Periodicity on poly-Euler numbers and Vandiver type congruence for Euler numbers

By

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Abstract

Poly-Euler numbers are introduced as a generalization of classical Euler numbers. In this article, a periodic property for poly-Euler numbers and Vandiver type congruence for Euler numbers are discussed.

§ 1. Introduction

For every integer \( k \), we define poly-Euler numbers \( E_n^{(k)} \) \( (n = 0, 1, 2, \ldots) \) by

\[
\frac{\text{Li}_k(1 - e^{-4t})}{4t \cosh t} = \sum_{n=0}^{\infty} \frac{E_n^{(k)}}{n!} t^n,
\]

where

\[
\text{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \quad (|x| < 1, \ k \in \mathbb{Z})
\]

is the \( k \)-th polylogarithm. Poly-Euler numbers are a generalization of classical Euler numbers \( E_n \) defined by

\[
\frac{1}{\cosh t} = \sum_{n=0}^{\infty} \frac{E_n}{n!} t^n.
\]
Indeed, we easily see that $E_{n}^{(1)} = E_{n}$. The manner of generalization using the polylogarithm is due to Kaneko [3]. He introduced the poly-Bernoulli numbers $B_{n}^{(k)}$ by

$$\frac{\text{Li}_{k}(1 - e^{-t})}{1 - e^{-t}} = \sum_{n=0}^{\infty} \frac{B_{n}^{(k)}}{n!} t^{n} \quad (k \in \mathbb{Z})$$

and discovered many meaningful properties of that. Furthermore, combinatorial interpretations for poly-Bernoulli numbers were discovered by Brewbaker [2] and Launois [6] up to the present (see also [10]). In the previous research, the authors presented many properties of poly-Euler numbers ([7] and [8]), for example, explicit formulae, Clausen-von Staudt type formula, and a parity formula. Further we also found certain combinatorial interpretations for poly-Euler numbers.

We should mention that the research on poly-Euler numbers relates to that of multiple $L$ values. In fact, poly-Euler numbers are introduced as special values of a generalized Dirichlet $L$-function which relates to multiple $L$-functions (see [9] and [7, 8]).

We should also mention Arakawa-Kaneko’s zeta-function. Arakawa and Kaneko [1] introduced a zeta-function which relates to the poly-Bernoulli numbers and the multiple zeta-functions. This property has been applied to the research on the duality for multiple zeta-star values (see [4] and [5]). Our $L$-function would also play a key role in the research on multiple $L$ values.

In this article, we treat a periodic property for poly-Euler numbers with negative index and the Vandiver type congruence for Euler numbers. From the numerical data (see Tables 1 and 2 below), we can find that the one’s digits of poly-Euler numbers $(n + 1)E_{n}^{(-k)} (k, n \geq 0)$ change periodically with respect to $k$ and $n$. We prove this periodicity in the next section. In Section 3, we give the Vandiver type congruence for Euler numbers. Tables 1 and 2 below are the lists of numerical values of poly-Euler numbers $(n + 1)E_{n}^{(-k)}$.

§ 2. Periodicity for the one’s digits of poly-Euler numbers

From the numerical data, we can observe many interesting information of poly-Euler numbers. Here, we focus on the one’s digits of poly-Euler numbers and prove the periodicity. We start with reviewing an explicit formula for poly-Euler numbers which will be used in the following sections. Hereafter, we put $E_{n}^{(k)} := (n + 1)E_{n}^{(k)}$.

**Theorem 2.1** (Theorems 3.1 and 6.1 in [7]). For any non-negative integer $n$ and any integer $k$, we have

$$E_{n}^{(k)} = \frac{1}{2} \sum_{m=0}^{n+1} \binom{n+1}{m} B_{n-m+1}^{(k)} 4^{n-m+1} ((-1)^{m} - (-3)^{m}).$$

(2.1)
### Table 1. Poly-Euler numbers $\tilde{E}_n^{(-k)} = (n+1)E_n^{(-k)}$ ($0 \leq k \leq 4$)

<table>
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<tr>
<th>$n \backslash k$</th>
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<th>3</th>
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</tbody>
</table>

### Table 2. Poly-Euler numbers $\tilde{E}_n^{(-k)} = (n+1)E_n^{(-k)}$ ($5 \leq k \leq 7$)

<table>
<thead>
<tr>
<th>$n \backslash k$</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
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</table>
In particular, for poly-Euler numbers with non-positive index, the above formula can be rewritten as

\begin{equation}
\tilde{E}_n^{(-k)} = \frac{(-1)^k}{2} \sum_{l=0}^{k} (-1)^l l! \binom{k}{l} \left( (4l + 3)^{n+1} - (4l + 1)^{n+1} \right) \quad (k \geq 0).
\end{equation}

The following theorem gives the periodicity for the one’s digits of poly-Euler numbers:

**Theorem 2.2.** For any non-negative integers \( n, n', k \) and \( k' \) with \( n \equiv n' \pmod{4} \), we have

\[ \tilde{E}_n^{(-k)} \equiv \tilde{E}_{n'}^{(-k')} \pmod{10}. \]

In particular, we have \( \tilde{E}_n^{(-k)} \equiv 6\mathcal{E}_n^{(-k)} + 5\delta_n \pmod{10} \), where \( \delta_n \) takes 1 if \( n \) is even and 0 otherwise, and

\[ \mathcal{E}_n^{(-k)} = \begin{cases} 
-(3^n + 3) & \text{for } k \equiv 0 \pmod{4}, \\
2^n & \text{for } k \equiv 1 \pmod{4}, \\
-2^n + (1 + (-1)^n) & \text{for } k \equiv 2 \pmod{4}, \\
(2^n + 2)(1 + (-1)^n) & \text{for } k \equiv 3 \pmod{4}. 
\end{cases} \]

**Proof.** The authors had shown the parity of poly-Euler numbers in [8]. Namely, we have \( \tilde{E}_n^{(-k)} \equiv \delta_n \pmod{2} \). Therefore we need to show that \( \tilde{E}_n^{(-k)} \equiv \mathcal{E}_n^{(-k)} \pmod{5} \). From (2.2), we have

\begin{equation}
2\tilde{E}_n^{(-k)} \equiv (-1)^k \sum_{l=0}^{k} (-1)^l l! \binom{k}{l} \alpha(n, l) \pmod{5},
\end{equation}

where \( \alpha(n, l) := (3 - l)^{n+1} - (1 - l)^{n+1} \). Note that \( \binom{k}{l} = 0 \) for \( 0 \leq k < l \) and

\[ \binom{k}{l} \equiv \binom{k'}{l} \pmod{p} \]

holds for any odd prime \( p \) and any non-negative integers \( l, k \) and \( k' \) with \( k \equiv k' \pmod{p - 1} \). Therefore, when we put \( k \equiv a, n \equiv b \pmod{4} \) \((a, b \in \{0, 1, 2, 3\})\), the above formula becomes

\[ 2\tilde{E}_n^{(-k)} \equiv (-1)^a \sum_{l=0}^{a} (-1)^l l! \binom{a}{l} \alpha(b, l) \pmod{5}, \]

which gives \( \tilde{E}_n^{(-k)} \equiv \mathcal{E}_n^{(-k)} \pmod{5} \). It follows that \( \tilde{E}_n^{(-k)} \equiv 6\mathcal{E}_n^{(-k)} + 5\delta_n \pmod{10} \) from the Chinese remainder theorem. Furthermore, we have \( \tilde{E}_n^{(-k)} \equiv \tilde{E}_{n'}^{(-k')} \pmod{10} \).
for any non-negative integers $n, n', k$ and $k'$ with $n \equiv n', k \equiv k' \pmod{4}$, since $\widetilde{E}_n^{(-k)} \equiv \widetilde{E}_{n'}^{(-k')} \pmod{2}$ and $\widetilde{E}_n^{(-k)} \equiv \widetilde{E}_{n'}^{(-k')} \pmod{5}$. Thus we obtain Theorem 2.2.

\section{Vandiver type congruence for the Euler numbers}

Kaneko \cite{3} showed the following congruence for the Bernoulli numbers which is originally due to Vandiver from the viewpoint of the poly-Bernoulli numbers: For any odd prime $p$ and positive integer $n \leq p - 2$,

$$B_n \equiv \sum_{l=1}^{p-2} H_l (l + 1)^n \pmod{p},$$

where $H_n := \sum_{i=1}^{n} i^{-1}$ is the $n$-th harmonic number and $B_n$ is the $n$-th Bernoulli number defined by

$$\frac{te^t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}.$$

Similarly, we obtain the following Vandiver type congruence for the Euler numbers from the viewpoint of poly-Euler numbers.

\textbf{Theorem 3.1.} For any odd prime $p$ and non-negative integer $n$ not exceeding $p - 3$, we have

$$(n + 1)E_n \equiv \sum_{l=1}^{(p-1)/2} H_l \widetilde{A}(n, l) \pmod{p},$$

where

$$\widetilde{A}(n, l) = \begin{cases} 0 & \text{when } n \text{ is odd}, \\ 1 & \text{when } l = (p - 1)/2 \text{ and } n \text{ is even}, \\ 2 \sum_{j=0}^{n/2} \binom{n+1}{2j+1} (4l + 2)^{n-2j} & \text{otherwise}. \end{cases}$$

\textbf{Proof.} In \cite{3}, Kaneko showed an explicit formula for the poly-Bernoulli numbers: For any integers $k$ and $n \geq 0$, we have

$$\mathcal{B}_n^{(k)} = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m! \binom{n}{m} \{m\}}{(m + 1)^k}.$$

From this formula, we see that $\mathcal{B}_n^{(1)} \equiv \mathcal{B}_n^{(2-p)} \pmod{p}$ for $n = 0, 1, \ldots, p - 2$ and any odd prime $p$. Consequently, $\widetilde{E}_n^{(1)} \equiv \widetilde{E}_n^{(2-p)} \pmod{p}$ for $n = 0, 1, \ldots, p - 3$ from Theorem 2.1.
Hereafter we use another explicit formula for poly-Euler numbers which is a modified version of (2.2) (see Corollary 6.6 in [7]):

$$\tilde{E}_n^{(-k)} = (-1)^k \sum_{l=0}^{k} (-1)^l \left\{ \frac{k}{l} \right\} \sum_{m=1}^{n+1} \frac{1}{m; \text{odd}} \left( n + 1 \right) (4l + 2)^{n+1-m}.$$ 

Since $\tilde{E}_n^{(1)} \equiv \tilde{E}_n^{(2-p)} \pmod{p}$, we have

(3.1) $$\tilde{E}_n^{(1)} \equiv - \sum_{l=0}^{p-2} (-1)^l \left\{ \frac{p-2}{l} \right\} A(n, l) \pmod{p}.$$ 

Here, we have put

$$A(n, l) := \sum_{j=0}^{[n/2]} \left( \frac{n + 1}{2j + 1} \right) (4l + 2)^{n-2j}.$$ 

Since $4(p-l-1) + 2 \equiv -(4l + 2) \pmod{p}$, we have

$$A(n, l) \equiv \begin{cases} 
0 & \text{for } l = (p-1)/2 \text{ and } n \text{ is odd}, \\
1 & \text{for } l = (p-1)/2 \text{ and } n \text{ is even}, \\
(-1)^n A(n, p-l-1) & \text{otherwise}.
\end{cases}$$ 

Furthermore we remark that

$$(-1)^l \left\{ \frac{p-2}{l} \right\} \equiv (-1)^{p-l-1} (p-l-1)! \left\{ \frac{p-2}{p-l-1} \right\} \pmod{p}$$

holds for any positive integer $l$ less than $p-1$ (see Lemma 7.6 in [7]). Hence (3.1) is rewritten as

$$\tilde{E}_n^{(1)} \equiv - \sum_{l=1}^{(p-1)/2} (-1)^l \left\{ \frac{p-2}{l} \right\} \tilde{A}(n, l) \pmod{p}.$$ 

Thus the theorem is proved by using the following lemma:

**Lemma 3.2** (Lemma 2 in [3]). Suppose $p$ is an odd prime, and $1 \leq l \leq p-2$. Then,

$$(-1)^{l-1} \left\{ \frac{p-2}{l} \right\} \equiv H_l \pmod{p}.$$ 

\[\square\]

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