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<td>Takagi, Hiromichi</td>
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ON CLASSIFICATION OF Q-FANO 3-FOLDS OF GORENSTEIN INDEX 2 AND FANO INDEX $\frac{1}{2}$

HIROMICHI TAKAGI

Notation and Conventions.

$\sim$ linear equivalence
$\equiv$ numerical equivalence

ODP ordinary double point, i.e., singularity analytically isomorphic to \{xy + $z^2 + u^2 = 0 \subset \mathbb{C}^4$\}

QODP singularity analytically isomorphic to \{xy+$z^2 + u^2 = 0 \subset \mathbb{C}^4/\mathbb{Z}_2(1,1,1,0)$\}

$\mathbb{F}_n$ Hirzebruch surface of degree $n$

$\mathbb{F}_{n,0}$ surface which is obtained by the contraction of the negative section of $\mathbb{F}_n$

$\mathbb{Q}_3$ smooth 3-dimensional quadric.

$B_i$ (1 $\leq i \leq 5$) Q-factorial Gorenstein terminal Fano 3-fold of Fano index 2, and with Picard number 1 and $(-K)^3 = 8i$, where $K$ is the canonical divisor

$A_{2i}$ (1 $\leq i \leq 11$ and $i \neq 10$) Q-factorial Gorenstein terminal Fano 3-fold of Fano index 1, and with Picard number 1 and $(-K)^3 = 2i$

contraction of $(m,n)$-type extremal contraction whose exceptional locus has dimension $m$ and the image of the exceptional locus has dimension $n$

0. INTRODUCTION

In this article, we will work over $\mathbb{C}$, the complex number field.

Definition 0.0 (Q-Fano variety). Let $X$ be a normal projective variety. We say that $X$ is a Q-Fano variety (resp. weak Q-Fano variety) if $X$ has only terminal singularities and $-K_X$ is ample (resp. nef and big).

Let $I(X) := \min\{I|IK_X$ is a Cartier divisor$\}$ and we call $I(X)$ the Gorenstein index of $X$.

Write $I(X)(-K_X) \equiv \tau(X)H(X)$, where $H(X)$ is a primitive Cartier divisor and $\tau(X) \in \mathbb{N}$. (Note that $H(X)$ is unique since Pic$X$ is torsion free.) Then we call $\frac{\tau(X)}{I(X)}$ the Fano index of $X$ and denote it by $F(X)$.

Remark 0.1.

(1) We can allow that a Q-Fano variety or a weak Q-Fano variety has worse singularities than terminal. When we have to treat such a variety in this paper, we indicate singularities which we allow, e.g., 'a Q-Fano 3-fold with only canonical singularities';

(2) if $X$ is Gorenstein in Definition 0.0, we say that $X$ is a Fano variety (resp. a weak Fano variety).

Key words and phrases. Q-Fano 3-fold, Extremal contraction.
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For the classification theory of varieties, a Q-factorial Q-Fano variety with Picard number 1 is important because it is an output of the minimal model program. Here we mention the known result about the classification of Q-Fano 3-folds:

1. G. Fane started the classification of smooth Fano 3-folds and it was completed by V. A. Iskovskih [I1] ~ [I4], V. V. Shokurov [Sh1], [Sh2], T. Fujita [Fu1] ~ [Fu3], S. Mori and S. Mukai [MM1] ~ [MM3];
2. S. Mukai [Mu] classified indecomposable Gorenstein Fano 3-folds with canonical singularities by using vector bundles;
3. T. Sano [San1] and independently F. Campana and H. Flenner [CF] classified non Gorenstein Fano 3-folds of Fano indices > 1;
4. T. Sano [San2] classified non Gorenstein Fano 3-folds of Fano indices 1 and with only cyclic quotient terminal singularities. Recently T. Minagawa [Mi1] proved that non Gorenstein Q-Fano 3-folds with Fano indices 1 can be deformed to one with only cyclic quotient terminal singularities;
5. A. R. Fletcher [Fl] gave the classification of Q-Fano 3-folds which are weighted complete intersections of codimension 1 or 2. Recently S. Altinok [Al] (see also [RM2]) obtained a list of Q-Fano 3-folds which are subvarieties in a weighted projective space of codimension 3 or 4.

On the other hand K. Takeuchi [T1] simplified and amplified V. A. Iskovskih's method of classification by using the theory of the extremal ray. In particular he reproved the Shokurov's theorem [Sh2], the existence of lines on a smooth Fano 3-fold of Fano index 1 and with Picard number 1 by simple numerical calculations.

We formulate a slight generalization of Takeuchi's construction for a Q-factorial Q-Fano 3-fold $X$ with $\rho(X) = 1$ and give a classification of a Q-factorial Q-Fano 3-fold with the following properties:

**Main Assumption 0.2.**

1. $\rho(X) = 1$;
2. $I(X) = 2$;
3. $F(X) = \frac{1}{2}$;
4. $h^0(-K_X) \geq 4$;
5. there exists an index 2 point $P$ such that

\[(X, P) \simeq (xy + z^2 + u^a = 0)/\mathbb{Z}_2(1, 1, 1, 0), 0)\]

for some $a \in \mathbb{N}$.

**Takeuchi's construction 0.3.** Here we explain a slight generalization of Takeuchi's construction. Let $X$ be a Q-factorial Q-Fano 3-fold with $\rho(X) = 1.$ Suppose that we are given a birational morphism $f : Y \to X$ with the following properties:

1. $Y$ is a weak Q-Fano 3-fold;
2. $f$ is an extremal divisorial contraction such that $f$-exceptional locus $E$ is a prime Q-Cartier divisor.

Then we obtain the following diagram:

\[ Y_0 := Y \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \ldots \xrightarrow{g_{k-1}} Y_k \]

\[ X \xrightarrow{f} Y \xrightarrow{f'} X' \]

9
where

1. $Y_0 \to Y_1$ is a flop or a flip and $Y_i \to Y_{i+1}$ is a flip for $i \geq 1$;
2. $f'$ is a crepant divisorial contraction (in this case, $i = 0$) or an extremal contraction which is not isomorphic in codimension 1.

We use the following notation:

$Y' := Y_k$;

$E_i :=$ the strict transform of $E$ on $Y_i$;

$E :=$ the strict transform of $E$ on $Y'$;

$e := E^3 - E_1^3$ if $Y_0 \to Y_1$ is a flop or $:= 0$ otherwise;

$d_i := (-K_{Y_i})^3 - (-K_{Y_{i+1}})^3$ (resp. $a_i := \frac{E \cdot l_i}{(-K_{Y_i})^3}$) if $Y_i \to Y_{i+1}$ is a flip, where $l_i$ is a flipping curve, or $:= 0$ (resp. $:= 0$) otherwise;

$z$ and $u$ is defined as follows:

If $f'$ is birational, then let $E'$ be the exceptional divisor of $f'$ and set $E' \equiv z(-K_{Y'}) - uE$ or if $f'$ is not birational, then let $L$ be the pull back of an ample generator of $Pic X'$ and set $L \equiv z(-K_{Y'}) - uE$.

We note the following:

1. 

\[ (-K_{Y'})^2 E = (-K_Y)^2 E - \sum a_i d_i; \]

\[ (-K_{Y'}) E^2 = (-K_Y) E^2 - \sum a_i^2 d_i; \]

\[ E^3 = E^3 - e - \sum a_i^3 d_i; \]

2. On the other hand the value or the relation of the value (expressed with $z$ and $u$) of $(-K_{Y'})^3$, $(-K_{Y'})^2 E$, $(-K_{Y'}) E^2$ and $E^3$ are restricted by the properties of $f'$.

By these (1) and (2), we obtain equations of Diophantine type.

Under Main Assumption 0.2, Construction 0.3 works for a suitable choice of $f$ and we can solve the equations as noted above.

**Main Theorem.** Let $X$ be as in Main Assumption 0.2. Let $f : Y \to X$ be the weighted blow up at $P$ with weight $\frac{1}{2}(1, 1, 1, 2)$. Then $Y$ is a weak $Q$-Fano 3-fold.

Consider the diagram as in 0.3. Let $h := h^\ell (-K_X)$, $N := aw(X)$ and $n := \sum aw(Y_i, P_{ij})$ (the summation is taken over the index 2 points on flipping curves), where $aw(X)$ is the number of $\frac{1}{2}(1, 1, 1)$-singularities which we obtain by deforming non Gorenstein points of $X$ locally and $aw(Y_i, P_{ij})$ is defined similarly. Then we can solve the equations above and obtain a geometric classification of $X$ as below (in the table means that we don't know the existence of an example):
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<table>
<thead>
<tr>
<th>$(-K_X)^3$</th>
<th>$N$</th>
<th>$e$</th>
<th>$n$</th>
<th>$z$</th>
<th>$(-K_Y\cdot C)$</th>
<th>$f', X'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{3}{2}$</td>
<td>1</td>
<td>15</td>
<td>0</td>
<td>1</td>
<td>/</td>
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</tr>
<tr>
<td>$\frac{5}{2}$</td>
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<td>/</td>
<td>/</td>
<td>1</td>
<td>/</td>
<td>crep. div., $(-K_X)^3 = 2$, $I(X') = 1$</td>
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<tr>
<td>3</td>
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</tr>
<tr>
<td>4</td>
<td>3</td>
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<td>1</td>
<td>1</td>
<td>$(2,0)_1$, $A_6$</td>
</tr>
<tr>
<td>4</td>
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<td>0</td>
<td>1</td>
<td>2</td>
<td>$(2,1)_4$, $A_8$</td>
</tr>
<tr>
<td>$\frac{9}{2}$</td>
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<td>9</td>
<td>3</td>
<td>1</td>
<td>/</td>
<td>$(2,0)<em>{14}, A</em>{10}$</td>
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<tr>
<td>$\frac{9}{2}$</td>
<td>5</td>
<td>6</td>
<td>0</td>
<td>1</td>
<td>3</td>
<td>$(2,1)<em>1$, $A</em>{10}$</td>
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<tr>
<td>$\frac{9}{2}$</td>
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<td>8</td>
<td>3</td>
<td>1</td>
<td>/</td>
<td>$(2,0)<em>{16}, A</em>{16}$</td>
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<tr>
<td>?$\frac{9}{2}$</td>
<td>5</td>
<td>9</td>
<td>0</td>
<td>2</td>
<td>/</td>
<td>$(3,1), \deg F = 6$</td>
</tr>
<tr>
<td>?5</td>
<td>6</td>
<td>4</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>$(2,1), A_{12}$</td>
</tr>
</tbody>
</table>

$z = u$ if $f'$ is not a crepant divisorial contraction.

$u = 2$ if $f'$ is a crepant divisorial contraction.

$F :=$ a general fiber of $f'$ if $f'$ is (3,1)-type.

See Appendix for $(2,0)_4$.

$g(C) = 0$ in case $f'$ is of type $E_1$ and every singularity of $Y$ is a $\frac{1}{2}(1,1,1)$—singularity.

<table>
<thead>
<tr>
<th>$(-K_X)^3$</th>
<th>$N$</th>
<th>$e$</th>
<th>$n$</th>
<th>$z$</th>
<th>deg $\Delta$</th>
<th>deg $F$</th>
<th>$f', X'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{3}{2}$</td>
<td>1</td>
<td>9</td>
<td>0</td>
<td>1</td>
<td>/</td>
<td>3</td>
<td>$(3,1)$</td>
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<tr>
<td>5</td>
<td>2</td>
<td>8</td>
<td>1</td>
<td>1</td>
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<td>4</td>
<td>$(3,1)$</td>
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<tr>
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<td>3</td>
<td>7</td>
<td>2</td>
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<td>$(3,1)$</td>
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<tr>
<td>?$\frac{11}{2}$</td>
<td>3</td>
<td>8</td>
<td>0</td>
<td>2</td>
<td>8</td>
<td>/</td>
<td>$(3,2)<em>2$, $\mathbb{P}</em>{2,0}$</td>
</tr>
<tr>
<td>?6</td>
<td>4</td>
<td>7</td>
<td>1</td>
<td>2</td>
<td>6</td>
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<td>$(3,2)<em>2$, $\mathbb{P}</em>{2,0}$</td>
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<tr>
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<td>5</td>
<td>6</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>/</td>
<td>$(3,2)<em>2$, $\mathbb{P}</em>{2,0}$</td>
</tr>
</tbody>
</table>

$z = u$.

$\Delta :=$ the discriminant divisor of $f'$ if $f'$ is (3,2)-type.

$F :=$ a general fiber of $f'$ if $f'$ is (3,1)-type.

<table>
<thead>
<tr>
<th>$(-K_X)^3$</th>
<th>$N$</th>
<th>$e$</th>
<th>$n$</th>
<th>$z$</th>
<th>deg $\Delta$</th>
<th>$(-K_Y\cdot C)$</th>
<th>$f', X'$</th>
</tr>
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<tr>
<td>$\frac{13}{2}$</td>
<td>1</td>
<td>7</td>
<td>0</td>
<td>1</td>
<td>7</td>
<td>/</td>
<td>$(3,2), \mathbb{P}^2$</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>7</td>
<td>0</td>
<td>4</td>
<td>/</td>
<td>35</td>
<td>$(2,1), [5]$</td>
</tr>
<tr>
<td>?7</td>
<td>2</td>
<td>6</td>
<td>1</td>
<td>1</td>
<td>6</td>
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<td>$(3,2), \mathbb{P}^2$</td>
</tr>
<tr>
<td>$\frac{15}{2}$</td>
<td>3</td>
<td>7</td>
<td>0</td>
<td>2</td>
<td>/</td>
<td>9</td>
<td>$(2,1), [2], I(X') = 2$</td>
</tr>
<tr>
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<td>1</td>
<td>4</td>
<td>/</td>
<td>30</td>
<td>$(2,1), [5]$</td>
</tr>
<tr>
<td>?$\frac{15}{2}$</td>
<td>3</td>
<td>5</td>
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<td>/</td>
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</tbody>
</table>
Q-FANO 3-FOLDS

Type [i] means the Q-Fano 3-fold of type [i] which was classified by T. Sano in [San 2].

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
(-K_X)^3 & \frac{19}{2} & N & e & n & z & (-K_Y . C) & f', X' \\
\hline
& 1 & 6 & 0 & 3 & 36 & (2, 1), P^3 \\
& 2 & 6 & 0 & 2 & 18 & (2, 1), Q_3 \\
& 2 & 5 & 1 & 3 & 32 & (2, 1), P^3 \\
& 2 & 5 & 1 & 2 & 15 & (2, 1), Q_3 \\
\hline
\end{array}
\]

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\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
(-K_X)^3 & \frac{1}{2} & N & e & n & z & (-K_Y . C) & f', X' \\
\hline
& 1 & 6 & 0 & 3 & 36 & (2, 1), P^3 \\
& 2 & 6 & 0 & 2 & 18 & (2, 1), Q_3 \\
& 2 & 5 & 1 & 3 & 32 & (2, 1), P^3 \\
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& 2 & 5 & 1 & 3 & 32 & (2, 1), P^3 \\
& 2 & 5 & 1 & 2 & 15 & (2, 1), Q_3 \\
\hline
\end{array}
\]

In particular we have \((-K_X)^3 \leq 15\) and \(h^0(-K_X) \leq 10\).

Based on this result, we can derive the following properties for \(X\) as in the main theorem:

**Theorem A.** if any index 2 point satisfies the assumption (5) of 0.2, then \(\mid -K_X \mid \) has a member with only canonical singularities.

So the general elephant conjecture by M. Reid is affirmative for such an \(X\).
Theorem B. Let $X$ be a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold with $(1)\sim(4)$ of $0.2$. Let $N := \text{aw}(X)$. Then if $N > 1$ (resp. $N = 1$), $X$ can be transformed to a $\mathbb{Q}$-factorial $\mathbb{Q}$-Fano 3-fold $\tilde{Z}'$ with $(1)\sim(4)$ of $0.2$ and with only QODP's or $\frac{1}{2}(1,1,1)$-singularities as its singularities and $h^0(-K_{\tilde{Z}'}) = h$ and $\text{aw}(\tilde{Z}') = N - 1$ (resp. a smooth Fano 3-fold $\tilde{Z}'$ with $\rho(\tilde{Z}') = 1$, $F(\tilde{Z}') = 1$ and $h^0(-K_{\tilde{Z}'}) = h$) as follows:

\[ \begin{array}{c}
\tilde{Y} \\
X \xrightarrow{\text{def}} \tilde{X} \xleftarrow{\tilde{f}} \tilde{Z} \xrightarrow{\text{def}} \tilde{Z}',
\end{array} \]

where $* \xrightarrow{\text{def}} **$ means that ** is a small deformation of *;

$\tilde{X}$ is a $\mathbb{Q}$-Fano 3-fold as in $0.2$ and with only ODP's, QODP's or $\frac{1}{2}(1,1,1)$-singularities as its singularities;

$\tilde{f} : \tilde{Y} \to \tilde{X}$ is chosen as $f$ in the main theorem;

$\tilde{g} : \tilde{Y} \to \tilde{Z}$ be the anti-canonical model.

This is an analogue to the Reid’s fantasy about Calabi-Yau 3-folds [RM1].

Theorem C. If any index 2 point is a $\frac{1}{2}(1,1,1)$-singularity, $X$ can be embedded into a weighted projective space $\mathbb{P}(h,2N)$, where $h := h^0(-K_X)$ and $N$ is the number of $\frac{1}{2}(1,1,1)$-singularities on $X$.

We hope that this fact can be used for the classification of Mukai’s type (see [Mu]).

1. Examples

We consider the case that $h^0(-K_X) = 4$ and $N = 4$. By the table of the main theorem, there are two possibilities of $X$ in this case. We assume that every singularity of $Y$ is a $\frac{1}{2}(1,1,1)$-singularity. Then one of the following holds:

[1]. $f'$ is an extremal divisorial contraction which contracts a divisor $E'$ to a curve $C$ and $| - K_Y - E'| \neq \phi$. $X'$ is a $(2,2,2)$-complete intersection in $\mathbb{P}^6$ and satisfies the following properties:

(1) $X'$ is factorial;

(2) $C$ is a smooth conic;

(3) $X'$ has 3 singularities $P_0 \sim P_2$ on $C$ and $P_3$ is an ODP or the singularity analytically isomorphic to the origin of $\{xy + z^2 + w^3 = 0\} \subset \mathbb{C}^4$. Outside $P_i$'s, $X'$ is smooth.

[2]. $f'$ is blowing up at a smooth point $Q := f'(E')$ and $| - K_Y - E'| \neq \phi$. $X'$ is smooth, isomorphic to $A_{10}$ and there exist exactly three lines through the point $Q$.

We will construct examples for these cases by the following three steps:

Step 1. We construct $X'$ satisfying the properties as stated as in [1] or [2];

Step 2. We construct $f'$ satisfying the properties as stated as in [1] or [2];

Step 3. We construct $f : Y \to X$ as in the main theorem from $Y'$.

[1].

Step 1 for [1]. We construct $X'$ with only ODP's.
Claim 1. Let $V$ (resp. $X'$) be a $(2,2)$-complete intersection in $\mathbb{P}^6$ (resp. a quadric section of $V$) with the following properties:

1) $V$ (resp. $X'$) contains a smooth conic $C$;

2) $V$ (resp. $X'$) has three ODP's $P_0 \sim P_2$ on $C$ and outside $P_i$'s, $V$ (resp. $X'$) is smooth.

Then $X'$ is factorial.

Proof. We claim that $V$ contains the plane $P$ spanned by $C$. Let $\sigma$ be the pencil which consists of quadrics in $\mathbb{P}^6$ containing $V$. Since $P_i$ is an ODP on $V$, there is a quadric in $\sigma$ which is singular at $P_i$. If there is a quadric in $\sigma$ which is singular at all $P_i$'s, then it is singular on $P$ and hence $V$ is singular along $C$, a contradiction. So $\sigma$ is generated by two quadrics which are singular at some $P_i$. But such quadrics contains $P$ and hence $V$ contains $P$.

Let $\nu : \tilde{V} \to V$ be the composition of the blowing ups at $P_0 \sim P_2$ and $F_i$ the exceptional divisor over $P_i$. Let $\tilde{X}'$ be the strict transform of $X'$ on $\tilde{V}$ and $H$ the total transform of a hyperplane section of $V$. Then $\tilde{X}' \sim 2H - F_0 - F_1 - F_2$. Note that $|H - F_i - F_j|$ is free outside the strict transform $l_{ij}$ of the line through $P_i$ and $P_j$ and $|H - F_k|$ is free (note that $l_{ij}$ is contained in $V$ since $l_{ij} \subset P$). By this, we can easily see that $|\tilde{X}'|$ is free and $\tilde{X}'$ is numerically trivial only for $l_{ij}$'s $((i,j) = (0,1), (1,2), (2,0))$.

Let $\phi$ be the morphism defined by $|\tilde{X}'|$. Then $\phi$-exceptional curves are $l_{ij}$'s. We will prove that $\text{Leff}(\tilde{V}, \tilde{X}')$ holds and $\tilde{X}'$ meets every effective divisor on $\tilde{V}$. By [H, p.165, Proposition 1.1] and the argument of [H, p.172, the proof of Theorem 1.5], it suffices to prove that $\text{cd}(\tilde{V} - \tilde{X}') < 3$, i.e., for any coherent sheaf $F$ on $\tilde{V} - \tilde{X}'$, $H^i(\tilde{V} - \tilde{X}', F) = 0$ for all $i \geq 3$. Let $\bar{V} := \phi(\tilde{V})$ and $\bar{X}' := \phi(\tilde{X}')$. Consider the Leray spectral sequence

$$E^{p,q}_2 = H^p(\tilde{V} - \tilde{X}', R^q\phi_* F) \Rightarrow E^{p+q} = H^{p+q}(\bar{V} - \bar{X}', F),$$

where $\phi' := \phi|_{\tilde{V} - \tilde{X}'}$. Since $\bar{V} - \bar{X}'$ is affine and the dimension of every fiber of $\phi$ is 1, we have $E^{p,q}_2 = 0$ for $p \geq 1$ or $q \geq 2$ whence $E^{p+q} = 0$ for $p + q \geq 2$. So the assertion follows.

Furthermore since $\tilde{X}'$ is nef and big, $H^i(\tilde{V}, \mathcal{O}(-n\tilde{X}')) = 0$ for $n \geq 1$ and $i = 1, 2$ by KKV vanishing theorem. Hence by the Grothendieck-Lefschetz theorem [G, p.135, 3.18] (or [H, p.178, Theorem 3.1]), we have $\text{Pic}\tilde{X}' \simeq \text{Pic}\tilde{V} \simeq \mathbb{Z}^4$. So $\rho(\tilde{X}'/X') = 3$ which imply that $X'$ is factorial. \qed

We will give a pair $(V, X')$ satisfying the condition of Claim 1. Let $C$ be a smooth conic in $\mathbb{P}^6$ and $P_0 \sim P_2$ three points on $C$. We can choose a coordinate of $\mathbb{P}^6$ such that $C = \{x_0x_1 + x_1x_2 + x_2x_0 = x_3 = x_4 = x_5 = x_6 = 0\}$ and $P_i = \{x_j = 0 \text{ for } j \neq i\}$.

Claim 2. Let $X'$ be a $(2,2,2)$-complete intersection in $\mathbb{P}^6$ satisfying the following conditions:

1) $X'$ is factorial;

2) $X'$ contains a smooth conic $C$;

3) $X'$ has three ODP's $P_0 \sim P_2$ on $C$ and outside $P_i$'s, $X'$ is smooth.

Then $X'$ is the intersection of three quadrics $Q_1 \sim Q_3$ of the following forms by permuting $P_i$'s if necessary:
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\[ Q_1 := \{m_0x_0 + m_1x_1 + q_1 = 0\}; \]
\[ Q_2 := \{pm_1x_1 + m_2x_2 + q_2 = 0\}; \]
\[ Q_3 := \{x_0x_1 + x_1x_2 + x_2x_0 + \sum_{i=3}^{6} l_i x_i = 0\}, \]

where \( p \in \mathbb{C} \), \( m_i \) (resp. \( q_i \)) is a linear form (resp. a quadratic form) of \( x_3 \sim x_6 \) and \( l_i \) is a linear form of \( x_0 \sim x_6 \).

Conversely if \( X' = Q_1 \cap Q_2 \cap Q_3 \), where \( Q_i \) is of the form as above and \( m_i, q_i \)
and \( l_i \) are suitably general, then \( X' \) satisfies (1) \sim (3).

**Proof.** Let \( \gamma \) be the net which consists of quadrics containing \( X' \). \( \gamma \) contains a member \( Q_1 \) which is singular at \( P_2 \). Then \( Q_1 \) is of the form as above. If \( m_1 = m_2 = 0 \), then \( Q_1 \) is singular on the plane \( P \) spanned by \( C \) and hence \( X' \) is singular along \( C \), a contradiction. Hence \( m_1 \neq 0 \) or \( m_2 \neq 0 \). By permuting \( P_1 \) and \( P_2 \) if necessary, we may assume that \( m_1 \neq 0 \). \( \gamma \) contains a member \( Q_2 \) which is singular at \( P_0 \). \( Q_2 \) is of the form as

\[ \{m_1' x_1 + m_2 x_2 + q_2 = 0\}, \]

where \( m_1' \) and \( m_2 \) (resp. \( q_2 \)) are linear forms (resp. a quadratic form) of \( x_3 \sim x_6 \).

\( \gamma \) also contains a member \( Q' \) which is singular at \( P_1 \). If \( Q_1, Q_2 \) and \( Q' \) generate \( \gamma \), then \( X' \) contains the plane \( P \), a contradiction to the factoriality and \( F(X') = 1 \).

Hence \( Q' \) is contained in the pencil generated by \( Q_1 \) and \( Q_2 \). So \( m_1' = pm_1 \) for some \( p \in \mathbb{C} \) and

\[ Q = \{-pm_0x_0 + m_2x_2 + (q_2 - pq_1) = 0\}. \]

Since \( X' \) does not contain \( P \) as noted above, \( \gamma \) contains a member \( Q_3 \) of the form as in the statement. \( Q_3 \) is not contained in the pencil generated by \( Q_1 \) and \( Q_2 \) and hence \( Q_i \)'s generate \( \gamma \).

Conversely let \( X' := Q_1 \cap Q_2 \cap Q_3 \), where \( Q_i \) is of the form as above and \( m_i, q_i \)
and \( l_i \) are suitably general. We can easily check that \( X' \) satisfies (2) and (3). Set \( V := Q_1 \cap Q_2 \). We may assume that \( V \) satisfies the condition of Claim 1.

Hence by Claim 1, \( X' \) is factorial. \( \square \)

**Step 2 for [1].** Let \( \nu' : \tilde{X}' \to X' \) be the composition of the blowing ups at \( P_0 \sim P_{N-2} \) and \( F_i' \) the exceptional divisor over \( P_i \). Let \( \mu' : \tilde{X}' \to \tilde{X}' \) be the blowing up along the strict transform \( \tilde{C} \) of \( C \) and \( F' \) the \( \mu' \)-exceptional divisor. We will denote the strict transforms of the two fibers of \( F_i \simeq \mathbb{P}^1 \times \mathbb{P}^1 \) through \( F_i \cap \tilde{C} \) by \( l_{ij} \) \((i = 1, 2)\). Note that \(-K_{\tilde{X}}, l_{ij} = 0\). We can easily see that \( | -K_{\tilde{X}} | \) is free by \( P \cap X' = C \), where \( P \) is the plane spanned by \( C \) and \(-K_{\tilde{X}} \) is big. Hence \( l_{ij} \)'s are flopping curves on \( \tilde{X}' \) and we can see that the classes of \( l_{i1} \) and \( l_{i2} \) belong to the same ray. Let \( \tilde{X}' \to \tilde{X}'^+ \) be the flop. Then the strict transforms of \( F_i \)'s on \( \tilde{X}'^+ \)
are \( \mathbb{P}^2 \)'s and we can contract them to \( \frac{1}{2}(1, 1, 1) \)-singularities. Let \( g' : \tilde{X}'^+ \to Y' \) be the contraction morphism, \( f' : Y' \to X' \) the natural morphism and \( E' \) the strict transform of \( F' \).

We will see that \( | -K_{\tilde{X}'}, -E'| \neq \phi \). Let \( F'^+ \) be the strict transform of \( F' \) on \( \tilde{X}'^+ \). Then \(-K_{\tilde{X}'^+} - F'^+ = g'^*(-K_{Y'}, -E') \).
Furthermore \( h^0(-K_{\tilde{X}'^+} - F'^+) = \)
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\( h^0(-K_{\mathcal{X}}, -F') \). Hence it suffices to prove that \( h^0(-K_{\mathcal{X}}|_{F'}) \leq 3 \) since \( h^0(-K_{\mathcal{X}}) = 4 \). Since there is a smooth member of \( |-K_{\mathcal{X}}| \), we have \( \mathcal{N}_{\mathcal{C}/\mathcal{X}} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-2) \). Hence \( F' \cong F \) and \( -K_{\mathcal{X}}|_{F'} \cong C_0 + l \), where \( C_0 \) is the minimal section of \( F' \) and \( l \) is a fiber of \( F' \). So we are done.

**Step 3 for [1].** Since \( Y' \) has only \( \frac{1}{2}(1,1,1) \)-singularities and \( -K_{Y'} \) is nef and big, we can construct a similar diagram \( Y_0' := Y' \rightarrow Y_1' \rightarrow \cdots \rightarrow Y_i' \rightarrow \cdots \rightarrow Y := Y' \rightarrow X \rightarrow Y_{0'} \). By considering extremal rays, where \( Y_i' \rightarrow Y_{i+1}' \) is a flop or a flip for \( i = 0 \) and a flip for \( i \geq 1 \). Let \( E_i \) (resp. \( E \)) be the strict transform of \( E \) on \( Y_i' \) (resp. \( Y \)). Let \( R_i \) be the extremal ray which is other than the ray associated to \( f' \) for \( i = 0 \) or the \( K_{Y_i} \)-negative extremal ray for \( i \geq 1 \). By similar calculations to 0.3, we have

\[
\begin{align*}
(1) \quad (-K_{Y})^2 E &= 1 + \sum a_i' d_i' ; \\
(2) \quad (-K_{Y})E^2 &= -2 - \sum a_i'^2 d_i' ; \\
(3) \quad E^3 &= -6 + \sum a_i'^3 d_i' + e',
\end{align*}
\]

where \( e', a_i' \) and \( d_i' \) are similarly defined to 0.3 with respect to \( -K_{Y_i} \) and \( E_i \) and furthermore we can see that \( a_i' \) is a non negative integer.

**Claim 3.** \( E_i.R_i < 0 \).

**Proof.** We can prove the assertion by induction. For \( i = 0 \), \( E_0.R_0 < 0 \) can be directly checked. Assume that the assertion holds for the numbers less than \( i \). So the other extremal ray than \( R_i \) is positive for \( E_i \). Since \( -K_{Y_i} \) is free outside a finite number of curves, \( -K_{Y_i}|_{E_i} \) is numerically equivalent to an effective 1-cycle. Hence by \( -K_{Y_i'}E_i^2 \leq -K_{Y_i}E_i^2 = -2 \), we have \( E_i,R_i < 0 \). □

By this claim, we know that \( f \) is an divisorial contraction whose exceptional divisor is \( E \). If \( f \) is a crepant divisorial contraction, then \( l = 0 \). But \( (-K_{Y_i})^2 E = 1 \), a contradiction. Hence \( f \) is a \( K_{Y_i} \)-negative contraction. Assume that \( f \) is a \( (2,1) \)-type which contracts \( E \) to a curve \( C' \). Then \( (-K_{X}.C') = (-K_Y + E)(-K_Y)E = -1 - \sum d_i' a_i'(a_i' - 1) < 0 \), a contradiction since \( X \) is a Q-Fano 3-fold.

By the classification of a \( (2,0) \)-type contraction from a 3-fold with only index 2 terminal singularities (see Appendix), if \( f \) is such an contraction, then we have \( -K_{Y_i} E^2 \geq -2 \). On the other hand \( -K_{Y_i} E^2 \leq -K_{Y_i'}E_i^2 = -2 \). Hence there is no flop. So \( (-K_{Y_i})^2 E = (-K_{Y_i})^2 E = 1 \) and hence again by the classification of a contraction as above, \( f \) is the blow up at a \( \frac{1}{2}(1,1,1) \)-singularity or the weighted blow up at a QODP with weight \( (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1) \). (we use the coordinate as stated in the definition of QODP). In any case \( X \) is a Q-Fano 3-fold with \( I(X) = 2 \). We can easily check that \( (-K_{X})^3 = 4 \) and \( \text{aw}(X) = 4 \). Furthermore by this, \( F(X) \) must be \( \frac{1}{2} \). So \( X \) is what we want.

[2]
Step 1 for [2]. The Grassmannian \( G(2,5) \) (parameterizing 2-dimensional subspaces of 5-dimensional vector space) can be embedded into \( \mathbb{P}^9 \) by the Plücker embedding. Its defining equations are \( x_{ij}x_{kl} - x_{ik}x_{jl} + x_{jk}x_{il} = 0 \) for all \( 1 \leq i < j < k < l \leq 5 \), where \( x_{pq} \) (\( 1 \leq p < q \leq 5 \)) is a Plücker coordinate.

Let \( Q \) be the point defined by \( x_{pq} = 0 \) for any \( (p,q) \neq (1,2) \). Let \( l_1 \) (resp. \( l_2 \)) be the line \( \subset G(2,5) \) defined by \( x_{pq} = 0 \) for any \( (p,q) \neq (1,2),(1,3) \) (resp. \( (p,q) \neq (1,2),(2,4) \)). Let \( l_3 \) be the line \( \subset G(2,5) \) defined by the equations \( x_{pq} = r_{pq}x_{12} \) for \( (p,q) \neq (1,2) \) such that \( r_{34} = r_{35} = r_{45} = 0, r_{13}r_{24} - r_{23}r_{14} = 0, r_{13}r_{25} - r_{23}r_{15} = 0, r_{14}r_{25} - r_{24}r_{15} = 0 \) and \( r_{15}r_{25} \neq 0 \). Let \( H \) be the 3-plane spanned by \( l_1, l_2 \) and \( l_3 \). Then \( G(2,5) \cap H = l_1 \cup l_2 \cup l_3 \). Hence by [MM3, Proposition 6.8], there are two hyperplane \( H_1, H_2 \) and a quadric \( Q \) such that \( X' := G(2,5) \cap H_1 \cap H_2 \cap Q \) is smooth and \( X' \) contains \( l_1, l_2 \) and \( l_3 \). Since the tangent space of \( X' \) at \( Q \) also contains all the lines on \( X' \) through \( Q \), it is equal to \( H \). Hence there are only three lines on \( X' \) through \( Q \).

Step 2 for [2]. Let \( f' : Y' \rightarrow X' \) be the blow up at \( Q \) and \( E' \) the exceptional divisor. Let \( l_1', l_2' \) and \( l_3' \) be the transforms of \( l_1, l_2 \) and \( l_3 \) on \( Y' \). Since \( \text{Bs}(-K_{Y'}) = l_1' \cup l_2' \cup l_3' \), the rank of the natural map \( H^0(-K_{Y'}) \rightarrow H^0(\mathcal{O}(-K_{Y'}|_{E'})) \) is 3. Hence there is a unique member \( \tilde{E} \) of \( -K_{Y'} - E' \) such that \( h^0(-K_{Y'}) = 4 \).

Step 3 for [2]. Since \( -K_{Y'} + E' \) is free and \( -K_{Y'} + E' \) is numerically trivial only for \( l_1', l_2' \) and \( l_3' \) and positive for a curve in \( E' \), they are numerically equivalent and span an extremal ray \( R \) of \( \overline{NE}(Y') \). Since \( \text{Bs}(-K_{Y'}) = l_1' \cup l_2' \cup l_3' \) and \( -K_{Y'}l_4' < 0 \), \( R = l_1' \cup l_2' \cup l_3' \). Furthermore by \( \text{Bs}(-K_{Y'}) = l_1' \cup l_2' \cup l_3' \), there is a smooth anti-canonical divisor \( D \) ([MM3, Proposition 6.8]). Hence the contraction of \( l_1', l_2' \) and \( l_3' \) is a log flopping contraction for the pair \((Y', D)\) and the log flop exists. Let \( Y' \rightarrow Y'_0 \) be the log flop. Since \( D.l_i' = -1 \), the normal bundle of \( l_i' \) is of type \((-1,-2)\). Hence \( Y'_0 \) has three \( \frac{1}{2}(1,1,1) \)-singularities. Since \( -K_{Y'_0} \) is nef and big, we can construct a similar diagram \( Y'_0 \rightarrow Y'_1 \rightarrow \ldots \rightarrow Y'_i \rightarrow \ldots \rightarrow Y'_i+1 \rightarrow \ldots Y := Y'_i \rightarrow X \) to Lemma 3.2 by considering extremal rays, where \( Y'_i \rightarrow Y'_{i+1} \) is a flop or a flip for \( i = 0 \) and a flip if \( i \geq 1 \). Let \( E_i \) be the strict transform of \( E \) on \( Y'_i \).

Similarly to Step 3 for [1], we can see that \( f \) is the blow up at a \( \frac{1}{2}(1,1,1) \)-singularity or the weighted blow up at a QODP with weight \((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)\). In any case \( X \) is a Q-Fano 3-fold with \( I(X) = 2 \). Since \(-K_X)^3 = 4 \) and \( N = 4 \), \( F(X) \) must be \( \frac{1}{2} \). So \( X \) is what we want.

**APPENDIX**

In this appendix, we give the table of a \((2,0)\)-type contraction from a 3-fold with only index 2 terminal singularities.

**Proposition.** Let \( X \) be a 3-fold with only index 2 terminal singularities and \( f : X \rightarrow (Y,Q) \) a contraction of \((2,0)\)-type to a germ \((Y,Q)\) which contracts a prime divisor \( E \) to \( Q \). Then the following holds:

1. Assume that \( E \) contains no index 2 point. Then one of the following holds:

   \[(2,0)_1 : (E, -E|_E) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \text{ and } Q \text{ is a smooth point}; \]

   \[(2,0)_2 : (E, -E|_E) \simeq (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1)) \text{ and } (Y, Q) \simeq (\{(xy+zw = 0) \subset \mathbb{C}^4\}, o); \]

2. Assume that \( E \) contains an index 2 point. Then one of the following holds:

   \[(2,0)_3 : (E, -E|_E) \simeq (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)) \text{ and } Q \text{ is a smooth point}; \]

   \[(2,0)_4 : (E, -E|_E) \simeq (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1)) \text{ and } (Y, Q) \simeq (\{(xy+zw = 0) \subset \mathbb{C}^4\}, o); \]

   \[(2,0)_5 : (E, -E|_E) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \text{ and } Q \text{ is a smooth point}; \]
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\( (2,0)_3 : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, \mathcal{O}_{\mathbb{P}^2}(1)|_{\mathbb{F}_{2,0}}) \) and \((Y, Q) \simeq (((xy + z + w^k = 0) \subset C^4), o)(k \geq 3);\)

\( (2,0)_4 : (E, -E|_E) \simeq (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \) and \(Q\) is a \(\frac{1}{2}(1,1,1)\)-singularity.

Furthermore for all cases, \(f\) is the blow up of \(Q\).

(2) Assume that \(E\) contains an index 2 point. Then one of the following holds:

\( (2,0)_5 : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, l) \), where \(l\) is a ruling of \(\mathbb{F}_{2,0} \).

\(Q\) is a smooth point and \(f\) is a weighted blow up with weight \((2,1,1)\).

In particular we have \(K_X = f^*K_Y + 3E;\)

\( (2,0)_6 : K_X = f^*K_Y + E\) and \(Q\) is a Gorenstein singular point. \(E^3 = \frac{1}{2};\)

\( (2,0)_7 : K_X = f^*K_Y + E\) and \(Q\) is a Gorenstein singular point. \(E^3 = 1;\)

\( (2,0)_8 : K_X = f^*K_Y + E\) and \(Q\) is a Gorenstein singular point. \(E^3 = \frac{3}{2};\)

\( (2,0)_9 : K_X = f^*K_Y + E\) and \(Q\) is a Gorenstein singular point. \(E^3 = 2;\)

\( (2,0)_{10} : (E, -E|_E) \simeq (((xy + w^2 = 0) \subset \mathbb{F}(1,1,2,1)), \mathcal{O}(2)).\)

\((Y, Q) \simeq (((xy + z^k + w^2 = 0) \subset C^4/\mathbb{Z}_2(1,1,0,1)), o).\)

\(f\) is a weighted blow up with a weight \((\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}).\)

In particular we have \(K_X = f^*K_Y + \frac{1}{2}E;\)

\( (2,0)_{11} : (E, -E|_E) \simeq (\mathbb{F}_{2,0}, 3l).\)

\(Q\) is a \(\frac{1}{3}(2,1,1)\)-singularity and \(f\) is a weighted blow up with a weight \(\frac{1}{3}(2,1,1).\)

In particular we have \(K_X = f^*K_Y + \frac{1}{3}E;\)
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[T3] ———, a private letter to the author.

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