Order of operators determined by operator mean

Masaru Nagisa
Graduate School of Science,
Chiba University

1 Introduction

This is a joint work with Prof. M. Uchiyama.

Let $J$ be an open interval of $\mathbb{R}$. We define $H_n$, $H_n(J)$ and $H_n^+$ as follows:

\[
H_n = \{A \in \mathbb{M}_n(\mathbb{C}) \mid A = A^*\}
\]

\[
H_n(J) = \{A \in H_n \mid \text{Sp}(A) \subset J\}
\]

\[
H_n^+ = H_n([0, \infty)).
\]

We call $f$ an operator monotone function on $J$ if we have $f(A) \leq f(B)$ for any $A, B \in H_n(J)$ with $A \leq B$. The following functions are well known as typical examples of operator monotone functions:

\[
f(t) = t^p \quad (0 \leq p < 1) \quad \text{on } J = [0, \infty),
\]

\[
f(t) = \frac{at + b}{ct + d} \quad (a, b, c, d \in \mathbb{R}, ad - bc = 1) \quad \text{on } J = (-\infty, -d/c) \text{ or } (-d/c, \infty).
\]

For the operator monotone function $f$ on $J$, it does not necessarily follow that $A, B \in H_n(J)$, $f(A) \leq f(B) \Rightarrow A \leq B$.

So we consider the following condition for $C \in H_n(J)$ and $A, B \in H_n$:

\[
f(C + tA) \leq f(C + tB) \quad \text{for any } 0 < t < \epsilon,
\]

(*)

where $\epsilon$ is a sufficiently small positive number. One of our problems is to determine the condition for $f$ or for $C$, which deduces $A \leq B$ from the condition(*).

By Kubo-Ando theory [5], it is known that an operator mean $\sigma$ is related to the operator monotone function $f$ on $[0, \infty)$ with $f(1) = 1$, that is, for $A, B \in H_n([0, \infty))$, the operator mean $A\sigma B$ of $A$ and $B$ is represented as the following form:

\[
A\sigma B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}.
\]

So we can naturally consider the following condition for $X, Y \in H_n((0, \infty))$ and $A, B \in H_n$ which is similar to above problem:

\[
Y\sigma(tA + X) \leq Y\sigma(tB + X) \quad \text{for any } 0 < t < \epsilon,
\]

(**)

where $\epsilon$ is a sufficiently small positive number. Our results is as follows:
Theorem 1. The condition (**) implies $A \leq B$ is equivalent to that $X$ is a scalar multiple of $Y$ or the operator monotone function $f$ associated with $\sigma$ has the form $f(t) = \frac{at+b}{ct+d}$.

2 Outline of Proof

We show the following:

Fact 1. When $X = cY$ for some positive scalar $c$, (**) implies $A \leq B$.

Fact 2. When the operator monotone function $f$ has the following form:

$$f(t) = \frac{at+b}{ct+d} \quad a, b, c, d \in \mathbb{R}, \ ad-bc > 0,$$

(**) implies $A \leq B$.

Fact 3. When $X$ is not scalar multiple of $Y$ and $f$ does not have the form $f(t) = \frac{at+b}{ct+d}$, then there exist positive operators $A$ and $B$ such that $A \not\preceq B$ and they satisfy the condition (**) for $X$, $Y$ and $f$.

Combining these facts, we can get Theorem 1. So we will explain these facts.

Let $f$ be an operator monotone function on $J$. For $A \in H_n(J)$, we denote the Fréchet derivative of $f$ at $A$ by $Df(A)$, that is,

$$\lim_{\|H\| \to 0} \frac{\|f(A+H) - f(A) - D(f)(A)\|}{\|H\|} = 0.$$ 

We remark $Df(A)$ a bounded real linear operator on $H_n$. We also denote the directional derivative of $f$ at $A$ in the direction $B$ by $Df(A)(B)$, that is,

$$Df(A)(B) = \frac{d}{dt} f(A + tB)_{t=0}.$$ 

We choose some unitary $U$ such that

$$\Lambda = U^*AU = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$ 

Then it is known that

$$Df(A)(B) = U(f^{[1]}(\Lambda) \circ (U^*BU))U^*,$$

where $f^{[1]}(\Lambda) = (f^{[1]}(\lambda_i, \lambda_j))$,

$$f^{[1]}(\lambda_i, \lambda_j) = \begin{cases} \frac{f(\lambda_i) - f(\lambda_j)}{\lambda_i - \lambda_j} & \lambda_i \neq \lambda_j \\ f'(\lambda_i) & \lambda_i = \lambda_j \end{cases}.$$
and the notation $\circ$ means Schur product of matrices.

Since $f$ is operator monotone, $f^{[1]}(\Lambda)$ becomes positive. When $A = cI$,

$$f^{[1]}(cI) = \begin{pmatrix} f'(c) & \cdots & f'(c) \\ \vdots & \ddots & \vdots \\ f'(c) & \cdots & f'(c) \end{pmatrix}$$

is positive and of rank 1. It is also known that the operator monotone function $f$ has the form

$$f(t) = \frac{at + b}{ct + d},$$

if $f^{[1]}(\Lambda)$ is of rank 1 for some $\Lambda \neq cI$ (see [3]).

The following proposition is a key idea of this paper:

**Proposition 2.** For $A = (a_{ij}) \in H_n^+$, we consider the map $S_A : H_n \ni B \mapsto A \circ B \in H_n$. Then the following are equivalent:

1. For $B \in H_n$, $S_A(B) \geq 0 \Rightarrow B \geq 0$.
2. $A$ is of strict rank 1, that is, there exists $\gamma = (\gamma_1 \gamma_2 \cdots \gamma_n)$ such that $A = \gamma^*\gamma$ and $\gamma_1\gamma_2\cdots\gamma_n \neq 0$.
3. $S_A(H_n^+) = H_n^+$.

(4) For any $k, l$ ($1 \leq k, l \leq n$), $a_{kk} > 0$ and $a_{kk}a_{ll} - a_{kl}a_{lk} = 0$.

We can prove $(1) \Rightarrow (4) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$. This proof has been written in [6]. Here we give only the part $(1) \Rightarrow (4) \Rightarrow (2)$, because the rest part of proof is not so difficult.

**Proof.** $(1) \Rightarrow (4)$ When $a_{kk} = 0$, we define $B = (b_{ij})$ as follows:

$$b_{ij} = \begin{cases} -1 & \text{if } (i, j) = (k, k) \\ 0 & \text{otherwise} \end{cases}$$

Since $B \not\simeq 0$ and $S_A(B) = A \circ B = 0 \geq 0$, this contradicts to the assumption. So $a_{kk} > 0$ for all $k$.

The positivity of $A$ implies that

$$\begin{pmatrix} a_{kk} & a_{kl} \\ a_{lk} & a_{ll} \end{pmatrix} \geq 0,$$

in particular, $a_{kk}a_{ll} - a_{kl}a_{lk} \geq 0$. We assume that $a_{kk}a_{ll} - a_{kl}a_{lk} > 0$. We define $B = (b_{ij})$ as follows:

$$b_{ij} = \begin{cases} \frac{|a_{kl}|}{a_{kk}} & \text{if } (i, j) = (k, k) \\ \frac{|a_{kl}|}{a_{kk}} & \text{if } (i, j) = (l, l) \\ \frac{|a_{kl}|}{a_{ul}} & \text{if } (i, j) = (k, l) \text{ or } (l, k) \\ 1 & \text{if } (i, j) = (k, l) \text{ or } (l, k) \\ 0 & \text{otherwise} \end{cases}.$$
Since $|a_{kl}|^2 = a_{kl}a_{lk} < a_{kk}a_{ll}$, we have $B \neq 0$. But we have

$$(A \circ B)_{ij} = \begin{cases} |a_{kl}| & \text{if } (i, j) = (k, k) \text{ or } (l, l) \\ a_{kl} & \text{if } (i, j) = (k, l) \\ a_{lk} & \text{if } (i, j) = (l, k) \\ 0 & \text{otherwise} \end{cases},$$

and $A \circ B \geq 0$. This contradicts the assumption. So we can get the following:

$$a_{kk}, a_{ll} > 0, \quad a_{kk}a_{ll} = a_{kl}a_{lk} (= |a_{kl}|^2).$$

(4) $\Rightarrow$ (2) Define $r_k > 0 \ (k = 1, 2, \ldots, n)$ by the following relation:

$$a_{kk} = r_k^2.$$

Then, for any $k$ and $l$, we can choose $\theta(k, l) \in \mathbb{R}$ such that

$$a_{kl} = r_k r_l e^{i\theta(k, l)},$$

and we may assume that the following relation:

$$e^{i\theta(k,l)} = e^{-i\theta(l,k)}, \quad e^{i\theta(k,k)} = 1.$$

If we show the relation

$$e^{i\theta(k,l)}e^{i\theta(l,m)} = e^{i\theta(k,m)}$$

for any $k, l$ and $m$, then we can see that $A$ is of strict rank 1 as follows:

$$\begin{pmatrix} r_1 \\ r_2 e^{-i\theta(1,2)} \\ \vdots \\ r_n e^{-i\theta(1,n)} \end{pmatrix} (r_1 \quad r_2 e^{i\theta(1,2)} \quad \cdots \quad r_n e^{i\theta(1,n)}) = \begin{pmatrix} r_1 \\ r_2 e^{i\theta(2,1)} \\ \vdots \\ r_n e^{i\theta(n,1)} \end{pmatrix} (r_1 \quad r_2 e^{i\theta(1,2)} \quad \cdots \quad r_n e^{i\theta(1,n)}) = \begin{pmatrix} r_1 \quad r_2 e^{i\theta(1,2)} \quad \cdots \quad r_1 r_n e^{i\theta(1,n)} \\ r_2 r_1 e^{i\theta(2,1)} \quad r_2^2 e^{i\theta(2,1)} e^{i\theta(1,2)} \quad \cdots \quad r_2 r_n e^{i\theta(2,1)} e^{i\theta(1,n)} \\ \vdots \quad \vdots \quad \ddots \quad \vdots \\ r_n r_1 e^{i\theta(n,1)} \quad r_n r_2 e^{i\theta(n,1)} e^{i\theta(1,2)} \quad \cdots \quad r_n^2 e^{i\theta(n,1)} e^{i\theta(1,n)} \end{pmatrix} = A.$$
It suffices to show the relation \( e^{i\theta(k,l)}e^{i\theta(l,m)} = e^{i\theta(k,m)} \) in the case of each two of \( k, l, m \) are different. By the positivity of \( A \), we have

\[
\begin{pmatrix}
  a_{kk} & a_{kl} & a_{km} \\
  a_{lk} & a_{ll} & a_{lm} \\
  a_{mk} & a_{ml} & a_{mm}
\end{pmatrix} \geq 0.
\]

Since

\[
\begin{pmatrix}
  a_{kk} & a_{kl} & a_{km} \\
  a_{lk} & a_{ll} & a_{lm} \\
  a_{mk} & a_{ml} & a_{mm}
\end{pmatrix} = \begin{pmatrix}
  r_k^2 & r_k r_l e^{i\theta(k,l)} & r_k r_m e^{i\theta(k,m)} \\
  r_l r_k e^{i\theta(l,k)} & r_l^2 & r_l r_m e^{i\theta(l,m)} \\
  r_m r_k e^{i\theta(m,k)} & r_m r_l e^{i\theta(m,l)} & r_m^2
\end{pmatrix}
\]

and

\[
\alpha = e^{-i\theta(k,l)}e^{-i\theta(l,m)}e^{i\theta(k,m)},
\]

we have

\[
\begin{pmatrix}
  1 & 1 & \alpha \\
  1 & 1 & 1 \\
  \bar{\alpha} & 1 & 1
\end{pmatrix} \geq 0.
\]

Remarking that \( |\alpha| = 1 \) and

\[
0 \leq \langle \begin{pmatrix}
  1 & 1 & \alpha \\
  1 & 1 & 1 \\
  \bar{\alpha} & 1 & 1
\end{pmatrix}, \begin{pmatrix}
  -1 & 2 \\
  2 & -1 \\
  -1 & 2
\end{pmatrix} \rangle = \alpha + \bar{\alpha} - 2,
\]

we can get \( \alpha = 1 \). So we have the desired relation. \( \square \)

We now consider the condition, for \( C \in H_n(J) \) and \( A, B \in H_n \):

\[
f(C + tA) \leq f(C + tB) \quad \text{for any } 0 < t < \epsilon. \tag{\star}
\]

Since

\[
\frac{f(C + tA) - f(C)}{t} \leq \frac{f(C + tB) - f(C)}{t},
\]

we have \( Df(C)(A) \leq Df(C)(B) \), i.e., \( Df(C)(B-A) \geq 0 \). As stated above \( f^{[1]}(C) \) is of strict rank 1 when \( C = cI \) or \( f(t) \) has the form \( (at+b)/(ct+d) \). Using the property (1) in Proposition 2, we have the following:

**Fact 1'**: When \( C = cI \) for some scalar in \( J \), \( \star \) implies \( A \leq B \).

**Fact 2'**: When the operator monotone function \( f \) on \( J \) has the following form:

\[
f(t) = \frac{at + b}{ct + d} \quad a, b, c, d \in \mathbb{R}, \quad ad - bc > 0,
\]

\( \star \) implies \( A \leq B \).
When \( f \) does not have the form \((at+b)/(ct+d)\), \( f^{[1]}(\Lambda) \) is not of rank 1 for 
\[ \Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \] (\( \lambda \neq \mu \in J \)). This means \( f'(\lambda)f'(\mu) > f^{[1]}(\lambda,\mu)^2 \). So we choose 
\[ H = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \in H_2 \] with \( h_{11}, h_{22} > 0 \) and 
\[ h_{11}h_{22} < |h_{12}|^2 < \frac{f'(\lambda)f'(\mu)}{f^{[1]}(\lambda,\mu)^2}h_{11}h_{22}. \]

Then \( H \not\simeq 0 \) and \( Df(\Lambda)(H) = f^{[1]}(\Lambda) \circ H > 0 \). Let \( A, B \geq 0 \) with \( H = B - A \). Since 
\[
0 < Df(\Lambda)(H) = Df(\Lambda)(B) - Df(\Lambda)(A) \\
= \lim_{t\to 0} \left( \frac{f(tB + \Lambda) - f(\Lambda)}{t} - \frac{f(tA + \Lambda) - f(\Lambda)}{t} \right) \\
= \lim_{t\to 0} \frac{f(tB + \Lambda) - f(tA + \Lambda)}{t},
\]
there exists \( \epsilon > 0 \) such that 
\[ f(tB + \Lambda) - f(tA + \Lambda) \geq 0 \]
for \( 0 < t < \epsilon \). In the case, \( A \not\leq B \) because \( H \not\simeq 0 \).

Using the embedding
\[
H_2 \ni \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & x_{12} & 0 & \cdots & 0 \\ x_{21} & x_{22} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in H_n,
\]
we can prove the following:

**Fact 3'**. When \( C \) is not scalar operator in \( H_n(J) \) and \( f \) does not have the form 
\[ f(t) = \frac{at+b}{ct+d}, \] then there exist positive operators \( A \) and \( B \) such that \( A \not\leq B \) and they satisfy the condition \((*)\).

Using the relation of an operator monotone function \( f \) on \((0, \infty)\) with \( f(1) = 1 \) and the operator mean \( \sigma \) related with \( f \), i.e.,
\[ A\sigma B = B^{1/2}f(A^{-1/2}BA^{-1/2})B^{1/2}, \]
we can prove **Fact i** from **Fact i'** \((i = 1, 2, 3)\).

**References**


Graduate School of Science
Chiba University
Chiba 263-8522
JAPAN
E-mail address: nagisa@math.s.chiba-u.ac.jp