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<td>KAMISHIMA, YOSHINOBU</td>
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Kyoto University
CONFORMALLY FLAT LORENTZ PARABOLIC MANIFOLD

YOSHINOBU KAMISHIMA

ABSTRACT. The purpose of this note is to introduce Lorentz parabolic structure on smooth manifolds. First we revisit $(G, X)$-structure on manifolds. Secondly we study Lorentz similarity structure and Fefferman-Lorentz parabolic structure.

1. INTRODUCTION

In the first part of this paper we review $(G, X)$-structure introduced by Thurston, Kulkarni et al. Many results are known when $(G, X)$ is a homogeneous Riemannian geometry. In 1980-90s non-Riemannian homogeneous geometries have been studied intensively. Specifically conformally flat geometry, spherical $CR$-geometry and flat quaternionic $CR$-geometry. Those geometries are obtained on the projective limit of the isometric actions of hyperbolic spaces. Similarly, another kind of non-Riemannian homogeneous geometry is obtained as the boundary behavior of the isometric actions on pseudo-hyperbolic spaces. The typical example is conformally flat Lorentz geometry. In the second part of this paper, we introduce conformally flat Lorentz parabolic geometry. A Lorentz parabolic structure contains Lorentz similarity structure and Fefferman-Lorentz structure. It is explained that the fundamental group of a compact complete Lorentz similarity manifold $M$ is virtually polycyclic. It turns out that a finite cover of $M$ admits a Lorentz parabolic structure. We discuss Fefferman-Lorentz parabolic geometry. The conformally flat Lorentz geometry $(\mathbb{O}(2n + 2), S^1 \times S^{2n+1})$ contains this as a subgeometry $(\mathbb{U}(n+1, 1), S^1 \times S^{2n+1})$. Let $\Gamma$ be a discrete subgroup of $\mathbb{U}(n+1, 1)$ acting properly discontinuously on a domain $\#$ of $S^1 \times S^{2n+1}$. We present a classification of compact conformally flat Fefferman-Lorentz parabolic manifolds $\#/\Gamma$ admitting a 1-parameter group $H \leq \text{Conf}(\#/\Gamma)$. This

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class contains $S^1 \times \mathcal{N}/\Delta$ where $\mathcal{N}$ is a 3-dimensional Heisenberg nilmanifold. Finally we discuss the deformation space of conformally flat Fefferma-Lorentz parabolic structures on the product $S^1 \times \mathcal{N}/\Delta$.

2. $(G,x)$-structure

Our geometry is a pair $(G,X)$ where $G$ is a finite dimensional Lie group with finitely many components and $X$ is an $n$ dimensional homogeneous space of $G$. A geometric structure $((G,X)$-structure) on a smooth $n$ dimensional manifold $M$ is a maximal collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ whose coordinate changes belong to $G$. More precisely, $M = \bigcup_{\alpha \in \Lambda} U_\alpha$, $\phi_\alpha : U_\alpha \to X$ is a diffeomorphism onto its image. If $U_\alpha \cap U_\beta \neq \emptyset$ then it satisfies that there exists a unique element $g_{\alpha \beta} \in G$ such that $g_{\alpha \beta} \cdot \phi_\alpha = \phi_\beta$ on $U_\alpha \cap U_\beta$. We say that $M$ is uniformized over $X$ with respect to $G$ (or simply, $M$ is locally modelled on $(G,X)$). An $n$-manifold $M$ is called a $(G,X)$-manifold if $M$ is uniformized over $X$ with respect to $G$. Using a collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ we can construct a geometric invariant $(\rho, \text{dev})$ called a developing pair of $M$. (See [5].)

Lemma 2.1. Given a $(G,X)$-structure on a smooth $n$-manifold $M$, there exists a pair $(\rho, \text{dev}) : (\pi_1(M), \tilde{M}) \to (G,X)$ unique up to conjugation of elements of $G$, where $\text{dev}$ is a $(G,X)$-structure preserving immersion and $\rho$ is a homomorphism such that the diagram is commutative for each element $\gamma \in \pi_1(M)$:

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\text{dev}} & X \\
\gamma \downarrow & & \downarrow \rho(\gamma) \\
\tilde{M} & \xrightarrow{\text{dev}} & X 
\end{array}
\]

(2.1)

Proof. Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in \Lambda}$ be a geometric structure on $M$. In the union $\bigcup_{\alpha \in \Lambda} (U_\alpha \times X)$, we define the following equivalence relation; for $(p,x) \in U_\alpha \times X$, $(q,y) \in U_\beta \times X$, then

\[
(p,x) \sim (q,y) \text{ if and only if } p = q \in U_\alpha \cap U_\beta, \ g_{\alpha \beta}x = y,
\]

(2.2)

\[g_{\alpha \beta} \in G\].

Put $E = \bigcup_{\alpha}(U_\alpha \times X)/ \sim$. Let $\pi : E \to M$ be the map defined by $\pi([p,x]) = p$ if $p \in U_\alpha$. Then it is easy to see that $E \xrightarrow{\pi} M$ is a fiber bundle with fiber $X$. Recall that $E$ is determined by the transitive functions $\{g_\alpha\}$. Since $g_{\alpha \alpha} = 1$ and $g_{\alpha \beta} : g_{\beta \gamma} = g_{\alpha \gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, $\{g_{\alpha \beta}\}$ is a 1-cocycle in the first cohomology $H^1(M;G)$. Here $G$ is
viewed as the sheaf of germs of \(G\)-valued functions. Since \(H^1(M; G) \approx \text{Hom}(\pi_1(M), G)\), \(\{g_{\alpha\beta}\}\) determines a homomorphism \(\rho : \pi_1(M) \to G\). More precisely it follows that \(E \approx \tilde{M} \times X\) in which each element \(\gamma \in \pi_1(M)\) acts on \(\tilde{M} \times X\) by \((\gamma, (b, x)) = (\gamma b, \rho(\gamma)x)\).

We construct a developing map. Let \(s : M \to E\) be a section defined by \(s(p) = [p, \phi(p)]\) if \(p \in U_\alpha\). Consider the pull back of the bundle:

\[
\begin{align*}
\pi_1(M) & \to P^*E \to E \\
\downarrow & \downarrow \\
\pi_1(M) & \to \tilde{M} \to M.
\end{align*}
\]

As before the bundle \(P^*E\) is determined by a lift \(\{\tilde{g}_{\alpha\beta}\}\) of \(\{g_{\alpha\beta}\}\). Since \(H^1(\tilde{M}; G) = \{1\}\), the bundle \(P^*E\) is trivial. Choose a trivialization \(\Psi : P^*E \to \tilde{M} \times X\). The section \(s\) extends to a section \(\tilde{s} : \tilde{M} \to P^*E\).

Put \(\text{dev} = \text{Pr}_2 \cdot \Psi \cdot \tilde{s} : \tilde{M} \to X\). It is an immersion and preserves the \((G, X)\)-structure. The map \(\text{dev}\) depends on the choice of sections and trivializations, however \(\text{dev}\) is unique up to elements of \(G\).

On the other hand, we note that for \((\tilde{p}, x) \in \tilde{U}_\alpha \times X\), \((\tilde{q}, y) \in \tilde{U}_\beta \times X\) in \(P^*E = \bigcup_{\alpha} (\tilde{U}_\alpha \times X)\), it follows that \((\tilde{p}, x) \sim (\tilde{q}, y)\) iff \(\gamma \tilde{p} = \tilde{q}\), \(\rho(\gamma)y = g_{\alpha\beta}x\) and \(p = q \in U_\alpha \cap U_\beta\) for some \(\gamma \in \pi_1(M)\) and \(g_{\alpha\beta} \in G\). It is easy to see that

\[\text{dev} \cdot \gamma = \rho(\gamma) \cdot \text{dev}\] for every \(\gamma \in \pi_1(M)\).

\[\square\]

If \(\text{Aut}(\tilde{M})\) is the group of all \((G, X)\)-structure preserving diffeomorphisms on \(\tilde{M}\). Then note that \(\pi_1(M) \leq \text{Aut}(\tilde{M})\) and \(\rho\) extends naturally to a continuous homomorphism \(\rho : \text{Aut}(\tilde{M}) \to G\).

**Definition 2.2.** The map \(\text{dev}\) is called a developing map for a \((G, X)\)-manifold \(M\) and the map \(\rho\) is called a holonomy homomorphism of \(M\).

Let \(\#(M)\) be the space consisting of all possible developing pairs \((\rho, \text{dev})\). A topology on \(\#(M)\) is given by the following subbasis.

- \(\mathcal{N}(U) = \{U\}\) where \(U\) is an open subset of \(\text{Map}(\tilde{M}, X)\) in the compact open topology of \(\text{Map}(\tilde{M}, X)\).
- \(\mathcal{N}(K) = \{\text{dev} \in \#(M) \mid \text{dev}|K \text{ is embedding}\}\) for a compact subset \(K \subset \tilde{M}\).

(Compare [1].) Recall that the deformation space \(\mathcal{T}(M)\) is a space of \((G, X)\)-structures on marked manifolds homeomorphic to \(M\). \(\mathcal{T}(M)\) consists of equivalence classes of diffeomorphisms \(f : M \to M'\) from
$M$ to a $(G, X)$-manifolds $M'$. Two such diffeomorphisms $f_i : M \to M_i$ ($i = 1, 2$) are equivalent if and only if there is an isomorphism (i.e. a $(G, X)$-structure preserving diffeomorphism) $h : M_1 \to M_2$ such that $h \circ f_1$ is isotopic to $f_2$.

\[
\begin{array}{c}
M \xrightarrow{f_1} M_1 \\
\text{f}_2 \searrow \simeq \downarrow h \\
M_2
\end{array}
\]

(2.4)

Denote by $\text{Diff}^0(M)$ the subgroup of $\text{Diff}(M)$ whose elements are isotopic to the identity map. Put $\pi = \pi_1(M)$. Consider the following exact sequences of the diffeomorphism groups, where $N_{\text{Diff}(\tilde{M})}(\pi)$ (resp. $C_{\text{Diff}(\tilde{M})}(\pi)$) is the normalizer (resp. centralizer) of $\pi$ in $\text{Diff}(\tilde{M})$

\[
1 \longrightarrow \pi \longrightarrow N_{\text{Diff}(\tilde{M})}(\pi) \xrightarrow{\eta} \text{Diff}(M) \longrightarrow 1
\]

\[
\begin{array}{c}
C_{\text{Diff}(\tilde{M})}(\pi) \longrightarrow \text{Diff}^0(M)
\end{array}
\]

Put $\widehat{\text{Diff}}(M) = \eta^{-1}(\text{Diff}(M))$ and let $\widehat{\text{Diff}}^0(M)$ be the identity component. Then $\eta(\widehat{\text{Diff}}(M)) = \text{Diff}^0(M)$ and $\widehat{\text{Diff}}^0(M) \leq C_{\text{Diff}(\tilde{M})}(\pi)$. The natural right action of $\widehat{\text{Diff}}(M)$ and the left action of $G$ on $\#(M)$ are given by

\[
(\rho, \text{dev}) \circ \tilde{f} = (\rho \circ \mu(\tilde{f}), \text{dev} \circ \tilde{f}),
\]

\[
g \circ (\rho, \text{dev}) = (g \circ \rho \circ g^{-1}, g \circ \text{dev}),
\]

where $\mu(\tilde{f}) : \pi \to \pi$ is an isomorphism defined by $\mu(\tilde{f})(\gamma) = \tilde{f} \circ \gamma \circ \tilde{f}^{-1}$. Obviously both actions commute.

It is noted that two developing pairs $(\rho_i, \text{dev}_i)$ ($i = 1, 2$) represent the same structure on $M$ if and only if there exists an element $g \in G$ such that $g \circ \text{dev}_1 = \text{dev}_2$. Put

\[
\#(M) = \#(M)/\widehat{\text{Diff}}^0(M).
\]

The action of $G$ induces an action of $\#(M)$. Then it is easy to show that

Lemma 2.3. The elements of $\mathcal{T}(M)$ are in one-to-one correspondence with the orbits of $G \setminus \#(M)$.

If $f : M \to M'$ is a representative element of $\mathcal{T}(H, M)$ then there is a developing pair $(\rho, \text{dev}) : (\pi_1(M'), \tilde{M}') \to (G, X)$. We have the
holonomy representation $\rho \circ f_{\#} : \pi \to G$ up to conjugate by an element of $G$. We then obtain a map $\text{hol} : \mathcal{T}(M) \to \text{Hom}(\pi, G)/G$ which assigns to a marked structure its holonomy representation. By the definition $\text{hol}$ lifts to a map $\widehat{\text{hol}} : \#(M) \to \text{Hom}(\pi, G)$ which makes the following diagram commute.

\[
\begin{array}{ccc}
\#(M) & \xrightarrow{\text{hol}} & \text{Hom}(\pi, G) \\
\downarrow & & \downarrow \\
\mathcal{T}(M) & \xrightarrow{\text{hol}} & \text{Hom}(\pi, G)/G.
\end{array}
\]

Thurston has shown the following. (See [Lo],[J-M],[Th] for the proof.)

**Theorem 2.4 (Holonomy Theorem).** $\widehat{\text{hol}} : \#(M) \to \text{Hom}(\pi, G)$ is a local homeomorphism.

### 3. Examples of non-Riemannian homogeneous geometry

3.1. **Homogeneous Riemannian geometry.** Let $X = G_{x}\backslash G$ be the simply connected homogeneous space ($x \in X$). If $G_{x}$ is compact, then $(G, X)$ is called homogeneous Riemannian geometry. If $M$ is a compact manifold which admits a $(G, X)$-structure, then it follows that $M = X/\rho(\pi)$ where $\rho : \pi = \pi_{1}(M) \to G$ is a discrete faithful representation. This is obtained by the following lemma.

**Lemma 3.1.** If $f : M \to N$ is a Riemannian immersion and $M$ is complete, then $f$ is a covering map.

A Riemannian manifold is complete if every Cauchy sequence converges relative to the Riemannian metric. Specifically a compact Riemannian manifold is complete.

Thus the deformation space $\mathcal{T}(M)$ is identified with the set of equivalence classes of discrete faithful representations $\text{R}(\pi, G)/G$. For example, when we take $G = \text{Isom}(\mathbb{H}_{K}^{n})$ the full isometry group of the $K$-hyperbolic space where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$. The Mostow rigidity theorem says that $\text{R}(\pi, G)/G$ is a single point. By Margulis-Mostow rigidity, the same result holds for a noncompact semisimple Lie group $G$ of $\mathbb{R}$-rank $\geq 2$. If $X = K\backslash G$, then $X/T$ is a compact nonpositively curved Riemannian manifold. On the other hand, if $M$ is noncompact, there occurs a remarkably distinct feature, one is Thurston bending while the other is Margulis super rigidity. After Thurston's hyperbolization theory several non-Riemannian homogeneous geometry surrounding hyperbolic geometry came to our interest in 1980s~1990s. The $K$-hyperbolic space $\mathbb{H}_{K}^{n+1}$ has the projective compactification $\partial \mathbb{H}_{K}^{n+1}$ which
is diffeomorphic to the sphere $S^{[\mathbb{K}]}(n+1)^{-1}$. It is well known that the isometric action $\text{Isom}(\mathbb{H}_{\mathbb{K}}^{n})$ extends to a smooth action on $S^{[\mathbb{K}]}(n+1)^{-1}$. This phenomenon occurs also for Hadamard manifolds (complete simply connected Riemannian manifold of nonpositive curvature). In general, an extended action on the boundary sphere is topological. But the above actions on $\partial \mathbb{H}_{\mathbb{K}}^{n+1}$ are known to be analytic. Denote $\text{Aut}(S^{[\mathbb{K}]}(n+1)^{-1})$ the (extended) action of Isom$(\mathbb{H}_{\mathbb{K}}^{n})$ on $S^{[\mathbb{K}]}(n+1)^{-1}$. It is known that $\text{Aut}(S^{[\mathbb{K}]}(n+1)^{-1})$ acts transitively on $S^{[\mathbb{K}]}(n+1)^{-1}$ with noncompact stabilizer $\text{Aut}(S^{[\mathbb{K}]}(n+1)^{-1})_{\infty}$ such that

$$S^{[\mathbb{K}]}(n+1)^{-1} = \text{Aut}(S^{[\mathbb{K}]}(n+1)^{-1})_{\infty} \setminus \text{Aut}(S^{[\mathbb{K}]}(n+1)^{-1})$$

where $\infty \in S^{[\mathbb{K}]}(n+1)^{-1}$. Hence we have a non-Riemannian homogeneous geometry $(\text{Aut}(S^{[\mathbb{K}]}(n+1)^{-1}), S^{[\mathbb{K}]}(n+1)^{-1})$. According to whether $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, it is said to be

$$\text{Conformally flat geometry} \quad (\text{PO}(n+1, 1), S^{n})$$

(3.1)

$$\text{Spherical CR-geometry} \quad (\text{PU}(n+1, 1), S^{2n+1})$$

$$\text{Quaternionic flat CR-geometry} \quad (\text{PSp}(n+1, 1), S^{4n+3})$$

It is an excellent result by Gromov-Lawson-Yau that a nontrivial $S^{1}$-bundle $M^{3}$ over a closed surface $\Sigma_{g}$ of genus $g > 1$ admits a conformally flat structure. It is trivial that the product $S^{1} \times \Sigma_{g}$ is a conformally flat manifold. On the other hand, in spherical CR-geometry $(\text{PU}(2, 1), S^{3})$, the complement of geometric circle $S^{3} - S^{1}$ has an invariant subgroup $U(1, 1) = \text{P}(U(1, 1) \times U(1))$. Choosing a discrete cocompact subgroup $\Gamma \leq U(1, 1)$, we get a spherical CR-manifold $S^{3} - S^{1}/\Gamma$ which is a nontrivial $S^{1}$-bundle: $S^{1} \rightarrow U(1) \setminus U(1, 1)/\Gamma \rightarrow U(1) \setminus \text{P}(U(1, 1)/P(\Gamma)$. Here $U(1) \setminus \text{P}(U(1, 1)/P(\Gamma) = \mathbb{H}_{\mathbb{C}}^{1}/P(\Gamma) = \Sigma_{g}$. However, to our knowledge, the following problem hasn’t been yet proved rigorously.

**Problem.** Does the product $S^{1} \times \Sigma_{g}$ admit a spherical CR-structure?

4. **Conformally flat Lorentz geometry**

It is natural to consider how the isometry group of the pseudo-hyperbolic space acts on the compactification. Put $V_{-}^{m+2,2} = \{x \in \mathbb{R}^{m+4} \mid \mathcal{B}(x, x) = x_{1}^{2} + \cdots + x_{m+2}^{2} - x_{m+3}^{2} - x_{m+4}^{2} < 0\}$. If $P_{\mathbb{R}}: \mathbb{R}^{m+4} - \{0\} \rightarrow \mathbb{RP}^{m+3}$ is the canonical projection, then the real pseudo-hyperbolic space $\mathbb{H}_{\mathbb{R}}^{m+2,1}$ is defined to be $P_{\mathbb{R}}(V_{-}^{m+2,2})$. For this reason, the $m+3$-dimensional quadrics $V_{-1}^{m+2,2} = \{x \in \mathbb{R}^{m+4} \mid x_{1}^{2} + \cdots + x_{m+2}^{2} - x_{m+3}^{2} - x_{m+4}^{2} = -1\}$ with Lorentz metric $g$ is the complete pseudo-Riemannian manifold of signature $(m+1, 1)$ and of constant curvature $-1$ such that $P_{\mathbb{R}}(V_{-}^{m+2,2}) = P_{\mathbb{R}}(V_{-1}^{m+2,2})$. Since $P_{\mathbb{R}}: V_{-1}^{m+2,2} \rightarrow \mathbb{H}_{\mathbb{R}}^{m+2,1}$
is a two-fold covering, so $\mathbb{H}_{\mathbb{R}}^{m+2,1}$ is a complete pseudo-hyperbolic space form. The action $O(m + 2, 2)$ on $V_{-}^{m+2,2}$ induces an action on $\mathbb{H}_{\mathbb{R}}^{m+2,1}$. The kernel of this action is the center $\mathbb{Z}/2 = \{\pm 1\}$ whose quotient is called real pseudo-hyperbolic group $PO(m + 2, 2)$. The projective compactification of $\mathbb{H}_{\mathbb{R}}^{m+2,1}$ is obtained by taking the closure $\overline{\mathbb{H}_{\mathbb{R}}^{m+2,1}}$ in $\mathbb{R}\mathbb{P}^{m+3}$. Consider the commutative diagram:

$$(\text{GL}(m + 4, \mathbb{R}), \mathbb{R}^{m+4} - \{0\}) \rightarrow_{P} (\text{PGL}(m + 4, \mathbb{R}), \mathbb{R}\mathbb{P}^{m+3})$$

$$(O(m + 2, 2), V_{-}^{m+2,2} \cup V_{0}) \rightarrow_{P} (PO(m + 2, 2), \mathbb{H}_{\mathbb{R}}^{m+2,1} \cup S^{m+1,1})$$

Here $V_{0} = V_{0}^{m+2,1} = \{x \in \mathbb{R}^{m+4} | x_{1}^{2} + \cdots + x_{m+2}^{2} - x_{m+3}^{2} - x_{m+4}^{2} = 0\}$. It follows that

$$\overline{\mathbb{H}_{\mathbb{R}}^{m+2,1}} = \mathbb{H}_{\mathbb{R}}^{m+2,1} \cup S^{m+1,1}.$$

From this viewpoint, the pseudo-hyperbolic action of $PO(m + 2, 2)$ on $\mathbb{H}_{\mathbb{R}}^{m+2,1}$ extends to conformal action of $S^{m+1,1}$. We obtain conformally flat Lorentz geometry $(PO(m + 2, 2), S^{m+1,1})$. This is of course non-Riemannian homogeneous geometry.

Let $(1, 0, \ldots, 0, 1) \in V_{0}$ be a null vector. Put $\infty = P(1, 0, \ldots, 0, 1) \in S^{m+1,1}$ which is called the point at infinity. The stabilizer $PO(m+2, 2)_{\infty}$ is $\mathbb{R}^{m+2} \times (O(m+1, 1) \times \mathbb{R}^{+})$ up to conjugacy. When $h \in PO(m+2, 2)_{\infty}$, the differential map $h_{*} : T_{\infty}S^{m+1,1} \rightarrow T_{\infty}S^{m+1,1}$ is an isomorphism, $h_{*} \in \text{Aut}(T_{\infty}S^{m+1,1}) = O(m+1, 1) \times \mathbb{R}^{+}$. Thus the structure group of $(PO(m+2, 2), S^{m+1,1})$ is $O(m+1, 1) \times \mathbb{R}^{+}$. Originally as a $G$-structure, conformal Lorentz structure is an $O(m+1, 1) \times \mathbb{R}^{+}$-structure. In addition, an integrable $O(m+1, 1) \times \mathbb{R}^{+}$-structure is conformally flat Lorentz structure. (Equivalently, the Weyl conformal curvature tensor vanishes.) When $\{\infty\}$ is the point at infinity of $S^{m} = \partial \mathbb{H}_{\mathbb{R}}^{m+1}$, we can consider the minimal parabolic group $O(m+1, 1)_{\infty}$ which is an amenable Lie subgroup of $O(m+1, 1)$. We remark that $O(m+1, 1)_{\infty}$ is isomorphic to the similarity group $\text{Sim}(\mathbb{R}^{m})$.

**Definition 4.1.** If the structure group of a conformally flat Lorentz $(m + 2)$-manifold $M$ belongs to $O(m+1, 1)_{\infty} \times \mathbb{R}^{+}$, then $M$ is said to be a conformally flat Lorentz parabolic manifold.

We study a special class of conformally flat Lorentz parabolic manifolds called Lorentz similarity manifold of dimension $m+2$ and Fefferman-Lorentz manifold of dimension $2n+2$.
5. LORENTZIAN SIMILARITY GEOMETRY

Recall that $\mathbb{R}^{m+2}$ is the euclidean space with Lorentz inner product sitting in $\mathbb{S}^{m+1,1} - \{\infty\}$. Then $\text{PO}(m+2,2)_\infty = \mathbb{R}^{m+2} \rtimes (O(m+1,1) \times \mathbb{R}^+)$. We define $\text{Sim}_L(\mathbb{R}^{m+2}) = \mathbb{R}^{m+2} \rtimes (O(m+1,1) \times \mathbb{R}^+)$. The pair $(\text{Sim}_L(\mathbb{R}^{m+2}), \mathbb{R}^{m+2})$ is said to be Lorentz similarity geometry. In [3] we proved the following.

**Theorem 5.1.** If $M$ is a compact complete Lorentz similarity manifold of dimension $m + 2$, then the fundamental group of $M$ is virtually polycyclic. Furthermore, $M$ is diffeomorphic to an infrasolvmanifold.

This theorem is originally proved by T. Aristide. Once $\pi_1(M)$ turns out to be virtually polycyclic, the holonomy group $L(\pi_1(M))$ belongs to either $O(m+1,1)_\infty \rtimes \mathbb{R}^+$ or $O(m+1) \times O(1) \times \mathbb{R}^+$. Here $O(m+1,1)_\infty = \text{Sim}(\mathbb{R}^m) = \mathbb{R}^m \rtimes (O(m) \times \mathbb{R}^*)$. Since $\Gamma$ acts freely as a' ne motions on $\mathbb{R}^{m+2}$, the matrix of holonomy group has no eigenvalue 1. The latter case shows that $L(\pi_1(M)) \leq O(m+1) \times O(1)$ so that $M$ reduces to a compact euclidean space form. Then $\pi_1(M)$ is a Bieberbach group.

**Corollary 5.2.** A finite cover of a compact complete Lorentz similarity manifold $M$ is a conformally flat Lorentz parabolic manifold.

We shall give a sketch of proof of Theorem 5.1. Put $M = \mathbb{R}^{m+2}/\Gamma$ where $\Gamma \leq \text{Sim}_L(\mathbb{R}^{m+2})$. There is the exact sequence: $1 \to \mathbb{R}^{m+2} \to \text{Sim}_L(\mathbb{R}^{m+2}) \overset{L}{\to} O(m+1,1) \times \mathbb{R}^+ \to 1$. If $\mathbb{R}^{m+2} \cap \Gamma$ is nontrivial, say $\mathbb{Z}^k$, then a properly discontinuous action of $\Gamma$ induces a properly discontinuous action of $L(\Gamma)$ on $\mathbb{R}^{m-k}$ as in the same argument of [3, (1) Proposition 2.2]. Then $\Gamma$ is virtually polycyclic by induction. So we assume

$$\mathbb{R}^{m+2} \cap \Gamma = \{1\}.$$  

Note also that $(\mathbb{R}^{m+2} \rtimes \mathbb{R}^+) \cap \Gamma = \{1\}$ because each element has the form $(a, \lambda \cdot I)$. As $\Gamma$ acts freely on $\mathbb{R}^{m+2}$, $\lambda = 1$. It follows $(\mathbb{R}^{m+2} \rtimes \mathbb{R}^+) \cap \Gamma = \mathbb{R}^{m+2} \cap \Gamma$.

Consider the following exact sequence:

$$1 \to \mathbb{R}^{m+2} \times \mathbb{R}^+ \to \text{Sim}_L(\mathbb{R}^{m+2}) \overset{\rho}{\to} O(m+1,1) \to 1.$$  

If $\rho(\Gamma)$ is discrete in $O(m+1,1)$, then the cohomological dimension $\text{cd} \rho(\Gamma) \leq m+1$. As $\mathbb{R}^{m+2}/\Gamma$ is compact, $\text{cd} \Gamma = m+2$. On the other hand, $\Gamma \cong \rho(\Gamma)$ by (5.1), $\text{cd} \Gamma = \text{cd} \rho(\Gamma)$ which yields a contradiction.

Suppose that $\rho(\Gamma)$ is indiscrete in $O(m+1,1)$. Then the identity component of the closure $\overline{\rho(\Gamma)}^0$ is solvable in $O(m+1,1)$. 

Case I. If it is noncompact, then it belongs to the maximal amenable subgroup \( \text{Sim}(\mathbb{R}^m) \) up to conjugate. The normalizer of \( \overline{p(\Gamma)}^0 \) is contained in \( \text{Sim}(\mathbb{R}^m) \). In particular, \( p(\Gamma) \leq \text{Sim}(\mathbb{R}^m) \). (5.2) induces an exact sequence:

\[
1 \rightarrow \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \rightarrow p^{-1}(\text{Sim}(\mathbb{R}^m)) \overset{p}{\rightarrow} \text{Sim}(\mathbb{R}^m) \rightarrow 1
\]
in which \( p^{-1}(\text{Sim}(\mathbb{R}^m)) \) is an amenable Lie subgroup. Any discrete subgroup of an amenable Lie group is virtually polycyclic so is \( \Gamma \).

Case II. Suppose that \( \overline{p(\Gamma)}^0 \) is compact, say \( T^\ell \). We consider actions of subgroups of \( O(m+1,1) \) on \( \mathbb{H}_\mathbb{R}^{m+1} \cup S^m \). If \( T^\ell \) has no fixed point in \( S^m \), then \( T^\ell \) has a unique fixed point \( 0 \in \mathbb{H}_\mathbb{R}^{m+1} \) so that \( p(\Gamma) \leq O(m+1) \times O(1) \). Thus \( \Gamma \leq \text{Sim}(\mathbb{R}^{m+2}) \). \( \mathbb{R}^{m+2}/\Gamma \) turns out to be a compact complete similarity manifold and so \( \Gamma \) is virtually abelian (a Bieberbach group).

Suppose that \( T^\ell \) has the fixed point set \( S^k \) in \( S^m \) for some \( k < m \). As \( p(\Gamma) \) leaves invariant the complement \( S^m - S^k = \mathbb{H}_\mathbb{R}^{k+1} \times S^{m-k-1} \). It follows \( \overline{p(\Gamma)} \leq O(k+1,1) \times O(m-k) \) for which \( T^\ell = \overline{p(\Gamma)}^0 \leq O(m-k) \). If \( \text{Pr} : O(k+1,1) \times O(m-k) \rightarrow O(k+1,1) \) is the canonical projection, then \( \text{Pr}(p(\Gamma)) \) is discrete. Note that \( \text{Ker \text{Pr} \circ p} = \mathbb{R}^{m+2} \rtimes (O(m-k) \times \mathbb{R}^+) \). Put

\[
(5.3) \quad \Delta = (\mathbb{R}^{m+2} \rtimes (O(m-k) \times \mathbb{R}^+)) \cap \Gamma.
\]

It is nontrivial, because if trivial, \( \Gamma \cong \text{Pr}(p(\Gamma)) \) so \( m + 2 = \text{cd} \Gamma = \text{cd} \text{Pr}(p(\Gamma)) \) but \( \text{cd} \text{Pr}(p(\Gamma)) \leq k + 1 \) which is impossible by the inequality \( k < m \).

Since \( T^\ell \) is a maximal torus in \( O(m-k) \), \( N_{O(m-k)}(T^\ell)/T^\ell \) is finite for the normalizer \( N_{O(m-k)}(T^\ell) \). As \( \overline{p(\Gamma)} \) normalizes \( T^\ell \), there exists a finite index normal subgroup \( H \) of \( p(\Gamma) \) which centralizes \( T^\ell \) with \( H^0 = T^\ell \). Note that \( H \cap p(\Gamma) \) is of finite index in \( p(\Gamma) \). Put \( \Gamma_1 = p^{-1}(H \cap p(\Gamma)) \) which is a finite index subgroup of \( \Gamma \).

Let \( p_1 : \mathbb{R}^{m+2} \rtimes (O(m-k) \times \mathbb{R}^+) \rightarrow O(m-k) \) be the projection. Then \( p_1(\Delta) \leq O(m-k) \) such that \( \overline{p_1(\Delta)}^0 \) is a torus in \( O(m-k) \) from (5.3). Since \( \overline{p_1(\Delta)} \) is a finite extension of \( \overline{p_1(\Delta)}^0 \), we choose \( \Delta_1 \) such that \( p_1(\Delta_1) = \overline{p_1(\Delta)}^0 \cap p_1(\Delta) \). As \( p_1(\Delta_1) \leq p(\Gamma) \), \( p_1(\Delta_1) \leq \overline{p(\Gamma)}^0 \). Noting that \( p(\Gamma) \leq H \) and \( p(\Delta_1) \leq \overline{p(\Gamma)}^0 \) for which \( H \) centralizes \( T^\ell = \overline{p(\Gamma)}^0 \) as above, it follows that \( p(\Gamma_1) \) centralizes \( p(\Delta_1) \).

Note that \( p_1 : \Delta \rightarrow p_1(\Delta) \) is injective. In fact, if not, then \( \Delta \cap \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \neq \{1\} \), so \( \Gamma \cap \mathbb{R}^{m+2} \rtimes \mathbb{R}^+ \neq \{1\} \) which is impossible by the remark below (5.1). Since \( \Gamma \) normalizes \( \Delta \), it is easy to see that \( \Gamma_1 \) centralizes \( \Delta_1 \). Consider the exact sequences:
$1 \longrightarrow \mathbb{R}^{m+2} \longrightarrow \mathbb{R}^{m+2} \times O(m-k) \overset{p}{\longrightarrow} O(m-k) \longrightarrow 1$

\[1 \longrightarrow \Delta_1 \overset{p}{\longrightarrow} p(\Delta_1) \longrightarrow 1\]

where $p(\Delta_1)^0 = T^s$ for some $s \leq m - k$. It is well known that the abelian discrete subgroup belongs to the following group (cf. [8]):

\[(5.4) \quad \Delta_1 \leq V \times T^s = \{\left(\begin{array}{l} a \\ 0 \end{array}\right), \left(\begin{array}{ll} I & 0 \\ 0 & C \end{array}\right)\} | C \in T^s, a \in V\}

such that $V \times T^s/\Delta_1$ is compact. Here $V \cong \mathbb{R}^{k+2}$.

Let $\Gamma_1 \leq \mathbb{R}^{m+2} \times (O(k+1,1) \times O(m-k) \times \mathbb{R}^+)$ be as before and choose an arbitrary element $\gamma = \left(\begin{array}{l} x \\ y \end{array}\right), \lambda \cdot \left(\begin{array}{ll} A & 0 \\ 0 & B \end{array}\right)$ and take an element $\alpha = \left(\begin{array}{l} a \\ 0 \end{array}\right), \left(\begin{array}{ll} I & 0 \\ 0 & C \end{array}\right)$ from $\Delta_1$. As $\Gamma_1$ centralizes $\Delta_1$, the equation $\gamma \alpha \gamma^{-1} = \alpha$ implies that

$\lambda \cdot Aa = a, BCB^{-1} = C$ and $y - BCB^{-1}y = 0$.

The projection $P : \mathbb{R}^{k+2} - \{0\} \rightarrow \mathbb{R}P^{k+1}$ maps the cone $V_0$ onto $S^k$. We observe that if $(a,a) = 0$ with respect to the Lorentz inner product, then $P(a) = [a] \in S^k$. Put $[a] = \infty \in S^k$ up to conjugacy. The equality $\lambda \cdot Aa = a$ implies $A\infty = \infty$ so $A \in O(k+1,1)_\infty$. This holds for arbitrary elements of $\Gamma_1$. It follows

\[(5.5) \quad \Gamma \leq \mathbb{R}^{m+2} \times (O(k+1,1)_\infty \times O(m-k) \times \mathbb{R}^+)\]

which is an amenable Lie subgroup. Thus $\Gamma$ is virtually polycyclic. When $\langle a, a \rangle \neq 0$, as $\langle a, a \rangle = \langle \lambda Aa, \lambda Aa \rangle = \lambda^2 \langle a, a \rangle$, it follows $\lambda = 1$.

Thus

\[(5.6) \quad \Gamma \leq \mathbb{R}^{m+2} \times (O(k+1,1) \times O(m-k)) \leq E(m+1,1)\]

$\mathbb{R}^{m+2}/\Gamma$ becomes a compact complete Lorentz flat space form. It is well known that $\Gamma$ is virtually polycyclic. This proves the theorem 5.1.

6. Conformally flat Fefferman-Lorentz geometry

Let $(O(2n+2,2), S^1 \times S^{2n+1})$ be the conformally flat Lorentz geometry (which is a 2-fold cover.) There is the natural embedding $U(n+1,1) \rightarrow O(2n+2,2)$. $U(n+1,1)$ acts transitively on $S^1 \times S^{2n+1}$ so we have a subgeometry $(U(n+1,1), S^1 \times S^{2n+1})$.

**Proposition 6.1.** A manifold locally modelled on $(U(n+1,1), S^1 \times S^{2n+1})$ admits a Lorentz prabolic structure.
Proof. We see that
\[ \hat{U}(n+1,1) \cap O(2n+2,2)_{\infty} = \mathbb{R}^{2n+2} \rtimes (O(2n+1,1) \times \mathbb{R}^*) \cap \mathcal{N} \times (U(n) \times \mathbb{R}^+) \]
which is amenable. Here \( \mathcal{N} \) is the Heisenberg Lie group. So \( \hat{U}(n+1,1)_{\infty} \) belongs to the maximal amenable group \( \mathbb{R}^{2n+2} \rtimes (O(2n+1,1)_{\infty} \times \mathbb{R}^*) \). Thus the structure group of \( (U(n+1,1), S^1 \times S^{2n+1}) \) belongs to the parabolic group \( O(2n+1,1)_{\infty} \times \mathbb{R}^* \). So any manifold modelled on \( (U(n+1,1), S^1 \times S^{2n+1}) \).

\[ \square \]

Definition 6.2. A manifold locally modelled on \( (U(n+1,1), S^1 \times S^{2n+1}) \) is said to be a conformally flat Fefferman-Lorentz parabolic manifold.

To the rest of this section we shall give our recent results concerning compact conformally flat Fefferman-Lorentz parabolic manifolds. The details will be given elsewhere.

Recall that the center \( S^1 \) acts freely on the 2-fold covering \( S^1 \times S^{2n+1} \) of \( S^{2n+1,1} \), there is the equivariant principal bundle:

\[ (6.1) \quad (S^1, S^1) \to (U(n+1,1), S^1 \times S^{2n+1}) \xrightarrow{(P,p)} (PU(n+1,1), S^{2n+1}). \]

Let \( X \) be a domain of \( S^1 \times S^{2n+1} \). If \( h \) is an element of the group of conformal Lorentz transformations \( \text{Conf}(X) \), then \( h : X \to X \) extends uniquely to a conformal diffeomorphism of \( S^1 \times S^{2n+1} \) by Liouville’s theorem. We assume that

\[ (6.2) \quad \text{Conf}(X) \leq U(n+1,1). \]

Suppose that a discrete subgroup \( \Gamma \) of \( U(n+1,1) \) acts properly discontinuously on \( X \) such that the quotient \( X/\Gamma \) is compact. Note that there is a covering group extension:

\[ (6.3) \quad 1 \longrightarrow \Gamma \longrightarrow N_{\text{Conf}(X)}(\Gamma) \xrightarrow{\nu} \text{Conf}(X/\Gamma) \longrightarrow 1. \]

We shall determine \( X/\Gamma \) when \( X/\Gamma \) admits a 1-parameter subgroup \( H \) whose lift \( H \) to \( U(n+1,1) \) is not the center \( \mathcal{Z}U(n+1,1) \).

Theorem 6.3. Let \( X/\Gamma \) be a 2n + 2-dimensional compact conformally flat Fefferman-Lorentz parabolic manifold. If \( X/\Gamma \) admits a 1-parameter subgroup \( H \) whose lift \( H \) to \( U(n+1,1) \) is not the center \( \mathcal{Z}U(n+1,1) \), then \( X/\Gamma \) is a Seifert fiber space over a spherical CR-orbifold. Moreover \( X/\Gamma \) is either one of (i), \ldots, (v). As a consequence, a finite covering of such \( X/\Gamma \) is a Fefferman-Lorentz manifold.

(i) \( X/\Gamma = S^1 \times_{\mathbb{Z}_k} S^{2n+1} \) where \( k \leq T^{n+1} \).

(ii) \( S^1 \to X/\Gamma \to \mathcal{N}/Q \) where \( Q \leq \mathcal{N} \times U(n) \).
(iii) \( S^1 \rightarrow X/\Gamma \rightarrow S^{2n} \times_{F} S^1 \) where \( F \leq U(n) \).
(iv) \( S^1 \rightarrow X/\Gamma \rightarrow S^{2n+1} / F \) where \( F \leq T^{n+1} \).
(v) \( S^1 \rightarrow X/\Gamma \rightarrow (S^{2n+1} - L(Q))/Q \) where \( Q \leq P(U(k, 1) \times U(n-k+1)) \) \((k = 1, \ldots, n)\).

The idea of proof is as follows. Let \( S^1 = ZU(n+1,1) \) be the center of \( U(n+1,1) \). Then \( S^1 \cdot \tilde{H} \leq U(n+1,1) \). There is an equivariant fibration:

\[(S^1, S^1) \rightarrow (S^1 \cdot \tilde{H}, \Gamma, X) \rightarrow^{(P,p)} (G, Q, W)\]

where we put \( G = S^1 \cdot \tilde{H} / S^1 \), \( Q = \Gamma / S^1 \cap \Gamma \) and \( W = X / S^1 \). As \( Q, G \leq PU(n+1,1) \), the quotient \( W/Q \) is a spherical \( CR \)-orbifold with \( CR \)-action \( G \). To determine \( X/\Gamma \) reduces to the classification of \( CR \)-manifolds \( (Q, W) \) with the 1-parameter group \( G \) of \( CR \)-transformations. The classification is accomplished by the result in [4].

When \( \dim X/\Gamma = 4 \), then \( Q \leq U(1,1) \) so that \( L(Q) \subset S^1 \) \((k = 1)\). According to whether \( L(Q) \) is a Cantor set in \( S^1 \) or \( L(Q) = S^1 \), it is well known that \( S^3 - L(Q)/Q = S^1 \times S^2 \# \cdots \# S^1 \times S^2 \) or some finite cover of \( S^3 - L(Q)/Q = V^3_{-1}/Q \) is a principal \( S^1 \)-bundle with nonzero euler class over a closed surface of genus \( g \geq 2 \).

### 6.1. Non Fefferman-Lorentz manifold

It is conceivable whether some finite cover of any compact conformally flat Fefferman-Lorentz parabolic manifold is a Fefferman-Lorentz manifold. It is not true in general. It will be shown

**Proposition 6.4.** There exists a compact conformally flat Fefferman-Lorentz parabolic manifold \( P \) of dimension \( 2n + 2 \) \((n \geq 1)\) but no finite covering is a Fefferman-Lorentz manifold.

This manifold \( P \) supports a principal fiber space: \( T^2 \rightarrow P \rightarrow \mathbb{H}_c^2/Q_0 \) where \( \mathbb{H}_c^2/Q_0 \) is a compact complex hyperbolic manifold.

### 7. Representation space

Let \( X/\Gamma \) be a compact conformally flat Lorentz manifold with \( S^1 \)-action so that \( X \subset \mathbb{R} \times S^{2n+1}, \Gamma, \tilde{S}^1 \leq O(m+2,2) \). If \( p : O(m+2,2) \rightarrow O(m+2,2) \) is the covering homomorphism, put \( G = p(\tilde{S}^1) \leq O(m+2,2) \). We will prove that

- If \( G \) is compact, then \( m = 2n \) and \( G = S^1 \), \( C_{O(m+2,2)}(S^1) = U(n+1,1) \). \((S^1, X/\Gamma) \) is locally modelled on \((U(n+1,1), S^1 \times S^{2n+1}) \) where \( S^1 = ZU(n+1,1) \), i.e. \( X/\Gamma \) is a conformally flat Fefferman-Lorentz parabolic manifold.
• If $G$ is noncompact, the either \( \Gamma \leq \mathbb{R}^{m+2} \rtimes O(m+1,1) \) and \( G = \mathbb{R} \) or \( \Gamma \leq O(m+1,1) \times \mathbb{R}^+ \) and \( G = \mathbb{R}^+ \).

**Proposition 7.1.** Let \( X/\Gamma = S^1 \times N^3/\Delta \) which is a conformally flat Lorentz parabolic manifold and \( \Gamma = \mathbb{Z} \times \Delta \leq O(4,2), \mathbb{N} \subset \mathbb{R} \times S^3 \). There are exactly two distinct faithful representations up to conjugate in \( O(4,2) \):

\[
\begin{align*}
\rho_1 : \Gamma &\rightarrow \mathbb{R} \times (N \rtimes U(1)), \quad S^1 \text{ is lightlike.} \\
\rho_2 : \Gamma &\rightarrow \mathbb{R}^3 \times (R^2 \rtimes O(2)) \leq \mathbb{R}^4 \rtimes O(3,1), \quad S^1 \text{ is spacelike.}
\end{align*}
\]

(7.1)

Then the space of discrete faithful representations \( R(\Gamma, O(4,2)) \) consists of two components \( R(\Gamma, \mathbb{R} \times (N \rtimes U(1))), R(\Gamma, \mathbb{R}^3 \times (R^2 \rtimes O(2))) \).

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DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY, MINAMI-OHSAWA 1-1, HACHIOJI, TOKYO 192-0397, JAPAN

E-mail address: kami@tmu.ac.jp