Selberg type inequalities on Hilbert $C^*$-modules

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1. INTRODUCTION

This paper is based on [15].

We briefly review the Selberg inequality and its generalization in a Hilbert space.

Let $H$ be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. The Selberg inequality [2, 17] states that if $y_1, y_2, \ldots, y_n$ and $x$ are nonzero vectors in $H$, then

$$\sum_{i=1}^{n} \frac{|\langle y_i, x \rangle|^2}{\sum_{j=1}^{n} |\langle y_j, y_i \rangle|} \leq \|x\|^2. \tag{1}$$

Moreover, Furuta [10] posed conditions enjoying the equality: The equality in (1) holds if and only if $x = \sum_{i=1}^{n} a_i y_i$ for some scalars $a_1, a_2, \ldots, a_n \in \mathbb{C}$ such that for arbitrary $i \neq j$

$$\langle y_i, y_j \rangle = 0 \quad \text{or} \quad |a_i| = |a_j| \quad \text{with} \quad \langle a_i y_i, a_j y_j \rangle \geq 0, \tag{2}$$

also see [7]. Note that the Selberg inequality is simultaneous extensions of the Bessel inequality and the Cauchy-Schwarz inequality. As a matter of fact, if $n = 1$ and $y = y_1$, then we have the Cauchy-Schwarz inequality $|\langle y, x \rangle| \leq \|y\| \|x\|$. If $\{y_i\}$ is an orthonormal system, then we have the Bessel inequality $\sum_{i=1}^{n} |\langle y_i, x \rangle|^2 \leq \|x\|^2$.

Fujii and Nakamoto [9] showed a refinement of the Selberg inequality (1): If $\langle y, y_i \rangle = 0$ for given nonzero vectors $y_1, \ldots, y_n \in H$, then

$$|\langle x, y \rangle|^2 + \sum_{i=1}^{n} \frac{|\langle x, y_i \rangle|^2}{\sum_{j=1}^{n} |\langle y_j, y_i \rangle|} \|y\|^2 \leq \|x\| \|y\|^2 \tag{3}$$

holds for all $x \in H$. Also, Bombieri [1] showed the following generalization of the Bessel inequality: If $x, y_1, \ldots, y_n$ are nonzero vectors in $H$, then

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \leq \|x\|^2 \max_{1 \leq i \leq n} \sum_{j=1}^{n} |\langle y_j, y_i \rangle|. \tag{4}$$

Moreover, Mitrinović, Pecarić and Fink [17, Theorem 5 in pp394] mentioned the following inequality equivalent to Bombieri’s type (4): If $x, y_1, \ldots, y_n$ are nonzero vectors in $H$ and $a_1, \ldots, a_n \in \mathbb{C}$, then

$$|\sum_{i=1}^{n} a_i \langle x, y_i \rangle|^2 \leq \|x\|^2 \sum_{i=1}^{n} |a_i|^2 \sum_{j=1}^{n} |\langle y_j, y_i \rangle|. \tag{5}$$

In this paper, from a viewpoint of the operator theory, we propose a Selberg type inequality in a Hilbert $C^*$-module, which is simultaneous extensions of the Bessel inequality and the Cauchy-Schwarz inequality in a Hilbert $C^*$-module. As applications, we show Hilbert $C^*$-module versions of Fujii-Nakamoto type (3), Bombieri type (4) and Mitrinović, Pecarić and Fink type (5).
2. Preliminaries

Let $\mathcal{A}$ be a unital $C^*$-algebra with the unit element $e$. An element $a \in \mathcal{A}$ is called positive if it is selfadjoint and its spectrum is contained in $[0, \infty)$. For $a \in \mathcal{A}$, we denote the absolute value of $a$ by $|a| = (a^*a)^{\frac{1}{2}}$. For positive elements $a, b \in \mathcal{A}$, the operator geometric mean of $a$ and $b$ is defined by

$$a \bowtie b = a^\frac{1}{2} \left( a^{-\frac{1}{2}}ba^{-\frac{1}{2}} \right)^{\frac{3}{2}} a^\frac{1}{2}$$

for invertible $a$. If $a$ and $b$ are non-invertible, then $a \bowtie b$ belongs to the double commutant $\mathcal{A}''$ of $\mathcal{A}$ in general. In fact, since $a \bowtie b$ satisfies the upper semicontinuity, it follows that $a \bowtie b = \lim_{\epsilon \to +0} (a + \epsilon e) \bowtie (b + \epsilon e)$ in the strong operator topology. If $\mathcal{A}$ is monotone complete in the sense that every bounded increasing net in the self-adjoint part has a supremum with respect to the usual partial order, then we have $a \bowtie b \in \mathcal{A}$, see [12]. The operator geometric mean has the symmetric property: $a \bowtie b = b \bowtie a$. In the case that $a$ and $b$ commute, we have $a \bowtie b = \sqrt{ab}$. For more details on the operator geometric mean, see [11, 8].

A complex linear space $\mathcal{X}$ is said to be an inner product $\mathcal{A}$-module (or a pre-Hilbert $\mathcal{A}$-module) if $\mathcal{X}$ is a right $\mathcal{A}$-module together with a $C^*$-valued map $(x, y) \mapsto \langle x, y \rangle : \mathcal{X} \times \mathcal{X} \to \mathcal{A}$ such that

(i) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ $(x, y, x \in \mathcal{X}, \alpha, \beta \in \mathbb{C})$,
(ii) $\langle x, ya \rangle = \langle x, y \rangle a$ $(x, y \in \mathcal{X}, a \in \mathcal{A})$,
(iii) $\langle y, x \rangle = \langle x, y \rangle^*$ $(x, y \in \mathcal{X})$,
(iv) $\langle x, x \rangle \geq 0$ $(x \in \mathcal{X})$ and if $\langle x, x \rangle = 0$, then $x = 0$.

We always assume that the linear structures of $\mathcal{A}$ and $\mathcal{X}$ are compatible. Notice that (ii) and (iii) imply $\langle xa, y \rangle = a^* \langle x, y \rangle$ for all $x, y \in \mathcal{X}, a \in \mathcal{A}$. If $\mathcal{X}$ satisfies all conditions for an inner-product $\mathcal{A}$-module except for the second part of (iv), then we call $\mathcal{X}$ a semi-inner product $\mathcal{A}$-module.

In this case, we write $\| x \| := \sqrt{\| \langle x, x \rangle \|}$, where the latter norm denotes the $C^*$-norm of $\mathcal{A}$. If an inner-product $\mathcal{A}$-module $\mathcal{X}$ is complete with respect to its norm, then $\mathcal{X}$ is called a Hilbert $C^*$-module. In [6], from a viewpoint of operator theory, we presented the following Cauchy-Schwarz inequality in the framework of a semi-inner product $C^*$-module over a unital $C^*$-algebra: If $x, y \in \mathcal{X}$ such that the inner product $\langle x, y \rangle$ has a polar decomposition $\langle x, y \rangle = u|\langle x, y \rangle|$ with a partial isometry $u \in \mathcal{A}$, then

$$|\langle x, y \rangle| \leq u^* \langle x, x \rangle u \bowtie \langle y, y \rangle.$$

Under the assumption that $\mathcal{X}$ is an inner product $\mathcal{A}$-module and $\langle y, y \rangle$ is invertible, the equality in (6) holds if and only if $xu = yb$ for some $b \in \mathcal{A}$. As applications of the Cauchy-Schwarz inequality (6), we cite [5, 18].

An element $x$ of a Hilbert $C^*$-module $\mathcal{X}$ is called nonsingular if the element $\langle x, x \rangle \in \mathcal{A}$ is invertible. The set $\{x_i \} \subset \mathcal{X}$ is called orthonormal if $\langle x_i, x_j \rangle = \delta_{ij} e$. For more details on Hilbert $C^*$-modules, see [16].

3. Main theorem

Fiest of all, we show the following Selberg type inequality in a Hilbert $C^*$-module.
**Theorem 1.** Let \( \mathscr{X} \) be an inner product \( C^* \)-module over a unital \( C^* \)-algebra \( \mathscr{A} \). If \( x, y_1, \ldots, y_n \) are nonzero vectors in \( \mathscr{X} \) such that \( y_1, \ldots, y_n \) are nonsingular, then

\[
\sum_{i=1}^{n} \langle x, y_i \rangle \left( \sum_{j=1}^{n} |\langle y_j, y_i \rangle| \right)^{-1} \langle y_i, x \rangle \leq \langle x, x \rangle.
\]

The equality in (7) holds if and only if \( x = \sum_{i=1}^{n} y_i a_i \) for some \( a_i \in \mathscr{A} \) and \( i = 1, \ldots, n \) such that for arbitrary \( i \neq j \) \( \langle y_i, y_j \rangle = 0 \) or \( |\langle y_j, y_i \rangle| a_i = \langle y_i, y_j \rangle a_j \).

Theorem 1 is simultaneous extensions of the Bessel inequality [4] and the Cauchy-Schwarz inequality [6] in a Hilbert \( C^* \)-module. As a matter of fact, if \( \{y_1, \ldots, y_n\} \) is orthonormal in Theorem 1, then we have the Bessel inequality:

\[
\sum_{i=1}^{n} |\langle y_i, x \rangle|^2 \leq \langle x, x \rangle
\]

holds for all \( x \in \mathscr{X} \). If \( n = 1 \) and \( y = y_1 \) in Theorem 1 and \( \langle x, y \rangle \) has a polar decomposition \( \langle x, y \rangle = u|\langle x, y \rangle| \) with a partial isometry \( u \in \mathscr{A} \), then we have \( u|\langle x, y \rangle| |\langle y, y \rangle|^{-1}|\langle y, x \rangle| u^* \leq \langle x, x \rangle \) and hence

\[
|\langle x, y \rangle| = |\langle x, y \rangle| |\langle y, y \rangle|^{-1}|\langle y, x \rangle| \# |\langle y, y \rangle| \leq u^* \langle x, x \rangle u \# |\langle y, y \rangle|.
\]

This implies the Cauchy-Schwarz inequality (6).

To prove Theorem 1, we need the following two lemmas:

**Lemma 2.** If \( a \in \mathscr{A} \), then the operator matrix on \( \mathscr{A} \oplus \mathscr{A} \)

\[
A = \begin{pmatrix} |a^*| & -a \\ -a^* & |a| \end{pmatrix}
\]

is positive, and \( \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in N(A) \) if and only if \( |a^*| \xi = a \eta \), where \( N(A) \) is the kernel of \( A \).

**Lemma 3.** For any \( y_1, y_2, \ldots, y_n \in \mathscr{X} \)

\[
\begin{pmatrix} \langle y_1, y_1 \rangle & \cdots & \langle y_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle y_n, y_1 \rangle & \cdots & \langle y_n, y_n \rangle \end{pmatrix} \leq \begin{pmatrix} \sum_{j=1}^{n} |\langle y_j, y_1 \rangle| & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \sum_{j=1}^{n} |\langle y_j, y_n \rangle| \end{pmatrix}.
\]
Proof of Theorem 1  For each $i = 1, \ldots, n$, put $c_i = \sum_{j=1}^{n} |\langle y_j, y_i \rangle|$. Since $y_i$ is nonsingular, it follows that $c_i$ is invertible in $\mathcal{A}$. It follows from Lemma 3 that

$$\sum_{i=1}^{n} \langle x, y_i \rangle c_i^{-1} \langle y_i, y_j \rangle c_j^{-1} \langle y_j, x \rangle$$

$$= (\langle x, y_1 \rangle c_1^{-1} \cdots \langle x, y_n \rangle c_n^{-1}) \begin{pmatrix} \langle y_1, y_1 \rangle & \cdots & \langle y_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle y_n, y_1 \rangle & \cdots & \langle y_n, y_n \rangle \end{pmatrix} \begin{pmatrix} c_1^{-1} \langle y_1, x \rangle \\ \vdots \\ c_n^{-1} \langle y_n, x \rangle \end{pmatrix}$$

$$\leq (\langle x, y_1 \rangle c_1^{-1} \cdots \langle x, y_n \rangle c_n^{-1}) \begin{pmatrix} c_1 & 0 \\ \vdots & \ddots \\ 0 & c_n \end{pmatrix} \begin{pmatrix} \langle y_1, x \rangle \\ \vdots \\ \langle y_n, x \rangle \end{pmatrix}$$

$$= \sum_{i=1}^{n} \langle x, y_i \rangle c_i^{-1} \langle y_i, x \rangle$$

and this implies

$$0 \leq \langle x - \sum_{i=1}^{n} y_i c_i^{-1} \langle y_i, x \rangle, x - \sum_{i=1}^{n} y_i c_i^{-1} \langle y_i, x \rangle \rangle$$

$$= \langle x, x \rangle - 2 \sum_{i=1}^{n} \langle x, y_i \rangle c_i^{-1} \langle y_i, x \rangle + \sum_{i=1}^{n} \langle x, y_i \rangle c_i^{-1} \langle y_i, y_j \rangle c_j^{-1} \langle y_j, x \rangle$$

$$\leq \langle x, x \rangle - \sum_{i=1}^{n} \langle x, y_i \rangle c_i^{-1} \langle y_i, x \rangle.$$

Hence we have the desired inequality (7).

The equality in (7) holds if and only if the following (8) and (9) are satisfied:

(8) \hspace{1cm} x = \sum_{i=1}^{n} y_i c_i^{-1} \langle y_i, x \rangle

and for arbitrary $i \neq j$

(9) \hspace{1cm} \langle x, y_i \rangle c_i^{-1} \langle x, y_j \rangle c_j^{-1} \begin{pmatrix} |\langle y_j, y_i \rangle| & -\langle y_i, y_j \rangle \\ -\langle y_j, y_i \rangle & |\langle y_i, y_j \rangle| \end{pmatrix} \begin{pmatrix} c_i^{-1} \langle y_i, x \rangle \\ c_j^{-1} \langle y_j, x \rangle \end{pmatrix} = 0.

Put $A = \begin{pmatrix} |\langle y_j, y_i \rangle| & -\langle y_i, y_j \rangle \\ -\langle y_j, y_i \rangle & |\langle y_i, y_j \rangle| \end{pmatrix}$ and it follows that the condition (9) holds if and only if

$$A^{1/2} \begin{pmatrix} c_i^{-1} \langle y_i, x \rangle \\ c_j^{-1} \langle y_j, x \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff A \begin{pmatrix} c_i^{-1} \langle y_i, x \rangle \\ c_j^{-1} \langle y_j, x \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence it follows from Lemma 2 that the condition (9) is equivalent to the following (10) and (11): For arbitrary $i \neq j$

(10) \hspace{1cm} \langle y_i, y_j \rangle = 0

or

(11) \hspace{1cm} |\langle y_j, y_i \rangle| c_i^{-1} \langle y_i, x \rangle = \langle y_i, y_j \rangle c_j^{-1} \langle y_j, x \rangle.
Conversely, suppose that \( x = \sum_{i=1}^{n} y_{i} a_{i} \) for some \( a_{i} \in \mathcal{A} \) and for \( i \neq j \), \( \langle y_{i}, y_{j} \rangle a_{i} = \langle y_{i}, y_{j} \rangle a_{j} \). Then

\[
\sum_{i=1}^{n} \langle x, y_{i} \rangle \left( \sum_{j=1}^{n} |\langle y_{j}, y_{i} \rangle| \right)^{-1} \langle y_{i}, x \rangle = \sum_{i=1}^{n} \langle x, y_{i} \rangle \left( \sum_{j=1}^{n} |\langle y_{j}, y_{i} \rangle| \right)^{-1} \sum_{j=1}^{n} \langle y_{i}, y_{j} \rangle a_{j} = \sum_{i=1}^{n} \langle x, y_{i} \rangle \left( \sum_{j=1}^{n} |\langle y_{j}, y_{i} \rangle| \right)^{-1} \sum_{j=1}^{n} |\langle y_{j}, y_{i} \rangle| a_{i} = \sum_{i=1}^{n} \langle x, y_{i} \rangle \left( \sum_{j=1}^{n} |\langle y_{j}, y_{i} \rangle| \right)^{-1} \left( \sum_{j=1}^{n} |\langle y_{j}, y_{i} \rangle| \right) a_{i} = \sum_{i=1}^{n} \langle x, y_{i} \rangle a_{i} = \langle x, x \rangle.
\]

Whence the proof is complete.

**Remark 4.** (1) In the case that \( \mathcal{X} \) is a Hilbert space, the equality condition \( |\langle y_{j}, y_{i} \rangle| a_{i} = \langle y_{i}, y_{j} \rangle a_{j} \) in Theorem 1 implies the condition (2) in Introduction. In fact, for some scalars \( a_{i}, a_{j} \in \mathbb{C} \), it follows that \( \langle a_{i} y_{i}, a_{j} y_{j} \rangle = a_{i}^{*} \langle y_{i}, y_{j} \rangle a_{j} = a_{i}^{*} |\langle y_{j}, y_{i} \rangle| a_{i} \geq 0 \), and \( |\langle y_{j}, y_{i} \rangle| = |\langle y_{j}, y_{i} \rangle^{*}| \) implies \( |a_{i}| = |a_{j}| \).

(2) In the Hilbert space setting, K. Kubo and F. Kubo [14] showed another proof of Selberg’s inequality (1) using Gersgorin’s location of eigenvalues [13, Theorem 6.1.1] and a diagonal domination theorem of positive semidefinite matrix.

### 4. Applications

In [4], Dragomir, Khosravi and Moslehian showed a version of the Bessel inequality and some generalizations of this inequality in the framework of Hilbert \( C^{*} \)-modules. Moreover, in [3], Bounader and Chahbi showed a type and refinement of Selberg inequality in Hilbert \( C^{*} \)-modules. In this section, by using Theorem 1, we consider several Hilbert \( C^{*} \)-module versions of the Selberg inequality and the Bessel inequality.

Bounader and Chahbi in [3, Theorem 3.1] showed that if \( \mathcal{X} \) is an inner product \( C^{*} \)-module and \( y_{1}, \ldots, y_{n} \) are nonzero vectors in \( \mathcal{X} \), and \( x \in \mathcal{X} \), then

\[
\sum_{i=1}^{n} \frac{|\langle y_{i}, x \rangle|^{2}}{\sum_{j=1}^{n} \| \langle y_{j}, y_{i} \rangle \|} \leq \langle x, x \rangle.
\]

By Theorem 1, we have the following corollary, which is an improvement of (12):

**Corollary 5.** Let \( \mathcal{X} \) be an inner product \( C^{*} \)-module over a unital \( C^{*} \)-algebra \( \mathcal{A} \). If \( x, y_{1}, \ldots, y_{n} \) are nonzero vectors in \( \mathcal{X} \) such that \( y_{1}, \ldots, y_{n} \) are nonsingular, then

\[
\sum_{i=1}^{n} \frac{|\langle y_{i}, x \rangle|^{2}}{\| \sum_{j=1}^{n} \| \langle y_{j}, y_{i} \rangle \|} \leq \langle x, x \rangle.
\]
Moreover, Bounader and Chahbi showed a Hilbert $C^*$-module version of Fujii-Nakamoto type (3), which is a refinement of (12): If $y$ and $y_1, \ldots, y_n$ are nonzero vectors in $\mathcal{X}$ such that $\langle y, y_i \rangle = 0$ for $i = 1, \ldots, n$, and $x \in \mathcal{X}$, then

$$
|\langle y, x \rangle|^2 + \sum_{i=1}^{n} \frac{|\langle y_i, x \rangle|^2}{\sum_{j=1}^{n} \| \langle y_i, y_j \rangle \|} \| \langle y, y \rangle \| \leq \| \langle y, y \rangle \| \langle x, x \rangle .
$$

We show a Hilbert $C^*$-module version of a refinement of the Selberg inequality due to Fujii and Nakamoto, which is another version of (13):

**Theorem 6.** Let $\mathcal{X}$ be an inner product $C^*$-module over a unital $C^*$-algebra $\mathcal{A}$. If $x, y, y_1, \ldots, y_n$ are nonzero vectors in $\mathcal{X}$ such that $y_1, \ldots, y_n$ are nonsingular, $\langle y, y_i \rangle = 0$ for $i = 1, \ldots, n$ and $\langle x, y \rangle = u \langle x, y \rangle$ is a polar decomposition in $\mathcal{A}$, i.e., $u \in \mathcal{A}$ is a partial isometry, then

$$
|\langle y, x \rangle| \leq u^* \langle y, y \rangle u \# \left( \langle x, x \rangle - \sum_{i=1}^{n} \langle x, y_i \rangle \left( \sum_{j=1}^{n} |\langle y_j, y_i \rangle| \right)^{-1} \langle y_i, x \rangle \right)
$$

$$
\leq u^* \langle y, y \rangle u \# \langle x, x \rangle .
$$

In [3, Corollary 3.5], Bounader and Chahbi showed a Hilbert $C^*$-module version of Bombieri type (4): If $y_1, \ldots, y_n$ are nonzero vectors in $\mathcal{X}$ and $x \in \mathcal{X}$, then

$$
\sum_{i=1}^{n} |\langle y_i, x \rangle|^2 \leq \langle x, x \rangle \max_{1 \leq i \leq n} \sum_{j=1}^{n} \| \langle y_i, y_j \rangle \| .
$$

We show a Hilbert $C^*$-module version of Bombieri type, which is an improvement of (14):

**Theorem 7.** Let $\mathcal{X}$ be an inner product $C^*$-module over a unital $C^*$-algebra $\mathcal{A}$. If $x, y_1, \ldots, y_n$ are nonzero vectors in $\mathcal{X}$ such that $y_1, \ldots, y_n$ are nonsingular, then

$$
\sum_{i=1}^{n} |\langle y_i, x \rangle|^2 \leq \langle x, x \rangle \max_{1 \leq i \leq n} \sum_{j=1}^{n} |\langle y_j, y_i \rangle| .
$$

As a corollary, we have the following Boas-Bellman type inequality [3, Corollary 3.6]:

**Corollary 8.** Let $\mathcal{X}$ be an inner product $C^*$-module over a unital $C^*$-algebra $\mathcal{A}$. If $x, y_1, \ldots, y_n$ are nonzero vectors in $\mathcal{X}$ such that $y_1, \ldots, y_n$ are nonsingular, then

$$
\sum_{i=1}^{n} |\langle y_i, x \rangle|^2 \leq \langle x, x \rangle \left( \max_{1 \leq i \leq n} \| \langle y_i, y_i \rangle \| + (n-1) \max_{j \neq i} \| \langle y_j, y_i \rangle \| \right) .
$$

Finally, we show a Mitrinović-Pečarić-Fink type inequality [17, Theorem 5 in pp394] in Hilbert $C^*$-modules, which is another version of [4, Theorem 3.8]:

**Theorem 9.** Let $\mathcal{X}$ be an inner product $C^*$-module over a unital $C^*$-algebra $\mathcal{A}$. If $x, y_1, \ldots, y_n$ are nonzero vectors in $\mathcal{X}$ and $a_1, \ldots, a_n \in \mathcal{A}$ such that $y_1, \ldots, y_n$ are nonsingular and $\langle x, \sum_{i=1}^{n} y_i a_i \rangle = u \langle x, \sum_{i=1}^{n} y_i a_i \rangle$ is a polar decomposition in $\mathcal{A}$, i.e., $u \in \mathcal{A}$ is a partial isometry, then

$$
|\sum_{i=1}^{n} \langle x, y_i \rangle a_i | \leq u^* \langle x, x \rangle u \# \left( \sum_{i=1}^{n} a_i^* \left( \sum_{j=1}^{n} |\langle y_j, y_i \rangle| \right) a_i \right) .
$$
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