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On large time behavior of solutions to the compressible Navier-Stokes equation around a time periodic parallel flow

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1 Introduction

In this article we give a summary of recent results on the stability of time-periodic parallel flows of the compressible Navier-Stokes equation in an infinite layer.

We consider the system of equations

\[ \begin{align*}
\partial_t \tilde{\rho} + \text{div} (\tilde{\rho} \tilde{v}) &= 0, \\
\tilde{\rho} (\partial_t \tilde{v} + \tilde{v} \cdot \nabla \tilde{v}) - \mu \Delta \tilde{v} - (\mu + \mu') \nabla \text{div} \tilde{v} + \nabla \tilde{P}(\tilde{\rho}) &= \tilde{\rho} \tilde{g},
\end{align*} \]

in an \( n \) dimensional infinite layer \( \Omega_\ell = \mathbb{R}^{n-1} \times (0, \ell) \):

\[ \Omega_\ell = \{ \tilde{x} = (\tilde{x}', \tilde{x}_n) ; \tilde{x}' = (\tilde{x}_1, \ldots, \tilde{x}_{n-1}) \in \mathbb{R}^{n-1}, 0 < \tilde{x}_n < \ell \}. \]

Here \( n \geq 2; \tilde{\rho} = \tilde{\rho}(\tilde{x}, \tilde{t}) \) and \( \tilde{v} = T(\tilde{v}_1(\tilde{x}_n, \tilde{t}), \ldots, \tilde{v}_n(\tilde{x}_n, \tilde{t})) \) denote the unknown density and velocity at time \( \tilde{t} \geq 0 \) and position \( \tilde{x} \in \Omega_\ell \), respectively; \( \tilde{P} \) is the pressure that is assumed to be a smooth function of \( \tilde{\rho} \) satisfying \( \tilde{P}'(\rho_*) > 0 \) for a given constant \( \rho_* > 0 \); \( \mu \) and \( \mu' \) are the viscosity coefficients that are assumed to be constants satisfying \( \mu > 0, \frac{2}{n} \mu + \mu' \geq 0 \); \text{div}, \( \nabla \) and \( \Delta \) denote the usual divergence, gradient and Laplacian with respect to \( \tilde{x} \), respectively. Here and in what follows \( T \) denotes the transposition.

Concerning the external force \( \tilde{g} \), we assume that \( \tilde{g} \) takes the form

\[ \tilde{g} = T(\tilde{g}_1(\tilde{x}_n, \tilde{t}), 0, \ldots, 0, \tilde{g}_n(\tilde{x}_n)) \]

with \( \tilde{g}_1(\tilde{x}_n, \tilde{t}) \) being a \( \tilde{T} \)-periodic function in \( \tilde{t} \), where \( \tilde{T} > 0 \).
The system (1.1)–(1.2) is considered under the boundary condition
\[ \tilde{v}|_{\tilde{x}_n=0} = \tilde{V}^1(\tilde{t})e_1, \quad \tilde{v}|_{\tilde{x}_n=\ell} = 0, \] (1.3)
and initial condition
\[ (\tilde{\rho}, \tilde{v})|_{\tilde{t}=0} = (\tilde{\rho}_0, \tilde{v}_0), \] (1.4)
where $\tilde{V}^1(\tilde{t})$ is a $\tilde{T}$-periodic function of $\tilde{t}$ and $e_1 = T(1,0,\ldots,0) \in \mathbb{R}^n$.

If $\tilde{g}^n$ is suitably small, problem (1.1)–(1.3) has a smooth time-periodic solution $\overline{u}_p = T(\overline{\rho}_p, \overline{v}_p)$, so called time-periodic parallel flow, satisfying
\[ \overline{\rho}_p = \overline{\rho}_p(\overline{x}_n) \geq \underline{\rho}, \quad \frac{1}{\ell} \int_0^\ell \overline{\rho}_p(\overline{x}_n) d\overline{x}_n = \rho_*, \]
\[ \overline{v}_p = T(\overline{v}_p(\overline{x}_n, \tilde{t}), 0,\ldots,0), \quad \overline{v}_p(\overline{x}_n, \tilde{t}+\tilde{T}) = \overline{v}_p(\overline{x}_n, \tilde{t}) \]
for a positive constant $\tilde{\rho}$.

Our aim is to study the stability of the time-periodic parallel flow $\overline{u}_p$. We will give a summary of the results on the large time behavior of perturbations to $\overline{u}_p$ when Reynolds and Mach numbers are sufficiently small, which were recently obtained in [1, 2, 3].

To formulate the problem for perturbations, we introduce the following dimensionless variables:
\[ \tilde{x} = \ell x, \quad \tilde{t} = \frac{\ell}{V} t, \quad \tilde{v} = Vv, \quad \tilde{\rho} = \rho_* \rho, \quad \tilde{P} = \rho_* V^2 P, \quad \tilde{V}^1 = V V^1, \quad \tilde{g} = \frac{\mu V}{\rho_* \ell^2} g \]
with $g = \tau(g^1(x_n, t), \cdots, g^n(x_n))$. Here
\[ \gamma = \frac{\sqrt{P'(\rho_*)}}{V}, \quad V = \frac{\rho_* \ell^2}{\mu} \left\{ |\partial_t \tilde{V}^1|_{C(\mathbb{R})} + |\tilde{g}^1|_{C(\mathbb{R} \times [0,\ell])} \right\} + |\tilde{V}^1|_{C(\mathbb{R})} > 0. \]

Under this change of variables the domain $\Omega_\ell$ is transformed into $\Omega = \mathbb{R}^{n-1} \times (0,1)$; and $g^1(x_n, t)$ and $V^1(t)$ are periodic in $t$ with period $T > 0$, where $T$ is defined by
\[ T = \frac{V}{\ell} \tilde{T}. \]

The time-periodic parallel flow $\overline{u}_p$ is transformed into $u_p = T(\rho_p, v_p)$ satisfying
\[ \rho_p = \rho_p(x_n) \geq \underline{\rho}, \quad \int_0^1 \rho_p(x_n) dx_n = 1, \]
for a positive constant $\rho_*$ and
\[ v_p = T(v_p^1(x_n, t), 0,\ldots,0), \quad v_p^1(x_n, t+T) = v_p^1(x_n, t) \]
It then follows that the perturbation $u(t) = T(\phi(t), w(t)) := T(\gamma^2(\rho(t) - \rho_p), v(t) - v_p(t))$ is governed by the following system of equations
\begin{equation}
\partial_t \phi + v_p^1 \partial_{x_1} \phi + \gamma^2 \text{div} (\rho_p w) = f^0,
\end{equation}
\begin{equation}
\partial_t w - \frac{\nu}{\rho_p} \Delta w - \frac{\tilde{\nu}}{\rho_p} \nabla \text{div} w + \nabla \left( \frac{P'(\rho_p)}{\gamma^2 \rho_p^2} \phi \right)
+ \frac{\nu}{\gamma^2 \rho_p^2} (\partial_{x_n} v_p^1) \phi e_1 + v_p^1 \partial_{x_1} w + (\partial_{x_n} v_p^1) w^n e_1 = f,
\end{equation}
\begin{equation}
w|_{\partial \Omega} = 0,
\end{equation}
\begin{equation}((\phi, w)|_{t=0} = (\phi_0, w_0).
\end{equation}

Here \text{div}, \nabla and \Delta denote the usual divergence, gradient and Laplacian with respect to $x$, respectively; $\nu$ and $\tilde{\nu}$ are the non-dimensional parameters
\begin{equation}
\nu = \frac{\mu}{\rho \ell V}, \quad \tilde{\nu} = \nu + \nu', \quad \nu' = \frac{\mu'}{\rho \ell V};
\end{equation}
and $f^0$ and $f = T(f^1, \cdots, f^n)$ denote the nonlinearities:
\begin{equation}
f^0 = -\text{div} (\phi w),
\end{equation}
\begin{equation}
f = -w \cdot \nabla w + \frac{\nu \phi}{\Sigma \rho_p^2} \left( -\Delta w + \frac{\partial_{x_1}^2 v_p^1}{\gamma^2 \rho_p} \phi e_1 \right) - \frac{\nu \phi^2}{(\phi + \gamma^2 \rho_p) \gamma^2 \rho_p^2} \left( -\Delta w + \frac{\partial_{x_1}^2 v_p^1}{\gamma^2 \rho_p} \phi e_1 \right)
- \frac{\phi^2}{(\phi + \gamma^2 \rho_p) \rho_p} \nabla \left( \frac{P'(\rho_p)}{\gamma^2 \rho_p^2} \phi \right)
+ \frac{1}{2 \gamma^2 \rho_p} \nabla \left( P''(\rho_p) \phi^2 \right)
+ P_2(\rho_p, \phi, \partial_{x} \phi),
\end{equation}
where
\begin{equation}
P_2 = \frac{\phi^3}{(\phi + \gamma^2 \rho_p) \gamma^2 \rho_p^2} \nabla P(\rho_p) - \frac{1}{2 \gamma^4 (\phi + \gamma^2 \rho_p)} \nabla (\phi^3 P_3(\rho_p, \phi))
+ \frac{\phi \nabla P'(\rho_p) \phi^2}{2 \gamma^2 \rho_p (\phi + \gamma^2 \rho_p)}
- \frac{\phi^2 \nabla P'(\rho_p) \phi}{(\phi + \gamma^2 \rho_p) \gamma^4 \rho_p^2}
\end{equation}
with
\begin{equation}
P_3(\rho_p, \phi) = \int_0^1 (1 - \theta)^2 P''(\theta \gamma^{-2}\phi + \rho_p) d\theta.
\end{equation}
We note that the Reynolds number $Re$ and Mach number $Ma$ are given by $Re = \nu^{-1}$ and $Ma = \gamma^{-1}$, respectively.

As for the stability of parallel flows of the compressible Navier-Stokes equations, Iooss and Padula ([4]) studied the linearized stability of a stationary parallel flow in a cylindrical domain under the perturbations periodic in the unbounded direction of the domain. It was shown that the linearized operator generates a $C_0$-semigroup in $L^2$-space on the basic period cell with
zero mean value condition for the density-component. Using the Fourier series expansion, the authors of [4] showed that the linearized semigroup is written as a direct sum of an analytic semigroup and an exponentially decaying $C_0$-semigroup, which correspond to low and high frequency parts of the semigroup, respectively. It was also proved that the essential spectrum of the linearized operator lies in the left-half plane strictly away from the imaginary axis and the part of the spectrum lying in the right-half to the line $\{\text{Re } \lambda = -c\}$ for some number $c > 0$ consists of finite number of eigenvalues with finite multiplicities. In particular, if the Reynolds number is suitably small, then the semigroup decays exponentially.

On the other hand, the stability of a stationary parallel flow in the infinite layer $\Omega$ were considered in [5, 6, 7, 8] under the perturbations in some $L^2$-Sobolev space on $\Omega$. It was shown in [5, 8] that the asymptotic leading part of the low frequency part of the linearized semigroup is given by an $n - 1$ dimensional heat kernel and the high frequency part decays exponentially as $t \to \infty$, if the Reynolds and Mach numbers are sufficiently small and the density of the parallel flow is sufficiently close to a positive constant. As for the nonlinear problem, it was proved in [5, 6, 7] that the stationary parallel flow is asymptotically stable under sufficiently small initial perturbations in $H^m(\Omega) \cap L^1(\Omega)$ with $m \geq [n/2] + 1$. Furthermore, the asymptotic leading part of the perturbation is given by the same $n - 1$ dimensional heat kernel as in the case of the linearized problem when $n \geq 3$. In the case of $n = 2$, the asymptotic leading part is no longer described by linear heat equations but by a one-dimensional viscous Burgers equation ([7]).

These results on stationary parallel flows were extended to the time-periodic case in [1, 2, 3]. In section 2 we will give assumptions on the given data $\tilde{g}$ and $\tilde{V}^1$ and state some properties of time-periodic parallel flow. In section 3 we will consider the linearized problem and give a summary of the results obtained in [2, 3]. We will give a Floquet representation for a part of low frequency part of the linearized evolution operator, which plays an important role in the analysis of the nonlinear problem. In section 4 we will consider the nonlinear problem and state the results on the global existence and asymptotic behavior obtained by J. Brezina ([1]).

## 2 Time-periodic parallel flow

We assume the following regularity for $\tilde{g}, \tilde{V}^1$ and $\tilde{P}$.

**Assumption 2.1** Let $m$ be an integer satisfying $m \geq 2$. We assume that
\( \tilde{g} = T(\tilde{g}^1(\tilde{x}_n, \tilde{t}), 0, \ldots, 0, \tilde{g}^n(\tilde{x}_n)), \tilde{V}^1(\tilde{t}) \) and \( \tilde{P} \) belong to the spaces

\[ \tilde{g}^1 \in \bigcap_{j=0}^{m/2} C^{j}_{\text{per}}([0, \tau]; H^{m-2j}(0, \ell)), \quad \tilde{g}^n \in C^{m}[0, \ell], \]

\[ \tilde{V}^1 \in C^{[m+1/2]}_{\text{per}}([0, \tilde{T}]), \]

and

\[ \tilde{P} \in C^{m+1}(\mathbb{R}). \]

It is easily verified that \( g, V^1 \) and \( P \) belong to similar spaces as \( \tilde{g}, \tilde{V}^1 \) and \( \tilde{P} \).

Let us consider the time-periodic parallel flow. The dimensionless form of problem (1.1)–(1.3) is written as

\[ \partial_t \rho + \text{div}(\rho v) = 0, \]

\[ \rho(\partial_t v + v \cdot \nabla v) - \nu \Delta v - \tilde{v} \nabla \text{div} v + \nabla P(\rho) = \nu \rho g, \]

\[ v|_{x_n = 0} = V^1(t)e_1, \quad v|_{x_n = 1} = 0. \]

The following result was shown in [2].

**Proposition 2.2 ([2])** There exists \( \delta_0 > 0 \) such that if

\[ \nu |g^n|_{C^m([0,1])} \leq \delta_0, \]

then the following assertions hold true.

There exists a time-periodic solution \( u_p = T(\rho_p(x_n), v_p(x_n, t)) \) of (2.1)–(2.3) that satisfies

\[ v_p \in \bigcap_{j=0}^{[m+1/2]} C^{j}_{\text{per}}(J_T; H^{m+2-2j}(0,1)), \quad \rho_p \in C^{m+1}[0,1], \]

and

\[ 0 < \underline{\rho} \leq \rho_p(x_n) \leq \overline{\rho}, \quad \int_0^1 \rho_p(x_n)dx_n = 1, \quad v_p(x_n, t) = v^1_p(x_n, t)e_1 \]

with

\[ P'(\rho) > 0 \text{ for } \rho \leq \rho \leq \overline{\rho}, \]
\[
|\rho_p - 1|_{C^{m+1}([0,1])} \leq \frac{C}{\gamma^2} \nu |P'\rho_{p}|_{C^m([\underline{\rho}, \overline{\rho}])} + |g^n|_{C^m([0,1])},
\]
\[
|P'(\rho_p) - \gamma^2|_{C([0,1])} \leq \frac{C}{\gamma^2} \nu |g^n|_{C([0,1])},
\]
and
\[
\frac{\rho_p P'(\rho_p)}{\gamma^2} \geq a_0
\]
(2.4)
for some constants \(0 < \underline{\rho} < 1 < \overline{\rho}\) and \(a_0 > 0\).

### 3 The linearized problem

In this section we consider the linearized problem

\[
\partial_t u + L(t)u = 0, \hspace{1cm} t > s, \hspace{1cm} w|_{\partial \Omega} = 0, \hspace{1cm} u|_{t=s} = u_0.
\]
(3.1)

Here \(L(t)\) is the operator given by

\[
L(t) = \begin{pmatrix}
    v^1_p(t) \partial_{x_1} & \gamma^2 \text{div} (\rho_p \cdot) \\
    \nabla \left( \frac{P'\rho_{p}}{\gamma^2 \rho_p} \cdot \right) & -\frac{\nu}{\rho_p} \Delta I_n - \frac{\nu}{\rho_p} \nabla \text{div}
\end{pmatrix}
\]

\[
+ \begin{pmatrix}
    0 & v^1_p(t) \partial_{x_1} I_n + (\partial_{x_n} v^1_p(t)) e_1^T e_n \\
    0 & 0
\end{pmatrix}
\]

Note that \(L(t)\) satisfies \(L(t + T) = L(t)\).

We introduce the space \(Z_s\) defined by

\[
Z_s = \{ u = (\phi, w); \phi \in C_{loc}([s, \infty); H^1(\Omega)), \hspace{1cm} \partial' x' w \in C_{loc}([s, \infty); L^2(\Omega)) \cap L^2_{loc}([s, \infty); H^1_0(\Omega) \hspace{1cm} (|\alpha'| \leq 1), \hspace{1cm} w \in C_{loc}((s, \infty); H^1_0(\Omega)) \}
\]

It was shown in [2] that for any initial data \(u_0 = (\phi_0, w_0)\) satisfying \(u_0 \in (H^1 \cap L^2)(\Omega)\) with \(\partial' x' w_0 \in L^2(\Omega)\) there exists a unique solution \(u(t)\) of linear problem (3.1) in \(Z_s\). We denote \(U(t, s)\) the solution operator for (3.1) given by

\[
u(t) = U(t, s)u_0.
\]

To investigate problem (3.1) we consider the Fourier transform of (3.1) with respect to \(x' \in \mathbb{R}^{n-1}
\]

\[
\frac{d}{dt} \hat{u} + \hat{L}'(t) \hat{u} = 0, \hspace{1cm} t > s, \hspace{1cm} \hat{u}|_{t=s} = \hat{u}_0.
\]
(3.2)
Here \( \hat{\phi} = \hat{\phi}(\xi', x_n, t) \) and \( \hat{w} = \hat{w}(\xi', x_n, t) \) are the Fourier transforms of \( \phi = \phi(x', x_n, t) \) and \( w = w(x', x_n, t) \) in \( x' \in \mathbb{R}^{n-1} \) with \( \xi' = (\xi_1, \cdots, \xi_{n-1}) \in \mathbb{R}^{n-1} \) being the dual variable; \( \hat{L}_{\xi'}(t) \) is the operator on \((H^1 \times L^2)(0,1)\) defined as

\[
D(\hat{L}_{\xi'}(t)) = (H^1 \times [H^2 \cap H^1_0])(0,1),
\]

\[
\hat{L}_{\xi'}(t) = \begin{pmatrix}
i\xi_1 v_p^1(t) & i\gamma^2 \rho_p^T \xi' & \gamma^2 \partial_{x_n}(\rho_p \cdot) \\
i\xi' P'(\rho_p) \xi' \gamma^2_{\rho_p} & \nu \rho_p |\xi'|^2 - \partial_{x_n}^2 I_{n-1} + \frac{\bar{\nu}}{\rho_p} \xi'^T \xi' & -i\frac{\bar{\nu}}{\rho_p} \xi' \partial_{x_n} \\
\partial_{x_n} \left( \frac{P'(\rho_p)}{\gamma^2_{\rho_p}} \right) & -i\frac{\bar{\nu}}{\rho_p} \xi' \partial_{x_n} & \frac{\nu}{\rho_p} (|\xi'|^2 - \partial_{x_n}^2 - \bar{\nu} \partial_{x_n}^2) \\
0 & 0 & 0 \\
0 & i\xi_1 v_p^1(t) & 0 \\
0 & 0 & i\xi_1 v_p^1(t) 
\end{pmatrix}.
\]

For each \( t \in \mathbb{R} \) and \( \xi' \in \mathbb{R}^{n-1} \), \( \hat{L}_{\xi'}(t) \) is sectorial on \((H^1 \times L^2)(0,1)\). We denote the solution operator for (3.2) by \( \hat{U}_{\xi'}(t, s) \). We note that it holds that

\[
U(t, s)u_0 = \mathcal{F}^{-1} [\hat{U}_{\xi'}(t, s)\hat{u}_0]
\]

for \( u_0 \in (H^1 \cap L^2)(\Omega) \) with \( \partial_{x'} w_0 \in L^2(\Omega) \).

We also need to investigate the adjoint problem

\[-\partial_s u + \hat{L}_{\xi}^*(s)u = 0, \quad s < t, \quad u|_{s=t} = u_0.\]

Here \( \hat{L}_{\xi}^*(s) \) is a formal adjoint operator defined by

\[
D(\hat{L}_{\xi}^*(s)) = (H^1 \times [H^2 \cap H^1_0])(0,1),
\]

\[
\hat{L}_{\xi}^*(s) = \begin{pmatrix}
-i\xi_1 v_p^1(s) & -i\gamma^2 \rho_p^T \xi' & -\gamma^2 \partial_{x_n}(\rho_p \cdot) \\
-i\xi' P'(\rho_p) \xi' \gamma^2_{\rho_p} & \nu \rho_p |\xi'|^2 - \partial_{x_n}^2 I_{n-1} + \frac{\bar{\nu}}{\rho_p} \xi'^T \xi' & -i\frac{\bar{\nu}}{\rho_p} \xi' \partial_{x_n} \\
-\partial_{x_n} \left( \frac{P'(\rho_p)}{\gamma^2_{\rho_p}} \right) & -i\frac{\bar{\nu}}{\rho_p} \xi' \partial_{x_n} & \frac{\nu}{\rho_p} (|\xi'|^2 - \partial_{x_n}^2 - \bar{\nu} \partial_{x_n}^2) \\
\frac{\nu^2}{P'(\rho_p)} (\partial_{x_n}^2 v_p^1(s))^T e_1 & 0 \\
0 & -i\xi_1 v_p^1(s) I_{n-1} & 0 \\
0 & \partial_{x_n} (v_p^1(s))^T e_1 & -i\xi_1 v_p^1(s) 
\end{pmatrix}.
\]
We denote the solution operator for the adjoint problem by \( \hat{U}_{\xi}^{*}(s, t) \).

It holds that \( \hat{U}_{\xi'}(t, s) \) and \( \hat{U}_{\xi}^{*}(s, t) \) are defined for all \( t \geq s \) and

\[
\hat{U}_{\xi}(t + T, s + T) = \hat{U}_{\xi}(t, s), \quad \hat{U}_{\xi}^{*}(s + T, t + T) = \hat{U}_{\xi}^{*}(s, t).
\]

Since \( \hat{L}_{\xi'}(t) \) is \( T \)-periodic in \( t \), the spectrum of \( \hat{U}_{\xi'}(T, 0) \) plays an important role in the study of the large time behavior. The following results were established in [2].

We set

\[
X_{0} = (H^{1} \times L^{2})(0, 1).
\]

**Theorem 3.1 ([2])** There exist positive numbers \( \nu_{0} \) and \( \gamma_{0} \) such that if \( \nu \geq \nu_{0} \) and \( \gamma^{2}/(\nu + \tilde{v}) \geq \gamma_{0}^{2} \) then there exists \( r_{0} > 0 \) such that for each \( \xi' \) with \( |\xi'| \leq r_{0} \) there hold the following assertions.

(i) The spectrum of operator \( \hat{U}_{\xi'}(T, 0) \) on \( (H^{1} \times H_{0}^{1})(0, 1) \) satisfies

\[
\sigma(\hat{U}_{\xi'}(T, 0)) \subset \{ \mu_{\xi'} \} \cup \{ \mu : |\mu| \leq q_{0} \}
\]

for a constant \( q_{0} > 0 \) with \( \frac{3}{2}q_{0} < \Re \mu_{\xi'} < 1 \). Here \( \mu_{\xi'} = e^{\lambda_{\xi'}T} \) is a simple eigenvalue of \( \hat{U}_{\xi'}(T, 0) \) and \( \lambda_{\xi'} \) has an expansion

\[
\lambda_{\xi'} = -i\kappa_{0}\xi_{1} - \kappa_{1}\xi_{1}^{2} - \kappa''|\xi''|^{2} + O(|\xi'|^{3}),
\]

where \( \kappa_{0} \in \mathbb{R} \) and \( \kappa_{1} > 0, \kappa'' > 0 \).

Let \( \hat{\Pi}_{\xi'} \) be the eigenprojection for the eigenvalue \( \mu_{\xi'} \). Then there holds

\[
|\hat{U}_{\xi'}(t, s)(I - \hat{\Pi}_{\xi'})u|_{H^{1}} \leq Ce^{-d(t-s)}|(I - \hat{\Pi}_{\xi'})u|_{X_{0}}
\]

for \( u \in X_{0} \) and \( t - s \geq T \). Here \( d \) is a positive constant depending on \( r_{0} \).

(ii) The spectrum of operator \( \hat{U}_{\xi}^{*}(0, T) \) on \( H^{1} \times H_{0}^{1} \) satisfies

\[
\sigma(\hat{U}_{\xi}^{*}(0, T)) \subset \{ \overline{\mu}_{\xi'} \} \cup \{ \mu : |\mu| \leq q_{0} \}.
\]

Here \( \overline{\mu}_{\xi'} \) is a simple eigenvalue of \( \hat{U}_{\xi}^{*}(0, T) \).

Let \( \hat{\Pi}_{\xi}^{*} \) be the eigenprojection for the eigenvalue \( \overline{\mu}_{\xi'} \). Then there holds

\[
\langle \hat{\Pi}_{\xi'}u, v \rangle = \langle u, \hat{\Pi}_{\xi}^{*}v \rangle
\]

for \( u, v \in X_{0} \).
Theorem 3.1 can be proved by a perturbation argument from the case $\xi' = 0$. See [2] for details.

Based on Theorem 3.1 we can obtain a Floquet representation of a part of $U(t, s)$.

Let $\nu_0$, $\gamma_0$ and $r_0$ are the numbers given by Theorem 3.1. In the rest of this section we assume that $\nu \geq \nu_0$ and $\gamma^2/(\nu + \tilde{\nu}) \geq \gamma_0^2$.

We set

$$u^{(0)}(t) = \hat{U}_0(t, 0)u^{(0)}_0. \tag{3.5}$$

Here $u^{(0)}_0$ is an eigenfunction of the operator $\hat{U}_0(T, 0)$ for the eigenvalue $e^{\lambda_0 T} = 1$. Observe that

$$u^{(0)}(t + T) = u^{(0)}(t).$$

We also define the multiplier $\Lambda : L^2(\mathbb{R}^{n-1}) \rightarrow L^2(\mathbb{R}^{n-1})$ by

$$\Lambda \sigma = \mathcal{F}^{-1}[\hat{\chi}_1 \lambda_{\xi'} \hat{\sigma}].$$

Here $\hat{\chi}_1$ is defined by

$$\hat{\chi}_1(\xi') = \begin{cases}1, & |\xi'| < r_0, \\ 0, & |\xi'| \geq r_0 \end{cases}$$

for $\xi' \in \mathbb{R}^{n-1}$.

Clearly, $\Lambda$ is a bounded linear operator on $L^2(\mathbb{R}^{n-1})$. It then follows that $\Lambda$ generates a uniformly continuous group $\{e^{t\Lambda}\}_{t \in \mathbb{R}}$. Furthermore, it holds that

$$\|\partial_x^k e^{t\Lambda} \sigma\|_{L^2(\mathbb{R}^{n-1})} \leq C(1+t)^{-\frac{n-1}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k}{2}} \|\sigma\|_{L^p(\mathbb{R}^{n-1})}, \quad k = 0, 1, \ldots, 1 \leq p \leq 2.$$

We have the following Floquet representation for $U(t, s)$.

**Theorem 3.2 ([3])**

(i) There exist time periodic operators

$$\mathcal{Q}(t) : L^2(\mathbb{R}^{n-1}) \rightarrow L^2(\Omega), \quad \mathcal{Q}(t + T) = \mathcal{Q}(t),$$

$$\mathcal{P}(t) : L^2(\Omega) \rightarrow L^2(\mathbb{R}^{n-1}), \quad \mathcal{P}(t + T) = \mathcal{P}(t)$$

such that the operator $\mathbb{P}(t) := \mathcal{Q}(t) \mathcal{P}(t) : L^2(\Omega) \rightarrow L^2(\Omega)$ satisfies

$$\mathbb{P}(t)^2 = \mathbb{P}(t), \quad \mathbb{P}(t + t) = \mathbb{P}(t),$$
\[
\mathbb{P}(t)(\partial_t + L(t))u(t) = (\partial_t + L(t))(\mathbb{P}(t)u(t)) = \mathcal{D}(t)\left[(\partial_t - \Lambda)(\mathcal{P}(t)u(t))\right]
\]
for \(u \in L^2(0, T; \{H^1 \times [H^2 \cap H_0^1]\})(\Omega) \cap H^1(0, T; L^2(\Omega))\).

(ii) It holds that
\[
\mathbb{P}(t)U(t, s) = U(t, s)\mathbb{P}(s) = \mathcal{Q}(t)e^{(t-s)\Lambda}\mathcal{P}(s).
\]
Furthermore,
\[
\|\partial_t^j \partial_{x'}^k \partial_{x_n}^l \mathbb{P}(t)U(t, s)u\|_{L^2(\Omega)} \leq C(1+t-s)^{-\frac{n-1}{4}-\frac{k}{2}}\|u\|_{L^1(\Omega)}
\]
for \(0 \leq 2j + l \leq m, k = 0, 1, \ldots\).

(iii) Let \(\mathcal{H}(t)\) be a heat semigroup defined by
\[
\mathcal{H}(t) = \mathcal{F}^{-1}e^{-(i\kappa_0 \xi_1 + \kappa_1 \xi_1^2 + \kappa''|\xi|^2)t}\mathcal{F}.
\]
Suppose that \(1 \leq p \leq 2\). Then it holds that
\[
\|\partial_x^k \partial_{x_n}^l (\mathbb{P}(t)U(t, s)u - [\mathcal{H}(t-s)\sigma]u^{(0)}(t))\|_{L^2(\Omega)}
\]
\[
\leq C(1+t-s)^{-\frac{n-1}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+1}{2}}\|u\|_{L^p(\Omega)}
\]
for \(u = T(\phi, w), k = 0, 1, \ldots\), and \(0 \leq l \leq m\). Here \(u^{(0)}(t)\) is the function given in (3.5) and \(\sigma = \int_0^1 \phi(x', x_n) dx_n\).

(iv) \((I - \mathbb{P}(t))U(t, s) = U(t, s)(I - \mathbb{P}(s))\) satisfies
\[
\|(I - \mathbb{P}(t))U(t, s)u\|_{H^1(\Omega)} \leq Ce^{-d(t-s)}(\|u\|_{(H^1 \times L^2)(\Omega)} + \|\partial_{x'}w\|_{L^2(\Omega)})
\]
for \(t - s \geq T\). Here \(d\) is a positive constant.

4 The nonlinear problem

In this section we consider the nonlinear problem (1.5)–(1.8).

Brezina ([1]) recently proved the global existence and the asymptotic behavior for (1.5)–(1.8) when the Reynolds and Mach numbers are sufficiently small.

**Theorem 4.1** ([1]) Let \(n \geq 2\) and let \(m\) be an integer satisfying \(m \geq \lceil n/2 \rceil + 1\). Suppose that \(g, V^1\) and \(P\) satisfy Assumption 2.1 for \(m\) replaced by \(m + 1\). Then there are positive numbers \(v_1\) and \(\gamma_1\) such that the following assertions hold true, provided that \(\nu \geq v_1\) and \(\gamma^2/(\nu + \bar{\nu}) \geq \gamma_1^2\).
There is a positive number $\epsilon_0$ such that if $u_0 \in T(\phi_0, w_0) \in H^m \cap L^1(\Omega)$ satisfies a suitable compatibility condition and $\|u_0\|_{H^m \cap L^1(\Omega)} \leq \epsilon_0$, then there exists a global solution $u(t)$ of (1.5)–(1.8) in $C([0, \infty); H^m(\Omega))$ and $u(t)$ satisfies

$$\|\partial_{x}^{k}u(t)\|_{L^{2}(\Omega)} = O(t^{-\frac{n-1}{4}-\frac{k}{2}}), \quad k = 0, 1,$$

as $t \to \infty$.

Furthermore, there holds

$$\|u(t) - (\sigma u^{(0)})(t)\|_{L^{2}(\Omega)} = O(t^{-\frac{n-1}{4}-\frac{1}{2}}\eta_n(t))$$

as $t \to \infty$. Here $\eta_n(t) = 1$ for $n \geq 4$, $\eta_n(t) = \log t$ for $n = 3$ and $\eta_n(t) = t^\delta$ for $n = 2$, where $\delta$ is an arbitrarily positive number; $u^{(0)} = u^{(0)}(x_n, t)$ is the function given in (3.5); and $\sigma = \sigma(x', t)$ satisfies

$$\partial_{t}\sigma - \kappa_1\partial_{x_1}^{2}\sigma - \kappa''\Delta^{n}\sigma + \kappa_0\partial_{x_1}\sigma = 0, \quad \sigma|_{t=0} = \int_{0}^{1}\phi_{0}(x', x_n)dx_n$$

if $n \geq 3$, and

$$\partial_{t}\sigma - \kappa_1\partial_{x_1}^{2}\sigma + \kappa_0\partial_{x_1}\sigma + a_0\partial_{x_1}(\sigma^2) = 0, \quad \sigma|_{t=0} = \int_{0}^{1}\phi_{0}(x', x_n)dx_n$$

if $n = 2$, where $\Delta'' = \partial_{x_2}^{2} + \cdots + \partial_{n-1}^{2}$ for $n \geq 3$, and $a_0$ is a constant.

**Remark 4.2** A result similar to Theorem 4.1 also holds for the case of stationary parallel flows ([7]).

Theorem 4.1 is proved by the decomposition method based on the spectral analysis in section 3. We write problem (1.5)–(1.8) as

$$\partial_{t}u + L(t)u = F(u), \quad u(0) = u_0.$$ We decompose the solution $u(t)$ of (1.5)–(1.8) into

$$u(t) = u_1(t) + u_\infty(t),$$

where

$$u_1(t) = \mathbb{P}(t)u(t), \quad u_\infty(t) = (I - \mathbb{P}(t))u(t).$$

It then follows from Theorem 3.2 that

$$u_1(t) = \mathcal{D}(t) \left[ e^{tA} \mathcal{P}(0)u_0 + \int_{0}^{t} e^{(t-s)A} \mathcal{P}(s)F(u(s)) ds \right],$$

$$\partial_{t}u_\infty + L(t)u_\infty = (I - \mathbb{P}(t))F(u), \quad u_\infty(0) = (I - \mathbb{P}(t))u_0.$$ To estimate $u_1$, we use the estimates obtained in Theorem 3.2, while $u_\infty$ is estimated by a variant of the Matsumura-Nishida energy method ([9, 6, 7]). See [1] for details.
References


