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# Novel Lifshitz point for chiral transition in the magnetic field



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## ABSTRACT

Based on the generalized Ginzburg–Landau theory, chiral phase transition is discussed in the presence of magnetic field. Considering the chiral density wave we show that chiral anomaly gives rise to an inhomogeneous chiral phase for nonzero quark-number chemical potential. Novel Lifshitz point appears on the vanishing chemical potential line, which may be directly explored by the lattice QCD simulation.

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One of the recent developments in the studies of the QCD phase diagram is a possible formation of inhomogeneous chiral phases, which may have various implications on high-energy heavy-ion collisions or compact stars [1]. They are specified by the spatially inhomogeneous chiral condensates and quite similar to FFLO state in superconductivity [2] or the textured phase in magnetism [3]. Similar subject has been also discussed in the context of color superconductivity [4]. Considering spatial modulation of the  $\bar{q}q$  condensates in quark matter, they take form,

$$\langle \bar{\psi} \psi \rangle + i \langle \bar{\psi} i \gamma_5 \tau_3 \psi \rangle \equiv \Delta(\mathbf{x}) \exp(i\theta(\mathbf{x})), \quad (1)$$

within  $SU(2)_L \times SU(2)_R$  chiral symmetry, where state is the eigenstate of the electromagnetic charge. Various types of the condensates can be considered: two kinds of one-dimensional order are well known, in 1 + 3 dimensions, within the Nambu–Jona–Lasinio (NJL) model: one is called dual chiral density wave (DCDW) characterized by the uniform amplitude  $\Delta$  and  $\theta = \mathbf{q} \cdot \mathbf{x}$  [5], and the other is called real kink crystal (RKC) by the spatially periodic function  $\Delta$  without  $\theta$  [6]. These configurations can be also obtained by embedding the Hartree–Fock solutions in the 1 + 1 dimensional models; the general form of the condensates has been found through the studies of the phase structure of the NJL<sub>2</sub> model or Gross–Neveu (GN) model [7]. Similar subject has been also discussed in quarkyonic matter [8].

In this Letter we consider the DCDW-type configuration specified by  $\Delta(\mathbf{x})$  and  $\theta(\mathbf{x})$ . Non-vanishing  $\Delta$  implies spontaneous symmetry breaking (SSB) of chiral symmetry. Then we can easily observe that the DCDW state can be described by operating the local chiral rotation with the chiral angle  $\theta(\mathbf{x})$ ,  $U_{\text{DCDW}}(\theta(\mathbf{x})) = \exp[i \int \theta(\mathbf{x}) A_3^0(\mathbf{x}) d^3x]$ , on the quark matter with the condensate  $\Delta = \langle \bar{\psi} \psi \rangle$ ,  $|\text{DCDW}\rangle = U_{\text{DCDW}} |\text{QM}; \Delta\rangle$ , where  $A_3^\mu(\mathbf{x})$  is the isospin 3-rd component of the axial-vector current. Consider the Dirac Hamiltonian,  $H_D^0 = -i\boldsymbol{\alpha} \cdot \nabla + \gamma^0 m(\mathbf{x})$ , where we assume that the mass function  $m(\mathbf{x})$  is given by the scalar condensate,  $m(\mathbf{x}) = -2G\Delta(\mathbf{x})$  as in the NJL-like models,  $\mathcal{L} = \bar{\psi}(i\partial - m_c)\psi + G(\bar{\psi}\psi)^2 + \dots$ . Applying  $U_{\text{DCDW}}$  on the Hamiltonian, we find the effective one in the DCDW state,

$$\begin{aligned} H_D &= U_{\text{DCDW}}^\dagger H_D^0 U_{\text{DCDW}} \\ &= -i\boldsymbol{\alpha} \cdot \nabla + \gamma^0 \left[ m_c + \frac{1 + \gamma_5 \tau_3}{2} M(\mathbf{x}) \right. \\ &\quad \left. + \frac{1 - \gamma_5 \tau_3}{2} M^*(\mathbf{x}) \right] \end{aligned} \quad (2)$$

with  $M(\mathbf{x}) = m(\mathbf{x}) \exp(i\theta(\mathbf{x}))$ . The phase degree of freedom  $\theta(\mathbf{x})$  or the complex order parameter  $M(\mathbf{x})$  then gives rise to important features. Differently from the chiral limit  $m_c = 0$ , the ansatz (1) does not give a self-consistent solution of Eqs. (1) and (2) by putting  $\theta(\mathbf{x}) = \mathbf{q} \cdot \mathbf{x}$ . The ground state is no more degenerate and the pionic mode described by  $\theta(\mathbf{x})$  gets a finite mass  $m_\pi^*$  of  $O(140 \text{ MeV})$ , which should be compared with the wave vector  $q$ . Thus it becomes important near the Lifshitz point, while it should be negligible in the well-developed phase [9]. In the follow-

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ing we first consider the chiral limit to clarify our idea, and will give some comments separately. The 1 + 1 dimensional version of DCDW, called *chiral spiral*, has been studied in Ref. [7], where chiral anomaly and the nesting effect play important roles to establish chiral spiral: chiral anomaly gives baryon density as  $\rho_B = \mu/\pi$  for chemical potential  $\mu$ , and the nesting effect  $q = 2\mu$  [7]. In particular it should be interesting to observe the latter relation is similar to the one in charge density wave or spin density wave in quasi-one dimensional system in condensed matter physics [10]. Consequently it has been shown that the chiral spiral is the most favorite configuration among various form of the condensate [7]. In 1 + 3 dimensions anomalous relation  $\rho_B = \mu/\pi$  becomes irrelevant and the nesting effect becomes incomplete. However, it has been shown that DCDW appears in the limited region of chemical potential [5].

Recently the chiral transition or deconfinement transition has attracted much attention in the presence of the magnetic field. The magnetic field is familiar in QCD through phenomena of compact stars [11] or high-energy heavy-ion collisions [12]. Theoretically, SSB has been shown to be enhanced by the magnetic effect, sometimes called *magnetic catalysis*, and the chiral magnetic effect has been another interesting subject [12]. The lattice QCD simulations have started to explore the chiral phase diagram on the temperature ( $T$ )-magnetic field ( $B$ ) plane [13]. One of the great advantages may be then that it is free from the sign problem on this plane. The inhomogeneous chiral phase has been also discussed in the presence of the magnetic field [14,15].

In this letter we discuss some topological aspects of DCDW in the presence of the uniform magnetic field  $\mathbf{B}$ , and explore the critical point in the  $\mu - T - B$  space. Since the chiral condensates couple with the magnetic field in the DCDW state, one need a careful treatment in the evaluation of physical quantities; chiral anomaly may play an important role in this context. The energy levels of quarks are discretized in the plane perpendicular to the magnetic field to produce the Landau levels, and each level has twofold degeneracy with respect to the spin degree of freedom in the absence of DCDW, except the lowest Landau level (LLL). Once DCDW is taken into account the spectrum is modified by the chiral condensates. Quarks in LLL then behave like one dimensional gas to exhibit a peculiar energy spectrum [5]. Since the effect of chiral anomaly has already discussed in the inhomogeneous chiral condensates in 1 + 1 dimensions [7], one may expect a manifestation of chiral anomaly in the DCDW state in the presence of the magnetic field.

Here we demonstrate it by using the NJL-like model in the mean-field approximation. Consider the Dirac operator,

$$H_D = \boldsymbol{\alpha} \cdot \mathbf{P} + \gamma^0 \left[ \frac{1 + \gamma_5 \tau_3}{2} M(\mathbf{x}) + \frac{1 - \gamma_5 \tau_3}{2} M^*(\mathbf{x}) \right] \quad (3)$$

with the covariant derivative  $\mathbf{P} = -i\nabla + Q\mathbf{A}$ , where  $\mathbf{A}$  is the electromagnetic vector potential and  $Q = \text{diag}(2/3e, -1/3e)$  is the charge matrix. We take the direction of the magnetic field  $\mathbf{B}$  along  $z$  axis. Consider for a while a single flavor without color degree of freedom by putting  $\tau_3 = 1$  ( $u$  quarks),  $Q = \tilde{e} > 0$  and  $N_c = 1$ , and take a generic form of  $\theta(\mathbf{x})$ . Changing the basis by the Weinberg transformation (local chiral  $U(1)$ ),  $\psi \rightarrow \psi_W = \exp(i\gamma_5 \tau_3 \theta(\mathbf{x})/2)\psi$ , the Dirac operator can be written as

$$\tilde{H}_D = \boldsymbol{\alpha} \cdot \mathbf{P} + \gamma_0 m(\mathbf{x}) - \gamma_0 \gamma_5 \boldsymbol{\gamma} \nabla \theta(\mathbf{x})/2. \quad (4)$$

Considering the flavor symmetric quark matter,  $\mu_u = \mu_d (\equiv \mu)$ , the quark number then can be generally given as

$$\langle \hat{N} \rangle = -\frac{1}{2} \eta_H + \sum_k \text{sign}(\lambda_k) [\theta(\lambda_k) n_F(\lambda_k - \mu) + \theta(-\lambda_k) (1 - n_F(\lambda_k - \mu))], \quad (5)$$

where  $\lambda_k$  is the eigenvalue of  $H_D$  and  $n_F(\omega) = (1 + e^{\omega/T})^{-1}$  [16]. The first term is called the Atiyah–Patodi–Singer  $\eta$  invariant [17],

$$\eta_H = \lim_{s \rightarrow 0^+} \eta_H(s), \quad \eta_H(s) = \sum_k \text{sign}(\lambda_k) |\lambda_k|^{-s} \quad (6)$$

and measures the extent of *spectral asymmetry* about zero. We shall see that it is a topological quantity for the case,  $|\nabla \theta(\mathbf{x})/m(\mathbf{x})| \ll 1$ , and closely related to chiral anomaly. The second one is the standard expression given by the Fermi–Dirac distribution function. Using the Mellin transform,  $\eta_H(s)$  can be written as

$$\eta_H(s) = \frac{1}{\pi} \cos\left(\frac{s\pi}{2}\right) \int_0^\infty d\omega \omega^{-s} \int d^3x \text{tr} [R_E(\mathbf{x}, i\omega)] + \text{c.c.}, \quad (7)$$

where  $R_E$  is the Euclidean resolvent,

$$R_E(\mathbf{x}, i\omega) \equiv \langle \mathbf{x} | \frac{1}{\tilde{H}_D - i\omega} | \mathbf{x} \rangle = \langle \mathbf{x} | \gamma_0 S(i\omega) | \mathbf{x} \rangle, \quad (8)$$

with the propagator,  $S(i\omega)$ ,  $S^{-1}(i\omega) = S_A^{-1}(i\omega) + \delta S$  with  $\langle \mathbf{x} | \delta S | \mathbf{y} \rangle = \gamma_5 \boldsymbol{\gamma} \cdot \nabla \theta(\mathbf{x})/2\delta(\mathbf{x} - \mathbf{y})$ .  $S_A$  is the Green's function in the presence of the magnetic field without DCDW. For slowly varying  $\theta(\mathbf{x})$ , we can apply the adiabatic method of Goldstone and Wilczek [19]. We can approximate  $m(\mathbf{x}) = m + \dots$  in the lowest order. Writing  $S_A(x, y) = \exp(i\tilde{e} \int_y^x d\mathbf{x} \cdot \mathbf{A}) \tilde{S}_A(x - y)$ , the Fourier transform of  $\tilde{S}_A(x - y)$  can be decomposed over the Landau levels [18],

$$\tilde{S}_A(k) = ie^{-\mathbf{k}_\perp^2/(\tilde{e}B)} \sum_{n=0}^\infty \frac{(-1)^n D_n(\tilde{e}B, k)}{(k^0)^2 - (k^3)^2 - m^2 - 2|\tilde{e}B|n}, \quad (9)$$

with the denominator,

$$D_n(\tilde{e}B, k) = (k_0 \gamma^0 - k^3 \gamma^3 + m) \left[ \mathcal{P}_- L_n^0(u) - \mathcal{P}_+ L_{n-1}^0(u) \right] + 4(k^1 \gamma^1 + k^2 \gamma^2) L_{n-1}^1(u), \quad (10)$$

with  $u = 2\mathbf{k}_\perp^2/|\tilde{e}B|$ , where  $\mathcal{P}_\pm = (1 \pm i\gamma^1 \gamma^2 \text{sign}(\tilde{e}B))$  is the spin projection operator, and  $L_n^\alpha(x)$  the generalized Laguerre polynomial. Expanding  $\tilde{S}(i\omega)$  around  $\tilde{S}_A$ ,  $\tilde{S}(i\omega) = \tilde{S}_A(i\omega) - \tilde{S}_A(i\omega) \delta S \tilde{S}_A \times (i\omega) + \dots$ , we have

$$\text{tr} R_E(\mathbf{x}, i\omega) = -\tilde{e}/(4\pi) m^2/(m^2 + \omega^2)^{3/2} \mathbf{B} \cdot \nabla \theta(\mathbf{x}) + \dots \quad (11)$$

There are two remarks in order: only LLL contributes and the result includes only the inner product of  $\mathbf{B}$  and  $\nabla \theta$ . Substituting it into Eq. (7) we find

$$\eta_H = \lim_{s \rightarrow 0} \eta_H(s) = -\frac{\tilde{e}}{2\pi^2} \int d^3x \mathbf{B} \cdot \nabla \theta(\mathbf{x}) + \dots \quad (12)$$

Thus the quark-number density can be written as

$$\rho_B^{\text{anom}} = \frac{\tilde{e}}{4\pi^2} \mathbf{B} \cdot \nabla \theta(\mathbf{x}) + \dots \quad (13)$$

This formula is the same as the one given by Son and Stephanov by gauging the Wess–Zumino–Witten action [20]. For  $d$  quarks with  $\tau_3 = -1$  and  $\tilde{e} < 0$ ,  $\tilde{e}$  in Eq. (13) is replaced by  $|\tilde{e}|$ . Hence the coefficient is always positive for both flavors,  $\tau_3 = \pm 1$ . Thus the anomalous quark-number density can be written as  $\rho_B^{\text{anom}} = \sum_{f=u,d} |e_f|/(4\pi^2) \mathbf{B} \cdot \nabla \theta(\mathbf{x}) + \dots$ , with  $e_u = 2/3e$  and  $e_d = -1/3e$ , when the flavor degree of freedom is explicitly written. Thus we find that the leading term in  $\eta_H$  originates from chiral anomaly and model independent, while other terms are model dependent.

Note that our result only comes from the non-vanishing magnetic field and is irrespective of its strength. Here it is interesting to observe that  $\eta_H$  is independent of the dynamical mass  $m$ , which is one of the remarkable features of chiral anomaly. It should be worth mentioning that the anomalous baryon number has been evaluated in the chiral bag model for nucleon [21]: quarks inside the bag exhibit the spectral asymmetry, and the baryon number is then given by the sum of the quarks, skyrmion and the anomalous baryon number to be one. Since  $\lambda_k$  changes its sign under the  $CT$  transformation,  $\psi \rightarrow i\gamma_0\gamma_5\psi$ ,  $\lambda_k(M) \rightarrow -\lambda_k(M^*)$ , we can see  $\eta_H$  always vanishes for real order parameter: the spectrum of the Dirac operator is symmetric about the zero eigenvalue for  $M \in \mathbf{R}$ . The phase degree of freedom  $\theta(\mathbf{x})$  is important in our case.

Accordingly, the thermodynamic potential should include the anomalous term besides the usual piece  $\Omega_s$ ,  $\Omega = \Omega_s + \Omega_{\text{anom}}$ . By way of the thermodynamic relation,  $\rho_B^{\text{anom}} = -\partial\Omega_{\text{anom}}/\partial\mu$ , we have

$$\Omega_{\text{anom}} = -\frac{\tilde{e}\mu}{4\pi^2} \int d^3x \mathbf{B} \cdot \nabla\theta(\mathbf{x}) + \dots \quad (14)$$

Note that LLL contributes to  $\Omega_s$  as well. Taking  $\theta(\mathbf{x}) = \mathbf{q} \cdot \mathbf{x}$  for DCDW, we immediately find from Eq. (14) that the most favorite direction of the wave vector  $\mathbf{q}$  is parallel to  $\mathbf{B}$  in the weak magnetic field. The authors in Ref. [14] have also found that the effective energy increases by a small deviation from the parallel configuration.

It should be interesting to see that the  $\eta$  invariant or spectral asymmetry can be directly evaluated in the closed form without recourse to the derivative expansion for the case,  $\mathbf{B}/\mathbf{q}$ . Using the Landau gauge,  $\mathbf{A} = (0, Bx, 0)$ , the Dirac operator  $H_D$  can be reduced to  $4 \times 4$  matrix on the basis of the plane wave  $\exp(ik_3z + ik_2y)$  and the Hermite functions  $u_n(x)$  [14], where  $n$  specifies the Landau levels. However, for the lowest Landau level (LLL),  $n = 0$ ,  $H_D$  is reduced to  $2 \times 2$  matrix from the property of  $u_n(x)$ . Thus the energy spectrum of the Dirac Hamiltonian then can be obtained,

$$\lambda_{n,p,\zeta,\epsilon} = \epsilon \sqrt{\left(\zeta \sqrt{m^2 + k_3^2} + q/2\right)^2 + 2|\tilde{e}B|n},$$

$$(n = 1, 2, \dots),$$

$$\lambda_{n=0,p,\epsilon} = \epsilon \sqrt{m^2 + k_3^2} + q/2, \quad (\text{LLL}), \quad (15)$$

with  $\zeta = \pm 1, \epsilon = \pm 1$ . Note that the spectrum is the same form for both flavors  $\tau_3 = \pm 1$ . We can immediately see the spectrum is symmetric about zero except LLL: LLL exhibits spectral asymmetry in the DCDW state. Note that the spectrum becomes symmetric in the absence of the magnetic field [5]. The evaluation of  $\eta_H$  is straightforward in this case and results in the same value as (12) in the case of  $q/2 < m$ , without any higher-order term [22]. For the opposite case,  $q/2 > m$ , some portion of LLL with  $\epsilon = -1$  becomes positive, so that the spectral asymmetry becomes different from Eq. (12); e.g., taking  $\theta(\mathbf{x}) = \mathbf{q} \cdot \mathbf{x}$ ,  $\eta_H$  reads

$$\eta_H = \frac{|\tilde{e}B|}{2\pi} V \left[ -\frac{q}{\pi} + \frac{(q^2 - 4m^2)^{1/2}}{\pi} \right]. \quad (16)$$

Note that the second term is a non-topological contribution.

After taking  $\mathbf{q}$  along  $\mathbf{B}$ , we can see another implication of chiral anomaly. The minimum point of  $\Omega$  with respect to  $|\mathbf{q}|$  is always shifted from zero by the linear term. Thus we find the DCDW phase is favorite for  $\mu \neq 0$  in the presence of the magnetic field, irrespective of the dynamical mass. In the following we shall reveal another interesting aspect of spectral asymmetry around the transition point, invoking the generalized Ginzburg–Landau (gGL) theory.

Thermodynamic potential can be evaluated by using the energy spectrum (15) [14]. Consider the general expansion of the thermodynamic potential density near the transition point [6],

$$\omega(M) = \omega(0) + \frac{\alpha_2}{2} |M|^2 + \alpha_3 \text{Im}(MM'^*)$$

$$+ \frac{\alpha_{4a}}{4} |M|^4 + \frac{\alpha_{4b}}{4} |M'|^2 + \dots \quad (17)$$

with a shorthand notation  $M' \equiv dM/dz$ , where we used the property that  $\omega(M)$  is invariant under the *global* chiral rotation,  $M \rightarrow e^{i\theta}M$ . The coefficients  $\alpha_n$  are functions of thermodynamic variables,  $\mu, T, B$  [6,7]. If the Dirac operator is symmetric by exchanging  $M(z)$  and  $M^*(z)$ , the imaginary terms are absent. DCDW in the absence of the magnetic field satisfies this condition, while it breaks in the presence of the magnetic field. The Dirac operator is no more symmetric for  $M(z)$  and  $M^*(z)$ , and the  $\alpha_3$  term is generated through the spectral asymmetry; since higher Landau levels only generate the even power of  $q$  in the thermodynamic potential due to the  $q \leftrightarrow -q$  symmetry in the spectrum (15), the odd power terms of  $q$  may be generated by the spectral asymmetry of LLL. We can see that it is closely related to the anomalous term in the thermodynamic potential (14). One may worry about the absence of the term proportional to  $q$  but independent of  $|M|$  in Eq. (17), since the anomaly term should give such term. However, we can see that such terms do not appear near the critical point, where the amplitude  $|M|$  or mass  $m$  becomes vanishingly small. General argument for this statement should go as follows: the thermodynamic potential never depends on  $q$  in the limit of  $|M| = 0$  or  $m = 0$ , where the phase degree of freedom is meaningless. Hence, if the thermodynamic potential include the  $q$  dependent terms, they must be expressed as the product of power functions of  $M$  and its derivative near the transition point. We show how the  $\alpha_3$  term comes out by evaluating the quark number (5) for LLL, which is related to the thermodynamic potential through the thermodynamic relation  $\partial(\Omega/V)/\partial\mu = -\langle \hat{N}/V \rangle$ . First, we consider the second term  $n_{\text{th}}$  in (5),

$$n_{\text{th}} = \sum_f \frac{N_c |e_f B|}{2\pi} \int_{-\infty}^{\infty} dE \rho_{\text{LLL}}(E) \left[ \frac{\theta(E)}{1 + e^{\beta(E-\mu)}} \right.$$

$$\left. - \frac{\theta(-E)}{1 + e^{-\beta(E-\mu)}} \right], \quad (18)$$

with the density of state,  $\rho_{\text{LLL}}(E) = |E - q/2|/\pi \sqrt{(E - q/2)^2 - m^2}$ . It is convenient for our purpose to rewrite  $n_{\text{th}}$  as an infinite series over the Matsubara frequency,  $\omega_n = (2n + 1)\pi T$ ,

$$n_{\text{th}} = \frac{1}{2} \eta_H - \sum_f \frac{N_c |e_f B|}{2\pi} \int_{-\infty}^{\infty} dE \rho_{\text{LLL}}(E)$$

$$\times T \sum_{n=0}^{\infty} \left[ \frac{1}{E - \mu - i\omega_n} + \frac{1}{E - \mu + i\omega_n} \right]. \quad (19)$$

Note that the first term apparently cancels the first term in (5), but information of the  $\eta$  invariant or the anomaly is not lost and still included in the remaining infinite series; actually one can see that it is reduced to be the pure anomalous contribution at  $T = 0$ ,  $\sum_f N_c |e_f B| q/4\pi^2$ , for  $q/2 < m$  and  $\mu < m + q/2$ . We shall see that such a pure anomalous term does not appear for small  $m$ . Expanding the density of state  $\rho_{\text{LLL}}$ , the remaining infinite series  $c_r$  can be easily evaluated for small  $m$ ,

$$c_r = \sum_f \frac{N_c |e_f B| q}{16\pi^4 T} \operatorname{Re} \sum_{n=0}^{\infty} \frac{1}{(n + 1/2 + i\mu\beta/2\pi)^3} m^2 + O(q^3) \quad (20)$$

up to  $O(m^2)$ , except the  $q$  independent terms. We'd like to note again here that the thermodynamic potential must not depend on  $q$  for  $m = 0$ , and it should be reasonable for  $m^2 q$  dependence to appear as the leading term in the thermodynamic potential. Thus we find the relevant term exactly corresponding to the  $\alpha_3$  one in the gGL expansion (17) (see Eq. (24)).

In the absence of the magnetic field, the spectrum becomes symmetric about zero and the coefficient  $\alpha_3(\mu, T, 0)$  vanishes. Thus the Lifshitz point, where the inhomogeneous state just appears, is given by looking at the leading-order contributions [23],  $\alpha_2(\mu, T, 0) = \alpha_{4b}(\mu, T, 0)$ . For further discussion we need a definite model to evaluate  $\alpha_i$ . Within the NJL model, which may be one of the effective models of QCD at low energy scale,  $\alpha_{4a}(\mu, T, 0) = \alpha_{4b}(\mu, T, 0)$ , so that the Lifshitz point coincides with the tricritical point for the chiral transition with the uniform condensate [6].

On the other hand, we can see that  $\alpha_3(\mu, T, B)$  becomes non-vanishing in the presence of the magnetic field. Thus gGL theory should bring about qualitatively different consequences. Most important and interesting one may be the appearance of the novel Lifshitz point. This point is defined as the tricritical one where the two lowest nontrivial coefficients vanish:

$$\alpha_2(\mu, T, B) = \alpha_3(\mu, T, B) = 0, \quad (21)$$

for given  $B$ .<sup>1</sup> First we evaluate  $\alpha_2(\mu, T, B)$  in the presence of the magnetic field in 1 + 3 dimensions, by using the two-flavor NJL model. Since it includes divergence, we need some regularization. Applying the proper-time regularization with cutoff  $\Lambda$ , we have

$$\alpha_2(\mu, T, B) = - \sum_{f; m \geq 0, n} \frac{N_c |e_f B|}{\pi^2} T (2 - \delta_{n,0}) \times \operatorname{Im} \int_{\Lambda^{-2}}^{\infty} d\tau \sqrt{\frac{\pi}{i\tau}} e^{i\tau[(\omega_m + i\mu)^2 + 2|e_f B|n]} + \frac{1}{2G} \quad (22)$$

with the Matsubara frequency,  $\omega_m = (2m + 1)\pi T$ , where we revive the flavor dependence by explicitly using  $e_{f=u,d}$  instead of  $\tilde{e}$ . In particular, for  $\mu = 0$ , the first term reads

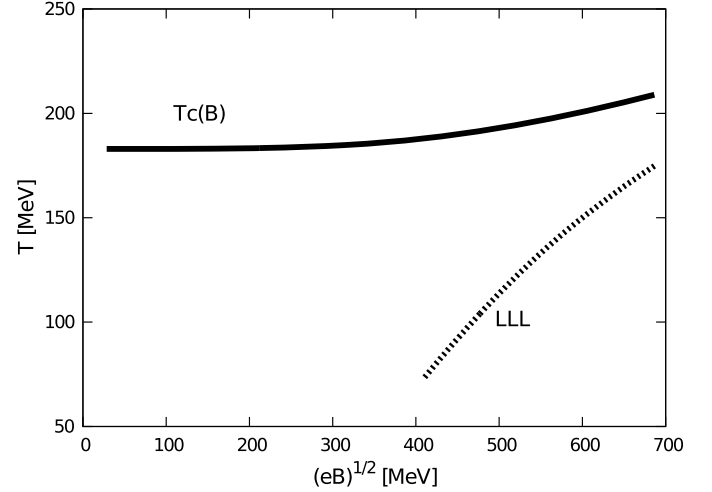
$$-4N_c \sum_{f; m \geq 0, n} \frac{|e_f B|}{(2\pi)^2} T \sqrt{\pi} \lambda_{m,n}^{-1} \Gamma\left(\frac{1}{2}, \frac{\lambda_{m,n}^2}{\Lambda^2}\right) \quad (23)$$

with  $\lambda_{m,n}^2 = \omega_m^2 + 2|e_f B|n$ , where  $\Gamma(a, x)$  is the incomplete Gamma function. For  $x \rightarrow \infty$ ,  $|\arg x| < 3\pi/2$ , it behaves  $\Gamma(a, x) = e^{-x} x^{a-1} [\sum_{n=0}^{N-1} (1-a)_n (-x)^{-n} + O(|x|^{-N})]$  [24], so that  $\alpha_2$  becomes finite.

The coefficient  $\alpha_3(\mu, T, B)$  includes no divergence. To evaluate  $\alpha_3(\mu, T, B)$  it should be sufficient to consider the LLL contribution,

$$\alpha_3(\mu, T, B) = - \sum_f \frac{N_c |e_f B|}{16\pi^3 T} \operatorname{Im} \psi^{(1)}\left(\frac{1}{2} + i\frac{\mu}{2\pi T}\right). \quad (24)$$

since other contributions vanish, where  $\psi^{(1)}$  is the trigamma function. We can easily check  $\alpha_3 \rightarrow \sum_f N_c |e_f B| / (8\pi^2 \mu)$  as  $T \rightarrow 0$ , which coincides with the discussion given below Eq. (17). Note



**Fig. 1.** Critical temperature (Lifshitz point) on the  $\mu = 0$  plane as a function of  $B$ . The same values are used for the parameters as in Ref. [5]:  $G\Lambda^2 = 6.35$ . The dotted curve is given by using only LLL contribution, which indicates the dimensional reduction in large  $B$ .

that  $\alpha_3(\mu, T, B) \geq 0$ . Then  $\alpha_3(\mu, T, B) = 0$  implies  $\mu = 0$ : the Lifshitz point resides on this plane. Note that this result does not depend on the detail of the model, but comes from spectral asymmetry: vanishing of spectral asymmetry simply means symmetry with respect to charge conjugation, which should be trivial in the case of  $\mu = 0$ . In Fig. 1 we present one example of the Lifshitz line on the  $B - T$  plane, determined by the equation,  $\alpha_2(0, T, B) = 0$ , within the NJL model, for readers to grasp its energy scale. Note that the critical temperature increases as  $B$  does in this example, while the recent lattice QCD simulation has suggested its decrease [13]. This is one of the controversial problems. It may be plausible that thermal fluctuations may become important at high-temperature, as suggested in Ref. [25]. If this is the case, we must take into account thermal fluctuations beyond the mean-field theory. However, our conclusion of the appearance of the Lifshitz point on the  $B - T$  plane is rather model-independent, reflecting spectral asymmetry, which should be little affected by thermal fluctuations.

We have shown that the Lifshitz point for the inhomogeneous chiral phase should reside on  $B - T$  plane given by  $\mu = 0$ . This conclusion may be model-independent and lead by chiral anomaly. A clear evidence may be obtained for small  $\mu$ , where the wave vector is proportional to the strength of the magnetic field  $B$  and chemical potential  $\mu$ . For  $\alpha_{4a,b}(\mu, T, B) > 0$ , the optimum values of the amplitude  $m$  and wave vector  $q$  are determined by the conditions  $\partial\omega/\partial m = \partial\omega/\partial q = 0$ , and we find  $q = -2\alpha_3(\mu, T, B)/\alpha_{4b}(\mu, T, B)$ . Since  $\alpha_3(\mu, T, B)$  should be proportional to  $\mu B$ ,  $q$  is as well. The critical line on the  $\mu - T$  plane, where the amplitude vanishes but wave vector necessarily does not, is given by the equation,

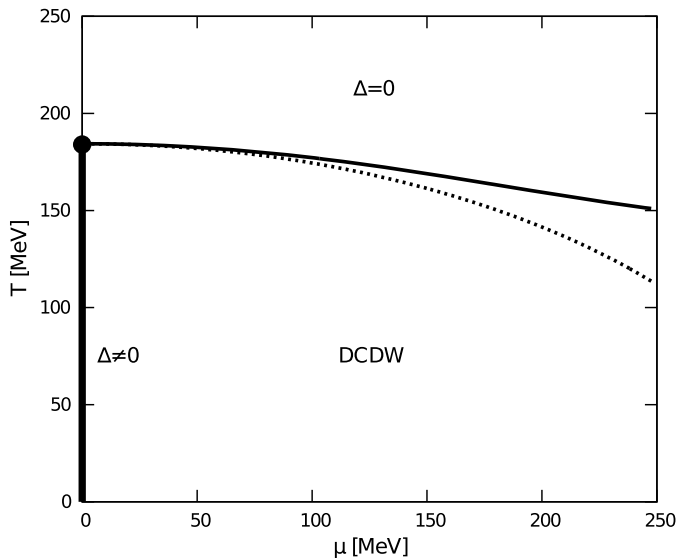
$$\alpha_2(\mu, T, B)\alpha_{4b}(\mu, T, B) = 4\alpha_3^2(\mu, T, B) \quad (25)$$

for given  $B$ . The critical line is then shifted upward from the usual chiral transition given by the uniform condensate,  $\alpha_2(\mu, T, B) = 0$ , assuming SSB at low  $T$  and small  $\mu$  (see Fig. 2 for example). Since  $\mu \simeq 0$  region is free from the sign problem, this critical line can be examined by the lattice QCD simulation. Note again that our result does not require the strong magnetic field.

In the presence of the small current mass  $m_c$  the chiral transition becomes cross-over for the usual chiral transition. In the gGL expansion (17) the linear terms should be added,  $\alpha_1(M + M^*)$  with  $\alpha_1 \propto m_c$ . Thus finite  $q$  is disfavored due to this term. On the other

<sup>1</sup> The same conditions also hold in 1 + 1 dimensions to give the critical temperature  $T_c$ ,  $T_c = e^\gamma/\pi$  with  $\gamma$  being the Euler constant on the line  $\mu = 0$  [7].





**Fig. 2.** Phase diagram in the  $\mu - T$  plane near the Lifshitz point for  $(eB)^{1/2} \simeq 300$  MeV, where we approximate  $\alpha_{4b}(\mu, T, B)$  and  $\alpha_2(\mu, T, B)$  by their values at  $B = 0$ . The uniform phase with  $\Delta \neq 0$  is confined on the  $\mu = 0$  axis (bold line). Solid line shows the boundary between the DCDW phase and the chiral restored phase ( $\Delta = 0$ ), while dotted line corresponds to the usual chiral transition.

hand, recalling that the  $\alpha_3$  term favors finite  $q$ , we can see some competition between these terms, depending on  $B$  and  $\mu$ . Hence there may appear no inhomogeneous phase in the  $\mu = 0$  plane, and the Lifshitz point is shifted from the  $\mu = 0$  plane. In the  $\mu - T$  plane one may expect there exists the Lifshitz point and the transition line for the uniform to nonuniform transition in the presence of the magnetic field. These features will be discussed in another paper.

Finally we briefly discuss the relation of DCDW with RKC in the presence of the magnetic field, leaving full discussion in another paper [26]. Considering the hybrid condensate,

$$M(z) = m \left( \frac{2\sqrt{\nu}}{1 + \sqrt{\nu}} \right) \operatorname{sn} \left( \frac{2mz}{1 + \sqrt{\nu}}; \nu \right) \exp(iqz), \quad (26)$$

we can discuss two phases simultaneously, where  $\operatorname{sn}(x; \nu)$  is the Jacobian elliptic function with modulus  $\nu$ . One can easily check this is one of the Hartree–Fock solutions in the  $1 + 1$  dimensional

$\text{NJL}_2$  model. We can immediately see that the anomalous term arises in the thermodynamic potential from the wave vector  $q$  even in this case. Hence the non-vanishing  $q$  is always favorite and pure RKC phase never appears in the presence of the magnetic field.

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