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Existence of eigenvalues and eigenfunctions for radially symmetric fully nonlinear elliptic operators

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1 Introduction

This note is based on a joint work [13] with H. Ishii and we take a slightly different approach in the radial case from the one in [13]. See also the comments after Theorem 1.2.

In this note, we consider the eigenvalue problem for fully nonlinear elliptic operator $F$:

\[
\begin{aligned}
F(D^2 u, Du, u, x) + \mu u &= 0 \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\] (1)

Here $\Omega \subset \mathbb{R}^N$ is an open interval $(a, b)$ with $-\infty < a < b < \infty$ when $N = 1$, or an open ball $B_R = B_R(0)$ when $N \geq 2$, $u : \bar{\Omega} \to \mathbb{R}$ and $\mu \in \mathbb{R}$ represent the unknown function (eigenfunction) and constant (eigenvalue), respectively, and $F : \mathbb{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given function, where $\mathbb{S}^N$ denotes the space of real symmetric $N \times N$ matrices.

The study of the eigenvalue problem for fully nonlinear elliptic operator goes back to the work of P.-L. Lions [16] and for the developments we refer to [1, 4, 5, 14, 17, 20] and to [2, 8, 9] for some earlier related works.

Recently, Busca, Esteban and Quaas [5] and Esteban, Felmer and Quaas [11] showed the existence of higher eigenvalues and of the corresponding eigenfunctions in the one-dimensional or the radially symmetric problem. In this note we extend the results of [11] into the $L^p$ framework.

Before giving our assumptions (F1)-(F4) on the function $F$, we introduce the Pucci operators $P^\pm$. Given constants $\lambda \in (0, \infty)$ and $\Lambda \in [\lambda, \infty]$, $P^\pm$ denote the Pucci operators defined as the functions on $\mathbb{S}^N$ given, respectively, by $P^+(M) = P^+(M; \lambda, A) = \sup\{ \text{tr } AM : A \in \mathbb{S}^N, \lambda I_N \leq A \leq \Lambda I_N \}$ and $P^-(M) = -P^+(-M)$, where $I_N$ denotes the $N \times N$ identity matrix and the relation, $X \leq Y$, is the standard order relation between $X, Y \in \mathbb{S}^N$. We remark that in the case $\Lambda = \infty$, $P^+(M) = \infty$ if $M \not\leq 0$ and $P^+(M) = \lambda \sum_{j=1}^N \nu_j$ if $M \leq 0$.

(F1) The function $F : \mathbb{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a Carathéodory function, i.e., the function $x \mapsto F(M, p, u, x)$ is measurable for any $(M, p, u) \in \mathbb{S}^N \times \mathbb{R}^{N+1}$ and the function $(M, p, u) \mapsto F(M, p, u, x)$ is continuous for a.a. $x \in \Omega$.

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(F2) There exist constants $\lambda \in (0, \infty)$, $\Lambda \in [\lambda, \infty]$, $q \in [1, \infty]$ and functions $\beta, \gamma \in L^q(\Omega)$ such that

$$F(M_1, p_1, u_1, x) - F(M_2, p_2, u_2, x) \leq P^+(M_1 - M_2) + \beta(x)|p_1 - p_2| + \gamma(x)|u_1 - u_2|$$

for all $(M_1, p_1, u_1), (M_2, p_2, u_2) \in S^N \times \mathbb{R}^{N+1}$ and a.a. $x \in \Omega$.

(F3) $F(tM, tp, tu, x) = tF(M, p, u, x)$ for all $t \geq 0$, all $(M, p, u) \in S^N \times \mathbb{R}^{N+1}$ and a.a. $x \in \Omega$.

Here we remark that if $\Lambda = \infty$ and $M_1 \leq M_2$, then the inequality in condition (F2) is trivially satisfied since $P^+(M_1 - M_2) = \infty$.

The next condition concerns the radial symmetry in the multi-dimensional case.

(F4) The function $F$ is radially symmetric in the sense that for any $(m, l, q, u) \in \mathbb{R}^4$ and a.a. $r \in (0, R)$, the function

$$\omega \mapsto F(m\omega \otimes \omega + l(I_N - \omega \otimes \omega), q\omega, u, r\omega)$$

is constant on the unit sphere $S^{N-1} \subset \mathbb{R}^N$. Here and henceforth $x \otimes x$ denotes the matrix in $\mathbb{S}^N$ with the $(i, j)$ entry given by $x_i x_j$ if $x \in \mathbb{R}^N$.

We study the eigenvalue problem (1) in the Sobolev space $W^{2, q}(\Omega)$. For any pair $(\mu, \varphi) \in \mathbb{R} \times (W^{2, 1}(\Omega) \cap W_0^{1, 1}(\Omega))$ which satisfies (1) in the almost everywhere sense, we call $\mu$ and $\varphi$ an eigenvalue and eigenfunction of (1), respectively, provided $\varphi(x) \neq 0$. We call such a pair an eigenpair of (1).

We state our main results in this note.

**Theorem 1.1.** Let $N = 1$ and $\Omega = (a, b)$, and assume that (F1), (F2) with $\Lambda = \infty$, and (F3) hold. Then

(i) For any $n \in \mathbb{N}$, there exist eigenpairs $(\mu^+_n, \varphi^+_n) \in \mathbb{R} \times W^{2, q}(a, b)$ of (1) and sequences $(x^+_n)_{n=0}^\infty \subset [a, b]$ such that

$$\begin{cases} 
\alpha = x^+_n < \cdots < x^+_n \not\equiv 0, \\
\varphi^+_n(x) > 0 \text{ in } (x^+_n, x^+_n) \text{ for } j = 1, \ldots, n, \\
\varphi^+_n(x) > 0 \text{ in } (x^+_n, x^+_n) \text{ for } j = 1, \ldots, n.
\end{cases}$$

(ii) The eigenpairs $\{(\mu^+_n, \varphi^+_n)\}_{n=1}^\infty$ are complete in the sense that for any eigenpair $(\mu, \varphi) \in \mathbb{R} \times W^{2, q}(a, b)$ of (1), there exist $n \in \mathbb{N}$ and $\theta > 0$ such that either $(\mu, \varphi) = (\mu^+_n, \theta \varphi^+_n)$ or $(\mu, \varphi) = (\mu^-_n, \theta \varphi^-_n)$ holds.

For $q \in [1, \infty]$, let $W^{2,q}(0, R)$ denote the space of those functions $\varphi \in W^{2, q}(B_R)$ which are radially symmetric. We may identify any function $f$ in $W^{2, q}(0, R)$ with a function $g$ on $[0, R]$ such that $f(x) = g(|x|)$ for a.a. $x \in B_R$ and we employ the standard abuse of notation: $f(x) = f(|x|)$ for $x \in B_R$. We set $\lambda_* = \lambda / \Lambda$ and $q_* = N/(\lambda_* N + 1 - \lambda_*)$ if $\Lambda < \infty$. Note that $0 < \lambda_* \leq 1$ and $q_* \in [1, N)$. 

Theorem 1.2. Let $N \geq 2$, $\Omega = B_R$, and assume that (F1)-(F4) with $\Lambda < \infty$ hold. Assume also $q \in (\max\{N/2, q_*\}, \infty]$ and $\beta \in L^N(B_R)$ if $q < N$. Then:

(i) For each $n \in \mathbb{N}$, there exist eigenpairs $(\mu_n^\pm, \varphi_n^\pm) \in \mathbb{R} \times W_{r}^{2,q}(0, R)$ of (1) and sequences $(r_{n,j}^\pm)_{j=0}^n \subset [0, R]$ such that

$$0 = r_{0,n}^\pm < r_{2,n}^\pm < \cdots < r_{n,n}^\pm = R,
\begin{cases}
(-1)^{j-1}\varphi_n^+(r) > 0 & \text{in } (r_{n,j-1}^+, r_{n,j}^+) \text{ for } j = 1, \ldots, n, \\
(-1)^{j}\varphi_n^-(r) > 0 & \text{in } (r_{n,j-1}^-, r_{n,j}^-) \text{ for } j = 1, \ldots, n,
\end{cases}
\varphi_n^+(0) > 0 > \varphi_n^-(0).
$$

(ii) The eigenpairs $\{(\mu_n^\pm, \varphi_n^\pm)\}_{n=1}^\infty$ are complete in the sense that for any eigenpair $(\mu, \varphi) \in \mathbb{R} \times W_{r}^{2,q}(0, R)$ of (1), there exist $n \in \mathbb{N}$ and $\theta > 0$ such that either $(\mu, \varphi) = (\mu_n^+, \theta \varphi_n^+)$ or $(\mu, \varphi) = (\mu_n^-, \theta \varphi_n^-)$ is valid.

In this note we only treat the case where $N \geq 2$, i.e., Theorem 1.2. As mentioned before, we will give a slightly different approach from the one in [13]. In [13], we take the following approach. For any $\epsilon > 0$ and $n \geq 1$, first we show the existence of solutions of

$$F(D^2u_{\epsilon,n}, Du_{\epsilon,n}, u_{\epsilon,n}) + \mu_{\epsilon,n}u_{\epsilon,n} = 0 \text{ in } A(\epsilon, R),
\begin{cases}
u_{\epsilon,n}^\pm \in W_0^{2,q}(A(\epsilon, R)),
(u_{\epsilon,n}^\pm)'(\epsilon) = 0, u_{\epsilon,n}^\pm(R) = 0, \pm u_{\epsilon,n}^\pm(\epsilon) > 0
\end{cases}
$$

which have $n - 1$ zeroes in $[\epsilon, R]$. Here $A(\epsilon, R) := \{x \in \mathbb{R}^N : \epsilon < |x| < R\}$ and $W_0^{2,q}(\epsilon, R)$ denotes the set consisting of all radial functions in $W_{r}^{2,q}(A(\epsilon, R))$. Then let $\epsilon \to 0$ and observe that we can extract a subsequence whose limit is an eigenpair of (1) with the desired properties.

However, in this note, we will show the existence of eigenpairs through the unique solvability of

$F(D^2u, Du, u, x) - \kappa u + f(x) = 0 \text{ in } B_R(0), u \in W_0^{2,q}(B_R(0)) \cap W_{1,q}^{2,q}(B_R(0))$, for some $\kappa \in \mathbb{R}$ and any radial function $f \in L^q(B_R(0))$. See, for instance, sections 5 and 6 (Theorems 5.1 and 6.1).

Lastly, we give a remark about the condition on $\beta$ in Theorem 1.2. Our requirement on $\beta$ in Theorem 1.2 is only that $\beta \in L^q(B_R) \cap L^N(B_R)$. This condition seems relatively sharp from the known results in a priori estimates of solutions to (1). We refer to [6, 7, 10, 12, 15, 18]. See also Proposition 3.6 in this connection.

## 2 Preliminaries

Throughout this note, we suppose $N \geq 2$. First, we introduce the notations. For $0 \leq a < b \leq R$ and $q \in [1, \infty]$,

$$A(a, b) := \{x \in \mathbb{R}^N : a < |x| < b\}, \quad A(0, b) := B_b(0),
L^q_t(a, b) := \{u \in L^q(A(a, b)) : u \text{ is radial}\},
W_0^{2,q}(a, b) := \{u \in W^{2,q}(A(a, b)) : u \text{ is radial}\},
\begin{aligned}
\|u\|_{L^q_t(a, b)}^q &:= \int_a^b r^{N-1}|u(r)|^qdr &\text{ if } q \in [1, \infty) \quad \text{and} \quad \|u\|_{L^\infty_t(a, b)} := \|u\|_{L^\infty(a, b)}.
\end{aligned}$$
Note that $C^\infty_r(A(a, b)) := \{u \in C^\infty(A(a, b)) : u \text{ is radial}\} \text{ is dense in } W^{2,q}_r(a, b)$.

Let $u$ be a smooth radial function and we identify $u(x)$ with $u(|x|)$. Then it is easy to see

$$
(2) \quad Du(x) = u'(|x|) \frac{x}{|x|}, \quad D^2u(x) = u''(|x|)P_x + \frac{u'(|x|)}{|x|}(I_N - P_x) \quad \text{for } x \neq 0
$$

where $P_x$ denotes the matrix $x \otimes x/|x|^2 = (x_ix_j/|x|^2)$ which represents the orthogonal projection in $\mathbb{R}^N$ onto the one-dimensional space spanned by the vector $x$.

Next, we introduce a norm in $W^{2,q}_r(a, b)$ which is equivalent to the usual norm $\|\cdot\|_{W^{2,q}(A(a,b))}$.

**Lemma 2.1.** The following norm is equivalent to $\|\cdot\|_{W^{2,q}_r(A(a,b))}$ in $W^{2,q}_r(a, b)$:

$$
\|u\|_{W^{2,q}_r(a,b)} := \|u\|_{L^q_r(a,b)} + \|u'/r\|_{L^q_r(a,b)} + \|u''\|_{L^q_r(a,b)}.
$$

**Proof.** First, noting that $C^\infty_r(A(a, b))$ is dense in $W^{2,q}_r(a, b)$, (2) holds for any $u \in W^{2,q}_r(a, b)$ and a.a. $x \in A(a,b)$. On the other hand, we have

$$
|D^2u(x)| := \left(\sum_{i,j} \left| \frac{\partial^2 u}{\partial x_i \partial x_j}(x) \right|^2 \right)^{1/2} = \left( \left| u''(|x|) \right|^2 + (N - 1) \frac{|u'(|x|)|^2}{|x|^2} \right)^{1/2}.
$$

Thus it is easy to see that $\|\cdot\|_{W^{2,q}_r(a,b)}$ and $\|\cdot\|_{W^{2,q}(A(a,b))}$ is equivalent.

In the rest of this note, we use $\|\cdot\|_{W^{2,q}_r(a,b)}$ instead of the usual norm $\|\cdot\|_{W^{2,q}(A(a,b))}$.

Next, we rewrite (1) in the radial form and give some remarks. Assume that $F$ satisfies (F1), (F2) with $\Lambda < \infty$ and (F4). We fix a point $\omega_0 \in S^{N-1}$ and define the function $\mathcal{F} : \mathbb{R}^4 \times (0, R) \rightarrow \mathbb{R}$ by

$$
\mathcal{F}(m, l, p, u, r) := F(m\omega_0 \otimes \omega_0 + (I_N - \omega_0 \otimes \omega_0)l, p\omega_0, u, r\omega_0).
$$

We write $\mathcal{F}[u](r)$ for $\mathcal{F}(u''(r), u'(r)/r, u'(r), u(r), r)$. Thanks to (F4) and (2), (1) is equivalent to

$$
(3) \quad \mathcal{F}[u] + \mu u = 0 \quad \text{a.e. in } (0, R), \quad u \in W^{2,q}_r(0, R), \quad u(R) = 0.
$$

We also introduce radial versions $\mathcal{P}^\pm : \mathbb{R}^2 \rightarrow \mathbb{R}$ of the Pucci operators by

$$
\mathcal{P}^+(m, l) := \max\{\Lambda(m_+ + (N-1)l_+), -\lambda(m_- + (N-1)l_-)\}, \quad \mathcal{P}^-(m, l) = -\mathcal{P}^+(m, -l).
$$

and $m_\pm := \max\{m, 0\}$. By (F2), we have

$$
(4) \quad \mathcal{F}(m_1, l_1, p_1, u_1, r) - \mathcal{F}(m_2, l_2, p_2, u_2, r) \leq \mathcal{P}^+(m_1 - m_2, l_1 - l_2) + \beta(r\omega)|p_1 - p_2| + \gamma(r\omega)|u_1 - u_2|
$$

for all $(m_i, l_i, p_i, u_i) \in \mathbb{R}^4$, $i = 1, 2$, and a.a. $(r, \omega) \in (0, R) \times S^{N-1}$. In view of Fubini's theorem in the polar coordinates, we can choose a $\omega \in S^{N-1}$ which has the
properties that the inequality (5) holds for all \((m_i, l_i, p_i, u_i) \in \mathbb{R}^4, i = 1, 2\), and a.a. \(r \in (0, R)\), and the functions \(r \mapsto r^{N-1}(\beta(r\omega))^q, r \mapsto r^{N-1}(\gamma(r\omega))^q\) are integrable in \((0, R)\). We fix such an \(\omega\), call it \(\omega_1\), and, with abuse of notation, we write \(\beta\) and \(\gamma\) the functions \(r \mapsto \beta(r\omega_1)\) and \(r \mapsto \gamma(r\omega_1)\), respectively. In other words, under the assumptions (F1), (F2) and (F4), we conclude the following:

(F5) There exist functions \(\beta, \gamma \in L_r^q(0, R)\) such that

\[
\mathcal{F}(m_1, l_1, p_1, u_1, r) - \mathcal{F}(m_2, l_2, p_2, u_2, r) \\
\leq \mathcal{P}^+(m_1 - m_2, l_1 - l_2) + \beta(r)|p_1 - p_2| + \gamma(r)|u_1 - u_2|
\]

for all \((m_i, l_i, p_i, u_i) \in \mathbb{R}^4, i = 1, 2\), and a.a. \(r \in (0, R)\).

Since \(\mathcal{P}^-(m, l) = -\mathcal{P}^+(m, l, -l)\), it holds from (F5) that for all \((m_i, l_i, p_i, u_i) \in \mathbb{R}^4\) and a.a. \(r \in (0, R)\),

\[
\mathcal{F}(m_1, l_1, p_1, u_1, r) - \mathcal{F}(m_2, l_2, p_2, u_2, r) \\
\geq \mathcal{P}^-(m_1 - m_2, l_1 - l_2) - \beta(r)|p_1 - p_2| - \gamma(r)|u_1 - u_2|.
\]

For later use, we rewrite the conditions in terms of \(\mathcal{F}\):

(r-F1) The function \(\mathcal{F}\) is a Carathéodory function.

(r-F2) There exist \(\beta, \gamma \in L_r^q(0, R)\) such that

\[
\mathcal{F}(m_1, p_1, u_1, r) - \mathcal{F}(m_2, p_2, u_2, r) \\
\leq \mathcal{P}^+(m_1 - m_2, p_1 - p_2, r) + \beta(r)|p_1 - p_2| + \gamma(r)|u_1 - u_2|
\]

for all \((m_i, p_i, u_i) \in \mathbb{R}^3, i = 1, 2\), and a.a. \(r \in (0, R)\).

(r-F3) \(\mathcal{F}(tm, tl, tp, tu, r) = t\mathcal{F}(m, l, p, u, r)\) for every \((m, l, p, u) \in \mathbb{R}^4\) and a.a. \(r \in (0, R)\).

In what follows, we shall prove the existence of solutions to (3) under (r-F1)–(r-F3). In order to show the existence of eigenpairs to (3), the solvability of the following equations plays an important role under (r-F1), (r-F2) and \(\mathcal{F}[0] \in L_r^q(0, R)\): for each \(0 \leq a < b \leq R\),

\[
\mathcal{F}_\kappa[u] = 0 \quad \text{a.e. in } (a, b), \quad u \in W_r^{2,q}(a, b), \quad u(b) = 0, \quad u'(a) = 0 \text{ if } a > 0
\]

where \(\mathcal{F}_\kappa(m, l, p, u, r) := \mathcal{F}(m, l, p, u, r) - \kappa u\) and \(\kappa \in \mathbb{R}\). The constant \(\kappa\) is fixed later.

To rewrite (7) in the normal form, we use the following lemma (See Lemma 2.1 in [11]).

Lemma 2.2. Under the conditions (r-F1) and (r-F2), the following hold:

(i) There is a unique \(g = g_\kappa(l, p, u, d, r) \in \mathbb{R}\) such that \(\mathcal{F}(g, l, p, u, r) = d\) for any \((l, p, u, d) \in \mathbb{R}^4\) and a.a. \(r \in (0, R)\)
(ii) For all \((m, l, p, u, d) \in \mathbb{R}^5\) and a.a. \(r \in (0, R)\), \(m < g_x(l, p, u, d, r)\) (resp. \(m > g_x(l, p, u, d, r)\)) if and only if \(\mathcal{F}(m, l, p, u, r) < d\) (resp. \(\mathcal{F}(m, l, p, u, r) > d\)).

(iii) The function \(g_x\) satisfies the following Lipschitz condition:

\[
|g_x(l_1, p_1, u_1, d_1, r) - g_x(l_2, p_2, u_2, d_2, r)| \\
\leq \lambda^{-1} L(r) (|l_1 - l_2| + |p_1 - p_2| + |u_1 - u_2| + |d_1 - d_2|)
\]

for every \((l_1, p_1, u_1, d_1) \in \mathbb{R}^4\) and a.a. \(r \in (0, R)\) where \(L(r) := \max\{\Lambda(N - 1), \beta(r), \gamma(r), 1\}\).

Furthermore, it holds that for any \(d \in \mathbb{R}\),

\[
|g_x(0, 0, 0, d, r)| \leq \lambda^{-1} |\mathcal{F}(0, 0, 0, 0, r) - d|
\]

Proof. (i) Let \(m_1 < m_2\). Then for each \((l, p, u) \in \mathbb{R}^3\) and a.a. \(r \in (0, R)\), it follows from (4) and (r-F2) that

\[
\mathcal{F}(m_1, l, p, u, r) - \mathcal{F}(m_2, l, p, u, r) \leq \mathcal{P}^+(m_1 - m_2, 0) = -\lambda (m_2 - m_1) < 0.
\]

Thus for any \((l, p, u) \in \mathbb{R}^3\) and a.a. \(r \in (0, R)\), we see from (8) that the function \(m \mapsto \mathcal{F}(m, l, p, u, r)\) is strictly increasing in \(m\) and \(\lim_{m \to \pm \infty} \mathcal{F}(m, l, p, u, r) = \pm \infty\).

By the intermediate value theorem yields that for all \(d \in \mathbb{R}\) there exists a unique \(g = g_x(l, p, u, d, r) \in \mathbb{R}\) satisfying \(\mathcal{F}(g, l, p, u, r) = d\).

The assertion (ii) holds from the strict monotonicity of \(\mathcal{F}(m, l, p, u, r)\) in \(m\).

Next we show the assertion (iii). Let \((l_1, p_1, u_1, d_1), (l_2, p_2, u_2, d_2) \in \mathbb{R}^4\), \(g_i = g_x(l_i, p_i, u_i, d_i, r)\) and \(g_1 < g_2\). Then it follows from (r-F2) that

\[
d_1 - d_2 \leq \mathcal{P}^+(g_1 - g_2, l_1 - l_2) + \beta(r)|p_1 - p_2| + \gamma(r)|u_1 - u_2|
\\
= \lambda (g_1 - g_2) + \Lambda(N - 1)|l_1 - l_2| + \beta(r)|p_1 - p_2| + \gamma(r)|u_1 - u_2|.
\]

Therefore we obtain \(0 < g_2 - g_1 \leq \lambda^{-1} L(r)(|l_1 - l_2| + |p_1 - p_2| + |u_1 - u_2| + |d_1 - d_2|)\).

This ensures the Lipschitz continuity of \(g_x\). Moreover if \(g = g_x(0, 0, 0, d, r) > 0\), then by (6) we have \(\mathcal{P}^-(g, 0) \leq \mathcal{F}(g, 0, 0, 0, r) - \mathcal{F}(0, 0, 0, 0, r) = d - \mathcal{F}(0, 0, 0, 0, r)\).

Hence \(0 < g \leq \lambda^{-1} d - \mathcal{F}(0, 0, 0, 0, r)\). We can also prove in the case where \(g = g_x(0, 0, 0, d, r) < 0\). \(\square\)

By Lemma 2.2, it is easy to see that \(\mathcal{F}[u](r) = 0\) for a.e. \(r \in (a, b)\) is equivalent to \(u''(r) = g_x(u'(r)/r, u'(r), u(r), 0, r)\) for a.e. \(r \in (a, b)\). Since \(g_x\) satisfies the Lipschitz continuity, by the contraction mapping argument, we can show

**Proposition 2.3.** Under the assumptions (r-F1), (r-F2) and \(\mathcal{F}[0] \in L^q_l(0, R)\), for each \(0 < a < b \leq R\), \(\alpha_1, \alpha_2 \in \mathbb{R}\), \(q \geq 1\), there is a unique solution \(u \in W^{2,q}_l(a, b)\) of \(\mathcal{F}[u](r) = 0\) a.e. in \((a, b)\) with \(u(a) = \alpha_1\) and \(u'(a) = \alpha_2\).

**Remark 2.4.** The similar results to Lemma 2.2 and Proposition 2.3 hold for \(\mathcal{F}_x\).
3 Estimates on radial functions

In this section we establish a priori type estimates on functions in $W_{r}^{2,q}(a, b)$, motivated by the boundary value problem (7) under $(r\text{-}F1)$, $(r\text{-}F2)$ and $\mathcal{F}[0] \in L_{r}^{q}(0, R)$.

Throughout this note we set $\lambda_{*} = \lambda/\Lambda \in (0, 1]$ and $q_{*} = N(1 + \lambda_{*}(N - 1)) = N/(\lambda_{*}N + (1 - \lambda_{*})) < N$.

The following two lemmas play important roles to derive a priori estimates of (7). For a proof, see [13].

**Lemma 3.1.** Let $0 \leq a < b \leq R$, $q \in (q_{*}, \infty]$, $\beta \in L_{r}^{N}(0, R)$ and $f \in L_{r}^{q}(a, b)$.

Let $v$ be a measurable function on $[a, b]$ such that for each $c \in (a, b)$, $v$ is absolutely continuous on $[c, b]$. Assume that $f \geq 0$ a.e. in $(a, b)$, $v/r \in L_{r}^{q}(a, b)$, $v \geq 0$ in $[a, b]$, $v(a) = 0$ if $a > 0$ and

$$v'(r) + \lambda_{*}(N - 1)\frac{v(r)}{r} \leq \lambda^{-1}\beta(r)v(r) + \lambda^{-1}f(r) \quad \text{for a.a. } r \in (a, b).$$

Then there exists a constant $C_{1} > 0$, depending only on $\lambda_{*}$, $q$, $\|\lambda^{-1}\beta\|_{L_{r}^{N}(0, R)}$ and $N$, such that

$$\|v/r\|_{L_{r}^{q}(a, b)} \leq C_{1}\lambda^{-1}\|f\|_{L_{r}^{q}(a, b)}.$$

An important point of the above estimate is that the constant $C$ can be chosen independently of the parameter $a$.

**Lemma 3.2.** Let $q \in (N/2, \infty]$ and $0 \leq a < b \leq R$. Let $u$ be a function on $[a, b]$ such that for each $c \in (a, b]$, the function $u$ is absolutely continuous on $[c, b]$, $u(b) \leq 0$ and $\|(u')_{-}/r\|_{L_{r}^{q}(a, b)} < \infty$. Then there exists a constant $C_{2} > 0$, depending only on $q$ and $N$, such that

$$\sup_{(a, b]} u \leq C_{2}(b^{(2q-N)/(q-1)}-a^{(2q-N)/(q-1)})^{(q-1)/q}\|(u')_{-}/r\|_{L_{r}^{q}(a, b)}.$$

The next lemma concerns the embedding $W_{r}^{2,q}(0, b) \subset C^{1}([0, b])$. Note that if $a > 0$, then $W_{r}^{2,q}(a, b) \subset C^{1}([a, b])$ for any $q \geq 1$. For instance, see Berestycki and Lions [3], Strauss [19].

**Lemma 3.3.** Let $q \geq N$, $0 \leq a < b \leq R$ and $u \in W_{r}^{2,q}(a, b)$. Assume in addition that $u'(a) = 0$ if $a > 0$. Then

$$\|u'/r\|_{L_{r}^{q}(a, b)} \leq R^{1-N/q}q^{1/q}\|u'/r\|_{L_{r}^{q}(a, b)}^{1-q/1}||u''||_{L_{r}^{q}(a, b)}^{1/q}.$$

In particular, $W_{r}^{2,N}(0, b) \subset C^{1}([a, b])$ and $u'(0) = 0$ hold for all $u \in W_{r}^{2,N}(0, b)$.

**Proof.** It is enough to show the above inequality when $u$ is smooth by the density of $C_{r}^{\infty}(A(a, b))$ in $W_{r}^{2,q}(a, b)$. Thus we may assume $u'(a) = 0$.

For any $a \leq r \leq R$, we have

$$|u'(r)|^{q} \leq \int_{a}^{r} q|u'(t)|^{q-1}|u''(t)|dt \leq R^{q-N}q \int_{a}^{r} |u'(t)/t|^{q-1}|u''(t)t^{N-1}|dt \leq R^{q-N}q \|u'/r\|_{L_{r}^{q}(a, b)}^{q-1}||u''||_{L_{r}^{q}(a, b)}.$$
The next lemma is about the estimate of $\|\beta u'|_{L^{q}(a,b)}$.

**Lemma 3.4.** Let $1 < q$, $0 \leq a < b \leq R$ and $u \in W^{2,q}_{r}(a,b)$. Assume that $u'(a) = 0$ if $a > 0$ and $\beta \in L^{N}_{r}(0,R)$. Then there exists a constant $C > 0$, depending only on $q$, $N$ and $R$, such that

$$
\|\beta u'|_{L^{q}_{r}(a,b)} \leq C \max\{\|\beta\|_{L^{q}_{r}(0,R)}, \|\beta\|_{L^{N}_{r}(0,R)}\} \left(\frac{1}{q} \|u'\|_{L^{q}_{r}(a,b)}^{1/q} \|\beta u''\|_{L^{q}_{r}(a,b)}^{1/q} + \frac{1}{q} \|u'\|_{L^{q}_{r}(a,b)}^{1/q}\right).
$$

**Proof.** When $1 < q < N$, see [13]. In the case where $q \geq N$, the claim holds from Lemma 3.3 since $u' \in L^{\infty}(a, b)$.

The following lemma is an Alexandrov-Bakelman-Pucci type inequality.

**Lemma 3.5.** Let $q \in (\max\{N/2, q^{*}\}, \infty]$, $0 \leq a < b \leq R$, $\beta \in L^{q}(0,R) \cap L^{N}_{r}(0,R)$, $u \in W^{2,q}_{r}(a,b)$ and $f \in L^{q}_{r}(a,b)$. Assume that $u(b) = 0$, $u'(a) = 0$ if $a > 0$ and $u$ satisfies

$$
\mathcal{P}^{+}[u](r) + \beta(r)|u'(r)| + f(r) \geq 0 \quad \text{a.e. in } (a, b).
$$

Then there exists a constant $C_{3} > 0$, depending only on $\lambda$, $\Lambda$, $q$, $N$ and $\|\beta\|_{L^{q}(0,R)}$, such that

$$
\max_{[a,b]} u \leq C_{3} \left(b^{(2q-N)/(q-1)} - a^{(2q-N)/(q-1)}\right)^{(q-1)/q} \|f_{+}\|_{L^{q}_{r}(a,b)}.
$$

**Proof.** Fix any $(m, l, d) \in \mathbb{R}^{3}$ such that $\mathcal{P}^{+}(m, l) + d \geq 0$ and $d \geq 0$. Assume that $l \leq 0$. We have $0 \leq \lambda m + \lambda(N-1)l + d$ if $m \leq 0$ and $0 \leq \Lambda m + \lambda(N-1)l + d$ if $m > 0$. Noting $l \leq 0$, we obtain

$$
(10) \quad m + \lambda_{*}(N-1)l + \lambda^{-1}d \geq 0 \quad \text{for any } (m, l, d) \in \mathbb{R}^{3} \text{ with } l \leq 0 \text{ and } d \geq 0.
$$

If we set $v = (u')_{-}$, then we have $v(r) = -u'(r)$ and $v'(r) = -u''(r)$ a.e. if $v(r) > 0$, and $v(r) = 0$ and $v'(r) = 0$ a.e. if $v(r) \leq 0$. Using (10), we get

$$
-v' - \lambda_{*}(N-1)^{-1}v + \lambda^{-1}\beta v + \lambda^{-1}f_{+}(r) \geq 0 \quad \text{a.e. in } (a, b).
$$

By Lemma 3.1, there exists a constant $C_{1} > 0$, depending only on $\lambda_{*}$, $q$, $N$ and $\|\lambda^{-1}\beta\|_{L^{q}_{r}(0,R)}$, such that

$$
\|u'/r\|_{L^{q}_{r}(a,b)} \leq C_{1} \|f_{+}\|_{L^{q}_{r}(a,b)}.
$$

On the other hand, by Lemma 3.2 and $u \in C([a, b])$, there is a $C_{2} > 0$ such that

$$
\max_{[a,b]} u(r) \leq C_{2} \left(b^{(2q-N)/(q-1)} - a^{(2q-N)/(q-1)}\right)^{(q-1)/q} \|u'_{-}/r\|_{L^{q}_{r}(a,b)}.
$$

Combining the above two inequalities, we can show our claim.
Proposition 3.6. Let \(0 \leq a < b \leq R\), \(q \in (\max\{N/2, q_\ast\}, \infty]\), \(\beta \in L_r^N(0, R) \cap L_r^2(0, R)\), \(f^1, f^2 \in L_r^2(a, b)\) and \(u \in W_r^{2,q}(a, b)\). Assume that
\[
\begin{cases}
\mathcal{P}^+[u](r) + \beta|u'| + f^1 \geq 0 \text{ a.e. in } (a, b), \\
\mathcal{P}^-[u](r) - \beta|u'| - f^2 \leq 0 \text{ a.e. in } (a, b), \\
u'(a) = 0 \text{ if } a > 0, \quad \text{and} \quad u(b) = 0.
\end{cases}
\]
Then there exists a constant \(C > 0\), depending only on \(q, \Lambda, N, R, \|\beta\|_{L_r^N(0,R)}\) and \(\|\beta\|_{L_r^2(0,R)}\) such that
\[
\|u\|_{W_r^{2,q}(a,b)} \leq C(\|f^+_1\|_{L_r^2(a,b)} + \|f^+_2\|_{L_r^2(a,b)}).
\]

Proof. First note that by the assumption, we have
\[
\mathcal{P}^-[\lambda^{-1}u](r) + \beta|\lambda^{-1}u'| + f^2(r) \geq 0.
\]
Thus as in the proof of Lemma 3.5, it holds that
\[
\|\lambda^{-1}u'/r\|_{L_r^2(a,b)} \leq C_1 \|\lambda^{-1}f^+_2\|_{L_r^2(a,b)}
\]
where \(C_1\) depends only on \(\lambda_\ast, q, N\) and \(\|\lambda^{-1}\beta\|_{L_r^N(0,R)}\). Hence, setting \(M = \|\lambda^{-1}f^+_1\|_{L_r^2(a,b)} + \|\lambda^{-1}f^+_2\|_{L_r^2(a,b)}\), we have
\[
\|u'/r\|_{L_r^2(a,b)} \leq C_1 M.
\]

Secondly, for each \((m, l, d) \in \mathbb{R}^3\) with \(m \leq 0\) and \(\mathcal{P}^+(m, l) + d \geq 0\), we have
\[
m + \lambda_\ast^{-1}(N-1)|l| + \lambda^{-1}d \geq 0.
\]
Using (12), \(\mathcal{P}^+[u](r) + \beta(r)|u'(r)| + f_1(r) \geq 0\) and \(\mathcal{P}^-[\lambda^{-1}u](r) + \beta(r)|\lambda^{-1}u'| + f_2(r) \geq 0\), we observe that
\[
|u''| \leq \lambda_\ast^{-1}(N-1)\frac{|u'|}{r} + \lambda^{-1}\beta|u'| + \lambda^{-1}(f^+_1 + f^+_2) \quad \text{a.e. in } (a, b).
\]
By Lemma 3.2 and (11), we can choose a constant \(C_2 > 0\), depending only on \(q, R\) and \(N\), for which we have
\[
\|u\|_{L^\infty(a,b)} \leq C_1 C_2 M.
\]
Also, by Lemmas 3.3, 3.4, (11) and Young’s inequality, for each \(\epsilon > 0\), we find a constant \(C_4 > 0\), depending only on \(\epsilon, q, N, R, \|\lambda^{-1}\beta\|_{L_r^N(0,R)}\) and \(\|\lambda^{-1}\beta\|_{L_r^2(0,R)}\), for which we have
\[
\|\lambda^{-1}\beta u'/r\|_{L_r^2(a,b)} \leq \epsilon \|u''\|_{L_r^2(a,b)} + C_1 C_4 M.
\]
Combining this, with \(\epsilon = 1/2\), and (13), we get
\[
\frac{1}{2}\|u''\|_{L_r^2(a,b)} \leq \lambda_\ast^{-1}(N-1)\|u'/r\|_{L_r^2(a,b)} + C_1 C_4 M + \|\lambda^{-1}(f_+ + g_+)|_{L_r^2(a,b)}
\]
\[
\leq (\lambda_\ast^{-1}(N-1)C_1 + C_1 C_4 + 1)M.
\]
This inequality together with (14) and (15) yields an estimate on \(\|u\|_{W_r^{2,q}(a,b)}\) with the desired properties. \(\square\)
Next, for $\kappa \in \mathbb{R}$, we recall the definition of $\mathcal{F}_\kappa$: $\mathcal{F}_\kappa (r) := F(m, l, p, u, r) - \kappa u$. By the definition, we remark that $\mathcal{F}[0](r) = \mathcal{F}_\kappa [0](r)$ holds. Noting (r-F2), if $u(r) - v(r) \geq 0$, then we have

$$\mathcal{F}_\kappa [u](r) - \mathcal{F}_\kappa [v](r) \leq \mathcal{F}[0](r) - \kappa u(r) + (\gamma (r) - \kappa) u(r) - v(r)).$$

Next we define a constant $\sigma_\kappa$ by

$$\sigma_\kappa := C_3 \lambda^{-1} R^{2-N/q} \|\gamma - \kappa\|_{L^q_r(0,R)}.$$

Here $C_3$ appears in Lemma 3.5 and we remark that $\sigma_\kappa \to 0$ as $\kappa \to \infty$.

**Proposition 3.7.** Suppose (r-F1), (r-F2) and $\mathcal{F}[0] \in L^q_r(0,R)$. Assume also that $q \in (\max\{N/2, q_\ast\}, \infty)$, $\sigma_\kappa < 1$, $0 < a < b \leq R$ and $u \in W^{2,q}_r(a, b)$ is a solution of (7). Then there exists a $C$ depending only on $q$, $\lambda$, $N$, $R$, $\|\beta\|_{L^q_r(0,R)}$, $\|\beta\|_{L^q_\infty(0,R)}$, $\|\gamma\|_{L^q_r(0,R)}$, $\kappa$ such that

$$\|u\|_{W^{2,q}_r(a,b)} \leq C \|\mathcal{F}[0]\|_{L^q_r(a,b)}.$$

**Proof.** If $u_+ \not\equiv 0$, then let $r^+ \in (a, b)$ be a maximum point of $u_+$, respectively. Furthermore, let

$$b^+ := \inf\{r \in (r^+, b^-) : u_+(r) = 0\} > r^+.$$

Noting $u \geq 0$ in $[r^+, b^+]$, if follows from (16) that for a.a. $r \in (r^+, b^+)$,

$$0 = \mathcal{F}_\kappa[u](r) = \mathcal{F}_\kappa[u](r) - \mathcal{F}_\kappa[0](r) + \mathcal{F}[0](r)
\leq \mathcal{F}[0](r) + \beta(r)|u'(r)| + (\gamma (r) - \kappa) u(r) + \|\mathcal{F}[0]\|_{L^q_r(r^+, b^+)}.$$

By Lemma 3.5, we have

$$u(r^+) = \max_{r^+ \leq r \leq b^+} u(r) \leq C_3 R^{2-N/q} \|\gamma - \kappa\|_{L^q_r(r^+, b^+)} u + C_3 R^{2-N/q} \|\mathcal{F}[0]\|_{L^q_r(a,b)}.$$

From $\sigma_\kappa < 1$, it holds that

$$\|u_+\|_{L^\infty(a,b)} \leq C_3 R^{2-N/q} (1 - \sigma_\kappa)^{-1} \|\mathcal{F}[0]\|_{L^q_r(a,b)}.$$

Similarly, if $u_- \not\equiv 0$, then we set $u_-(r^-) = \max_{a \leq r \leq b} u_-(r) > 0$, $u_-(b^-) = 0$ and $-u \geq 0$ in $[r^-, b^-]$. Furthermore we can show

$$0 \leq \mathcal{F}_\kappa[-u](r) + \beta(r)|u'(r)| + (\gamma (r) - \kappa) u_-(r) + \|\mathcal{F}[0]\|_{L^q_r(r^-, b^-)}.$$

Repeating the argument in the above, one obtains

$$\|u_-\|_{L^\infty(a,b)} \leq C_3 R^{2-N/q} (1 - \sigma_\kappa)^{-1} \|\mathcal{F}[0]\|_{L^q_r(a,b)}.$$

Thus it holds that

$$\|u\|_{L^\infty(a,b)} \leq C_3 R^{2-N/q} (1 - \sigma_\kappa)^{-1} \|\mathcal{F}[0]\|_{L^q_r(a,b)}.$$
Next, by (r-F2), we have

\[
0 = \mathcal{F}_\kappa[u](r) \leq \mathcal{P}^+[u](r) + \beta(r)|u'(r)| + (\gamma(r) + |\kappa|)|u(r)| + |\mathcal{F}[0](r)| \quad \text{a.e. in } (a,b),
\]

\[
0 \geq \mathcal{P}^-[u](r) - \beta(r)|u'(r)| - (\gamma(r) + |\kappa|)|u(r)| - |\mathcal{F}[0](r)| \quad \text{a.e. in } (a,b).
\]

Therefore, Proposition 3.6 and (18) ensure

\[
\|u\|_{W^{2,q}_r(a,b)} \leq C(\|\gamma + |\kappa|u\|_{L^q_r(a,b)} + \|\mathcal{F}[0]\|_{L^q_r(a,b)}) \leq C(\|u\|_{L^\infty(a,b)} \|\gamma + |\kappa|\|_{L^q_r(a,b)} + \|\mathcal{F}[0]\|_{L^q_r(a,b)}) \leq C\|\mathcal{F}[0]\|_{L^q_r(a,b)}
\]

where \(C\) depends only on \(q\), \(\lambda\), \(\Lambda\), \(N\), \(R\), \(\|\beta\|_{L^N_r(0,R)}\), \(\|\beta\|_{L^q_r(0,R)}\), \(\|\gamma\|_{L^q_r(0,R)}\) and \(\kappa\).

\[\square\]

4 Comparison theorem

In this section, we prove a weak maximum principle and strong maximum principle, respectively. A weak maximum principle for \(\mathcal{F}_\kappa\) is stated as follows.

**Proposition 4.1.** Let \(q \in (\max\{N/2, q^*\}, \infty\], \sigma_\kappa < 1\) appearing in (17), \(0 \leq a < b \leq R\), \(u, v \in W^{2,q}_r(a, b)\) and \(f, g \in L^q_r(a, b)\). Furthermore, suppose that \(u, v, f, g\) satisfy

\[
\mathcal{F}_\kappa[v] + g \leq \mathcal{F}_\kappa[u] + f \quad \text{a.e. in } (a, b)
\]

and \(v'(a) \leq u'(a)\) and \(u(b) \leq v(b)\). Then it follows that

\[
\max_{[a,b]}(u-v) \leq C_3(1-\sigma_\kappa)^{-1}(b^{(2q-N)/(q-1)}-a^{(2q-N)/(q-1)})^{(q-1)/q}\|\!(f-g)_{+}\!\|_{L^q_r(a,b)}.
\]

**Proof.** Set \(w(r) := u(r) - v(r)\). We may assume \(\max_{[a,b]}w(r) > 0\). Let \(r_0 \in [a, b)\) be a maximum point of \(w\). Furthermore, set \(r_1 = \min\{r \in [r_0, b] : w(r) = 0\}\). By the assumptions, \(u'(r_0) = 0\).

On the other hand, it follows from (16) that

\[
0 \leq \mathcal{P}^+[u] + \beta|u'| + (\gamma - \kappa)_{+}w + (f - g)_{+} \quad \text{a.e. in } (r_0, r_1).
\]

Applying Lemma 3.5, we obtain

\[
\max_{[a,b]} w \leq C_3 \left( b^{(2q-N)/(q-1)} - a^{(2q-N)/(q-1)} \right)^{(q-1)/q} \|\!(\gamma - \kappa)_{+}w + (f - g)_{+}\!\|_{L^q_r(r_0,r_1)}
\]

\[
\leq \sigma_\kappa \max_{[a,b]} w + C_3 \left( b^{(2q-N)/(q-1)} - a^{(2q-N)/(q-1)} \right)^{(q-1)/q} \|\!(f - g)_{+}\!\|_{L^q_r(a,b)}.
\]

Since \(\sigma_\kappa < 1\), we have the conclusion. \[\square\]

The next proposition is a version of the strong maximum principle for radial functions.

**Proposition 4.2.** Let \(0 \leq a < b \leq R\), \(q \in (\max\{N/2, q^*\}, \infty\], \(u \in W^{2,q}_r(a, b)\), \(\beta \in L^N_r(a, b)\) and \(\gamma \in L^q_r(a, b)\). Assume that \(u \geq 0\) in \([a, b]\) and

\[
\mathcal{P}^-[u] - \beta|u'| - \gamma u \leq 0 \quad \text{a.e. in } (a,b).
\]

Then either \(u \equiv 0\) in \([a, b]\) or \(u > 0\) in \((a, b)\). Furthermore, \(\max\{u(b), -u'(b)\} > 0\) and \(\max\{u(a), u'(a)\} > 0\) holds if \(a > 0\). When \(a = 0\), \(u(0) > 0\) holds.
Proof. First we show that if $u'(r_0) = 0$ and $u(r_0) = 0$ for some $r_0 \in [a, b]$ with $r_0 > 0$, then $u \equiv 0$ in $[a, b]$. Set $v = (u')_-$ and $w = (u')_+$. Since $u$ satisfies $P^+[u] + \beta|u'| + \gamma u \geq 0$ a.e. in $(a, b)$, we observe that

$$-\hat{\gamma}u - \hat{\beta}v \leq v' \quad \text{and} \quad w' \leq \hat{\beta}w + \hat{\gamma}u \quad \text{a.e. in} \ (a, b)$$

where $\hat{\beta}(r) = \lambda^{-1}(\beta + \Lambda(N - 1)/r)$ and $\hat{\gamma}(r) = \lambda^{-1}\gamma(r)$. Thus by Gronwall's inequality, we have

(19) \( (u')_- (t) \leq \int_{t}^{r_0} \hat{\gamma}(s)u(s)\exp(\int_{t}^{s}\hat{\beta}(\tau)d\tau)ds \) for all $t \in (a, r_0]$.

(20) \( (u')_+ (t) \leq \int_{r_0}^{t} \hat{\gamma}(s)u(s)\exp(\int_{s}^{t}\hat{\beta}(\tau)d\tau)ds \) for all $t \in [r_0, b]$.

We fix $\epsilon \in (a, r_0)$ arbitrarily. Then for each $r \in [\epsilon, r_0]$, it follows from (19) that

$$u(r) - u(r_0) \leq \int_{r}^{r_0} (u')_-(t)dt \leq (r_0 - \epsilon)\exp(\|\hat{\beta}\|_{L^1(\epsilon,r_0)})\int_{\epsilon}^{r_0} \hat{\gamma}(s)u(s)ds.$$  

Using Gronwall's inequality again, we get $u \equiv 0$ in $[\epsilon, r_0]$. Since $\epsilon > 0$ is arbitrary, $u \equiv 0$ in $[a, r_0]$. Similarly $u \equiv 0$ in $[r_0, b]$ holds from (20). Hence $u \equiv 0$ in $[a, b]$. Moreover, by the above arguments, we see that if $u \not\equiv 0$, then $\max\{u(b), -u'(b)\} > 0$. Furthermore, $\max\{u(a), u'(a)\} > 0$ holds if $a > 0$.

Next we treat the case where $a = 0$. In this case, it is enough to show that $u \equiv 0$ provided $u(0) = 0$. We choose $a > 0$ so small that $C_1C_2a^{2-N/q}\|\gamma\|_{L^q(0,a)} < 1$ where $C_1$ and $C_2$ appear in Lemmas 3.1 and 3.2.

As in the above, if we set $v = (u')_+$, then we have

$$v' + \lambda_+(N - 1)\frac{u}{r} \leq \lambda^{-1}(\beta v + \gamma u) \quad \text{a.e. in} \ (0, b).$$

By Lemma 3.1, we get

$$\|(u')_+/r\|_{L^q(0,a)} \leq C_1\|\gamma u\|_{L^q(0,a)} \leq C_1\|\gamma\|_{L^q(0,a)} \sup_{0\leq r \leq a} u$$

where $C_1 > 0$ is a constant independent of $a$. Applying Lemma 3.2 to the function $r \mapsto u(c) - u(r)$, with $0 < c \leq a$, we get

$$\max_{0 \leq r \leq c} (u(c) - u(r)) \leq C_2c^{(2q-N)/q}\|(u')_+/r\|_{L^q(0,c)},$$

where $C_2 > 0$ is a constant independent of $c$ and $a$. In particular, since $u(0) = 0$, we have

$$\max_{0 \leq r \leq a} u(c) \leq C_2a^{(2q-N)/q}\|(u')_+/r\|_{L^q(0,a)}.$$  

Thus, we get

$$\max_{0\leq r \leq a} u \leq C_1C_2a^{(2q-N)/q}\|\gamma\|_{L^q(0,a)} \sup_{0\leq r \leq a} u.$$  

Since $C_1C_2a^{(2q-N)/q}\|\gamma\|_{L^q(0,a)} < 1$, we find $\max_{[0,a]} u = 0$, which implies $u \equiv 0$ in $[0,a]$. Using the previous argument, we can conclude $u \equiv 0$ in $[0,b]$. \qed
5 Solvability of (7)

This section is devoted to proving that (7) has a unique solution in $W^{2,q}_{r}(a,b)$ under (r-F1), (r-F2) and $\mathcal{F}[0] \in L^{2}_{r}(0,R)$.

Theorem 5.1. Assume $\mathcal{F}$ satisfies (r-F1), (r-F2) and $\mathcal{F}[0] \in L^{2}_{r}(0,R)$. Let $\sigma_{\kappa} < 1$ and $q \in [\max\{2/N, q_{*}\}, \infty]$. Then for each $0 \leq a < b \leq R$, the equation (7) has a unique solution $u$ and $u$ satisfies

$$\|u\|_{W^{2,q}_{r}(a,b)} \leq C\|\mathcal{F}_{\kappa}[0]\|_{L^{2}_{r}(0,R)}$$

where $C$ depends only on $q$, $N$, $\lambda$, $\Lambda$, $R$, $\kappa$, $\|\beta\|_{L^{N}_{r}(0,R)}$, $\|\beta\|_{L^{q}_{r}(0,R)}$ and $\|\gamma\|_{L^{q}_{r}(0,R)}$.

To prove Theorem 5.1, we prepare the next lemma concerning a supersolution to $\mathcal{P}^{+}$.

Lemma 5.2. Let $0 < a < b \leq R$, $q \in (\max\{2/N, q_{*}\}, \infty]$ and $f \in L^{q}_{r}(a, b)$. Then there exists a $\phi \in W^{2,q}_{r}(a, b)$ such that $\phi \geq 0$ in $[a, b]$ and

$$\mathcal{P}^{+}[\phi] + \beta|\phi'| + \gamma\phi + |f| \leq 0 \text{ a.e. in } (a, b), \quad \phi(b) = 0, \quad \phi'(r) < 0.$$

Proof. Let $\eta > 0$ and define

$$\phi(r) := \int_{r}^{b} e^{A(t)} dt \quad \text{where} \quad A(t) := \int_{a}^{t} \eta(\beta(s) + \gamma(s) + |f(s)|) ds.$$

Then it is easy to see

$$\phi(b) = 0, \quad \phi'(r) = -e^{A(r)} < 0, \quad \phi(r) \leq (b-a)e^{A(b)},$$

$$\phi''(r) = -\eta(\beta(r) + \gamma(r) + |f(r)|)e^{A(r)}.$$

Thus $\phi \in W^{2,q}_{r}(a, b)$ and it holds that

$$\mathcal{P}^{+}[\phi](r) + \beta|\phi'(r)| + \gamma\phi + |f(r)| \leq (1 - \eta\lambda)\beta(r)e^{A(r)} + ((b-a)e^{A(b)} - \eta\lambda)\gamma(r) + (1 - \eta\lambda)|f(r)|.$$

Hence, taking $\eta > 0$ sufficiently large, we obtain $\mathcal{P}^{+}[\phi] + \beta|\phi'| + \gamma\phi + |f| \leq 0 \text{ a.e. in } (a, b)$, which completes the proof.

Proof of Theorem 5.1. The uniqueness follows from Proposition 4.1. Furthermore, the estimates for $u$ also hold from Proposition 3.7. So it is sufficient to show the existence.

First we assume $a > 0$. Let $\phi$ be the function appearing in Lemma 5.2 with $f(r) = |\mathcal{F}[0]|(r)$ and set $v^{\pm}(r) := \pm \phi(r)$. Then we see that $\mathcal{F}_{\kappa}[v^+] \leq 0 \leq \mathcal{F}_{\kappa}[v^-]$ a.e. in $(a, b)$, $v^{-}(a) < 0 < v^{+}(a)$ and $(v^+)'(a) < 0 < (v^-)'(a)$.

For any $d \in \mathbb{R}$, we denote by $u(r : d)$ the unique solution of $\mathcal{F}_{\kappa}[u] = 0 \text{ a.e. in } (a, b)$ with $u(a : d) = d$ and $u'(a : d) = 0$ where $u'$ stands for $\partial u/\partial r$. Such a solution exists from Remark 2.4. Next we shall prove the following claim:

$$v^{+}(r) < u(r : d) \quad (\text{resp. } u(r : d) < v^{-}(r)) \text{ in } [a, b] \quad \text{if } d > v^{+}(a) \quad (\text{resp. } d < v^{-}(a)).$$
First we suppose $d > v^+(a)$. Then we can take a neighborhood $U$ of $a$ such that $u(r : d) > v^+(r)$ for all $r \in U$. Next set $r_0 := \inf\{r \in (a, b) : u(r : d) = v(r)\}$. We argue by contradiction and assume $r_0 \in (a, b]$. Since $\mathcal{F}_\kappa[u] = 0 \geq \mathcal{F}_\kappa[v^+]$ a.e. in $(a, r_0)$ and $v'(a) < 0 = u'(a)$, $v(r_0) = u(r_0)$, it follows from Proposition 4.1 that $u - v \leq 0$ in $[a, r_0]$, which is a contradiction. Thus $v^+(r) < u(r : d)$ in $[a, b]$ if $d > v^+(a)$. For the other claim, one can prove similarly.

Noting that the function $d \mapsto u(b : d)$ is continuous, we can choose a $d_0 \in [v^-(a), v^+(a)]$ such that $u(a : d_0) = 0$. Thus the existence result holds in the case where $a > 0$.

Next we consider the case where $a = 0$. Let $(u_k) \subset W^2_t(1/k, b)$ be a solution of (7) in $(1/k, b)$. Furthermore, we extend $u_k$ by

$$v_k(r) := \begin{cases} u_k(r) & \text{if } 1/k \leq r \leq b, \\ u_k(1/k) & \text{if } 0 \leq r < 1/k. \end{cases}$$

Then $v_k \in W^2_t(0, b)$ since $v_k'(1/k) = 0$. Moreover, by Proposition 3.7 and Lemma 3.2, $(u_k)$ is bounded in $W^2_t(0, b)$.

Now suppose $q \neq \infty$. Taking a subsequence if necessary, we may assume $u_{k_\ell} \to v_0$ weakly in $W^2_t(0, b)$. Note also that $v_{k_\ell} \to v_0$ strongly in $C^1([\epsilon, b])$ for each $\epsilon \in (0, b)$. Let $0 < s < t$ and $1/k_\ell \leq s$. Then the from the property of $g_{\mathcal{F}_\kappa}$, we have

$$v_{k_\ell}'(t) - v_{k_\ell}'(s) = \int_s^t g_{\mathcal{F}_\kappa}(v_{k_\ell}'(\tau)/\tau, v_{k_\ell}'(\tau), v_{k_\ell}(\tau), 0, \tau)d\tau.$$

Let $k_\ell \to \infty$, then we observe from Lemma 2.2 that

$$v_0'(t) - v_0'(s) = \int_s^t g_{\mathcal{F}_\kappa}(v_0'(\tau)/\tau, v_0'(\tau), v_0(\tau), 0, \tau)d\tau$$

for every $0 < s < t \leq b$. This means

$$v_0''(r) = g_{\mathcal{F}_\kappa}(v_0'(r)/r, v_0'(r), v_0(r), 0, r) \quad \text{a.a. } r \in (0, b).$$

Therefore, $v_0$ is a solution of (7).

In the case where $q = \infty$, then for any $p < \infty$, $(v_k)$ is bounded in $W^2_t(0, b)$. Thus we may assume $u_{k_\ell} \to v_0$ weakly in $W^2_t(0, b)$. Then as in the above, we can show $v_0$ is a solution of (7). Moreover, since $\|v_0\|_{W^2_t(0, b)} \leq C_\kappa \sup_{k \geq 1} \|v_k\|_{W^2_t(0, b)}$ holds for all $p \in (N, \infty)$, we have $v_0 \in W^2_t(0, b)$. Thus we complete the proof.

\section{Existence of Principal Eigenpairs}

In this section, we prove the existence of principal eigenpairs for (3).

\textbf{Theorem 6.1.} Let $\mathcal{F}$ satisfy (r-F1)–(r-F3), $q \in (\max\{N/2, q_*\}, \infty)$ and $0 \leq a < b \leq R$. Then there exist pairs $(\varphi^+_{N_1}, \varphi^+_{N_2}) \in \mathbb{R} \times W^2_t(a, b)$ satisfying $\mathcal{F}[\varphi^+_{N_1}] + \mu^+_{N_1} \varphi^+_{N_2} = 0$ a.e. in $(a, b)$, $\pm \varphi^+_{N_1} > 0$ in $(a, b)$, $\varphi^+_{N_2}(b) = 0$ and $(\varphi^+_{N_1})'(a) = 0$ if $a > 0$. 

First we fix a $\kappa \in \mathbb{R}$ so that
\[ \sigma_{\kappa} = C_{3}\lambda^{-1}R^{2-N/q} \| (\gamma - \kappa)_{+} \|_{L_{r}^{q}(0, R)} < 1. \]

Next, for every $f \in L_{r}^{q}(a, b)$, we consider
\[ (21) \quad \mathcal{F}_{\kappa}[u] + f = 0 \quad \text{a.e. in (a, b), u(b) = 0, u} \in W_{r}^{2,q}(a, b), u'(a) = 0 \text{ if } a > 0. \]

Put $\mathcal{F}(m, l, p, u, r) := \mathcal{F}_{\kappa}(m, l, p, u, r) + f(r)$. Then it is easy to see that $\mathcal{F}$ satisfies (r-F1), (r-F2) and $\mathcal{F}[0] \in L_{r}^{q}(a, b)$. Hence according to Theorem 5.1, there is a unique solution $u \in W_{r}^{2,q}(a, b)$ to (21). We introduce the solution mapping $T_{N}$: $L_{r}^{q}(a, b) \to W_{r}^{2,q}(a, b)$ by $T_{N}f(r) := u(r)$. Noting $\mathcal{F}[0] = f$, $T_{N}$ satisfies
\[ (22) \quad \| T_{N}f \|_{W_{r}^{2,q}(a, b)} \leq C\| f \|_{L_{r}^{q}(a,b)} \]
for every $f \in L_{r}^{q}(a, b)$.

**Lemma 6.2.** The following hold:
(i) If $f \geq 0$ a.e. (a, b), then $(T_{N}f) \geq 0$ in [a, b]. Furthermore, if $f \not\equiv 0$, then $T_{N}f > 0$ in [a, b], $(T_{N}f)'(b) < 0$.
(ii) Let $f_{k} \to f_{0}$ strongly in $L_{r}^{q}(a, b)$. Then $T_{N}f_{k} \to T_{N}f_{0}$ strongly in $W_{r}^{2,q}(a, b)$.

**Proof.** (i) Set $u(r) = T_{N}f$. Since $f$ is nonnegative, $\mathcal{F}_{\kappa}[u] + f = 0 \leq \mathcal{F}_{\kappa}[0] + f$ in (a, b). Thus by Proposition 4.1, we have $0 \leq u$ in [a, b]. Furthermore, if $f \not\equiv 0$, then $u$ satisfies $\mathcal{P}^{-}[u] - \beta |u'| - (\gamma + |\kappa|)u \leq 0 \text{ a.e. in (a, b).}$ Thus Proposition 4.2 shows $u > 0$ in [a, b] and $u'(b) < 0$.

(ii) Next let $f_{k} \to f_{0}$ strongly in $L_{r}^{q}(a, b)$ and set $u_{k}(r) := (T_{N}f_{k})(r)$. For each $k, \ell \in \mathbb{N}$, we obtain, $u_{k}''(a) = u_{\ell}''(a) = 0$ if $a > 0$, $u_{k}(b) = u_{\ell}(b) = 0$ and
\[ 0 = \mathcal{F}_{\kappa}[u_{k}] + f_{k} - \mathcal{F}_{\kappa}[u_{\ell}] - f_{\ell} \]
\[ \leq \mathcal{P}^{+}[u_{k} - u_{\ell}] + \beta |u_{k}' - u_{\ell}'| + (\gamma + |\kappa|)|u_{k} - u_{\ell}| + |f_{k} - f_{\ell}|, \]
\[ 0 \geq \mathcal{P}^{-}[u_{k} - u_{\ell}] - \beta |u_{k}' - u_{\ell}'| - (\gamma + |\kappa|)|u_{k} - u_{\ell}| - |f_{k} - f_{\ell}| \quad \text{a.e. in (a, b).} \]

We apply Proposition 3.6 to get
\[ (23) \quad \| u_{k} - u_{\ell} \|_{W_{r}^{2,q}(a, b)} \leq C(\| (\gamma + \kappa) \|_{L_{r}^{q}(a, b)} \| u_{k} - u_{\ell} \|_{L^{\infty}(a, b)} + \| f_{k} - f_{\ell} \|_{L_{r}^{q}(a,b)}). \]

It follows from (22) that $(u_{k})$ is bounded in $W_{r}^{2,q}(a, b)$. Taking a subsequence, we may assume $u_{k_{j}} \rightharpoonup u$ weakly in $W_{r}^{2,q}(a, b)$ and strongly in $L^{\infty}(a, b)$. Hence, by (23), $u_{k_{j}} \rightarrow u$ strongly in $W_{r}^{2,q}(a, b)$.

Next we show $u$ solves $\mathcal{F}_{\kappa}[u] + f_{0} = 0$ in (a, b). If we showed this claim, then by the uniqueness, $u = u_{0}$ holds. Thus the uniqueness of the weak limit implies $u_{k} \rightharpoonup u_{0}$ weakly in $W_{r}^{2,q}(a, b)$. Therefore $u_{k} \rightarrow u_{0}$ strongly in $W_{r}^{2,q}(a, b)$ from (23).

Since $\mathcal{F}_{\kappa}[u_{k}] + f_{k} = 0$ in (a, b), we have
\[ u_{k_{j}}''(r) = g_{\kappa}(u_{k_{j}}'(r) / r, u_{k_{j}}(r), u_{k_{j}}(r), f_{k_{j}}(r), r). \]
Thus for every $a < s < t < b$, it holds

$$u'_{k_j}(t) - u'_{k_j}(s) = \int_{s}^{t} g_{\mathcal{F}_{\kappa}}(u'_{k_j}(\tau)/\tau, u'_{k_j}(\tau), u_{k_j}(\tau), f_{k_j}(\tau), \tau) d\tau.$$ 

Noting that $u_{k_j} \to u$ strongly in $C^{1}_{loc}(a, b)$, from Lemma 2.2 and Lebesgue’s dominated convergence theorem, we obtain

$$u'(t) - u'(s) = \int_{s}^{t} g_{\mathcal{F}_{\kappa}}(u(\tau)/\tau, u'(\tau), u(\tau), f_{0}(\tau), \tau) d\tau$$

for each $a < s < t < b$. This means $u''(r) = g_{\mathcal{F}_{\kappa}}(u'(r)/r, u'(r), u(r), f_{0}(r), r)$, so does $\mathcal{F}_{\kappa}[u] + f_{0} = 0$. \qed

Define $X_{N} \subset W_{r}^{2,q}(a, b)$ by

$$X_{N} := \{ f \in W_{r}^{2,q}(a, b) : f > 0 \text{ in } [a, b), f(b) = 0, f'(b) < 0 \}.$$ 

We equip $W_{r}^{2,q}(a, b)$ norm into $X_{N}$. Then, in view of Lemma 6.2, we see that $T_{N}f \in X_{N}$ if $f \in X_{N}$ and $T_{N} : X_{N} \to X_{N}$ is continuous.

Next for each $f \in X_{N}$, we define $R_{N}$ by

$$R_{N}f(r) := \begin{cases} 
\frac{T_{N}f(r)}{f(r)} & \text{if } r \in [a, b), \\
\frac{(T_{N}f)'(b)}{f'(b)} & \text{if } r = b,
\end{cases}$$

It follows from (r-F3) that for any $t \geq 0$ and $f \in X_{N}$, 

(24)

$$R_{N}(tf)(r) = R_{N}f(r).$$

**Lemma 6.3.** The following hold:

(i) If $f \in X_{N}$, then $R_{N}f \in C([a, b])$ and $0 < \min_{[a, b]} R_{N}f \leq \max_{[a, b]} R_{N}f < \infty$.

(ii) The map $R_{N} : X_{N} \to C([a, b])$ is continuous.

**Proof.** Noting L'Hôpital's rule, it is easy to see that the assertion (i) holds. We turn to the assertion (ii). Let $f_{n}, f_{0} \in X_{N}$ satisfy $f_{n} \to f_{0}$ strongly in $W_{r}^{2,q}(a, b)$. By Lemma 6.2, $T_{N}f_{n} \to T_{N}f_{0}$ strongly in $W_{r}^{2,q}(a, b)$. In particular, we have $f_{n} \to f_{0}$ and $T_{N}f_{n} \to T_{N}f_{0}$ strongly in $C^{1}_{loc}([a, b])$ and $C([a, b])$. Since $f_{0}(0) > 0$, $R_{N}f_{n} \to R_{N}f_{0}$ uniformly in $[a, a + \delta]$ for some $\delta > 0$.

On the other hand, we see that

$$R_{N}f_{n}(r) = \left( \int_{a+\delta}^{r} (T_{N}f_{n})'(s) ds + T_{N}f_{n}(a + \delta) \right) / \left( \int_{a+\delta}^{r} f_{n}'(s) ds + f_{n}(a + \delta) \right),$$

$$R_{N}f_{0}(r) = \left( \int_{a+\delta}^{r} (Tf_{0})'(s) ds + T_{N}f_{0}(a + \delta) \right) / \left( \int_{a+\delta}^{r} f_{0}'(s) ds + f_{0}(a + \delta) \right).$$

From these expressions, $R_{N}f_{n} \to R_{N}f_{0}$ uniformly in $[a + \delta, b]$. Thus we complete the proof. \qed
Lemma 6.4. Let \( f \in X_N \) and \( u = T_Nf \). Then
\[
\min_{[a,b]} R_N f \leq \min_{[a,b]} R_N u \leq \max_{[a,b]} R_N u \leq \max_{[a,b]} R_N f.
\]
Moreover, if \( \min_{[a,b]} R_N f = \min_{[a,b]} R_N u \), then
\[
T_N u = \left( \min_{[a,b]} R_N f \right) u \quad \text{in} \ [a,b].
\]

Proof. Set \( v := T_N u \) and \( \theta = \min_{[a,b]} R_N f \). Since \( \theta f \leq u \) in \([a,b]\), it follows from (r-F3) that \( I_N[v] + \theta f \leq 0 = I_N[\theta u] + \theta f \) in \((a,b)\). Thus Proposition 4.1 yields \( \theta u(r) \leq v(r) \) for all \( r \in [a,b] \), which implies \( \min_{[a,b]} R_N u = \theta \leq \min_{[a,b]} R_N f \). In a similar way, one can show \( \max_{[a,b]} R_N u \leq \max_{[a,b]} R_N f \).

Next we suppose \( \theta = \min_{[a,b]} R_N f = \min_{[a,b]} R_N u \). Setting \( v := T_N u \), then we have \( \theta u \leq v \).

On the other hand, by (r-F2) and \( \theta f \leq u \) in \([a,b]\), we can prove
\[
0 = I_N[v] + u - I_N[\theta u] - \theta f \geq \mathcal{P}^{-}[w] - \beta|w| - (\gamma + |\kappa|)w \quad \text{in} \ (a,b)
\]
where \( w(r) := v(r) - \theta u(r) \geq 0 \). Thus by Proposition 4.2, it holds either \( w \equiv 0 \) in \([a,b]\) or \( w(r) > 0 \) for any \( r \in [a,b] \) and \( w'(r) < 0 \). If the latter case happens, then we obtain \( \theta < \min_{[a,b]} R_N u \). This is a contradiction, hence \( v \equiv \theta u \) holds. \( \square \)

Proof of Theorem 6.1. First we remark that it is sufficient to prove for \((\mu_N^+, \varphi_N^+)\). Indeed, let \( \mu = (m, l, p, u, r) := -I(m, -l, -p, -u, r) \). Then \( \mu \) satisfies (r-F1)–(r-F3) if and only if \( \mathcal{F} \) satisfies (r-F1)–(r-F3). Furthermore, let \((\nu^+, \psi^+) \in \mathbb{R} \times W_{r}^{2,q}(a, b)\) satisfy \( \mathcal{G}[\psi^+] + \nu^+ \psi^+ = 0 \) in \((a,b)\) with \( \psi(b) = 0 \) and \( \psi'(a) = 0 \) if \( a > 0 \). Then it is easily seen that \((\nu^+, -\psi^+)\) is a negative eigenpair of \( \mathcal{F} \). Therefore, it is enough to show for \((\mu_N^+, \varphi_N^+)\).

Now we prove the existence of \((\mu_N^+, \varphi_N^+)\). Let \( f_0 \in X_N \) satisfy \( \|f\|_{L^\infty(a,b)} = 1 \) and define \( u_n \) and \( f_n \) as follows:
\[
 u_n(r) := T_N f_{n-1}(r) \quad \text{and} \quad f_n(r) := u_n(r) / \|u_n\|_{L^\infty(a,b)}.
\]
Set also \( \theta_n := \min_{[a,b]} R_N u_n \) and \( \Theta_n := \max_{[a,b]} R_N u_n \). First, note that \( (u_n) \) is bounded in \( W_r^{2,q}(a, b) \) from (22). Second, by Lemma 6.4, we have \( 0 < \theta_n \leq \theta_{n+1} \leq \Theta_{n+1} \leq \Theta_n \). So we assume \( \theta_n \to \theta > 0 \). Furthermore, noting \( R_N u_n = R_N f_n \) by (24), it holds that
\[
\theta_n f_n(r) \leq u_{n+1}(r) \leq \Theta_n f_n(r) \quad \text{for all} \ r \in [a,b],
\]
which implies \( \theta_n \leq \|u_n\|_{L^\infty(a,b)} \leq \Theta_n \).

Now we assume \( q < \infty \). Taking a subsequence if necessary, we may suppose that there exists a \( u \in W_r^{2,q}(a, b) \) such that \( u_{n_k} \rightharpoonup u \) weakly in \( W_r^{2,q}(a, b) \). Furthermore, \( \theta \leq \|u\|_{L^\infty(a,b)} \) holds, which implies \( f_{n_k} = u_{n_k} / \|u_{n_k}\|_{L^\infty(a,b)} \to u / \|u\|_{L^\infty(a,b)} \) strongly in \( L^\infty(a,b) \). Thus \( u_{n_k+1} = T_N f_{n_k} \to T_N u / \|u\|_{L^\infty(a,b)} =: v \) strongly in \( W_r^{2,q}(a, b) \) from Lemma 6.2. By Lemma 6.3, we obtain
\[
\min_{[a,b]} R_N u = \lim_{k \to \infty} \min_{[a,b]} R_N u_{n_k} = \lim_{n_k \to \infty} \theta_{n_k+1} = \theta.
\]
Since $R_N(T_N u_{n_k+1}) = R_N(T_N f_{n_k+1}) = R_N u_{n_k+2}$ holds, we also have
\[ \min_{[a,b]} R_N(T_N v) = \lim_{n_k \to \infty} \min_{[a,b]} R_N u_{n_k+2} = \theta. \]

Hence, by Lemma 6.4, one can show $T_N v \equiv \theta v$ in $[a,b]$, which implies that $(\mu^+, \varphi^+) = (\theta^{-1} + \kappa, \nu)$ is a positive eigenpair of (3).

When $q = \infty$, from the boundedness of $(u_n)$ in $W^{2,\infty}_r(a,b)$, there exist a subsequence $(u_{n_k})$ and $u$ such that $u_{n_k} \rightharpoonup u$ weakly in $W^{2,m}_r(a,b)$ for each $m \in \mathbb{N}$ with $m \geq N$. We remark that $T_N$ and $R_N$ depend on $q$ and to stress it, here we write $R_{N,q}$ and $T_{N,q}$. If $f \in W^{2,q_1}_r(a,b) \cap W^{2,q_2}_r(a,b)$ with $q_1 < q_2$, then we can prove $T_{N,q} f = T_{N,q} f$. Thus repeating the above argument, the pair $(\theta^{-1} + \kappa, T_N u/\|u\|_{L^\infty(a,b)})$ is a positive eigenpair in $\mathbb{R} \times W^{2,m}_r(a,b)$ for every $m \geq N$. Moreover, since $\|u\|_{W^{2,-m}_r(a,b)} \leq C \sup_{n \geq 1} \|u_n\|_{W^{2,\infty}_r(a,b)}$ for all $m \geq N$, we have $u \in W^{2,\infty}_r(a,b)$, which completes the proof.

Next, we prove the simplicity of the principal eigenpairs.

**Proposition 6.5.** Let $0 < b \leq R$, $(\mu, \varphi) \in W^{2,q}_r(0,b)$ satisfy $\mathcal{F}[\varphi] + \mu \varphi = 0$ a.e. in $(0,b)$, $\varphi \geq 0$, $\varphi \equiv 0$ and $\varphi(b) = 0$. Then there exists a $\theta > 0$ such that $(\mu, \varphi) = (\mu^+_N, \theta \varphi^+_N)$ holds. Similarly, the simplicity of $(\mu^-_N, \varphi^-_N)$ also holds.

**Proof.** First we remark that for any $\kappa \in \mathbb{R}$, $(\mu, \varphi)$ satisfies $\mathcal{F}^\kappa[\varphi] + (\kappa + \mu) \varphi = 0$ a.e. in $(0,b)$. Furthermore, taking $\kappa > 0$ sufficiently large, we may assume $\kappa + \mu > 0$, $\kappa + \mu^+_N > 0$ and $\sigma_\kappa < 1$ defined in (17). Since $\varphi \equiv 0$, it follows from Lemma 6.2 that $\varphi \equiv 0$ in $[0,b)$ and $\varphi'(b) < 0$.

Now we assume $\mu^-_N \leq \mu$ and set $\theta := \inf_{[0,b)} \varphi/\varphi^+$. Noting $\theta \varphi^+ \leq \varphi$ in $[0,b)$ and (r-F3), we obtain
\[ \mathcal{F}^\kappa[\varphi] = - (\kappa + \mu) \varphi \leq - (\kappa + \mu^+_N) \theta \varphi_N = \mathcal{F}^\kappa[\theta \varphi^+_N] \quad \text{a.e. in } (0,b). \]

Thus
\[ \mathcal{P}^- [w] - \beta |w'| - (\gamma + \kappa) w \leq 0 \quad \text{a.e. in } (0,b) \]
where $w := \varphi - \theta \varphi^+_N$. By Proposition 4.2, we see either $w \equiv 0$ in $[0,b)$ or $w > 0$ in $[0,b)$ and $w'(b) < 0$ holds. If the latter case happens, then $\theta < \inf_{[0,b)} \varphi/\varphi^+_N$ holds, which is a contradiction. Thus $\varphi \equiv \theta \varphi^+_N$ and $\mu = \mu^+_N$ hold.

In the case where $\mu < \mu^+_N$, exchanging the role of $\varphi$ and $\varphi^+_N$ in the above, we get the same conclusion. For the negative eigenpair, it is reduced to the positive case by using the function $G(m, l, p, u, r) = - \mathcal{F}(-m, -l, -p, -u, r)$.

By Proposition 6.5, the positive and negative eigenvalue of $\mathcal{F}$ in $[0,b)$ are unique for each $b \in (0, R]$. Thus we denote them by $\mu^+_N(0,b)$ and $\mu^-_N(0,b)$, respectively.

**Proposition 6.6.** Let $0 < b_1 < b_2 \leq R$. Then $\mu^+_N(0,b_2) < \mu^+_N(0,b_1)$ holds. Furthermore, the functions $b \mapsto \mu^+_N(0,b)$ are continuous in $[0,R]$ and $\mu^+_N(0,b) \to \infty$ as $b \to 0$. 

Proof. We only show for $\mu_N^+(0,b)$. Now we argue by contradiction. Suppose $\mu_2 := 
abla N(0,b_2) \leq \mu_N^+(0,b_1) := \mu_1$ and denote the corresponding eigenfunctions by $\varphi_1$ and $\varphi_2$, respectively. Put $\theta := \inf_{[0,b_1]} \varphi_2/\varphi_1$. Then $\theta \varphi_1 \leq \varphi_2$ in $[0,b_1]$. Thus as in the above,

$$P^-[w] - \beta |w'| - (\kappa + \gamma)w \leq 0 \quad \text{a.e. in } (0,b_1)$$

and $w(b) > 0$ where $w := \varphi_2 - \theta \varphi_1$. So Proposition 4.2 tells us that $w > 0$ in $[0,b_1]$, which contradicts to the definition of $\theta$. Thus we get $\mu_1 > \mu_2$.

Next, we show the continuity of $\mu_N^+$. Let $b_n \to b_0 > 0$, $\mu_n := \mu_N^+(0,b_n)$ and $\varphi_n$ be a corresponding positive eigenfunction with $\|\varphi_n\|_{L^\infty([0,b_n])} = 1$. Furthermore, by extending $\varphi_n$ into $[b_n,R]$ appropriately, we suppose $\varphi_n \in W^{2,q}_t(0,R)$. We may also assume $b_0/2 \leq b_n \leq R$ without loss of generality.

By the monotonicity of $\mu_N^+$, we have $\mu_N^+(0,R) \leq b_n \leq \mu_N^+(0,b_0/2)$. So it follows from (22) that $(\varphi_n)$ is bounded in $W^{2,q}_t(0,R)$. Thus in the case where $q < \infty$, taking a subsequence if necessary, we may suppose $\varphi_{n_k} \to \varphi_0$ weakly in $W^{2,q}_t(0,R)$ and $\mu_{n_k} \to \mu_0$. As in the proof of Proposition 6.1, one can show that $(\mu_0,\varphi_0)$ is an eigenpair with $\|\varphi_0\|_{L^\infty([0,b_0])} = 1$ and $\varphi_0 > 0$ in $[0,b_0]$. Thus it holds from the simplicity of the positive eigenvalue, $\mu_0 = \mu_N^+(0,b_0)$ holds. Therefore the uniqueness of the limit implies $\mu_n \to \mu_N^+(0,b_0)$. The case $q = \infty$ can also be treated similarly.

Lastly, we show $\mu_N^+(0,b) \to \infty$ as $b \to 0$. Let $(\mu,b,\varphi)$ be a positive eigenpair with $\|\varphi\|_{L^\infty([0,b])} = 1$. Then we have $P^{+}[\varphi] + \beta|\varphi'| + (\gamma + |\mu_b|)\varphi \geq 0$ a.e. in $(0,b)$. Using Lemma 3.5, we obtain

$$1 = \max_{[0,b]} \varphi_b \leq C_3 b^{(2q-N)/q}\|\gamma + |\mu_b|\|_{L_t^q(0,b)} \leq C_3 b^{(2q-N)/q}\|\gamma + |\mu_0|\|_{L_t^q(0,b)}.$$ 

The above inequality shows $|\mu_b| \to \infty$ as $b \to 0$. Furthermore, it follows the monotonicity of $\mu_b$ that $\mu_b \to \infty$ as $b \to 0$. \hfill \square

7 Existence of general Eigenpairs

In this section, we shall prove Theorem 1.2. First we prove the existence and simplicity of general eigenpairs.

Theorem 7.1. Assume $N \geq 2$, $q \in (\max\{N/2,q_*\}, \infty]$, (r-F1)--(r-F3) with $\Lambda < \infty$ and $\beta \in L_t^N(0,R)$ if $q < N$.

(i) For each $n \in \mathbb{N}$, there exist eigenpairs $(\mu_n^\pm, \varphi_n^\pm) \in \mathbb{R} \times W^{2,q}_t(0,R)$ of (3) and sequences $(r_{n,j}^\pm)_{j=0}^n \subset [0,R]$ such that

$$0 = r_{n,0}^+ < r_{n,1}^+ < \cdots < r_{n,n}^+ = R, \quad 0 = r_{n,0}^- < r_{n,1}^- < \cdots < r_{n,n}^- = R,$$

$$(-1)^{j-1} \varphi_n^+(r) > 0 \text{ in } (r_{n,j-1}^+, r_{n,j}^+) \text{ for } j = 1, \ldots, n,$$

$$(-1)^{j-1} \varphi_n^-(r) > 0 \text{ in } (r_{n,j-1}^-, r_{n,j}^-) \text{ for } j = 1, \ldots, n,$$

$$\varphi_n^+(0) > 0 > \varphi_n^-(0).$$

(ii) Let $(\mu,\varphi) \in \mathbb{R} \times W^{2,q}_t(0,R)$ be an eigenpair of (3) and have $n-1$ zeroes $(t_j)_{j=1}^{n-1}$ in $(0,R)$. Then there exists a $\theta > 0$ such that either $(\mu,\varphi) = (\mu_n^+, \theta \varphi)$ or $(\mu,\varphi) = (\mu_n^-, \theta \varphi)$ holds.
To prove Theorem 7.1, we introduce the following eigenvalue problems: for each $0 < a < b \leq R$,

(25) $\mathcal{F}[u] + \mu u = 0$ a.e. in $(a, b)$, $u \in W^{2,q}_{r}(a, b)$, $u > 0$ in $(a, b)$, $u(a) = u(b) = 0$.

Now we define $\mathcal{H}$ by

$$\mathcal{H}(m, p, u, x) := \mathcal{F}(m, p/x, p, u, x) : \mathbb{R}^{3} \times (a, b) \to \mathbb{R}. $$

Note that $\mathcal{H}$ satisfies (F1)-(F3) in $(a, b)$ and for $u(x) = u(|x|)$, $\mathcal{F}[u] + \mu u = 0$ in $(a, b)$ if and only if $\mathcal{H}(u''(x), u'(x), u(x), x) + \mu u(x) = 0$ in $(a, b)$. Thus we can apply Theorem 1.1 and obtain the following result.

**Proposition 7.2.** For any $0 < a < b \leq R$, (25) has positive and negative eigenpairs $(\mu_{D}^{\pm}, \varphi_{D}^{\pm})$ which are simple. If we denote the unique positive and negative eigenvalues on $[a, b]$ by $\mu_{D}^{+}(a, b)$ and $\mu_{D}^{-}(a, b)$, then

(i) $\mu_{D}^{+}(a_{1}, b_{1}) < \mu_{D}^{+}(a_{2}, b_{2})$ if $[a_{2}, b_{2}] \subset [a_{1}, b_{1}]$ and $[a_{2}, b_{2}] \neq [a_{1}, b_{1}].$

(ii) The maps $(a, b) \mapsto \mu_{D}^{\pm}(a, b) : \{(a, b) \in \mathbb{R}^{2} : 0 < a < b < R\} \to \mathbb{R}$ are continuous.

(iii) As $\varepsilon \to 0$, $\inf\{\mu_{D}^{\pm}(a, b) : 0 < a < b \leq R, b - a < \varepsilon\} \to \infty.$

The following two lemmas can be shown as in [13], so we omit a proof.

**Lemma 7.3.** Let $h : (0, R) \to (0, R)$ be a nondecreasing continuous function such that $f(s) \leq s$ in $(0, R)$. Then there exists unique functions $\tau^{\pm} : (0, R) \to (0, R)$ such that $\tau^{\pm}(t) < t$ and $\mu_{D}^{\pm}(0, h(\tau^{\pm}(t))) = \mu_{D}^{\pm}(\tau^{\pm}(t), t)$ for each $t \in (0, R]$. Furthermore, the functions $\tau^{\pm}$ are continuous and strictly increasing in $(0, R]$.

**Lemma 7.4.** Let $n \in \mathbb{N}$ and $(r_{j})_{j=0}^{n} : (s_{j})_{j=0}^{n} \subset [0, R]$ be increasing sequences such that $r_{0} = s_{0} = 0$ and $r_{n} = s_{n} = R$. Then there exist $j, k \in \{1, \ldots, n\}$ such that $[r_{j-1}, r_{j}] \subset [s_{j-1}, s_{j}]$ and $[r_{k-1}, r_{k}] \subset [s_{k-1}, s_{k}].$

Now we give a proof of Theorem 7.1.

**Proof of Theorem 7.1.** As in Proposition 6.1, it is enough to show only for $(\mu_{n}^{+}, \varphi_{n}^{+})$. First we treat the existence.

We show that for any $n \in \mathbb{N}$, there is a sequence $(r_{n,j}(t))_{j=1}^{n}$ of functions on $(0, R]$ such that

(26) $a < r_{n,1}(t) < r_{n,2}(t) < \ldots < r_{n,n}(t) = t$ for every $t \in (0, R],$

(27) $r_{n,j}(t)$ is continuous and strictly increasing on $(0, R],$

(28) $\mu_{D}^{+}(r_{n,j-1}(t), r_{n,j}(t)) = \mu_{n}^{+}(0, r_{n,1}(t))$ for all $t \in (0, R]$ and $j \geq 2.$

Here $s_{j}$ stands for the symbol $+$ if $j$ is odd and $-$ if $j$ is even.

For $n = 1$, the function $r_{1,1}(t) = t$ clearly satisfies (26)–(28). We show by induction, so suppose that there is a sequence $(r_{n,j})_{j=1}^{n}$ satisfying (26)–(28). We apply Lemma 7.3 to obtain an increasing continuous function $\tau$ on $(0, R]$ such that $\tau(t) < t$ and $\mu_{D}^{+}(0, r_{n,1}(\tau(t))) = \mu_{n+1}^{+}(\tau(t), t)$ for all $t \in (0, R]$. Now define $r_{n+1,j}(t) = r_{n,j} \circ \tau(t)$ for every $1 \leq j \leq n$ and $r_{n+1,n+1}(t) = t$. Then it is easily seen that (26) and
(27) hold. Furthermore, since \( r_{n,n}(t) = t \) and \( \mu_{n}^{+}(0, r_{n,1}(t)) = \mu_{D}^{s_{j}}(r_{n,j-1}(t), r_{n,j}(t)) \) for each \( 2 \leq j \leq n \) and \( t \in (0, R] \), we have

\[
\mu_{D}^{s_{j}+1}(r_{n,n} \circ \tau(t), t) = \mu_{D}^{s_{j}+1}(0, r_{n,1} \circ \tau(t)) = \mu_{D}^{s_{j}}(r_{n,j-1} \circ \tau(t), r_{n,j} \circ \tau(t))
\]

for any \( t \in (0, R] \) and \( 2 \leq j \leq n \). Hence \((r_{n+1,j}(t))_{j=2}^{n+1}\) satisfies (26)–(28).

Now we prove the existence for \( n \geq 2 \). Set \( r_{n,0}^+ = 0, r_{n,j}^+ = r_{n,j}(R) \) for each \( j = 1, \ldots, n \) and \( \mu_{n}^{+} = \mu_{n}^{+}(0, r_{n,1}^+). \) Then by (28), \( \mu_{n}^{+} = \mu_{D}^{s_{j}}(r_{n,j-1}^+, r_{n,j}^+) \) holds for all \( 2 \leq j \leq n \). Let \( \varphi_{n,1} \in W_{r}^{2,q}(0, r_{n,1}^+) \) be a positive eigenfunction corresponding to \( \mu_{n}^{+}(r_{n,0}^+, r_{n,1}^+) \) and \( \varphi_{n,j} \in W_{r}^{2,q}(r_{n,j-1}^+, r_{n,j}^+) \) an eigenfunction corresponding to \( \mu_{D}^{s_{j}}(r_{n,j-1}^+, r_{n,j}^+) \). Then we obtain \(-1)^{j-1}\varphi_{n,j} > 0 \) in \((r_{n,j-1}^+, r_{n,j}^+)\) and

\[
(-1)^{j}\varphi_{n,j}(r_{n,j}^+ - r_{n,j}^-) > 0 \quad \text{and} \quad (-1)^{j-1}\varphi_{n,k}(r_{n,k}^+ - 0) > 0
\]

for every \( 1 \leq j \leq n \) and \( 2 \leq k \leq n \). Thus we can find a sequence \((\theta_{j})_{j=1}^{n}\) of positive numbers such that

\[
\theta_{1} = 1, \quad \theta_{j-1}\varphi_{j-1}^{+}(r_{n,j-1}^+ - 0) = \theta_{j}\varphi_{j}^{+}(r_{n,j}^+ - 0) \quad \text{for any} \quad j = 2, \ldots, n.
\]

Define \( \varphi_{n}^{+} \) by

\[
\varphi_{n}^{+}(r) := \theta_{j}\varphi_{n,j}(r) \quad \text{if} \quad r \in [r_{n,j}^+, r_{n,j}^-] \text{ and } 1 \leq r \leq n.
\]

From (30), \( \varphi \in W_{r}^{2,q}(0, R) \) and \((\mu_{n}^{+}, \varphi_{n}^{+})\) is an eigenpair of (3) with \((-1)^{j-1}\varphi_{n}^{+}(r) > 0 \) in \((r_{n,j-1}^+, r_{n,j}^-) \) and \( \varphi_{n}^{+}(0) > 0 \).

Next we deal with the assertion (ii). When \( n = 1 \), the claim holds from Proposition 6.5, hence let \( n \geq 2 \) and \((\mu, \varphi) \in \mathbb{R} \times W_{r}^{2,q}(0, R) \) be an eigenpair of (3) with \( n - 1 \) zeroes \( 0 < t_{1} < \cdots < t_{n-1} < R \). Set \( t_{0} = 0 \) and \( t_{n} = R \). It is enough to show the claim in the case where \( \varphi > 0 \) in \([t_{0}, t_{1}]\).

By Lemma 7.4, there exist \( j, k \in \{1, \ldots, n\} \) satisfying \([r_{n,j-1}, r_{n,j}] \subset [t_{j-1}, t_{j}] \) and \([t_{k-1}, t_{k}] \subset [r_{n,k-1}, r_{n,k}] \). Note that \((-1)^{m-1}\varphi_{n,m} > 0 \) in \((r_{n,m-1}, r_{n,m})\) and \((-1)^{m-1}\varphi > 0 \) in \((t_{m-1}, t_{m})\) for all \( 1 \leq m \leq n \). We also remark that \( \mu = \mu_{N}^{+}(0, t_{1}) = \mu_{D}^{s_{j}}(t_{m-1}, t_{m}) \) and \( \varphi \) is an eigenfunction on \((0, t_{1})\) and \((t_{m-1}, t_{m})\) corresponding to \( \mu_{N}^{+}(0, t_{1}) \) and \( \mu_{D}^{s_{j}}(t_{m-1}, t_{m}) \) for \( 2 \leq m \leq n \). Hence by Propositions 6.6 and 7.2, we obtain \( \mu_{n}^{+} \leq \mu \) and \( \mu \leq \mu_{n}^{+} \), which implies \( \mu = \mu_{n}^{+} \). Furthermore, again by Propositions 6.6 and 7.2, we see that \( r_{n,j} = t_{j} \) for all \( 1 \leq j \leq n \) and there exists a sequence \((\theta_{j})_{j=1}^{n}\) of positive numbers satisfying \( \varphi = \theta_{j}\varphi_{n,j} \) on \([r_{n,j-1}, r_{n,j}]\) for each \( j = 1, \ldots, n \). Noting that \( \varphi \) is of class \( C^{1} \) and (29), \( \theta_{j} \equiv 0 > 0 \) holds for \( 1 \leq j \leq n \).

This completes the proof.

\( \square \)

**Proof of Theorem 1.2.** By Proposition 7.1, it is sufficient to prove the completeness of \((\mu_{n}^{\pm}, \varphi_{n}^{\pm})_{n=1}^{\infty}\). Let \((\mu, \varphi) \in \mathbb{R} \times W_{r}^{2,q}(0, R) \) be an eigenpair of (3). Then in view of Proposition 7.1, we only show that \( \varphi(0) \neq 0 \) and \( \varphi \) has finitely many zeroes in \((0, R)\).

First we show that there is no accumulation point in \((0, R)\) of zeroes of \( \varphi \). We argue by contradiction and suppose that \((r_{n})_{n=1}^{\infty}\) is a sequence of zeroes of \( \varphi \) satisfying \( r_{n} \neq r_{m} \) if \( n \neq m \) and \( r_{n} \to r_{0} \in (0, R) \). Then by Rolle's theorem, we see that
\( \varphi(r_0) = \varphi'(r_0) = 0 \). Then \( \varphi \equiv 0 \) holds from Proposition 2.3, which is a contradiction. Hence there is no accumulation point of zeroes of \( \varphi \) in \((0, R)\).

Next we consider the case where 0 is an accumulation point of zeroes of \( \varphi \). Let \((r_n)_{n=1}^{\infty}\) be a sequence of zeroes of \( \varphi \). From the above argument, we may assume \( r_1 > r_2 > \ldots > 0 \). Now choose \( n \) so large that

\[
C_3 r_n^{(2q-N)/(q-1)} \| (\gamma + |\mu|) \|_{L^q_r(0,R)} < 1
\]

where \( C_3 \) appears in Lemma 3.5. We may also suppose that \( \varphi > 0 \) in \((r_{n+1}, r_n)\) and \( \varphi(t_n) = \max_{[r_{n+1}, r_n]} \varphi = 1 \) for some \( t_n \in (r_{n+1}, r_n) \). Then \( \varphi'(t_n) = 0, \varphi(r_n) = 0 \) and \( \mathcal{P}^+[\varphi] + \beta |\varphi'| + (\gamma + |\mu|) \varphi \geq 0 \) a.e. in \((t_n, r_n)\). It follows from Lemma 3.5 and the choice of \( r_n \) that

\[
1 = \max_{[t_n, r_n]} \varphi \leq C_3 r_n^{(2q-N)/(q-1)} \| (\gamma + |\mu|) \varphi \|_{L^q_r(t_n, r_n)}
\]

\[
\leq C_3 r_n^{(2q-N)/q} \| (\gamma + |\mu|) \|_{L^q_r(0,R)} < 1,
\]

which is a contradiction. Thus \( \varphi \) has finitely many zeroes in \([0, R]\).

Lastly we show \( \varphi(0) > 0 \). If \( \varphi(0) = 0 \), then \( \mathcal{P}^-[\varphi] - \beta |\varphi| - (\gamma + |\mu|) \varphi \leq 0 \) a.e. in \((0, s)\) for sufficiently small \( s > 0 \) since \( \varphi \) has finitely many zeroes. Then Proposition 4.2 yields \( \varphi \equiv 0 \) on \([0, s]\), which is a contradiction. Hence \( \varphi(0) > 0 \) holds and we complete the proof. \( \square \)

References


