Introduction

The principal objectives of structural models are to model credit risk and price securities of a specific firm. Under the structural model framework, credit risk is modelled by the movement of the firm's asset value V_t relative to a default-triggering threshold b_t called the default barrier. Default occurs when the firm's asset value hits this default barrier. The structural model provides economic reasoning behind a firm's default. Also under the structural model framework, the firm's asset value V is the driving force behind the dynamics of the prices of all the securities issued by the firm (equity and debt). Thus a major issue in this framework is to model the capital structure (the mixture of equity and debt) of the firm. From the structural model one can model both credit risk and equity, the fundamental link being the firms' asset value V. This is the reason why we will focus on structural models.

In this chapter we analyse the seminal structural model established by Merton (1974). We then examine extensions of the Merton model, and study what value these extensions provide to security pricing and credit risk modelling. Different methods of estimating structural model parameters are reviewed. Finally, we develop a suitable structural model to hedge credit default swaps with equity and equity derivatives.

4.2 The Merton Model

Merton (1974) was the first to model credit risk by modelling the evolution of the firm's asset value V_t relative to some default barrier b_t. Merton's paper focused on the effects that the risk-neutral probability (probability measure $\mathbb{Q}$) of default has on the yield and price of corporate bonds and the validity of credit spread as a measure of credit risk. The Merton model assumes a very simplistic capital structure for the firm, with equity only consisting of ordinary shares and liabilities consisting of only one zero-coupon bond. Equity and debt issued by the firm are claims against the assets of the firm. Debt has priority over all other claims on the firm, and equity is a residual claim, after the debt has been paid off.

Along with the standard Black & Scholes (1973) assumptions Merton made the following fundamental assumptions:

- The asset value process V_t, follows an one dimensional standard geometric Brownian motion under measure $\mathbb{P}$ (real world measure)

  $$\mathrm{d}V_t = \mu_V V_t \,\mathrm{d}t + \sigma V_t \,\mathrm{d}W_t,$$

  which has a solution

  $$V_t = V_0 \exp\left[\left(\mu_V - \frac{\sigma^2}{2}\right)t + \sigma W_t\right]. \tag{4.1}$$

  The constants μ_V and σ are the instantaneous expected rate of return and the instantaneous standard deviation[1] of the return (also known as the *volatility*) of the firm's assets, respectively and W_t is a standard Brownian motion under measure $\mathbb{P}$.

[1] We omit the subscript for σ, to simplify our notation, since σ will appear many times in the dissertation, and to conform with academic literature.

- The value of the firm's equity (E_t) is a function of the firm's asset value and time, $E_t = g(V_t, t)$. It is essentially implied that the dynamics of the equity markets are fully induced by the stochastic behaviour of the asset values and that there is no further source of uncertainty in the equity markets, as for instance speculation or imperfect aggregation of information.
- The default-free (risk-free) instantaneous interest rate is constant, $r_t = r$.
- The firm has a simple capital structure. The firm's only outstanding debt is a zero-coupon bond (ZCB). Let $\widetilde{D}(t,T)$ denote the value at time t of a defaultable zero-coupon bond maturing at time T. The principal[2] on this bond is equal to P and is payable at maturity T. The firm will not issue any new debt before this ZCB matures. In addition no intermediate payments (such as cash dividends or share repurchases) to equity holders are allowed prior to the maturity of the debt.
- Bankruptcy and reorganisation costs are negligible. This is to ensure the Modigliani-Miller theorem holds. The Modigliani-Miller theorem states that in the absence of bankruptcy, reorganisation costs and taxes, the value of the firm V_t is invariant to its capital structure. If this theorem did not hold then, for example, the value of equity would require a nonlinear solution for $E_t = g[V(E_t), t]$.

Using Girsanov's Theorem 2.2.3, under the risk-neutral measure $\mathbb{Q}$ the stochastic process for the firm's asset value is

$$V_t = V_0 \exp\left[\left(r - \frac{\sigma^2}{2}\right)t + \sigma W_t^{\mathbb{Q}}\right], \tag{4.2}$$

where $W^{\mathbb{Q}}$ is a Brownian motion process under the measure $\mathbb{Q}$. We focus on the above dynamics (4.2) for V, since pricing is done under measure $\mathbb{Q}$.

Let our ZCB mature at some time point in the future T. At this time point if our assets are greater than the principal amount P due on the ZCB, then the bondholders (owners of the ZCB) receive the promised payment P. If the firm's assets are less than or equal to the principal amount P at time T, the firm is incapable of honouring the debt payment, and the firm has defaulted. The default barrier in this model is a point with value P at time T, i.e.

$$b_t = \begin{cases} P & \text{if } t = T \\ 0 & \text{otherwise.} \end{cases}$$

The bondholders take over the firm and the value of their bond will be the value of the remaining assets, $V_T \leq P$. In mathematical notation the ZCB payoff can be written as

$$\widetilde{D}(T,T) = \min(V_T,\, P). \tag{4.3}$$

Note that the ZCB payoff (4.3) is equivalent to $V_T - \max(V_T - P,\, 0)$. From this we can see that a defaultable ZCB is equivalent to a portfolio that consists of a long position on the firm's assets and a short position on a call written on the firm's assets with strike P. It can also be interpreted that the bondholders are essentially the primarily holders of the firm's assets, and when the promised principal payment P is due at time T, the assets remaining after this payment (if any) are sold to the equityholders. Equity is a residual claim on the firm's assets once debt has been paid, thus if $V_T > P$ the equity is worth $V_T - P$, if $V_T \leq P$ default has occurred and equity is worthless i.e.

$$E_T = \max\left(V_T - P,\, 0\right).$$

Equity can also be interpreted as a call option to buy the firm's asset's back from the bondholders, at an exercise price of P. The limited liability property of equity holds, since the value of equity is never negative, meaning equityholders are not liable for more than what they have invested.

The payoff of the ZCB (4.3) is also equivalent to $P - \max(P - V_T,\, 0)$. Since this payoff is always less than or equal to the payoff of a corresponding non-defaultable bond $D(T,\, T) = P$ (since $\max(P - V_T,\, 0)$ is non-negative), the value of $\widetilde{D}(t,T)$ must always be below or equal the

[2]Note in Chapter 3 the expression $\widetilde{D}(t,T)$ represented a zero-coupon bond with principal 1, however for this chapter the bond has a principal of P.

price of a non-defaultable bond $D(t,T)$, to avoid arbitrage opportunities. Using the risk-neutral pricing methodology, the value of the firm's ZCB and equity are given by the following equations:

$$\begin{aligned} \widetilde{D}(t,T) &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\min(V_T,\, P)\middle|\mathscr{F}_t\right] \\ &= Pe^{-r(T-t)}\Phi(d_2) + V_t\Phi(-d_1), \qquad (4.4) \\ E_t &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\max(V_T - P,\, 0)\middle|\mathscr{F}_t\right] \\ &= V_t\Phi(d_1) - Pe^{-r(T-t)}\Phi(d_2). \qquad (4.5) \end{aligned}$$

Where

$$d_{1,2} = \frac{\ln\left(\frac{V_t}{P}\right) + (r \pm \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

and $\Phi(\cdot)$ is the standard normal cumulative distribution function.

Note this valuation method preserves the balance sheet equation that states that the total assets V of the firm is equal to the sum of the firm's total equity E and total liabilities L, i.e.

$$V_t = E_t + L_t.$$

In the Merton model, the arrival risk, which is described by the risk-neutral probability[3] of default, is given by $\mathbb{Q}(V_T \leq P) = \Phi(-d_2)$. The recovery rate is determined by how low V_T falls below P. Timing risk is neglected as default can only happen at the time horizon T.

Under the structural model framework, an option on equity is a compound option (an option on an option). An equity option is an option on equity, and equity is an option on the firm's asset value. Let us consider a call option, maturing at time T^* written on a firm with η shares. The value of the share price at time t is given by

$$S_t = \frac{E_t}{\eta}.$$

The value of a call option written on share S at maturity T^* is

$$c_{T^*} = \max(S_{T^*} - K,\, 0),$$

where K is the strike price of the equity option. Note, that the stochastic behavior of S is driven by the dynamics of V, since S is a function of V, i.e. $S_t = E_t(V_t)/\eta$. Let V^* denote the value of the firm's assets that makes the option-holder indifferent between exercising and not exercising the option, V^* is the solution to the integral equation $S_{T^*}(V) - K = 0$. Using the risk-neutral pricing methodology, the value of the call option at a time $t \leq T^*$ is

$$\begin{aligned} c_t &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T^*-t)}\max(S_{T^*} - K,\, 0)\middle|\mathscr{F}_t\right] \qquad (4.6) \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T^*-t)}(S_{T^*}(V) - K)\mathbb{1}_{\{V_{T^*} > V^*\}}\middle|\mathscr{F}_t\right]. \end{aligned}$$

In order to evaluate (4.6), we need to express the expectation in terms of V, since it contains all the stochastic behaviour of the call option. The closed form solution for (4.6) can be found in Geske (1979).

The structural model perspective incorporates the firm's leverage into equity option pricing and consequently the instantaneous variance of the rate of return on the stock is not constant as Black & Scholes assumed, but rather it is a function of the firm's leverage. This can be seen by comparing the Black & Scholes dynamics for the share price with the share price dynamics implied by the Merton model. Black & Scholes assumed that the stock price follows a geometric Brownian motion[4]

$$\mathrm{d}S_t = S_t\mu_S\,\mathrm{d}t + S_t\sigma_S\,\mathrm{d}W_t, \qquad (4.7)$$

[3]This probability does not give the actual probability of default, however it is useful as it can be compared to others to distinguish which firm is more likely to default.

[4]Note that the Brownian motion process W_t is equivalent as in the dynamics of V_t, since under the Merton model the dynamics of the share price are fully induced by the stochastic behaviour of the asset value V.

where μ_S, σ_S are constants. Under the Merton model, securities issued by a firm are thought of as derivatives on the firm's asset value. It is assumed firm's asset value follows a standard geometric Brownian motion process

$$\mathrm{d}V_t = V_t \mu_V \,\mathrm{d}t + V_t \sigma \,\mathrm{d}W_t. \tag{4.8}$$

Using Itô's formula (see Appendix A.1) on the explicit expression of equity (4.5) and substituting the firm's asset value dynamics (4.8), we can write the share dynamics under Merton's model as

$$\begin{aligned}
\mathrm{d}S_t &= \frac{\partial S_t}{\partial t}\,\mathrm{d}t + \frac{\partial S_t}{\partial V_t}\,\mathrm{d}V_t + \frac{1}{2} \times \frac{\partial^2 S_t}{\partial V_t^2}\,\mathrm{d}[V, V]_t \\
&= \left(\frac{\partial S_t}{\partial t} + \frac{\sigma^2 V_t^2}{2} \times \frac{\partial^2 S_t}{\partial V_t^2}\right)\mathrm{d}t + \frac{\Phi(d_1)}{\eta}\,\mathrm{d}V_t \\
&= \left(\frac{\partial S_t}{\partial t} + \frac{\sigma^2 V_t^2}{2} \times \frac{\partial^2 S_t}{\partial V_t^2} + \frac{\Phi(d_1)}{\eta} V_t \mu_V\right)\mathrm{d}t + \frac{\Phi(d_1)}{\eta} V_t \sigma \,\mathrm{d}W_t.
\end{aligned} \tag{4.9}$$

Let $\sigma_S^{(M)}$ denote the stock's proportional volatility, under the Merton model. By comparing the last term (diffusion component) of (4.7) and (4.9), we realise that under the Merton framework the stock's proportional volatility, $\sigma_S^{(M)}$ is not a constant but a function of the V_t and t,

$$\begin{aligned}
\frac{\Phi(d_1)}{\eta} V_t \sigma &= \sigma_S^{(M)} S_t \\
\sigma_S^{(M)} &= \frac{\Phi(d_1) V_t}{\eta S_t} \sigma.
\end{aligned} \tag{4.10}$$

The leverage of a firm is defined as the ratio of the firm's liabilities to the firm's assets:

$$lev = \frac{L_t}{V_t} = 1 - \frac{\eta S_t}{V_t},$$

where L_t is the total value of the firm's liabilities a time t. Under the Merton model $L_t = \widetilde{D}(t, T)$. Writing Equation (4.10) in terms of the leverage ratio lev we obtain

$$\sigma_S^{(M)} = \frac{\Phi(d_1)}{(1 - lev)} \sigma. \tag{4.11}$$

It can be seen from (4.11) that the firm's financial leverage ratio alters the stock's riskiness (volatility), $\sigma_S^{(M)}$. If the firm's financial leverage rises (resp. lowers), stock volatility increases (resp. decreases)[5].

Black & Scholes assumed that equity volatility is not a function of the firm's stock price, however in the Merton structural model, stock price volatility is inversely related to the stock price. Thus equity volatility will be larger when stock prices have fallen, than when they have risen. Since the value of an equity option is monotonic increasing with respect to equity volatility, if the stock price has fallen (resp. risen), the increased (resp. decreased) equity volatility will act to raise (resp. lower) the equity option value. In the market, it is observed that the Black-Scholes formula underprices deep-out-the-money stock options and overprices deep-in-the-money stock options[6]. Options are commonly issued near-the-money, thus the stock prices must undergo considerable movement before stock options are either deep-in or deep-out-of-the-money. In the Merton model the stock price movement will cause the equity volatility to change in the direction necessary to reduce this mispricing (see Geske (1979)).

Yield-to-maturity (YTM) is the continuous rate of return anticipated on a bond if it is held until the maturity date. We denote the YTM at time t of a default-free and a defaultable bond with maturity T by $Y(t, T)$ and $\widetilde{Y}(t, T)$, respectively. Mathematically, the YTM on a default-free ZCB with a principal of P, can be defined as

$$Y(t, T) = \frac{\ln[P/D(t, T)]}{T - t},$$

[5] See Geske (1979) for the result $\frac{\partial \sigma_S^{(M)}}{\partial\, lev} > 0$.

[6] Black (1975) discusses some of the discrepancies between the Black & Scholes (1973) stock option prices and market prices.

and the YTM on a defaultable ZCB with a principal of P, can be defined as

$$\widetilde{Y}(t,T) = \frac{\ln[P/\widetilde{D}(t,T)]}{T-t}.$$

The *credit spread* of a defaultable bond is defined as the difference between its YTM and the YTM of a default-free bond with an equivalent cashflow. We denote the credit spread at time t until the maturity of the bond T by

$$\tilde{s}(t,T) := \widetilde{Y}(t,T) - Y(t,T).$$

The dynamics of the firm's asset value and the simple capital structure assumed in the Merton model leads to undesirable credit spread properties. Under the Merton model the YTM at time t of a defaultable zero-coupon bond, with maturity T and a principal P, is

$$\widetilde{Y}(t,T) = -\frac{\ln\left[(V_t/P)\Phi(-d_1) + e^{-r(T-t)}\Phi(d_2)\right]}{T-t}.$$

and the YTM on an equivalent default free ZCB is $Y(t,T) = r$ (the riskless rate). Thus under the Merton model, the credit spread at time t of a defaultable ZCB with maturity T and principal P, is

$$\tilde{s}(t,T) = -\frac{\ln\left[(V_t/P)e^{r(T-t)}\Phi(-d_1) + \Phi(d_2)\right]}{T-t}.$$

As time converges to the debt's maturity the credit spread under the Merton model tends to zero[7], i.e. $\lim_{t\uparrow T}\tilde{s}(t,T) = 0$. In Figure 4.1 we plot credit spreads and risk-neutral default probabilities, which were calculated under the Merton model[8]. Figure 4.1 illustrates that credit spreads and risk-neutral default probabilities tend to zero for short maturities. This qualitative behaviour for credit spreads does not hold in the market. The credit spreads in the market are significantly higher than yields implied from the Merton model (see Jones et al. (1984)), especially for shorter term bonds (see Gemmill (2003)). If we assume that the difference between the yield of a corporate bond and a Treasury bond (credit spread) is driven by the probability of default, then the Merton model underestimates the probability of default. This is considered as a major shortcoming of the the Merton model.

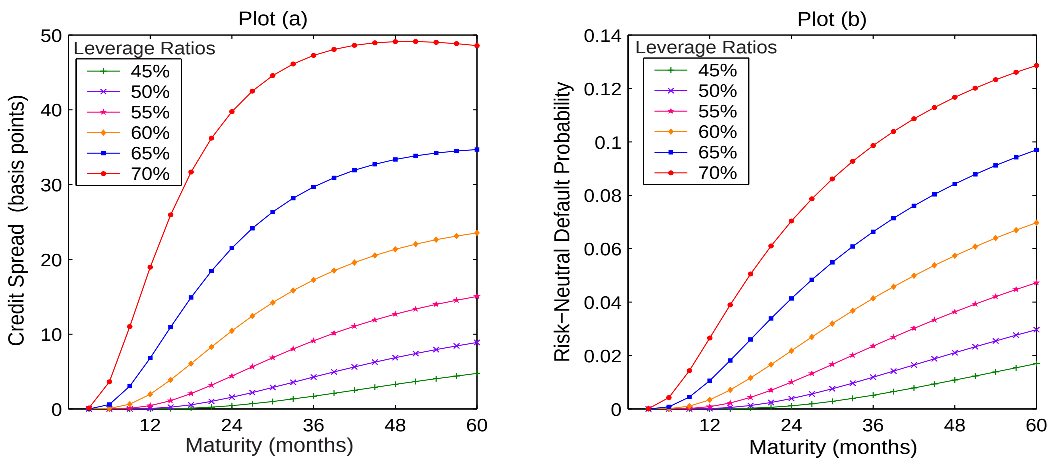

Figure 4.1: Credit spreads and default probabilities implied by the Merton (1974) model, for different leverage ratios. This leverage ratio is not the true firm leverage ratio; this ratio is the ratio of the principal value of the firm's debt P to the initial firm asset value V_0.

The Merton model assumes that the firm has a very simplistic capital structure (a single ZCB constitutes the firm's total liabilities). It is therefore difficult to implement this model for firms which have numerous bonds and loans with different indentures, maturing at different times. A possible ad-hoc method to overcome this, is to value all debt to a single time point, but it is unclear what time point and discount rate to use to discount all the scheduled debt payments.

[7] This can be calculated by using l'Hôpital's rule.

[8] The values of the parameters used to plot Figure 4.1 are: $V_0 = 100$, $\sigma = 0.2$ and $r = 0.05$.

Timing risk is not taken into account in this model, since default can only occur at one time point. Timing risk is essential when pricing credit default swaps, since at the time of default the CDS premium payments become void and the default payment becomes payable. The cash-flows of a CDS are influenced by the time of default. Another major drawback of the Merton model, and of all structural models in general, is the non-observability of the firm's asset value which causes difficulty in the estimation of the model's parameters. Estimation of parameters will be discussed in Section 4.4.

4.3 Extensions to the Merton Model

The Merton (1974) model was the first structural model and after this seminal model there have been many extensions. Merton's model assumed a simplistic capital structure of the firm, which lead to default only being possible at a single point in time[9]. This is unrealistic as a firm can default before its debt is paid off, and so the Merton model does not model timing[10] risk accurately. The time of default is important to this research as it has a great influence on the price of a CDS. Another major shortcoming of the Merton model is that is underpredicts credit spreads, which implies it underpredicts default probabilities, especially for short term maturities. The Merton model also does not take into account other factors that could have a significant influence on credit risk, such as interest rate risk. We now present a number of extensions of the Merton model that attempt to address these problems.

4.3.1 Capital Structure

Geske (1977) was the first to relax the simplistic capital structure assumption made in Merton (1974), he allowed the firm to be financed with several coupon-bearing bonds, of different priorities. This allows for a more realistic model of capital structure and timing risk, since the firm can now default on any of the bond payment dates, principal or coupon payments, and not just on one aggregate payment [11]. In Geske's paper equity is also priced as a compound option. Lets assume that there are n different debt payment dates, and at each of these payment dates t_1, t_2, ..., t_n a respective payment of $P_1, P_2, ..., P_n$ [12] is due. Geske assumed that these debt payments are financed by issuing new equity, and not by selling the firm's assets, this ensures our firm's asset value dynamics are continuous across debt payment dates [13]. At all debt payment dates t_i for t_1, t_2, ..., t_n shareholders have the choice of buying the next option, which matures at t_{i+1}, by paying the debt payment P_i or forfeiting the firm to the bondholders. The final option maturing at t_n is to repurchase the assets of the firm, from the bondholders by paying the last debt payment P_n.

Let $V^*_{t_i}$ be the root of $E_{t_i^+}(V) = P_i$, where $E_{t_i^+}$ denotes the value of equity immediately after debt payment P_i. The declaration of bankruptcy will occur if the value of the firm's assets immediately before the ith debt payment, V_{t_i}[14], is less than $V^*_{t_i}$. The value of equity immediately before debt payment P_i is

$$E_{t_i^-} = \begin{cases} E_{t_i^+} - P_i & \text{, if } V_{t_i} > V^*_{t_i} \\ 0 & \text{, if } V_{t_i} \leq V^*_{t_i}. \end{cases}$$

This can be interpreted as saying that the shareholders will not commit any additional capital to finance the debt payment if the value of the equity after the debt payment is less than the debt payment. The value of the equity at time t_n is

$$E_{t_n} = \begin{cases} V_{t_n} - P_n & \text{, if } V_{t_n} > P_n \\ 0 & \text{, if } V_{t_n} \leq P_n. \end{cases}$$

[9] The maturity of the firms debt, which consists of a single ZCB.

[10] Default time.

[11] When implementing the Merton (1974) model to a firm financed by several bonds maturing at different times, the following arbitrary approach is used: all debt payments are discounted to a critical time point, usually calculated by the duration (see Hull (2003) for a definition of duration) of the debt.

[12] Note that P_i (for $i = 1, ..., n$) could represent either one coupon/principal payment or a combination of coupon/principal payments.

[13] However, the equity process E_t will be discontinuous across the debt payment dates.

[14] This should technically be $V_{t_i^-}$, however since V_t is continuous we can simply use V_{t_i}.

At time t_n the shareholders will buy the firm's assets from the bondholders if the value of the firm's assets are above the last debt payment i.e. $V_{t_n} > P_n$. If $V_{t_n} \leq P_n$, shareholders will forfeit the firm's assets to the bondholders.

The value of equity[15] at t can be calculated using a recursive procedure starting at time t_n, and working backwards to t. Lets assume that $t_{i-1} \leq t \leq t_i$. Using the risk neutral expectation pricing methodology the value of equity is[16]

$$E_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(t_n-t)}\max\left(V_{t_n} - P_n, 0\right)\mathbb{1}_{\{V_{t_j} > V^*_{t_j} \text{ for } j=i,i+1,\ldots,n-1\}}\Big|\mathscr{F}_t\right] \tag{4.12}$$

Using the tower property of conditional expectations, (4.12) can be written as

$$\begin{aligned} & e^{-r(t_n-t)}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{V_{t_i} > V^*_{t_i}\}}\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{V_{t_{i+1}} > V^*_{t_{i+1}}\}} \;\cdots\; \mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{V_{t_{n-1}} > V^*_{t_{n-1}}\}}\right.\right.\right. \\ \times\; & \left.\left.\left.\mathbb{E}^{\mathbb{Q}}\left[\max\left(V_{t_n} - P_n, 0\right)\Big|\mathscr{F}_{t_{n-1}}\right]\Big|\mathscr{F}_{t_{n-2}}\right] \;\cdots\; \Big|\mathscr{F}_{t_i}\right]\Big|\mathscr{F}_t\right]. \end{aligned} \tag{4.13}$$

The above expression (4.13) can be evaluated, by first calculating the inner-most conditional expectation and working towards the outer-most conditional expectation. European equity options can be considered as a further option on the compound options, since equity under this framework is considered to be a compound option.

The credit spread implied by this model also has the same problem as the Merton model: the credit spread and default probability tend to zero as you get closer to the final payment. Under the Geske's framework, timing risk is modelled more accurately as it includes all debt payment dates as possible default times. However, default can occur on any financial obligation the firm has i.e. tax, wages, creditor's payments. Default can also occur due to the breaching of a bond covenant, which can stipulate that if the firm's asset value drops below a specified value, the bondholders must be paid immediately. Debt payments are not the only default triggering payments. A credit risk model that models the timing of default (timing risk) accurately is needed in order to price a CDS efficiently. We now introduce models that attempt to include all payment obligations and debt covenants into the default barrier.

4.3.2 Default Barrier

Default in the Merton model only occurs at maturity of the bond; it does not allow for premature default. The Merton and Geske models only allow for stock-based insolvency[17] and do not allow for flow-based insolvency[18]. Under the Merton model the firm's assets can dwindle to almost nothing without triggering a default. Several academics have put forward a structural model that relaxes this unrealistic credit risk feature. They have proposed modelling default as the first time the firm's assets value V_t crosses a default triggering barrier b_t, which is defined over a time interval and not a single point in time as in the Merton model. Thus default time can be mathematically expressed as

$$\tau^* = \inf\{s \geq 0 : V_s \leq b_s\}. \tag{4.14}$$

Default can now occur prior to debt maturity. The barrier b_t, is the critical firm asset value (at time t) at which the firm becomes insolvent because the firm's assets at this value can not generate sufficient cash flow to meet the firm's current obligations, or because the firm's asset value is below the outstanding debt value. Brockman & Turtle (2003) set up an hypothesis test to investigate whether the default barrier is different from zero. If it is equal to zero then the model becomes equivalent to the Merton model. Their results imply that the default barrier implied by market securities prices are statistically different from zero, i.e. default barriers are significant.

There are two types of barriers: exogenous and endogenous. An exogenous barrier is a barrier that is defined outside the model; it is a parameter in the model. An endogenous barrier is a barrier that is calculated within the structure of the model. The default barrier is chosen to maximise the equity value. Since equity holders are the owners of the firm they will decide when to default. Equity holders will choose the barrier level optimally to maximise their equity value.

[15]We focus in the value of equity and how default is modelled and not the value of a bond, since we aim to hedge a CDS with equity/options. Geske (1977) provides the price of the firm's debt for the interested reader.

[16]Note that Geske also assumed a constant risk-free rate r.

[17]When the total assets of the firm are less then the total value of the liabilities.

[18]When a firm's current operating cash flow is unable to satisfy current obligations i.e. taxes, wages, expenses, interest payments and invoices from suppliers.

Exogenous Default Barrier

Black & Cox (1976) were the first to add a continuous barrier to a structural model. They were interested in bond indentures, specifically the effect of bond safety covenants. Safety covenants are contractual provisions which entitle the bondholders to force bankruptcy or impose reorganisation of the firm, if it is underperforming according to some standard. A natural standard is the value of the firm's assets. If the value of the firm's assets falls below a specified level, then bondholders are entitled to force bankruptcy on the firm, and obtain the ownership of the firm's assets while shareholders receive nothing. Black & Cox specified the barrier as

$$b_t = \begin{cases} \gamma P e^{-r(T-t)} & \text{if } t < T \\ P & t = T \\ 0 & \text{otherwise} \end{cases} \tag{4.15}$$

with $0 \leq \gamma \leq 1$. The parameter γ can be interpreted as the recovery rate of the firm's bond. Thus the value of a defaultable ZCB[19], at time t, with a principal value of P, maturing at time T, expressed as a discounted risk-neutral expectation is given by:

$$\widetilde{D}(t,T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} P \mathbb{1}_{\{\tau^* > T\}} + e^{-r(\tau^* - t)} V_{\tau^*} \mathbb{1}_{\{\tau^* \leq T\}} \middle| \mathscr{F}_t \right].$$

Since the assumed dynamics of the firm's asset value are continuous, the firm's asset value at default will be equal to the barrier level, i.e. $V_{\tau^*} = \gamma P e^{-r(T-\tau^*)}$. Note that if $\gamma = 1$, the value of the defaultable ZCB will be equivalent to a non-defaultable ZCB, since at all possible payoff times $t \leq T$ the payoff is equivalent to a non-defaultable bond. The value of equity at time t, expressed as a discounted risk-neutral expectation is given by

$$E_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \max(V_T - P,\, 0) \mathbb{1}_{\{\tau^* > T\}} \middle| \mathscr{F}_t \right].$$

Under this framework, equity is a down-and-out barrier option. The effect of adding this barrier in the structural model to equity prices, is that the value of equity decreases compared to the Merton model equity price. This can be seen by comparing the Black and Cox equity payoff $\max(V_T - P,\, 0)\mathbb{1}_{\{\tau^* > T\}}$ to the Merton equity payoff $\max(V_T - P,\, 0)$. It can be readily seen that the Merton equity payoff dominates the Black and Cox equity payoff. By using the balance sheet equation (4.2) we notice that the Black & Cox value of the firm's debt is greater than the Merton value of debt.

Black and Cox's intention of including a continuous barrier into a structural model, was to incorporate the effect of bond indentures on the price of corporate securities. Subsequent research interpreted the barrier as an arbitrary critical asset value at which the firm enters into financial distress. When financial distress occurs the firm defaults on all its obligations and the bondholders take over the firm's assets, and decide whether to liquidate or reorganise the firm. Note that these models assume the firm's liability is made up of several debt instruments and not one bond. There are many different specifications of the default barrier. Let us mention the following: Kim et al. (1993) and Ericsson & Reneby (1998) assume the barrier is a function of the debt coupon rate (default will be more prevalent if the coupon rate is high). Longstaff & Schwartz (1995) and Briys & de Varenne (1997) assume the barrier is a proportion of the face value of debt (the barrier is the recovery value of the firm's debt). Collin-Dufrense & Goldstein (2001) and Nielsen et al. (1993) model the barrier as a stochastic process. This takes into account varying leverage ratios as time passes, which could be a result of assets being sold, the firm taking on new debt, settling debt or the market value of debt changing. Collin-Dufrense & Goldstein (2001) model the default barrier process as a mean reverting process that reverts to some target leverage ratio. Finger (2002) incorporated a stochastic barrier to model the uncertainty of the exact level at which the firm will default. It also represents the stochastic behaviour of the recovery rates for bonds.

Endogenous Default Barrier

Debt interest payments reduce the amount a firm is taxed. This is an incentive for firms to take on more debt. However, bankruptcy costs are an incentive for firms to raise capital through equity. Under endogenous barrier structural models, bankruptcy costs and tax rates influence the value

[19] As in the Merton model, Black & Cox assume the firm has one zero-coupon bond. They also extend their analysis by including discrete interest payments, and dividend payments.

of equity, debt and the level of the default barrier. Shareholders are the owners of the firm and they will ultimately decide when it is unfavourable to service their payment obligations. The shareholders choose the default point such that it will maximise their value. This is the viewpoint of endogenous barrier structural models. Under endogenous barrier models the default barrier b is calculated to maximise the equity value. Leland (1994) concluded that the value of equity increases as corporate taxes increase and the default barrier decreases as the corporate taxes increase. The equity value and barrier level is independent of bankruptcy costs which decrease the value of debt. For more details on endogenous barrier models see Leland (1994), Leland & Toft (1996), Mella-Barral & Perraudin (1997) and Anderson & Sundaresan (1996).

In the presence of bankruptcy costs and taxes the value of the firm is not equivalent to the value of the firm's assets; the value of the firm is equal to the value of the firms assets minus potential bankruptcy costs plus the benefits of tax on debt payments. A key objective of endogenous barrier models (especially in the papers by Leland (1994) and Leland & Toft (1996)), is to find an optimal capital structure that maximises the value of the firm.

4.3.3 Stochastic Interest Rates

The focus of structural models, and in general of credit risk models, in academic literature is to attempt to price fixed income securities that are subject to credit risk i.e. corporate bonds. These instruments are subject to both interest rate and credit risk. This is especially true of floating coupon corporate bonds, whose coupons are strongly linked to the prevailing risk-free rate. Longstaff & Schwartz (1995) and Briys & de Varenne (1997) assume the Vasicek (1977) model to govern the risk-neutral (under measure $\mathbb{Q}$) dynamics of the risk-free interest rates[20]

$$\mathrm{d}\, r_t = a\,(b - r_t)\,\mathrm{d}\, t + \sigma_r \mathrm{d}\, W_t^{r\,\mathbb{Q}},$$

where $W_t^{r\,\mathbb{Q}}$ is a Brownian motion process for the interest rate process r_t. Since the Vasicek model allows for negative interest rates, Kim et al. (1993) assume the risk-free interest rate under measure $\mathbb{Q}$ is governed by the CIR model of Cox et al. (1985):

$$\mathrm{d}\, r_t = a\,(b - r_t)\,\mathrm{d}\, t + \sigma_r \sqrt{r_t}\mathrm{d}\, W_t^{r\,\mathbb{Q}}.$$

Under suitable restrictions for a, b and σ_r, the CIR model does not allow negative interest rates. Longstaff & Schwartz (1995) and Kim et al. (1993) assume that the Brownian motion for the firm's asset value $W_t^{\mathbb{Q}}$, and $W_t^{r\,\mathbb{Q}}$ are correlated, with instantaneous correlation coefficient ρ. Briys & de Varenne (1997) assume these Brownian motions are independent but a correlation factor ρ is incorporated into their assumed risk-neutral firm asset value dynamics. Briys & de Varenne (1997) found that an increase in instantaneous correlation coefficient ρ will increase credit spreads. The intuitive reason why this occurs is because the distribution of future values for V_t depend on ρ. Thus if ρ is positive the covariance term will add to the variance of the changes in the value of the firm, making it more probable that V_t will reach the barrier (default probability becomes higher) thus widening (increasing) the credit spread. An important implication of the addition of stochastic interest rates into structural models, is that credit spreads can vary among firms with similar default risk[21], depending on the correlation of the firm's assets V_t with the risk-free rate r_t. This is a possible explanation why firms with similar credit ratings but from different market sectors have significantly different credit spreads[22]. By incorporating stochastic interest rates we are able to model the influences on interest rate risk on credit risk. Ericsson et al. (2004) found that interest rate changes have a statistically significant effect on CDS premium changes. By linearly regressing the change in market credit spreads on the change of the risk free rate and of stock prices, Longstaff & Schwartz (1995) found that the change of the risk-free interest rate accounts for the majority of the variation in market credit spreads. This is expected since credit spread is defined as the difference between the yield on a defaultable bond and the risk free rate. However, the addition of stochastic interest rates does not resolve near zero credit spreads and default probabilities for short maturities.

[20] Briys & de Varenne (1997) assumed the Vasicek Model, while Longstaff & Schwartz (1995) assumed the generalised Vasicek model where the parameters a, b and σ_r are functions of time.

[21] Longstaff & Schwartz (1995) measure a firm's default risk by the ratio of the initial asset value V_0 over a constant endogenous default barrier b.

[22] See Longstaff & Schwartz (1995) for empirical evidence of this.

4.3.4 Unpredictable Default Time

A stopping time[23] τ is *predictable* if there exists a sequence of stopping times $\{\tau_n\}_{n\geq 1}$ that has the the following three properties: τ_n is an increasing sequence, $\tau_n \leq \tau$ on $\{\tau > 0\}$ $\forall$ n and $\lim_{n\uparrow\infty} \tau_n = \tau$. Thus, a predictable stopping time τ is known prior to time τ, since it is announced by an increasing sequence of stopping times. A drawback of the pre-mentioned structural models, is that in general defaults are predictable[24]. Default in a structural model is defined as the first time the firm's asset value hits the default barrier. Time of default can be mathematically represented as (4.14). In the preceding structural models, the firm's asset value is modelled by a continuous process and the barrier by a deterministic function. This causes the time of default τ^* to be a predictable stopping time. The predictability of the default time τ^* causes low short term credit spreads and default probabilities.

Default is an uncertain event and thus technically should be a surprise (unpredictable). The time a firm defaults is uncertain since the market has incomplete information on the ability of the firm to pay its obligations. Market participants receive periodic and sometimes imperfect accounting reports on the firm and thus cannot be certain when a default is going to occur. The implication of predictable default times in structural models, is that the models produce credit spreads and default probabilities that tend to zero for short maturities (see Giesecke (2005) [Prop. 3.2, p.7-8] for proof of this). Eom et al. (2004) find that extended structural models (such as Geske (1977), Leland & Toft (1996), Longstaff & Schwartz (1995), and Collin-Dufrense & Goldstein (2001)) are able to generate higher credit spreads than the Merton (1974) model, but still underpredict the market spread on short-term and high credit quality bonds. The term structure of market credit spreads can take on a variety of shapes: upward sloping, hump-backed shaped, flat and downward sloping[25]. However if we assume a structural model with a continuous firm asset value process and a deterministic barrier process, the model is unable to account for flat or downward sloping credit spread term structures since it cannot predict non zero credit spreads for the short term.

There exist essentially three ways to resolve this issue of predictable defaults. The first method, developed by Duffie & Lando (2001), focuses on the incomplete and inaccurate information available to the investor about the precise value of the firm's assets. Duffie & Lando (2001) infers a distribution for the firm's asset value conditional on survivorship and inaccurate accounting reports[26]. The second method assumes the default barrier is stochastic. This is to account for the incomplete knowledge of all the firm's payment obligations (see Nielsen et al. (1993) and Finger (2002)). The third method incorporates randomly occurring jumps into the firm's asset value process. Zhou (1997) assumes that the firm's asset value process follows a jump-diffusion process. Under this jump-diffusion process a default can happen expectedly from steady declines in the firm's value. Default can also occur unexpectedly from a sudden drop in the firm's value. These jumps are due to new information causing non-marginal changes in the firm's asset value i.e. the release of unexpected financial results, the detection of fraud, or a market crash. All of these extensions cause the default time to be unpredictable, consequently generating non-zero short term credit spreads.

4.3.5 Discontinuous Firm's Asset Value Process

As previously mentioned one of the major shortcomings of structural models (with a continuous process for V_t and a deterministic barrier) is that they generate zero instantaneous default probabilities for healthy firms, which causes the models to underpredict market credit spreads and CDS premiums, (Ericsson et al. (2006)[27]). To remedy this Zhou (1997) (exogenous barrier) and Hilberink & Rogers (2002) (endogenous barrier) included jumps into their assumed firm's asset value process.

[23] See Definition 2.3.1

[24] Excluding structural models with a stochastic default barrier.

[25] Upward sloping, humpbacked and flat term structures are predominate in investment- grade bonds and downward sloping term structures appear in the junk bond rating (see Sarig & Warga (1989)).

[26] Duffie & Lando (2001) incorporate an extra stochastic variable to the assumed distribution for firm asset value observations. This is to account for the accounting noise in reported firm asset values.

[27] Ericsson et al. (2006) compare both model credit spreads and CDS premiums to market observations, to assess the performance of structural models. The reason why CDS premiums are also compared is because credit spreads are influenced by non-default factors that are not captured in structural models, such as tax differentials, liquidity, and other market risk factors (see Delianedis & Geske (2001)). Credit default swaps are commonly thought to capture credit risk most efficiently, making them an interesting source of data for evaluating models of credit risk.

If the firm's asset value process V_t, is modelled with a jump-diffusion process, then default could occur with the marginal fluctuations (the diffusion part of V_t) or with sudden shocks (the jump part of V_t). If default is caused by the diffusion part, the value of the firm's asset at default will equal the barrier. However, if default occurs by a jump, it could fall below the default barrier. The usual assumption for the recovery rate is that it is a proportion of the remaining assets after default (see Black & Cox (1976) and Longstaff & Schwartz (1995)). Thus using a jump-diffusion process for V_t one can naturally incorporate the randomness of recovery rates.

The following points summarise the advantages of including jumps into V:

- Higher credit spreads and CDS premiums (see Zhou (1997) and Eom et al. (2004)).
- Flexible credit spread term structures that can take on shapes observed in the market (upward sloping, hump-backed shaped, flat and downward sloping, see Sarig & Warga (1989)).
- Non-zero short-term default probabilities and credit spreads of healthy firms (see figure 4.2 below).
- Stochastic recovery rates, that are naturally defined as a percentage of the value of the firm after default.

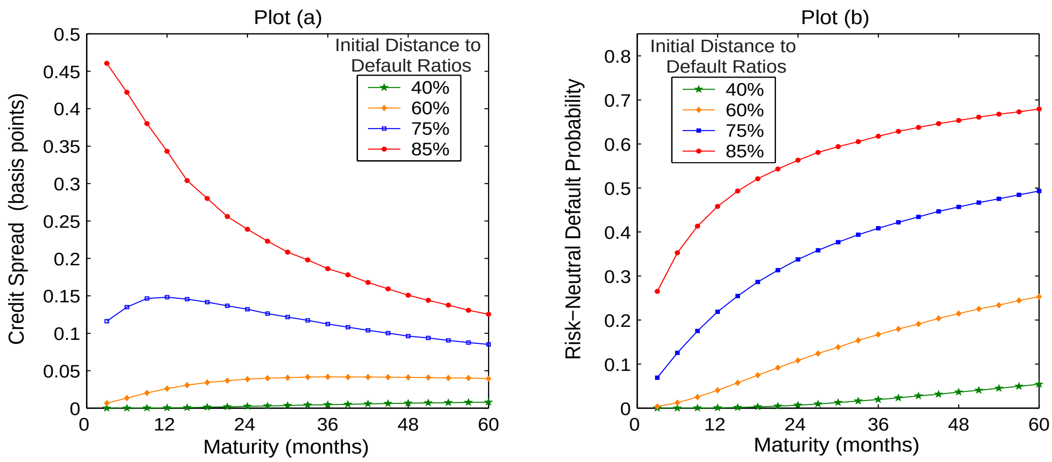

Figure 4.2: Credit spreads and default probabilities implied by the Zhou (1997) jump-diffusion credit risk model, for different initial distance to default ratios $\frac{b}{V_0}$.

Figure 4.2 illustrates that, by including a jump component for V_t we are able to produce significantly higher credit spreads, non-zero short-term default probabilities and credit spreads and flexible credit spread term structure shapes. The values of the parameters[28] used to plot Figure 4.2 are: $\sigma = 0.2$, $r = 0.05$, $\lambda = 1$, $\mu_A = -0.05$, $\sigma_A = 0.1$, $w_0 = 0.4$ and $w_1 = 1$.

4.4 Parameter estimation

A major drawback of structural models is that the firm's asset value V is not a publicly traded asset, thus V is only partially unobservable[29]. As a result of the non-observability of V_t, the parameters[30] that characterise V are difficult to estimate; we do not have a rich empirical time series of the firm's asset values to estimate its associated parameters. The default barrier b of structural models are commonly estimated using the liability data of a firm. Most firms have complex debt structures, which consist of publicly traded debt and private debt (loans). Thus we come across the same problem of partial non-observability when estimating b too. Depending on the accounting system used by a firm, the firm's asset and liability value is revealed annually, biannually, or quarterly in the firm's financial statements. The firm's asset and liability value would

[28] See Section 4.5 for the interpretation of all the parameters.

[29] It is not totally unobservable. The firm's asset value is observable in financial statements only at large intervals (quarterly or semi annually). In the rest of the dissertation, we omit the word 'partially' and just describe it as unobservable.

[30] Depending on the stochastic process chosen to model V, the number of parameters will differ.

be found in the balance sheet section of the financial statements. However these figures could be unreliable due to questionable accounting methods[31].

Academic literature offers at least four methods to estimate the unobservable parameters[32] of a structural model. We name them as: the *simple* method, the *system of equations* method, the *maximum likelihood* method, and the *calibration* method.

Simple Method

The simple method involves estimating the parameters that characterise V directly from balance sheet data. For example if we assumed V follows the geometric Brownian motion process as in Equation (4.1), the parameters that characterise V are the initial asset value V_0, μ_V and σ. We do not consider μ_V since we price under the risk-neutral measure $\mathbb{Q}$, where μ_V does not appear in the dynamics of V. Thus we only consider the initial asset value and the firm's asset volatility[33]. The initial asset value V_0 is estimated with the most recent total asset value figure in the balance sheet. The asset volatility σ is estimated by calculating the standard deviation of the percentage changes in the firm's asset value V_t, observed from periodic balance sheet data (see Brockman & Turtle (2003) for an example of this method).

The default barrier is estimated as a proportion of the present liability value of the firm[34]. The present liability value is estimated by the total liability value observed from periodic balance sheet data. The total liability value is not a very accurate estimate for the present liability value, since it is the sum of all the principal values of debt and not the current value of all debt. The proportion of the present liability value is estimated by the recovery rate of other firms with similar credit ratings as the firm under consideration. The simple method is easy to implement, but has the drawback of only having a few data points for V and the liability value to estimate with. Thus parameters can only be updated when the next financial statements are released, which is not suitable for our purposes since we intend to hedge regularly.

System of Equations Method

The most popular stochastic process assumed for V, under measure $\mathbb{Q}$, is the geometric Brownian motion (GBM) process, given by Equation (4.2). This GBM process has two characterising parameters of V than need to be estimated: V_0, and σ. A common method to estimate these two unobservable parameters involves setting up a system of two equations, that relate equity E_t with the firm asset value V_t and equity volatility to asset volatility (see for example Jones et al. (1984) and Hull (2003)). This method enables one to relate the two unobservable parameters to observable parameters. The first equation is (4.5) and the other is similar to (4.10), except that we are working with total equity and not just a single share,

$$V_t\Phi(d_1) - Pe^{-r(T-t)}\Phi(d_2) = E_t \tag{4.16}$$
$$\Phi(d_1)V_t\sigma = \sigma_E E_t. \tag{4.17}$$

Equity is publicly traded and observable. Equity volatility σ_E can be estimated by calculating the standard deviation of the percentage changes in E. The estimate of the equity volatility and the observed market price for equity will be represented respectively by $\hat{\sigma}_E$ and E_t^{obs}. Substituting the observed market value for equity and the estimated equity volatility into (4.16) and (4.17) we get

$$\Phi(d_1)V_t\sigma = \hat{\sigma}_E E_t^{obs} \tag{4.18}$$
$$E_t^{obs} = V_t\Phi(d_1) - Pe^{-r(T-t)}\Phi(d_2). \tag{4.19}$$

Note that $\Phi(d_1)$ and $\Phi(d_2)$ are both functions of V_t and σ. Since these equations cannot be explicitly written for σ and V_t, this system of equations must be solved numerically to attain estimates for σ and V_t.

Duan (1994) pointed out that there are several theoretical inconsistencies with this method. The equity volatility is estimated assuming that it is constant. However (4.18) implies that it

[31]In this dissertation we assume that these figures are accurate.

[32]We focus on the major structural model parameters: the parameters that characterise V and the default barrier. These parameters encompass a majority of the parameters of a general structural model. Some structural models do have other parameters, for example in the paper written by Briys & de Varenne (1997), they introduce a correlation coefficient ρ between V and r.

[33]We do not discuss the risk-free rate r, since this can be relatively easily estimated by the prevailing observable risk-free rate in the country, e.g we would estimate r with the LIBOR in the United Kingdom.

[34]For an example see Black & Cox (1976) and Finger (2002).

is function of V_t and t. Furthermore equation (4.19) is redundant since it is used to derive the equity volatility equation (4.18). Note that if the estimate of the equity volatility were correct (meaning that if we were to substitute our estimate for σ, derived from $\hat{\sigma}_E$, into the equity formula (4.5) it would result in the market price for equity), then one equation from the above system of equations would be redundant. Thus the first theoretical inconsistency (constant equity volatility) is needed to provide a unique solution for the parameters we are solving for. The default barrier b is estimated with the same method used in the above simple method. This estimation method is restricted to structural models that only have V_0 and σ to estimate, such as the Merton model. These methods do not account for estimating other parameters that feature in more stylised models (such as the model proposed by Zhou (1997), which has three more parameters characterising the jump component of V). The following two methods we present, are able estimate all parameters that feature in more stylised models.

Transformed-Data Maximum Likelihood Estimation

Duan (1994) was the first to use the transformed-data maximum likelihood estimation method to estimate structural model parameters. The aim of maximum likelihood estimation (MLE) is to find the parameter value that makes the observed data most likely. Let $\boldsymbol{\theta}$ denote the vector of parameters we need to estimate. For example in the Merton (1974) model $\boldsymbol{\theta} = (\mu_V, \sigma)$ and in the Black & Cox (1976) model $\boldsymbol{\theta} = (\mu_V, \sigma, b)$. The reason why we need to estimate μ_V in this method, is that we are using the observed firm asset values which are realisations under the real world measure $\mathbb{P}$. Under the measure $\mathbb{P}$, V is assumed to follow Equation 4.1, which includes the parameter μ_V. The initial asset value V_0 is implied from equity prices. The MLE method involves finding the parameter set that maximises the joint probability density function (pdf) of the observed data, $\mathbf{X} = (X_0^{obs}, X_1^{obs}, X_2^{obs}, ..., X_n^{obs})$, given the parameter set, $\boldsymbol{\theta}$. We denote the joint pdf of $\mathbf{X}$ given the parameter set $\boldsymbol{\theta}$ with

$$f(\mathbf{X}|\boldsymbol{\theta}). \tag{4.20}$$

The above expression (4.20) is also called the likelihood function of the observed data set $\mathbf{X}$. It is usually easier to maximise the logarithm of the likelihood function. Thus we will often use the logarithm of the likelihood function (log-likelihood function), $L(\boldsymbol{\theta}; \mathbf{X})$. Note that the parameters that maximise the likelihood function will also maximise the log-likelihood function.

For structural models we are dealing with unobservable data, thus Duan (1994) proposed a transformed-data maximum likelihood estimation method. For illustrative purposes we assume the Merton model, however this estimating method is applicable to more complex models. We assume the Merton model and for now the firm's asset values are directly observable. Let the set $\{V_0^{obs}, V_h^{obs}, V_{2h}^{obs}, ..., V_{nh}^{obs}\}$ denote observed asset values at equally spaced time intervals of h. Then the likelihood function for the observed asset values is

$$\begin{aligned} f(V_0^{obs}, V_h^{obs}, V_{2h}^{obs}, ..., V_{nh}^{obs}|\mu_V, \sigma) &= \prod_{k=1}^{n} f(V_{kh}^{obs}|\mu_V, \sigma) \\ &= \prod_{k=1}^{n} \frac{1}{V_{kh}^{obs}\sqrt{2\pi\sigma^2 h}} \\ &\quad \times \exp\left[\frac{-\left(\ln V_{kh}^{obs} - (\mu_V - \frac{\sigma^2}{2})h - \ln V_{(k-1)h}^{obs}\right)^2}{2\sigma^2 h}\right] \end{aligned} \tag{4.21}$$

The above likelihood function (4.21) is obtained from the dynamics of the firm's asset value (4.1) and the property of independent increments of V_t. The log-likelihood function is given by

$$\begin{aligned} L(\sigma, \mu_V; V_0^{obs}, V_h^{obs}, V_{2h}^{obs}, ..., V_{nh}^{obs}) &= -\sum_{k=1}^{n} \ln V_{kh}^{obs} - \frac{n}{2}\ln(2\pi\sigma^2 h) \\ &\quad -\frac{1}{2}\sum_{k=1}^{n} \frac{\left(\left(\ln V_{kh}^{obs} \middle/ \ln V_{(k-1)h}^{obs}\right) - \left(\mu_V - \frac{\sigma^2}{2}\right)h\right)^2}{\sigma^2 h}. \end{aligned}$$

See Theorem C.0.1, in Appendix C. This theorem (which is adapted from Bain & Engelhardt (1991)) shows that if there exists a one-to-one relationship between two variables then one can write the probability distribution function (pdf) of one variable in terms of the other. This is needed to find the transformed log-likelihood function.

Using the above Theorem (C.0.1), the monotonic relationship between E and V (given by (4.5)), and the log-normal properties of V_t (seen from (4.1)), one can obtain the marginal pdf of equity $f(E_t)$, in terms of V_t, μ_V and σ:

$$\begin{aligned} f(E_t) &= f(V_t) \Big/ \left| \frac{\partial E(V_t;\sigma)}{\partial V_t} \right| \\ &= \frac{1}{V_t\sqrt{2\pi\sigma^2 t}} \exp\left[\frac{-\left(\ln V_t - (\mu_V - \frac{\sigma^2}{2})h - \ln V_0\right)^2}{2\sigma^2 t} \right] \Phi(d_1) \end{aligned} \tag{4.22}$$

Let us denote the market observed equity data by $E_0^{obs}, E_h^{obs}, E_{2h}^{obs}, \ldots, E_{nh}^{obs}$. Using Equation (4.22), the log-likelihood function for the observed equity data is given by

$$\begin{aligned} L^E(\sigma, \mu_V; E_0^{obs}, E_h^{obs}, E_{2h}^{obs}, ..., E_{nh}^{obs}) &= L(\sigma, \mu_V; \hat{V}_0(\sigma), \hat{V}_h(\sigma), \hat{V}_{2h}(\sigma), ..., \hat{V}_{nh}(\sigma)) \\ &\quad - \sum_{k=1}^{n} \ln\left(\Phi\left(\hat{d}_1(\sigma, kh)\right)\right), \end{aligned} \tag{4.23}$$

where

$$\hat{d}_1(\sigma, kh) = \frac{\ln\left(\frac{\hat{V}_{kh}(\sigma)}{P}\right) + (r + \frac{1}{2}\sigma^2)(T - kh)}{\sigma\sqrt{T - kh}}$$

and $\hat{V}_{kh}(\sigma) = g^{-1}(E_{kh}^{obs}; \sigma)$. Now it is possible to find maximum likelihood estimates for μ_V and σ, respectively denoted by $\hat{\mu}_V$ and $\hat{\sigma}$. This is achieved by maximising Equation (4.23) numerically for μ_V and σ. One can obtain an estimate for the initial asset value V_0 by applying the inversion $\hat{V}_0 = g^{-1}(E_0; \hat{\sigma})$.

The maximum likelihood approach provides a consistent approach to parameter estimation problems. Maximum likelihood estimates can be obtained for a large variety of estimation situations, thus this approach is suited for estimating more technical structural models. This approach allows for the straightforward derivation of the distributions of the estimates which could be used to generate confidence bounds and hypothesis tests for the parameters or for default probabilities which can be useful in risk management applications. The disadvantages of the transformed likelihood approach are that in some cases it is not straight forward to find the likelihood function, and also if a closed-form solution does not exist for the transformation, then it presents the difficulty of numerically maximising a likelihood function which consists of a non-analytical derivative term.

Calibration

Another possible method to estimate parameters is by calibrating the model to market prices. The estimation method implies estimates for the model's parameters from market instrument prices. These calibrated estimates, once substituted into the model's theoretical valuation formulae for securities, produce prices roughly equal to the market prices of these securities. To calibrate parameters, a measure must be chosen to distinguish how well our calibration procedure is calibrating our parameters to market prices. A popular calibration measure is

$$\sum_{i=1}^{n} \left(U_i(\boldsymbol{\theta}) - U_i^{obs}\right)^2 \tag{4.24}$$

where U_i^{obs} is the observed theoretical market price of the ith calibrating instrument and $U_i(\boldsymbol{\theta})$ is the theoretical price given by the model for this instrument. Our aim is to find estimates $\hat{\boldsymbol{\theta}}$ for our parameters $\boldsymbol{\theta}$ that minimise this calibration measure (4.24) i.e.

$$\hat{\boldsymbol{\theta}} = \arg\inf_{\boldsymbol{\theta}} \sum_{i=1}^{n} \left(U_i(\boldsymbol{\theta}) - U_i^{obs}\right)^2 \tag{4.25}$$

Examples of financial instruments that can be used in the structural model case are: shares, share options, corporate bonds and credit default swaps.

The benefits of this approach are that the minimisation procedure is simple to set up and our calibrated model parameters produce market prices. There are a few disadvantages using this method. If we calibrate the parameters we exclude any chance of relative trading between model prices and market prices. Due to the non-convex nature of (4.24), there may be several combinations of parameters that achieve the same precision. It is difficult to find the global minimum of (4.24). Often the problem with calibration is that estimates achieved from calibration of $\hat{\boldsymbol{\theta}}$ are unstable. When a small change in the market prices occur, the estimated parameters oscillate excessively. This may indicate that the global minimum has not been identified and only a local minimum has been reached (see Cont & Tankov (2004*b*)). One must consider this before opting for the estimates from a calibration process. If the estimates do oscillate excessively one should consider a different estimation procedure.

4.5 Our Model

This section lays down the framework for our credit risk model, which will be used in the rest of this dissertation to obtain our research objectives. After stating our assumptions we explain and justify these assumptions. Our assumptions parallel those of Black & Scholes (1973), Merton (1974), Longstaff & Schwartz (1995) and Zhou (1997).

Consider a continuous time financial market setting with time period $[0, T]$. Given on this time interval is a filtered probability space $(\Omega, \mathscr{F}_T, \mathbb{F}, \mathbb{P})$, $\mathbb{P} \in \mathscr{P}$, where Ω is the sample space set containing all possible outcomes, $\mathbb{F}$ is a right continuous filtration $\mathbb{F} = \{\mathscr{F}_t : 0 \leq t \leq T\}$ (a dynamically evolving information structure), $\mathbb{P}$ is the real world (or statistical) probability measure assigned to event sets $A \in \mathbb{F}$, and $\mathscr{P}$ is a class of equivalent probability measures on $(\Omega, \mathscr{F}_T)$. There exists an equivalent martingale measure $\mathbb{Q} \in \mathscr{P}$. This implies our market is arbitrage-free (from Theorem 2.2.1). The information set is generated by the sample path realizations of the firm's asset value, denoted by V_t, over a subinterval $[0, t] \leq [0, T]$; viz.

$$\mathscr{F}_t = \sigma(V_s,\, 0 \leq s \leq t)$$

The following assumptions characterise our credit risk model:

Assumption 1. *The firm's asset value process V_t, follows a jump-diffusion process under the real-world measure $\mathbb{P}$, i.e.*

$$\mathrm{d}V_t/V_{t^-} = \mu_V\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + Y_t\,\mathrm{d}N_t, \tag{4.26}$$

where N_t is a time homogeneous Poisson process with intensity parameter λ, and W_t is a Brownian motion process. The Poisson process N_t represents how many jumps occur in the interval $[0, t]$. The constants μ_V and σ are the instantaneous expected rate of return and the instantaneous standard deviation of the return of the firm's assets, conditional that no jumps occur. The stochastic process Y_t represents the percentage changes in the firm's asset value at time t, (i.e. $Y_t = (V_{t^+} - V_{t^-})/V_{t^-}$). If a jump occurs at time t then $Y_t < 0$ or $Y_t > 0$, otherwise $Y_t = 0$. The processes W_t, N_t and Y_t are assumed to be jointly independent. Let $\tau_1, \tau_2, ...,$ denote the arrival times of the jumps, and A_i the logarithm of the ratio of the firm's asset value after and before the i^{th} jump, i.e.

$$A_i = \ln \frac{V_{\tau_i^+}}{V_{\tau_i^-}} = \ln(Y_{\tau_i} + 1).$$

We assume all A_i's to be independent and identically distributed with the following normal distribution $A_i \sim \mathrm{N}(\mu_A,\, \sigma_A^2)$.

Assumption 2. *There exists a threshold value $b_{0,T}$ (constant in the time interval $[0,\, T]$) for the value of the firm, at which the firm enters financial distress. When $V_t > b_{0,T}$ the firm is able to honour all its payment obligations, and when $V_t \leq b_{0,T}$ financial distress occurs and the firm defaults on all of its immediate and future payment obligations.*

Assumption 3. *The value of all securities issued by the firm are functions of the firm's asset value and time (i.e. equity can be expressed as $E_t = f(V_t, t)$).*

Assumption 4. *The riskless rate is constant, i.e. $r_t = r$.*

Assumption 5. *The market is frictionless, i.e. there are no transaction or bankruptcy costs or taxes, borrowing and lending can be done at the same rate of interest, short sales of assets are allowed. Arbitrage opportunities do not exist. The firm's securities are traded continuously in this market.*

Assumption 6. *The firm issues both debt and non-dividend paying equity. If it defaults, equityholders receive nothing and bondholders receive* $1 - w(V_{\tau^*}/b)$ *times the face value of the debt security. The notation* τ^* *represents the time of default and the function* w *represents the percentage writedown on the bonds. The function* w *has the following linear form linear form of* $w = w_0 - (V_{\tau^*}/b)$*. The recovery rate* R *equals* $1 - w$*.*

The usual assumption that the dynamics of the firm's asset value follow a geometric Brownian motion (Black & Scholes (1973) and Merton (1974)), is not sufficient to account for outliers that exist in financial security prices. Fama (1965) investigated the behaviour of stock prices and found considerable non-marginal movements in stock prices that could not be captured by a geometric Brownian motion process. The logarithm of the returns of an asset that follows a geometric Brownian motion process has a normal (Gaussian) distribution[35]. However, Fama (1965) and Hull (2003) found that the market implied distribution for stock returns display leptokurtosis[36]. Note that under the structural model, securities issued by the firm are assumed to have values which are a function of the firm's asset value. In order to capture this evidence of outliers in securities, we have assumed a jump-diffusion process, similar to the stochastic process proposed by Merton (1976). Pan (2002) found that assuming jumps in their asset distribution, plays an important role in explaining the time-series behaviour of option prices. Cremers et al. (2005) found that a structural model with jumps improves the fit of the equity distribution and option prices considerably, and predicts market credit spreads more accurately too. Zhang et al. (2005) found that the jump effect is a significant factor in explaining the variation in CDS premiums. In order to hedge the value of a CDS with equity and equity options, we need a model that can capture the changes in these security values accurately. The aforementioned papers provide evidence showing that by assuming a jump-diffusion model for the firm's asset value, improvement in the accuracy of prediction of price changes in these securities occurs.

The notation $b_{t,T}$ represents the default triggering barrier. The default barrier $b_{t,T}$ can be financially interpreted as the value of all future payment obligations, averaged over the time interval $[t, T]$. If we let C_t denote the time-t value of all the future payment obligations, then $b_{t,T} = \int_t^T C_t \, \mathrm{d}t/(T-t)$. Note as t and the choice of T changes, $b_{t,T}$ will differ. We will choose T to be the maturity of the CDS we are trying to hedge. In this way we only need to estimate one barrier. For notational brevity, we will denote $b_{t,T}$ by b; it will be clear from the pricing function it appears in, what the values of t and T are.

By calculating equity values from a structural model with a barrier and without (the Merton (1974) model) and comparing them to market equity prices, Brockman & Turtle (2003) found that barriers in structural models are statistically significant. The barrier is defined as the value of the firm's assets at which the firm is unable to meet its payment obligations. Debt payments are a major constituent of these payments. If the default barrier is presumed to be a monotonic function of the outstanding debt, then by assuming the default barrier to be constant we predict expected leverage ratios to decline exponentially over time[37]. However, in practice, firms have target leverage ratios that they attempt to keep (see Liu (2005)) and thus expected leverage ratios in practice are stationary (see Collin-Dufrense & Goldstein (2001)). Collin-Dufrense & Goldstein (2001) model the default barrier as a mean-reverting[38] stochastic process to capture temporary fluctuations in leverage ratios and stationary expected leverage ratios. It is the ratio of the firm's assets and the default barrier rather than the actual value of barrier, that is critical in our analysis of credit risk. We choose a constant barrier for simplicity. Our analysis can easily be extended to accommodate more complicated default barriers. The assumption that when the firm defaults,

[35]If the asset follows a GBM, for example (4.1), then the logarithm of the asset's returns follow a normal distribution, i.e. $\ln[(V_t - V_0)/V_0] \sim \mathrm{N}([r - \sigma^2/2]t, \sigma^2 t)$.

[36]Leptokurtosis is the property of a probability density function having fatter tails and a higher peak at the mean than a Gaussian distribution. This implies that there are more values at the extreme tails and at the mean than a Gaussian distribution.

[37]By assuming the firm's asset value dynamics follows (4.26), the expected value of the firm's asset value increases exponentially over time. Thus if the barrier is assumed to be constant, the expected leverage ratios will decline. Additionally, the debt level will remain constant since we presume the level of debt to be a monotonic function of the default barrier.

[38]The mean is the target leverage ratio.

it defaults on all its payments simultaneously, is realistic since there exist cross-default provisions and injunctions to service coupons on debt.

Securities issued by a firm are claims against the firm's assets. For example bondholders will receive their promised payments from the firm's assets, which is either financed by the firm's cashflows, or proceeds of sold assets, or issuing equity or debt, all of which are constituents of the firm's asset value. If the payments are not honoured, the remaining assets are handed over to the bondholders. Shareholders are the owners of the firm's assets once all debt payments have been made. Thus equity is a residual claim on the firm's assets. Intuitively, it is evident that the firm's securities are functions of the firm's asset value. We use a similar equity pricing formula as was used in Finger (2002)

$$E_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\max(V_T - b_{t,T}\,,\,0)\mathbb{1}_{\{\tau^*>T\}}|\mathscr{F}_t\right]. \tag{4.27}$$

Commonly, the focal point of credit risk models in academic literature, is to price defaultable fixed income securities. These instruments are subject to both interest rate and credit risk. This is the argument for introducing stochastic interest rates into structural models. However, pricing defaultable fixed income securities is not the primary concern in this dissertation. Our focus is to hedge a CDS with equity and equity options. Ericsson et al. (2004) regressed levels and changes in CDS prices with theoretical determinants, the firm's leverage, volatility, and the riskless rate. They found that all have statistically significant explanatory power in the changes and levels of CDS premia. However, the riskless rate has the least explanatory power. We have ignored interest rate risk in our model by assuming the riskless rate to be constant. Our model can easily be adjusted to include stochastic interest rates.

The assumptions in *Assumption 5* are the usual Black & Scholes (1973) assumptions that allow us to calculate arbitrage-free prices. We exclude bankruptcy costs and taxes since our barrier is exogenous, thus bankruptcy costs and taxes are less influential than for endogenous barrier models. Special mention must be made of the assumption that the firm's securities must be traded continuously. This assumption is necessary, since arbitrage-free prices are obtained on the principle that the value of the security we are pricing can be replicated with tradeable assets. A common assumption in structural models is that the firm's assets are tradeable. Although convenient, such an assumption is difficult to uphold from a practical viewpoint, since in practice the assets are often not tradeable and the value of the firm is not observable. To circumvent, at least on the theoretical level, the issue of non-tradeability of the firm's assets, Ericsson & Reneby (1999) argue that if at least one of the firm's securities (e.g. shares) is traded, it is sufficient to postulate that the firm's asset value can be replicated by dynamic trading in the firm's tradeable securities. To understand the intuition of this argument consider an analogy with an ordinary stock option model. Fundamentally, the stock option can be priced because we can replicate its payoff using the stock and risk free bonds. However, we can just as well value the stock by replicating its payoff using the traded option and risk-free bonds. In the same fashion we can value the firm's asset using any of the firm's traded securities (e.g. shares, corporate bonds) and risk free bonds.

In practice the recovery rates, $R = 1 - w$, vary among firms, seniority of issue and even time (see Altman (1992), Franks & Torous (1994) and Altman & Bencivenga (1995)). By adding jumps to our model we can include the randomness of recovery rates in a natural way, that depends on the remaining assets of the firm after default. Since our firm's asset value follows a jump-diffusion process, the value of the firm's assets at default (i.e. V_τ) is stochastic. The writedown percentage function is a decreasing function of (V_{τ^*}/b) and is bond specific, thus a senior bond will have a distinct w function from a junior bond. The recovery rate, $1 - w$ can also be thought of as the result of a bargaining process among the claimants. Similar to Zhou (1997), we make the simple assumption that the writedown function has the linear form of $w = w_0 - (V_{\tau^*}/b)$. We will estimate w_0, by equating $\mathbb{E}^{\mathbb{Q}}(1 - [w_0 - (V_{\tau^*}/b)])$ with the average market recovery rate, and then solving for w_0. This market estimation is done by calculating the average recovery rate of defaulted firms with the same credit rating at issuance.

The absolute priority rule is the rule used in bankruptcy proceedings which states that creditors' claims take priority over shareholders' claims in the event of liquidation. Thus in the event of a default on a bond payment, shareholders receive nothing. In practice this does not always occur, and there are often violations of the absolute priority rule, especially in U.S.. However, such violations are not common in Europe (see Franks & Torous (1989)). We assume shareholders receive nothing once the firm defaults.

4.6 Discussion

In this chapter we reviewed structural models in detail. We firstly looked at the seminal Merton (1974) model. The Merton model has simplistic assumptions which results in the model underpredicting market credit spreads and default probabilities. This is considered as a major drawback of the Merton model. We then reviewed extensions to the Merton model, and analysed their effect on equity and credit risk. We established that predictable default times are the cause of low short term credit spreads and default probabilities. By assuming the firm's asset value process V_t follows a discontinuous process, the issue of predictable default times is resolved. Estimating the structural model parameters are difficult since a firm's asset value is not traded and unobservable. We discuss four approaches to estimate these unobservable parameters. Lastly, the mathematical framework and assumptions of our structural model are stated, with justification. We now use our proposed model to price equity, equity options and credit default swaps.

Chapter 5

Pricing and Estimation

5.1 Introduction

Since our model does not admit closed form expressions for equity, equity options and credit default swaps we apply Monte Carlo simulation to price theses securities. In this chapter we introduce Monte Carlo simulation. We use a modified Monte Carlo simulation procedure to calculate our prices: the Brownian bridge Monte Carlo hybrid method (Metwally & Atiya (2002)). We use this procedure since it is computationally faster and more accurate than the standard Monte Carlo method. We will discuss the convergence rate of our calculated Brownian bridge Monte Carlo prices. Since our model produces non-analytical solutions for the prices of the securities under investigation, we will use a calibration procedure to estimate our parameters. We discuss calibration and the Nelder & Mead (1965) minimising algorithm which will be used in our calibration procedure. The following explanation of Monte Carlo methods and Brownian bridge is adapted from Šelić (2006).

5.2 Principles of Monte Carlo Methods

Some mathematical models do not admit closed-form solutions for financial product prices, and we have to resort to numerical or Monte Carlo methods. Monte Carlo methods are popular methods that easily enable us to calculate estimates for these non-analytical prices. The topic of Monte Carlo methods is a field of *experimental mathematics*, in which random numbers are used to perform experiments. Typically these experiments are carried out on a computer using anywhere from hundreds to billions of random numbers. By performing simulated experiments with computers, complex mathematical models can rapidly be studied in a manner which is relatively simple and inexpensive.

To calculate the no arbitrage prices for market instruments, we need to evaluate the discounted payoff of the instrument, under the risk-neutral measure $\mathbb{Q}$. By using Monte Carlo methods we can compute estimates for these expectations. Suppose we need to find the expectation of a function g, of a stochastic process[1] X_t, $\xi = \mathbb{E}\left[g(X_t)\right]$, where X_t has pdf $f_{X_t}(x)$. The Monte Carlo method for estimating ξ is done by randomly sampling n points $X_t^{(1)}, X_t^{(2)}, ..., X_t^{(n)}$ from the distribution[2] of X_t, and then computing the mean of the function values of these sampled points,

$$\hat{\xi}_n = \frac{g\left(X_t^{(1)}\right) + g\left(X_t^{(2)}\right) + ... + g\left(X_t^{(n)}\right)}{n}.$$

According to the Law of Large Numbers (LLN), if g is integrable over the domain of X_t then the sample mean $\hat{\xi}_n$ converges almost surely to the distribution mean ξ:

$$\hat{\xi}_n \to \xi \quad \text{almost surely as } n \to \infty.$$

[1]Note that we introduce this topic by estimating a financial contract whose value only depends on the underlying asset at one particular time point. This can easily be extended to a contract whose value depends on the underlying asset at various time points and for a contract whose value is dependent on more than one underlying asset.

[2]See Glasserman (2004) for methods to randomly sample points from specific distributions. Note that we sample from the distribution for the measure under which the expectation is taken.

Theorem 5.2.1. ***Central Limit Theorem (CLT).*** *If $X_1, X_2, ..., X_n$ is a random sample from a distribution with mean μ and variance $\sigma^2 < \infty$ then the limiting distribution of*

$$Z_n = \frac{\sum_{i=1}^n (X_i/n) - \mu}{\sigma/\sqrt{n}}$$

is the standard normal distribution, i.e. $Z_n \to Z$ in distribution as $n \to \infty$, where $Z \sim N(0,1)$

Proof. See Bain & Engelhardt (1991) [Thm. 7.3.2, p. 238-240]. □

Our sampled points are independent and identically distributed. Thus by the Central Limit Theorem 5.2.1

$$\left(\hat{\xi}_n - \xi\right) \to \frac{\sigma_g}{\sqrt{n}} Z \text{ in distribution as } n \to \infty, \tag{5.1}$$

where $Z \sim N(0,1)$ and σ_g is the standard deviation of $g(X_t)$, $\sigma_g^2 = \mathbb{V}\text{ar}[g(X_t)]$. The standard error of an estimator is defined as the standard deviation of the difference between the estimator $(\hat{\xi}_n)$ and the true value (ξ). The standard error for the Monte Carlo estimator $\hat{\xi}_n$, can be seen from (5.1) to be $\sigma_g/\sqrt{n}$, which has a convergence rate $\mathscr{O}(n^{-1/2})$.

The expectation $\mathbb{E}\left[g(X_t)\right]$ is defined as the integral,

$$\int_{-\infty}^{\infty} g(x) f_{X_t}(x)\, \mathrm{d}x.$$

The Monte Carlo method is in fact a method of estimating integrals. Another method to estimate integrals is by using the trapezoidal rule. If the function g is twice differentiable then the simple trapezoidal rule can be used to numerically evaluate the above integral. The simple trapezoidal rule has a convergence rate of $\mathscr{O}(n^{-2})$, thus the Monte Carlo method has a slower convergence rate than the simple trapezoidal rule. However, Monte Carlo excels when the problem involves higher dimensions of the domain. If $\mathbf{X}_t \in \mathbb{R}^d$, then $\mathbb{E}\left(g(\mathbf{X}_t)\right)$ would involve evaluating an d-dimensional integral. The convergence rate for the Monte Carlo method is independent of the dimension of the problem, however the convergence of the product trapezoidal rule for a d-dimensional integral is $\mathscr{O}(n^{-2/d})$, if the integrand is twice differentiable. Thus the Monte Carlo method is advantageous, with respect to the convergence rate, when the problem involves dimensions greater than three.

The CLT enables the construction of approximate confidence intervals for the estimate. The CLT also enables us to calculate the number of sample points, that will make us $100(1-\alpha)\%$ confident[3] the error of our Monte Carlo estimate $\hat{\xi}_n$ is accurate to the k^{th} decimal point

$$\begin{aligned} \mathbb{P}\left(|\hat{\xi}_n - \xi| < 10^{-k}\right) &= 1-\alpha \\ \Leftrightarrow \mathbb{P}\left(\left(-10^{-k}\right)\frac{\sqrt{n}}{\sigma_g} < \frac{\hat{\xi}_n - \xi}{\sigma_g/\sqrt{n}} < \left(10^{-k}\right)\frac{\sqrt{n}}{\sigma_g}\right) &= 1-\alpha \\ \Rightarrow \mathbb{P}\left(\left(-10^{-k}\right)\frac{\sqrt{n}}{\sigma_g} < Z < \left(10^{-k}\right)\frac{\sqrt{n}}{\sigma_g}\right) &\simeq 1-\alpha. \end{aligned}$$

In terms of standard normal percentiles:

$$\mathbb{P}\left(-z_{1-\alpha/2} < Z < z_{1-\alpha/2}\right) = 1-\alpha,$$

where $z_{1-\alpha/2}$ represents the $(1-\alpha/2)^{\text{th}}$ percentile of a standard normal distribution i.e. $\Phi(z_{1-\alpha/2}) = 1-\alpha/2$. Then, by using the standard normal distribution table, we can equate $z_{1-\alpha/2}$ with $(10^{-k})\sqrt{n}/\sigma_g$, and can calculate how many samples are needed, to be $100(1-\alpha)\%$ confident that our estimate is accurate to the k^{th} decimal place,

$$n \approx (10^{2k})\sigma_g^2 (z_{1-\alpha/2})^2.$$

However, the parameter σ_g would typically be unknown in a setting in which ξ is unknown. However by the LLN, applied to the sample variance:

$$s_g^2 = \frac{1}{n-1}\sum_{i=1}^n \left(g\left(X_t^{(i)}\right) - \hat{\xi}_n\right)^2,$$

[3]To conform with standard statistical literature we use $(1-\alpha)$ and not just α.

we have that s_g^2 converges almost surely to the true variance σ_g^2, for large n. Thus we can use s_g as an estimate for σ_g in the argument above[4]. In most practical problems of this type, the sample variance is a good estimate for the distribution variance, and can be used in place of the distribution variance to determine approximate confidence levels for $\hat{\xi}_n$.

We can increase the efficiency of the estimator $\hat{\xi}_n$ by reducing its variance. There exist many methods of reducing the variance of the Monte Carlo estimator $\hat{\xi}_n$ (see Glasserman (2004) for reviews on several of these variance reduction methods). The following briefly outlines two popular variance reduction methods: control variates and antithetic variates.

The control variate method uses the price of another financial security that has a closed-form solution, to reduce the variance error of the estimator. Lets denote the closed-form expression for the price of another financial security[5] as $\xi^* = \mathbb{E}\left[h(Y_t)\right]$. The Monte Carlo estimate for ξ using control variates is

$$\begin{aligned}\hat{\xi}_n^{cv} &= \frac{1}{n}\sum_{i=1}^{n}\left[g\left(X_t^{(i)}\right) - \left[h\left(Y_t^{(i)}\right) - \xi^*\right]\right]\\ &= \hat{\xi}_n - \left[\hat{\xi}_n^* - \xi^*\right],\end{aligned}$$

where

$$\hat{\xi}_n^* = \frac{h\left(Y_t^{(1)}\right) + h\left(Y_t^{(2)}\right) + ... + h\left(Y_t^{(n)}\right)}{n}.$$

The error $\hat{\xi}_n^* - \xi^*$ serves as a control in estimating ξ. The variance of the Monte Carlo control variate estimator is

$$\begin{aligned}\mathbb{V}\mathrm{ar}(\hat{\xi}_n^{cv}) &= \mathbb{V}\mathrm{ar}\left(\hat{\xi}_n - \left[\hat{\xi}_n^* - \xi^*\right]\right)\\ &= \frac{1}{n}(\sigma_g^2 - 2\sigma_{g,h} + \sigma_h^2),\end{aligned}$$

where $\sigma_h^2 = \mathbb{V}\mathrm{ar}[h(Y_t)]$ and $\sigma_{g,h} = \mathbb{C}\mathrm{ov}[g(X_t), h(Y_t)]$. Hence, the control variate estimator $\hat{\xi}_n^{cv}$ has a smaller variance than the standard estimator $\hat{\xi}_n$ if

$$2\sigma_{g,h} > \sigma_h^2. \tag{5.2}$$

A good control variate would be one that is highly correlated (thus a high covariance) to the financial product we are trying to estimate, in order for (5.2) to hold. In general a hedge portfolio is a good control variate.

To simulate random a variable X_t from a specific density $f_{X_t}(x)$, with cdf $F_{X_t}(x)$, a uniformly distributed random variable over $[0, 1]$, i.e. $U \sim \mathrm{UNIF}(0, 1)$, is generated. Next one applies the inverse of the cdf to the uniformly generated random variable, $X_t = F_{X_t}^{-1}(u)$. The antithetic method is based on the observations that if U is uniformly distributed over $[0, 1]$, then $1 - U$ is too, and also $X_t = F_{X_t}^{-1}(U)$ and $\tilde{X}_t = F_{X_t}^{-1}(1 - U)$ both have the same CDF (they are distributed identically). The estimate for ξ using antithetic variates is

$$\hat{\xi}_n^{av} = \frac{1}{n}\left[\sum_{i=1}^{n}\frac{g\left(X_t^{(i)}\right) + g\left(\tilde{X}_t^{(i)}\right)}{2}\right].$$

To compare the estimator variances of the antithetic variate method with the standard method, we will assume that the standard method has $2n$ observations in its random sample. Using antithetic

[4]The confidence intervals are approximate since convergence only occurs for large n.

[5]Note that Y_t and X_t do not have to be different. Usually these underlying assets would be the same because we would want the covariance $\sigma_{g,h}$ of these two financial securities prices ξ and ξ^* to be high in order to reduce the estimator's variance.

variates reduces variance if

$$\begin{aligned}
& \mathbb{V}\text{ar}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{g\left(X_t^{(i)}\right)+g\left(\tilde{X}_t^{(i)}\right)}{2}\right] &<& \quad \mathbb{V}\text{ar}\left[\frac{1}{2n}\sum_{i=1}^{2n} g\left(X_t^{(i)}\right)\right] \\
\text{i.e}\quad & \mathbb{V}\text{ar}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{g\left(X_t^{(i)}\right)+g\left(\tilde{X}_t^{(i)}\right)}{2}\right] &<& \quad \mathbb{V}\text{ar}\left[\frac{1}{n}\sum_{i=1}^{n}\frac{g\left(X_t^{(i)}\right)+g\left(X_t^{(n+i)}\right)}{2}\right] \\
\text{i.e}\quad & \mathbb{V}\text{ar}\left[g\left(X_t^{(i)}\right)+g\left(\tilde{X}_t^{(i)}\right)\right] &<& \quad 2\mathbb{V}\text{ar}\left[g\left(X_t^{(i)}\right)\right] \text{ for all } i=1,\ldots,n.
\end{aligned}$$

The variance on the left can be written as

$$\begin{aligned}
\mathbb{V}\text{ar}\left[g\left(X_t^{(i)}\right)+g\left(\tilde{X}_t^{(i)}\right)\right] &= \mathbb{V}\text{ar}\left[g\left(X_t^{(i)}\right)\right]+\mathbb{V}\text{ar}\left[g\left(\tilde{X}_t^{(i)}\right)\right]+2\mathbb{C}\text{ov}\left[g\left(X_t^{(i)}\right),g\left(\tilde{X}_t^{(i)}\right)\right] \\
&= 2\mathbb{V}\text{ar}\left[g\left(X_t^{(i)}\right)\right]+2\mathbb{C}\text{ov}\left[g\left(X_t^{(i)}\right),g\left(\tilde{X}_t^{(i)}\right)\right].
\end{aligned}$$

This uses the fact that $X_t^{(i)}$ and $\tilde{X}_t^{(i)}$ have the same distribution and so $\mathbb{V}\text{ar}\left[g\left(X_t^{(i)}\right)\right]$ is equal to $\mathbb{V}\text{ar}\left[g\left(\tilde{X}_t^{(i)}\right)\right]$. It can be seen that the condition to reduce the variance σ_g^2 is

$$\mathbb{C}\text{ov}\left[g\left(X_t^{(i)}\right),g\left(\tilde{X}_t^{(i)}\right)\right]<0.$$

In a financial setting, a sufficient condition for antithetic variates to guarantee a reduction in the variance of the estimator, is if the payoff function g of the derivative is monotone in the underlying state variable X_t.

5.3 Brownian Bridge

The Brownian bridge method is an alternative method for constructing Brownian motion paths. Let W_t be a Brownian motion at time $t \geq 0$. The Brownian bridge technique generates Brownian motion points W_{t_j} given $W_{t_i}=w_{t_i}$ and $W_{t_k}=w_{t_k}$, where $t_i<t_j<t_k$. From the definition of conditional densities,

$$f_{X|Y}(x|y)=\frac{f_{X,Y}(x,\,y)}{f_Y(y)}$$

we can determine the conditional density[6] of $W_{t_j}=W=w$ given $W_{t_i}=w_{t_i}$ and $W_{t_k}=w_{t_k}$:

$$\begin{aligned}
f(W|W_{t_i},\,W_{t_k}) &= \frac{f(w,w_{t_i},w_{t_k})}{f(w_{t_i},w_{t_k})} \\
&= \frac{\frac{1}{\sqrt{2\pi t_i(t_j-t_i)(t_k-t_j)}}\exp\left(-\frac{1}{2}\left[\left(\frac{w_{t_i}}{t_i}\right)^2+\left(\frac{w-w_{t_i}}{t_j-t_i}\right)^2+\left(\frac{w_{t_k}-w}{t_k-t_j}\right)^2\right]\right)}{\frac{1}{\sqrt{2\pi t_i(t_k-t_i)}}\exp\left(-\frac{1}{2}\left[\left(\frac{w_{t_i}}{t_i}\right)^2+\left(\frac{w_{t_k}-w_{t_i}}{t_k-t_i}\right)^2\right]\right)}.
\end{aligned}$$

This can be simplified to:

$$f\left(W=w|W_{t_i}=w_{t_i},W_{t_k}=w_{t_k}\right)=\frac{1}{\sqrt{2\pi\sigma_{bb}^2}}\exp\left(-\frac{1}{2\sigma_{bb}^2}\left[(w-\mu_{bb})^2\right]\right), \tag{5.3}$$

where

$$\sigma_{bb}^2=\frac{(t_k-t_j)(t_j-t_i)}{(t_k-t_i)},\quad \mu_{bb}=\left(\frac{t_k-t_j}{t_k-t_i}\right)w_{t_i}+\left(\frac{t_j-t_i}{t_k-t_i}\right)w_{t_k}.$$

To simulate Brownian motion paths, we can generate W_{t_j} given realisations $W_{t_i}=w_{t_i}$ and $W_{t_k}=w_{t_k}$ using the following linear interpolation

$$W_{t_j}=\left(\frac{t_k-t_j}{t_k-t_i}\right)w_{t_i}+\left(\frac{t_j-t_i}{t_k-t_i}\right)w_{t_k}+\sqrt{\left(\frac{(t_k-t_j)(t_j-t_i)}{(t_k-t_i)}\right)}Z, \tag{5.4}$$

[6] We will omit the subscripts for the following density functions for notational simplicity.

where Z is a $N(0,1)$ distributed random variable. The above linear interpolation (5.4) has an equivalent distribution as the conditional distribution (5.3).

Consider the construction of a Brownian motion path with n steps $W_{t_0}, W_{t_1}, W_{t_2}, ..., W_{t_n}$. The Brownian bridge algorithm first generates the endpoints $W_{t_n} = \sqrt{t_n}Z$ and $W_{t_0} = 0$ of the Brownian motion path, and then sequentially "fills in" the remaining points in the path, conditional on the previously generated realisations. The next sample point to be generated in the Brownian bridge algorithm is $W_{t_{[n/2]}}$. It is generated using the realised endpoints w_{t_0} and w_{t_n}. The sequence is generated in the following order, using the closest two realised Brownian motion points to the one that is being generated

$$W_{t_0}, W_{t_n}, W_{t_{[n/2]}}, W_{t_{[n/4]}}, W_{t_{[3n/4]}}, W_{t_{[n/8]}}, \dots$$

For example, if the $n = 10$, the sequence of Brownian motion points will be generated in the following order

$$W_{t_0}, W_{t_{10}}, W_{t_5}, W_{t_2}, W_{t_7}, W_{t_1}, W_{t_3}, W_{t_6}, W_{t_8}, W_{t_4}, W_{t_9}.$$

The standard method of generating a Brownian motion point W_{t_j}, given the realisation $W_{t_i} = w_{t_i}$ where $t_j > t_i$, is done by the following function of a standard normal random variable

$$W_{t_j} = w_{t_i} + \sqrt{(t_j - t_i)}Z. \tag{5.5}$$

The variance of the points of a Brownian bridge constructed Brownian motion path is smaller than that of the points of a standard constructed Brownian motion path. This can be seen by comparing the variances of expressions (5.4) and (5.5):

$$(t_j - t_i) > \frac{(t_k - t_j)(t_j - t_i)}{(t_k - t_i)}.$$

Furthermore the majority of the variance of the points of the Brownian Bridge constructed path is contained in the initial steps.

5.4 A Brownian Bridge Simulation Procedure for Pricing CDS Premiums

The difficulty in a jump-diffusion framework is the derivation of analytical expressions for exotic financial derivatives prices. We often need to evaluate these prices using numerical or Monte Carlo methods. In the structural model framework, default is modelled as the firm's asset value process, V_t, hitting a default barrier b_t. In our case, we are modeling V_t using a jump-diffusion process and the default barrier is a constant b. In order to find analytical expressions for the value of the CDS premiums we need a closed-form solution for the risk-neutral probability of V_t crossing the barrier b before the time T, maturity of the CDS:

$$\mathbb{Q}(\inf V_s \leq b, \, 0 \leq s \leq T). \tag{5.6}$$

Closed form solutions do exist for expression (5.6), but only when the jump size distribution follows specific distributions such as the double-exponential distribution, Kou & Wang (2003), the mixed exponential distribution, Mordecki (2002), or when jumps are only positive, Blake & Lindsey (1973). In general closed form solutions do not exist. Monte Carlo methods are able to provide numerical solutions for expression (5.6) for general jump-diffusion processes. We will use in particular a Monte Carlo Brownian bridge hybrid simulation procedure developed by Metwally & Atiya (2002). This method significantly increases speed of convergence and reduces bias[7] compared to the standard Monte Carlo approach. The speed of convergence is important for us, since we estimate our model parameters by calibration, and this procedure entails calculating Monte Carlo estimates several times.

The standard Monte Carlo method for evaluating financial derivatives with a barrier feature, when the underlying state variable follows a jump-diffusion process, requires simulating the risk-neutral jump-diffusion process of the underlying state variable at short time steps until the derivatives' maturity is reached. Thus, under the proposed structural model, in order to price the

[7] The *bias* of an estimator $\hat{\xi}$ is given by: $b(\hat{\xi}) = \mathbb{E}(\hat{\xi}) - \xi$.

premiums of a CDS initiated at $t_0 = 0$ with maturity T, we must divide the interval $[0, T]$ into small time steps, and then simulate the jump-diffusion process V_t along these time steps, under the risk-neutral measure $\mathbb{Q}$. This would be done by firstly generating the jump times $\tau_1, \tau_2, ..., \tau_{N(T)}$ using the distribution of the jump times ($N(T)$ represents the number of jumps up to time T). Once the jump times are generated, then one can simulate the diffusion section between the jumps along the short time steps. When a jump time is reached, one simulates the jump size according to the assumed distribution of the jump sizes. This is done until the maturity of the CDS. The value of the CDS premium is obtained by averaging the discounted CDS payoff profile from each simulated path, and then calculating the value of the premium that makes the average of the discounted CDS payoff profiles equal to zero.

The major problem with the standard Monte Carlo approach, is that by simulating the underlying stochastic process (V_t) at short time steps, we introduce bias into the estimate of the CDS. To reduce this bias, one must further divide the time interval into smaller time steps, however this increases the amount of times one needs to generate random values, increasing computation time. The Monte Carlo Brownian bridge hybrid method uses the fact there exists a closed form solution for the conditional probability that a process, that follows a GBM, remains above a barrier for a certain time interval, conditional on the process starting and ending on this time interval at specified values. Therefore, once one has generated the jump times $\tau_1, \tau_2, \ldots, \tau_{N(T)}$, the asset values at these jump times and jump sizes, one merely can use these conditional probabilities to calculate the value of a CDS. This method is computationally quicker than the standard Monte Carlo method and also eliminates the bias completely.

5.4.1 Model Description and the CDS Pricing Formula

Our firm's value process V_t is assumed to follow a jump-diffusion process, under the real-world measure $\mathbb{P}$,

$$\frac{\mathrm{d}V_t}{V_{t^-}} = \mu_V \,\mathrm{d}t + \sigma \,\mathrm{d}W_t + Y_t \,\mathrm{d}N_t, \tag{5.7}$$

where N_t is a Poisson process with intensity parameter λ, and $Y_t > 0$ represents the percentage change in V at time t. If there is a jump at time $t = \tau_i$ then $Y_t = (V_{t^+} - V_{t^-})/(V_{t^-})$, otherwise $Y_t = 0$. The stochastic processes W_t, N_t, and Y_t are mutually independent. Let A_i be the logarithm of the ratio of V after and before the i^{th} jump:

$$A_i = \ln V_{\tau_i^+} - \ln V_{\tau_i^-} = \ln(Y_{\tau_i} + 1). \tag{5.8}$$

We assume A_i is normally distributed:

$$A_i \sim \mathrm{N}(\mu_A, \sigma_A^2).$$

Let J_t be the sum of the logarithms of the ratio of jump sizes in the interval [0, t]

$$J_t = \sum_{i=1}^{N(t)} A_i.$$

The solution of the SDE (5.7) under the risk neutral measure $\mathbb{Q}$ is given by[8]

$$V_t = V_0 \exp\left[\left(r - \frac{\sigma^2}{2} - \lambda\kappa\right) t + \sigma W_t^{\mathbb{Q}} + J_t\right],$$

where

$$\kappa = \mathbb{E}^{\mathbb{P}}(Y_t) = \exp\left(\mu_A + \frac{\sigma_A^2}{2}\right) - 1,$$

and $W_t^{\mathbb{Q}}$ is a Brownian motion process under measure $\mathbb{Q}$.

If we let $X(t) = \ln V_t$, then the transformed process $X(t)$ follows a normal distribution i.e.

$$X(t) \sim \mathrm{N}\left(X(0) + \left[r - \frac{\sigma^2}{2} - \lambda\kappa + \mu_A\lambda\right] t,\ \sigma^2 t + \left[\sigma_A^2 + \mu_A^2\right] \lambda t\right).$$

[8]See Appendix A.3 to understand why V_t has this form under measure $\mathbb{Q}$.

The proposed Monte Carlo algorithms will be applied to the transformed process $X(t)$, since it is computationally faster than using the process V_t.

To determine the fair premium $\tilde{p}_{0,T}$, that is paid periodically at times $t_1, t_2, \ldots, t_n = T$, one needs to find the discounted risk-neutral expectation of the CDS payoffs, set it to 0 and solve for $\tilde{p}(0,T)$

$$\begin{aligned}\phi(0,0,T) \quad &= \quad \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n}\left(e^{-rt_i}\tilde{p}(0,T)(t_i - t_{i-1})N\mathbb{1}_{\{\tau^*>t_i\}} + e^{-r\tau^*}\tilde{p}(0,T)(\tau^* - t_{i-1})N\mathbb{1}_{\{t_{i-1}<\tau^*<t_i\}}\right)\right.\\ &\qquad \left. - \, e^{-r\tau^*}N(1-R)\mathbb{1}_{\{\tau^*\leq T\}}\right] = 0 \end{aligned} \tag{5.9}$$

where[9]

$$\tau^* = \inf\{t > 0 : V_t \leq b\},$$

R is the recovery rate[10], N is the notional amount of the CDS and $\phi(0,0,T)$ represents the value at time $t = 0$ (present time) of a CDS, initiated at time $t = 0$ with maturity T. Let $\tau^* = s$ and rewrite the expectations in terms of integrals, then the RHS of equation (5.9) becomes

$$\begin{aligned} & \tilde{p}_{0,T}N\sum_{i=1}^{n} e^{-rt_i}(t_i - t_{i-1})\mathbb{E}^{\mathbb{Q}}\left[\mathbb{1}_{\{s>t_i\}}\right] + \tilde{p}_{0,T}N\sum_{i=1}^{n}\mathbb{E}^{\mathbb{Q}}\left[e^{-rs}(s - t_{i-1})\mathbb{1}_{\{t_{i-1}<s<t_i\}}\right] \\ & -N(1-R)\mathbb{E}^{\mathbb{Q}}\left[e^{-rs}\mathbb{1}_{\{s\leq T\}}\right] \\ = \quad & \tilde{p}_{0,T}N\sum_{i=1}^{n}\left[e^{-rt_i}(t_i - t_{i-1})\int_0^\infty \mathbb{1}_{\{s>t_i\}}h(s)\,\mathrm{d}s + \int_0^\infty e^{-rs}s\mathbb{1}_{\{t_{i-1}<s<t_i\}}h(s)\,\mathrm{d}s\right] \\ & -N\left[\sum_{i=1}^{n}\tilde{p}_{0,T}t_{i-1}\int_0^\infty e^{-rs}\mathbb{1}_{\{t_{i-1}<s<t_i\}}h(s)\,\mathrm{d}s + (1-R)\int_0^\infty e^{-rs}\mathbb{1}_{\{s\leq T\}}h(s)\,\mathrm{d}s\right] \\ = \quad & \tilde{p}_{0,T}N\sum_{i=1}^{n}\left[e^{-rt_i}(t_i - t_{i-1})\int_{t_i}^\infty h(s)\,\mathrm{d}s + \int_{t_{i-1}}^{t_i} e^{-rs}sh(s)\,\mathrm{d}s\right] \\ & -N\left[\sum_{i=1}^{n}\tilde{p}_{0,T}t_{i-1}\int_{t_{i-1}}^{t_i} e^{-rs}h(s)\,\mathrm{d}s + (1-R)\int_0^T e^{-rs}h(s)\,\mathrm{d}s\right] \end{aligned} \tag{5.10}$$

where $h(\cdot)$ is the pdf of the random variable τ^*. Since V_t is a jump-diffusion process with log-normal jump sizes, we unfortunately do not have a closed form expression for the pdf $h(\cdot)$. In order to find the premium $\tilde{p}_{0,T}$ that makes the value of a newly minted CDS contract equal to 0, we set equation (5.10) equal to 0 and make $\tilde{p}_{0,T}$ subject of the formula

$$\tilde{p}_{0,T} = \frac{(1-R)\int_0^T e^{-rs}h(s)\,\mathrm{d}s}{\sum_{i=1}^{n}\left(e^{-rt_i}(t_i - t_{i-1})\int_{t_i}^\infty h(s)\,\mathrm{d}s + \int_{t_{i-1}}^{t_i} e^{-rs}sh(s)\,\mathrm{d}s - t_{i-1}\int_{t_{i-1}}^{t_i} e^{-rs}h(s)\,\mathrm{d}s\right)}. \tag{5.11}$$

The proposed hybrid Monte Carlo method numerically evaluates these non-analytical integrals.

5.4.2 Outline of the Monte Carlo Brownian Bridge Method to Price CDS premiums

The simulation procedure starts by firstly generating the jump times using the inter-arrival jump time distributions, which are i.i.d. with an exponential distribution[11] with mean $1/\lambda$ i.e.

$$(\tau_i - \tau_{i-1}) \sim \mathrm{Exp}(1/\lambda), \ \forall i \in \mathbb{N}.$$

In between jump times, for $t \in [\tau_{i-1}, \tau_i]$, the risk-neutral asset value process follows a pure GBM diffusion process

$$\frac{\mathrm{d}V_t}{V_t} = (r - \lambda\kappa)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{\mathbb{Q}}, \tag{5.12}$$

[9] Since we want to find the fair premium at $t = 0$ for a CDS with maturity T, the barrier we are working with is $b_{0,T}$. For notational brevity we just denote it as b.

[10] To condense the following analysis we assume the recovery rate is constant. The recovery rate can be easily changed in this Monte Carlo framework to our model assumption, that states that the $R = 1 - [w_0 - (V_{\tau^*}/b)]$.

[11] Since the number of jumps N_t has a Poisson distribution with mean λ.

Using Itô's Lemma (see Appendix A.1.1) the dynamics of the transformed process $X(t) = \ln V_t$, for $t \in [\tau_{i-1}, \tau_i]$, can be represented in the following SDE form

$$dX(t) = \left(r - \frac{\sigma^2}{2} - \lambda\kappa\right) dt + \sigma dW_t^{\mathbb{Q}},$$

We can generate the transformed asset values immediately before the jump times, $X(\tau_i^-)$ where $i = 1, 2, ..., N(T)$, using its normal distribution

$$X(\tau_i^-) \sim \mathrm{N}\left(X(\tau_{i-1}^+) + \left[r - \frac{\sigma^2}{2} - \lambda\kappa\right](\tau_i - \tau_{i-1}),\, \sigma^2(\tau_i - \tau_{i-1})\right).$$

Note that to generate a value for $X(\tau_i^-)$ we need a value for the transformed asset value immediately after the previous jump $X(\tau_{i-1}^+)$. This will be calculated by generating the transformed asset jumps, $A_{i-1} = X(\tau_{i-1}^+) - X(\tau_{i-1}^-)$, which are i.i.d. with a Normal distribution with mean μ_A and variance σ_A^2. Once this is generated we add A_{i-1} to $X(\tau_{i-1}^-)$ to get the value of the transformed asset value immediately after the jump time τ_{i-1}. Let $\tau_{N(T)+1} = T$. If any one of these generated transformed asset values $X(\tau_i^-)$ and $X(\tau_i^+)$, for $i = 1, 2, ..., N(T), N(T) + 1$, are less than $\ln b$ then the firm has defaulted and the CDS buyer stops paying his periodic payments and the CDS seller pays out the default payment at the time of the default barrier crossing[12]. However, the firm's asset value process could have fallen below the barrier in between the jump times. To accommodate for this, we use the conditional probability of the firm's asset value process crossing the barrier in these inter-jump intervals (note that the process follows a GBM in between these jumps), conditional on the asset values taking on known values at the end of these inter-jump intervals. We then use this conditional probability with the concept of uniform sampling in order to sample a hitting time in between jumps. Now, that we've covered all possibilities of reaching the default barrier b, either by jumps or by diffusion, we sequentially generate these possible default barrier crossings, until a crossing or maturity, whichever comes first. Finally we average all the payoffs of a CDS resulting from these simulated paths, equate it to 0 and calculate the fair premium $\tilde{p}_{0,T}$.

5.4.3 Conditional First Passage Time Distribution

The stopping time $\tau_i^* = \inf\{\tau_{i-1} \le s \le \tau_i : V_s \le b\}$, is the first-passage time of the GBM process V_s to the constant barrier b on the interval $[\tau_{i-1}, \tau_i]$. Note that $\inf \emptyset = +\infty$. If we let $\tau_i^* = t$, the conditional probability density function of the first passage time τ_i^*, conditional on the end-values of the process on the interval $[\tau_{i-1}, \tau_i]$, is defined as[13]:

$$g_i(t) = f\left(t | X(\tau_{i-1}^+) = x(\tau_{i-1}^+), X(\tau_i^-) = x(\tau_i^-)\right). \tag{5.13}$$

Using Bayes' rule[14] the above equation (5.13) can be expressed as

$$\begin{aligned} g_i(t) &= \frac{f\left(t, x(\tau_i^-) | x(\tau_{i-1}^+)\right)}{f\left(x(\tau_i^-) | x(\tau_{i-1}^+)\right)} \\ &= \frac{f\left(t | x(\tau_{i-1}^+)\right) f\left(x(\tau_i^-) | t, x(\tau_{i-1}^+)\right)}{f\left(x(\tau_i^-) | x(\tau_{i-1}^+)\right)}. \end{aligned} \tag{5.14}$$

Then by the definition of τ_i^* and by the Markov property of $X(t)$, the above equation (5.14) becomes

$$\frac{f\left(t | x(\tau_{i-1}^+)\right) f\left(x(\tau_i^-) | x(t) = \ln b\right)}{f\left(x(\tau_i^-) | x(\tau_{i-1}^+)\right)}. \tag{5.15}$$

From the assumed Gaussian distribution of $X(t)$ on the inter-jump interval $[\tau_{i-1}, \tau_i]$, the following two expressions can be rewritten:

$$f\left(x(\tau_i^-) | x(t) = \ln b\right) = \frac{1}{\sqrt{2\pi(\tau_i - t)\sigma^2}} \exp\left(-\frac{[x(\tau_i^-) - \ln b - c(\tau_i - t)]^2}{2\sigma^2(\tau_i - t)}\right), \tag{5.16}$$

$$f\left(x(\tau_i^-) | x(\tau_{i-1}^+)\right) = \frac{1}{\sqrt{2\pi(\tau_i - \tau_{i-1})\sigma^2}} \exp\left(-\frac{[x(\tau_i^-) - x(\tau_{i-1}^+) - c(\tau_i - \tau_{i-1})]^2}{2\sigma^2(\tau_i - \tau_{i-1})}\right), \tag{5.17}$$

[12] We must also check if barrier crossing has occurred at maturity $T = \tau_{N(T)+1}$.

[13] We will omit the subscripts for the following density functions for notation simplicity.

[14] See Bain & Engelhardt (1991) for the Bayes' rule Theorem.

where $c = r - \lambda\kappa - \sigma^2/2$. The remaining component of expression (5.15) is derived from the first passage time (FPT) distribution[15] of a GBM process

$$f\left(t|x(\tau_{i-1}^+)\right) = \frac{x(\tau_{i-1}^+) - \ln b}{\sqrt{2\pi\sigma^2}}(t - \tau_{i-1})^{-\frac{3}{2}} \exp\left(-\frac{[x(\tau_{i-1}^+) - \ln b - c(t - \tau_{i-1})]^2}{2\sigma^2(t - \tau_{i-1})}\right). \tag{5.18}$$

Now substituting (5.16), (5.17) and (5.18) into (5.15), and simplifying we get:

$$\begin{aligned} g_i(t) &= \frac{x(\tau_{i-1}^+) - \ln b}{2\pi\sigma^2 y}(t - \tau_{i-1})^{-\frac{3}{2}}(\tau_i - t)^{-\frac{1}{2}} \times \\ &\quad \exp\left[-\frac{[x(\tau_i^-) - \ln b - c(\tau_i - t)]^2}{2(\tau_i - t)\sigma^2} - \frac{[x(\tau_{i-1}^+) - \ln b - c(t - \tau_{i-1})]^2}{2(t - \tau_i)\sigma^2}\right], \end{aligned}$$

where

$$y = \frac{1}{\sqrt{2\pi(\tau_i - \tau_{i-1})\sigma^2}} \exp\left(-\frac{[x(\tau_{i-1}^+) - x(\tau_i^-) + c(\tau_i - \tau_{i-1})]^2}{2\sigma^2(\tau_i - \tau_{i-1})}\right).$$

The conditional probability (under the risk-neutral measure $\mathbb{Q}$) that the risk-neutral GBM asset value process V_t (5.12) will hit the barrier b on the interval $[\tau_i, \tau_{i-1}]$, conditional on the values of the asset process at the end-points on this interval is computed by integrating the FPT distribution[16]:

$$\begin{aligned} p_i &= \mathbb{Q}\left(\inf_{\tau_{i-1}\le s\le\tau_i} V_s \le b \,\Big|\, V_{\tau_{i-1}^+} = v_{\tau_{i-1}^+}, V_{\tau_i^-} = v_{\tau_i^-}\right) \\ &= \mathbb{Q}\left(\inf_{\tau_{i-1}\le s\le\tau_i} X(s) \le \ln b \,\Big|\, X(\tau_{i-1}^+) = x(\tau_{i-1}^+), X(\tau_i^-) = x(\tau_i^-)\right) \\ &= \mathbb{Q}\left(\tau_{i-1} < \tau_i^* < \tau_i\right) \\ &= \int_{\tau_{i-1}}^{\tau_i} g_i(t)\,\mathrm{d}t \\ &= \begin{cases} \exp\left(\frac{-2[\ln b - x(\tau_{i-1}^+)][\ln b - x(\tau_i^-)]}{(\tau_i - \tau_{i-1})\sigma^2}\right) & \text{if } x(\tau_i^-) \ge \ln b \\ 1 & \text{otherwise} \end{cases} \end{aligned} \tag{5.19}$$

The conditional probability of not reaching the barrier in the interval $[\tau_i, \tau_{i-1}]$ is defined as $\bar{p}_i := 1 - p_i$.

5.4.4 Decomposition of the CDS Pricing Formula

In this section we decompose the RHS of equation (5.10), in order to observe that this Monte Carlo Brownian bridge algorithm indeed evaluates $\phi(0, 0, T)$. For notational brevity we set $\tilde{p}_{0,T} := \tilde{p}$ for this section 5.4.4 only.

By the Tower property we can express $\phi(0, 0, T)$ as

$$\begin{aligned} \phi(0,0,T) = \mathbb{E}^{\mathbb{Q}}\Bigg[\mathbb{E}^{\mathbb{Q}}\Bigg[\sum_{i=1}^{n}\Big(e^{-rt_i}\tilde{p}(t_i - t_{i-1})N\mathbb{1}_{\{\tau^*>t_i\}} \\ + e^{-r\tau^*}\tilde{p}(\tau^* - t_{i-1})N\mathbb{1}_{\{t_i<\tau^*<t_i\}}\Big) - e^{-r\tau^*}N(1-R)\mathbb{1}_{\{\tau^*\le T\}}\Bigg|\mathscr{F}^*\Bigg]\Bigg] \end{aligned} \tag{5.20}$$

where

$$\mathscr{F}^* := \sigma\left\{N(T); 0 < \tau_1 < \ldots < \tau_{N(T)} < T; X\left(\tau_1^-\right), X\left(\tau_1^+\right), \ldots, X\left(\tau_{N(T)}^-\right), X\left(\tau_{N(T)}^+\right), X(T)\right\}$$

is the σ-algebra representing the information set containing the number of jumps, the location of the jump times, the values of X immediately before and after jump times, and the value of X at

[15] See Appendix B for the calculation of the first passage time distribution for a GBM process. Note in the appendix the FPT distribution is calculated with a drift term μ_V, in this section our drift term for our GBM process is $r - \lambda\kappa$.

[16] See Metwally & Atiya (2002) [§2, p. 47] for more details.

maturity. Note that $X\left(\tau_i^+\right) = X\left(\tau_i^-\right) + A_i$ (from (5.8)), and so the information set generated by the values $X\left(\tau_i^-\right)$ and $X\left(\tau_i^+\right)$ is equivalent to the information set generated by the values $X\left(\tau_i^-\right)$ and A_i. If we let

$$N(T) = k,$$

$$\tau_1 = z_1, \ldots \tau_k = z_k,$$

$$X\left(\tau_1^-\right) = x_1, \ldots, X\left(\tau_k^-\right) = x_k, X(T) = x_{k+1},$$

$$A_1 = y_1, \ldots, A_k = y_k,$$

then by the definition of the information set $\mathscr{F}^*$ and from the assumption that $X(t)$, τ_i and $N(t)$ are mutually independent $\forall\, i \in \mathbb{N}$ and $\forall\, t \in \mathbb{R}^+$, we can rewrite the outer expectation of expression (5.20) in terms of a summation over the discrete random variable k, and integrals of the continuous random variables $z_1, \ldots z_k$; $x_1, \ldots, x_k, x_{k+1}$ and $y_1, \ldots, y_k$:

$$\sum_{k=0}^{\infty} \int\limits_{\substack{(z_1,\ldots,z_k) \\ \in[0,T]^k}} \int\limits_{\substack{(x_1,\ldots,x_{k+1}) \\ \in[-\infty,\infty]^{k+1}}} \int\limits_{\substack{(y_1,\ldots,y_k) \\ \in[-\infty,\infty]^k}} \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n}\left(e^{-rt_i}\tilde{p}(t_i - t_{i-1})N\mathbb{1}_{\{\tau^*>t_i\}}\right.\right.$$
$$\left.\left. + e^{-r\tau^*}\tilde{p}(\tau^* - t_{i-1})N\mathbb{1}_{\{t_i<\tau^*<t_i\}}\right) - e^{-r\tau^*}N(1-R)\mathbb{1}_{\{\tau^*\le T\}}\right|\mathscr{F}^*\Bigg] \tag{5.21}$$
$$\times f_{A_1,\ldots,A_k}(y_1,\ldots,y_k)\,\mathrm{d}y_1\cdots\mathrm{d}y_k\, f_{X(\tau_1^-),\ldots,X(\tau_k^-),X(T)}(x_1,\ldots,x_{k+1})\,\mathrm{d}x_1\cdots\mathrm{d}x_{k+1}$$
$$\times\, \mathbb{1}_{\{\tau_1<\cdots<\tau_k<T\}} f_{\tau_1,\ldots,\tau_k}(z_1,\ldots,z_k|N(T)=k)\,\mathrm{d}z_1\cdots\mathrm{d}z_k\, f_{N(T)}(k).$$

The terms $f_{A_1,\ldots,A_k}(y_1,\ldots,y_k)$ and $f_{X(\tau_1^-),\ldots,X(\tau_k^-),X(T)}(x_1,\ldots,x_{k+1})$ are the joint probability density functions for $A_1,\ldots,A_k$ and $X(\tau_1^-),\ldots,X(\tau_k^-),X(T)$, respectively. Furthermore, the expression $f_{N(T)}(k)$ is the probability mass function[17] (pmf) for the discrete random variable $N(T)$ and the expression $f_{\tau_1,\ldots,\tau_k}(z_1,\ldots,z_k|N(T)=k)$ is the conditional joint probability density function of $\tau_1,\ldots,\tau_k$ conditional that $N(T)=k$.

Since we assumed the number of jumps to follow a Poisson process with intensity parameter λ, $N(T) \sim \mathrm{POI}(\lambda T)$. The pmf for $N(T)$ is

$$f_{N(T)}(k) = \frac{e^{-\lambda T}(\lambda T)^k}{k!}. \tag{5.22}$$

A consequence of assuming the number of jumps following a homogenous (constant intensity parameter) Poisson process, is that the inter-arrival jump times $\tilde{\tau}_i = \tau_i - \tau_{i-1}$, $\forall i \in \mathbb{N}$ are independent and have an exponential distribution with mean $1/\lambda$:

$$\tilde{\tau}_i \sim \mathrm{EXP}(1/\lambda).$$

Suppose that $0 < \tau_1 < \tau_2 < \cdots < \tau_k < T$. Then the conditional joint probability density function of $\tau_1,\ldots,\tau_n$ given $N(T)=k$ can be written in terms of $\tilde{\tau}_i$:

$$f_{\tau_1,\ldots,\tau_k}(z_1,\ldots,z_k|N(T)=k) = \frac{\mathbb{Q}(\tilde{\tau}_{k+1} < T-\tau_k)}{f_{N(T)}(k)}\prod_{i=1}^{k} f_{\tilde{\tau}_i}(\tilde{z}_i), \tag{5.23}$$

where $f_{\tilde{\tau}_i}(\tilde{z})$ is the pdf of the random variable[18] $\tilde{\tau}_i$, and $\tilde{z}_i = z_i - z_{i-1}$. Since $\tilde{\tau}_i \sim \mathrm{EXP}(1/\lambda)$ equation (5.23) can also be expressed as

$$\begin{aligned} f_{\tau_1,\ldots,\tau_k}(z_1,\ldots,z_k|N(T)=k) &= \frac{\lambda e^{\lambda z_1}\lambda e^{\lambda(z_2-z_1)}\ldots\lambda e^{\lambda(z_k-z_{k-1})}e^{\lambda(T-z_k)}}{e^{\lambda T}(\lambda T)^k/k!} \\ &= \frac{k!}{T^k}. \end{aligned} \tag{5.24}$$

[17]The probability mass function is $f_{N(T)}(k) = \mathbb{Q}[N(T)=k]$.
[18]Note $\tau_0 = t_0 = 0$ and thus $z_0 = 0$.

We have assumed that $A_i\ \forall i \in \mathbb{N}$, are independent and have a normal distribution with mean μ_A and variance σ_A^2. Thus the explicit form of the joint probability density function $f_{A_1,\ldots,A_k}(y_1,\ldots,y_k)$ is

$$\begin{aligned} f_{A_1,\ldots,A_k}(y_1,\ldots,y_k) &= \prod_{i=1}^{k} f_{A_i}(y_i) \\ &= \left(2\pi\sigma^2\right)^{-k/2} \exp\left(\sum_{i=1}^{k} \frac{(y_i-\mu_A)^2}{2\sigma_A^2}\right), \end{aligned} \tag{5.25}$$

where $f_{A_i}(y_i)$ is the probability density function of the random variable A_i. From equation (5.12) it can be seen that the firm's asset value process V follows a GBM in between jump times. Hence $\tilde{X}(\tau_i) := X(\tau_i^-) - X(\tau_{i-1}^+)$, for all $0 \le \tau_i \le T$, are mutually independent and have a normal distribution, i.e.

$$\tilde{X}(\tau_i) \sim \mathrm{N}\left(\left[r - \frac{\sigma^2}{2} - \lambda\kappa\right](\tau_i - \tau_{i-1}),\, \sigma^2(\tau_i - \tau_{i-1})\right).$$

Note that $X(\tau_{i-1}^+) = X(\tau_{i-1}^-) + A_{i-1}$, so $\tilde{X}(\tau_i) = x_i - (x_{i-1} + y_{i-1})$. The joint probability density function $f_{X(\tau_1^-),\ldots,X(\tau_k^-),X(T)}(x_1,\ldots,x_{k+1})$ can be written in terms of $\tilde{X}(\tau_i)$:

$$\begin{aligned} f_{X(\tau_1^-),\ldots,X(\tau_k^-),X(T)}(x_1,\ldots,x_{k+1}) &= \prod_{i=1}^{k+1} f_{\tilde{X}(\tau_i)}(\tilde{x}_i) \\ &= \left[\prod_{i=1}^{k+1} \left(2\pi\sigma^2\tilde{\tau}_i\right)^{-1/2}\right] \exp\left(\sum_{i=1}^{k+1} \frac{\tilde{x}_i - \left(r - \sigma^2/2 - \lambda\kappa\right)\tilde{\tau}_i}{2\sigma^2\tilde{\tau}_i}\right) \end{aligned} \tag{5.26}$$

where $\tilde{x}_i = x_i - (x_{i-1} + y_{i-1})$ and $f_{\tilde{X}(\tau_i)}(\tilde{x}_i)$ is the pdf of the random variable $\tilde{X}(\tau_i)$. Note that $\tau_0 = t_0 = 0$ and $\tau_{k+1} = T$. Substituting (5.22), (5.24), (5.25), and (5.26) into (5.21) results in

$$\begin{aligned} &\sum_{k=0}^{\infty} \int\limits_{\substack{(z_1,\ldots,z_k)\\ \in[0,T]^k}} \int\limits_{\substack{(x_1,\ldots,x_{k+1})\\ \in(-\infty,\infty)^{k+1}}} \int\limits_{\substack{(y_1,\ldots,y_k)\\ \in(-\infty,\infty)^k}} \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n}\left(e^{-rt_i}\tilde{p}(t_i - t_{i-1})N\mathbb{1}_{\{\tau^* > t_i\}}\right.\right. \\ &\left.\left.+\, e^{-r\tau^*}\tilde{p}(\tau^* - t_{i-1})N\mathbb{1}_{\{t_i < \tau^* < t_i\}}\right) - e^{-r\tau^*}N(1-R)\mathbb{1}_{\{\tau^* \le T\}}\middle|\mathscr{F}^*\right] \prod_{i=1}^{k} f_{A_i}(y_i)\,\mathrm{d}y_i \\ &\times \prod_{i=1}^{k+1} f_{\tilde{X}(\tau_i)}(\tilde{x}_i)\,\mathrm{d}x_i\,\mathbb{1}_{\{\tau_1 < \cdots < \tau_k < T\}}\,\mathrm{d}z_i\,e^{-\lambda T}\lambda^k. \end{aligned} \tag{5.27}$$

Now define the following:

$$Q_i := \max\{q \in \{0,1,\ldots,n\} : t_q < \tau_i\}, \quad \text{i.e. } \tau_i \in (t_{Q_i}, t_{Q_i+1}],$$

$$J := \min\left\{j \in \{1,\ldots,N(T)\} : V_{\tau_j^+} \le b\right\}, \quad \min\emptyset := N(T)+1,$$

$$G_{\tau_i^*} := \max\{g \in \{0,1,\ldots,n\} : t_g < \tau_i^*\}, \quad \text{i.e. } \tau_i^* \in (t_{G_{\tau_i^*}}, t_{G_{\tau_i^*}+1}],$$

and remember that $\tau_i^* = \inf\{\tau_{i-1} \le s \le \tau_i : V_s \le b\}$, (the first hitting time of the barrier in the i^{th} inter-jump interval). Taking into account the information set $\mathscr{F}^*$ and that V has independent increments, the conditional expectation in expression (5.27) can be expressed as:

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\Bigg[\sum_{i=1}^{n}\Big(e^{-rt_i}\tilde{p}(t_i-t_{i-1})N\mathbb{1}_{\{\tau^*>t_i\}}+e^{-r\tau^*}\tilde{p}(\tau^*-t_{i-1})N\mathbb{1}_{\{t_i<\tau^*<t_i\}}\Big)\\
&-e^{-r\tau^*}N(1-R)\mathbb{1}_{\{\tau^*\le T\}}\Bigg|\mathscr{F}^*\Bigg]\\
&=N\mathbb{E}^{\mathbb{Q}}\Bigg[\sum_{j=1}^{J}\Bigg(\prod_{l=1}^{j-1}\mathbb{1}_{\{\tau_l^*\notin(\tau_{l-1},\tau_l)\}}\Bigg)\mathbb{1}_{\{\tau_j^*\in(\tau_{j-1},\tau_j)\}}\Bigg(\tilde{p}\sum_{i=1}^{Q_j}(t_i-t_{i-1})e^{-rt_i}+\tilde{p}(\tau_j^*-t_{G_{\tau_j^*}})e^{-r\tau_j^*}\\
&-(1-R)e^{-r\tau_j^*}\Bigg)\Bigg|\mathscr{F}^*\Bigg]+N\mathbb{E}^{\mathbb{Q}}\Bigg[\mathbb{1}_{\{J\neq0\}}\prod_{l=1}^{J}\mathbb{1}_{\{\tau_l^*\notin[\tau_{l-1},\tau_l]\}}\Bigg(\tilde{p}\sum_{i=1}^{Q_J}(t_i-t_{i-1})e^{-rt_i}+\tilde{p}(t_{Q_J}-\tau_J)e^{-r\tau_J}\\
&-(1-R)e^{-r\tau_J}\Bigg)\Bigg|\mathscr{F}^*\Bigg]+\tilde{p}N\mathbb{E}^{\mathbb{Q}}\Bigg[\mathbb{1}_{\{J=0\}}\prod_{l=1}^{N(T)+1}\mathbb{1}_{\{\tau_l^*\notin[\tau_{l-1},\tau_l]\}}\Bigg(\sum_{i=1}^{n}(t_i-t_{i-1})e^{-rt_i}\Bigg)\Bigg|\mathscr{F}^*\Bigg]\\
&=N\sum_{j=1}^{J}\prod_{l=1}^{j-1}\Bigg(\bar{p}_l p_j\tilde{p}\sum_{i=1}^{Q_j}(t_i-t_{i-1})e^{-rt_i}+\bar{p}_l\tilde{p}\mathbb{E}^{\mathbb{Q}}\Bigg[\mathbb{1}_{\{\tau_j^*\in[\tau_{j-1},\tau_j]\}}(\tau_j^*-t_{G_{\tau_j^*}})e^{-r\tau_j^*}\\
&-(1-R)e^{-r\tau_j^*}\Bigg|\mathscr{F}^*\Bigg]\Bigg)+\mathbb{1}_{\{J\neq0\}}N\prod_{l=1}^{J}\Bigg(\bar{p}_l\tilde{p}\sum_{i=1}^{Q_J}(t_i-t_{i-1})e^{-rt_i}+\bar{p}_l\tilde{p}(t_{Q_J}-\tau_J)e^{-r\tau_J}\\
&-\bar{p}_l(1-R)e^{-r\tau_J}\Bigg)+\mathbb{1}_{\{J=0\}}\tilde{p}N\prod_{l=1}^{N(T)+1}\bar{p}_l\Bigg(\sum_{i=1}^{n}(t_i-t_{i-1})e^{-rt_i}\Bigg)\\
&=N\sum_{j=1}^{J}\prod_{l=1}^{j-1}\Bigg(\bar{p}_l p_j\tilde{p}\sum_{i=1}^{Q_j}(t_i-t_{i-1})e^{-rt_i}+\bar{p}_l\tilde{p}\int_{\tau_j^*}^{\tau_j}\Big((s-t_{G_s})e^{-rs}-(1-R)e^{-rs}\Big)g_j(s)\,\mathrm{d}s\Bigg)\\
&+\mathbb{1}_{\{J\neq0\}}N\prod_{l=1}^{J}\Bigg(\bar{p}_l\tilde{p}\sum_{i=1}^{Q_J}(t_i-t_{i-1})e^{-rt_i}+\bar{p}_l\tilde{p}(t_{Q_J}-\tau_J)e^{-r\tau_J}\\
&-\bar{p}_l(1-R)e^{-r\tau_J}\Bigg)+\mathbb{1}_{\{J=0\}}\tilde{p}N\prod_{l=1}^{N(T)+1}\bar{p}_l\Bigg(\sum_{i=1}^{n}(t_i-t_{i-1})e^{-rt_i}\Bigg).
\end{aligned}
\tag{5.28}
$$

Essentially three events can occur: V does not reach the barrier in the lifetime of the CDS $[0,T]$, V reaches the barrier by the diffusion part of V, or V reaches the barrier by a jump. The above expression (5.28) covers all three of these possible events. The first part of expression (5.28) deals with reaching the barrier by the diffusion part of V, the second part deals with V reaching the barrier by a jump, and the final part deals with no barrier crossing.

5.4.5 Sampling a First Hitting Time in an Inter-Jump Interval

The integral in expression (5.28) cannot be evaluated in closed form. In order to evaluate this non-analytical integral, we perform a uniform sampling method. Consider an inter-jump interval $[\tau_{i-1},\tau_i]$, this sampling method samples the first hitting time τ_i^*, conditional on the asset values at the beginning and end of the inter-jump intervals. The conditional probability that the asset value will hit the barrier during this interval is given by equation (5.19). In order to make sure that the probability of the sampled first hitting time s falls in this interval is equal to p_i, we sample from a uniform distribution over the interval $[\tau_{i-1},\tau_{i-1}+l]$, where $l=(\tau_i-\tau_{i-1})/p_i$. In order to see that these probabilities are equal, let $S\sim\text{UNIF}(\tau_{i-1},\tau_{i-1}+l)$, then the probability that a realisation[19] of S will fall in the interval $[\tau_{i-1},\tau_i]$ is given by:

$$
\begin{aligned}
\mathbb{Q}(\tau_{i-1}\le s\le\tau_i) &= F_S(\tau_i)-F_S(\tau_{i-1})\\
&= \frac{\tau_i-\tau_{i-1}}{\tau_{i-1}+l-\tau_{i-1}}-\frac{\tau_{i-1}-\tau_{i-1}}{\tau_{i-1}+l-\tau_{i-1}}\\
&= \frac{\tau_i-\tau_{i-1}}{l}\\
&= p_i,
\end{aligned}
$$

[19] A sampled point from $\text{UNIF}(\tau_{i-1},\tau_{i-1}+l)$ distribution is denoted by s.

where $F_S(s)$ is the cdf of $S \sim \text{UNIF}(\tau_{i-1}, \tau_{i-1} + l)$. Thus the probability of the random variable S falling in the inter-jump interval is equal to the conditional probability of the first hitting time in the interval $[\tau_{i-1}, \tau_i]$.

We sequentially work through each of the intervals generating a point s each time. If s falls in the inter-jump interval under consideration, then a barrier crossing has occurred. If s falls outside the inter-jump interval then the barrier has not been reached in the interval.

5.4.6 The Simulation Algorithm to Price the Premium of a CDS

The following is an algorithm to price the fair premium[20] of a newly created CDS contract, using the Monte Carlo Brownian bridge simulation method. Premiums are paid periodically until maturity T or default whichever comes first and the default payment is paid at default. The steps of the algorithm are as follows[21]:

1. For $w = 1$ to W, perform Monte Carlo simulations according to steps 2 to 4:
2. Generate the jump times τ_i for $i = 1, 2, ..., N(T) = K$, using the inter-arrival jump time distributions, $(\tau_i - \tau_{i-1}) \sim \text{Exp}(1/\lambda)$.
3. For $j = 1$ to $K + 1$ (loop through all inter-jump intervals, including the final interval $[\tau_K, \tau_{K+1} = T]$):
   a. Generate the j^{th} jump size a_j, from its distribution, $A_j \sim \text{N}(\mu_A, \sigma_A^2)$.
   b. Generate the transformed asset value the instant before the j^{th} jump time $x(\tau_j^-)$, according to its normal distribution:
   $X(\tau_j^-) \sim \text{N}\left(x(\tau_{j-1}^+) + \left[r - \sigma^2/2 - \lambda\kappa\right](\tau_j - \tau_{j-1}),\ \sigma^2(\tau_j - \tau_{j-1})\right)$.
   c. Calculate the transformed asset value the instant after the j^{th} jump, $x(\tau_j^+) = x(\tau_j^-) + a_j$. Note that $a_{k+1} = 0$.
   d. Calculate the conditional probability p_j according to equation (5.19).
   d. Let $l = (\tau_j - \tau_{j-1})/p_j$.
   e. Generate a point s_j from a uniform distribution in the interval $[\tau_{j-1},\ \tau_{j-1} + l]$, $\text{UNIF}(\tau_{j-1},\ \tau_{j-1} + l)$
   f. If $s_j \in [\tau_{j-1}, \tau_j]$, default has occurred, then:
   - Evaluate the conditional first passage time density $g_j(s_j)$.
   - Let DiscDefaultPayment$(w) = (1 - R)lg_j(s_j)e^{-rs_j}$.
   - Let DiscPremiums$(w) = \sum_{i=1}^{Q(j)} e^{-rt_i}(t_i - t_{i-1}) + (s_j - t_{Q(j)})lg_j(s_j)e^{-rs_j}$, where $Q(j) := \max\{q \in \{0, 1, \ldots, n\} : t_q < s_j\}$.
   - Exit the j loop, return to step 1, now $w = w + 1$, and perform another Monte Carlo simulation.

   g. If $x(\tau_j^+) \leq \ln b$, then the j^{th} jump has crossed the barrier and default has occurred. Then:
   - Let DiscDefaultPayment$(w) = (1 - R)e^{-r\tau_j}$.
   - Let DiscPremiums$(w) = \left(\sum_{i=1}^{H} e^{-rt_i}(t_i - t_{i-1}) + (\tau_j - t_H)e^{-r\tau_j}\right)$, where $H := \max\{h \in \{0, 1, \ldots, n\} : t_h < \tau_j\}$.
   - Exit j loop , return to step 1, now $w = w + 1$, and perform another Monte Carlo simulation.

   h. If $x(\tau_j^+) > \ln b$, no default has occurred at the j^{th} jump. Proceed to the next inter-jump interval. This is done by returning to step 3, where now $j = j + 1$.

[20] The fair premium is the premium $\tilde{p}$ that makes $\phi(0, 0, T) = 0$.

[21] Note that in this chapter we assumed the recovery rate is constant. If we wanted to include our original assumption on the recovery rate, we would have to simulate (V_{τ^*}/b). This can be done by firstly simulating the first hitting time, then one can simulate the firm's value at the first hitting time.

4. If default does not happen until the maturity T then:
   - LetDiscDefaultPayment$(w) = 0$.
   - LetDiscPremiums$(w) = \sum_{i=1}^{n} e^{-rt_i}(t_i - t_{i-1})$.
   - Exit j loop, return to step 1, now $w = w + 1$, and perform another Monte Carlo simulation.
5. If $w = W$, we have completed W Monte Carlo simulations. Calculate the Monte Carlo estimate for the fair premium of a CDS, with notional N:
   - $\tilde{p} = N \frac{1}{W} \sum_{w=1}^{W} \frac{\text{DiscDefaultPayment}(w)}{\text{DiscPremiumPayments}(w)}$.

5.4.7 The Brownian Bridge Simulation Algorithm to Value a CDS

The simulation algorithm to value a CDS contract after the CDS issue date follows similarly as the algorithm to price the fair CDS premium. Lets assume the CDS contract we are valuing, was issued at $t = 0$, and we are valuing it at some future time point t_f. The value of this CDS at time t_f is

$$\begin{aligned}\phi(0, t_f, T) =& \mathbb{E}^{\mathbb{Q}} \Bigg([\tilde{p}(t_f, T) - \tilde{p}(0, T)] \sum_{i=1}^{n} \Big[\mathbb{1}_{\{t_i \geq t_f\}} e^{-r(t_i - t_f)} (t_i - t_{i-1}) \mathbb{1}_{\{\tau^* > t_i\}} \\ &+ \mathbb{1}_{\{t_i \geq t_f\}} e^{-r(\tau^* - t_f)} (\tau^* - t_{i-1}) N \mathbb{1}_{\{t_{i-1} < \tau^* < t_i\}} \Big] \Bigg).\end{aligned} \tag{5.29}$$

See Section 3.8 for the derivation of equation (3.7) which reduces to equation (5.29) on setting $t = 0$. Note that the present time was equal to 0 when calculating the fair premium, we now wish to value the CDS at time t_f. The decomposition of this formula is very similar to the CDS premium decomposition, except that we do not include the protection seller's default payoff. We now present a Brownian bridge algorithm to value a CDS at time t_f after time of issue 0 i.e. the value of $\phi(0,\, t_f,\, T)$. Steps from 1 to 3e are the same as for the algorithm for the fair premiums, so we start from 3f:

3. For $j = 1$ to $K + 1$ (loop through all inter-jump intervals, including the final interval $[\tau_K, \tau_{K+1} = T]$):

   f. If $s_j \in [\tau_{j-1}, \tau_j]$, default has occurred. Then:
   - Let DiscDiffPremiums$(w) = [\tilde{p}(t_f, T) - \tilde{p}(0, T)] \Big(\sum_{i=1}^{Q(j)} e^{-r(t_i - t_f)} \times (t_i - t_{i-1}) + (s_j - t_{Q(j)}) lg_j(s_j) e^{-r(s_j - t_f)} \Big)$.
   - Exit the j loop, return to step 1, now $w = w + 1$, and perform another Monte Carlo simulation.

   g. If $x(\tau_j^+) \leq \ln b$, then the j^{th} jump has crossed the barrier and default has occurred. Then:
   - Let DiscDiffPremiums$(w) = [\tilde{p}(t_f, T) - \tilde{p}(0, T)] \Big(\sum_{i=1}^{H} e^{-r(t_i - t_f)} \times (t_i - t_{i-1}) + (\tau_j - t_H) e^{-r(\tau_j - t_f)} \Big)$.
   - Exit j loop , return to step 1, now $w = w + 1$, and perform another Monte Carlo simulation.

   h. If $x(\tau_j^+) > \ln b$, no default has occurred at the j^{th} jump. Proceed to the next inter-jump interval. This is done by returning to step 3, where now $j = j + 1$.
4. If default does not happen until the maturity T then:
   - Let DiscDiffPremiums$(w) = [\tilde{p}(t_f, T) - \tilde{p}(0, T)] \sum_{i=1}^{n} e^{-r(t_i - t_f)} \times (t_i - t_{i-1})$.
   - Exit j loop, return to step 1, now $w = w + 1$, and perform another Monte Carlo simulation.

5. If $w = W$, we have completed W Monte Carlo simulations. Calculate the Monte Carlo estimate for the fair premium of a CDS, with notional N:
- $\phi(0,\, t_f,\, T) = \frac{1}{W}\sum_{w=1}^{W}$ DiscDiffPremiums(w).

5.4.8 The Brownian Bridge Simulation Algorithm to Price Equity

In a similar fashion we can price the firm's equity using this Monte Carlo Brownian bridge method. Under our structural model framework the price of equity at time t is given by

$$E_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}\max(V_T - b_{\,t,\,T}\,,\,0)\mathbb{1}_{\{\tau^*>T\}}\middle|\mathscr{F}_t\right], \tag{5.30}$$

where $b_{t,T}$ is the barrier for the interval[22] $[t,\, T]$. If there are η number of outstanding shares then the price of a single share at time t is

$$S_t = \frac{E_t}{\eta}. \tag{5.31}$$

From (5.30) we can see that to obtain an analytical expression for the price of equity one needs a closed form solution for the risk-neutral probability of default. Under our assumed jump-diffusion framework this closed form solution does not exist. We will estimate the price of equity E_0 using Monte Carlo simulation. Specifically we will use the Monte Carlo Brownian bridge algorithm as we did for the CDS premiums. Thus for[23] $t = 0$ equation (5.30) can be written similarly to equation (5.21), with the inner conditional expectation equal to

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-rT}\max(V_T - b_{0,T}\,,\,0)\mathbb{1}_{\{\tau^*>T\}}|\mathscr{F}^*\right]. \tag{5.32}$$

The above conditional expectation (5.32) can be decomposed into[24]

$$\mathbb{1}_{\{J=0\}}\prod_{l=1}^{N(T)+1}\tilde{p}_l(\max(V_T - b_{t,T}\,,\,0)). \tag{5.33}$$

The steps of the Monte Carlo Brownian bridge algorithm will be as for the algorithm for CDS premiums, except for steps 3f, 3g, 4 and 5. These steps are altered to evaluate the equity payoff profile:

3. For $j = 1$ to $K+1$ (loop through all inter-jump intervals, including the final interval $[\tau_K, \tau_{K+1} = T]$):
   f. If $s_j \in [\tau_{j-1}, \tau_j]$, default has occurred. Then:
   - Let DiscEquityPayoff$(w) = 0$.
   - Exit the j loop, return to step 1, now $w = w+1$, and perform another Monte Carlo simulation.

   g. If $x(\tau_j^+) \le \ln b$, then the j^{th} jump has crossed the barrier and default has occurred. Then:
   - Let DiscEquityPayoff$(w) = 0$.
   - Exit j loop , return to step 1, now $w = w+1$, and perform another Monte Carlo simulation.

   h. If $x(\tau_j^+) > \ln b$, no default has occurred at the j^{th} jump. Proceed to the next inter-jump interval. This is done by returning to step 3, where now $j = j+1$.
4. If default does not happen until the maturity T then:
   - Let DiscEquityPayoff$(w) = e^{-rT}\max(e^{X_T} - b_{0,T}\,,\,0)$.
   - Exit j loop, return to step 1, now $w = w+1$, and perform another Monte Carlo simulation.

[22] Note that we use T to denote a generic maturity for the security under consideration. We do this for notational simplicity. The maturity of the CDS and equity need not be the same. When we are dealing with more than one security we will distinguish between the maturities.

[23] To correspond with the notation in the algorithm for the CDS premium, we will consider the price of equity at time 0. The algorithm is equivalent for $t > 0$, except that our starting time points differ.

[24] The notation here is the same as for the decomposition of the CDS premium formula (5.28).

5. If $w = W$, we have completed W Monte Carlo simulations. Calculate the Monte Carlo estimate for the fair premium of a CDS, with notional N:

- $E_0 = \frac{1}{W}\sum_{w=1}^{W} \text{DiscEquityPayoff}(w)$.

5.4.9 Numerical Results of the Brownian Bridge Simulation Algorithm

We perform the Monte Carlo Brownian bridge simulation algorithm on two examples for both CDS premiums and share prices. We numerically investigate the convergence rate of this algorithm for both CDS premiums[25] $\tilde{p}$ and share prices S. For the CDS premium examples we keep parameters V_0, b, R, r and T unchanged through the two examples. The values of these parameters are: $V_0 = 10000$, $b = 4000$, $R = 0.4$, $r = 0.05$ and $T = 5$ years. We choose to use different values for σ, λ, μ_A and σ_A for the different examples since these values together with maturity[26] determine the variance of the CDS premiums[27], and from expression (5.1) it can be seen that the convergence rate of our Monte Carlo estimate for $\tilde{p}$ is a function of $\mathbb{V}\text{ar}(\tilde{p})$. The values of these parameters for each of the CDS examples are as follows:

Example	σ	λ	μ_A	σ_A
CDS Example 1	0.3	1	-0.05	0.02
CDS Example 2	0.5	2	-0.07	0.1

Table 5.1: The different parameter values for the Monte Carlo Brownian bridge simulation procedure, to calculate the fair CDS premium.

For the share price examples we keep the values for the parameters V_0, b and r the same as for the above CDS cases. The number of shares equals $\eta = 1000$. We choose to use different values for T, σ, λ, μ_A and σ_A for the different share examples for the same reason as stipulated in the CDS case[28]. Their values for each example are:

Example	σ	λ	μ_A	σ_A	T
Share Example 1	0.3	1	-0.05	0.02	5
Share Example 2	0.5	2	-0.07	0.1	10

Table 5.2: The different parameter values for the Monte Carlo Brownian bridge simulation procedure, to calculate the share price.

There are two different set of parameters for CDS premiums and share prices: low volatility parameter set[29] (example 1) and high volatility parameter set (example 2). We name them so, since lower values of the parameter set produce a lower variance for V and higher values produces higher variance for V.

From expression (5.1), it can be seen that a higher number of simulations produces a more accurate Monte Carlo estimate. However, we need to find a balance between computation time and accuracy. We wish to determine how many simulations are required to achieve a subjectively reasonable convergence. It can be seen from Figure 5.1, that the low parameter set converges quicker than the high parameter set. For example 1 for CDS and equity, it can be seen that a reasonable convergence occurs at the 500 000 and 200 000 simulation mark, respectively. This can be seen from Plot (a.1) and Plot (b.1) from Figure 5.1. For example 2 for CDS and equity, a reasonable convergence occurs at the 700 000 and 300 000 simulation mark, respectively (see Plot (a.2) and Plot (b.2) from Figure 5.1). The dotted lines in Figure 5.1, represent the maximum and minimum values after the chosen simulation mark of reasonable convergence. We need to calibrate these parameters eventually, which is computationally intensive under the Monte Carlo framework.

[25] For this section of numerical analysis the premiums are paid twice a year: mid-year and at the end of the year.

[26] We do not use different maturities, since the standard maturity for a CDS contract is 5 years. Also note that maturity values are measured in years.

[27] These parameters determine the variances of all the stochastic elements in our model.

[28] Note that different maturities are used in the share price case since, unlike the CDS case, there is no standard maturity.

[29] The majority of the estimated parameters in the empirical tests done in Chapter 6, fall under the low volatility parameter set.

Thus we need to know how many simulations are needed under a parameter value set, which will achieve reasonable convergence.

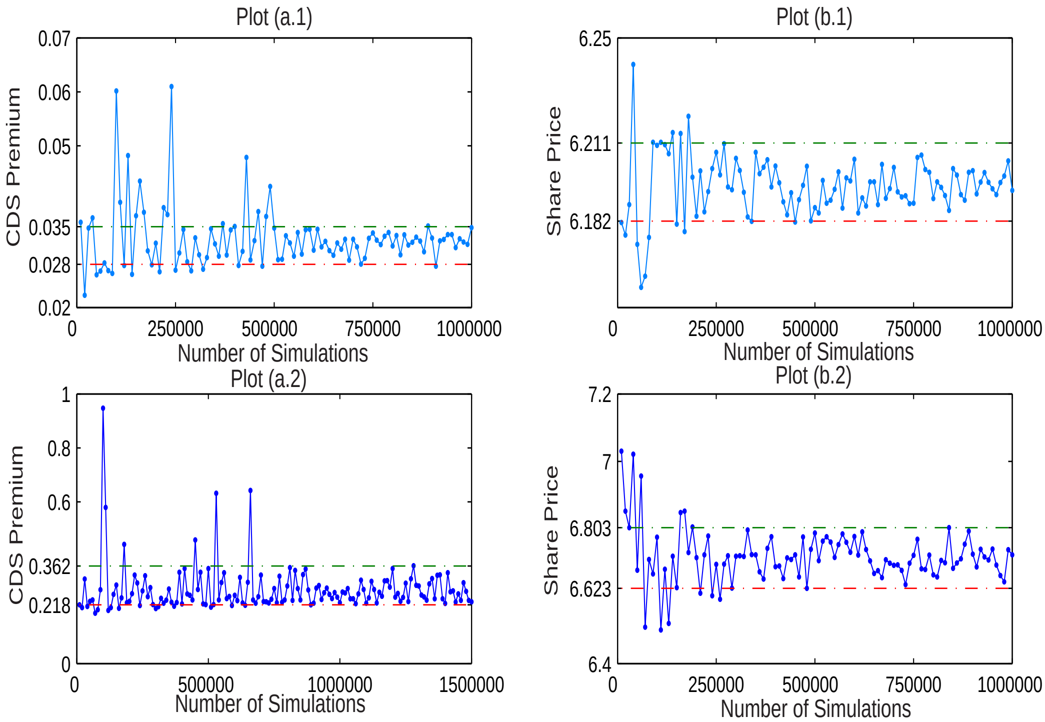

Figure 5.1: Plot (a.1) and (a.2) refer to CDS example 1 and 2 respectively. Plot (b.1) and (b.2) refer to share example 1 and 2 respectively.

We also compare the effectiveness of the Brownian bridge Monte Carlo algorithm against the standard Monte Carlo algorithm for a jump diffusion framework. We run MATLAB programs on a Pentium IV 3.20 GHz computer. The number of simulations for the particular example are specified in the tables. The number of simulations is chosen from the previous analysis on convergence. We know that the Brownian bridge method has zero bias, and therefore use its simulation result to obtain the true value of the CDS premium and share price. We use 10 million iterations to obtain the true vales. We record the bias between the simulated results and the true value and the computation CPU time (measured in seconds). The results are displayed in Table 5.3 and Table 5.4:

CDS Example 1 Comparisons

Method 500 000 simulations	**Bias (abs)**	**Bias (%)**	**CPU Time**
Standard Monte Carlo $\Delta t = 0.01$	0.0162295	-0.46206	1368.782
Standard Monte Carlo $\Delta t = 0.001$	0.0156827	-0.44649	57789.512
Brownian Bridge Monte Carlo	0.0017029	-0.04848	34.828

True value: 0.03512

CDS Example 2 Comparisons

Method 700 000 simulations	**Bias (abs)**	**Bias (%)**	**CPU Time**
Standard Monte Carlo $\Delta t = 0.01$	0.1616869	-0.54997	5092.516
Standard Monte Carlo $\Delta t = 0.001$	0.1587716	-0.54006	64758.438
Brownian Bridge Monte Carlo	0.0154644	-0.05261	64.414

True value: 0.29399

Table 5.3: Numerical comparison of the standard Monte Carlo method versus the Monte Carlo Brownian bridge method applied to pricing CDS premiums, for different parameter sets.

Share Example 1 Comparisons

Method 200 000 simulations	Bias (abs)	Bias (%)	CPU Time
Standard Monte Carlo $\Delta t = 0.01$	0.0555575	-0.00896	1202.251
Standard Monte Carlo $\Delta t = 0.001$	0.0445763	-0.00719	11277.391
Brownian Bridge Monte Carlo	0.0063537	-0.00102	12.313

True value: 6.19724

Share Example 2 Comparisons

Method 300 000 simulations	Bias (abs)	Bias (%)	CPU Time
Standard Monte Carlo $\Delta t = 0.01$	0.1138508	0.0171	2156.546
Standard Monte Carlo $\Delta t = 0.001$	0.0861704	0.01287	46151.515
Brownian Bridge Monte Carlo	0.0136171	0.00203	23.204

True value: 6.69756

Table 5.4: Numerical comparison of the standard Monte Carlo method versus the Monte Carlo Brownian bridge method applied to pricing share prices, for different parameter sets.

From the tabled results it can be seen that the standard Monte Carlo method becomes more accurate as our time discretisations Δt become smaller. However this significantly increases computation time. The proposed Brownian bridge algorithm is not only considerably faster but also more accurate as measured by the bias of the estimate. In both equity and CDS examples, both methods take more time to complete and are less accurate for the high volatility parameter set. The reason why the estimates are less accurate is because the variance increases with higher values of the parameter set, thus our estimates have a higher standard error. In order to increase accuracy one needs to increase the number of simulations. The Monte Carlo Brownian bridge method is superior in both accuracy and computation time for all scenarios tested.

5.5 Valuing Equity Options by a Monte Carlo Linear Regression Approach

It is difficult to price an equity option in our structural model framework because equity options are in fact compound options. In order to price this compound option we have to evaluate a conditional expectation, which is a non-trivial problem within a Monte Carlo framework.

Under our structural model framework the price of an equity call option at time 0, with strike K and maturity $T^* < T$, is given by[30]

$$\varphi_0 = \mathbb{E}^{\mathbb{Q}}\left[e^{-rT^*}\max(S_{T^*} - K,\, 0)\right]. \tag{5.34}$$

The stock price S_{T^*} itself is an option under our structural model framework. From (5.30) and (5.31) S_{T^*} can be written as

$$S_{T^*} = \eta^{-1}\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-T^*)}\max(V_T - b_{T^*,T}\,,\, 0)\mathbb{1}_{\{\tau^*>T\}}\middle|\mathscr{F}_{T^*}\right]. \tag{5.35}$$

Thus in order to evaluate (5.34) we need to evaluate the conditional expectation (5.35). The naive approach to obtain a Monte Carlo estimate for equation (5.34) would be to simulate a number of paths for the firm's asset value from 0 until the option maturity T^*, and then for each of these paths simulate again a number of firm's asset value paths from T^* to T (called resimulation). This resimulation procedure requires simulating a large number of sample paths which causes high computation time. For example if we use 100 000 simulated paths until T^* and then for each of these paths simulate another 100 000 paths from T^* to T, we end up simulating 10 000 000 000 paths. Figure 5.2 illustrates how the simulated paths[31] have to be simulated to evaluate the conditional expectation (5.35).

[30]In practice, following a bankruptcy announcement by a firm, trading in it's underlying stock is suspended by the exchange that lists the firm. When trading in the underlying stock has been halted, trading on the options is also halted. Equity option positions are usually then immediately closed out by the clearinghouse. For simplicity,

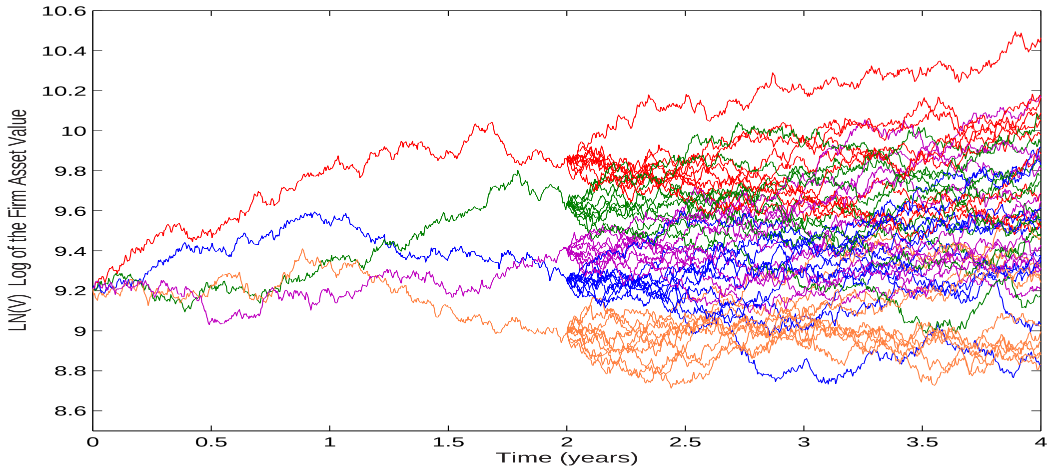

Figure 5.2: Example of simulated sample paths of a resimulation procedure.

Longstaff & Schwartz (2001) provide an efficient alternative to obtain a Monte Carlo estimate for options that are functions of conditional expectations such as (5.34). They name it the Least Squares Monte Carlo (LSM) approach. For our problem, the LSM approach involves simulating one set a sample paths from 0 until T. The conditional expectation (5.35) is estimated from the cross-sectional information of the simulated paths, at time T^* by using least squares regression. This is done by regressing the discounted payoffs[32] on a set of basis functions[33] of the various values of the firm's asset value at time T^*.

5.5.1 The LSM Algorithm

For example, say we need to find a Monte Carlo estimate for the following expression

$$\mathbb{E}^{\mathbb{Q}}\left[g\left(\mathbb{E}^{\mathbb{Q}}\left[h(V_T)|\mathscr{F}_{T^*}\right]\right)\right]. \tag{5.36}$$

The LSM approach uses least squares regression to approximate the inner conditional expectation within expression (5.36). Let

$$f = \mathbb{E}^{\mathbb{Q}}\left[h(V_T)|\mathscr{F}_{T^*}\right].$$

Note that since V is a Markov process, f is function of V_{T^*} and not of any past realisations of V. Let ω_i represent the i^{th} sample path of V. If we simulate W number of paths for V, we will have W number of realisations for f. We can write f as $f(V_{T^*}(\omega))$ to show its dependence on the realisations of $V_{T^*}(\omega)$. Longstaff and Schwartz assumed that f can be represented as a linear combination of a countable set of $\mathscr{F}_{T^*}$-measurable orthonormal[34] basis functions[35]. There are many possible choices of basis functions. Here are a few types: Laguerre, Hermite, Legendre, Chebyshev and Jacobi polynomials. Section 5.5.2 has numerical comparisons of two different choices of basis function: Laguerre, and a simple order 3 polynomial function. Let $\{L_n\}$ represent the general form of our choice of basis functions. With this specification, $f(V_{T^*})$ can be represented as

$$f(V_{T^*}) = \sum_{j=0}^{\infty} \beta_j L_j(V_{T^*}),$$

where the coefficients β_j are constants.

To implement the LSM approach, we approximate $f(V_{T^*})$ using the first $M < \infty$ basis functions and denote this truncated approximation by $f_M(V_{T^*})$. If we simulate W number of paths of V from

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we will assume that the equity options are closed out at the options maturity.

[31] The parameters used to generate Figure 5.2 are: $V_0 = 10000$, $r = 0.05$, $\sigma = 0.2$, $\lambda = 1$, $\mu_A = -0.05$, $\sigma_A = 0.2$, $T^* = 2$ and $T = 4$. The number of simulated paths are 5 and 10 for the interval [0, 2] and [2, 4], respectively.

[32] For our case, the discounted payoffs are $e^{-r(T-T^*)}\max(V_T - b_{T^*,T}, 0)\mathbb{1}_{\{\tau^*>T\}}$.

[33] This will be explained in Section 5.5.2.

[34] See Fraleigh & Beauregard (1995) for the definition of orthonormal functions.

[35] Some integrability conditions must hold for this assumption to hold: see Longstaff & Schwartz (2001) for further explanation on this assumption.

0 until T, we can then estimate $f_M(V_{T^*})$ by regressing realised values of $h(V_T(\omega_i))$ on $f_M(V_{T^*}(\omega_i))$ for each $i = 1, \ldots, W$. This involves finding the least squares estimates for β_j, $j = 1, \ldots, M$, which we will denote as $\hat{\beta}_j$. By substituting $\hat{\beta}_j$ for β_j in $f_M(V_{T^*})$, we get a least squares approximation for $f_M(V_{T^*})$, which we will denote by $\hat{f}_M(V_{T^*})$[36]. The conditional expectation function f is estimated by $\hat{f}_M(V_{T^*})$. We then estimate (5.36) by averaging the realisations $g\left[\hat{f}_M(V_{T^*}(\omega_i))\right]$, i.e.

$$W^{-1}\sum_{i=1}^{W} g\left[\hat{f}_M(V_{T^*}(\omega_i))\right].$$

In some cases it may be more efficient to use alternative regression techniques, such as weighted least squares or generalised least squares in estimating the conditional expectation function. For example, if the process V has volatility that is a function of V, then the residuals from regression may be heteroskedastic. In this case these alternative regression techniques may have advantages[37]. Longstaff and Schwartz point out that numerical tests indicate that the results from the LSM algorithm are remarkably robust to the choice of basis functions. They also notice that few basis functions are needed to closely approximate the conditional expectation function (they use $M = 3$ for their numerical analysis). It is important to note the numerical implications of the choice basis functions. It could lead to computation overflows depending on the how large the realised values of V_{T*} are. This can be resolved by normalising V.

For our case[38],

$$h(V_T) = e^{-r(T-T^*)}\max(V_T - b_{T^*,T}\,,\,0)\mathbb{1}_{\{\tau^* > T\}}$$

and

$$g = e^{-rT^*}\max(h(V_T) - K,\,0).$$

We calculate all our equity option prices φ_0, using the LSM algorithm, with Laguerre polynomials as our choice of basis functions (with $M = 3$).

5.5.2 Numerical Comparison of Basis Functions

In this section we compare two choices of basis function to illustrate that this choice has a negligible impact on our numerical results. We chose a Laguerre polynomial

$$\beta_0 + \beta_1 e^{-V/2} + \beta_2 e^{-V/2}(1-V) + \beta_3 e^{-V/2}(1 - 2V + V^2/2), \tag{5.37}$$

and a simple order 3 polynomial

$$\beta_0 + \beta_1 V + \beta_2 V^2 + \beta_3 V^3, \tag{5.38}$$

for our two choices of basis function. We then apply both these basis functions to price equity options using the described Longstaff and Schwartz method.[39]. Figure 5.3 illustrates the convergence and variability of the results achieved from the two basis functions. The strike price is $K = 3$ for all the plots. The parameters used for Plot (a.1) and (b.1) are equivalent to those used in Section 5.4.9 for equity example 1, the equity option maturity is $T^* = 0.25$. Similarly Plots (a.2) and (b.2) have equivalent parameters to equity example 2, with option maturity $T^* = 0.5$.

It can be seen from Figure 5.3 that both basis functions produce very similar results. There is a significant difference when the high volatility parameter set is used. From Figure 5.3 it can be seen that the simple polynomial basis functions produces estimates with a higher variance. The low and high parameter set examples converge reasonably after the 500 000 and 600 000 mark respectively. The dotted lines represent the maximum and minimum values after the respective reasonable convergence mark.

[36] Note that $\hat{f}_M(V_{T^*})$ converges in probability to $f_M(V_{T^*})$ as W tends towards to infinity.

[37] See Longstaff & Schwartz (1995) for more details.

[38] Note that the function $h(V_T)$ is also a function of τ^*, which depends on other realisations of V. This indicator can easily be evaluated within the LSM algorithm by noting if any simulated paths cross the barrier.

[39] We use the Monte Carlo Brownian bridge algorithm to simulate sample paths.

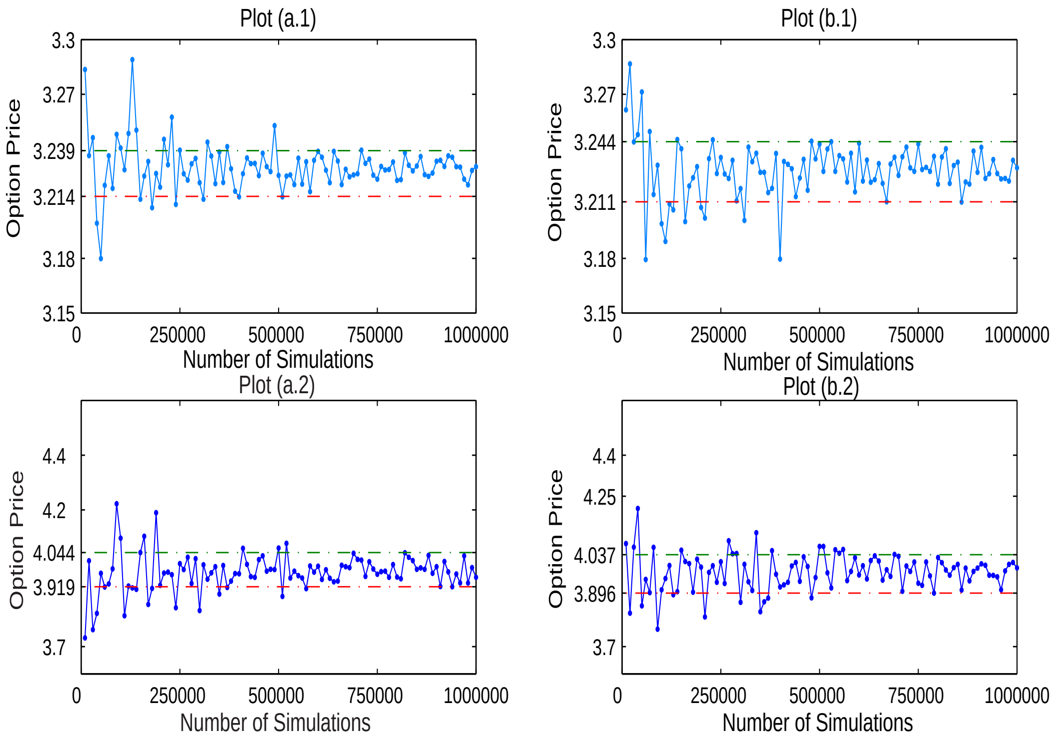

Figure 5.3: Plot (a.1) and (a.2) correspond to the Longstaff Schwartz method using the Laguerre basis function, with different parameter sets. Plot (b.1) and (b.2) correspond to the Longstaff Schwartz method using the simple polynomial basis function, with different parameter sets.

5.6 Estimation of parameters

The parameters in our model that need to be estimated are $\boldsymbol{\theta} = (V_0, \sigma_V, b, \lambda, \mu_A, \sigma_A)$. We choose the *calibration* method in order to estimate the parameter set $\boldsymbol{\theta}$, and the *simple* estimating method for the initial guess for our calibration procedure[40]. We choose the calibration method since the other estimation methods need analytical solutions for the share price. Our model does not omit closed-form solutions for the share price. We are going to estimate our parameters $\boldsymbol{\theta}$ by calibrating our model prices to market prices. We are going to use the sum of the squared differences[41] a as measure of how well our calibration procedure is performing. Define

$$g(\boldsymbol{\theta}) = \sum_{i=1}^{n} \left(U_i(\boldsymbol{\theta}) - U_i^{obs}\right)^2 , \qquad (5.39)$$

where U_i^{obs} is the observed market price of the ith calibrating instrument[42], and $U_i(\theta)$ is the theoretical price given by the model for this instrument. We denote the estimates for our parameters by $\hat{\boldsymbol{\theta}}$. Our objective is to search for the values of the parameters that minimise $g(\boldsymbol{\theta})$ i.e.

$$\hat{\boldsymbol{\theta}} = \arg\inf_{\boldsymbol{\theta}} \sum_{i=1}^{n} \left(U_i(\boldsymbol{\theta}) - U_i^{obs}\right)^2 . \qquad (5.40)$$

We will use the Nelder & Mead (1965) direct search method to locate the parameters that minimise $g(\boldsymbol{\theta})$. This is a standard method to find the minimum. We use this method since it is simple to implement in our framework. The following sections on the Nelder-Mead simplex method is derived from the paper Lagarias et al. (1998).

[40] See Section 4.4 for details on the different available estimating methods for structural models.
[41] See Hull (2003) for alternative more complex measures.
[42] In our case this would be either shares, CDS premiums, or share options of different strikes and maturities. The number of market instruments used will be equal to the number of parameters we are estimating.

5.6.1 The Nelder-Mead Simplex Method

Nelder & Mead (1965) developed a direct search method[43] that aims to find the minimum[44] of a function $g(\boldsymbol{\theta})$ of k variables $\boldsymbol{\theta} = (\theta_1, \theta_2, ..., \theta_k)$. It depends solely on comparing the function values at the $(k+1)$ vertices of a general simplex[45], and replacing the vertex with the highest value achieved with another created vertex with a lower function value. We will denote a simplex by $\triangle$. This algorithm terminates when the simplex structure contracts sufficiently. It is necessary to specify a convergence criterion[46]. This simplex adapts itself to the local landscape of the function $g(\boldsymbol{\theta})$, by using four operations: *reflection*, *expansion*, *contraction* and *shrinkage*. Four parameters relating to these operations must firstly be specified to define the Nelder-Mead method: coefficients of *reflection* ρ, *expansion* χ, *contraction* γ and *shrinkage* κ. According to Nelder & Mead these coefficients must satisfy the following inequalities

$$\rho > 0, \quad \chi > 1, \quad 0 < \gamma < 1 \quad \text{and} \quad 0 < \kappa < 1.$$

The standard values for these coefficients used in the Nelder-Mead algorithm are:

$$\rho = 1, \quad \chi = 2, \quad \gamma = \frac{1}{2} \quad \text{and} \quad \kappa = \frac{1}{2}.$$

5.6.2 The Nelder-Mead Algorithm

At the beginning of the j^{th} iteration, $j \geq 0$ of the Nelder-Mead algorithm, a new simplex is given by $\triangle_j$, along with its $k+1$ vertices, each of which are points in $\mathbb{R}^k$. The first step at the j^{th} iteration is to order and label the vertices as $\boldsymbol{\theta}_{(1)}^{(j)}, \boldsymbol{\theta}_{(2)}^{(j)}, \ldots, \boldsymbol{\theta}_{(k+1)}^{(j)}$, such that

$$g\left(\boldsymbol{\theta}_{(1)}^{(j)}\right) \leq g\left(\boldsymbol{\theta}_{(2)}^{(j)}\right) \leq \cdots \leq g\left(\boldsymbol{\theta}_{(k+1)}^{(j)}\right).$$

For notational brevity, let $g\left(\boldsymbol{\theta}_{(i)}^{(j)}\right) = g_{(i)}^{(j)}$. The j^{th} iteration of this algorithm generates another set of $k+1$ vertices that defines a different simplex for the next iteration, the $(j+1)^{th}$ iteration, so that $\triangle_j \neq \triangle_{j+1}$. The following steps illustrate a generic iteration. We omit the superscript j to simplify the notation:

1. **Order.** The $k+1$ vertices must be ordered to satisfy to $g_{(1)} \leq g_{(2)} \leq ... \leq g_{(k+1)}$.
2. **Reflect.** Calculate the *reflection point* $\boldsymbol{\theta}_r$ from

   $$\boldsymbol{\theta}_r = \bar{\boldsymbol{\theta}} + \rho(\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}_{(k+1)})$$

   where $\bar{\boldsymbol{\theta}} = \sum_{i=1}^{k} \boldsymbol{\theta}_{(i)}/k$ is the centroid of the vertices excluding $\boldsymbol{\theta}_{(k+1)}$. Substitute the above *reflection point* into the function g and evaluate $g_r = g(\boldsymbol{\theta}_r)$. If $g_{(1)} \leq g_r < g_{(k)}$, replace $\boldsymbol{\theta}_{(k+1)}$ with the *reflected point* $\boldsymbol{\theta}_r$ and skip to the last step (step 6) of the iteration. Otherwise proceed on to the next step.
3. **Expand.** If $g_r < g_{(1)}$, calculate the *expansion point* $\boldsymbol{\theta}_e$ from

   $$\boldsymbol{\theta}_e = \bar{\boldsymbol{\theta}} + \chi(\boldsymbol{\theta}_r - \bar{\boldsymbol{\theta}})$$

   Substitute the above *expansion point* into the function g and evaluate $g_e = g(\boldsymbol{\theta}_e)$. If $g_e < g_r$, replace $\boldsymbol{\theta}_{(k+1)}$ with the *expanded point* $\boldsymbol{\theta}_e$ and skip to last step (step 6) of the iteration. Otherwise proceed on to the next step.

[43] The expression *direct search method* means that the gradient (derivative) of the function is not used to determine the minimum of the function.

[44] Note that this algorithm finds local minima. To find the global minimum you would to have run this algorithm several times to cover the whole domain.

[45] A simplex is a geometric figure in k-dimensions of non-zero volume.

[46] The convergence criterion is usually a measure of the distance of the vertices from a central point.

4. **Contract.** If $g_r \geq g_{(k)}$, perform a *contraction* with either $\boldsymbol{\theta}_r$ or $\boldsymbol{\theta}_{(k+1)}$ depending on which parameters gives us the lowest function value
   a. **Outside** . If $g_{(k)} \leq g_r < g_{(k+1)}$ calculate the *outside contraction point* $\boldsymbol{\theta}_{oc}$ from

$$\boldsymbol{\theta}_{oc} = \bar{\boldsymbol{\theta}} + \gamma(\boldsymbol{\theta}_r - \bar{\boldsymbol{\theta}})$$

   Substitute the above *outside contraction point* into the function g and evaluate $g_{oc} = g(\boldsymbol{\theta}_{oc})$. If $g_{oc} \leq g_r$, replace $\boldsymbol{\theta}_{(k+1)}$ with the *outside contracted point* $\boldsymbol{\theta}_{oc}$ and skip to last step (step 6) of the iteration. Otherwise proceed on to the next step.
   b. **Inside** . If $g_r \geq g_{(k+1)}$ calculate the *inside contraction point* $\boldsymbol{\theta}_{ic}$ from

$$\boldsymbol{\theta}_{ic} = \bar{\boldsymbol{\theta}} - \gamma(\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}_{k+1})$$

   Substitute the above *inside contraction point* into the function g and evaluate $g_{ic} = g(\boldsymbol{\theta}_{ic})$. If $g_{ic} < g_{(k+1)}$, replace $\boldsymbol{\theta}_{(k+1)}$ with the *inside contracted point* $\boldsymbol{\theta}_{ic}$ and skip to last step (step 6) of the iteration. Otherwise proceed on to the next step.
5. **Shrink.** Construct k new vertices by the following equation

$$\boldsymbol{\theta}_i = \boldsymbol{\theta}_{(1)} + \kappa(\boldsymbol{\theta}_{(i)} - \boldsymbol{\theta}_{(1)}),$$

   $i = 2, ..., n+1$. The new set of unordered vertices for the new simplex consists of $\boldsymbol{\theta}_{(1)}, \boldsymbol{\theta}_2, ..., \boldsymbol{\theta}_{k+1}$. Note that we keep the vertex $\boldsymbol{\theta}_{(1)}$, which produces the lowest value when substituted in g. Evaluate g at these vertices, and proceed to the next step.

6. **Stopping Criterion.** Evaluate the following

$$\sum_{i=1}^{k+1} (g_i - \bar{g})^2 / k.$$

   where $\bar{g} = \sum_{i=1}^{k+1} g_i/(k+1)$. The above measure indicates how close the vertices are amongst themselves. When the measure is below the preset value, the local minimum has been reached. In this case stop the algorithm, otherwise start another iteration beginning at step 1.

The result of each iteration is a new simplex with either a single new vertex that replaces $\boldsymbol{\theta}_{(k+1)}$ or, if a *shrink* is performed, a new set of k vertices along with $\boldsymbol{\theta}_{(1)}$. The variables above are assumed to be individually unconstrained. However, this algorithm also holds for individually constrained variables. One method to deal with constraints on the variables, is by assigning a large function value to g when the variables violate their constraints. It will be generally inaccessible to reach an actual minimum at these constraints, though arbitrarily close approaches could be made to it.

To initiate the Nelder-Mead algorithm we need an initial guess for $\boldsymbol{\theta}$. We use the simple estimating method for this. The initial asset value V_0 is estimated with the most recent total asset value figure in the balance sheet. The asset volatility σ is estimated by calculating the standard deviation of the percentage changes in the firm asset value V_t. The default barrier is estimated as the midpoint between the long term liabilities and the current liabilities of the firm (Crosbie & Bohn (2003)). We calculate the initial estimate for λ by counting the number of non-marginal percentage changes per year. We consider a non-marginal change to be a jump greater than 2σ. We estimate μ_A and σ_A by calculating the mean and standard deviation of the natural logarithm of the percentage changes, respectively.

No assumptions are made on the surface of the function g except that it is continuous and it has a unique minimum around the area of the search. A general problem common with all minimisation problems is the false convergence to a point other than the minimum. Refining the convergence criterion could just add needless evaluations and it could still end up being ineffective. A method to overcome this is to continue after the first convergence for a specified number of iterations, and then evaluate the convergence criterion again. If the second set of functions meet the convergence criterion, then compare the two sets of *converged* function values. If they are sufficiently close, convergence is accepted. Note that since we are working in a jump-diffusion framework, we have an incomplete market which has several equivalent martingale measures. Thus there exist several possible parameter set values that will achieve the same prices[47]. Cont & Tankov (2004*b*) studied calibration under a jump-diffusion framework. They extend the calibration measure (5.39) to include a relative entropy term. A possible criterion[48] to choose a set of parameter values is to choose the parameter set values that do not oscillate excessively when a small change in the market prices occurs.

5.7 Discussion

Our jump-diffusion framework does not admit analytical solutions for the values of a CDS, equity or equity options. We use Monte Carlo methods to estimate these values. In this chapter we introduce Monte Carlo simulation methods, and a few variance reduction methods. We use the Brownian bridge Monte Carlo method (Metwally & Atiya (2002)) to price these financial securities. The advantages of this method are that it is computationally faster and has a smaller bias than the standard Monte Carlo method. We numerically investigated the convergence rate of this method for different parameter sets. In order to calculate the price of an equity option, under our framework, we need to calculate the conditional expectation (5.35). We do this by using the Monte Carlo linear regression method developed by Longstaff & Schwartz (2001). Lastly, in this chapter we discussed how the parameters of our model are estimated. We use a calibration procedure to estimate the model's parameters. We discussed the Nelder-Mead minimising algorithm, which is implemented in our calibration procedure in order to find the parameters that minimise the calibration measure (5.39).

[47]See Appendix A.
[48]See Cont & Tankov (2004*b*).

Chapter 6

Numerical Analysis

6.1 Introduction

In this chapter we summarise the effects of credit risk on the values of credit default swaps, stocks and stock options. An explanation of hedging and an examination of the difference in hedging in complete and incomplete markets, is included. We focus only on one particular hedging method, termed *delta hedging*. A hedging efficiency measure is constructed, so the empirical hedging results can be tested and compared. An outline of the procedure used to perform the hedges is provided. To begin with, we test the hedging mechanism on simulated data, and then apply the hedges to market data. The results are presented, and a discussion of the results follows. Finally, we draw a conclusion from the results.

6.2 Effects of Credit Risk on CDS, Equity and Equity Derivatives

Risk is related to the variability of the future value of a financial position, due to market changes or more generally to uncertain events. Credit risk is the uncertainty of the ability of an obligor to honour payment obligations[1]. Credit risk of a firm affects the value of all the securities issued by the firm, debt and equity. As a firm's credit risk increases, the firm's financial health decreases (its propensity to default increases). In this case investors in the firm will become sceptical of realising a return on their investment, and so the value of their investment will decrease.

The value of a CDS contract, with maturity T, at some time $t_f \geq t$, after it was initially issued at time t, is equal to[2]

$$\begin{aligned}\phi(t,t_f,T) =& N\left(\tilde{p}_{t_f,T}-\tilde{p}_{t,T}\right)\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n}\mathbb{1}_{\{t_i\geq t_f\}}\left(e^{-r(t_i-t_f)}(t_i-t_{i-1})\mathbb{1}_{\{\tau^*>t_i\}}\right.\right.\\ &\left.\left.+\,e^{-r(\tau^*-t_f)}(\tau^*-t_{i-1})\mathbb{1}_{\{t_{i-1}<\tau^*<t_i\}}\right)\right],\end{aligned} \tag{6.1}$$

where

$$\tilde{p}_{t_f,T}=\frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-r(\tau^*-t_f)}(1-R_{\tau^*})\mathbb{1}_{\{\tau^*\leq T\}}\right]}{\mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n}\mathbb{1}_{\{t_i\geq t_f\}}\left(e^{-r(t_i-t)}(t_i-t_{i-1})\mathbb{1}_{\{\tau^*>t_i\}}+e^{-r(\tau^*-t_f)}(\tau^*-t_{i-1})\mathbb{1}_{\{t_{i-1}<\tau^*<t_i\}}\right)\right]}. \tag{6.2}$$

By construction, credit default swaps reflect a firm's credit risk, since the value of a CDS $\phi(t,t_f,T)$ is primarily affected by all the major constituents of credit risk *arrival risk*, *timing risk* and *recovery risk*. At time of issue t the CDS spread $\tilde{p}_{t,T}$ is chosen such that the value of the CDS contract is equal to 0. At a future time point $t_f > t$, the credit risk of the reference entity (the firm which the CDS contract was written on) may have changed. Thus the CDS spread for providing protection from the future time point t_f until the same maturity T will change if the market's perception of the reference entity's credit risk has also changed. If at a future time t_f the credit risk has

[1] See Section 3.2, for a detailed description of credit risk.

[2] See Section 3.8 for the derivation of equations (6.1) and (6.2).

increased, the probability of default would also have *increased*, thus the CDS spread $\tilde{p}_{t,T}$ also *increases*. This can be seen from Equation (6.2). If the spread for time interval $[t_f, T]$ is higher than the spread for the time interval $[t, T]$, i.e. $\tilde{p}_{t,T} < \tilde{p}_{t_f,T}$, then the original contract issued at t, has a positive value for the CDS buyer and a negative value for the CDS seller at time t_f. The reason for this is that the spread the CDS buyer is paying for default protection, is cheaper than the spread being offered at time t_f. The CDS seller has a negative value since he is receiving a spread lower than the current market spread.

The main objective of this dissertation is to hedge one's exposure on a CDS position, with an equity position (either using an equity option or the underlying equity). We chose a structural credit risk model since it enables us to link the value of a CDS with the value of equity via the firm's asset value V. The default time τ^* in time interval $[t_f, T]$ is modelled as

$$\tau^* = \inf\{s \geq 0 : V_s \leq b_{t_f,T}\}.$$

Let $t \leq t_f \leq t_g \leq T$. Then under our model assumptions, the t_f–time value of a share[3] is:

$$S_{t_f} = \eta^{-1}\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t_f)}\max(V_T - b_{t_f,T},\, 0)\mathbb{1}_{\{\tau^*>T\}}\Big|\mathscr{F}_{t_f}\right]. \tag{6.3}$$

A European share call option[4] maturing at $t_g \geq t_f$, has value:

$$\varphi_{t_f} = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(t_g-t_f)}\max(S_{t_g} - K,\, 0)\Big|\mathscr{F}_{t_f}\right], \tag{6.4}$$

and a European share put option maturing at $t_g \geq t_f$, has value:

$$\overline{\varphi}_{t_f} = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(t_g-t_f)}\max(K - S_{t_g},\, 0)\Big|\mathscr{F}_{t_f}\right]. \tag{6.5}$$

If the credit risk of a firm *increases*, the probability of default *increases*, and so the firm's share price S *decreases*. This can be seen from Equation (6.3). Thus an *increase* in credit risk will *decrease* the value of a call option and *increase* the value of a put option. We summarise the effects of credit risk on the value of a long position in a CDS, a share and a European share option (call & put), in the following table[5]:

Credit Risk	**CDS** (ϕ)	**Share** (S)	**Call Option** (φ)	**Put Option** ($\overline{\varphi}$)
Increase	Decrease	Decrease	Decrease	Increase
Decrease	Increase	Increase	Increase	Decrease

Table 6.1: The effects of credit risk on the value of a long position in a CDS, a share and a European share option (call & put).

6.3 Hedging

A hedge is a trading strategy $\boldsymbol{\psi}_t$ that is designed to reduce risk. Let $\xi(V_t, t)$ denote the price at time t of a contingent claim that is dependent on the firm's asset value process V_t. To hedge this contingent claim we need to find a trading strategy $\boldsymbol{\psi}_t$ that reduces the variability of the future values of the contingent claim. In a complete market a *perfect hedging* strategy can be constructed that eliminates all the risk of the contingent claim, since in a complete market all contingent claims can be replicated exactly. Thus by taking an exact opposite position in the replication strategy all the uncertainty of the future values of the contingent claim will be completely eliminated. However, in an incomplete market it is impossible to replicate a contingent claim exactly. Thus it is impossible to hedge perfectly in an incomplete market. To hedge in an incomplete market, one needs to find a hedging strategy that minimises a specified risk measure[6]. We now provide an example of a perfect hedging strategy in a complete market, and then apply this hedging strategy to an incomplete market.

[3]See Section 4.5 for the derivation of the share formula.

[4]See Section 5.5 for the derivation of the share option pricing formula.

[5]For a visualisation of the relationship of credit risk on these different financial products, see Figure 6.1. Note that in these graphs, we model credit risk as the distance of the firm's asset value V from the default barrier b. If b is constant, the higher V is, the lower the credit risk; and the higher V is, the higher the credit risk.

[6]See Cont & Tankov (2004*b*) and McWalter (2006) for a detailed analysis of hedging in an incomplete market.

Example: Complete Market Hedging

Suppose we are modelling under a structural framework, and the assumptions of this structural model are equivalent to the assumptions in Section 4.5, except for the assumption of the dynamics of the firm's asset value V_t. In this example let, V_t follow a geometric Brownian motion (GBM) process

$$\mathrm{d}V_t/V_t = \mu_V\,\mathrm{d}t + \sigma\,\mathrm{d}W_t. \tag{6.6}$$

Using Girsanov's Theorem, it can be seen that the model admits a unique equivalent martingale measure[7] $\mathbb{Q}$. This implies the market is complete (see Theorem 2.2.4). Suppose we construct a self-financing portfolio, consisting of a single contingent claim and ψ_t units of the underlying asset V_t. The value of the contingent claim and the portfolio at time t is denoted by $\xi_t := \xi(V_t, t)$ and ϑ_t, respectively. Mathematically we can represent the value of the portfolio as

$$\vartheta_t = \xi_t + \psi_t\,V_t. \tag{6.7}$$

Applying Itô's lemma (see Appendix A.1) we can express the portfolio ϑ_t by the following stochastic differential equations

$$\begin{aligned}\mathrm{d}\vartheta_t &= \mathrm{d}\xi_t + \psi_t\,\mathrm{d}V_t \\ &= \frac{\partial \xi_t}{\partial t}\,\mathrm{d}t + \frac{\partial \xi_t}{\partial V_t}\,\mathrm{d}V_t + \frac{\sigma^2 V_t^2}{2} \times \frac{\partial^2 \xi_t}{\partial V_t^2}\,\mathrm{d}t + \psi_t\,\mathrm{d}V_t. \end{aligned} \tag{6.8}$$

There are two stochastic terms in Equation (6.8): $\frac{\partial \xi_t}{\partial V_t}\,\mathrm{d}V_t$ and $\psi_t\,\mathrm{d}V_t$. By selecting the asset weight ψ_t to be equal to $-\frac{\partial \xi_t}{\partial V_t}$, we are able to eliminate all the randomness in the portfolio's dynamics. If

$$\psi_t = -\frac{\partial \xi_t}{\partial V_t},$$

then

$$\mathrm{d}\vartheta_t = \left(\frac{\partial \xi_t}{\partial t} + \frac{\sigma^2 V_t^2}{2} \times \frac{\partial^2 \xi_t}{\partial V_t^2}\right)\mathrm{d}t. \tag{6.9}$$

Equation (6.9) has no diffusion term. By letting the asset weight $\psi_t = -\frac{\partial \xi_t}{\partial V_t}$ we have hedged the contingent claim's risk perfectly. Furthermore, the portfolio's return must be equal to the risk-free rate to avoid arbitrage opportunities:

$$\begin{aligned} \mathrm{d}\vartheta_t &= r\vartheta_t\,\mathrm{d}t \\ \Rightarrow \quad \vartheta_t &= \vartheta_0 e^{rt} \\ \Rightarrow \quad \xi_t + \psi_t\,V_t &= (\xi_0 + \psi_0\,V_0)\,e^{rt} \\ \Rightarrow \quad \xi_t &= (\xi_0 + \psi_0\,V_0)\,e^{rt} - \psi_t\,V_t, \end{aligned} \tag{6.10}$$

where

$$\psi_t = -\frac{\partial \xi_t}{\partial V_t}.$$

On the right-hand side of Equation (6.10) we have the the replicating portfolio for the contingent claim: $(\xi_0 + \psi_0\,V_0)$ units invested at the risk-free rate and a short position of ψ_t units of the underlying asset V_t.

Example: Incomplete Market Hedging

Suppose now, we are modelling under a structural framework, and the assumptions of this structural model are equivalent to the assumptions in Section 4.5. Then the firm's asset value V follows a jump-diffusion process

$$\mathrm{d}V_t/V_{t^-} = \mu_V\,\mathrm{d}t + \sigma\,\mathrm{d}W_t + Y_t\,\mathrm{d}N_t. \tag{6.11}$$

See Section 4.5, for the interpretation of the parameters in Equation (6.11). Using Girsanov's Theorem, it can be seen that the model does not admit an unique equivalent martingale measure[8]

[7] From Theorem A.2.1 it can be seen the Radon-Nikodým derivative ρ_t is unique, where $Y_t = 0$, $N_t = 0$ and $\Theta = \frac{u_V - r}{\sigma}$.

[8] From Theorem A.2.1 it can be seen the Radon-Nikodým derivative ρ_t is not unique. Different combinations of Θ and β characterise a different equivalent martingale measure $\mathbb{Q}$.

$\mathbb{Q}$. This implies that the market is incomplete (see Theorem 2.2.4). Let us examine the dynamics of the portfolio

$$\vartheta_t = \xi(V_t,\, t) + \psi_t\, V_t$$

where

$$\psi_t \;= -\frac{\partial \xi_t}{\partial V_t},$$

and compare it to the previous portfolio's dynamics when V_t followed a GBM process. Applying Itô's lemma (see Appendix A.1) we can express the portfolio ϑ_t by the following stochastic differential equations

$$\begin{aligned}
\mathrm{d}\vartheta_t &= \mathrm{d}\xi_t - \frac{\partial \xi_t}{\partial V_t}\mathrm{d}V_t \\
&= \left(\frac{\partial \xi_t}{\partial t} + \mu_V V_{t-}\frac{\partial \xi_t}{\partial V_t} + \frac{\sigma^2 V_{t-}^2}{2} \times \frac{\partial^2 \xi_t}{\partial V_t^2}\right)\mathrm{d}t + \sigma V_{t-}\frac{\partial \xi_t}{\partial V_t}\,\mathrm{d}W_t + [\xi(V_{t-}+\Delta V_t) - \xi(V_{t-})] \\
&\quad - \mu_V V_{t-}\frac{\partial \xi_t}{\partial V_t}\,\mathrm{d}t - \sigma V_{t-}\frac{\partial \xi_t}{\partial V_t}\,\mathrm{d}W_t - \frac{\partial \xi_t}{\partial V_t}\Delta V_t \\
&= \left(\frac{\partial \xi_t}{\partial t} + \frac{\sigma^2 V_{t-}^2}{2} \times \frac{\partial^2 \xi_t}{\partial V_t^2}\right)\mathrm{d}t + [\xi(V_{t-}+\Delta V_t) - \xi(V_{t-})] - \frac{\partial \xi_t}{\partial V_t}\Delta V_t, \qquad (6.12)
\end{aligned}$$

where

$$\Delta V_t = V_t - V_{t-}$$

and the time of jumps have a Poisson distribution with intensity parameter λ. There are two stochastic terms in Equation (6.12): $[\xi(V_{t-} + \Delta V_t) - \xi(V_{t-})]$ and $\frac{\partial \xi_t}{\partial V_t}\Delta V_t$. Thus the portfolio still has random behaviour due to the jump component of V. The two stochastic jump terms in Equation (6.12) can not be eliminated since portfolio mixing is a linear operation and the contingent claim price is not necessarily a linear function of V_t (Merton (1976)). By selecting the asset weight $\psi_t := -\frac{\partial \xi_t}{\partial V_t}$, we are able to eliminate the randomness caused by the marginal fluctuations in V_t, modelled by W_t (diffusion risk), but we are unable to eliminate the risk of the rare non-marginal jumps[9].

The firm's asset value is not a publicly traded asset. Thus to hedge contingent claim $\xi(V_t,\, t)$ we need to use another publicly traded financial instrument whose value depends on the firm's asset value V_t. Let $\xi_t^1 = \xi^1(V_t,\, t)$ denote the time-t value of another contingent claim whose value depends on V_t. Suppose we are still under the jump-diffusion framework. Let us construct a portfolio at time t, consisting of a single unit of contingent claim ξ_t (which we want to hedge) and ψ_t units of another contingent claim ξ_t^1 (which will serve as our hedging instrument)

$$\vartheta_t = \xi(V_t,\, t) + \psi_t\, \xi^1(V_t,\, t). \qquad (6.13)$$

Let[10]

$$\psi_t \;:= -\frac{\partial \xi_t}{\partial \xi_t^1} = -\frac{\partial \xi_t}{\partial V_t}\,\frac{\partial V_t}{\partial \xi_t^1}.$$

Then applying Itô's lemma (see Appendix A.1) we can express the portfolio ϑ_t by the following stochastic differential equations

$$\begin{aligned}
\mathrm{d}\vartheta_t &= \mathrm{d}\xi_t - \frac{\partial \xi_t}{\partial \xi_t^1}\mathrm{d}\xi_t^1 \\
&= \left(\frac{\partial \xi_t}{\partial t} + \mu_V V_{t-}\frac{\partial \xi_t}{\partial V_t} + \frac{\sigma^2 V_{t-}^2}{2} \times \frac{\partial^2 \xi_t}{\partial V_t^2}\right)\mathrm{d}t + \sigma V_{t-}\frac{\partial \xi_t}{\partial V_t}\,\mathrm{d}W_t + [\xi(V_{t-}+\Delta V_t) - \xi(V_{t-})] \\
&\quad - \frac{\partial \xi_t}{\partial \xi_t^1}\left(\frac{\partial \xi_t^1}{\partial t} + \mu_V V_{t-}\frac{\partial \xi_t^1}{\partial V_t} + \frac{\sigma^2 V_{t-}^2}{2} \times \frac{\partial^2 \xi_t^1}{\partial V_t^2}\right)\mathrm{d}t - \sigma V_{t-}\,\frac{\partial \xi_t}{\partial \xi_t^1}\,\frac{\partial \xi_t^1}{\partial V_t}\,\mathrm{d}W_t \\
&\quad - \frac{\partial \xi_t}{\partial \xi_t^1}[\xi^1(V_{t-}+\Delta V_t) - \xi^1(V_{t-})] \\
&= \left[\left(\frac{\partial \xi_t}{\partial t} + \frac{\sigma^2 V_{t-}^2}{2} \times \frac{\partial^2 \xi_t}{\partial V_t^2}\right) - \frac{\partial \xi_t}{\partial \xi_t^1}\left(\frac{\partial \xi_t}{\partial t} + \frac{\sigma^2 V_{t-}^2}{2} \times \frac{\partial^2 \xi_t}{\partial V_t^2}\right)\right]\mathrm{d}t \qquad (6.14) \\
&\quad + [\xi(V_{t-}+\Delta V_t) - \xi(V_{t-})] - \frac{\partial \xi_t}{\partial \xi_t^1}[\xi^1(V_{t-}+\Delta V_t) - \xi^1(V_{t-})]. \qquad (6.15)
\end{aligned}$$

[9]See Cont & Tankov (2004*a*) for more details on this hedging inaccuracy, and different methods to hedge under a jump-diffusion framework.

[10]We assume there exists a one-to-one relationship between $\xi^1(V_t,\, t)$ and V_t.

Thus the portfolio still has random behaviour due to the jump component of V. However, by selecting the asset weight

$$\psi_t = -\frac{\partial \xi_t}{\partial V_t}\frac{\partial V_t}{\partial \xi_t^1},$$

we are able to eliminate the frequent marginal fluctuations, modelled by W_t (diffusion risk). Similarly, if V_t follows a the GBM process (6.6) it can be shown that by selecting asset weight

$$\psi_t = -\frac{\partial \xi_t}{\partial V_t}\frac{\partial V_t}{\partial \xi_t^1},$$

we perfectly hedge contingent claim ξ_t.

In our case we are interested in hedging the CDS value ϕ with either shares S, call options φ or put options $\overline{\varphi}$. We will apply this simple hedging tactic whereby the weight of the hedging instrument is the negative of the rate of change of the contingent claim price with respect to the hedging instrument's price. This is known as *delta hedging.*

6.4 Delta Hedging

Suppose we construct a self-financing portfolio with value:

$$\begin{aligned}\vartheta_t(\boldsymbol{\psi}) = \boldsymbol{\psi}_t \,.\, \boldsymbol{\xi}_t &= \sum_{i=1}^{n-1} \psi_t^i\, \xi^i(V_t,\, t) - \psi_t^n\, \xi^n(V_t,\, t) \\ &= \sum_{i=1}^{n-1} \psi_t^i\, \xi_t^i - \psi_t^n\, \xi_t^n, \end{aligned} \tag{6.16}$$

where $\xi_t^i := \xi^i(V_t,\, t)$ is the time-t value of the ith contingent claim and where the underlying asset is the firm's asset value V_t. We assume that $y_t^i := g_i(V_t) := \xi^i(V_t,\, t)$ is a one-to-one transformation from $A = \{V_t > 0\}$ on to $B = \{y_t^i \in \mathbb{R}\}$ for $i = 1,\, 2,\, \ldots,\, n$. The inverse transformation is defined as $V_t = g_i^{-1}(y_t^i)$. The notation ψ_t^i represents the number of units of contingent claim ξ_t^i held at time t. Contingent claim ξ_t^n will represent our *hedging instrument.* The objective of a hedging instrument in a portfolio is to reduce the randomness of the portfolio's behaviour. The *delta of a portfolio* ϑ_t , denoted by $\Delta_{\vartheta_t/\xi_t^n}$, is defined as the rate of change of the portfolio's value with respect to the hedging instrument, with all the other model parameters remaining fixed. Thus $\Delta_{\vartheta_t/\xi_t^n}$ of portfolio (6.16) is

$$\begin{aligned}\Delta_{\vartheta_t/\xi_t^n} &:= \frac{\partial \vartheta_t}{\partial \xi_t^n} = \sum_{i=1}^{n-1} \psi_t^i\,\frac{\partial \xi_t^i}{\partial \xi_t^n} - \psi_t^n\,\frac{\partial \xi_t^n}{\partial \xi_t^n} \\ &= \sum_{i=1}^{n} \psi_t^i\,\frac{\partial \xi_t^i}{V_t}\,\frac{V_t}{\partial \xi_t^n} - \psi_t^n. \end{aligned} \tag{6.17}$$

Thus the delta of a portfolio with respect to the hedging instrument is given by the summation of each rate of change of the price of the contingent claims in the portfolio with respect to the hedging instrument. Let us denote the rate of change of the ith contingent claim ξ_t^i with respect to the hedging instrument ξ_t^n by

$$\Delta_{\xi_t^i/\xi_t^n} := \frac{\partial \xi_t^i}{\partial \xi_t^n} = \frac{\partial \xi_t^i}{V_t}\,\frac{V_t}{\partial \xi_t^n}. \tag{6.18}$$

We refer to $\Delta_{\xi_t^i/\xi_t^n}$ as the delta of the ith contingent claim ξ_t^i with respect to the hedging instrument ξ_t^n. Using the introduced notation (6.18), we can express Equation (6.17) as

$$\Delta_{\vartheta_t/\xi_t^n} = \sum_{i=1}^{n-1} \psi_t^i\,\Delta_{\xi_t^i/\xi_t^n} - \psi_t^n\,\Delta_{\xi_t^n/\xi_t^n}.$$

The objective of *delta hedging* is to choose the weight of the hedging instrument such that the delta of the portfolio, with respect to the hedging instrument, is equal to zero

$$\Delta_{\vartheta_t/\xi_t^n} = 0.$$

If the delta of a portfolio with respect to a hedging instrument is zero, then the portfolio is referred to as *delta neutral*. For portfolio (6.16) to be delta neutral, the hedging instrument weight ψ_t^n must be equal to

$$\psi_t^n = \sum_{i=1}^{n-1} \psi_t^i \frac{\partial \xi_t^i}{\partial \xi_t^n} = \sum_{i=1}^{n-1} \psi_t^i \Delta_{\xi_t^i/\xi_t^n}.$$

Example: Delta Hedging with the Underlying Asset V

Suppose, we construct a portfolio consisting of a long position on one unit of contingent claim $\xi^1(V_t, t)$ and a short position of

$$\Delta_{\xi_t^1/V_t} := \frac{\partial \xi_t^1}{\partial V_t}$$

units of underlying asset V_t (our hedging instrument). Let

$$\begin{aligned} \vartheta_t &= \xi_t^1 - \Delta_{\xi_t^1/V_t} V_t \\ &= \xi_t^1 - \frac{\partial \xi_t^1}{\partial V_t} V_t. \end{aligned} \tag{6.19}$$

The delta of portfolio (6.19), with respect to the hedging instrument V_t, is equal to 0:

$$\begin{aligned} \Delta_{\vartheta_t/V_t} &= \Delta_{\xi_t^1/V_t} - \Delta_{\xi_t^1/V_t} \Delta_{V_t/V_t} \\ &= \frac{\partial \xi_t^1}{\partial V_t} - \frac{\partial \xi_t^1}{\partial V_t} \frac{\partial V_t}{\partial V_t} \\ &= 0. \end{aligned} \tag{6.20}$$

From Section 6.3 we have seen that this delta neutral portfolio (6.19) perfectly hedges contingent claim ξ_1, when V follows a GBM process (6.6). Under our structural model, where V follows a jump-diffusion process (6.11), by constructing the delta neutral portfolio (6.19), we eliminate the diffusion risk but the randomness, due to the jumps in V, still remains. The same results hold for the general case when the delta neutral portfolio ϑ_t consists of $n-1$ contingent claims and one hedging instrument V_t, i.e.

$$\vartheta_t = \sum_{i=1}^{n-1} \psi_t^i \xi^i(V_t, t) - \sum_{i=1}^{n-1} \psi_t^i \Delta_{\xi_t^i/V_t} V_t$$

Example: Delta Hedging with Another Contingent Claim

Since V is not a publicly traded asset, consider the following delta neutral portfolio with two contingent claims, $\xi^1(V_t, t)$ and $\xi^2(V_t, t)$ (which will act as our hedging instrument):

$$\begin{aligned} \vartheta_t &= \xi^1(V_t, t) - \Delta_{\xi_t^1/\xi_t^2} \xi^2(V_t, t) \qquad (6.21) \\ &= \xi^1(V_t, t) - \frac{\partial \xi_t^1}{\partial \xi_t^2} \xi^2(V_t, t), \end{aligned}$$

Using $\Delta_{\xi_t^1/\xi_t^2}$ as the weight of our short position in hedging instrument $\xi^2(V_t, t)$, it was shown under the jump-diffusion framework (Section 6.3), that it hedges the diffusion risk but not the jump risk. It can similarly be shown, that the asset weight $\Delta_{\xi_t^1/\xi_t^2}$ hedges contingent claim $\xi^1(V, t)$ perfectly, when V follows the GBM process (6.6). The same results hold for the general case when the delta neutral portfolio ϑ_t consists of $n-1$ contingent claims and one hedging instrument $\xi^n(V_t, t)$

$$\vartheta_t = \sum_{i=1}^{n-1} \psi_t^i \xi^i(V_t, t) - \sum_{i=1}^{n-1} \psi_t^i \Delta_{\xi_t^i/\xi_t^n} \xi^n(V_t, t)$$

Example: Simple Numerical Example

Consider delta neutral portfolio (6.21). Suppose $\Delta_{\xi_t^1/\xi_t^2} = 0.75$, this means that when contingent claim (hedging instrument) $\xi^2(V_t, t)$ changes by a small amount $\delta\xi_t^2$, contingent claim $\xi^1(V_t, t)$ changes by 75% of that amount:

$$\delta\xi_t^1 = 0.75\delta\xi_t^2.$$

If the price of ξ^2 goes up \$10, then contingent claim ξ^1 will tend to increase by $0.75 \times \$10 = \$7.50$. The delta neutral portfolio (6.21) will make a profit of \$7.50 from the long position of one unit in contingent claim ξ^1. However this will be offset by the loss of \$7.50 from the short position $-\Delta_{\xi_t^1/\xi_t^2}\xi^2$. From this simplistic example it can be seen that this method reduces the future randomness of the value of contingent claim ξ^1.

6.4.1 A Measure for the Efficiency of a Delta Hedge

Note that when the underlying asset value V_t changes in time, the delta of the portfolio will change too. In practice, we cannot trade continuously but only discretely. Thus a delta neutral portfolio will only remain delta neutral for a *short period of time*, and will only be delta neutral again the instant we adjust the hedge such that the delta of the portfolio is zero. These periodic hedge adjustments are known as *rebalancing*. Delta hedging is a *dynamic-hedging scheme* since the hedge is regularly adjusted[11]. The delta hedge will be more efficient, as the number of times that rebalancing occurs increases (Hull (2003)). We measure the *efficiency* of a delta hedge by measuring the difference between the accumulated change in the values of the assets which we want to hedge, and the accumulated change in value in the hedging instrument.

Suppose we are interested in hedging contingent claim $\xi^1(V_t,\, t)$ with contingent claim $\xi^2(V_t,\, t)$ (hedging instrument). We construct the following delta neutral portfolio

$$\begin{aligned} \vartheta_t &= \xi^1(V_t,\, t) - \Delta_{\xi_t^1/\xi_t^2}\,\xi^2(V_t,\, t) \\ &= \xi^1(V_t,\, t) - \frac{\partial \xi_t^1}{\partial \xi_t^2}\,\xi^2(V_t,\, t). \end{aligned}$$

If the risk of this portfolio is perfectly hedged, the instantaneous return on the portfolio is totally deterministic (see Section 6.3). To avoid arbitrage opportunities this deterministic return must be equal to the risk-free rate

$$\mathrm{d}\vartheta_t = r\vartheta_t\,\mathrm{d}t. \tag{6.22}$$

Suppose we are interested in hedging until some time point T. Let

$$\begin{aligned} \hat{\vartheta}_t &= \vartheta_t e^{r(T-t)} \\ \hat{\xi}_t^1 &= \xi_t^1 e^{r(T-t)} \\ \hat{\xi}_t^2 &= \xi_t^2 e^{r(T-t)}. \end{aligned}$$

Then from the perfect hedge property (6.22), we can write

$$\begin{aligned} \mathrm{d}\vartheta_t &= r\vartheta_t\,\mathrm{d}t \\ \mathrm{d}\hat{\vartheta}_t &= 0 \qquad\qquad (6.23) \\ \mathrm{d}\hat{\xi}_t^1 - \Delta_{\xi_t^1/\xi_t^2}\,\mathrm{d}\hat{\xi}_t^2 &= 0 \\ \mathrm{d}\hat{\xi}_t^1 &= \Delta_{\xi_t^1/\xi_t^2}\,\mathrm{d}\hat{\xi}_t^2 \\ \int_t^T \mathrm{d}\hat{\xi}_s^1 &= \int_t^T \Delta_{\xi_s^1/\xi_s^2}\,\mathrm{d}\hat{\xi}_s^2 \\ \hat{\xi}_T^1 - \hat{\xi}_t^1 &= \int_t^T \Delta_{\xi_s^1/\xi_s^2}\,\mathrm{d}\hat{\xi}_s^2 \\ \xi_T^1 - e^{r(T-t)}\xi_t^1 &= \int_t^T \Delta_{\xi_s^1/\xi_s^2}\,\mathrm{d}\hat{\xi}_s^2 \end{aligned}$$

Thus if the delta hedge eliminates all random fluctuations in contingent claim $\xi^1(V_t,\, t)$ then

$$\xi_T^1 - e^{r(T-t)}\xi_t^1 - \int_t^T \Delta_{\xi_s^1/\xi_s^2}\,\mathrm{d}\hat{\xi}_s^2 = 0. \tag{6.24}$$

The closer the right hand side (RHS) of Equation (6.24) is to zero, the better the delta hedge is performing. However since it is impossible to trade continuously, we will anaylse the discrete trading case of Equation (6.24). If we divide interval $[t,\, T]$ into N subintervals such that $t_0,\, t_1,\, t_2\, \ldots,\, t_N$,

[11] See Hull (2003) for examples of *static-hedging schemes*.

where $t_0 = t$, $t_N = T$ and $t_{i-1} \leq t_i, \forall i = 1, 2 \ldots N$. Using this interval, the discrete form of the RHS of Equation (6.24) becomes

$$\begin{aligned}
& \xi^1_{t_N} - e^{r(t_N - t_0)} \xi^1_{t_0} - \sum_{i=1}^{N} \Delta_{\xi^1_{t_{i-1}}/\xi^2_{t_{i-1}}} \left(\hat{\xi}^2_{t_i} - \hat{\xi}^2_{t_{i-1}} \right) \\
= \; & \xi^1_{t_N} - e^{r(t_N - t_0)} \xi^1_{t_0} - \sum_{i=1}^{N} \Delta_{\xi^1_{t_{i-1}}/\xi^2_{t_{i-1}}} \left(e^{r(t_N - t_i)} \xi^2_{t_i} - e^{r(t_N - t_{i-1})} \xi^2_{t_{i-1}} \right) \\
= \; & \xi^1_{t_N} - e^{r(t_N - t_0)} \xi^1_{t_0} - \left(\Delta_{\xi^1_{t_{N-1}}/\xi^2_{t_{N-1}}} \xi^2_{t_N} - \Delta_{\xi^1_{t_0}/\xi^2_{t_0}} \xi^2_{t_0} e^{r(t_N - t_0)} \right) \\
& - \sum_{i=1}^{N-1} \left(\Delta_{\xi^1_{t_{i-1}}/\xi^2_{t_{i-1}}} - \Delta_{\xi^1_{t_i}/\xi^2_{t_i}} \right) \left(e^{r(t_N - t_i)} \xi^2_{t_i} \right). \qquad (6.25)
\end{aligned}$$

For a perfect hedge (6.25) will be equal to zero. Let

$$\Lambda(t_0, t_N) = a + b, \qquad (6.26)$$

where

$$a = \xi^1_{t_N} - e^{r(t_N - t_0)} \xi^1_{t_0}, \qquad (6.27)$$

$$b = -\left(\Delta_{\xi^1_{t_{N-1}}/\xi^2_{t_{N-1}}} \xi^2_{t_N} - \Delta_{\xi^1_{t_0}/\xi^2_{t_0}} \xi^2_{t_0} e^{r(t_N - t_0)} \right) - \sum_{i=1}^{N-1} \left(\Delta_{\xi^1_{t_{i-1}}/\xi^2_{t_{i-1}}} - \Delta_{\xi^1_{t_i}/\xi^2_{t_i}} \right) \left(e^{r(t_N - t_i)} \xi^2_{t_i} \right). \qquad (6.28)$$

In fact, $\Lambda(t_0, T_N)$ is the accumulated change in value in the hedging portfolio ϑ, from inception t_0 till the hedging maturity t_N. The term a in Equation (6.26) is the accumulated change of the value in the instrument we wish to hedge ξ^1. The term b in Equation (6.26) is the accumulated change of the value in the hedging position $-\frac{\partial \xi^1_t}{\partial \xi^2_t} \xi^2_t$. A financial position is well hedged, when the change of value in the financial position we wish to hedge is mirrored by the change of value of the hedging position. For our empirical delta hedging tests, we will measure the effectiveness of the delta hedge using $\Lambda(t, T)$. The closer $\Lambda(t, T)$ is to zero, the better our delta hedge is performing. The purpose of a hedging strategy is to reduce the randomness of the value of the financial position we wish to hedge. A hedging strategy is functioning if the absolute value of the change in value of our financial hedged position $\lambda(t_0, t)$ is less than the absolute value of the change in value of the financial position without the hedging strategy a. Mathematically, this condition can be expressed as $|\lambda(t_0, t)| < |a|$. Thus a hedging is functioning if $|\lambda(t_0, t)| < |a|$, and is performing well if $\lambda(t_0, t) \approx 0$. These conditions can be restated as the following $\frac{|a_t| - |\lambda(t_0, t)|}{|a_t|} \leq 1$, the closer the ratio is to 1 the better the hedge is performing. This ratio can be interpreted as the percentage reduction in the change of the value of the position we wish to hedge, achieved by the hedge.

Our empirical tests do not take into account transaction costs. If transaction costs are taken into account, the investor has to balance the effectiveness of a delta hedge, achieved by increasing the number of times rebalancing occurs (Hull (2003)), with the increase in transactions costs accompanied along with this. This can be done by measuring the sensitivity of the delta of the portfolio with respecting to the hedging instrument.

6.4.2 Gamma Hedge

The *gamma* of a portfolio ϑ_t, with respect to a hedging instrument $\xi^n(V_t, t)$, $\Gamma_{\vartheta_t/\xi^n_t}$, is the rate of change of the portfolios's delta $\Delta_{\vartheta_t/\xi^n_t}$, with respect to the hedging instrument

$$\Gamma_{\vartheta_t/\xi^n_t} := \frac{\partial \left(\Delta_{\vartheta_t/\xi^n_t} \right)}{\partial \xi^n_t} = \frac{\partial^2 \vartheta_t}{\partial^2 \xi^n_t}.$$

A small portfolio gamma, implies the delta of the portfolio changes slightly with respect to changes in the hedging instrument. Thus portfolio rebalancing needs to be done relatively infrequently, to keep the portfolio delta neutral. A large portfolio gamma, implies the delta of the portfolio is highly sensitive to the changes in the hedging instrument. Thus portfolio rebalancing needs to be done frequently. A portfolio is gamma neutral if $\Gamma_{\vartheta_t/\xi^n_t} = 0$. The procedure of keeping a portfolio gamma neutral, is similar to the procedure of keeping a portfolio delta neutral, except we use second order partial derivatives. Note, by adjusting a delta neutral portfolio such that it is gamma neutral, will affect the delta of the portfolio, thus a position in the underlying asset is needed in order to maintain delta neutrality (Hull (2003)). By increasing the number of times a delta neutral portfolio is rebalanced, increases the efficiency of the delta hedge (Hull (2003)). However, by increasing the number of times rebalancing occurs, increases the transaction costs incurred. An investor wishing to keep his portfolio delta neutral, can decide how often to rebalance, by analysing the gamma of the portfolio. By doing this he also curbs transaction costs.

6.5 Empirical Tests

The main objective of this dissertation is it to hedge credit default swaps with shares and European share options. Suppose a long position is taken in a CDS with maturity T. The value of the CDS at its inception is $\phi(t_0, t_0, T) = 0$. As we move towards maturity, market's perception of the reference entity's credit risk may change, altering the value of the CDS (see Section 6.2). We wish to take a position in either shares or share options to hedge the randomness of the future values of a CDS. To achieve this we will delta hedge. We will construct the following delta neutral portfolios:

$$\vartheta_t^1 = \phi(t_0, t, T) - \Delta_{\phi(t_0,t,T)/S_t}\, S_t \tag{6.29}$$

$$\vartheta_t^2 = \phi(t_0, t, T) - \Delta_{\phi(t_0,t,T)/\varphi_t}\, \varphi_t \tag{6.30}$$

$$\vartheta_t^3 = \phi(t_0, t, T) - \Delta_{\phi(t_0,t,T)/\overline{\varphi}_t}\, \overline{\varphi}_t, \tag{6.31}$$

where

$$\begin{aligned}
\Delta_{\phi(t_0,t,T)/S_t} &= \frac{\partial\left(\phi(t_0,t,T)\right)}{\partial S_t} \\
\Delta_{\phi(t_0,t,T)/\varphi_t} &= \frac{\partial\left(\phi(t_0,t,T)\right)}{\partial \varphi_t} \\
\Delta_{\phi(t_0,t,T)/\overline{\varphi}_t} &= \frac{\partial\left(\phi(t_0,t,T)\right)}{\partial \overline{\varphi}_t}.
\end{aligned}$$

We name these partial derivatives the *delta hedge ratios.* We use delta hedging for our purposes, since it was shown in Section 6.4 that this hedging method hedges all the marginal fluctuations in the value of a contingent claim, and it is a widely used and understood hedging scheme.

Under our assumed structural model (see Section (4.5)), the time of default is modelled as

$$\tau^* = \inf\{s \geq 0 : V_s \leq b_{t_f,T}\}.$$

Thus the value of a CDS (Equation (6.1)) is dependent on the firm's asset value V. Also, under our model, all of the securities issued by a firm are dependent on the firm's asset value V. Thus the share price of the firm is dependent on V. The relation is given by Equation (6.3). This implies that option prices are also dependent on V. The relation is given by Equation (6.4) for a European call, and by Equation (6.5) for a European put.

Figure 6.1 shows that there is one-to-one relationship between CDS values, share prices, share option prices and the firm's asset value[12]. From this one-to-one relationship we can write the delta hedge ratios as

$$\begin{aligned}
\Delta_{\phi(t_0,\,t,\,T)/S_t} &= \frac{\partial\left(\phi(t_0,\,t,\,T)\right)}{\partial V_t}\,\frac{\partial V_t}{\partial S_t} \\
\Delta_{\phi(t_0,\,t,\,T)/\varphi_t} &= \frac{\partial\left(\phi(t_0,\,t,\,T)\right)}{\partial V_t}\,\frac{\partial V_t}{\partial \varphi_t} \\
\Delta_{\phi(t_0,\,t,\,T)/\overline{\varphi}_t} &= \frac{\partial\left(\phi(t_0,\,t,\,T)\right)}{\partial V_t}\,\frac{\partial V_t}{\partial \overline{\varphi}_t}.
\end{aligned}$$

We perform delta hedges, according to portfolios (6.29), (6.30) and (6.31), with daily, weekly and monthly rebalancing. We also calculate $\Lambda(t,\,T)$ (6.25), to determine how efficient these delta hedges are performing.

[12]Note that there are no explicit solutions for equations (6.1), (6.3), (6.4) and (6.5). Thus the graphs in Figure 6.1 were calculated using Monte Carlo simulation. Specifically, we used the Monte Carlo methods described in Chapter 5. We used the following parameter values when calculating Monte Carlo estimates for equations (6.1), (6.3), (6.4) and (6.5): $t = 0$, $t_f = 1$, $t_g = 1.25$, $T = 5$, $V_1 = \$1000000000$, $b_{1,5} = 500000000$, $r = 0.05$, $\lambda = 1$, $\mu_A = -0.05$, $\sigma_A = 0.1$, $K = \$75$, $N = \$1000000$, $\eta = 5000000$, $w_0 = 0.4$, $\tilde{p}_{0,5} = 0.0324$ and $n = 20$ (number of CDS premium payments from inception until maturity, four per year). We use Equation (5.37) as the basis function for the calculation of the European options.

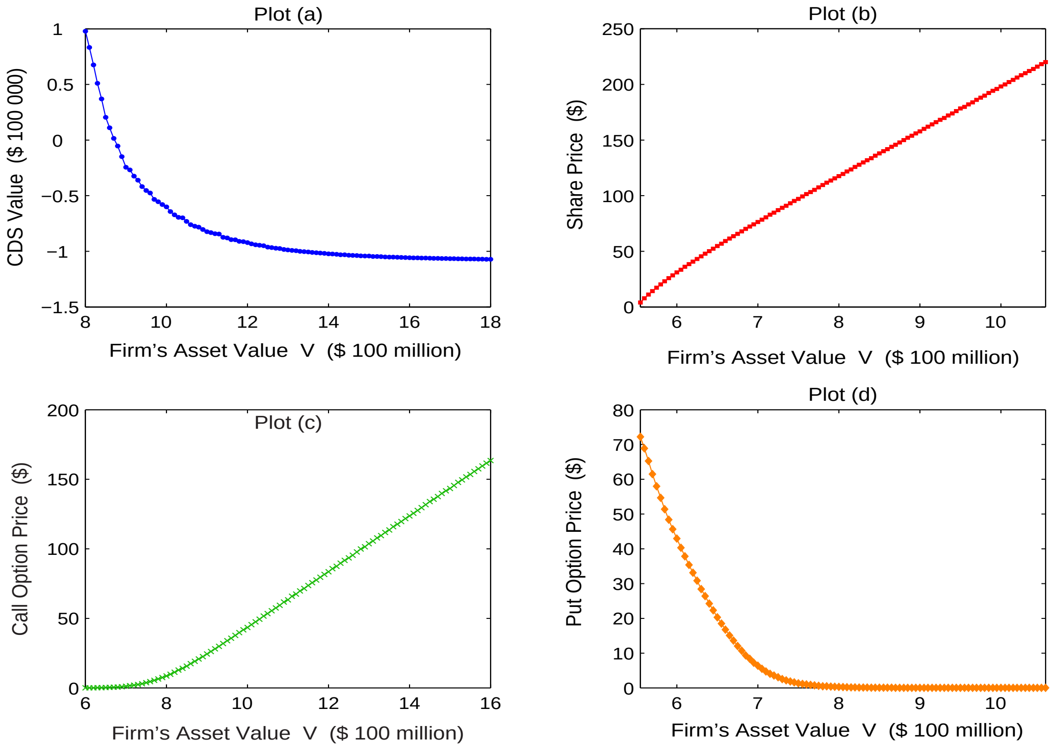

Figure 6.1: Plot (a) is a plot of CDS values versus different values of the firm's asset value V. Plot (b) is a plot of share prices versus different values of the firm's asset value V. Plot (c) is a plot of European call option prices versus different values of the firm's asset value V. Plot (d) is a plot of European put option prices versus different values for the firm's asset value V.

6.5.1 Method

We will now describe the procedure applied, under our mathematical and pricing framework, to create delta neutral portfolios, in order to hedge the value of a CDS with shares or European share options (portfolios (6.29), (6.30) and (6.31)). We explain our steps to perform delta hedge (6.29), where the hedging instrument is the firm's shares. The steps to perform the other two delta hedges (6.30) and (6.31) follow similarly. Suppose we enter into a long position in a CDS contract, initiated at time t, with a maturity of T. We wish to hedge this CDS position, from initiation time t_0 until some time $T_h < T$. We divide the time interval $[t, T_h]$ into N subintervals such that $t_0, t_1, t_2 \ldots, t_N$, where $t_0 = t$, $t_N = T_h$ and $t_{i-1} \leq t_i, \forall i = 1, 2 \ldots N$.

Firstly, at time t_0 we calibrate our model parameters $\boldsymbol{\theta}_{t_0} = (V_{t_0}, \sigma_V, b_{t_0,T}, \lambda, \mu_A, \sigma_A)$ to market prices. See Section 5.6 for a description of our calibration method. Once we have obtained the calibrated model parameters $\hat{\boldsymbol{\theta}}_{t_0}$, we use them to calculate the theoretical premium $\tilde{p}_{t_0,t_0,T}$ from Equation (6.2). We calculate the premium according to the method described in Section 5.4.6. The calculated premium $\tilde{p}_{t_0,t_0,T}$ will be approximately equal to the current market CDS premium $\tilde{p}^{obs}_{t_0,T}$. Next, we calculate the delta hedge ratio

$$\Delta_{\phi(t_0,\, t_0,\, T)/S_{t_0}} = \frac{\partial\left(\phi(t_0,\, t_0,\, T)\right)}{\partial V_{t_0}} \frac{\partial V_{t_0}}{\partial S_{t_0}}. \tag{6.32}$$

Our mathematical model, does not admit explicit solutions for the value of a CDS, the price of a share and the price of a share option. We use Monte Carlo methods to find the value of these instruments (see Chapter 5). Thus there is no explicit solution for the partial derivative (6.32). We estimate the delta hedge ratios numerically. Let the numerical estimate for $\Delta_{\phi(t_0,\, t_0,\, T)/S_{t_0}}$ be denoted by $\widehat{\Delta}_{\phi(t_0,\, t_0,\, T)/S_{t_0}}$.

To denote the functional dependence of the CDS and share price on the firm's asset value V let

$\phi_{t_0,\,t,\,T}(V_t) := \phi(t_0,\,t,\,T)$ and $S_t(V_t) := S_t$. We estimate the partial derivative $\frac{\partial(\phi(t_0,\,t_0,\,T))}{\partial V_{t_0}}$ with[13]

$$\widehat{\Delta}_{\phi(t_0,t_0,T)/V_{t_0}} := \frac{\phi_{t_0,\,t_0,\,T}(V_{t_0}+h) - \phi_{t_0,\,t_0,\,T}(V_{t_0})}{h}, \tag{6.33}$$

where h is *small.* In our numerical analysis we choose h to be 1% of V_{t_0}. Since there is an one-to-one relationship between $S_t(V_t)$ and V_t

$$\frac{\partial V_{t_0}}{\partial S_{t_0}} = 1 \Big/ \frac{\partial S_{t_0}}{\partial V_{t_0}}. \tag{6.34}$$

We estimate the partial derivative $\frac{\partial S_{t_0}}{\partial V_{t_0}}$ with

$$\widehat{\Delta}_{S_{t_0}/V_{t_0}} := \frac{S_{t_0}(V_{t_0}+h) - S_{t_0}(V_{t_0})}{h}. \tag{6.35}$$

We estimate partial derivative (6.32) with

$$\widehat{\Delta}_{\phi(t_0,\,t_0,\,T)/S_{t_0}} = \widehat{\Delta}_{\phi(t_0,t_0,T)/V_{t_0}} \times \widehat{\Delta}_{S_{t_0}/V_{t_0}}. \tag{6.36}$$

We calculate expression (6.36), using Monte Carlo simulation. For example to calculate $\widehat{\Delta}_{S_{t_0}/V_{t_0}}$: specify a small h (we choose $h = 0.01V_{t_0}$), then using Monte Carlo simulation calculate $S_{t_0}(V_{t_0})$, and with the same generated paths calculate $S_{t_0}(V_{t_0}+h)$. Subtract $S_{t_0}(V_{t_0})$ from $S_{t_0}(V_{t_0}+h)$ and divide it by h. Note Monte Carlo simulation is done with the calibrated parameters $\hat{\boldsymbol{\theta}}_{t_0}$.

Next, we construct the delta neutral portfolio (6.29), using our estimated delta hedge ratio $\widehat{\Delta}_{\phi(t_0,\,t_0,\,T)/S_{t_0}}$:

$$\vartheta^1_{t_0} = \phi^{obs}(t_0, t_0, T) - \widehat{\Delta}_{\phi(t_0,\,t_0,\,T)/S_{t_0}}\, S^{obs}_{t_0}. \tag{6.37}$$

Remember $S^{obs}_{t_0}$ is the observed market share price and $\phi^{obs}(t_0, t_0, T)$ is the observed *market* CDS value at time t_0. Note, the market value of a CDS is not a true market price since it is not observable in the market, only the CDS premiums $\tilde{p}_{t_0,t_0,T}$ are disclosed in the market. The market CDS value is calculated by inputting $\tilde{p}_{t_0,t_0,T}$ into the investor's mathematical model (called market-to-model value). In our case, the market value at time t_1 of a CDS initiated at time t_0, with maturity T, is

$$\begin{aligned}\phi^{obs}(t_0, t_1, T) =& N\left(\tilde{p}^{obs}_{t_1,T} - \tilde{p}^{obs}_{t_0,T}\right) \mathbb{E}^{\mathbb{Q}}\left[\sum_{i=1}^{n} \mathbb{1}_{\{t^p_i \geq t_1\}}\Big(e^{-r(t^p_i - t_1)}(t^p_i - t^p_{i-1})\mathbb{1}_{\{\tau^* > t^p_i\}}\right.\\ &\left.+\, e^{-r(\tau^* - t_1)}(\tau^* - t^p_{i-1})\mathbb{1}_{\{t^p_{i-1} < \tau^* < t^p_i\}}\Big)\right]\end{aligned} \tag{6.38}$$

where t^p_i is the time of the i^{th} premium payment.

At the next time point rebalancing occurs, t_1, the next step is to once again calibrate our model parameters $\boldsymbol{\theta}_{t_1} = (V_{t_1},\, \sigma_V,\, b_{t_1,T},\, \lambda,\, \mu_A,\, \sigma_A)$ to the current market prices. The instant before rebalancing takes place at time t_1, the market value of the delta neutral portfolio constructed at t_0, is equal to

$$\vartheta^1_{t_1} = \phi^{obs}(t_0, t_1, T) - \widehat{\Delta}_{\phi(t_0,\,t_0,\,T)/S_{t_0}}\, S^{obs}_{t_1}. \tag{6.39}$$

We then calculate the delta hedge ratio for time point t_1, $\widehat{\Delta}_{\phi(t_0,\,t_1,\,T)/S_{t_1}}$. This is done similarly to the method to calculate the delta ratio at time t_0, $\widehat{\Delta}_{\phi(t_0,\,t_0,\,T)/S_{t_0}}$. Now, the market value at time t_1, for the delta neutral portfolio the instant after rebalancing is

$$\vartheta^1_{t_1} = \phi^{obs}(t_0, t_1, T) - \widehat{\Delta}_{\phi(t_0,\,t_1,\,T)/S_{t_1}}\, S^{obs}_{t_1}. \tag{6.40}$$

We then calculate the *delta hedge efficiency measure* $\Lambda(t_0,\, t_1)$. These steps are done recursively until the hedge maturity T_h is reached. We then analyse $\Lambda(t_0,\, T_h)$.

[13]See Appendix C, Definition C.0.1, for the definition of a partial derivative. From this definition one can understand why we use the RHS of Equation (6.33) as an estimate for the partial derivatives.

6.5.2 Testing the Delta Hedging Strategy with Simulated Data

We now present simulated examples of the delta neutral portfolios (6.29) and (6.30). From these examples one can see the dynamics of the value of delta neutral portfolios and Λ as we reach the hedge maturity.

We wish to construct the following delta neutral portfolios

$$\vartheta_t^1 = \phi(t_0, t, T) - \Delta_{\phi(t_0,t,T)/S_t} S_t \tag{6.41}$$
$$\vartheta_t^2 = \phi(t_0, t, T) - \Delta_{\phi(t_0,t,T)/\varphi_t} \varphi_t, \tag{6.42}$$

at time $t_0 = 0$, and hold it until time $T_h = 60$ days. Rebalancing will be done daily for both portfolios. The CDS value $\phi(t_0, t, T)$, share price S_t and share option price φ_t will be simulated. This is done by simulating the real-world firm's asset value process

$$V_{t_i} = V_{t_{i-1}} \exp\left[\left(\mu_V - \frac{\sigma^2}{2}\right)(t_i - t_{i-1}) + \sigma W_{t_i - t_{i-1}} + J_{t_i} - J_{t_{i-1}}\right] \tag{6.43}$$

for 60 daily periods. Next, 60 daily CDS values, share and share option prices are calculated from each of these simulated firm asset values. Then we calculate the delta hedge ratios $\Delta_{\phi(t_0,t_{i-1},T)/S_{i-1}}$ and $\Delta_{\phi(t_0,t_{i-1},T)/\varphi_{t_{i-1}}}$ for $i = 1, \ldots, 60$, as illustrated in Section 6.5.1.

The following graphs (Figure 6.2) summarise our simulated[14] delta hedge results, when shares are used as the hedging instrument (portfolio (6.41)).

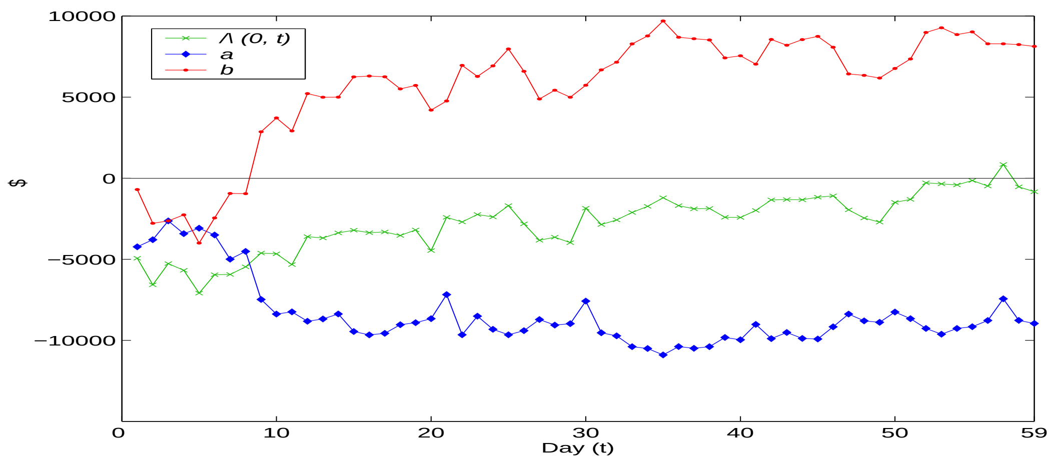

Figure 6.2: Time series plot of $\Lambda(0, t)$, and its constituents a and b. This is a simulated delta hedge, where a share is the hedging instrument.

The complete results are tabulated, and appear in Table D.1, in Appendix D.1. These tables include each rebalancing delta hedge ratio and $\Lambda(t_0, t_i)$. From Figure 6.2, one can see that the accumulated loss from the CDS position (a) is approximately mirrored by the profits from the share hedge position. The general trend is as a decreases (resp. increases), b increases (resp. decreases). We will call this property the *mirroring* property, and is an essential characteristic of a well performing hedge. The net result $\Lambda(0, t)$ fluctuates near zero. Note that the hedge becomes progressively better with time. This is because the accumulated losses and profits are averaged out as we progress in time. The end results, at Day 59, are: the accumulated loss in the CDS position is $-\$8961.30$, the accumulated profit in the share hedge position is \$8126 and $\Lambda(0, 59)$ is $-\$835.28$.

The following graphs (Figure 6.3) summarise our simulated delta hedge results, when call options are used as the hedging instrument (portfolio (6.41)):

The complete results are tabulated, and appear in Table D.2, in Appendix D.1. From Figure 6.3, one can see again that the accumulated loss from the CDS position (a) is approximately

[14]The parameters used for these simulations are: $t_0 = 0$, $t_f = 0$, $t_g = 0.25$, $T = 5$, $V_0 = \$62994450000$, $b_{0,5} = 28731768645$, $r = 0.045$, $\mu_V = 0.05$, $\lambda = 1.4350$, $\mu_A = -0.0329$, $\sigma_A = 0.0540$, $K = \$45$, $N = \$1000000$, $\eta = 791380000$, $w_0 = 0.4$ and $n = 20$ (number of CDS premium payments from inception until maturity, four per year). We use Equation (5.37) as the basis function for the calculation of the European options.

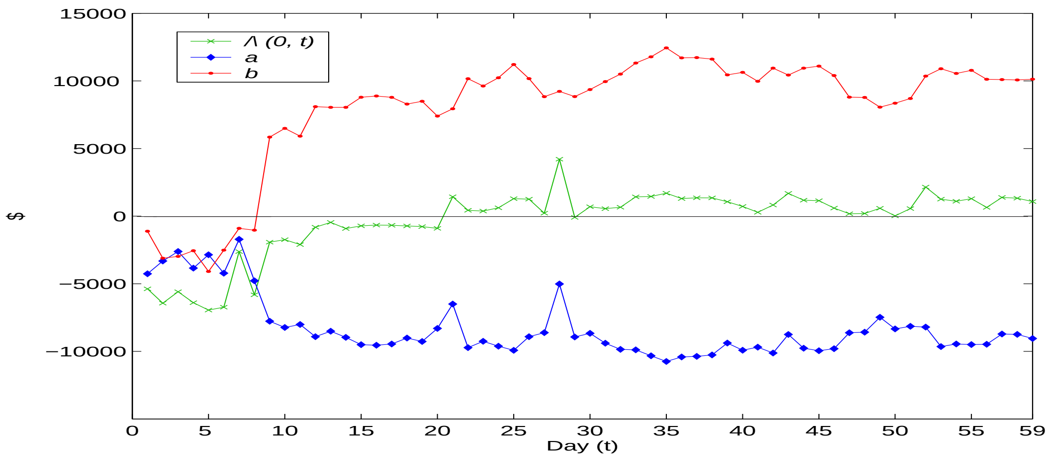

Figure 6.3: Time series plot of $\Lambda(0, t)$, and its constituents a and b. This is a simulated delta hedge, where a call option is the hedging instrument.

mirrored by the profits from the call option hedge position. The end results, at Day 59, are: the accumulated loss in the CDS position is −\$9044.90, the accumulated profit in the call option hedge position is \$10127 and $\Lambda(0, 59)$ is \$1082.20.

From both the above delta hedges one can see that the changes in the value of the delta neutral portfolios are significantly less than the change in value of the CDS position. Both delta neutral portfolios fluctuate near zero, but neither are exactly equal to zero. There are several reasons why these simulated delta hedges do not perfectly hedge the value of a CDS. Firstly, we simulated under a jump-diffusion framework, thus the delta hedge will only eliminate the diffusion risk and not the jump risk (see Section 6.4). Another reason is that we do not have explicit solutions for the values of the financial products, thus we need to numerically estimate their values and the delta hedge ratios. Finally, it is impossible to continuously trade, thus the discrete nature of trading, also makes our hedge imperfect.

Both delta hedges, from simulated data, performed well, since $|\lambda(t_0, t)| < |a|$ and $\lambda(t_0, t) \approx 0$. This simulated example shows theoretically that our methods to calculate the price and delta hedge ratios are acceptable, since the simulated delta hedges perform well. Thus if the estimation of the model parameters is correct and our model assumptions are not significantly different to reality, our hedging results from market data should follow similarly.

6.5.3 Data

We chose six companies to test our hedge: Boeing, Daimler Chrysler, Deutsche Telekom, Ford, General Motors and Vodafone. We chose companies that have widespread credit ratings to test if our model can produce a hedging mechanism that can manage firms with different credit risk profiles. We chose popular companies, since their data is easily available.

The balance sheet and equity (stock and stock option) data was obtained from Bloomberg. The daily closing prices were used for the stock and stock option prices. In order to simplify the analysis we chose stock options with long maturities, so we only need to use one particular option contract throughout the hedging time interval. If we chose short maturity option contracts, we may have to use several different option contracts since these options could mature before the end point of our hedging time interval. The most liquid[15] long maturity option contracts were chosen. A liquid asset has a high probability that the next trade is executed at a price near to the most recent price. Since European type stock options are rare, we use corresponding American type stock options to approximate the value European stock options[16].

All of the CDS contracts we hedge have a maturity of 5 years and a notional value of 1 000 000. Consider the CDS valuation formula (6.1). The maturity of the CDS contract is $T = 5$ years and the time of initiation of the CDS is the present time $t = 0$. To calculate the value of the CDS

[15]Liquidity was measured by the volume of contracts traded.

[16]The difference in value between American and European options is neglige in our case. Since not many dividends are paid out in our hedging intervals and most of the options are deep out of the money. The early exercise premium of an American option is an increasing function of the moneyness of the option (see Engström & Nordén (2000)).

contract after one month $\phi(0, \frac{1}{12}, 5)$ we need the market premium $\tilde{p}_{\frac{1}{12},5}$. However, in the market they do not quote premiums for all time intervals. The most common time intervals are 5, 3, and 1 year time intervals. The 5 year CDS premiums are the most liquid and are available for most companies. To calculate $\phi(0, \frac{1}{12}, 5)$ at time $t = \frac{1}{12}$, we estimate the market premium $\tilde{p}_{\frac{1}{12},5}$ with the current 5 year market premium $\tilde{p}_{\frac{1}{12},5\frac{1}{12}}$. For all the CDS valuation in our empirical tests, we use the 5 year CDS market premiums to estimate the premium for the time interval starting from the valuation time till CDS maturity. We used the daily mid price for the 5 year CDS premiums, obtained from Bloomberg.

For each company we perform an equity and an equity option hedge. For each equity and equity option hedge, we perform three hedges where rebalancing is done daily, weekly and monthly. The reason why we do this, is that the relationship between the credit and equity markets might be weak between daily data, but stronger for larger intervals. The currency used for equity hedging is US Dollars. However the currency used for the equity option hedging is the currency that particular option is quoted in.

The time interval for each hedge is given in Appendix D.2.1, Table D.10. The descriptions of the stock options is given in Appendix D.2.1, Table D.9.

6.5.4 Results

We sequentially analyse the hedging results of each company and conclude at the end of the chapter. See the tables in Appendix D.2, for a detailed description of all the hedging results.

Boeing

In the case where shares are used as the hedging instrument, the delta hedge, for the Boeing company, did not perform well. Figure 6.4 (Plot (a.1), (a.2) and (a.3)), shows the small losses incurred by the CDS position is excessively hedged by the position in shares, for all three cases where rebalancing is done daily, weekly and monthly. The equity hedge positions make disproportionate profits relative to the CDS losses. These equity delta hedges do not have the mirroring property seen in the simulated equity hedge Figure 6.2. The accumulated changes in the CDS positions (a) are relatively stationary. However, the accumulated changes in the equity positions (b) increase steadily for the daily and the monthly cases; for the weekly case there is a temporary decline of b to appropriate levels at weeks 27 to 33 ($\lambda(0, 32) = -\$82.81$ is the best value achieved), this is short lived as b rises towards the end of the hedge period to disproportionate levels again. The final equity delta hedge results for Boeing are:

Boeing	Daily Rebalancing	Weekly Rebalancing	Monthly Rebalancing
λ	\$7605.01	\$6682.39	\$47608.30
a	−\$847.67	−\$1643.59	−\$3024.01
b	\$8452.68	\$8325.98	\$50632.31

Table 6.2: The final results of the equity delta hedge for Boeing.

A possible explanation for the poor results of the equity delta hedge is the non-intuitive co-movement of market CDS spreads and share prices. Intuitively, if credit risk increases (resp. decreases), CDS spreads increase (resp. decrease) and share prices decrease (resp. increase). For all three equity delta hedges this intuitive relationship did not hold for more than half of the data. The data of the Boeing company, suggests the link between the CDS (credit) and the stock (equity) market dynamics is weak.

In the case where share options are used as the hedging instrument, the delta hedge results, for the Boeing company, are similar to those when shares were used as a hedging instrument. Again, the excessive hedge positions in the share options nullify the mirroring property needed for a good hedge result. For the daily rebalancing case, $\lambda(0, t)$ is only stable around 0 for a short period (between Day 14 and Day 16). For the rest of the hedging period $\lambda(0, t)$ is considerably distant from 0 (see Figure 6.4, Plot (b.1)). The weekly rebalancing hedging case, performs better than the daily case, since $\lambda(0, t)$ remains at the level 0 longer (between weeks 8 and 28) however increases to disproportionate levels at the end of the hedging interval (see Figure 6.4, Plot (b.2)). The monthly rebalancing case, is similar to the weekly case, in the beginning $\lambda(0, t)$ hovers around 0 but increases dramatically towards the end of the hedge (see Figure 6.4, Plot (b.3)). For all

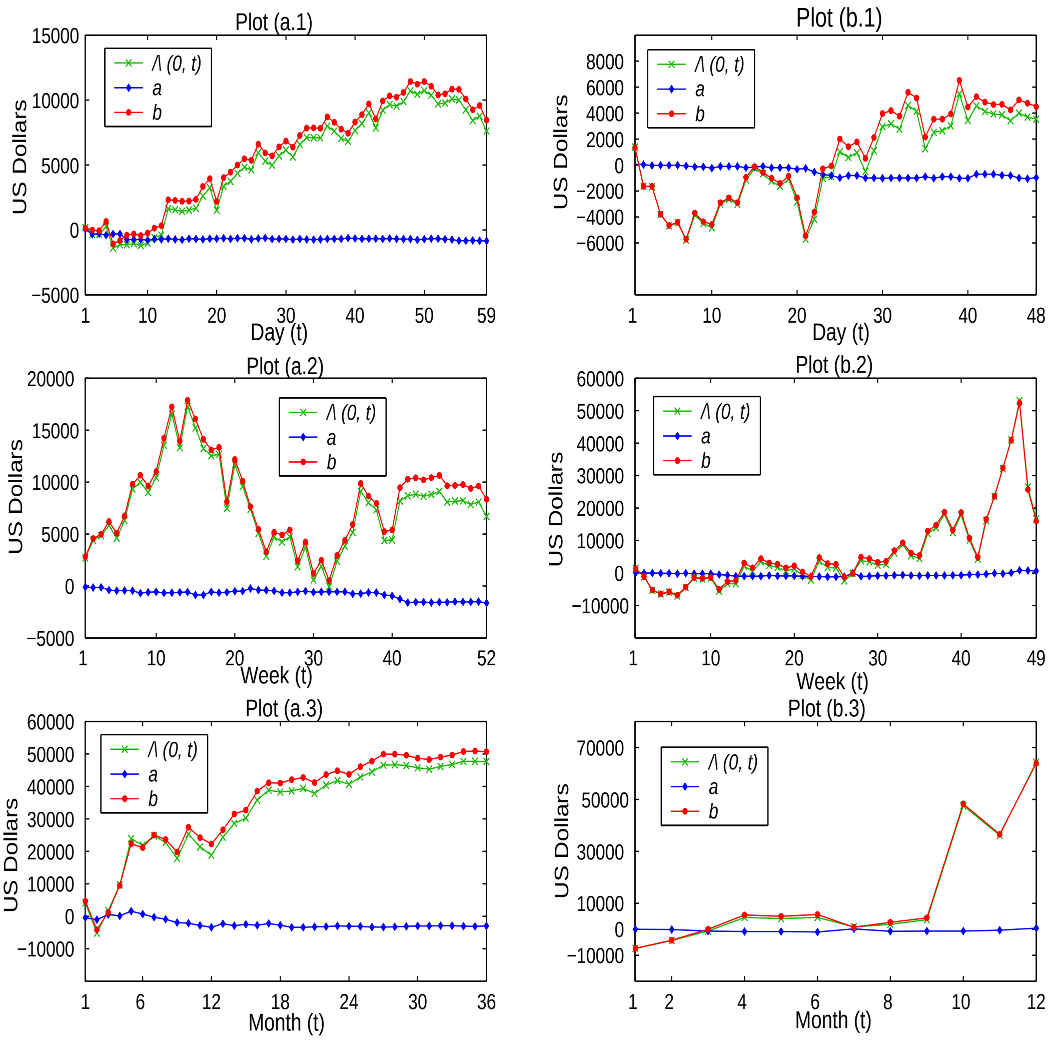

Figure 6.4: Plot (a.1), (a.2) and (a.3) are the results of the equity delta hedges, for the Boeing company, for daily, weekly and monthly rebalancing, respectively. Plot (b.1), (b.2) and (b.3) are the results of the equity option delta hedges, for the Boeing company, for daily, weekly and monthly rebalancing, respectively.

three rebalancing cases, the hedge position is too large, relative to the CDS value. The final equity option delta hedge results for Boeing are:

Boeing	Daily Rebalancing	Weekly Rebalancing	Monthly Rebalancing
λ	\$3523.38	\$16650.63	\$64245.41
a	−\$966	−\$584.86	−\$421.28
b	\$4489.38	\$16065.77	\$63824.13

Table 6.3: The final results of the equity option delta hedge for Boeing.

Similarly to the equity hedging cases, a weak link between the CDS (credit) and the stock option (equity) market dynamics is observed. Again, more than 50% of the data does not follow the intuitive credit/equity relationship. This could be a possible reason for the poor performance of the equity option hedge results.

Daimler Chrysler

In the case where shares are used as the hedging instrument, the delta hedge performs well for the weekly rebalancing case, however the daily and monthly cases do not perform well. From Figure 6.5 Plot (a.1), we can see for the daily rebalancing case, there is a mirroring property between the accumulated changes in the CDS position and the accumulated changes in the equity hedge position. However, this mirroring property is not symmetric about the the level 0, thus $\lambda(0, t)$ is not close to zero. This same property holds for the monthly rebalancing case.

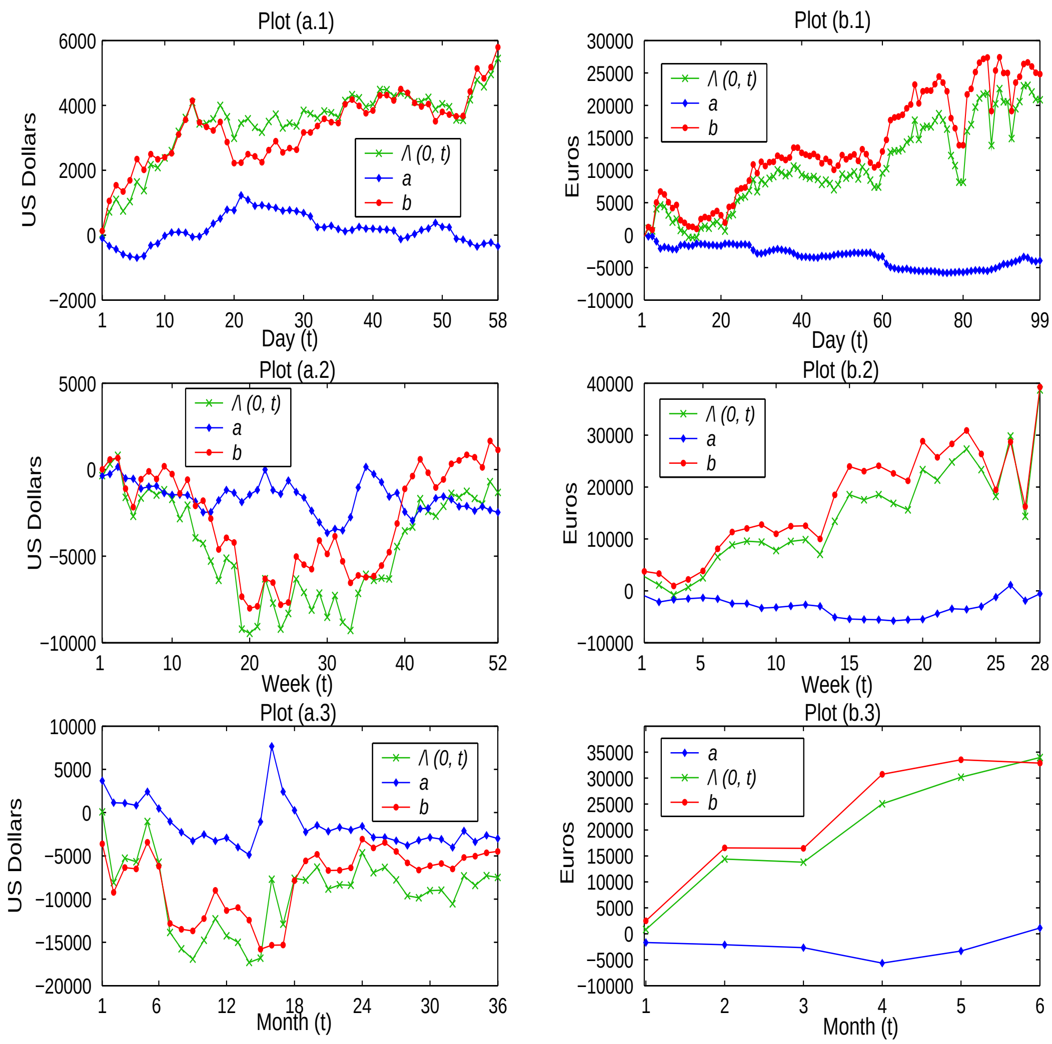

Figure 6.5: Plot (a.1), (a.2) and (a.3) are the results of the equity delta hedges, for the Daimler Chrysler company, for daily, weekly and monthly rebalancing, respectively. Plot (b.1), (b.2) and (b.3) are the results of the equity option delta hedges, for the Daimler Chrysler company, for daily, weekly and monthly rebalancing, respectively.

The weekly case displays a mirroring property, that is approximately symmetric about 0 (see Figure 6.5, Plot (a.2)). For the weekly rebalancing case, the hedge performs well initially (since $\lambda(0, t)$ is close to zero). During the middle of the hedging interval the hedge does not perform too well, (from Week 11 to Week 33) it diverges away from zero. However, at the end the hedge performs well since $|\lambda(0, t)| < |a_t|$, and $\lambda(0, t)$ stabilises around the level 0. At the end of the hedging time interval, the weekly equity hedge successfully reduces the change in value of the CDS position by $1 - |\lambda(0, T_h)|/|a_{T_h}| = 46\%$. The final equity delta hedge results for Daimler Chrysler are:

Daimler Chrysler	Daily Rebalancing	Weekly Rebalancing	Monthly Rebalancing
λ	\$4303.83	−\$1321.62	−\$7497.30
a	−\$477.62	−\$2461.82	−\$3000.89
b	\$4781.54	\$1140.20	−\$4496.41

Table 6.4: The final results of the equity delta hedge for Daimler Chrysler.

Again for the majority of the data, the intuitive relationship between credit risk and equity does not hold for all three equity hedges. However, the relationship holds more often than with the data for Boeing. The daily and monthly equity hedge cases did not fare well. The weekly case was successful at the initial and end period of the hedging time interval. This could be due to the strong credit/equity relationship during these periods. For the weekly case the relationship held for the majority of the data at the beginning and end of the hedging time interval. The weekly rebalancing equity hedge was the only successful hedge.

When a stock option is used as a hedging instrument, the hedging for all three rebalancing cases did not fare well. The accumulated changes in the hedge position (b_t), are disproportionate to the changes in the CDS value (a_t). The hedge positions are too large, and it eliminates the mirroring property. For all rebalancing cases $\lambda(0, t)$ is considerably far from the level 0 (see Figure 6.5, Plot(b.1), Plot(b.2) and Plot(b.3)). The profits from the equity hedge positions are excessive relative to the losses from the CDS positions. The final equity option delta hedge results for Daimler Chrysler are: All of the equity option hedges did not perform well. This could be due to

Daimler Chrysler	Daily Rebalancing	Weekly Rebalancing	Monthly Rebalancing
λ	€20858.65	€38681.54	€33957.82
a	- €3955.38	- €579.67	€1086.29
b	€24814.03	€39261.21	€32871.53

Table 6.5: The final results of the equity option delta hedge for Daimler Chrysler.

the weak link between between the CDS (credit) and the stock option (equity) market. Similar to the equity hedging scenario, the majority of the stock data, for all rebalancing cases, does not follow the intuitive credit/equity relationship. However, this relationship holds more often in the monthly and weekly than the daily case.

Deutsche Telekom

For the instance where shares are used as the hedging instrument, the hedge performs well for the daily and weekly rebalancing cases. All three rebalancing cases, have the mirroring property between the accumulated changes in the CDS position (a) and the equity hedge position (b) (See Figure 6.6, Plot (a.1), (a.2) and (a.3)).

For the daily and weekly rebalancing cases, the accumulated changes (a and b), are approximately symmetric about the level 0, since the net results $\lambda(0, t)$ for most of the time points are between a and b. The hedge performs well for the daily and weekly cases as the absolute value of the accumulated changes in the delta neutral portfolios $\lambda(0, t)$ is less than the absolute value of the accumulated changes in the CDS position a, i.e. $|\lambda(0, t)| < |a|$. The equity hedge successfully reduced the change in value of the CDS position by 44% and 25% for the daily and weekly case respectively. The equity hedge did not perform well for the monthly case. The monthly case has the mirroring property, but is not symmetric about the level 0, consequently the absolute value of the accumulated change in the delta neutral portfolio ($\lambda(0, t)$) is greater than the absolute value of the accumulated change in the CDS value (b), at the end of the hedge period.

The final equity delta hedge results for Deutsche Telekom are:

Deutsche Telekom	Daily Rebalancing	Weekly Rebalancing	Monthly Rebalancing
λ	\$892.87	\$2143.60	−\$6690.01
a	−\$1607.45	−\$2841.69	−\$1358.42
b	\$2500.32	\$4985.29	−\$5331.59

Table 6.6: The final results of the equity delta hedge for Deutsche Telekom.

The majority of the data points for the daily and weekly cases follow the intuitive relationship between CDS spreads and share prices. However, the majority of the data for the monthly case does not. This is a possible reason for why the hedge results perform well for the daily and weekly case but not for the monthly case.

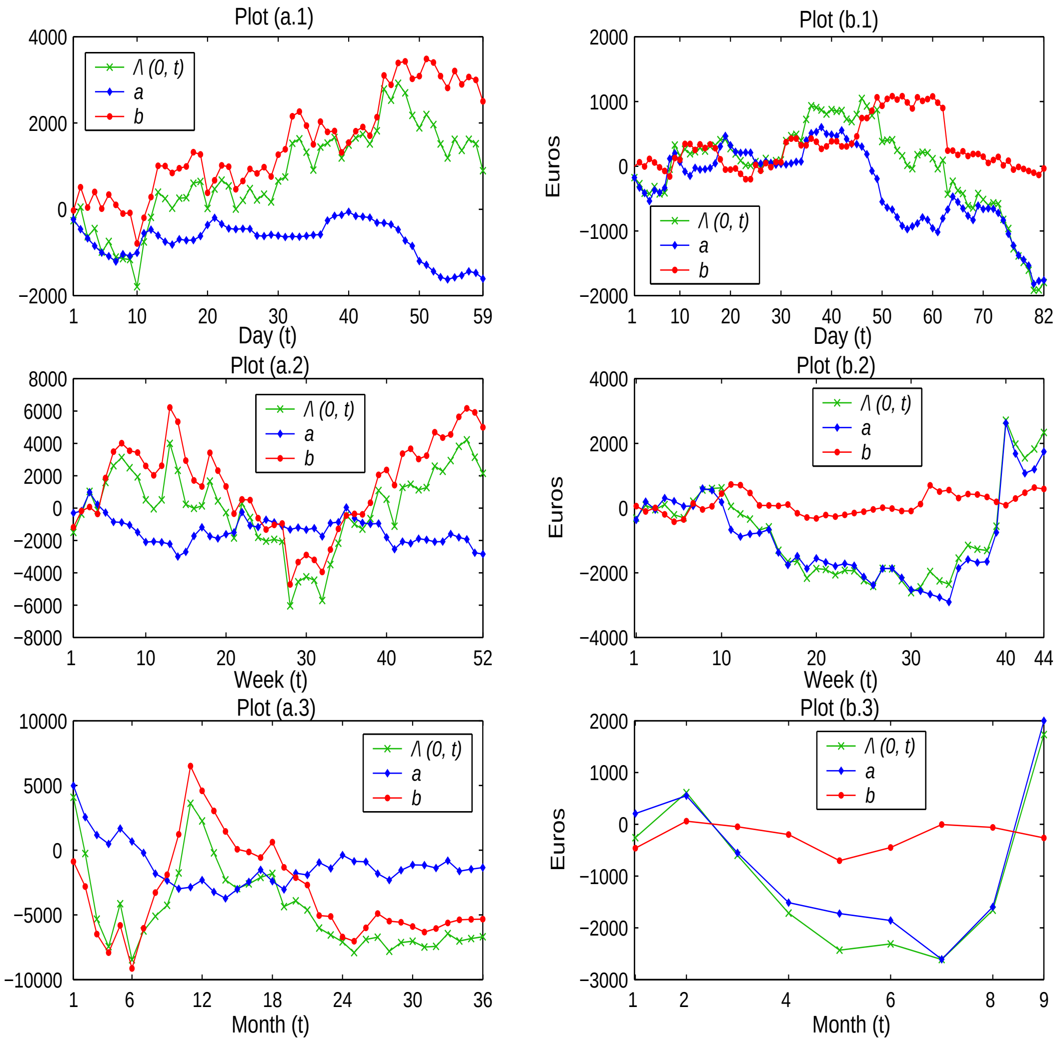

Figure 6.6: Plot (a.1), (a.2) and (a.3) are the results of the equity delta hedges, for the Deutsche Telekom company, for daily, weekly and monthly rebalancing, respectively. Plot (b.1), (b.2) and (b.3) are the results of the equity option delta hedges, for the Deutsche Telekom company, for daily, weekly and monthly rebalancing, respectively.

For the instance where share options are used as a hedging instrument, the daily and weekly rebalancing cases did not perform well since $|\lambda(0, T_h)| > |a_{T_h}|$. There exists no mirroring property between a_t and b_t for the weekly case, but there is a weak mirroring property for the daily case. The monthly cases at the end of the hedging interval is successful in reducing the change in value of the CDS position by 13%. This is only a slight reduction. However, for the monthly case the hedge before the end did not perform well. The final equity option delta hedge results for Deutsche Telekom are:

Deutsche Telekom	Daily Rebalancing	Weekly Rebalancing	Monthly Rebalancing
λ	- €1798.47	€2333.09	€1732.74
a	- €1763.00	€1741.66	€1999.15
b	- €35.47	€591.43	- €266.41

Table 6.7: The final results of the equity option delta hedge for Deutsche Telekom.

For most part the stock option data did not follow the intuitive co-movement between CDS spreads and equity values. This could be a reason for the poor hedge results.

Ford

For the instance where shares are used as the hedging instrument, the monthly rebalancing case is the only hedge that is successful. For the daily and weekly cases, both equity delta hedges have the mirroring property between the accumulated changes in the CDS position (a) and the accumulated changes in the equity hedge position (b). However, they are not symmetric about the level 0. Thus the final hedge results are not satisfactory (since $|\lambda(0, t)| > |a|$), but they perform well for the period 0 to 25 days and 0 to 10 weeks (see Figure 6.7, Plot (a.1), (a.2)). The monthly hedge is an example of a very successful hedge. It has the mirroring property and it is approximately symmetric about the level 0. Throughout the monthly equity hedge case, the accumulated changes in the delta neutral portfolio $\lambda(0, t)$ lies between a and b, and it ends close to zero (see Figure 6.7, Plot (a.3)). At the end of the hedging interval, the monthly equity hedge successfully reduced the change in value of the CDS position by 75%.

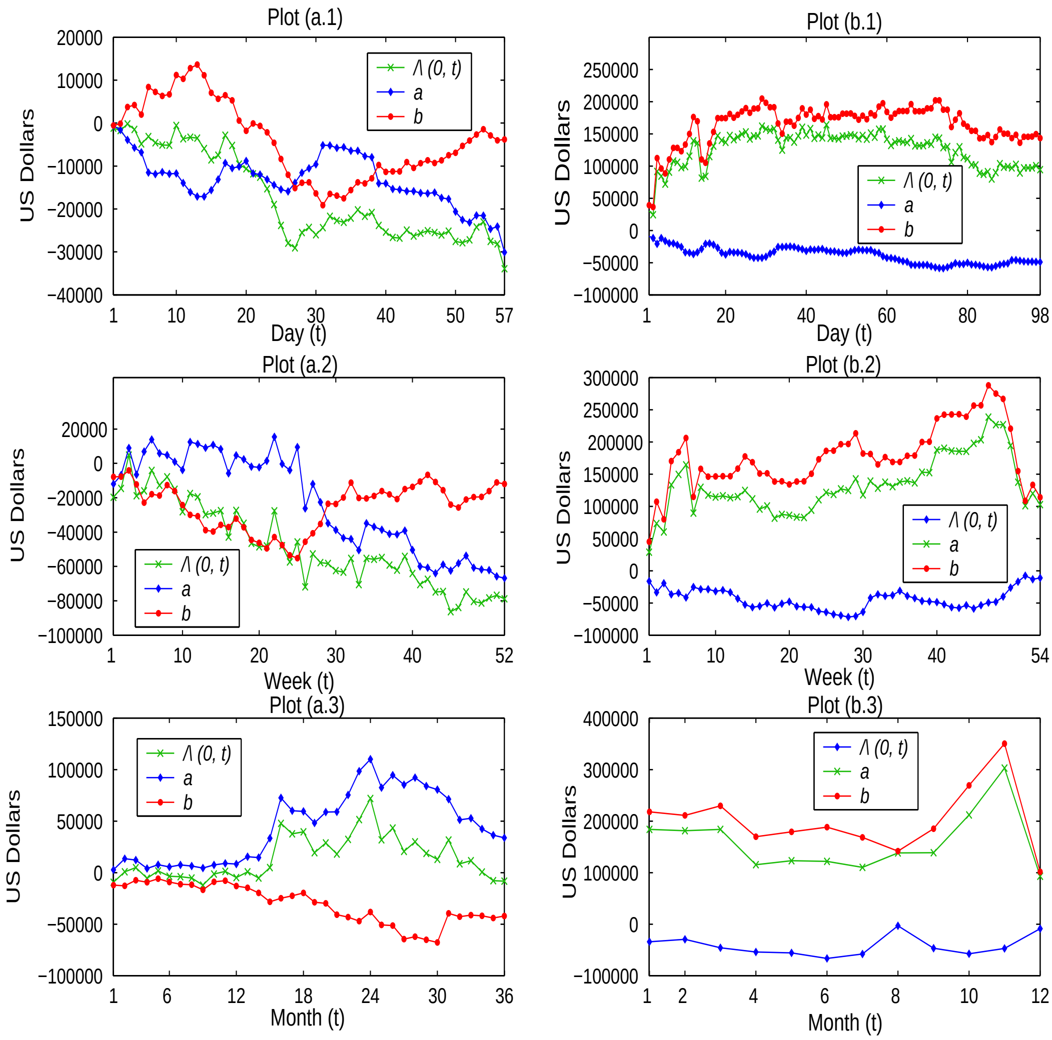

Figure 6.7: Plot (a.1), (a.2) and (a.3) are the results of the equity delta hedges, for the Ford company, for daily, weekly and monthly rebalancing, respectively. Plot (b.1), (b.2) and (b.3) are the results of the equity option delta hedges, for the Ford company, for daily, weekly and monthly rebalancing, respectively.

The final equity delta hedge results for Ford are:

Ford	Daily Rebalancing	Weekly Rebalancing	Monthly Rebalancing
λ	−\$33922.20	−\$78867.60	−\$8278.23
a	−\$30113.10	−\$66827.30	−\$33774.77
b	−\$3809.10	−\$12040.30	−\$42053

Table 6.8: The final results of the equity delta hedge for Ford.

A possible explanation for the poor results of the daily and weekly equity delta hedges is the non-intuitive co-movement of market CDS spreads and share prices, especially for at the end of the hedging periods. This hedge shows that the link between changes in CDS values and share prices is stronger between monthly intervals than daily and weekly intervals.

For the case where share options are used as the hedging instrument, all three rebalancing hedges do not perform well. The equity option hedge position is too large and the profits from this hedge position are disproportionate to the CDS losses. Throughout the equity option hedges there is no strong mirroring property. For all three rebalancing cases $|\lambda(0,\, t)| > |a|$ throughout all of the hedging intervals. The final equity option delta hedge results for Ford are:

Ford	Daily Rebalancing	Weekly Rebalancing	Monthly Rebalancing
λ	\$94304.75	\$102662.04	\$92808.47
a	−\$49088.81	−\$11258.11	−\$8728.33
b	\$143393.55	\$113920.15	\$101536.80

Table 6.9: The final results of the equity option delta hedge for Ford.

The majority of the stock option data for Ford, did follow the intuitive co-movement of market CDS spreads and stock option prices, however all three equity option hedges failed.

General Motors

For the instance where shares are used as the hedging instrument, the weekly and monthly rebalance cases succeed in hedging the value of the position in the CDS; since at the end of the hedging time intervals $|\lambda(0, T_h)| < |a_{T_h}|$, for both situations. However, this does not occur for the daily rebalancing case, and for the most part $|\lambda(0,t)| > |a_t|$ throughout the hedging interval (see Figure 6.8, Plot (a.1)). For the weekly case, the majority of the time $|\lambda(0,t)| > |a_t|$ holds, however by not much. Thus the $\lambda(0,t)$ does not stay close to 0. The weekly equity hedging case only manages to reduce the change in value of the CDS position by 14%. For the weekly case, the profits from the equity hedge position are minor compared to the CDS losses (see Figure 6.8, Plot (a.2)). The monthly equity hedge performs well: throughout the hedging time interval $|\lambda(0,t)| < |a_t|$ and at the end of the time interval the hedge successfully reduced the change in value of the CDS position by 51%. For the monthly case there exists a mirroring property between a_t and b_t at the beginning of the period however the equity hedge position is unable to mirror the volatile profits in the latter stages (see Figure 6.8, Plot (a.3)). The final equity delta hedge results for General Motors are:

General Motors	Daily Rebalancing	Weekly Rebalancing	Monthly Rebalancing
λ	−\$25894.80	−\$124092.31	\$6678.52
a	−\$ − 24667.30	−\$143667.45	\$13606.02
b	−\$1227.50	\$19575.14	−\$6927.50

Table 6.10: The final results of the equity delta hedge for General Motors.

For the daily and weekly rebalancing cases, the majority of stock data did not follow the intuitive co-movement pattern between market CDS spreads and stock prices. This could be a reason for the daily hedging rebalancing case's failure and the minor hedge achievement from the weekly case. However, 61% of the monthly stock data followed the intuitive credit/equity relationship.

When stock options are used as the hedging instrument, the daily rebalancing case is the only hedge that succesfully hedges since $|\lambda(0, T_h)| < |a_{T_h}|$. At the end of the interval, the hedge reduces

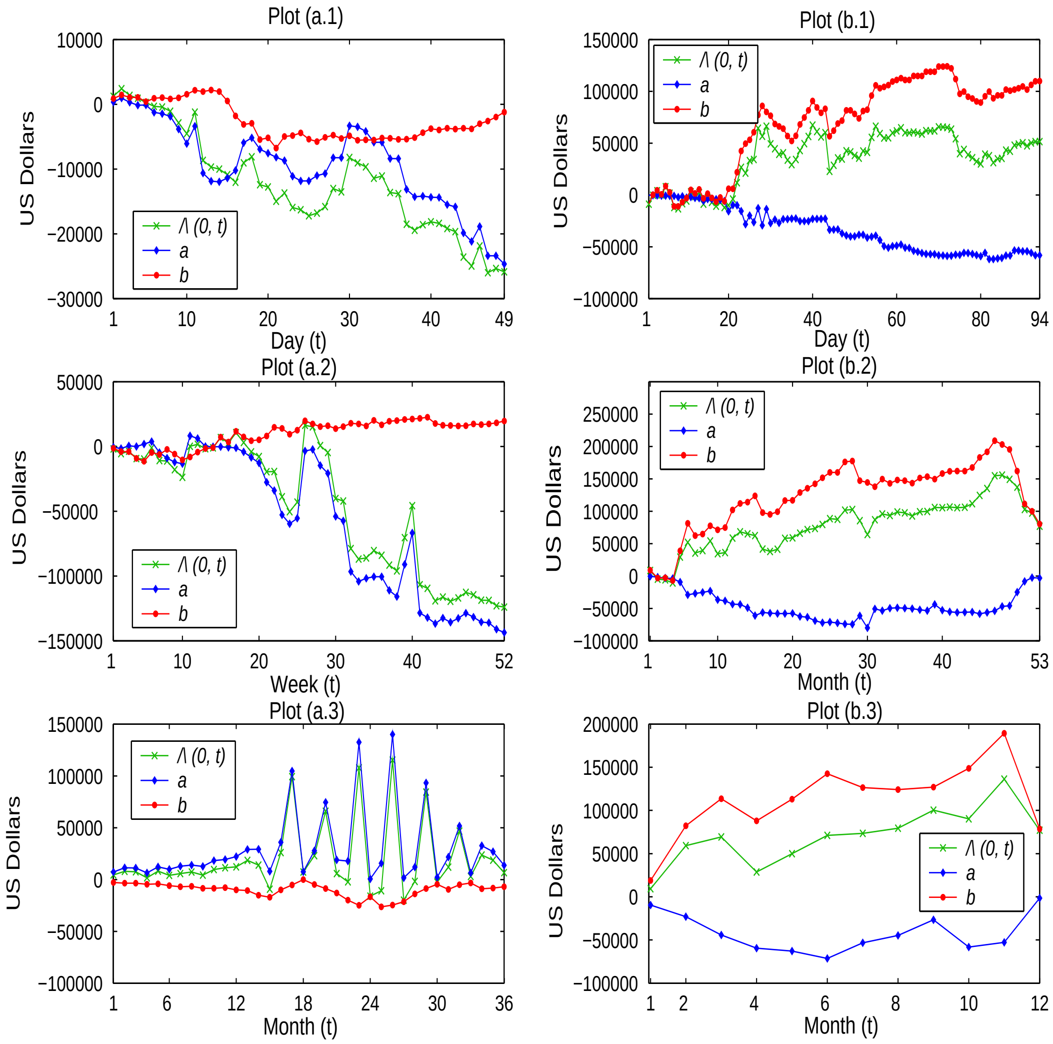

Figure 6.8: Plot (a.1), (a.2) and (a.3) are the results of the equity delta hedges, for the General Motors company, for daily, weekly and monthly rebalancing, respectively. Plot (b.1), (b.2) and (b.3) are the results of the equity option delta hedges, for the General Motors company, for daily, weekly and monthly rebalancing, respectively.

the absolute value of the change in the CDS position a_{T_h} by 11%. The weekly and monthly cases do not meet this hedging condition. For these two rebalancing cases, the profits from the equity option hedge position are too large compared to the losses of the CDS positions (see Figure 6.8, Plot (b.2) and Plot (b.3)). The final equity option delta hedge results for General Motors are:

General Motors	Daily Rebalancing	Weekly Rebalancing	Monthly Rebalancing
λ	\$51396.76	\$76840.89	\$77068.19
a	−\$58321.93	−\$3310.04	−\$1581.05
b	\$109718.69	\$80150.93	\$78649.24

Table 6.11: The final results of the equity option delta hedge for General Motors.

The majority of the equity option data did follow the intuitive co-movement of market CDS spreads and share prices, however the daily hedge was the only successful hedge.

Vodafone

The equity delta hedge results for Vodafone company are poor for the daily and weekly rebalancing cases. From Figure 6.9 (Plot (a.1) and (a.2)), it can be seen that the accumulated changes in the CDS positions (a) are very small compared to the large accumulated changes in the equity hedge positions (b). There is no mirroring property, the equity hedge position seems to move independently and at disproportionate levels relative to the CDS position value. For these two equity hedges $|\lambda(0, T_h)| > |a_{T_h}|$, and thus they did not perform well. The monthly rebalancing case performs well as there is the mirroring property between a_t and b_t, and at the end of the hedging interval $|\lambda(0, T_h)| < |a_{T_h}|$. The monthly equity hedge successfully reduced the change in value of the CDS position by 74%. The final equity delta hedge results for Vodafone are:

Vodafone	Daily Rebalancing	Weekly Rebalancing	Monthly Rebalancing
λ	\$65884.71	\$77173.21	\$ − 28668.56
a	−\$49.51	−\$1691.60	−\$ − 96.42
b	\$65934.22	\$78864.81	\$ − 28572.10

Table 6.12: The final results of the equity delta hedge for Vodafone.

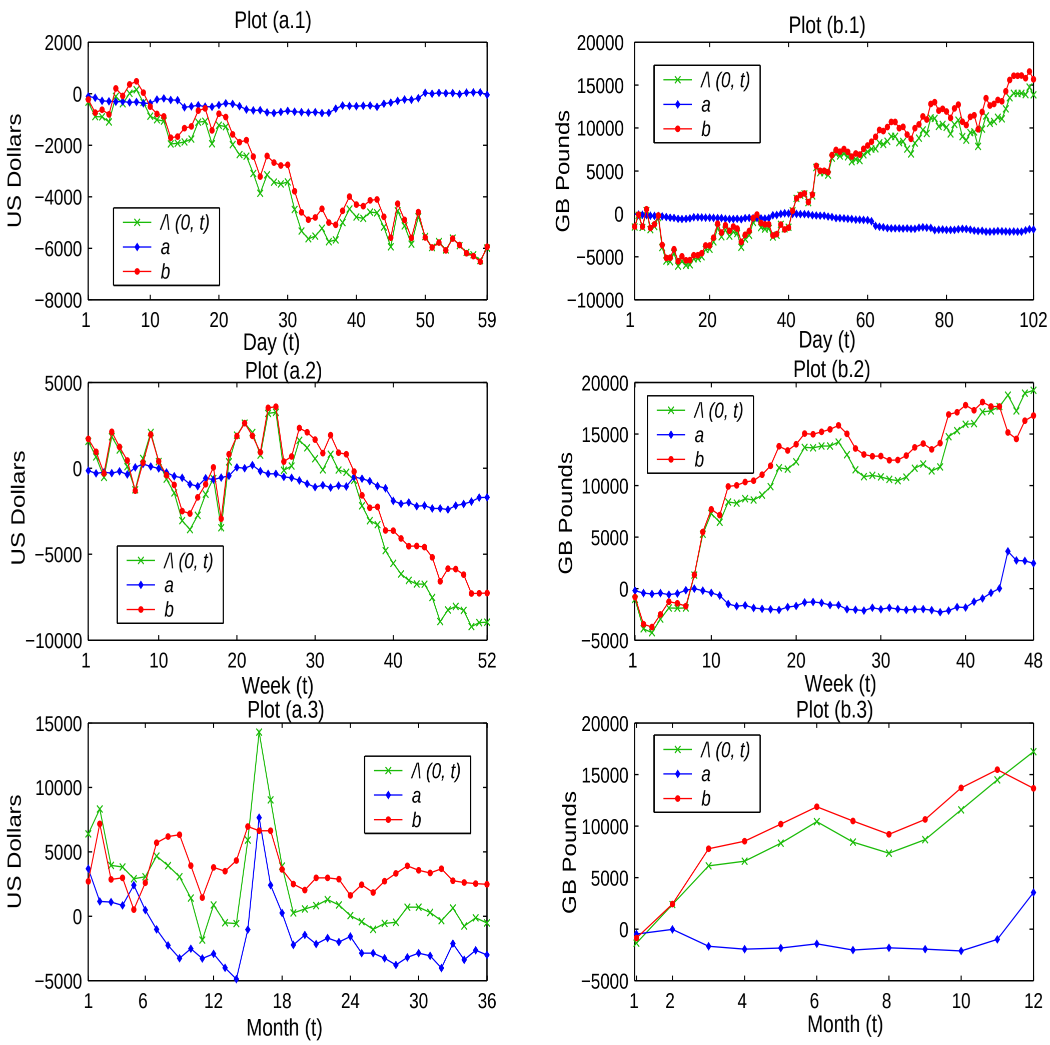

Figure 6.9: Plot (a.1), (a.2) and (a.3) are the results of the equity delta hedges, for the Vodafone company, for daily, weekly and monthly rebalancing, respectively. Plot (b.1), (b.2) and (b.3) are the results of the equity option delta hedges, for the Vodafone company, for daily, weekly and monthly rebalancing, respectively.

For the daily and weekly rebalancing cases the intuitive relationship between CDS spreads and share prices did not hold for more than 50% of the data. Surprisingly, for the monthly data, 67% of the data did follow the intuitive relationship. However, our equity delta hedge scheme was unable to predict the correct market equity delta ratios, which quantifies the sensitivity between the market CDS value and the market share price.

For the instance where stock options are used as hedging instruments, all three rebalancing cases did not fare well. The accumulated changes in the value of equity option hedge positions are too large compared to the accumulated changes in the value of the CDS position. This nullifies the mirroring property. For all cases $|\lambda(0, T_h)| > |a_{T_h}|$. The final equity option delta hedge results for Vodafone are: The majority of stock option data did not follow the intuitive credit/equity relationship. This could be a reason for the poor performance of the equity option hedges.

Vodafone	Daily Rebalancing	Weekly Rebalancing	Monthly Rebalancing
λ	£13848.94	£19227.33	£17198.71
a	- £1809.66	£2447.17	£3554.44
b	£15658.60	£16780.16	£13644.27

Table 6.13: The final results of the equity option delta hedge for Vodafone.

6.6 Conclusion

The simulated delta hedge examples (see Section 6.5.2) verify this hedge procedure is able to theoretically hedge the value of a CDS position with an equity or equity option position. From the market data, we show that it is possible to hedge the value of a CDS position with an equity position, however it is more difficult to do with an equity option position. In the market the link between credit risk and equity is more robust for firms that have a higher credit risk, this is evident from our hedge results. The firms that produced successful hedges all have a credit rating lower than Baa1 (Moody's), only Deutsche Telekom being the exception[17]. The successful hedge results are summarised in Table 6.14. It can be seen from Table 6.14 that most of the successful hedges occurs when rebalancing is done with intervals longer than a day. This suggests that the link between the credit and equity market dynamics is weak for daily intervals but stronger for larger intervals. The majority of the equity data did not follow the intuitive co-movement relationship between CDS spreads and equity prices. This suggests that equity and credit markets seem to move independently or that information is absorbed into the different markets at different rates. This inconsistency occurs more often with firms with higher credit ratings. Our hedge results confirms this reasoning, since most of the successful hedges occur for firms with a low credit rating and where rebalancing is done at long intervals. Another explanation for why our hedges performeed better for firms with a low credit rating, is that the value of equity is influenced by several factors (e.g. dividend rates, interest rates, credit risk). However, credit risk is likely to be the dominant factor for firms with a low credit rating.

Company	Rebalancing	Hedging Instrument	$\|a_{T_h}\| - \|\lambda(0, T_h)\|/\|a_{T_h}\|$
Daimler Chrysler	Weekly	Shares	46%
Deutsche Telekom	Daily	Shares	44%
Deutsche Telekom	Weekly	Shares	25%
Deutsche Telekom	Monthly	Call Option	13%
Ford	Monthly	Shares	75%
General Motors	Weekly	Shares	14%
General Motors	Monthly	Shares	51%
General Motors	Daily	Put Option	11%
Vodafone	Monthly	Shares	74%

Table 6.14: Summary of the successful hedge results.

[17] Deutsche Telekom official credit rating is A3, however Moody's equity implied credit rating for Deutsche Telekom is Baa3.

Only two equity option hedge results were successful. The inaccuracy in the equity hedges are magnified in the equity option hedging cases. From our analysis it can be seen that is possible to hedge the value of a CDS with equity, but it is more difficult to hedge with equity options.

6.7 Further Research Directions

The main reason for why some hedges do not produce good results is the non-intuitive co-movement relationship between CDS spreads and equity prices. Further investigations can be made, to construct a model that is able to explain this characteristic that is more evident for low credit risk firms. Since the market is incomplete, using a combination of shares and share options simultaneously in the hedging procedure should produce better hedge results, since this is a step towards completing the market. Our calibration procedure is a very simple local minimisation algorithm. A more stable and global calibration procedure could possibly provide more accurate results, an example of this is the non-parametric calibration procedure by Cont & Tankov (2004*a*). Since we use the jump-diffusion framework the market is incomplete. We use the simple delta hedge technique. However one could also use more complex incomplete market risk minimising techniques such as: *utility maximisation, quadratic hedging, optimal martingale measures* (see McWalter (2006) and Cont & Tankov (2004*b*)). Since our model does not admit closed-form solutions for prices of the financial instruments under consideration, we need to calculate the prices and delta hedge ratios using Monte Carlo simulation. Mallivian calculus (*theory of variational stochastic calculus*) provides a much more efficient[18] method for calculating derivatives (hedge ratios) of random variables (see Sanz-Solé (2005)). In order to make a stronger conclusion for which type of firms this hedging model will perform well for, more firm's should be analysed.

[18]Convergence is quicker, especially when calculating the delta ratios of financial instruments with discontinuous payoffs.

Appendix A

Jump Processes: Miscellaneous Results

A.1 Itô's Formula for Diffusions with Jumps

Consider a jump diffusion process

$$X_t = \mu + \sigma W_t + J_t = X_t^c + J_t,$$

where J is a compound Poisson process and X^c is the continuous part of X:

$$J_t = \sum_{i=1}^{N_t} \Delta X_i, \qquad X_t^c = \mu + \sigma W_t,$$

where N_t is a Poisson process, representing the number of jumps in the interval $[0, t]$. Let $G_t = f(X_t)$, where $f \in C^2(\mathbb{R})$ and τ_i, $i = 1, \ldots, N_T$ the jump times of X. The notation C^2 represents the space of continuous, twice differentiable functions. On the interval (τ_i, τ_{i+1}), X follows the following dynamics

$$\mathrm{d}X_t = \mathrm{d}X_t^c = \mu \, \mathrm{d}t + \sigma \, \mathrm{d}W_t.$$

Applying Itô's formula on the above diffusion part, we obtain

$$G_{\tau_{i+1}^-} - G_{\tau_i} = \int_{\tau_i}^{\tau_{i+1}^-} \frac{\sigma^2}{2} f''(X_t) \, \mathrm{d}t + \int_{\tau_i}^{\tau_{i+1}^-} f'(X_t) \, \mathrm{d}X_t.$$

If a jump of size ΔX_t, occurs then the resulting change in G_t is given by $f(X_{t^-} + \Delta X_t) - f(X_{t^-})$. The total change in G_t can therefore be written as the sum of these two contributions:

$$\begin{aligned} f(X_t) - f(X_0) = & \int_0^t \frac{\sigma^2}{2} f''(X_s) \, \mathrm{d}s + \int_0^t f'(X_s) \, \mathrm{d}X_s^c \\ & + \sum_{0 \le s \le t, \, \Delta X_s \ne 0} [f(X_{s^-} + \Delta X_s) - f(X_{s^-})]. \end{aligned}$$

The following proposition is derived from Cont & Tankov (2004*b*).

Proposition A.1.1. ***Itô's formula for jump-diffusion processes***. *Let X be a diffusion process with jumps:*

$$X_t = X_0 + \int_0^t \mu_s \, \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}W_s + \sum_{i=1}^{N_t} \Delta X_i,$$

where μ_s and σ_s are continuous processes with

$$\mathbb{E}\left[\int_0^T \sigma_t^2 \mathrm{d}t\right] < \infty.$$

Then, for any $C^{1,2}$ function, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$, the processe $G_t = f(t, X_t)$ can be represented as:

$$\begin{aligned} f(t, X_t) - f(0, X_0) = & \int_0^t \left[\frac{\partial f}{\partial s}(s, X_s) + \frac{\partial f}{\partial x}(s, X_s)\mu_s \right] \mathrm{d}s \\ & + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_s)\sigma_s^2 \, \mathrm{d}s + \int_0^t \frac{\partial f}{\partial x}(s, X_s)\sigma_s \, \mathrm{d}W_s \\ & + \sum_{i \geq 1,\, \tau_i \leq t} [f(X_{\tau_i^-} + \Delta X_i) - f(X_{\tau_i^-})]. \end{aligned}$$

In differential notation:

$$\begin{aligned} dG_t = & \frac{\partial f}{\partial t}(t, X_t)\, \mathrm{d}t + \mu_t \frac{\partial f}{\partial x}(t, X_t)\, \mathrm{d}t \\ & + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}(t, X_t)\, \mathrm{d}t + \frac{\partial f}{\partial x}(t, X_t)\sigma_t \, \mathrm{d}W_t \\ & + [f(X_{t^-} + \Delta X_t) - f(X_{t^-})]. \end{aligned}$$

Proof. See Applebaum (2004) [Thm. 4.4.10, p. 229]. □

A.2 Girsanov Theorem for Jump-Diffusion Processes

We state the Girsanov theorem specifically for the jump process V_t. We assume V follows the following dynamics

$$\mathrm{d}V_t/V_{t^-} = \mu_V \, \mathrm{d}t + \sigma \, \mathrm{d}W_t + Y_t \, \mathrm{d}N_t, \tag{A.1}$$

where N_t is a Poisson process with intensity parameter λ, and $Y_t > 0$ represents the percentage change in V at time t. If there is a jump at time $t = \tau_i$ then $Y_t = (V_{t^+} - V_{t^-})/(V_{t^-})$, otherwise $Y_t = 0$. The stochastic processes W_t, N_t, and Y_t are mutually independent. Let A_i be the logarithm of the ratio of V after and before the i^{th} jump:

$$A_i := \ln V_{\tau_i^+} - \ln V_{\tau_i^-} = \ln(Y_{\tau_i} + 1).$$

We assume A_i is normally distributed:

$$A_i \sim \mathrm{N}(\mu_A, \sigma_A^2).$$

Let J_t be the sum of the logarithms of the ratio of jump sizes in the interval $[0, t]$

$$J_t = \sum_{i=1}^{N_t} A_i.$$

Let

$$\hat{V}_t = V_t e^{-rt},$$

where V_t follows a jump-diffusion process (A.1). Thus by using the Itô formula for jump-diffusion processes, the stochastic differential equation for $\hat{V}_t$ is

$$\mathrm{d}\hat{V}_t/\hat{V}_{t^-} = (\mu_V - r)\, \mathrm{d}t + \sigma \, \mathrm{d}W_t + Y_t \, \mathrm{d}N_t. \tag{A.2}$$

Equation (A.2) can be written as

$$\mathrm{d}\hat{V}_t/\hat{V}_{t^-} = (\mu_V - r)\, \mathrm{d}t + \sigma \, \mathrm{d}W_t + \int_{\mathbb{R}} Y_t \, \widetilde{N}(\mathrm{d}t, \, \mathrm{d}A), \tag{A.3}$$

where $\widetilde{N}(t, A) = N(t, A) - \lambda t$ is the compensated Poisson random measure, and the jump-size space is the real line $\mathbb{R}$ (see Last & Brandt (1995) for an explanation of this representation).

Theorem A.2.1. ***Girsanov theorem for jump-diffusion processes****. Let $\hat{V}_t$ be a jump-diffusion process of the form*

$$\mathrm{d}\hat{V}_t/\hat{V}_{t^-} = (\mu_V - r)\, \mathrm{d}t + \sigma \, \mathrm{d}W_t + \int_{\mathbb{R}} Y_t \, \widetilde{N}(\mathrm{d}t, \, \mathrm{d}A).$$

Assume there exist predictable processes $\Theta(t) = \Theta(t, \omega) \in \mathbb{R}$ *and* $\beta(t, A) = \beta(t, A, \omega) \in \mathbb{R}$ *such that*

$$\sigma\Theta(t) + \int_{\mathbb{R}} Y_t \beta(t, A)\lambda \, \mathrm{d}A = \mu_V - r,$$

and such that the process

$$\begin{aligned} \rho_t := \exp\Bigg[& -\int_0^t \Theta(s)\, \mathrm{d}W_s - \frac{1}{2}\int_0^t \Theta^2(s)\, \mathrm{d}s + \int_0^t \int_{\mathbb{R}} \ln(1-\beta(s, A))\, \widetilde{N}(\mathrm{d}s, \mathrm{d}A) \\ & + \lambda \int_0^t \int_{\mathbb{R}} \ln(1-\beta(s, A)) + \beta(s, A)\, \mathrm{d}A\, \mathrm{d}s \Bigg]; \qquad 0 \le t \le T \end{aligned} \tag{A.4}$$

is well-defined and satisfies $\mathbb{E}^{\mathbb{P}}[\rho_T] = 1$*. Define the probability measure* $\mathbb{Q}$ *on* $\mathscr{F}_T$ *by the Radon-Nikodým derivative*

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = \rho_t.$$

Then $\hat{V}_t$ *is a local martingale with respect to* $\mathbb{Q}$*, i.e.*

$$\mathbb{E}^{\mathbb{Q}}[\hat{V}_T | \mathscr{F}_t] = \mathbb{E}^{\mathbb{P}}[\hat{V}_T \rho_T | \mathscr{F}_t] = \hat{V}_t$$

Proof. See Øksendal & Sulem (2005) [Thm. 1.31, p. 15-16]. □

A.3 The Firm's Value Process under the Risk-Neutral Measure

In our constructed mathematical framework, firm's value process V_t is assumed to follow a jump-diffusion process, under the real-world measure $\mathbb{P}$,

$$\frac{\mathrm{d}V_t}{V_{t^-}} = \mu_V \, \mathrm{d}t + \sigma \, \mathrm{d}W_t + Y_t \, \mathrm{d}N_t.$$

See Section A.1 for the interpretation of the parameters. Recall that:

$$A_i = \ln V_{\tau_i^+} - \ln V_{\tau_i^-} = \ln(Y_{\tau_i} + 1),$$

and A_i is normally distributed:

$$A_i \sim \mathrm{N}(\mu_A, \sigma_A^2).$$

Let J_t be the sum of the logarithms of the ratio of jump sizes in the interval [0,t]:

$$J_t = \sum_{i=1}^{N_t} A_i.$$

Let

$$\kappa := \mathbb{E}^{\mathbb{P}}(Y_t) = \exp\left(\mu_A + \frac{\sigma_A^2}{2}\right) - 1.$$

Under the risk-neutral measure $\mathbb{Q}$ (see Section A.2 for the Radon-Nikodým derivative ρ_t that characterises measure $\mathbb{Q}$), the stochastic differential equation for V is

$$\frac{\mathrm{d}V_t}{V_{t^-}} = (r - \lambda^{\mathbb{Q}}\kappa^{\mathbb{Q}})\, \mathrm{d}t + \sigma \, \mathrm{d}W_t^{\mathbb{Q}} + Y_t^{\mathbb{Q}} \, \mathrm{d}N_t^{\mathbb{Q}}, \tag{A.5}$$

where $W_t^{\mathbb{Q}}$ is a Brownian motion process under the measure $\mathbb{Q}$ and $Y_t^{\mathbb{Q}}$ is the percentage jump sizes under the measure $\mathbb{Q}$. The distribution of $Y_t^{\mathbb{Q}}$ is determined by the risk-neutral distribution of $A^{\mathbb{Q}}$ which has pdf denoted by $f^{\mathbb{Q}}(A)$. Furthermore $\kappa^{\mathbb{Q}} := \mathbb{E}^{\mathbb{Q}}(Y_t^{\mathbb{Q}}) = \int_{\mathbb{R}} Y_t^{\mathbb{Q}} f^{\mathbb{Q}}(A)\, \mathrm{d}A$, and $N_t^{\mathbb{Q}}$ is a Poisson process under the measure $\mathbb{Q}$ with intensity $\lambda^{\mathbb{Q}}$ (see Metwally & Atiya (2002)).

The Radon-Nikodým derivative process $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\big|_{\mathscr{F}_t} := \rho_t$ is characterised by two processes $\Theta(t)$ and $\beta(t, A)$ (see Section A.2). If we let

$$\beta(t, A) = (1 - \Theta_A(t, A)) f(A),$$

where $\Theta_A(t,\, A) = \Theta_A(t,\, A,\, \omega) \in \mathbb{R}$ and $f(A)$ is the pdf of A under measure $\mathbb{P}$. We can relate the risk-neutral distributions of the stochastic elements in V to their real-world distributions[1]:

$$\begin{aligned}
\lambda^{\mathbb{Q}} &= \lambda \int_{\mathbb{R}} \Theta_A(t,\, A) f(A)\, \mathrm{d}A, \\
f^{\mathbb{Q}}(A) &= \frac{\Theta_A(t,\, A) f(A)}{\int_{\mathbb{R}} \Theta_A(t,\, A) f(A)\, \mathrm{d}A}, \\
W_t^{\mathbb{Q}} &= W_t + \Theta(t)\, t.
\end{aligned}$$

If we let

$$J_t^{\mathbb{Q}} = \sum_{i=1}^{N_t^{\mathbb{Q}}} A_i^{\mathbb{Q}},$$

then by using Itô's formula for jump-diffusion processes (see Section A.1.1), the solution for the stochastic differential equation (A.5) is

$$V_t = V_0 \exp\left[\left(r - \frac{\sigma^2}{2} - \lambda^{\mathbb{Q}} \kappa^{\mathbb{Q}}\right) t + \sigma W_t^{\mathbb{Q}} + J_t^{\mathbb{Q}}\right]. \tag{A.6}$$

From equation (A.4), we can see that the Radon-Nikodým derivative ρ_t is not unique. Thus the measure $\mathbb{Q}$ is not unique and differs for different combinations of $\Theta(t)$ and $\Theta_A(t,\, A)$. This implies that the market is incomplete (see Section 2.2.3). The term $\Theta(t)$ is named the market price of diffusion risk and the term $\Theta_A(t,\, A)$ is named the market price of jump-risk.

We apply the Merton (1976) simplistic approach to select the risk-neutral measures $\mathbb{Q}$ to price a firm's securities and credit default swaps written on a firm, our model can easily be changed to account for different choices of $\mathbb{Q}$. Merton proposed that *jump risk* is diversifiable[2], therefore no risk premium is attached to it: in mathematical terms $\Theta_A(t,\, A) = 1$, which implies that

$$\Theta(t) = \frac{u_V - r}{\sigma}.$$

This means that the risk neutral $\mathbb{Q}$ properties of the jump component of V_t are the same as its statistical (real-world) $\mathbb{P}$ properties. In particular, the distribution of jump times and jump sizes is unchanged:

$$\begin{aligned}
\lambda^{\mathbb{Q}} &= \lambda, \\
f^{\mathbb{Q}}(A) &= f(A), \\
\kappa^{\mathbb{Q}} &= \kappa.
\end{aligned}$$

Under this assumption the firm's value process under the risk-neutral measure $\mathbb{Q}$ is:

$$V_t = V_0 \exp\left[\left(r - \frac{\sigma^2}{2} - \lambda\kappa\right) t + \sigma W_t^{\mathbb{Q}} + J_t\right],$$

where $W_t^{\mathbb{Q}}$ is a Brownian motion process under the measure $\mathbb{Q}$. This approach has the effect of correcting the price of the contingent claim for the average effect of jumps[3].

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[1] See Metwally & Atiya (2002) for a discussion of these relationships.

[2] The jump component of V, represents important new information that has a nonmarginal impact on V. Merton proposed that this information is firm or industry specific. Thus the jump component is uncorrelated with the market (jump risk is *nonsystematic*) and is diversifiable.

[3] See Cont & Tankov (2004*a*) for more details on the effect on Merton's assumption on the price and hedging error, and different methods to hedge and price under a jump-diffusion framework.

Appendix B

First Passage Time

Let

$$V_t = V_0 \exp\left[\left(\mu_V - \frac{\sigma^2}{2}\right)t + \sigma W_t\right], \tag{B.1}$$

and

$$\tau^* = \inf\{s \geq 0 : V_s \leq b\}.$$

In order to find the probability density function of the *first hitting time* of a geometric Brownian motion process V to a constant level b, $f(\tau^*)$, we firstly investigate the distributions of the supremum and infimum of a standard Brownian motion process W_t. The following examination of first hitting times is derived from Glasserman (2004).

B.1 The Hitting Time of a Standard Brownian Motion Process

The maximum of a standard Brownian motion process on the interval $[0,\, t]$ is denoted by

$$M_t := \sup_{s \leq t} W_s\,.$$

Let us denote the first hitting time of an upper constant barrier $b > W_0$ by a standard Brownian motion process W_t by:

$$\begin{aligned} \tau^*_{M,\,b} &:= \inf\{s \geq 0 : M_s \geq b\} \\ &= \inf\{s \geq 0 : W_s \geq b\}. \end{aligned}$$

B.1.1 Distribution of the Pair $(W_t,\, M_t)$

From the following proposition we are able to obtain the cdf $F(\tau^*_{M,\,b})$.

Proposition B.1.1. *Let* $\Phi(\cdot)$ *be the standard normal cumulative distribution function. Then:*

$$\begin{aligned} &\mathbb{P}\left(W_t \leq b,\, M_t \leq y\right) = \Phi\left(\tfrac{b}{\sqrt{t}}\right) - \Phi\left(\tfrac{b-2y}{\sqrt{t}}\right) && \text{for} \quad y \geq 0,\, b \leq y, \\ &\mathbb{P}\left(W_t \leq b,\, M_t \leq y\right) = \Phi\left(\tfrac{y}{\sqrt{t}}\right) - \Phi\left(\tfrac{-y}{\sqrt{t}}\right) && \text{for} \quad y \geq 0,\, b \geq y, \\ &\mathbb{P}\left(W_t \leq b,\, M_t \leq y\right) = 0 && \text{for} \quad y \leq 0. \end{aligned}$$

Proof. The joint distribution of $(W_t,\, M_t)$ depends on which area of the plane we are in.
Let us show that for $y \geq 0$, $b \leq y$

$$\mathbb{P}\left(W_t \leq b,\, M_t \geq y\right) = \mathbb{P}(W_t \geq 2y - b). \tag{B.2}$$

Define the stopping time $\tau^*_{M,\,y}$ as the first hitting time of an upper level y by a standard Brownian motion process W_t:

$$\tau^*_{M,\,y} = \inf\{s \geq 0 : W_s \geq y\}.$$

Note that $\{\tau^*_{M,\,y} \leq t\} = \{M_t \geq y\}$ and by the continuity of the Brownian motion's paths, we have

$$\tau^*_{M,\,y} = \inf\{s \geq 0 : W_s = y\}$$

and

$$W_{\tau^*_{M,\,y}} = y.$$

Thus

$$\mathbb{P}\Big(W_t \leq b,\, M_t \geq y\Big) = \mathbb{P}\Big(W_t \leq b,\, \tau^*_{M,\,y} \leq t\Big) = \mathbb{P}\left(W_t - W_{\tau^*_{M,\,y}} \leq b - y,\, \tau^*_{M,\,y} \leq t\right).$$

By making use of the Multiplication Theorem[1] the above expression becomes

$$\mathbb{P}\left(W_t - W_{\tau^*_{M,\,y}} \leq b - y \,\middle|\, \tau^*_{M,\,y} \leq t\right) \mathbb{P}\Big(\tau^*_{M,\,y} \leq t\Big). \tag{B.3}$$

Using the equality $\mathbb{P}(A) = \mathbb{E}^{\mathbb{P}}(\mathbb{1}_A)$, the expression (B.3) becomes

$$\begin{aligned} & \mathbb{P}\left(W_t - W_{\tau^*_{M,\,y}} \leq b - y \,\middle|\, \tau^*_{M,\,y} \leq t\right) \mathbb{E}^{\mathbb{P}}\left(\mathbb{1}_{\tau^*_{M,\,y} \leq t}\right) \\ = \;& \mathbb{E}^{\mathbb{P}}\left(\mathbb{1}_{\tau^*_{M,\,y} \leq t}\, \mathbb{P}\left(W_t - W_{\tau^*_{M,\,y}} \leq b - y \,\middle|\, \tau^*_{M,\,y} \leq t\right)\right) \\ = \;& \mathbb{E}^{\mathbb{P}}\left(\mathbb{1}_{\tau^*_{M,\,y} \leq t}\, \mathbb{P}\left(W_{t+\tau^*_{M,\,y}-\tau^*_{M,\,y}} - W_{\tau^*_{M,\,y}} \leq b - y \,\middle|\, \tau^*_{M,\,y} \leq t\right)\right). \end{aligned} \tag{B.4}$$

Using the strong Markov property[2] of Brownian motion, the expression (B.4) reduces to

$$\mathbb{E}^{\mathbb{P}}\left(\mathbb{1}_{\tau^*_{M,\,y} \leq t}\, \mathbb{P}\left(W_t - W_{\tau^*_{M,\,y}} \leq b - y\right)\right) \tag{B.5}$$

also by noting that W and $-W$ have the same distribution, the expression (B.5) equals

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}}\left(\mathbb{1}_{\tau^*_{M,\,y} \leq t}\, \mathbb{P}\Big(-(W_t - W_{\tau^*_{M,\,y}}) \leq b - y\Big)\right) \\ = \;& \mathbb{E}^{\mathbb{P}}\left(\mathbb{1}_{\tau^*_{M,\,y} \leq t}\, \mathbb{P}\Big(W_t - W_{\tau^*_{M,\,y}} \geq y - b\Big)\right) \\ = \;& \mathbb{P}\Big(\tau^*_{M,\,y} \leq t\Big)\, \mathbb{P}\Big(W_t \geq 2y - b\Big) \\ = \;& \mathbb{P}\Big(\tau^*_{M,\,y} \leq t,\, W_t \geq 2y - b\Big). \end{aligned} \tag{B.6}$$

Finally, by noting that $(\tau^*_{M,\,y} \leq t) \subset (W_t \geq 2y - b)$, and that if an event $A \subset B$ then $\mathbb{P}(A \cap B) = \mathbb{P}(B)$, we can reduce (B.6) to

$$\mathbb{P}(W_t \geq 2y - b). \tag{B.7}$$

To reduce

$$\mathbb{P}(W_t \leq b,\, M_t \leq y), \tag{B.8}$$

we only need to observe that

$$\mathbb{P}(A \cap \bar{B}) = \mathbb{P}(A) - \mathbb{P}(A \cap B), \tag{B.9}$$

where $\bar{B}$ is the complement of B. Applying relation (B.9) to expression (B.8) results in

$$\mathbb{P}(W_t \leq b,\, M_t \leq y) = \mathbb{P}(W_t \leq y) - \mathbb{P}(W_t \leq y,\, \tau^*_{M,\,y} \leq t).$$

Noting that $W_t \sim \mathrm{N}(0,\, t)$, for $y \geq 0$, $b \leq y$,

$$\mathbb{P}\left(W_t \leq b,\, M_t \leq y\right) = \Phi\left(\frac{b}{\sqrt{t}}\right) - \Phi\left(\frac{b - 2y}{\sqrt{t}}\right).$$

Since $M_t \geq W_t$, for $0 \leq y \leq b$,

$$\mathbb{P}(W_t \leq b, M_t \leq y) = \mathbb{P}(W_t \leq y, M_t \leq y) = \mathbb{P}(M_t \leq y) = \Phi\left(\frac{y}{\sqrt{t}}\right) - \Phi\left(\frac{-y}{\sqrt{t}}\right).$$

Since $M_t \geq M_0$, for $y \leq 0$,

$$\mathbb{P}(W_t \leq b, M_t \leq y) = 0.$$

□

[1] The *Multiplication Theorem*: for any events A and B,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A|B)\mathbb{P}(B).$$

[2] See Karatzaz & Shreve (1991) for the definition of the strong Markov property.

B.1.2 Distribution of M_t and $\tau^*_{M,\,b}$

From the distribution of M_t, it is possible to attain the distribution of the first hitting time of an upper barrier b, $\tau^*_{M,\,b}$ by a standard Brownian motion process.

Proposition B.1.2. *The random variable M_t has the same distribution as the random variable $|W_t|$.*

Proof. Note that $\mathbb{P}(A) = \mathbb{P}[(A \cap B) \cup (A \cap \bar{B})]$, now for $b \geq 0$:

$$\mathbb{P}(M_t \geq b) = \mathbb{P}(M_t \geq b,\, W_t \geq b) + \mathbb{P}(M_t \geq b,\, W_t \leq b)$$

Substituting b for y into (B.2) we obtain

$$\begin{aligned} \mathbb{P}(M_t \geq b) &= \mathbb{P}(W_t \geq b) + \mathbb{P}(W_t \geq b) \\ &= \mathbb{P}(W_t \geq b) + \mathbb{P}(W_t \leq -b) \\ &= \mathbb{P}(|W_t| \geq b). \end{aligned}$$

□

We obtain the distribution of $\tau^*_{M,\,b} = \inf\{s \geq 0 : W_s \geq b\}$ by noting that

$$\mathbb{P}(\tau^*_{M,\,b} \leq t) = \mathbb{P}(M_t \geq b).$$

The cumulative distribution function for $\tau^*_{M,\,b}$ is denoted by $F(\tau^*_{M,\,b})$.

$$\begin{aligned} F(\tau^*_{M,\,b}) = \mathbb{P}(T^u_b \leq t) &= \mathbb{P}(M_t \geq b) \\ &= \mathbb{P}(W_t \geq b) + \mathbb{P}(W_t \geq b) \\ &= 2\mathbb{P}(W_t \geq b). \end{aligned} \tag{B.10}$$

Since $W_t \sim \mathrm{N}(0,\, t)$, the expression (B.10) can be written as

$$\frac{2}{\sqrt{2\pi t}} \int_b^\infty e^{\frac{-u^2}{2t}}\, \mathrm{d}u.$$

Then by change of variables $s = \frac{tb^2}{u^2}$ we can rewrite (B.10) as

$$\begin{aligned} &\frac{-1}{\sqrt{2\pi t}} \int_t^0 e^{\frac{-b^2}{2s}} b \sqrt{\frac{t}{s^3}}\, \mathrm{d}s \\ =\ &\frac{1}{\sqrt{2\pi t}} \int_0^t e^{\frac{-b^2}{2s}} b \sqrt{\frac{t}{s^3}}\, \mathrm{d}s. \end{aligned} \tag{B.11}$$

The above is the cumulative distribution function (cdf) for $\tau^*_{M,\,b}$, to obtain the probability density function $f(\tau^*_{M,\,b})$ we need to find the partial derivative of the cdf (B.11) with respect to t, thus the pdf of $\tau^*_{M,\,b}$ is

$$\begin{aligned} f(\tau^*_{M,\,b}) &= \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{2\pi t}} \int_0^t e^{\frac{-b^2}{2s}} b \sqrt{\frac{t}{s^3}}\, \mathrm{d}s\right) \\ &= e^{\frac{-b^2}{2t}} b \frac{1}{\sqrt{2\pi t^3}}. \end{aligned} \tag{B.12}$$

B.1.3 The Distribution of the pair $(W_t,\, m_t)$

Let us define m_t, as the minimum value that a standard Brownian motion process achieves in the time interval $[0,\, t]$, i.e.

$$m_t := \inf_{s \leq t} W_s.$$

Which can also be defined as

$$m_t := -\sup_{s \leq t}(-W_s).$$

Proposition B.1.3. *Let W be a Brownian motion starting at 0, then*

$$\begin{aligned}
\mathbb{P}(W_t \geq b,\, m_t \geq y) &= \Phi\left(\tfrac{-b}{\sqrt{t}}\right) - \Phi\left(\tfrac{2y-b}{\sqrt{t}}\right) && \text{for } \; y \leq 0,\, b \geq y,\\
\mathbb{P}(W_t \geq b,\, m_t \geq y) &= \Phi\left(\tfrac{-y}{\sqrt{t}}\right) - \Phi\left(\tfrac{y}{\sqrt{t}}\right) && \text{for } \; y \leq 0,\, b \leq y,\\
\mathbb{P}(W_t \geq b,\, m_t \geq y) &= 0 && \text{for } \; y \geq 0.
\end{aligned}$$

Proof. The proof follows the same principles as the proof for Theorem B.1.1. Remember that W and $-W$ have the same distribution properties. □

B.1.4 Distribution of m_t and $\tau^*_{m,b}$

Let us denote the first hitting time of a lower constant barrier $b < 0$ by a standard Brownian motion process W_t by:

$$\begin{aligned}
\tau^*_{m,b} &:= \inf\{s \geq 0 : m_s \leq b\}\\
&= \inf\{s \geq 0 : W_s \leq b\}.
\end{aligned}$$

Following the same procedure as in Section B.1.2 we can see that the distribution of $\tau^*_{m,b}$ is equivalent to the distribution of $\tau^*_{M,b}$,

$$f(\tau^*_{m,b}) = e^{\frac{-b^2}{2t}} b \frac{1}{\sqrt{2\pi t^3}}.$$

B.2 The First-Hitting Time for a Brownian Motion with Drift

Let $X(t) = \ln(V_t/V_0)$, where V_t is given by equation (B.1). Let $\mu := \mu_V - \frac{\sigma^2}{2}$, then the process $X(t)$ can be written as

$$X(t) = \mu t + \sigma W_t.$$

The process X is an example of a Brownian motion with drift μ (the drift term). Let us further define the maximum, M_t^X, and the minimum, m_t^X, process of the drifted Brownian motion process $X(t)$. The process M_t^X is mathematically defined as

$$M_t^X = \sup_{s \leq t} X(s)$$

and m_t^X as

$$m_t^X = \inf_{s \leq t} X(s).$$

Let

$$\widetilde{X}(t) := X(t)/\sigma,$$

then with the use of Girsanov's theorem, $\widetilde{X}(t)$ can be transformed into standard Brownian motion under an equivalent measure, which we denote as $\widetilde{\mathbb{Q}}$. This equivalent measure $\widetilde{\mathbb{Q}}$ is characterised by the following Radon-Nikodým derivative

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = \exp\left(-\frac{\mu}{\sigma} W_t - \frac{\mu^2}{2\sigma^2} t\right). \tag{B.13}$$

By Girsanov's theorem, under the measure $\widetilde{\mathbb{Q}}$ the process $\widetilde{X}(t) = \frac{\mu}{\sigma} t + W_t$ is a standard Brownian motion. We can write the Radon-Nikodým derivative (B.13) with respect to $\widetilde{X}(t)$,

$$\left.\frac{\mathrm{d}\widetilde{\mathbb{Q}}}{\mathrm{d}\mathbb{P}}\right|_{\mathscr{F}_t} = \exp\left(-\frac{\mu}{\sigma} \widetilde{X}(t) + \frac{\mu^2}{2\sigma^2} t\right). \tag{B.14}$$

If we define

$$M_t^{\widetilde{X}} := \sup_{s \leq t} \widetilde{X}_s,$$

then using the above Radon-Nikodým derivative (B.14) we can establish the following

$$\begin{aligned}\mathbb{P}(X(t) \le b,\, M_t^X \le y) &= E^{\mathbb{P}}\left[\mathbb{1}_{\{X(t)\le b,\, M_t^X \le y\}}\right]\\ &= E^{\mathbb{Q}}\left[\frac{d\mathbb{P}}{d\widetilde{\mathbb{Q}}}\mathbb{1}_{\{\widetilde{X}(t)\le b/\sigma,\, M_t^{\widetilde{X}} \le y/\sigma\}}\right]\\ &= E^{\mathbb{Q}}\left[\exp\left(\frac{\mu}{\sigma}\widetilde{X}(t) - \frac{\mu^2}{2\sigma^2}t\right)\mathbb{1}_{\{W_t\le b/\sigma,\, M_t^{\widetilde{X}} \le y/\sigma\}}\right].\end{aligned}$$

Similarly to Proposition B.1.1, it can be shown that:

$$\begin{aligned}&\mathbb{P}(X(t) \le b,\, M_t^X \ge y) = \exp\left(\tfrac{2\mu y}{\sigma^2}\right)\Phi\left(\tfrac{b-2y-\mu t}{\sigma\sqrt{t}}\right) && \text{for } y \ge 0,\, y \ge b,\\ &\mathbb{P}(X(t) \le b,\, M_t^X \le y) = \Phi\left(\tfrac{b-\mu t}{\sigma\sqrt{t}}\right) - \exp\left(\tfrac{2\mu y}{\sigma^2}\right)\Phi\left(\tfrac{b-2y-\mu t}{\sigma\sqrt{t}}\right) && \text{for } y \ge 0,\, y \ge b,\\ &\mathbb{P}(X(t) \ge b, M_t^X \ge y) = \Phi\left(\tfrac{-b+\mu t}{\sigma\sqrt{t}}\right) - \exp\left(\tfrac{2\mu y}{\sigma^2}\right)\Phi\left(\tfrac{-b+2y+\mu t}{\sigma\sqrt{t}}\right) && \text{for } y \le 0,\, y \le b.\end{aligned}$$

In particular, we can deduce from above the cumulative distribution function of the maximum as well as that of the minimum:

$$F(M_t^X) := \mathbb{P}(M_t^X \le y) = \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) - \exp\left(\frac{2\mu y}{\sigma^2}\right)\Phi\left(\frac{-y-\mu t}{\sigma\sqrt{t}}\right), \tag{B.15}$$

$$F(m_t^X) := \mathbb{P}(m_t^X \le y) = \Phi\left(\frac{y-\mu t}{\sigma\sqrt{t}}\right) + \exp\left(\frac{2\mu y}{\sigma^2}\right)\Phi\left(\frac{y+\mu t}{\sigma\sqrt{t}}\right). \tag{B.16}$$

Let us denote the first hitting time of a *lower* constant barrier $y < 0$ by process $X(t)$ by:

$$\begin{aligned}\tau^*_{X,\,m,\,y} &:= \inf\{s \ge 0 : m_s^X \le y\}\\ &= \inf\{s \ge 0 : X_s \le y\},\end{aligned}$$

and the first hitting time of an *upper* constant barrier $y < 0$ by process $X(t)$ by:

$$\begin{aligned}\tau^*_{X,\,M,\,y} &:= \inf\{s \ge 0 : M_s^X \ge y\}\\ &= \inf\{s \ge 0 : X_s \ge y\},\end{aligned}$$

The cumulative distribution function of $\tau^*_{X,\,m,\,y}$ and $\tau^*_{X,\,M,\,y}$ can be computed by using the equalities $\{\tau^*_{X,\,m,\,y} \le t\} = \{m_t^X \le y\}$ and $\{\tau^*_{X,\,M,\,y} \le t\} = \{M_t^X \ge y\}$. The probability distribution functions of $\tau^*_{X,\,m,\,y}$ and $\tau^*_{X,\,M,\,y}$ can be calculated as in Section B.1.2 by finding the derivatives of the respective cdf's with respect to t (see Musiela & Rutkowski (1997) for details).

B.3 The First-Hitting Time of a Geometric Brownian Motion Process V to a Lower Barrier b

Since $X(t) := \ln(V_t/V_0)$, where V_t is given by equation (B.1), we can determine the cdf of

$$\tau^* = \inf\{s \ge 0 : V_s \le b\}$$

from the cdf of $\tau^*_{X,\,m,\,y}$, given by Equation (B.16). Let

$$m_t^V := \inf_{s\le t} V_s$$

then from the definitions $X(t) := \ln(V_t/V_0)$ and $\mu := \mu_V - \frac{\sigma^2}{2}$ we can say

$$\begin{aligned}F(\tau^*) := \mathbb{P}(\tau^* \le t) &= \mathbb{P}(m_t^V \le b)\\ &= \mathbb{P}(V_t \le b,\, m_t^V \le b)\\ &= \mathbb{P}\left(X(t) \le \ln\left(\frac{b}{V_0}\right),\, m_t^X \le \ln\left(\frac{b}{V_0}\right)\right)\\ &= \mathbb{P}\left(m_t^X \le \ln\left(\frac{b}{V_0}\right)\right)\\ &= \Phi\left(\frac{\ln(b/V_0) - \mu t}{\sigma\sqrt{t}}\right) + \exp\left(\frac{2\mu\ln(b/V_0)}{\sigma^2}\right)\Phi\left(\frac{\ln(b/V_0) + \mu t}{\sigma\sqrt{t}}\right).\end{aligned}$$

To determine the pdf of τ^*, $f(\tau^*)$, we differentiate the cdf $F(\tau^*)$ with respect to t:

$$f(\tau^*) = (V_t/b)^{-2\mu} \frac{1}{\sqrt{2\pi}} \frac{\ln(V_t/b)}{\sigma t^{3/2}} \exp\left[-\frac{1}{2}\left(\frac{\mu\sigma^2 t - \ln(V_t/b)}{\sigma\sqrt{u}}\right)^2\right].$$

Appendix C

Miscellaneous

Theorem C.0.1. *Suppose X is a continuous random variable with marginal pdf $f_X(x)$, and that there exists a one-to-one transformation from $A = \{x \,|\, f_X(x) > 0\}$ on to $B = \{y \,|\, f_Y(y) > 0\}$ defined by $Y = g(X)$, with inverse transformation $x = g^{-1}(y)$. If the derivative $(d/dy)g^{-1}(y)$ is continuous and nonzero on B, then the pdf of Y is*

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, \qquad y \in B. \tag{C.1}$$

Proof. Since $y = g(x)$ is one-to-one, then it is either monotonic increasing or monotonic decreasing. Lets first assume that it is monotonic increasing, then $g(x) \leq y$ if and only if $x \leq g^{-1}(y)$. Thus[1]

$$F_Y(y) = \mathbb{P}[g(X) \leq y] = \mathbb{P}[X \leq g^{-1}(y)] = F_X(g^{-1}(y))$$

and

$$f_Y(y) = \frac{d}{dy} F_X(g^{-1}(y)) = \frac{d}{dg^{-1}(y)} F_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

because in this case $\frac{d}{dy} g^{-1}(y) > 0$.

In the monotonic decreasing case, $g(x) \leq y$ if and only if $x \geq g^{-1}(y)$, thus

$$F_Y(y) = \mathbb{P}[g(X) \leq y] = \mathbb{P}[X \geq g^{-1}(y)] = 1 - F_X(g^{-1}(y))$$

and,

$$f_Y(y) = -f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$$

because in this case $\frac{d}{dy} g^{-1}(y) < 0$. □

Definition C.0.1. *Let U be an open subset of $\mathbb{R}^n$ and let $g : U \to \mathbb{R}$ be a given functional. We define the partial derivative of g at the point $\boldsymbol{a} = (a_1, \ldots, a_n) \in U$ with respect to the i^{th} variable x_i as*

$$\frac{\partial f(\boldsymbol{a})}{\partial x_i} = \lim_{h \to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots a_n) - f(a_1, \ldots, a_n)}{h}.$$

[1] If $X \in \mathbb{R}$ is a random variable with pdf $f_X(x)$, then the cumulative density function (cdf) is denoted by $F_X(x) := \int_{\infty}^{x} x \, \mathrm{d}x$.

Appendix D

Hedging Results

This appendix contains all the results of the simulated and empirical delta hedges performed in this dissertation.

D.1 Simulated Delta Hedge Results

Hedging with Shares

Day t_i	Share Price ($) S_{t_i}	Delta Hedge Ratio $\Delta_{\phi(t_0,t_i,T)/S_{t_i}}$	Equity Position Value ($) $-\Delta_{\phi(t_0,t_i,T)/S_{t_i}} S_{t_i}$	CDS Position Value ($) $\phi(t_0, t_i, T)$	Delta Hedge Efficiency Measure ($) $\Lambda(t_0, t_i)$
0	50.55	-1441	72845.94	0.00	0.00
1	50.07	-1643	82270.18	-4235.90	-4936.56
2	48.81	-1764	86096.68	-3795.98	-6577.05
3	48.90	-1733	84741.12	-2636.83	-5270.10
4	49.12	-1784	87628.83	-3422.90	-5685.68
5	48.15	-1682	80985.84	-3085.67	-7090.01
6	49.08	-1501	73673.41	-3507.84	-5958.40
7	50.09	-1459	73077.07	-4994.70	-5938.64
8	50.09	-1059	53044.03	-4515.41	-5468.47
9	53.70	-1225	65776.57	-7484.13	-4620.86
10	54.40	-1315	71537.17	-8381.45	-4668.44
11	53.80	-1167	62780.67	-8243.97	-5328.32
12	55.78	-1276	71180.32	-8828.39	-3609.46
13	55.61	-1084	60276.58	-8680.53	-3686.65
14	55.62	-1310	72863.00	-8374.12	-3376.22
15	56.58	-1004	56807.49	-9456.92	-3209.78
16	56.64	-771	43670.40	-9666.84	-3365.70
17	56.58	-914	51714.36	-9568.58	-3318.30
18	55.77	-1249	69655.16	-9037.18	-3532.85
19	55.95	-1364	76320.07	-8916.09	-3194.85
20	54.84	-947	51937.57	-8665.43	-4466.93
21	55.44	-1530	84826.19	-7178.31	-2417.50
22	56.88	-1019	57960.69	-9657.19	-2703.05
23	56.22	-833	46829.92	-8512.62	-2237.31
24	57.01	-915	52162.60	-9324.78	-2396.40
25	58.15	-972	56517.29	-9659.63	-1693.72
26	56.74	-1284	72854.98	-9406.12	-2816.72
27	55.42	-1001	55471.89	-8719.78	-3833.43
28	55.96	-941	52653.69	-9060.19	-3639.54
29	55.51	-1146	63619.47	-8973.10	-3981.72
30	56.16	-1327	74524.74	-7582.24	-1853.19

Day t_i	Share Price (\$) S_{t_i}	Delta Hedge Ratio $\Delta_{\phi(t_0,t_i,T)/S_{t_i}}$	Equity Position Value (\$) $-\Delta_{\phi(t_0,t_i,T)/S_{t_i}}\, S_{t_i}$	CDS Position Value (\$) $\phi(t_0,t_i,T)$	Delta Hedge Efficiency Measure (\$) $\Lambda(t_0,\, t_i)$
31	56.88	-816	46414.91	-9536.77	-2860.76
32	57.47	-971	55803.34	-9726.73	-2574.18
33	58.64	-848	49722.87	-10395.24	-2112.62
34	59.22	-829	49094.70	-10507.75	-1738.40
35	60.34	-891	53764.64	-10901.57	-1208.71
36	59.22	-1025	60704.38	-10386.60	-1697.09
37	59.14	-645	38146.88	-10493.97	-1892.87
38	59.03	-723	42675.59	-10396.15	-1869.65
39	57.50	-474	27254.41	-9827.27	-2411.17
40	57.78	-500	28891.33	-9970.31	-2423.93
41	56.76	-1243	70550.95	-9023.02	-1989.27
42	57.99	-537	31138.09	-9893.84	-1339.03
43	57.33	-810	46438.42	-9517.37	-1319.77
44	57.77	-874	50495.18	-9884.87	-1335.58
45	58.00	-783	45417.60	-9917.96	-1172.82
46	57.15	-837	47831.81	-9165.98	-1090.92
47	55.19	-1008	55627.18	-8384.57	-1954.93
48	55.11	-190	10471.48	-8802.73	-2459.79
49	54.24	-1356	73552.60	-8887.73	-2710.60
50	54.68	-1101	60197.73	-8258.89	-1493.43
51	55.22	-1069	59034.12	-8665.89	-1312.48
52	56.75	-626	35523.04	-9267.01	-284.40
53	57.22	-801	45836.67	-9623.72	-350.16
54	56.70	-738	41843.36	-9270.27	-417.74
55	56.93	-823	46849.42	-9160.86	-142.65
56	56.05	-742	41586.44	-8776.94	-487.64
57	56.05	-1336	74884.19	-7440.77	844.42
58	56.02	-1288	72152.18	-8772.36	-535.46
59	55.94	-768	42964.21	-8961.26	-835.28

Table D.1: Hedging results for simulated stock prices and CDS values.

Hedging with Call Options

Day t_i	Share Price (\$) φ_{t_i}	Delta Hedge Ratio $\Delta_{\phi(t_0,t_i,T)/\varphi_{t_i}}$	Equity Position Value (\$) $-\Delta_{\phi(t_0,t_i,T)/\varphi_{t_i}}\, \varphi_{t_i}$	CDS Position Value (\$) $\phi(t_0,t_i,T)$	Delta Hedge Efficiency Measure (\$) $\Lambda(t_0,\, t_i)$
0	6.74	-3083	20784.96	0.00	0.00
1	6.38	-2040	13010.87	-4262.29	-5374.73
2	5.40	-2294	12393.03	-3318.25	-6431.63
3	5.46	-2354	12845.34	-2610.15	-5587.80
4	5.64	-2124	11969.48	-3843.54	-6399.43
5	4.92	-2190	10778.76	-2854.14	-6941.10
6	5.64	-2063	11641.03	-4217.66	-6729.65
7	6.42	-2127	13664.79	-1713.53	-2618.12
8	6.36	-2130	13551.13	-4778.92	-5812.93
9	9.59	-1287	12336.45	-7769.89	-1925.80
10	10.09	-1275	12858.64	-8226.57	-1739.78
11	9.64	-1254	12087.27	-8011.79	-2099.54
12	11.38	-1186	13496.85	-8908.53	-815.08
13	11.34	-549	6226.76	-8509.61	-464.26

Day t_i	Share Price ($) φ_{t_i}	Delta Hedge Ratio $\Delta_{\phi(t_0,t_i,T)/\varphi_{t_i}}$	Equity Position Value ($) $-\Delta_{\phi(t_0,t_i,T)/\varphi_{t_i}}\,\varphi_{t_i}$	CDS Position Value ($) $\phi(t_0,t_i,T)$	Delta Hedge Efficiency Measure ($) $\Lambda(t_0,\,t_i)$
14	11.34	-800	9069.51	-8962.90	-917.33
15	12.27	-839	10295.76	-9504.04	-714.59
16	12.38	-723	8950.37	-9541.04	-659.49
17	12.24	-670	8202.46	-9454.40	-674.08
18	11.50	-994	11430.03	-9008.16	-723.57
19	11.71	-1062	12437.20	-9264.92	-771.98
20	10.68	-976	10423.55	-8299.16	-900.56
21	11.23	-1523	17103.19	-6499.52	1435.51
22	12.69	-775	9835.12	-9723.24	434.24
23	12.00	-855	10257.70	-9249.40	373.36
24	12.71	-847	10765.18	-9614.19	615.55
25	13.87	-737	10220.10	-9922.72	1289.47
26	12.45	-1040	12947.57	-8911.34	1254.43
27	11.17	-735	8213.23	-8612.16	222.07
28	11.70	-1278	14950.44	-5013.07	4210.79
29	11.40	-923	10518.72	-8932.13	-92.38
30	11.96	-923	11039.36	-8660.67	695.75
31	12.60	-929	11707.01	-9392.37	554.57
32	13.20	-706	9320.51	-9846.57	657.55
33	14.35	-728	10444.12	-9885.51	1430.65
34	14.99	-652	9774.20	-10326.36	1455.83
35	16.00	-670	10722.72	-10743.05	1697.91
36	14.90	-642	9568.34	-10408.31	1295.86
37	14.93	-574	8570.22	-10367.36	1356.34
38	14.73	-786	11577.32	-10261.77	1347.51
39	13.25	-587	7776.03	-9382.73	1063.28
40	13.56	-687	9315.86	-9910.13	718.18
41	12.60	-759	9566.86	-9683.03	285.92
42	13.88	-662	9187.89	-10116.62	823.90
43	13.11	-1140	14944.07	-8749.32	1681.67
44	13.56	-739	10020.23	-9761.64	1181.80
45	13.76	-882	12135.62	-9946.65	1144.70
46	12.97	-853	11064.25	-9798.32	596.12
47	11.10	-832	9238.19	-8621.35	177.90
48	11.07	-863	9556.75	-8574.30	199.94
49	10.24	-657	6727.99	-7476.52	581.33
50	10.69	-769	8220.96	-8332.59	21.07
51	11.14	-1162	12940.06	-8142.32	557.41
52	12.56	-1192	14965.83	1004.04	2148.93
53	13.02	-681	8869.68	-9643.18	1253.82
54	12.51	-747	9347.36	-9450.98	1098.96
55	12.82	-717	9190.92	-9492.66	1289.00
56	11.89	-1400	16639.54	-3469.72	6453.23
57	11.88	-1040	12357.49	-8714.63	1385.61
58	11.85	-976	11561.64	-8743.11	1325.65
59	11.91	-1027	12235.95	-9044.91	1082.22

Table D.2: Hedging results for simulated European call share option prices and CDS values.

D.2 Empirical Delta Hedge Results

Boeing

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
t_i	Stock	Option	Stock	Option	Stock	Option	($) Stock	($) Option	($) Stock	($) Option	($) Stock	($) Option
1	-639	-3300	-1246	-3300	-3651	-3300	0.00	0.00	0.00	0	0.00	0.00
2	-988	-3289	-1093	-2700	-3700	-2275	188.20	1371.74	2690.80	1354.08	4118.40	-7426.59
3	-579	-751	-1136	-3180	-3420	-3311	-345.23	-1620.95	4390.23	-1185.78	-5188.29	-4348.75
4	-517	-2379	-953	-2824	-2710	-2410	-360.51	-1687.18	4831.70	-5331.1	1749.64	-741.38
5	-594	-2145	-1156	-662	-2450	-3351	252.72	-3822.55	5817.95	-6524.78	9619.77	4564.43
6	-427	-2344	-1246	-2659	-2962	-4006	-1408.17	-4683.36	4606.72	-5934.46	23869.82	4066.10
7	-550	-2577	-1015	-2509	-2690	-3113	-1142.05	-4452.18	6311.71	-7135.16	21801.15	4541.60
8	-437	-2003	-1065	-2485	-2087	-4595	-1143.94	-5789.89	9331.07	-4499.73	24691.07	870.50
9	-479	-2145	-1228	-2355	-2158	-2511	-1053.39	-3853.50	9959.77	-1644.19	22699.61	1814.03
10	-492	-2062	-880	-3048	-2109	-9627	-1202.81	-4500.14	8987.84	-1896.62	17821.58	3626.94
11	-409	-2086	-976	-2606	-1757	-4503	-1033.49	-4829.34	10420.73	-1613.66	25248.49	47519.36
12	-554	-1857	-868	-1891	-1605	-4347	-597.90	-3015.71	13530.59	-5601.35	21331.03	36125.46
13	-679	-1853	-1073	-2716	-1010	-7014	-401.96	-2647.05	16571.83	-3333.2	18805.42	64245.41
14	-415	-1400	-789	-1821	-1425		1628.84	-3020.43	13316.32	-3365.1	24323.55	
15	-610	-2783	-1167	-1383	-1148		1535.60	-1172.34	17267.19	2087.12	28567.07	
16	-633	-2227	-806	-1791	-1347		1426.90	-286.61	15201.75	541.23	30195.08	
17	-573	-2291	-862	-1861	-1225		1532.20	-680.29	13222.70	3398.16	35762.23	
18	-485	-1949	-1008	-2138	-1065		1661.43	-1251.95	12521.99	2290.91	38801.09	
19	-479	-1843	-1557	-2090	-1071		2594.95	-1643.70	12667.01	1692.62	38237.74	
20	-715	-1835	-820	-2076	-756		3258.27	-1093.53	7472.47	634.17	38617.62	

Table D.3: The values of delta hedge ratios and the results of the delta hedge for Boeing.

Boeing

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
t_i	Stock	Option	Stock	Option	Stock	Option	($) Stock	($) Option	($) Stock	($) Option	($) Stock	($) Option
21	-722	-2196	-940	-1531	-490		1514.03	-2857.75	11663.96	1217.15	39357.32	
22	-799	-2202	-1604	-1829	-695		3374.12	-5724.04	9588.89	-693.49	37838.49	
23	-499	-2565	-1094	-2260	-585		3740.03	-4173.33	7371.98	-2192.34	40408.20	
24	-494	-2188	-770	-2015	-571		4345.12	-1023.16	5043.56	3586.3	41781.62	
25	-579	-2598	-1067	-2108	-521		4819.49	-899.50	2858.58	1687.33	40638.89	
26	-639	-1488	-1100	-2067	-334		4608.77	1010.96	4677.94	1453.56	42856.69	
27	-440	-1847	-927	-2282	-380		5936.37	577.88	4243.06	-2460.9	44429.74	
28	-343	-2078	-825	-4203	-201		5293.08	945.62	4723.20	-206.48	46550.49	
29	-498	-1604	-1133	-2237	-285		4982.12	-503.09	1840.23	3771.04	46636.31	
30	-536	-1676	-811	-2679	-204		5696.07	1100.08	3718.84	3411.64	46388.23	
31	-401	-2302	-884	-2554	-185		6143.69	2937.22	566.30	2392.38	45704.39	
32	-455	-2114	-728	-3500	-183		5615.16	3168.90	1886.42	2672.27	45264.05	
33	-457	-1834	-1143	-3523	-607		6569.01	2753.04	-82.81	6153.18	46125.19	
34	-480	-1486	-661	-3864	-122		7110.56	4576.10	2376.87	8644.26	46745.24	
35	-471	-1869	-986	-3085	-103		7102.41	4129.32	3821.39	5294.8	47675.85	
36	-512	-1729	-827	-2700	-95		7083.22	1245.04	5145.90	4474.58	47714.02	
37	-502	-1617	-889	-3721	-90		7993.40	2500.54	9109.58	12102.05	47608.30	
38	-392	-1308	-985	-3129			7586.91	2625.29	8005.72	13919.24		
39	-718	-1992	-1316	-3078			7051.71	3015.92	7307.30	17988.04		
40	-816	-2278	-965	-3619			6816.63	5480.82	4363.56	12446.95		
41	-474	-1617	-726	-4159			7644.95	3428.42	4437.51	17874.9		
42	-535	-2126	-216	-5308			8187.68	4539.82	8181.69	10127.26		
43	-742	-1651	-616	-5577			9017.05	4112.23	8677.06	4261.85		
44	-547	-1466	-676	-4241			7851.66	3928.23	8832.62	16081.73		
45	-574	-2119	-407	-6810			9223.26	3849.86	8641.51	23524.85		
46	-500	-1543	-447	-5697			9646.23	3410.79	8813.69	32087.26		
47	-535	-758	-494	-9611			9524.78	3996.15	9083.59	40916.69		

Table C.3 (contd.): The values of delta hedge ratios and the results of the delta hedge for Boeing.

Boeing

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
t_i	Stock	Option	Stock	Option	Stock	Option	($) Stock	($) Option	($) Stock	($) Option	($) Stock	($) Option
48	-448	-1564	-548	-10586			9867.67	3684.87	8070.41	53083.7		
49	-359	-1712	-300	-11853			10702.71	3523.38	8146.98	26431.01		
50	-281		-327	-10750			10445.71		8199.40	16650.63		
51	-544		-490				10708.61		7842.97			
52	-407		-507				10378.94		8080.48			
53	-496		-600				9703.85		6682.38			
54	-578						9764.01					
55	-476						10080.19					
56	-508						9987.34					
57	-640						9251.27					
58	-337						8425.71					
59	-388						8746.25					
60	-349						7605.01					

Table C.3 (contd.): The values of delta hedge ratios and the results of the delta hedge for Boeing.

Daimler Chrysler

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
							($)	(€)	($)	(€)	($)	(€)
t_i	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option
1	-522	753	-1016	753	-1127	1055	0.00	0.00	0.00	0.00	0.00	0
2	-503	965	-713	958	-1673	1712	28.47	-248.80	-349.36	2756.79	82.80	792.27
3	-592	1056	-864	1101	-1161	2355	714.30	1005.85	330.91	1090.80	-8091.04	14388.63
4	-556	1018	-840	1198	-1444	3859	1100.66	633.37	834.34	-786.73	-5273.34	13767.65
5	-570	978	-715	1238	-1285	12791	739.95	4035.87	-1601.59	647.59	-5672.63	25019.03
6	-595	1086	-901	1361	-1173	10882	1030.63	4627.97	-2701.83	2485.05	-1022.50	30168.72
7	-565	1121	-780	1553	-2007	9519	1635.52	4409.03	-1643.43	6550.21	-5726.30	33957.82
8	-627	1228	-696	1792	-1204		1366.28	3088.32	-1092.20	8830.15	-13859.52	
9	-464	1293	-690	1725	-1129		2170.27	1977.92	-1477.52	9559.26	-15743.92	
10	-589	1380	-889	1972	-460		2078.16	2463.13	-1160.39	9375.18	-16948.03	
11	-624	1306	-931	2397	-1078		2373.30	717.70	-1714.36	7736.66	-14773.97	
12	-918	1111	-688	2146	-846		2600.55	436.70	-2823.38	9495.10	-12280.54	
13	-510	1238	-699	3095	-906		3198.06	-374.48	-2053.98	9843.69	-14242.78	
14	-527	1199	-789	2856	-905		3626.60	-380.41	-3937.13	6986.18	-14989.92	
15	-487	1138	-760	4228	-603		4084.81	-354.68	-4259.31	13362.72	-17323.36	
16	-530	1370	-870	4518	-679		3425.17	1073.00	-5288.38	18518.89	-16842.32	
17	-449	1075	-860	4506	-1222		3439.08	1362.77	-6392.87	17507.57	-7691.89	
18	-544	1393	-746	7082	-968		3581.67	1041.32	-5121.22	18499.53	-12913.50	
19	-515	1212	-706	5939	-752		3995.33	1792.83	-5548.18	16839.36	-7605.23	
20	-432	1425	-738	8255	-614		3646.25	2064.90	-9212.60	15587.46	-7817.99	
21	-525	1418	-717	9809	-560		2983.17	1408.05	-9463.18	23336.33	-6296.93	
22	-598	1382	-876	7883	-968		3453.23	596.32	-9074.32	21306.33	-8844.00	
23	-628	1403	-732	5712	-533		3583.80	2973.42	-6309.07	24823.17	-8358.35	
24	-587	1330	-821	7046	-548		3324.07	3153.22	-7716.18	27303.04	-8399.84	
25	-540	1272	-671	11644	-482		3169.67	5329.94	-9224.09	23321.38	-4652.65	
26	-567	1264	-720	13568	-364		3498.44	5757.08	-8316.36	18184.29	-6941.17	

Table D.4: The values of delta hedge ratios and the results of the delta hedge for Daimler Chrysler.

Daimler Chrysler

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
							(\$)	(€)	(\$)	(€)	(\$)	(€)
t_i	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option
27	-625	1217	-680	9000	-376		3730.99	5902.76	-6315.02	29794.44	-6328.06	
28	-543	1306	-836	16507	-524		3299.54	6833.17	-7100.28	14318.10	-7768.70	
29	-590	1454	-646	11137	-235		3451.01	8555.27	-8120.95	38681.54	-9608.25	
30	-506	1623	-546		-247		3363.60	6683.96	-7137.83		-9857.16	
31	-525	1659	-845		-315		3842.12	8448.66	-8534.46		-9027.51	
32	-540	1513	-668		-202		3740.58	7926.82	-7278.47		-8968.36	
33	-566	1426	-705		-198		3609.22	8750.37	-8817.13		-10546.38	
34	-489	1654	-658		-149		3821.50	9007.50	-9291.97		-7321.57	
35	-614	1498	-657		-146		3765.92	10067.67	-7153.37		-8430.78	
36	-665	1614	-604		-174		3629.39	9648.29	-6053.50		-7286.74	
37	-466	1820	-663		-116		4145.74	9167.27	-6404.15		-7497.30	
38	-529	1664	-667				4330.09	9488.48	-6267.13			
39	-515	1606	-663				4230.70	10635.14	-6323.58			
40	-543	1596	-641				3952.98	10291.00	-4452.45			
41	-579	1642	-706				4032.63	9304.92	-3539.87			
42	-487	1983	-673				4485.13	9005.33	-3325.82			
43	-420	1938	-618				4479.78	8746.58	-1675.13			
44	-571	1743	-582				4288.44	9030.27	-2416.06			
45	-471	1900	-616				4373.13	8560.80	-2690.18			
46	-549	2152	-575				4316.21	7812.89	-2113.79			
47	-488	2062	-583				4108.21	8458.74	-1371.17			
48	-571	2151	-586				4116.07	7947.43	-1604.49			
49	-489	1973	-608				4242.07	6982.52	-1252.84			
50	-505	2151	-742				3882.47	7772.86	-1662.58			
51	-532	2292	-523				4047.33	9386.60	-1994.09			
52	-508	2290	-556				3952.79	8792.91	-690.38			

Table C.4 (contd.): The values of delta hedge ratios and the results of the delta hedge for Daimler Chrysler.

Daimler Chrysler

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
							(\$)	(€)	(\$)	(€)	(\$)	(€)
t_i	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option
53	-522	2392	-518				3547.47	9236.15	-1321.62			
54	-558	2398					3531.70	9744.46				
55	-427	2076					4171.87	8652.43				
56	-467	2205					4783.21	10502.47				
57	-484	2076					4567.15	9729.42				
58	-480	2564					4947.69	8450.32				
59	-452	2462					5445.82	7406.11				
60	-484	2367					4303.83	7403.01				
61		2313						9569.03				
62		2930						10227.75				
63		3239						12762.98				
64		3830						12991.75				
65		3564						12998.26				
66		3755						13250.69				
67		4229						14338.22				
68		3916						14733.22				
69		4455						17716.95				
70		4439						14749.63				
71		4238						16597.20				
72		5185						16763.90				
73		4560						16686.71				
74		5280						17647.88				
75		4181						18714.68				
76		6522						17719.03				
77		5439						16316.16				
78		5845						12241.63				

Table C.4 (contd.): The values of delta hedge ratios and the results of the delta hedge for Daimler Chrysler.

Daimler Chrysler

Day / Week / Month t_i	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
	Stock	Option	Stock	Option	Stock	Option	Stock ($)	Option (€)	Stock ($)	Option (€)	Stock ($)	Option (€)
79		4231						10703.90				
80		7621						8151.50				
81		5581						8119.86				
82		7860						16032.51				
83		5501						17002.44				
84		5877						19703.13				
85		8432						21120.85				
86		6776						21699.39				
87		7373						21838.95				
88		7806						13824.93				
89		6736						20239.77				
90		8082						22559.50				
91		8313						20523.25				
92		9021						20535.66				
93		8229						14892.23				
94		7249						19448.87				
95		7088						20563.10				
96		5195						23014.76				
97		4716						23095.98				
98		4832						22040.02				
99		5458						20925.59				
100		6054						20858.65				

Table C.4 (contd.): The values of delta hedge ratios and the results of the delta hedge for Daimler Chrysler.

Deutsche Telekom

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$ ($)	(€)	Weekly $\Lambda(t_0, t_i)$ ($)	(€)	Monthly $\Lambda(t_0, t_i)$ ($)	(€)
t_i	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option
1	-2408	-3805	-4228	-5708	-2939	-5708	0.00	0.00	0.00	0	0.00	0.00
2	-2183	-4083	-2682	-4109	-1125	-7548	-263.35	-177.30	-1509.68	-314.42	4079.45	-1524.88
3	-2455	-4306	-1494	-3738	-3575	-5119	50.57	-271.37	-373.75	89.55	-265.68	-1096.21
4	-1673	-3946	-1118	-4941	-4625	-3667	-634.75	-419.29	1028.51	-53.38	-5344.31	-2455.21
5	-2580	-3856	-3924	-3755	-2664	-5564	-445.13	-427.22	-147.13	107.94	-7455.57	-3113.14
6	-2557	-1641	-4959	-3567	-2995	-2584	-992.02	-315.36	1583.39	-211.93	-4152.04	-3563.27
7	-2547	-3681	-2848	-6236	-3937	-2493	-747.08	-428.33	2618.22	-302.85	-8478.74	-4139.20
8	-2232	-1461	-2705	-6246	-2607	-2658	-1106.85	-410.81	3122.30	210.47	-6253.07	-4484.17
9	-2524	-4771	-2488	-4948	-2581	-1664	-1141.92	-46.59	2484.02	555.94	-5113.70	-3573.70
10	-2200	-1801	-3253	-3893	-1670	-3319	-1162.59	324.36	1930.76	604.35	-4268.38	-285.85
11	-1858	-4120	-2607	-3562	-3809		-1797.24	160.25	508.20	625	-1768.18	
12	-1954	-2908	-1303	-2010	-1705		-755.64	261.19	-48.89	53.09	3617.56	
13	-2592	-3179	-3107	-3422	-1851		-185.86	193.02	499.14	-179.37	2258.36	
14	-2467	-2937	-2419	-2761	-1629		398.69	222.47	3988.76	-341.15	-207.87	
15	-2535	-3781	-2441	-4317	-1109		249.23	281.41	2330.13	-687.49	-2300.71	
16	-2839	-3759	-2392	-575	-1011		23.84	231.46	224.73	-584.92	-2968.81	
17	-2297	-3829	-2189	-4360	-1427		256.48	305.61	-23.29	-1308.76	-2614.39	
18	-2583	-3859	-2649	-3900	-929		268.86	321.94	135.30	-1652.92	-2112.90	
19	-2356	-2603	-2216	-4340	-2518		605.90	409.05	1663.11	-1652.32	-1795.89	
20	-2593	-3228	-2033	-2333	-874		646.35	414.71	433.72	-2165.55	-4371.89	
21	-2459	-544	-7205	-3421	-912		20.33	271.52	-281.71	-1872.63	-3924.19	
22	-2339	-2752	-2421	-2313	-2004		472.28	186.80	-1855.28	-1897.69	-4621.20	
23	-2574	-2740	-3050	-2838	-2062		675.00	85.06	288.04	-2064.14	-6019.52	
24	-2072	-3573	-2076	-1692	-1859		543.08	10.69	-588.73	-1930.56	-6559.09	
25	-2488	-7402	-3769	-2161	-2517		2.95	7.47	-1815.01	-1943.39	-7099.90	
26	-2580	-6035	-949	-3504	-1092		205.27	71.12	-2054.83	-2245.53	-7916.63	

Table D.5: The values of delta hedge ratios and the results of the delta hedge for Deutsche Telekom.

Deutsche Telekom

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
							(\$)	(€)	(\$)	(€)	(\$)	(€)
t_i	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option
27	-2021	-3908	-2152	-2686	-1003		481.01	-31.74	-1931.19	-2426.42	-6908.86	
28	-2533	-4028	-1918	-2204	-313		217.84	118.27	-2062.56	-1865.36	-6725.41	
29	-2669	-2553	-1737	-2468	-242		353.27	35.63	-6050.80	-1882.44	-7815.31	
30	-2147	-2442	-1941	-3123	-542		167.97	83.07	-4551.79	-2246.54	-7140.38	
31	-1815	-4116	-2286	-1771	-423		657.16	93.93	-4249.34	-2618.14	-7051.28	
32	-2491	-3769	-1797	-2409	-254		751.72	395.39	-4461.59	-2441.18	-7484.37	
33	-2265	-3109	-2143	-1378	-299		1529.50	470.71	-5719.31	-1964.37	-7446.40	
34	-2311	-3500	-1980	-1167	-800		1631.22	492.54	-3496.72	-2252.91	-6451.12	
35	-2524	-2872	-3358	-2210	-170		1322.44	393.34	-2172.80	-2354.08	-7020.01	
36	-1607	-3472	-2185	-3062	-248		910.82	721.19	-435.81	-1546.99	-6831.58	
37	-1669	-3254	-1694	-1479	-207		1443.79	933.16	-995.61	-1158.51	-6690.01	
38	-2503	-3608	-3318	-1873			1533.85	905.17	-1301.01	-1271.54		
39	-2571	-2569	-1742	-2492			1662.43	868.73	-658.98	-1315.06		
40	-2599	-2645	-2861	-1288			1189.62	806.13	1088.33	-564.33		
41	-2230	-2501	-2505	-2595			1490.12	873.35	538.22	2714.55		
42	-2539	-5204	-2502	-3592			1653.14	848.22	-1127.21	1973.93		
43	-2164	-6309	-1976	-3851			1737.43	858.83	1278.32	1545.93		
44	-1800	-2734	-1861	-3566			1513.65	726.41	1474.78	1826.48		
45	-1944	-2495	-1981	-2574			1821.47	689.27	1129.87	2333.09		
46	-1801	-3165	-2803				2785.92	797.41	1268.96			
47	-2228	-2401	-1918				2529.95	1045.07	2585.10			
48	-2015	-2508	-1337				2920.66	927.96	2272.77			
49	-2199	-3481	-1805				2699.20	781.70	2936.88			
50	-1617	-2967	-1870				2177.40	869.33	3818.86			
51	-2246	-2427	-1602				1886.47	381.65	4209.24			
52	-1599	-2590	-1762				2196.36	400.75	3143.92			

Table C.5 (contd.): The values of delta hedge ratios and the results of the delta hedge for Deutsche Telekom.

Deutsche Telekom

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
							($)	(€)	($)	(€)	($)	(€)
t_i	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option
53	-1847	-3307	-654				1965.22	407.48	2143.60			
54	-1677	-3426					1514.20	242.66				
55	-1638	-1601					1189.96	159.82				
56	-1379	-3063					1623.73	10.76				
57	-2460	-2286					1364.00	-38.30				
58	-1605	-1220					1624.10	179.06				
59	-1541	-1711					1519.99	217.20				
60	-2064	-1374					892.87	204.86				
61		-1597						115.10				
62		-1797						-39.72				
63		-4399						92.40				
64		-2251						-432.74				
65		-2165						-232.87				
66		-3690						-378.94				
67		-1616						-423.81				
68		-2010						-610.61				
69		-1931						-645.56				
70		-2531						-423.66				
71		-6628						-515.76				
72		-3119						-606.58				
73		-3172						-565.98				
74		-2860						-580.58				
75		-2321						-821.45				
76		-3041						-964.95				
77		-2859						-1278.98				
78		-2037						-1388.26				

Table C.5 (contd.): The values of delta hedge ratios and the results of the delta hedge for Deutsche Telekom.

Deutsche Telekom

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
							(\$)	(€)	(\$)	(€)	(\$)	(€)
t_i	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option
79		-2190						-1488.94				
80		-1708						-1609.92				
81		-2469						-1919.65				
82		-1366						-1911.83				
83		-1547						-1798.47				

Table C.5 (contd.): The values of delta hedge ratios and the results of the delta hedge for Deutsche Telekom.

Ford

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
t_i	Stock	Option	Stock	Option	Stock	Option	($) Stock	($) Option	($) Stock	($) Option	($) Stock	($) Option
1	-13375	301262	-19632	446622	-4164	458929	0.00	0.00	0.00	0.00	0.00	0.00
2	-13678	303205	-19217	362293	-3075	440200	-1170.13	29998.74	-19783.91	28889.00	-9286.70	183622.29
3	-16246	348996	-20884	306127	-3149	341547	-1754.98	24472.97	-14532.84	73334.25	669.34	181186.09
4	-12356	234084	-22556	332911	-3305	411444	-173.28	91468.76	4789.41	60209.50	4777.27	183680.37
5	-12410	257075	-19929	273196	-4718	375311	-1513.11	84274.74	-18818.98	133439.59	-5415.99	115509.14
6	-18338	217589	-19142	272124	-3369	36563	-4840.97	72002.64	-15997.13	149507.06	1716.00	123275.54
7	-10956	353177	-21180	366772	-3084	426644	-3144.25	90421.27	-4105.08	164449.17	-3599.79	121645.48
8	-13067	284577	-22100	286606	-2871	355795	-4619.76	107925.69	-12795.74	89535.69	-3924.79	110340.64
9	-18208	257412	-25124	246640	-4625	522587	-5115.17	105524.08	-7822.24	129322.50	-5282.92	138001.06
10	-15518	255484	-22270	253486	-7041	548277	-5147.65	97402.56	-15149.55	117349.22	-12133.36	138482.80
11	-12108	331363	-22158	272711	-3033	656908	-552.90	99149.91	-28161.15	114304.53	-1472.81	211769.18
12	-12514	237442	-20347	211249	-3494	481845	-3674.63	115203.07	-17540.93	116372.26	1216.26	302775.43
13	-27847	214645	-21503	229214	-2853	530827	-3282.95	139152.25	-19541.53	113400.23	-4763.41	92808.47
14	-17379	298054	-27593	262191	-3612		-3528.50	135728.18	-29832.81	115164.64	683.24	
15	-18525	255932	-22425	305880	-3775		-5950.49	81340.35	-28888.03	124542.09	-5256.65	
16	-12442	248473	-15245	198534	-4427		-8558.68	84032.07	-27516.17	111656.93	4820.09	
17	-15981	255673	-18331	215217	-10530		-7467.40	114766.63	-42930.41	95709.07	47651.22	
18	-12730	264913	-23624	214160	-6476		-2853.16	130708.16	-27408.44	100396.83	37516.36	
19	-13026	264632	-21096	211488	-11096		-5196.54	146945.43	-34907.26	81484.64	39620.96	
20	-14928	288621	-21048	248038	-7574		-9541.31	139704.98	-46501.16	87520.54	19383.16	
21	-10092	218186	-18399	212768	-6881		-10652.86	136895.76	-48602.37	86045.29	28777.68	
22	-13409	197034	-16378	211786	-9284		-11857.12	147455.79	-48049.95	83370.82	17990.25	
23	-16684	217308	-22485	193261	-8848		-12623.80	140715.55	-27593.12	82475.58	32203.69	
24	-15955	169974	-16372	183699	-10920		-15320.69	145151.11	-47928.80	93820.89	51304.91	
25	-17878	263165	-15772	255588	-19258		-18986.54	149346.11	-57405.80	110173.80	71800.77	
26	-11529	185571	-15978	184258	-11474		-23819.15	152975.37	-45785.81	121682.94	31730.02	

Table D.6: The values of delta hedge ratios and the results of the delta hedge for Ford.

Ford

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
t_i	Stock	Option	Stock	Option	Stock	Option	($) Stock	($) Option	($) Stock	($) Option	($) Stock	($) Option
27	-15377	331726	-20738	198881	-12476		-27970.85	141990.61	-71844.35	118570.34	43110.86	
28	-17721	263419	-23838	302187	-14066		-29038.77	146758.14	-52760.51	126917.48	20665.65	
29	-15056	189375	-18871	324262	-10677		-25513.53	146864.95	-57780.28	124738.13	29955.02	
30	-17125	216045	-16937	212206	-8229		-24306.66	162006.93	-58358.45	142654.45	18730.41	
31	-13152	232229	-14396	131070	-17000		-26003.89	157385.30	-62393.46	117762.36	12740.00	
32	-11571	246249	-17572	236882	-9634		-24332.31	155267.19	-63314.53	139027.57	31663.71	
33	-12502	212003	-11715	223511	-10683		-21705.50	158225.96	-55282.47	128262.88	8533.42	
34	-12240	225063	-12468	269283	-2710		-22686.42	140210.15	-70635.87	137387.42	11568.00	
35	-12585	265520	-18590	266008	-3112		-23133.50	124633.51	-55363.36	130658.26	536.24	
36	-12409	208361	-13189	305111	-3147		-22109.14	143452.17	-55902.85	137692.25	-7796.67	
37	-12012	207008	-13166	328627	-3265		-20231.96	143911.39	-54781.31	139418.34	-8278.26	
38	-10837	198990	-16248	232544			-21719.08	137156.63	-59153.50	136317.38		
39	-19444	292890	-21471	252544			-20811.41	146851.98	-62183.82	152853.91		
40	-19601	317185	-15378	323752			-23835.28	160243.55	-54116.74	152412.35		
41	-10309	247613	-16180	291343			-25412.73	148035.83	-64066.44	187580.89		
42	-11316	287999	-12894	277154			-26627.23	158142.36	-70551.29	190070.98		
43	-11636	217665	-10921	255625			-26799.93	143062.90	-67422.76	185960.75		
44	-16962	269045	-9840	215631			-24942.61	148171.25	-74814.68	185359.33		
45	-9896	259181	-10177	245399			-26306.53	143521.96	-74590.74	185281.92		
46	-12053	223003	-11073	327498			-25655.88	163927.70	-86417.82	197597.49		
47	-10521	252864	-13892	307800			-25051.76	143005.10	-83853.69	203129.76		
48	-16470	259064	-17442	263961			-25489.52	143132.59	-74834.16	238575.55		
49	-9672	270990	-1978	283925			-26061.77	141944.85	-80461.12	226559.19		
50	-12101	253918	-12945	311922			-25191.07	146369.57	-81271.48	226688.76		
51	-16227	313686	-12508	300862			-27560.45	146294.72	-78282.77	194143.04		
52	-9005	177175	-11084	273436			-27883.36	148403.87	-76804.80	137563.19		

Table C.6 (contd.): The values of delta hedge ratios and the results of the delta hedge for Ford.

Ford

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
t_i	Stock	Option	Stock	Option	Stock	Option	($) Stock	($) Option	($) Stock	($) Option	($) Stock	($) Option
53	-11566	203338	-14920	350828			-27218.77	147256.93	-78867.58	100707.17		
54	-11333	207205		395524			-24191.49	142140.63		119922.97		
55	-13119	192082		397776			-23020.58	147461.09		102662.04		
56	-11094	186422					-27550.64	141631.13				
57	-11765	168341					-28146.64	151224.67				
58	-9863	194639					-33922.19	144595.54				
59		260140						157022.40				
60		337933						157405.94				
61		301445						141489.89				
62		201012						131917.68				
63		201424						137088.33				
64		275214						138988.79				
65		372137						137355.38				
66		214475						136422.08				
67		223190						142707.26				
68		376268						131271.68				
69		351268						131585.70				
70		208027						131503.85				
71		224650						135866.61				
72		248805						133767.56				
73		238811						144669.62				
74		294402						143395.16				
75		205017						128512.52				
76		299480						130222.45				
77		228406						105765.57				
78		250274						120984.01				

Table C.6 (contd.): The values of delta hedge ratios and the results of the delta hedge for Ford.

Ford

Day / Week / Month t_i	Daily Delta Hedge Ratio Stock	Daily Delta Hedge Ratio Option	Weekly Delta Hedge Ratio Stock	Weekly Delta Hedge Ratio Option	Monthly Delta Hedge Ratio Stock	Monthly Delta Hedge Ratio Option	Daily $\Lambda(t_0, t_i)$ (\$) Stock	Daily $\Lambda(t_0, t_i)$ (\$) Option	Weekly $\Lambda(t_0, t_i)$ (\$) Stock	Weekly $\Lambda(t_0, t_i)$ (\$) Option	Monthly $\Lambda(t_0, t_i)$ (\$) Stock	Monthly $\Lambda(t_0, t_i)$ (\$) Option
79		236665						129909.78				
80		220889						113324.68				
81		224689						111160.71				
82		251314						101732.22				
83		231843						101454.99				
84		385975						88819.37				
85		240838						87202.63				
86		216430						91069.37				
87		262225						79899.46				
88		231338						89812.44				
89		220379						103684.66				
90		223118						98193.57				
91		356791						99137.29				
92		261975						97129.61				
93		248371						102421.53				
94		306334						89276.18				
95		191909						97235.61				
96		432449						97035.07				
97		186944						97174.86				
98		302576						100512.91				
99		233113						94304.74				

Table C.6 (contd.): The values of delta hedge ratios and the results of the delta hedge for Ford.

General Motors

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
t_i	Stock	Option	Stock	Option	Stock	Option	($) Stock	($) Option	($) Stock	($) Option	($) Stock	($) Option
1	-1471	86140	-2130	86140	-442	77924	0.00	0.00	0.00	0.00	0.00	0.00
2	-1425	85958	-2025	81368	-491	71774	1215.78	-8885.08	-2068.47	8303.93	4339.47	9220.05
3	-1361	87723	-2258	79247	-525	76145	2391.15	-276.46	-5655.36	-4939.87	7897.47	59253.77
4	-1389	78923	-2719	84855	-521	66195	1357.67	3899.02	-3644.37	-6026.66	7524.13	69213.88
5	-1565	78979	-2764	79478	-506	65159	919.40	-95.96	-9335.55	-10725.76	1662.08	28543.36
6	-1405	75622	-2791	75562	-510	57376	271.51	7861.88	-9692.73	29300.87	8090.56	49998.67
7	-1390	80909	-2710	68311	-492	61860	-343.73	1574.11	-1128.78	51882.07	3887.70	70966.56
8	-1396	82101	-2629	68516	-433	65756	-439.22	-12083.91	-11011.75	35302.56	5793.11	73276.27
9	-1484	79284	-2613	73786	-452	63278	-1081.50	-13433.46	-11317.06	39266.59	7264.71	79231.39
10	-1336	76360	-2544	79610	-412	69571	-2873.44	-8407.34	-18044.90	54042.98	4195.02	100263.33
11	-1480	80889	-2618	64421	-496	72150	-4537.65	-5862.18	-23870.70	34297.54	9552.08	90266.92
12	-1195	72547	-2759	63561	-578	75314	-1231.37	2840.30	43.02	36136.50	11397.99	136294.48
13	-1209	76527	-2525	65444	-564	69358	-8685.38	-1665.41	1708.26	58429.16	12236.55	77068.18
14	-1323	85432	-2687	60918	-710		-9710.92	1992.85	-1766.96	67931.70	18385.57	
15	-1209	90501	-2793	61261	-657		-10030.99	-9021.65	-1160.44	64948.46	14003.44	
16	-1151	85314	-2649	59844	-1532		-10871.70	-2719.70	6829.77	62220.88	-9341.22	
17	-1205	75631	-2484	62796	-2016		-12045.02	-7204.11	2578.71	41386.69	25949.28	
18	-1253	82948	-2777	58929	-1911		-9065.89	-10962.87	10585.43	37651.75	99268.80	
19	-1292	74673	-2408	53612	-1686		-8118.95	-7224.99	3054.39	41317.51	7225.75	
20	-1244	79862	-2431	65313	-1056		-12430.25	-12163.04	-4141.06	58173.32	22964.44	
21	-1250	78361	-2212	69828	-1254		-12773.60	-10011.62	-7894.80	58934.23	65777.48	
22	-1236	88070	-2396	63207	-1253		-14978.86	-3892.05	-19593.02	66107.40	5812.09	
23	-1302	85146	-2328	63000	-1853		-13707.67	11823.01	-19369.43	71601.68	-2269.71	
24	-1216	88701	-2225	64218	-1814		-15945.38	26700.49	-38842.79	73290.52	107576.01	
25	-1230	71247	-2017	61124	-2537		-16281.51	21036.90	-50379.91	79360.85	-16004.35	
26	-1170	74659	-2185	63518	-2153		-17231.26	33126.86	-43027.07	88561.85	-10831.67	

Table D.7: The values of delta hedge ratios and the results of the delta hedge for General Motors.

General Motors

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
t_i	Stock	Option	Stock	Option	Stock	Option	($) Stock	($) Option	($) Stock	($) Option	($) Stock	($) Option
27	-1253	74639	-1924	68793	-2092		-16807.40	34158.34	16284.53	87445.24	115256.48	
28	-1266	80254	-2083	62857	-1974		-15811.70	64270.34	15023.66	101600.30	-19971.56	
29	-1293	72511	-1978	61153	-1820		-13020.50	56576.54	528.35	102692.87	-1782.17	
30	-1309	73453	-1807	60151	-1856		-13545.45	66344.33	-4746.22	85390.02	84402.53	
31	-1306	68176	-1679	67212	-1590		-8223.15	49194.99	-40224.43	63957.32	-2566.30	
32	-1355	79558	-1495	67211	-1148		-9058.64	44492.52	-42083.95	86878.69	12059.77	
33	-1244	70804	-1584	64824	-1081		-9688.53	39065.31	-78892.53	95621.42	46560.22	
34	-1104	71494	-1539	58621	-960		-11433.01	40608.97	-86963.38	93258.37	2929.47	
35	-1216	70109	-1668	62108	-580		-11099.68	33498.29	-86107.31	98952.63	23581.98	
36	-1225	70612	-1580	67584	-594		-13640.95	29130.54	-80491.72	97302.66	18544.51	
37	-1312	68862	-1496	60007	-476		-13803.34	34137.72	-84106.49	92737.61	6678.52	
38	-1283	73180	-1376	59494			-18543.63	42779.37	-91596.75	99062.90		
39	-1242	68434	-1392	65125			-19452.34	49421.82	-96066.09	99475.03		
40	-1177	70181	-1305	64162			-18608.62	56320.12	-70355.62	105598.17		
41	-1222	80121	-1296	67067			-18156.24	67444.52	-45822.85	105221.14		
42	-1173	74256	-1179	70097			-18401.56	61106.13	-106859.42	106232.46		
43	-1256	75804	-1129	69062			-19199.19	55956.65	-109652.02	105464.17		
44	-1137	75939	-974	71772			-19676.72	60015.10	-119156.61	106024.86		
45	-1109	68446	-1009	73955			-23598.24	22701.69	-116332.39	111435.73		
46	-1060	68907	-951	82291			-24982.00	28511.20	-119628.77	124473.60		
47	-1117	68921	-996	77226			-21885.55	35591.98	-116918.75	135099.94		
48	-1117	61407	-1069	85984			-26022.04	34530.92	-112729.65	154635.84		
49	-1142	62709	-982	70729			-25374.47	42588.17	-114536.49	155773.88		
50	-1142	71883	-884	80109			-25894.79	41405.74	-118707.79	149214.89		
51		72283	-1134	76228				38136.22	-118807.65	137006.82		
52		63620	-892	77483				35352.67	-122977.35	102587.68		

Table C.7 (contd.): The values of delta hedge ratios and the results of the delta hedge for General Motors.

General Motors

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
t_i	Stock	Option	Stock	Option	Stock	Option	($) Stock	($) Option	($) Stock	($) Option	($) Stock	($) Option
53		66724	-891	79207				42396.00	-124092.31	97196.43		
54		67376		78838				40948.80		76840.89		
55		65624						55409.08				
56		56251						66293.22				
57		63352						59189.97				
58		60584						54753.16				
59		65757						54984.30				
60		71768						59871.00				
61		62028						61842.91				
62		60512						64687.24				
63		61190						59937.80				
64		61655						59863.06				
65		61121						60727.78				
66		65654						59827.03				
67		57746						58669.69				
68		55602						61894.11				
69		55052						61731.72				
70		60988						61696.71				
71		60488						65614.92				
72		61154						65320.17				
73		60453						64957.44				
74		57796						63469.93				
75		64918						54105.07				
76		69878						39763.82				
77		62267						43823.00				
78		61953						38631.35				

Table C.7 (contd.): The values of delta hedge ratios and the results of the delta hedge for General Motors.

General Motors

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
t_i	Stock	Option	Stock	Option	Stock	Option	Stock ($)	Option ($)	Stock ($)	Option ($)	Stock ($)	Option ($)
79		56863						35902.36				
80		56063						32135.93				
81		61635						29908.77				
82		63842						39352.63				
83		55482						37732.82				
84		55812						31109.05				
85		55927						34762.61				
86		55629						35190.45				
87		58686						43137.62				
88		57711						42056.54				
89		57624						48016.68				
90		61304						49047.63				
91		64423						50314.56				
92		63823						47244.97				
93		59597						50067.16				
94		59117						51310.92				
95		60024						51396.76				

Table C.7 (contd.): The values of delta hedge ratios and the results of the delta hedge for General Motors.

Vodafone

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
t_i	Stock	Option	Stock	Option	Stock	Option	($) Stock	(£) Option	($) Stock	(£) Option	($) Stock	(£) Option
1	-1248	848	-1795	565	-844	424	0.00	0.00	0.00	0.00	0.00	0.00
2	-1453	926	-1588	666	-1327	495	-335.26	-1519.12	-18618.38	-1038.00	-8556.93	-1368.90
3	-1507	941	-1836	608	-1734	492	-896.30	-156.44	-10647.14	-3909.27	-26142.70	2380.10
4	-1171	895	-1836	602	-1299	336	-902.77	-1578.82	2987.30	-4279.75	-16469.69	6132.59
5	-1661	1090	-1868	605	-1025	373	-1103.23	395.98	-23342.09	-2964.28	-29769.21	6573.87
6	-1343	787	-1585	650	-885	355	-111.93	-1844.44	-13685.46	-1888.84	-49079.83	8339.64
7	-1683	825	-1807	631	-944	385	-396.20	-1470.13	-5248.34	-1901.96	-53354.31	10428.95
8	-1165	858	-2062	707	-899	227	16.13	-426.76	14206.27	-1885.58	-43156.74	8444.92
9	-1391	862	-1919	689	-964	223	152.38	-3897.40	-3254.21	1313.52	-33251.03	7349.56
10	-2048	898	-1507	581	-758	331	-338.83	-5480.68	-21271.26	5269.65	-16769.09	8684.56
11	-1844	979	-1896	595	-814	255	-879.05	-5555.74	-4362.33	7267.63	-5377.42	11569.65
12	-1670	924	-1894	649	-864	204	-1015.19	-4594.30	4063.09	6433.16	-6321.81	14464.18
13	-2156	1082	-1594	374	-743	306	-1068.07	-6077.04	10023.82	8399.00	-20532.69	17198.71
14	-1932	907	-1668	407	-519		-1956.89	-5543.30	26552.40	8288.25	-19722.91	
15	-2106	805	-1918	473	-477		-1924.20	-5977.56	27638.33	8711.14	-19086.10	
16	-1977	1115	-1668	439	-459		-1872.65	-5927.72	17333.74	8594.66	-21016.50	
17	-1479	762	-1590	375	-425		-1764.32	-5231.79	9489.33	9070.47	-26594.18	
18	-1558	931	-1810	441	-389		-1107.97	-5208.66	-1267.68	9886.23	-30960.13	
19	-1721	709	-1933	421	-377		-1077.40	-4997.95	31269.33	11712.21	-25028.83	
20	-1634	817	-1340	467	-383		-1944.41	-4130.39	-9315.80	11587.53	-19698.23	
21	-1302	717	-1662	409	-291		-1228.06	-4128.68	-20300.15	12303.77	-25142.86	
22	-2742	939	-1653	352	-195		-1281.56	-3258.41	-28456.09	13688.44	-24816.47	
23	-1947	842	-1807	434	-203		-1980.02	-1626.35	-20373.20	13659.08	-35103.98	
24	-2146	850	-2105	419	-208		-2370.12	-2662.17	-10322.68	13826.19	-35149.69	
25	-1286	883	-1580	374	-397		-2423.68	-1893.94	-38552.81	13828.08	-35537.80	
26	-1871	981	-2039	377	-358		-3100.44	-2578.75	-39233.96	14218.25	-43708.74	

Table D.8: The values of delta hedge ratios and the results of the delta hedge for Vodafone.

Vodafone

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
							(\$)	(£)	(\$)	(£)	(\$)	(£)
t_i	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option
27	-1893	1006	-2183	384	-317		-3864.61	-2088.28	-4659.86	12990.42	-38100.61	
28	-1336	884	-1172	299	-238		-3148.31	-2286.80	-8014.21	11514.23	-28724.51	
29	-1634	1090	-1697	350	-169		-3434.05	-3887.55	-26092.45	10852.38	-31458.73	
30	-1776	848	-1149	301	-170		-3500.07	-2922.68	-23694.64	10974.64	-34600.73	
31	-1597	597	-1351	435	-234		-3430.78	-2483.24	-19145.21	10839.56	-34048.33	
32	-1621	819	-1617	324	-102		-4495.11	-995.93	-10505.90	10585.45	-34630.61	
33	-1578	782	-1695	333	-138		-5328.29	-570.51	-22045.45	10470.23	-33826.79	
34	-1774	641	-1294	341	-137		-5629.57	-1585.64	-10889.31	10813.26	-29395.75	
35	-1649	801	-1127	294	-35		-5526.36	-1755.88	-9941.33	11668.83	-29708.72	
36	-1868	658	-1797	389	-53		-5227.71	-1702.97	1620.10	12088.80	-29464.15	
37	-1348	680	-1146	381	-41		-5748.46	-2693.63	16541.22	11411.43	-28668.56	
38	-1227	632	-1271	291			-5674.49	-2474.46	24237.30	11798.44		
39	-1392	470	-1358	287			-5008.18	-1230.55	23405.92	14725.11		
40	-1726	785	-1455	297			-4478.40	-1747.60	38168.82	15310.74		
41	-1393	604	-1039	301			-4789.33	-1563.02	37598.15	15943.87		
42	-1449	725	-1061	287			-4839.09	346.73	42288.85	16009.03		
43	-1403	757	-1136	261			-4602.90	1802.99	47216.00	17133.96		
44	-1485	673	-953	-41			-4623.30	2172.93	46879.69	17233.38		
45	-1147	663	-796	421			-5175.84	2325.16	47666.28	17670.67		
46	-1709	701	-1092	370			-5944.42	1327.38	53993.30	18760.95		
47	-1392	782	-1071	497			-4546.37	2101.47	69044.45	17236.39		
48	-1603	778	-662	483			-5135.88	5394.03	60967.02	18951.77		
49	-1200	765	-851	435			-5842.04	4798.40	61518.78	19227.33		
50	-1311	665	-1022				-4780.07	4807.04	65135.95			
51	-1487	722	-909				-5549.65	4528.49	77160.62			
52	-1424	806	-788				-5973.74	6475.39	77313.05			

Table C.8 (contd.): The values of delta hedge ratios and the results of the delta hedge for Vodafone.

Vodafone

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
t_i	Stock	Option	Stock	Option	Stock	Option	($) Stock	(£) Option	($) Stock	(£) Option	($) Stock	(£) Option
53	-1474	892	-1068				-5747.22	6952.18	77173.21			
54	-1407	400					-6068.71	6746.15				
55	-1022	685					-5607.71	7015.08				
56	-1811	711					-5899.20	6640.51				
57	-1914	683					-6161.19	6070.98				
58	-1254	565					-6255.83	6360.76				
59	-1364	893					-6479.68	6211.28				
60	-905	773					-5993.42	6870.62				
61		597						7224.88				
62		547						7542.41				
63		640						7568.82				
64		488						8275.45				
65		621						8102.28				
66		778						8460.36				
67		583						9033.10				
68		588						9026.80				
69		554						8302.32				
70		599						8414.96				
71		673						7535.51				
72		712						6986.93				
73		836						8243.33				
74		738						8812.80				
75		736						9780.56				
76		661						9364.01				
77		803						11173.05				
78		783						11118.79				

Table C.8 (contd.): The values of delta hedge ratios and the results of the delta hedge for Vodafone.

Vodafone

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
							(\$)	(£)	(\$)	(£)	(\$)	(£)
t_i	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option
79		770						10186.07				
80		616						10426.88				
81		616						10073.19				
79		770						10186.07				
80		616						10426.88				
81		616						10073.19				
82		741						9298.69				
83		574						10414.07				
84		884						10918.69				
85		795						9004.48				
86		640						8589.44				
87		679						9471.77				
88		814						9525.99				
89		889						7866.65				
90		706						9887.02				
91		675						11398.80				
92		689						10537.50				
93		597						10755.41				
94		626						11250.87				
95		589						11068.56				
96		744						12239.58				
97		651						13525.68				
98		641						14043.26				

Table C.8 (contd.): The values of delta hedge ratios and the results of the delta hedge for Vodafone.

Vodafone

Day / Week / Month	Daily Delta Hedge Ratio		Weekly Delta Hedge Ratio		Monthly Delta Hedge Ratio		Daily $\Lambda(t_0, t_i)$		Weekly $\Lambda(t_0, t_i)$		Monthly $\Lambda(t_0, t_i)$	
							(\$)	(£)	(\$)	(£)	(\$)	(£)
t_i	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option	Stock	Option
99		717						14012.66				
100		603						14031.03				
101		508						13891.04				
102		609						14775.08				
103		629						13848.94				

Table C.8 (contd.): The values of delta hedge ratios and the results of the delta hedge for Vodafone.

D.2.1 Option Descriptions Details and Hedging Time Intervals

Company	Put or Call	Strike Price	Maturity Date	Currency
Boeing	Call	100	17/01/2009	US Dollars
Daimler Chrysler	Put	64	21/12/2007	Euros
Deutsche Telekom	Call	20	19/06/2007	Euros
Ford	Put	5	17/01/2009	US Dollars
General Motors	Put	17.50	17/01/2009	US Dollars
Vodafone	Put	1.60	21/12/2007	GB Pound

Table D.9: Description details of the options used to hedge.

Company	Rebalancing	Equity Hedging Time Interval		Equity Option Hedging Time Interval	
		Beginning Date	Terminal Date	Beginning Date	Terminal Date
Boeing	Daily	27/10/2006	25/01/2007	01/08/2006	21/12/2006
	Weekly	25/01/2006	22/01/2007	01/08/2006	15/08/2007
	Monthly	05/01/2004	03/01/2007	01/08/2006	01/08/2007
Daimler Chrysler	Daily	27/10/2006	25/01/2007	07/02/2007	06/07/2007
	Weekly	25/01/2006	22/01/2007	07/02/2007	01/22/2007
	Monthly	05/01/2004	03/01/2007	07/02/2007	01/08/2007
Deutsche Telekom	Daily	27/10/2006	25/01/2007	01/11/2006	26/02/2007
	Weekly	25/01/2006	22/01/2007	01/11/2006	15/08/2007
	Monthly	05/01/2004	03/01/2007	01/11/2006	01/08/2007
Ford	Daily	27/10/2006	25/01/2007	01/08/2006	22/12/2006
	Weekly	25/01/2006	22/01/2007	01/08/2006	15/08/2006
	Monthly	05/01/2004	03/01/2007	01/08/2006	01/08/2007
General Motors	Daily	27/10/2006	25/01/2007	01/08/2006	22/12/2006
	Weekly	25/01/2006	22/01/2007	01/08/2006	15/08/2007
	Monthly	02/01/2004	03/01/2007	01/08/2006	01/08/2007
Vodafone	Daily	27/10/2006	25/ 01/2007	01/08/2006	22/12/2006
	Weekly	25/01/2006	22/01/2007	01/08/2006	15/08/2007
	Monthly	05/01/2004	03/01/2007	01/08/2006	01/08/2007

Table D.10: The hedging time intervals for our empirical testing

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